Jumbled up thoughts

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Abstract

Stray thoughts of a stray man. This is a collection of things that I thought was fun to do and also things that I always come back to after forgetting it :) There's no order or structure here just whatever my mind facies at the moment.

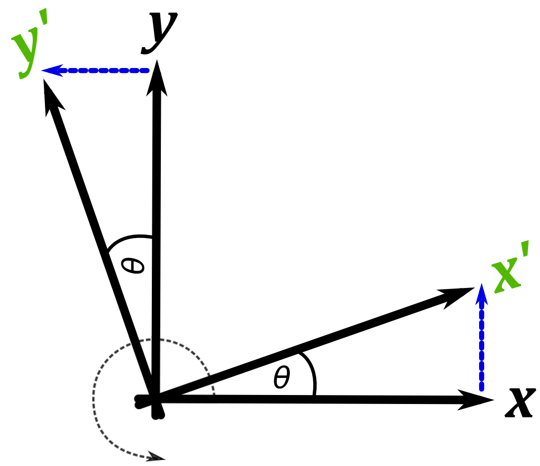

FIG. 1: A rotation of the coordinate axes around the $\hat{z}$ axis.

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How does a field and the associated Lagrangian change when we perform a rotation?

Intuitively say we perform an active rotation on the axes themselves and for simplicity we take a small angle rotation around $\hat{z}$, we'll see that $\hat{x}$ will get a contribution in $+\hat{y}$ direction and $\hat{y}$ will get a contribution in $-\hat{x}$ direction, see Fig. 1, which means

$$\vec{\delta}\hat{x} \Rightarrow +\hat{y} = \hat{z} \times \hat{x}$$
$$\vec{\delta}\hat{y} \Rightarrow -\hat{x} = \hat{z} \times \hat{y}$$

Note that this only works for infinitesimal rotation since for a finite rotation there will be contributions from multiple axes, *e.g.* $\vec{\delta}x \propto \hat{y} - \hat{x}$, if you can't see this just remember the rotation matrix in 2 dimension.

How about a generic rotation axis? say we pick an arbitrary rotation axis $\vec{r}$ and we want to rotate a vector $\vec{A}$, remember that the magnitude of $\vec{r}$ is the amount of rotation, *i.e.*

$$\vec{r} = a\hat{x} + b\hat{y} + c\hat{z} \longrightarrow |\vec{r}| = \sqrt{a^2 + b^2 + c^2} = \epsilon$$

Thus the change in $\vec{A}$ is given by

$$\vec{\delta}\vec{A} = \vec{r} \times \vec{A} = a(\hat{x} \times \vec{A}) + b(\hat{y} \times \vec{A}) + c(\hat{z} \times \vec{A})$$

Here we see why it only works for infinitesimal rotations since rotations w.r.t different axes do not commute but they only contribute at ϵ^2.

What if instead of a vector we have a scalar A? First we need to convert it into a vector, but how? there are many ways of doing it. If we look back to translation $A(x) \to A(x+\epsilon)$ the transformation we get is $A \to A + \epsilon\partial A$. Now, we just need to turn this into a vector, the most obvious way is $A \to A + \vec{\epsilon}\cdot(\vec{\nabla}A)$ (note: we always need a derivative to calculate δA thanks to taylor expansion).

Using the formula we have above

$$\vec{\delta}(\nabla A) = \vec{r} \times \nabla A = a(\hat{x} \times \nabla A) + b(\hat{y} \times \nabla A) + c(\hat{z} \times \nabla A)$$

however the result we have is a vector we need a scalar since A was initially a scalar. The solution is just to take a dot product with $\vec{x} = \frac{1}{\sqrt{3}}(\hat{x} + \hat{y} + \hat{z})$

$$\delta A = \vec{x} \cdot (\vec{r} \times \nabla A) = -\vec{r} \cdot (\vec{x} \times \nabla A)$$

by using a triple scalar product rule, the thing here now is that I don't know how to get rid of that minus sign LOL

So for a general scalar field you'll get $\delta\phi = \vec{\epsilon}\cdot(\vec{x} \times \nabla)\phi$, sans a minus sign, where $\vec{x}$ is just the coordinate axes and $\vec{\epsilon}$ is the magnitude and axis of rotation.

Now, how does the lagrangian change? Well, the lagrangian is also a scalar quantity thus we can expect it to also change as $\delta\mathcal{L} = \vec{\epsilon}\cdot(\vec{x} \times \nabla)\mathcal{L}$.

The goal here is to show that $\delta\mathcal{L}$ is related to the stress energy tensor *of translation* and for simplicity we will ignore $\vec{\epsilon}\cdot$ for now

$$\begin{aligned}(\vec{x} \times \nabla)\mathcal{L} = \epsilon^{ijk}x_j\partial_k\mathcal{L} &= \partial_\mu(\epsilon^{ijk}x_j\delta^\mu_k\mathcal{L}) \\ &= \partial_\mu\mathcal{F}^{\mu i} \\ \to \mathcal{F}^{\mu i} &= \epsilon^{ijk}x_j\delta^\mu_k\mathcal{L}\end{aligned}$$

Since this is a symmetry the change must be a total derivative.

Now from the generic formula for energy stress tensor (under rotation)

$$\begin{aligned}\widetilde{T}^{\mu i} &= \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi - \mathcal{F}^{\mu i} \\ &= \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}(\vec{x}\times\vec{\nabla})\phi - \epsilon^{ijk}x_j\delta^\mu_k\mathcal{L} \\ &= \epsilon^{ijk}x_j\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_k\phi - \delta^\mu_k\mathcal{L}\right) = \epsilon^{ijk}x_j(T^\mu_k) \\ &= \vec{x}\times\vec{T}^\mu\end{aligned}$$

Notes on energy stress tensor: Under symmetry transformations, the lagrangian should change by a total derivative (since a symmetry should not change the physics)

$$\delta\mathcal{L} = \partial_\mu\mathcal{F}^\mu$$

However the transformation is also given by

$$\begin{aligned}\delta\mathcal{L} &= \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta(\partial_\mu\phi) \\ &= \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right)\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\mu(\delta\phi) \\ \partial_\mu\mathcal{F}^\mu &= \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi\right) \\ 0 &= \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi - \mathcal{F}^\mu\right) = \partial_\mu J^\mu \\ \rightarrow J^\mu &= \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi - \mathcal{F}^\mu\end{aligned}$$

where we have used the equation of motion on the first term going into the second line, this is actually tricky if we apply wrongly we will get something like (doing Integration by parts on the second term at the same time)

$$\begin{aligned}\delta\mathcal{L} &= \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta(\partial_\mu\phi) = \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right)\delta\phi \\ &= \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right)\delta\phi - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right)\delta\phi \\ &= 0 \quad ???\end{aligned}$$

LOL indeed but what went wrong? One, we can't really use IBP here since we are not

really under an integral. However we can just use the usual trick, $u(dv) = d(uv) - (du)v$

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta(\partial_\mu\phi) &= \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi\right) - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right)\delta\phi \\ &= \left\{\frac{\delta\mathcal{L}}{\delta\phi}\delta\phi - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right)\delta\phi\right\} + \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi\right) \\ &= 0 + \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi\right)\end{aligned}$$

again using the equations of motion for ϕ.

