Math 5620 Numerical Solution of Differential Equation Homework 8

Staple your solutions to this sheet Score:________

Problem 1: Write separate routines to implement the following methods for the approximate solution of the following hyperbolic partial differential equation.

1. Upwinding,
2. the Lax-Wendorff Method, and
3. the Warming and Beam Method

Test the routines on

$$u_t + 2\ u_x = 0$$

Problem 2: Write out the details on the vonNeumann stability analysis for both the Lax-Wendroff method and the Warming and Beam method.

George Staples
P1-
https://github.com/georgest347/MATH-5620/blob/master/softwareManual/HW8/hyperLWM.md
https://github.com/georgest347/MATH-5620/blob/master/softwareManual/HW8/hyperSoln.md
https://github.com/georgest347/MATH-5620/blob/master/softwareManual/HW8/hyperWB.md
https://github.com/georgest347/MATH-5620/blob/master/softwareManual/HW8/initalCon.md
https://github.com/georgest347/MATH-5620/blob/master/softwareManual/HW8/step.md
https://github.com/georgest347/MATH-5620/blob/master/softwareManual/HW8/upwind.md
All methods were used to produce data. This data was compared to the exact solution. Figures 1-3 shows the Upwinding method plotted with the Exact Solution at different time step values.

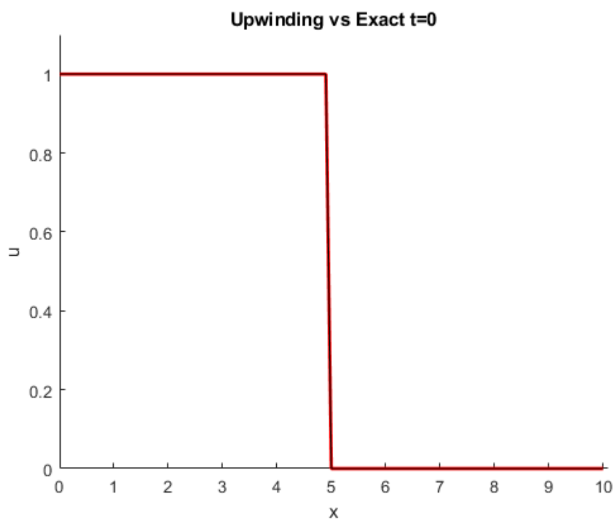
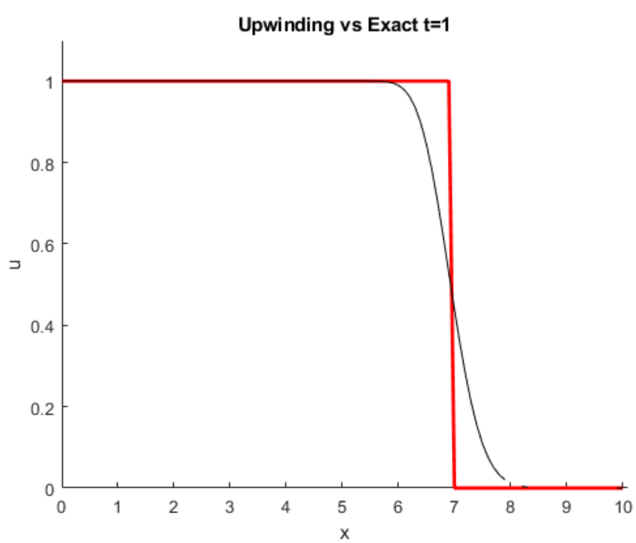


Figure 1: Upwinding solution vs Exact at t=0. This is the initial conditions for the Upwinding method.



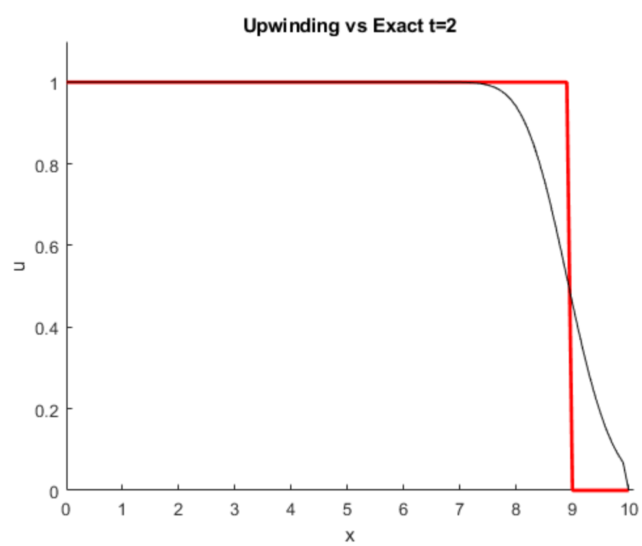


Figure 2(left) Figure 3(right): Upwinding at t=1. The Upwinding solution starts to smooth out the sharp corners of the wave. The “Smearing” increases as t increases. The Upwinding solution gets farther from the exact.

