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- If these spins only interact with their nearest neighbours and are placed in an external field, we can write a simple Hamiltonian for the system as

$$\mathcal{H} = -\frac{J}{2}\sum_{i=1}^{N}{\sum_{j}}' S_i S_j - H\sum_{i=1}^{N} S_i,$$

  - J is the coupling between the spins
  - H is an external field.

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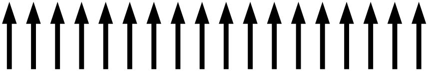
- $J > 0$ the first term is minimized when the spins on sites i and j are pointing in the same direction (ferromagnet)



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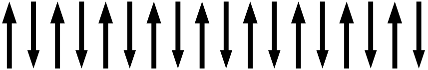

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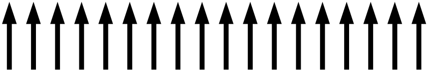

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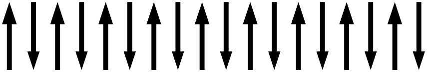

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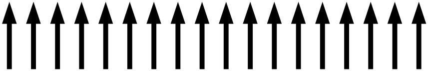

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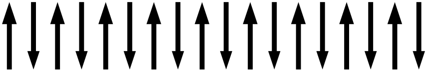

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- Finite temperatures: is there a phase transition? Does it depend on dimension?

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Clearly, F is perfectly analytic so there is *no phase transition in the one dimensional Ising model with* $H=0$.

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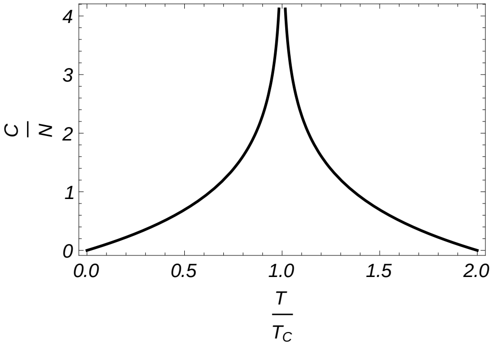
- Second order phase transition with temperature $T_C = 2.27J/k_B$
- Not analytically solved in 3d, but transition temperature found numerically to be $T_C = 4.52J/k_B$

Heat Capacity - 2d Zero field Ising Model

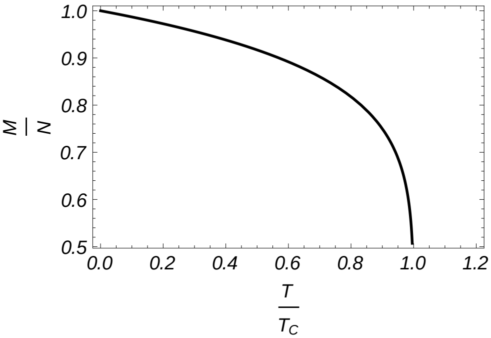
$$\frac{C}{N} = \frac{1}{N}\left(\frac{\partial \langle E \rangle}{\partial T}\right)_{H=0} \sim \frac{8k_B}{\pi}(\beta J)^2 \ln\left|\frac{1}{T - T_C}\right|. \quad (7)$$



The magnetization for $T < T_C$ is

$$\frac{M}{N} \propto (T_C - T)^{\beta}, \quad T < T_c \tag{8}$$

with $\beta = 1/8$.



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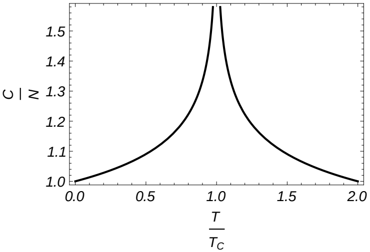

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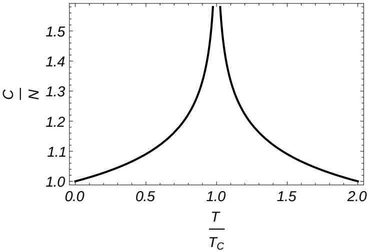

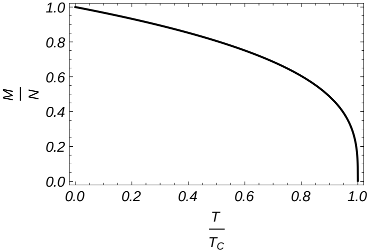

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- α and β are referred to as *critical exponents*
- critical exponents typically used to characterize the phase transition.

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- Need to define a Markov process such that it obeys the correct statistics - the states end up created according to e.g. the Boltzmann distribution

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- Unfortunately, satisfying this equation does not ensure that the probability distribution will go to p_{μ} from any state of the system if we run the process for long enough

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- Remove possibility of spurious limit cycles by introducing **detailed balance condition**

$$p_\mu P(\mu \to \nu) = p_\nu P(\nu \to \mu) \tag{18}$$

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- Want to find the choice that maximizes the acceptance ratios (and thus makes the algorithm most efficient)

Monte Carlo for Ising Model: Single Flip

Solution: The **Metropolis algorithm**

$$A(\mu \to \nu) = \begin{cases} e^{-\beta(E_\nu - E_\mu)} & E_\nu - E_\mu > 0 \\ 1 & \text{otherwise} \end{cases} \tag{25}$$

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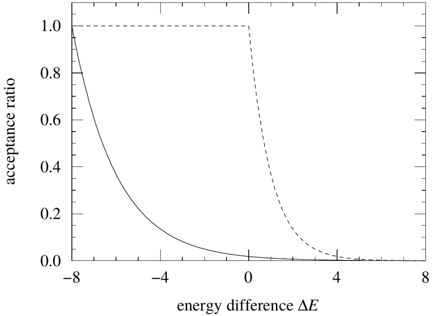

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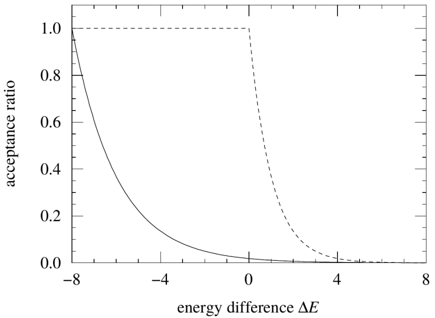

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Example: energy autocorrelation function:

$$A(\delta t) = \frac{\left\langle (E(t) - \bar{E})(E(t+\delta t) - \bar{E}) \right\rangle_t}{\left\langle (E(t) - \bar{E})^2 \right\rangle_t} \tag{26}$$

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- See section 3.1.1. in the book on blackboard for some tricks associated with implementing this algorithm