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In QFT, why does $\phi(x)$ create a particle at spacetime point x, *i.e.* $\phi(x)|0\rangle = |x\rangle$? For simplicity we'll set $t = 0 \to \phi(x) = \phi(\vec{x}, 0)$. To see this take the inner product of

$$\begin{aligned}\langle\vec{x}|\vec{p}\rangle &= \langle 0|\phi^\dagger(\vec{x},0)|\vec{p}\rangle \\ &= \langle 0|\left(\int\frac{d^3p'}{(2\pi)^3\sqrt{\omega_{p'}}}a^\dagger e^{-i\vec{p'}\vec{x}} + ae^{i\vec{p'}\vec{x}}\right)|\vec{p}\rangle \\ &= \left(\langle 0|\int\frac{d^3p'}{(2\pi)^3\sqrt{\omega_{p'}}}ae^{i\vec{p'}\vec{x}}\right)|\vec{p}\rangle, \quad (\langle 0|a^\dagger) = 0 \\ &= \int\frac{d^3p'}{(2\pi)^3\sqrt{\omega_{p'}}}e^{i\vec{p'}\vec{x}} \quad \{\langle\vec{p'}|\vec{p}\rangle\} \\ &= \int\frac{d^3p'}{(2\pi)^3\sqrt{\omega_{p'}}}e^{i\vec{p'}\vec{x}} \quad \left\{(2\pi)^3\sqrt{\omega_p}\delta(\vec{p'}-\vec{p})\right\} \\ \langle\vec{x}|\vec{p}\rangle &= e^{i\vec{p}\vec{x}}\end{aligned}$$

Thus $\phi(x)$ does indeed create a particle at spacetime point x :)

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What is the explicit formula for multiple integrations of a function, *i.e.* what is $\int\int\ldots\int f(x)dx$? The answer is the Cauchy formula for repeated integration

$$\begin{aligned}(Jf)(x) &= \int_0^x f(t)dt \\ (J^2f)(x) &= \int_0^x\left(\int_0^t f(s)ds\right)dt \\ (J^\alpha f)(x) &= \frac{1}{(\alpha-1)!}\int_0^x(x-t)^{\alpha-1}f(t)dt \\ &= \frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)dt\end{aligned}$$

Here we have used J as an integration operator and the identity $\Gamma(\alpha) = (\alpha - 1)!$, and also that gamma function can take non integer arguments compared to factorials.

So how do we derive this? The most intuitive derivation I saw was using Laplace transform, some preliminary Laplace transforms that we need are

1. Laplace transform of an integral of a function, $\mathcal{L}\{\int_0^x f(t)dt\}$

$$\mathcal{L}\left\{\int_0^x f(t)dt\right\} = \int_0^\infty e^{-sx}\left(\int_0^x f(t)dt\right)dx$$

we now use integration by parts, $u = \left(\int_0^x f(t)dt\right),\ dv = e^{-sx}dx$ such that

$$\begin{aligned}\mathcal{L}\left\{\int_0^x f(t)dt\right\} &= \left(\int_0^x f(t)dt\right)\frac{e^{-sx}}{(-s)}\Big|_{x=0}^{x=\infty} - \left(-\frac{1}{s}\right)\int_0^\infty e^{-sx}f(x)dx \\ &= 0 + \frac{1}{s}\int_0^\infty e^{-sx}f(x)dx \\ \mathcal{L}\{(Jf)(x)\} &= \frac{1}{s}\mathcal{L}\{f(x)\}\end{aligned}$$

the boundary terms are zero because $\int_0^0 f(t) = 0$ and $e^{-\infty} = 0$, assuming $f(t)$ is well behaved. Generalizing to $(J^\alpha f)(x)$ we have

$$\mathcal{L}\{(J^\alpha f)(x)\} = \frac{1}{s^\alpha}\mathcal{L}\{f(x)\}$$

2. Laplace transform of x^α, *i.e.* $\mathcal{L}\{x^\alpha\}$ (all function under Laplace transform are assumed to be valid only on the positive real axis)

$$\mathcal{L}\{x^\alpha\} = \int_0^\infty e^{-sx}x^\alpha dx$$

here we just need to do multiple integration by parts with $u = x^\alpha,\ dv = e^{-sx}dx$

$$\begin{aligned}\mathcal{L}\{x^\alpha\} &= x^\alpha e^{-sx}\big|_{x=0}^{x=\infty} - \frac{\alpha}{(-s)}\int_0^\infty x^{\alpha-1}e^{-sx}dx \\ &= \frac{\alpha}{s}\int_0^\infty x^{\alpha-1}e^{-sx}dx\end{aligned}$$

here we can do 2 things, one use the definition of gamma function $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ or repeat the integration by parts again (note that the boundary terms are again zero in

this case), if we repeat the IBP $\alpha - 1$ more times we get

$$\begin{aligned}
\mathcal{L}\{x^\alpha\} &= \frac{\alpha!}{s^\alpha}\int_0^\infty e^{-sx}dx \\
&= \frac{\alpha!}{s^\alpha}\frac{1}{(-s)}e^{-sx}\Big|_0^\infty \\
&= -\frac{\alpha!}{s^{\alpha+1}}\left(e^{-\infty} - e^0\right) \\
\mathcal{L}\{x^\alpha\} &= \frac{\alpha!}{s^{\alpha+1}}
\end{aligned}$$

which can be generalized to $\mathcal{L}\{x^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ *or* just by using the definition of gamma function from the beginning

$$\begin{aligned}
\frac{\alpha}{s}\int_0^\infty x^{\alpha-1}e^{-sx}dx &= \frac{\alpha}{s}\int_0^\infty \left(\frac{y}{s}\right)^{\alpha-1} e^{-y}\left(\frac{dy}{s}\right), \quad y = sx,\ dx = \frac{dy}{s} \\
&= \frac{\alpha}{s^{\alpha+1}}\int_0^\infty y^{\alpha-1}e^{-y}dy = \frac{\alpha\Gamma(\alpha)}{s^{\alpha+1}}, \quad \alpha\Gamma(\alpha) = \Gamma(\alpha+1) \\
\mathcal{L}\{x^\alpha\} &= \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}
\end{aligned}$$

which is the same result as above.

3. The infamous convolution formula

$$\begin{aligned}
\mathcal{L}\{f\}\mathcal{L}\{g\} &= \left(\int_0^\infty e^{-sx}f(x)dx\right)\left(\int_0^\infty e^{-sy}g(y)dy\right) \\
&= \int_0^\infty e^{-s(x+y)}\left(\int_0^\infty f(x)g(y)dy\right)dx, \quad z = x + y \\
&= \int_0^\infty e^{-sz}\left(\int_0^z f(z-y)g(y)dy\right)dz \\
\mathcal{L}\{f\}\mathcal{L}\{g\} &= \mathcal{L}\{f * g\}
\end{aligned}$$

where $*$ denotes convolution, some explanation on $dx \to dz$, inside the brackets of $\int dy$, x is constant while outside of this bracket y is *constant* and thus $dz = dx$.

We are now ready to derive the formula for $(J^\alpha f)(x)$

$$\begin{aligned}
(J^\alpha f)(x) &= \mathcal{L}^{-1}\{\mathcal{L}\{J^\alpha f\}\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\{f\}\right\} = \mathcal{L}^{-1}\left\{\frac{\Gamma(\alpha)}{\Gamma(\alpha)}\frac{1}{s^\alpha}\mathcal{L}\{f\}\right\} \\
&= \frac{1}{\Gamma(\alpha)}\mathcal{L}^{-1}\left\{\frac{\Gamma(\alpha)}{s^\alpha}\mathcal{L}\{f\}\right\} \\
&= \frac{1}{\Gamma(\alpha)}\left(\mathcal{L}^{-1}\left\{\frac{\Gamma(\alpha)}{s^\alpha}\right\} * f\right) \\
&= \frac{1}{\Gamma(\alpha)}\left(x^{\alpha-1} * f\right) \\
(J^\alpha f)(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt
\end{aligned}$$

which is the Cauchy formula we were looking for.