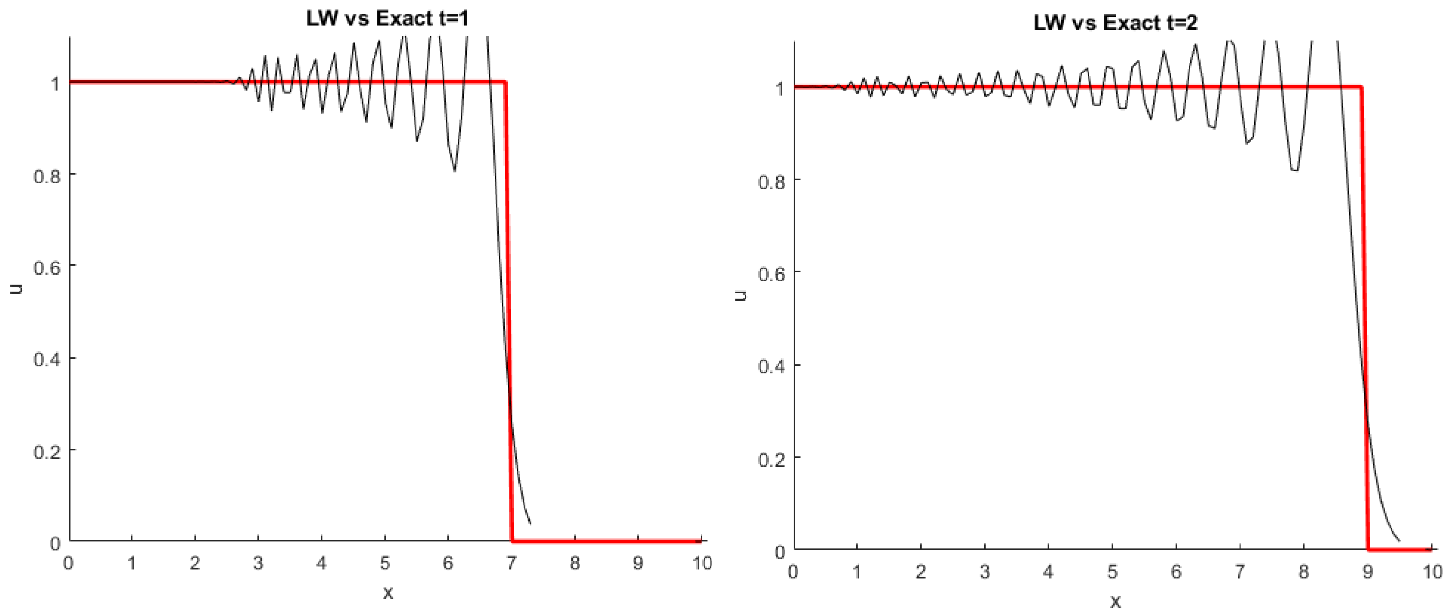


Figure 5(left) Figure 6 (right): Lax Wendroff Method at t=1. This method has numerical dispersion around the top of the wave. This is the oscillations about the solution. The Lax Wendroff method has less “smearing” than the Upwinding method. The Oscillations grow as t increases.

