Computational Finance
Exercise Set I

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Exercise 1

First, I am going to prove the property $\mathbb{E}[(X-\mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

$$\begin{aligned}\mathbb{E}[(X-\mathbb{E}[X])^2] &= \mathbb{E}\big[X^2 - 2\mathbb{E}[X]X + \mathbb{E}[X]^2\big] \\ &= \mathbb{E}[X^2] + \mathbb{E}\big[-2E[X]X\big] + \mathbb{E}\big[\mathbb{E}[X]^2\big] \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[X]^2 \\ &= \mathbb{E}[X^2] - \mathbb{E}[X]^2.\end{aligned}$$

In the second line, we used the property of expectation values $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$. In the third line, we used the property $\mathbb{E}[\alpha X] = \alpha\mathbb{E}[X]$.

Now, with this property, it is easy to show that $\mathbb{V}ar[\alpha X] = \alpha\mathbb{V}ar[X]$. By definition, $\mathbb{V}ar[X] = \mathbb{E}[(X-\mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$. Therefore,

$$\begin{aligned}\mathbb{V}ar[\alpha X] &= \mathbb{E}[(\alpha X)^2] - \mathbb{E}[\alpha X]^2 \\ &= \mathbb{E}[\alpha^2 X^2] - (\alpha\mathbb{E}[X])^2 \\ &= \alpha^2\mathbb{E}[X^2] - \alpha^2\mathbb{E}[X]^2 \\ &= \alpha^2(\mathbb{E}[X^2] - \mathbb{E}[X]^2) \\ &= \alpha^2\mathbb{V}ar[X],\end{aligned}$$

where, again, we have used the two properties of expectation values mentioned above.

Exercise 6

The goal of this exercise is to prove that the standard distribution function $F_{\mathcal{N}(0,1)}(x)$, define by

$$F_{\mathcal{N}(0,1)}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} ds\ \exp\left(\frac{-s^2}{2}\right), \tag{1}$$

can be calculated with the so-called error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_{0}^{x} ds\ \exp\big(-s^2\big), \tag{2}$$

with the help of the relationship

$$F_{\mathcal{N}(0,1)}(x) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right] \tag{3}$$

To demonstrate this, one can start from the very definition of the standard distribution function

$$
\begin{aligned}
F_{\mathcal{N}(0,1)} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} ds \;\exp\left(\frac{-s^2}{2}\right) \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} ds \;\exp\left(\frac{-s^2}{2}\right) + \int_{0}^{x} ds \;\exp\left(\frac{-s^2}{2}\right) \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\sqrt{\frac{\pi}{2}} + \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) \right] \\
&= \frac{1}{2} \left[1 + \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) \right] \qquad (4)
\end{aligned}
$$