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Problem 1.1 Srednicki, Show that the Dirac matrices must be even dimensional. Hint: show that the eigenvalues of β are all ± 1, and that $\mathrm{Tr}\beta = 0$. To show that $\mathrm{Tr}\beta = 0$, consider *e.g.* $\mathrm{Tr}\alpha_1^2\beta$**. Similarly, show that $\mathrm{Tr}\alpha_i = 0$.**

Some facts about Dirac matrices

$$\{\alpha^j, \alpha^k\}_{ab} = 2\delta^{jk}\delta_{ab} \quad \{\alpha^j, \beta\}_{ab} = 0 \quad (\beta)^2_{ab} = \delta_{ab}$$

For eigenvalues of β it follows immediately from $(\beta)^2_{ab} = \delta_{ab}$ its eigenvalues must be ± 1.

For $\mathrm{Tr}\beta$ we start with

$$\begin{aligned}
\mathrm{Tr}(\alpha_1^2\beta) &= \mathrm{Tr}(\alpha_1\beta\alpha_1), \quad \{\alpha^j, \beta\}_{ab} = 0 \\
&= \mathrm{Tr}(-\beta\alpha_1\alpha_1) \\
\mathrm{Tr}(\alpha_1^2\beta) &= \mathrm{Tr}(\beta\alpha_1\alpha_1), \quad \mathrm{Tr}(AB) = \mathrm{Tr}(BA) \\
\rightarrow \mathrm{Tr}(-\beta\alpha_1\alpha_1) &= \mathrm{Tr}(\beta\alpha_1\alpha_1) \\
\mathrm{Tr}(\beta\alpha_1\alpha_1) &= 0
\end{aligned}$$

Now from $\{\alpha^j, \alpha^k\}_{ab} = 2\delta^{jk}\delta_{ab} \rightarrow \{\alpha_i, \alpha_i\}_{ab} = 2\delta_{ab}$, it means that $(\alpha_i^2)_{ab} = \delta_{ab}$ and thus $\mathrm{Tr}(\beta\alpha_i\alpha_i) = \mathrm{Tr}(\beta) = 0$.

Now since $\mathrm{Tr}\beta = 0$ and its eigenvalues are ± 1, β must be even dimensional.

This argument applies for $\mathrm{Tr}\alpha_i$ as well by reversing the roles of β and α, $\mathrm{Tr}(\beta^2\alpha_i) = \mathrm{Tr}(\alpha_i) = 0$ and since $(\alpha_i^2)_{ab} = \delta_{ab} \rightarrow$ eigenvalues must be ± 1, α^i must be even dimensional. This means that *all* of Dirac matrices must be even dimensional.

Now we need to show that Dirac matrices must be bigger than 2×2. I'll start with the fact that any 2×2 matrix can be written as $a_0 1 + a_i \sigma^i$, *i.e.* a combination of Pauli matrices and the identity matrix. If you remember some sort of group theory, the group $su(2)$ spans a three sphere and the basis vectors (lie algebra $SU(2)$) are spanned by the Pauli matrices.

The commutation relations of Pauli matrices are

$$\{\sigma^i, \sigma^j\}_{ab} = 2\delta^{ij}\delta_{ab}$$
$$[\sigma^i, \sigma^j] = 2i\varepsilon^{ijk}\sigma^k$$

As we can see from their anticommutation relations, the pauli matrices can play the role of α^i. We now need to show that there are no fourth 2×2 matrix to play the role of β. To do this let's construct a generic 2×2 matrix $A = a_0 1 + a_i \sigma^i$ and do its commutation relation with the pauli matrices and see what coefficients (a_0, a_i) we need to fulfill the commutation relation $\{\alpha^j, \beta\}_{ab} = 0$

$$\{A, \sigma^j\} = \{a_0 1 + a_i \sigma^i, \sigma^j\}$$
$$0 = \{a_0 1, \sigma^j\} + \{a_i \sigma^i, \sigma^j\}$$
$$0 = 2a_0\sigma^j + a_i \delta^{ij} 1$$
$$0 = 2a_0\sigma^j + a_j 1$$

Recall that there's only one diagonal pauli matrix, *i.e.* if you use $\hat{z}$ as the main direction $\sigma_z = (+1, -1)$ which is just spin up and spin down and the other two pauli matrices will be off diagonal, if you choose any other axis the same will occur, only one will be diagonal and the other two will be off diagonal.

From $0 = 2a_0\sigma^j + a_j 1$, we need σ^j to be diagonal but since $\sigma^j = (+1, -1)$ the only solution is $a_0 = 0, a_j = 0$. However the anticommutation relation $\{A, \sigma^j\} = 0$ has to be true for all values of $j = 1, 2, 3$ which means that $a_1 = a_2 = a_3 = 0 \rightarrow A = 0$, so the only 2×2 matrix that can fulfill the role of β is the zero matrix. However, $\beta^2 = 1$, hence Dirac matrices must be at least 4×4.

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Srednicki Problem 4.1 Verify eq. (4.12). Verify its limit as $m \to 0$. Eq. 4.12 is $[\varphi^+(x), \varphi^-(x')] = (m/4\pi^2 r)K_1(mr) \equiv C(r)$ where $K_1(mr)$ is a modified Bessel function of the second kind.

The subject here is that for a transition amplitude to be Lorentz invariance, the above commutator must be zero, however it's not. Why does it have to be zero? because the transition amplitude is given by time ordered product of the fields and for spacelike separations, $x - x' > 0$, the time ordering might shuffle the order of the fields as for a spacelike separation different observers might see two events happen in different orders. Note that the metric used here is $(-+++)$. Hence the need to set $[\varphi^+(x), \varphi^-(x')] = 0$ to maintain equal transition amplitudes for all observers, $\langle\varphi(x)\varphi(y)\rangle = \langle\varphi(y)\varphi(x)\rangle$.

However, eq. 4.12 shows that $[\varphi^+(x), \varphi^-(x')] \neq 0$! Our task at hand is to derive this conclusion. First we need to choose a frame where $t - t' = 0$ as suggested by the text to simplify calculation.

$$
\begin{aligned}
\int \widetilde{dk}\; e^{ik(x-x')} &= \int \frac{d^3k}{(2\pi)^3 2\sqrt{k^2+m^2}} e^{(-ik_0\Delta t + i\vec{k}\cdot\Delta\vec{x})},\; \Delta t = 0 \\
&= \int \frac{d^3k}{(2\pi)^3 2\sqrt{k^2+m^2}} e^{(i\vec{k}\cdot\Delta\vec{x})} \\
&= \int \frac{d^3k}{(2\pi)^3 2\sqrt{k^2+m^2}} e^{i|k||r|\cos\theta},\; |r| = |\Delta\vec{x}| \\
&= \int \frac{k^2 dk\; d\phi\; d(\cos\theta)}{(2\pi)^3 2\sqrt{k^2+m^2}} e^{i|k||r|\cos\theta} \\
&= \int \frac{k^2 dk\; \cancel{2\pi}}{(2\pi)^{\cancel{3}2} 2\sqrt{k^2+m^2}} \int_{-1}^{1} d(\cos\theta)\; e^{i|k||r|\cos\theta} \\
&= \int \frac{k^2 dk}{(2\pi)^2 2\sqrt{k^2+m^2}} \frac{(e^{i|k||r|} - e^{-i|k||r|})}{ikr} \\
&= \int \frac{kdk}{(2\pi)^2 \cancel{2}\sqrt{k^2+m^2}} \frac{\cancel{2i}\sin(kr)}{\cancel{i}r} \\
&= \frac{1}{(2\pi)^2 r} \int_0^\infty dk \frac{k\sin(kr)}{\sqrt{k^2+m^2}}
\end{aligned}
$$

