Some Derivation

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Abstract

Some ideas

I. VARYING THE (INVERSE) TETRAD

When we are in curved space/coordinate system, the tetrad itself is now a field. We also need to vary this "field" just like any other field to get its constraint just like equation G4, $\partial\mathcal{L}/\partial\overline{\psi} = 0$. In Neil's paper, what's being used is actually the inverse tetrad, e_I^μ, rather than the tetrad. The constraint for the inverse tetrad will be called c_e here.

First we vary the lagrangian with respect of the inverse tetrad, but before we do that we'll rewrite the root of the determinant of the metric

$$\begin{aligned} g &= \sqrt{-\det(g_{\mu\nu})} = \sqrt{-\det(\eta_{IJ}e_\mu^I e_\nu^J)} \\ &= \sqrt{-\det(\eta_{IJ})\det(e_\mu^I)\det(e_\nu^J)} \\ &= \sqrt{-(-1)|e|^2} \\ g &= |e| \end{aligned}$$

The lagrangian is then given by $L = g\mathcal{L} \to |e|\mathcal{L}$ while the variation of $|e|$, *i.e.* $\delta|e|$, is given by $\delta|e| = -|e|e_\mu^I\delta e_I^\mu$, the minus sign is there because we are varying the inverse tetrad instead of the tetrad. Thus the variation of the lagrangian due to the variation in the inverse tetrad is given by

$$\begin{aligned} L + \delta L = (|e| - |e|e_\beta^J\delta e_J^\beta) \Big\{ & -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu\partial^\nu A^\mu \\ & + i\overline{\psi}(\gamma^I e_I^\mu + \gamma^I\delta e_I^\mu)\partial_\mu\psi \\ & - \overline{\psi}[(\gamma^I e_I^\mu + \gamma^I\delta e_I^\mu)A_\mu + m]\psi \Big\} \\ = |e| \Big\{ & -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu\partial^\nu A^\mu + i\overline{\psi}(\gamma^I e_I^\mu)\partial_\mu\psi - \overline{\psi}[(\gamma^I e_I^\mu)A_\mu + m]\psi \Big\} \\ -|e|e_\beta^J\delta e_J^\beta \Big\{ & -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu\partial^\nu A^\mu + i\overline{\psi}(\gamma^I e_I^\mu)\partial_\mu\psi - \overline{\psi}[(\gamma^I e_I^\mu)A_\mu + m]\psi \Big\} \\ +|e| & \left\{ i\overline{\psi}(\gamma^I\delta e_I^\mu)\partial_\mu\psi - \overline{\psi}(\gamma^I\delta e_I^\mu)A_\mu\psi \right\} + O(\delta^2 e_I^\mu) \\ = L - & |e|e_\beta^J\delta e_J^\beta\mathcal{L} + |e|\delta e_J^\beta \left\{ i\overline{\psi}\gamma^J\partial_\beta\psi - \overline{\psi}\gamma^J A_\beta\psi \right\} \end{aligned}$$

And so the constraint for the inverse tetrad is given by

$$c_e = \frac{\delta L}{\delta e_J^\beta} = -|e|e_\beta^J\mathcal{L} + |e| \left\{ i\overline{\psi}\gamma^J\partial_\beta\psi - \overline{\psi}\gamma^J A_\beta\psi \right\}$$

but $c_e = 0$ since there's no derivative of the inverse tetrad in the lagrangian (as is the case with $\overline{\psi}$) and hence

$$-|e|e^J_\beta \mathcal{L} + |e| \left\{ i\overline{\psi}\gamma^J \partial_\beta \psi - \overline{\psi}\gamma^J A_\beta \psi \right\} = 0$$

And this is exactly equation I64, if I rewrite it in my notation, which is the standard GR notation by the way :) I'll reproduce equation I64 here

$$-(\partial_\phi g)\mathcal{L} - ig\overline{\psi}\gamma^\alpha \left(\partial_\phi \frac{\partial r^\mu}{\partial \xi^\alpha} \right) \partial_\mu \psi + eg\overline{\psi}\gamma^\alpha \left(\partial_\phi \frac{\partial r^\mu}{\partial \xi^\alpha} \right) A_\mu \psi$$

The translation we need is

- $\partial_\phi g \neq 0$ as explained above but instead $\partial_\phi g = \partial_\phi |e| = -|e| e^I_\mu \partial_\phi e^\mu_I$
- Latin indices, $I, J, K, ...$ always indicate flat space while greek indices indicate curved space/coordinate system, thus $\gamma^\alpha \to \gamma^I$
- $\frac{\partial r^\mu}{\partial \xi^\alpha}$ is just the inverse tetrad e^μ_I
- set the electron charge $e \to 1$ to avoid confusion with the tetrad
- use the determinant of the tetrad $|e|$ instead of the root of the determinant of the metric g

Applying these trivial changes to equation I64

$$\begin{aligned}
\text{Eq I64} \to |e|e^I_\mu \partial_\phi e^\mu_I \mathcal{L} - i|e|\overline{\psi}\gamma^I \left(\partial_\phi e^\mu_I\right) \partial_\mu \psi + |e|\overline{\psi}\gamma^I \left(\partial_\phi e^\mu_I\right) A_\mu \psi \\
= -\partial_\phi e^\mu_I \left(-|e|e^I_\mu \mathcal{L} + i|e| \left\{ \overline{\psi}\gamma^I \partial_\mu \psi - \overline{\psi}\gamma^I A_\mu \psi \right\} \right) \\
= -\partial_\phi e^\mu_I \left\{ c_e \right\} \\
= 0
\end{aligned}$$

The Hamiltonian is thus conserved under ϕ (azimuthal?) rotations.