1. The graph[1] of Europe $\mathcal{G}^* = \langle V, E \rangle$ is defined as follows: each vertex $v \in V$ is a Europe country[2]; two vertices are adjacent ($\{u, v\} \in E$) if the corresponding countries share a land border. Let $\mathcal{G}$ be a maximum connected component of $\mathcal{G}^*$.
   (a) Prove that $\mathcal{G}^*$ is planar by drawing it on a plane without an intersection of edges.
   (b) Find $|V|$, $|E|$, $\delta(\mathcal{G})$, $\Delta(\mathcal{G})$, $\mathrm{rad}(\mathcal{G})$, $\mathrm{diam}(\mathcal{G})$, $\mathrm{girth}(\mathcal{G})$, $\mathrm{center}(\mathcal{G})$, $\kappa(\mathcal{G})$, $\lambda(\mathcal{G})$.
   (c) Find the minimum vertex coloring $Z : V \to \mathbb{N}$ of $\mathcal{G}$.
   (d) Find the minimum edge coloring $X : E \to \mathbb{N}$ of $\mathcal{G}$.
   (e) Find the maximum clique $Q \subseteq V$ of $\mathcal{G}$.
   (f) Find the maximum stable set $S \subseteq V$ of $\mathcal{G}$.
   (g) Find the maximum matching $M \subseteq E$ of $\mathcal{G}$.
   (h) Find the minimum vertex cover $R \subseteq V$ of $\mathcal{G}$.
   (i) Find the minimum edge cover $F \subseteq E$ of $\mathcal{G}$.
   (j) Find the shortest closed path (circuit) W that visits every vertex of $\mathcal{G}$.
   (k) Find the shortest closed path (circuit) U that visits every edge of $\mathcal{G}$.
   (l) Find all 2-vertex-connected components (blocks) and draw a block-cut tree of $\mathcal{G}^*$.
   (m) Find all 2-edge-connected components of $\mathcal{G}^*$.
   (n) Construct an SPQR tree of the largest biconnected component of $\mathcal{G}$.
   (o) Add the weight function $w : E \to \mathbb{R}$ denoting the distance[3] between capitals. Find the minimum (*w.r.t.* the total weight of edges) spanning tree T for the maximum connected component of the weighted Europe graph $\mathcal{G}^*_w = \langle V, E, w \rangle$.
   (p) Find $\mathrm{centroid}(T)$ (*w.r.t.* the edge weight function w).
   (q) Construct the Prüfer code for T.

2. Prove *rigorously* the following theorems:

   Theorem 1. (Triangle Inequality) For any connected graph $G = \langle V, E \rangle$:
   $$\forall x, y, z \in V : \mathrm{dist}(x, y) + \mathrm{dist}(y, z) \geq \mathrm{dist}(x, z)$$

   Theorem 2. For any connected graph G: $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\,\mathrm{rad}(G)$.

   Theorem 3. A connected graph $G = \langle V, E \rangle$ is a tree (*i.e.* acyclic graph) *iff* $|E| = |V| - 1$.

   Theorem 4. Given a connected graph G with n vertices, if $\delta(G) \geq \lfloor n/2 \rfloor$, then $\lambda(G) = \delta(G)$.

   Theorem 5. Every block of a block graph[4] is a clique.

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[1] Hereinafter, "graphs" are "simple undirected and unweighted", unless stated otherwise.

[2] Since the absolute geopolitical correctness is not necessary to accomplish this task, you can simply use https://simple.wikipedia.org/wiki/List_of_European_countries or any similar source as a reference.

[3] You can choose a geodesic or a road distance, to your preference.

[4] A block graph $H = \mathrm{B}(G)$ is an intersection graph of all blocks (biconnected components) of G, *i.e.* each vertex $v \in V(H)$ corresponds to a block of G, and there is an edge $\{v, u\} \in E(H)$ iff "blocks" v and u share a cut vertex.