We now need to massage this into a Bessel function, but first let's do a change of variable

$\rightarrow y = k/m$

$$= \frac{1}{(2\pi)^2 r} \int_0^\infty m \; d\left(\frac{k}{m}\right) \frac{m\left(\frac{k}{m}\right)\sin(\left(\frac{k}{m}\right)mr)}{m\sqrt{\left(\frac{k}{m}\right)^2+1}}$$

$$= \frac{m}{(2\pi)^2 r} \int_0^\infty dy \frac{y\sin(ymr)}{\sqrt{y^2+1}}$$

now, there are 2 facts we need about Bessel functions

$$K_0(z) = \int_0^\infty dy \frac{\cos(zy)}{\sqrt{y^2+1}}$$

$$\frac{d}{dz}K_\nu(z) = \frac{\nu}{z}K_\nu(z) - K_{\nu+1}(z)$$

$$\rightarrow \frac{d}{dz}K_0(z) = -K_1(z)$$

If we do the actual derivative on K_0 we get

$$\rightarrow \frac{d}{dz}K_0(z) = \int_0^\infty dy \frac{d}{dz}\left(\frac{\cos(zy)}{\sqrt{y^2+1}}\right)$$

$$-K_1(z) = -\int_0^\infty dy \frac{y\sin(zy)}{\sqrt{y^2+1}}$$

$$\rightarrow \frac{m}{(2\pi)^2 r}K_1(z) = \frac{m}{(2\pi)^2 r}\int_0^\infty dy \frac{y\sin(zy)}{\sqrt{y^2+1}}, \; z \rightarrow mr$$

$$\frac{m}{4\pi^2 r}K_1(mr) = \frac{m}{(2\pi)^2 r}\int_0^\infty dy \frac{y\sin(mry)}{\sqrt{y^2+1}}$$

$$\frac{m}{4\pi^2 r}K_1(mr) = \int \widetilde{dk} \; e^{ik(x-x')}$$

and that is the result that we want. How about its limit as $m \rightarrow 0$? One might be tempted to find or calculate the power series expansion of the Bessel function around $z \rightarrow 0$ but I found it extremely difficult (as I've forgotten most of the methods from my mathematical methods class) but (after looking at the problem long enough) I realized that we need not mess with Bessel functions at all.

You see $\int \widetilde{dk} \; e^{ik(x-x')}$ looks a lot like a fourier transform and setting $m \rightarrow 0$ makes it look like this

$$= \int \frac{d^3k}{(2\pi)^3 2\sqrt{k^2+m^2}} e^{i\vec{k}\cdot\vec{r}}, \; m \rightarrow 0$$

$$= \frac{1}{2(2\pi)^3}\int d^3k \frac{e^{i\vec{k}\cdot\vec{r}}}{|k|}$$

which looks a lot like the fourier transform of the Coulomb potential but this time it's going into the $\vec{x}$ domain instead of the $\vec{k}$ domain.

The evaluation of this integral is usually done using a regulator $e^{-\mu k}$ and we'll take the limit $\mu \to 0$ at the end.

$$
\begin{aligned}
&= \frac{1}{2(2\pi)^3} \int d^3k \frac{e^{-\mu k} e^{i\vec{k}\cdot\vec{r}}}{|k|} \\
&= \frac{1}{2(2\pi)^3} \int \cancel{k^2} dk\ d\phi\ \frac{e^{-\mu k}}{\cancel{k}} \int_{-1}^{1} d(\cos\theta) e^{i|k||r|\cos\theta} \\
&= \frac{1}{2(2\pi)^2} \int_0^\infty \cancel{k} dk\ e^{-\mu k} \frac{(e^{ikr} - e^{-ikr})}{i\cancel{k}r} \\
&= \frac{1}{2ir(2\pi)^2} \int_0^\infty dk\ e^{-\mu k}(e^{ikr} - e^{-ikr}) \\
&= \frac{1}{2ir(2\pi)^2} \left[\frac{1}{(-\mu + ir)}\ e^{(-\mu+ir)k}\Big|_0^\infty - \frac{1}{(-\mu - ir)}\ e^{(-\mu-ir)k}\Big|_0^\infty \right] \\
&= \frac{1}{2ir(2\pi)^2} \left[\frac{1}{(-\mu + ir)}(e^{-\infty} - e^0) - \frac{1}{(-\mu - ir)}(e^{-\infty} - e^0) \right] \\
&= \frac{1}{2ir(2\pi)^2} \left[\frac{1}{(-\mu - ir)} - \frac{1}{(-\mu + ir)} \right] = \frac{1}{2ir(2\pi)^2} \left[\frac{-\mu + ir + \mu + ir}{\{(-\mu)^2 - (ir)^2\}} \right] \\
&= \frac{1}{\cancel{2ir}(2\pi)^2} \frac{\cancel{2ir}}{\{(-\mu)^2 + r^2\}},\ \mu \to 0 \\
&= \frac{1}{4\pi^2 r^2}
\end{aligned}
$$

which is the result we want :) without messing with Bessel functions at all

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How do you do the integral of $\int_{-\infty}^{\infty} dx \frac{\sin x}{x}$? The common solution is to do a contour integration, but is there another way? This was what I attempted, the answer to that question is in short no, but I sure learned a hell lot thinking about this problem.

First, there's no function whose derivative is $\frac{\sin x}{x}$ which means that the indefinite integral $\int dx \frac{\sin x}{x}$ has no solution. Contour integral was a neat technique apropos to a numerical calculation.

My original approach was inspired by green's function method. This was because the integral $\int dx \frac{\sin x}{x}$ is akeen to $\int dk \frac{e^{ikx}}{k}$ and since $\cos k$ is even, only the $\sin k$ term will survive the integration $\to \int dk \frac{i \sin(kx)}{k}$ and by taking $x \to 1$ we'll get $\int dk \frac{i \sin k}{k}$ which is

what we want. The starting point is the unit step function $u(x)$

$$\begin{aligned} \frac{d}{dx}u(x) &= \delta(x) \\ \rightarrow ik\ u(k) &= 1 \\ u(k) &= \frac{1}{ik} \end{aligned}$$

We must take a pause here. First some trivial things, I have chosen the fourier transform pair to be $f(x) = \frac{1}{2\pi}\int dk e^{ikx}F(k)$ and $F(k) = \int dx e^{-ikx}f(x)$, *i.e.* we put all factor of normalization $\frac{1}{2\pi}$ in the inverse transform. We can choose other normalization schemes, but the result will be the same.