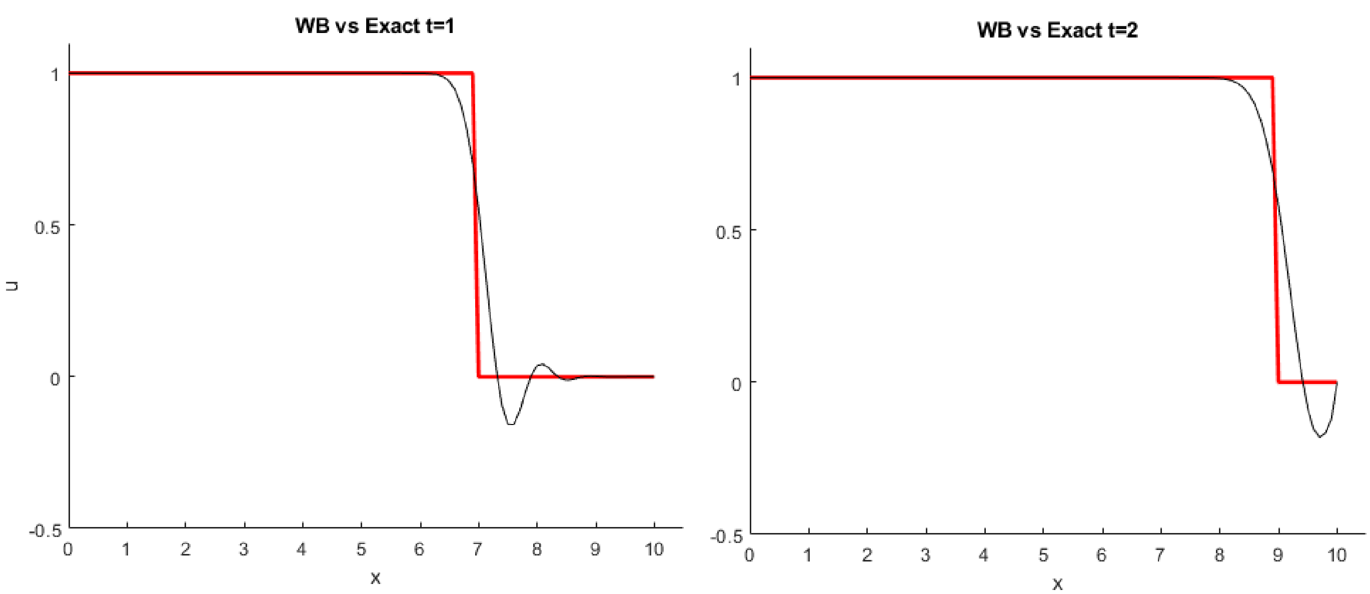


Figure 7(left) Figure 8(Right): Show how the Warming Beam method approximates the exact solution. The Warming and Beam solution has oscillations on the lower half of the wave.

#2

Von Neumann Stability for Lax-Wendroff

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{a^2k^2}{2h^2}\left(U_{j-1}^n - 2U_j^n + U_{j+1}^n\right) \quad 10.18 \ \boxed{1}_{pp} 209$$

$\boxed{1}$ "Finite Difference Methods for Ordinary and Partial Differential Equations" LeVeque

$U_j^n = \xi^n e^{i\theta x_j} \qquad \lambda = \frac{ak}{h}$

$$\xi^{n+1} e^{i\theta x_j} = \xi^n e^{i\theta x_j} - \frac{\lambda}{2}\left(\xi^n e^{i\theta x_{j+1}} - \xi^n e^{i\theta x_{j-1}}\right) + \frac{\lambda^2}{2}\left(\xi^n e^{i\theta x_{j-1}} - 2\xi^n e^{i\theta x_j} + \xi^n e^{i\theta x_{j+1}}\right)$$

Divide by $\xi^n e^{i\theta x_j}$

$$\frac{\xi^{n+1}}{\xi^n} = 1 - \frac{\lambda}{2}\left(\frac{e^{i\theta x_{j+1}}}{e^{i\theta x_j}} - \frac{e^{i\theta x_{j-1}}}{e^{i\theta x_j}}\right) + \frac{\lambda^2}{2}\left[\frac{e^{i\theta x_{j-1}}}{e^{i\theta x_j}} - 2 + \frac{e^{i\theta x_{j+1}}}{e^{i\theta x_j}}\right]$$

$$\xi = 1 - \frac{\lambda}{2}\overset{(a)}{\left(e^{i\theta\Delta x} - e^{-i\theta\Delta x}\right)} + \frac{\lambda^2}{2}\overset{(b)}{\left[e^{-i\theta\Delta x} - 2 + e^{i\theta\Delta x}\right]}$$