The second reason to pause here is more serious. The unit step function is a peculiar function. Note that

$$\frac{d}{dx}\left(u(x) + C\right) = \delta(x)$$

is still true for any constant C. But there's a particular value of C that has serious ramifications, *i.e.* $C = -\frac{1}{2}$. Why? because

$$\left(u(x) - \frac{1}{2}\right) = \frac{1}{2}\left(u(x) - u(-x)\right)$$

that is $\left(u(x) - \frac{1}{2}\right)$ is just the antisymmetric part of $u(x)$. Now there is a fact that we need to remember, an antisymmetric (real) function has as its fourier counterpart a purely imaginary function $\int dx e^{-ikx}f(x) = -i\int dx \sin(kx)f(x)$. We see above that $u(k) = \frac{1}{ik}$ (purely imaginary) and since $\frac{d}{dx}u(x) = \frac{d}{dx}\left(u(x) + C\right)$ what we were really doing was fourier transforming only the antisymmetric part of $u(x)$, *i.e.*

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{2}(u(x) - u(-x))\right) &= \frac{d}{dx}\left(u(x) - \frac{1}{2}\right) = \delta(x) \\ \mathcal{F}\left\{\frac{d}{dx}\left(u(x) - \frac{1}{2}\right)\right\} &= \mathcal{F}\left\{\delta(x)\right\} \\ ik\ \mathcal{F}\left\{\left(u(x) - \frac{1}{2}\right)\right\} &= 1 \\ ik\left(u(k) - \frac{1}{2}(2\pi\delta(k))\right) &= 1 \\ u(k) &= \frac{1}{ik} + \pi\delta(k) \end{aligned}$$

Note that $\mathcal{F}\{\delta(x)\} = 1$ and $\mathcal{F}\{1\} = 2\pi\delta(k)$ is affected by the normalization scheme, other schemes will produce different numerical factors. Now, what does it have to do with $\int dx \frac{\sin x}{x}$? Let's do an inverse fourier transform on the above function

$$\begin{aligned}
\mathcal{F}^{-1}\{u(k)\} &= \mathcal{F}^{-1}\left\{\frac{1}{ik} + \pi\delta(k)\right\} \\
u(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \frac{e^{ikx}}{ik} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{ikx}\left(\pi\delta(k)\right) \\
u(x) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk \frac{e^{ikx}}{k} + \frac{1}{2}
\end{aligned}$$

Now we set $x \to 1, u(1) = 1$

$$\begin{aligned}
u(1) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk \frac{e^{ik}}{k} + \frac{1}{2} \\
1 &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk \frac{\sin k}{k} + \frac{1}{2} \\
\to \int_{-\infty}^{\infty} dk \frac{\sin k}{k} &= \pi
\end{aligned}$$

In the second line we have used the fact that since $\frac{\cos k}{k}$ is an odd function, its integral is zero. And this is the origin of the factor of π. For the longest time I have wondered how this integral has anything to do with π. But wait, you might ask how in turn did the fourier transform pair acquire this factor of $\frac{1}{2\pi}$ in the first place?

The answer to that question is the gaussian integral $e^{-x^2/2}$. This gaussian function is the eigenfunction of the fourier transform. Let's see how it works

$$\begin{aligned}
\int_{-\infty}^{\infty} dx\ e^{-ikx} e^{-x^2/2} &= \sqrt{2\pi}\ e^{\frac{(-ik)^2}{4\left(\frac{1}{2}\right)}} \\
&= \sqrt{2\pi}\ e^{-k^2/2}
\end{aligned}$$

where we have used the gaussian integral formula $\int dx\ e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c}$ and if we do an inverse transform we get

$$\begin{aligned}
\int_{-\infty}^{\infty} dk\ e^{ikx}\left(\sqrt{2\pi}\ e^{-k^2/2}\right) &= \sqrt{2\pi}\sqrt{2\pi}\ e^{\frac{(ix)^2}{4\left(\frac{1}{2}\right)}} \\
&= 2\pi\ e^{-x^2/2}
\end{aligned}$$

Thus

$$\mathcal{F}^{-1}\left\{\mathcal{F}\left\{e^{-x^2/2}\right\}\right\} = 2\pi\ e^{-x^2/2}$$

or equivalently

$$\begin{aligned} e^{-(0)^2/2} &\stackrel{?}{=} \int_{-\infty}^{\infty} dk\ e^{ik(0)} \left(\sqrt{2\pi}\ e^{-k^2/2}\right) \\ &= \sqrt{2\pi} \int_{-\infty}^{\infty} dk\ e^{-k^2/2} = \sqrt{2\pi}\sqrt{2\pi} \\ 1 &\stackrel{?}{=} 2\pi \end{aligned}$$

this means that we need to absorb the extra factor of 2π somehow, the usual way is to put a $\frac{1}{2\pi}$ in the inverse transform, another common way is to put a factor of $\frac{1}{\sqrt{2\pi}}$ on both the fourier transform and its inverse. Note that my derivation of $\int \frac{\sin k}{k}$ does not depend on the normalization scheme, you might try it if you don't believe me ;)

But then again, how did the gaussian integral get a factor of π, well it's because we use a circle to compute it, remember?

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-x^2/2} &= \sqrt{\left(\int_{-\infty}^{\infty} e^{-x^2/2} \int_{-\infty}^{\infty} e^{-y^2/2}\right)} \\ &= \sqrt{\int_{-\infty}^{\infty} dxdy\ e^{-(x^2+y^2)/2}} \\ &= \sqrt{\left(\int_0^{2\pi} d\phi\right) \int_0^{\infty} rdr\ e^{-(r^2)/2}} \\ \int_{-\infty}^{\infty} e^{-x^2/2} &= \sqrt{2\pi} \end{aligned}$$

And that my friend, is how you get a factor of π in the first place :) Of course in the contour integration of $\int \frac{\sin x}{x}$ the factor of π is due to the infinitesimal semi circle around $x = 0$ but again, it's all due to a circle.

It took me two days to figure this out and I learned a lot about the step function and fourier transform in the process. It all started with a simple thought of whether or not it's possible to compute $\int \frac{\sin x}{x}$ wihout contour integration, turns out that it's not quite possible, sometimes we do need a new technique to solve an old problem and I gained a new respect for complex numbers.

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Why do we need limits and what are they? Let's start with the infamous $\lim_{x\to 0}$. Without messing with δ and ϵ as in the usual formal definition of a limit, what a limit amounts to

is (in the case $x \to 0^+$, *i.e.* we're approaching the limit from the right) $\lim_{x\to 0^+} \Rightarrow x > 0$ but x is smaller than any positive real number.

Now how can that be? If it is smaller than any positive real number (and it's not negative) then it has to be zero. One way it is possible if we are dealing with infinities, something like

$$x = 0.\underbrace{000000\ \ldots\ 00000}_{\text{infinite \# of zeros}}1$$

There are an infinite number of zeros before the digit 1 so that it doesn't matter how small the real number you present me, I can always produce a smaller one by adding more zeros in between and it has to be infinite for this to work because x is smaller than *any* positive real number. But this won't work since it is just zero! Remember that

$$1 = 0.\underbrace{999999\ \ldots\ 99999}_{\text{infinite \# of 9's}}$$

Thus limits are a (more severe?) manifestation of infinities. The real question is: why do we need it? The more obvious reason is in the calculation of a slope. To calculate a slope you need at least two points, say (x_1, y_1) and (x_2, y_2) and the slope is

$$\frac{\Delta y}{\Delta x} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

but then whose slope is it? Is it the slope at (x_1, y_1) or at (x_2, y_2)? because slopes at different points are usually different. What we want is to calculate the slope at *one* point but we need two points to calculate it. How do we solve this conundrum? The answer is of course limits! If we take the limit $\Delta x = (x_2 - x_1) \to 0^+$ we are practically using two distinct points located at the 'same' place because x_2 is *not* x_1 and yet it is closer to x_1 than any other points.

In closing, in the words of my high school math teacher, "limits bring into existence things that don't exist" and I believe that what real numbers do. And that might be the *real* reason behind its difficulties, real numbers were supposed to address the problem of continuum in mathematics although physical continuum may not actually exist, maybe "what God has not created we shouldn't bring about" :)

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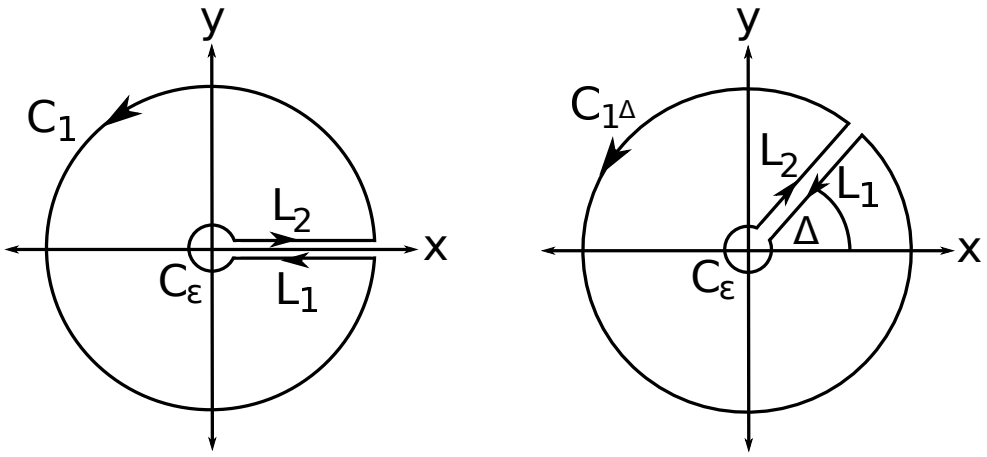

FIG. 2: A choice of contour that includes a cut.

Branch cuts, what are they for? and how do we use them? For an illustration see Fig. 2. What we want to do here is to calculate $\oint_C \sqrt{z}dz$ along a unit circle around the origin. The tricky thing here is that $1 = e^{i0} = e^{i2\pi}$ but $\sqrt{1} \to e^{i0/2} \neq e^{i\pi}$! So we need to provide a cut such that the contour doesn't go full circle. First we will calculate $\oint_C \sqrt{z}dz$ using the contour shown in Fig. 2 (left).

$$\oint_C \sqrt{z}dz = \oint_{C_1} \sqrt{z}dz + \int_{L_1} \sqrt{z}dz + \oint_{C_\varepsilon} \sqrt{z}dz + \int_{L_2} \sqrt{z}dz$$

$$\to \oint_C \sqrt{z}dz = 0, \quad \text{no pole inside the contour}$$

$$\to \oint_{C_1} \sqrt{z}dz = \int_\varepsilon^{2\pi-\varepsilon} \sqrt{1e^{i\theta}}d\theta = \int_0^{2\pi} \sqrt{e^{i\theta}}d\theta = \oint_C \sqrt{z}dz$$

$$\to \int_{L_1} \sqrt{z}dz = \int_1^\varepsilon \sqrt{xe^{(2\pi-\varepsilon)i}}dx = \int_1^0 \sqrt{x}e^{\pi i}dx = \int_0^1 \sqrt{x}\ dx$$

$$\to \oint_{C_\varepsilon} \sqrt{z}dz = \int_{2\pi-\varepsilon}^\varepsilon \sqrt{\varepsilon e^{i\theta}}d\theta = 0 \text{ as } \varepsilon \to 0$$

$$\to \int_{L_2} \sqrt{z}dz = \int_\varepsilon^1 \sqrt{xe^{\varepsilon i}}dx = \int_0^1 \sqrt{x}\ dx$$

and we take the limit of $\varepsilon \to 0$. Thus we have, after rearranging

$$\begin{aligned}
\oint_C \sqrt{z}dz &= \oint_{C_1} \sqrt{z}dz + \int_{L_1} \sqrt{z}dz + \oint_{C_\varepsilon} \sqrt{z}dz + \int_{L_2} \sqrt{z}dz \\
\to 0 &= \oint_C \sqrt{z}dz + \int_0^1 \sqrt{x}\ dx + 0 + \int_0^1 \sqrt{x}\ dx \\
\oint_C \sqrt{z}dz &= -\int_0^1 \sqrt{x}\ dx - \int_0^1 \sqrt{x}\ dx \\
&= -\frac{4}{3}
\end{aligned}$$

but what happens if we choose a cut that doesn't lie on the x axis? say the cut is at an angle Δ? as shown in Fig. 2 (right), note that we want L_1 and L_2 to differ by 2π otherwise there's no point in introducing a cut, here L_2 is at angle Δ and L_1 is at angle $\Delta + 2\pi$

$$\begin{aligned}
\oint_{C^\Delta} \sqrt{z}dz &= \oint_{C_{1\Delta}} \sqrt{z}dz + \int_{L_{1\Delta}} \sqrt{z}dz + \oint_{C_{\varepsilon\Delta}} \sqrt{z}dz + \int_{L_{2\Delta}} \sqrt{z}dz \\
\to \oint_{C^\Delta} \sqrt{z}dz &= 0, \quad \text{no pole inside the contour} \\
\to \oint_{C_{1\Delta}} \sqrt{z}dz &= \int_{\Delta+\varepsilon}^{\Delta+2\pi-\varepsilon} \sqrt{1e^{i\theta}}d\theta = \int_{\Delta}^{\Delta+2\pi} \sqrt{e^{i\theta}}d\theta = \oint_{C^\Delta} \sqrt{z}dz \\
\to \int_{L_{1\Delta}} \sqrt{z}dz &= \int_1^{\varepsilon} \sqrt{re^{(\Delta+2\pi-\varepsilon)i}}dr = \int_1^0 \sqrt{r}e^{i\Delta/2}e^{\pi i}dr = e^{i\Delta/2}\int_0^1 \sqrt{r}\ dr \\
\to \oint_{C_{\varepsilon\Delta}} \sqrt{z}dz &= \int_{\Delta+2\pi-\varepsilon}^{\Delta+\varepsilon} \sqrt{\varepsilon e^{i\theta}}d\theta = 0 \text{ as } \varepsilon \to 0 \\
\to \int_{L_{2\Delta}} \sqrt{z}dz &= \int_{\varepsilon}^1 \sqrt{re^{(\Delta+\varepsilon)i}}dr = e^{i\Delta/2}\int_0^1 \sqrt{r}\ dr
\end{aligned}$$

Now we have

$$\begin{aligned}
\oint_{C^\Delta} \sqrt{z}dz &= \oint_{C_{1\Delta}} \sqrt{z}dz + \int_{L_{1\Delta}} \sqrt{z}dz + \oint_{C_{\varepsilon\Delta}} \sqrt{z}dz + \int_{L_{2\Delta}} \sqrt{z}dz \\
0 &= \oint_{C^\Delta} \sqrt{z}dz + e^{i\Delta/2}\int_0^1 \sqrt{r}\ dr + 0 + e^{i\Delta/2}\int_0^1 \sqrt{r}\ dr \\
\oint_{C^\Delta} \sqrt{z}dz &= -2e^{i\Delta/2}\int_0^1 \sqrt{r}\ dr \\
&= e^{i\Delta/2}\left(-\frac{4}{3}\right)
\end{aligned}$$