(b) $\lambda^2\left[\frac{e^{i\theta\Delta x} + e^{-i\theta\Delta x}}{2} - 1\right] \Rightarrow \lambda^2\left[\cos(\theta\Delta x) - 1\right] \Rightarrow -\lambda^2\left[1 - \cos(\theta\Delta x)\right]$

$e^{i\theta\Delta x} = \cos(\theta\Delta x) + i\sin(\theta\Delta x)$

$e^{-i\theta\Delta x} = \cos(\theta\Delta x) - i\sin(\theta\Delta x)$

(a) $-\frac{\lambda}{2}\left(\cos(\theta\Delta x) + i\sin(\theta\Delta x) - \left[\cos(\theta\Delta x) - i\sin(\theta\Delta x)\right]\right)$

$$-\frac{\lambda}{2}\left(\cancel{\cos(\theta\Delta x)} + i\sin(\theta\Delta x) - \overset{0}{\cancel{\cos(\theta\Delta x)}} + i\sin(\theta\Delta x)\right) \Rightarrow -\frac{\lambda}{\cancel{2}}\left(\overset{1}{\cancel{2}}\, i\sin(\theta\Delta x)\right) \Rightarrow -\lambda i\sin(\theta\Delta x)$$

$$\xi = 1 - \lambda i\sin(\theta\Delta x) - \lambda^2\left[1 - \cos(\theta\Delta x)\right]$$

$|\xi|^2$ Satisfied $\quad |\xi|^2 = 1 - \lambda^2(1-\lambda^2)\left[1 - \cos^2(\theta\Delta x)\right]$

if $\quad \lambda^2 \le 1 \quad \therefore \boxed{\left(\frac{ak}{h}\right)^2 \le 1}$ ← Stability Condition

#2 Cont

Warming + Beam stability using Von Neumann

for $a>0$

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(3U_j^n - 4U_{j-1}^n + U_{j-2}^n\right) + \frac{a^2k^2}{2h^2}\left(U_j^n - 2U_{j-1}^n + U_{j-2}^n\right) \quad 10.26 \ \boxed{1} \ pp\ 212$$

$\boxed{\text{Define}}$

$$U_j^n = \xi^n e^{i\theta x_j} \qquad \lambda = \frac{ak}{h} \qquad k = \Delta t \quad h = \Delta x$$

$$\xi^{n+1} e^{i\theta x_j} = \xi^n e^{i\theta x_j} - \frac{\lambda}{2}\left[3\xi^n e^{i\theta x_j} - 4\xi^n e^{i\theta x_{j-1}} + \xi^n e^{i\theta x_{j-2}}\right] + \frac{\lambda^2}{2}\left[\xi^n e^{i\theta x_j} - 2\xi^n e^{i\theta x_{j-1}} + \xi^n e^{i\theta x_{j-2}}\right]$$

Divide by $\xi^n e^{i\theta x_j}$

$$\frac{\xi^{n+1}}{\xi^n} = 1 - \frac{\lambda}{2}\left[3 - 4\frac{e^{i\theta x_{j-1}}}{e^{i\theta x_j}} + \frac{e^{i\theta x_{j-2}}}{e^{i\theta x_j}}\right] + \frac{\lambda^2}{2}\left[1 - 2\frac{e^{i\theta x_{j-1}}}{e^{i\theta x_j}} + \frac{e^{i\theta x_{j-2}}}{e^{i\theta x_j}}\right]$$

$$\xi = 1 - \frac{\lambda}{2}\left[3 - 4e^{-i\theta h} + e^{-2i\theta h}\right] + \frac{\lambda^2}{2}\left[1 - 2e^{-i\theta h} + e^{-2i\theta h}\right]$$

$\boxed{\text{Define}}$

$$e^{+ix} = \cos x + i\sin x$$

$$e^{-ix} = \cos x - i\sin x$$

$$\xi = 1 - \overset{ⓐ}{\frac{\lambda}{2}}\left[3 - 4\left[\cos(\theta h) - i\sin(\theta h)\right] + \cos(2\theta h) - i\sin(2\theta h)\right] + \dots$$

$$\overset{ⓑ}{\frac{\lambda^2}{2}}\left[1 - 2\left[\cos(\theta h) - i\sin(\theta h)\right] + \cos(2\theta h) - i\sin(2\theta h)\right]$$

Scheme is stable if $|\xi|^2 < 1$

$$\left|\frac{ka}{h}\right| \le 2 \longleftarrow \text{Stability Condition}$$