So apparently we get a different result for different contours, but if we take a closer look

$$\oint_{C^\Delta} \sqrt{z}dz = \int_\Delta^{\Delta+2\pi} \sqrt{1e^{i\theta}}d\theta$$
$$\text{Let } \omega = \theta - \Delta \rightarrow d\omega = d\theta, \ \int_\Delta^{\Delta+2\pi} d\theta = \int_0^{2\pi} d\omega$$
$$\rightarrow \oint_{C^\Delta} \sqrt{z}dz = \int_0^{2\pi} \sqrt{1e^{i(\omega+\Delta)}}d\omega$$
$$= e^{i\Delta/2} \int_0^{2\pi} \sqrt{1e^{i\omega}}d\omega$$
$$\oint_{C^\Delta} \sqrt{z}dz = e^{i\Delta/2} \oint_C \sqrt{z}dz$$

so indeed

$$\oint_{C^\Delta} \sqrt{z}dz = e^{i\Delta/2} \oint_C \sqrt{z}dz = e^{i\Delta/2} \left(-\frac{4}{3}\right)$$
$$\rightarrow \oint_C \sqrt{z}dz = -\frac{4}{3}$$

Thus rotating the branch cut (by an angle Δ) does indeed rotate the original integral $\oint_C \sqrt{z}dz$ into $e^{i\Delta/2} \oint_C \sqrt{z}dz$, so we need to be careful.

A side note, how do we specify the contour if the circle is not at the origin but say is at z_0? Well, we can just parametrize the circle as $z = z_0 + re^{i\theta}$ (remember that additions in the complex plane are vectorial operations), thus a contour integral of z around a point z_0 is simply given by

$$\oint z \ dz \rightarrow \int_0^{2\pi} (z_0 + re^{i\theta})d\theta$$

Going back to branch cuts, what if we have more than one root, *e.g.* $\sqrt{(z-z_1)(z-z_2)}$? see Fig. 3. Then there are two choices, we can put the cut between the roots as shown in Fig. 3 (left) or we can put the cuts starting from the roots to infinity as shown in Fig. 3 (right), the point here is to prevent a full circle around each root.

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Throwback Tuesday! (today is Tuesday Oct 13 2015) High school favorite, how do we compute $\sin(18°)$ without approximation and without a calculator? Well, start with $\sin(18°) = \sin(90°/5)$, since we know that $\sin(90°) = 1$. However, our real starting point is actually $\cos(90°/5)$ because $\cos(90°) = 0$ and the end goal here is to transform $\sin(18°)$

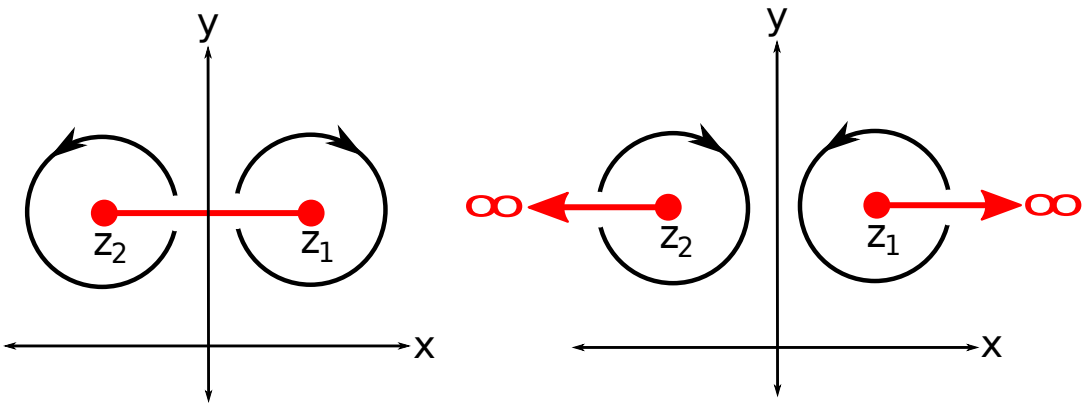

FIG. 3: Different choices of contours that include a cut with 2 roots.

into solving a polynomial and it's easier to find the zeros of a polynomial if there's no constant.

First we need the formula for $\cos(5x)$. The easiest way to do this is to use Euler's formula

$$e^{i5x} = \left(e^{ix}\right)^5$$
$$\rightarrow \cos(5x) + i\sin(5x) = (\cos(x) + i\sin(x))^5$$

In case you forget how to expand a polynomial, here's the Pascal's triangle

$$1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

the expansion is then

$$\begin{aligned}
(\cos(x) + i\sin(x))^5 &= \cos^5(x) + 5\cos^4(x)i\sin(x) + 10\cos^3(x)i^2\sin^2(x) + \\
&\quad 10\cos^2(x)i^3\sin^3(x) + 5\cos(x)i^4\sin^4(x) + i^5\sin^5(x) \\
\rightarrow \cos(5x) &= \mathrm{Re}\left\{(\cos(x) + i\sin(x))^5\right\} \\
&= \cos^5(x) - 10\cos^3(x)\sin^2(x) + 5\cos(x)\sin^4(x) \\
\rightarrow \sin(5x) &= \mathrm{Im}\left\{(\cos(x) + i\sin(x))^5\right\} \\
&= 5\cos^4(x)\sin(x) - 10\cos^2(x)\sin^3(x) + \sin^5(x)
\end{aligned}$$

simplifying

$$\begin{aligned}
\cos(5x) &= \cos^5(x) - 10\cos^3(x)\sin^2(x) + 5\cos(x)\sin^4(x) \\
&= \cos^5(x) - 10\cos^3(x)(1 - \cos^2(x)) + 5\cos(x)(1 - \cos^2(x))^2 \\
&= \cos^5(x) - 10\cos^3(x) + 10\cos^5(x) + 5\cos(x)(1 - 2\cos^2(x) + \cos^4(x)) \\
&= 11\cos^5(x) - 10\cos^3(x) + 5\cos(x) - 10\cos^3(x) + 5\cos^5(x) \\
\cos(5x) &= 16\cos^5(x) - 20\cos^3(x) + 5\cos(x)
\end{aligned}$$

while for sin

$$\begin{aligned}
\sin(5x) &= 5\cos^4(x)\sin(x) - 10\cos^2(x)\sin^3(x) + \sin^5(x) \\
&= 5(1 - \sin^2(x))^2\sin(x) - 10(1 - \sin^2(x))\sin^3(x) + \sin^5(x) \\
&= 5(1 - 2\sin^2(x) + \sin^4(x))\sin(x) - 10\sin^3(x) + 10\sin^5(x) + \sin^5(x) \\
&= 5\sin(x) - 10\sin^3(x) + 5\sin^5(x) - 10\sin^3(x) + 11\sin^5(x) \\
\sin(5x) &= 16\sin^5(x) - 20\sin^3(x) + 5\sin(x)
\end{aligned}$$

We can of course derive the above result using the usual formulae, for example

$$\begin{aligned}
\sin(5x) &= \sin(2x + 3x) \\
&= \sin(2x)\cos(3x) + \cos(2x)\sin(3x) \\
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(3x) &= \cos(x + 2x) = \cos(x)\cos(2x) - \sin(x)\sin(2x) \\
\cos(2x) &= \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x) \\
\sin(3x) &= \sin(x + 2x) = \sin(x)\cos(2x) + \cos(x)\sin(2x)
\end{aligned}$$

and working out the algebra, the result will be the same. Now we see as to why starting out with $\sin(18^\circ) = \sin(90^\circ/5)$ doesn't work well

$$\sin(5 \times 18^\circ) = 1 = 16\sin^5(18^\circ) - 20\sin^3(18^\circ) + 5\sin(18^\circ)$$
$$1 = 16\sin^5(18^\circ) - 20\sin^3(18^\circ) + 5\sin(18^\circ)$$

the LHS is not zero, while starting with $\cos(18^\circ) = \cos(90^\circ/5)$ does work

$$\begin{aligned}
\cos(5 \times 18^\circ) = 0 &= 16\cos^5(18^\circ) - 20\cos^3(18^\circ) + 5\cos(18^\circ) \\
0 &= \cos(18^\circ)(16\cos^4(18^\circ) - 20\cos^2(18^\circ) + 5), \quad \text{Let } x = \cos^2(18^\circ) \\
0 &= 16x^2 - 20x + 5 \\
x &= \frac{20 \pm \sqrt{400 - 320}}{32} = \frac{20 \pm 4\sqrt{5}}{32}
\end{aligned}$$

we just need to take the + of the $\pm$ since cos of small angles are closer to one. The result we want is then given by

$$\begin{aligned}
\sin(18^\circ) &= \sqrt{\sin^2(18^\circ)} = \sqrt{1 - \cos^2(18^\circ)} = \sqrt{1 - x} \\
&= \sqrt{1 - \frac{20 + 4\sqrt{5}}{32}} \\
\sin(18^\circ) &= \sqrt{\frac{3 - \sqrt{5}}{8}}
\end{aligned}$$

but wait, how do you calculate the square root by hand? The problem states that we should calculate it without a calculator.

To calculate square roots by hand we use the ubiquitous formula $(a+b)^2 = a^2+2ab+b^2$ and do it iteratively. Say we want to find the root of a number, *e.g.* $\sqrt{C}$, first we do a rough approximation $C_1^2 < C$ and then we find the next factor C_2 through

$$\begin{aligned}
C &= (C_1 + C_2)^2 = C_1^2 + 2C_1C_2 + C_2^2 \\
(C - C_1^2) &= (2C_1 + C_2)C_2
\end{aligned}$$

finding the largest C_2 such that $(2C_1 + C_2)C_2 < (C - C_1^2)$. Once we find C_2 we repeat the process

$$\begin{aligned}
C &= (\{C_1 + C_2\} + C_3)^2 = \{C_1 + C_2\}^2 + 2\{C_1 + C_2\}C_3 + C_3^2 \\
(C - \{C_1 + C_2\}^2) &= (2\{C_1 + C_2\} + C_3)C_3
\end{aligned}$$

by finding the largest C_3 such that $(2\{C_1 + C_2\} + C_3)C_3 < (C - \{C_1 + C_2\}^2)$. In other words

$$C_1 \to \text{largest number such that } C_1^2 < C$$

$$C_n \to \text{largest number such that } \left(2\left(\sum_{i=1}^{n-1} C_i\right) + C_n\right) C_n < \left(C - \left(\sum_{i=1}^{n-1} C_i\right)^2\right)$$

$$\to \sqrt{C} = \sum_{i=1}^{\infty} C_i$$

You can stop anytime you want, depending on the accuracy you need. The above prescription is correct but we want to make it more systematic by dividing the number into 'hundreds', partitioning it into 2-digit groupings (hundreds because our digits run from 0 to 9 which translate to 0 to < 100). Let's see how it works with a concrete example

$$\begin{array}{rl}
 & 2\ \ 3\ \ 8.\ \ \ 3\ \ \ 0\ \ 4\ \ 4 \\
 & \sqrt{5\ 67\ 89.\ 01\ 20\ 00\ 00} \\
2 \times 2 = & 4 \\
 & \hline
((2)(10)(2) + 3) \times 3 = & 1\ 67 \\
 & 1\ 29 \\
 & \hline
((2)(10)(23) + 8) \times 8 = & 38\ 89 \\
 & 37\ 44 \\
 & \hline
((2)(10)(238) + 3) \times 3 = & 1\ 45\ 01 \\
 & 1\ 42\ 89 \\
 & \hline
((2)(10)(2383) + 0) \times 0 = & 2\ 12\ 20 \\
 & 0\ 00\ 00 \\
 & \hline
((2)(10)(23830) + 4) \times 4 = & 2\ 12\ 20\ 00 \\
 & 1\ 90\ 64\ 16 \\
 & \hline
((2)(10)(238304) + 4) \times 4 = & 21\ 55\ 84\ 00 \\
 & 19\ 06\ 43\ 36
\end{array}$$

Thus $\sqrt{56789.012} = 238.3044\ldots$

What we need to take note above is that C_1 takes care only the first digit, this makes it a lot easier to start the procedure, rather than trying to figure out 238 from the start.

In the second iteration, we multiply C_1 by 10 and at the same time 'multiplying' the target by 100 by including the next 2 digits as

$$100C = 100(C_1 + C_2)^2 = 100C_1^2 + 200C_1C_2 + 100C_2^2$$

$$100(C - C_1^2) = (2(10)C_1 + C_2)C_2$$

Another important point is that $C_5 = 0$! Zero is apparently permitted. I actually learned this method when I was 10 but didn't understand the reasoning behind it.

Another way of computing square roots is using taylor expansion

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)!!}{2^n} x^{n+1}$$

for example

$$\sqrt{101} = 10\sqrt{1+0.01} = 10\left(1 + \frac{1}{2}(0.01) - \frac{1}{2}(0.01)^2 + \frac{(1)(3)}{2^2}(0.01)^3 + \dots\right)$$

but the convergence might be slow due to the alternating signs in the series. One important note here is that although the previous method looks a lot like the usual division algorithm, they are extremely different. In the case of roots, you need to keep using the previous results to compute the next one, not so with divisions, this is because divisions are linear while roots are not.

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Why does a number that is a multiple of 9 have digits tham sum up to be a multiple of 9? *e.g.*

$$3 \times 9 = 27 \to 2 + 7 = 9$$

$$13 \times 9 = 117 \to 1 + 1 + 7 = 9$$

$$5555 \times 9 = 49995 \to 4 + 9 + 9 + 9 + 5 = 36 = 4 \times 9$$

First of all, any number can be written as

$$a_N a_{N-1} a_{N-2} \dots a_2 a_1 a_0 = \sum_{i=0}^{N} a_i 10^i, \quad a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

e.g. $10937 \to a_4 = 1, a_3 = 0, a_2 = 9, a_1 = 3, a_0 = 7$, now say we have a number that is a multiple of 9, $b = 9n$, with n a positive integer

$$b = 9n = \sum_{i=0}^{N} a_i 10^i$$

$$\sum_{i=0}^{N} a_i 10^i = \sum_{i=1}^{N} a_i (10^i - 1) + \sum_{i=0}^{N} a_i$$

but we can rewrite

$$\begin{aligned}
\left(10^i - 1\right) &= \sum_{j=0}^{i-1} 9(10)^j = 9\sum_{j=0}^{i-1}(10)^j \\
\rightarrow \sum_{i=0}^{N} a_i 10^i &= \sum_{i=1}^{N} a_i \left(9\sum_{j=0}^{i-1}(10)^j\right) + \sum_{i=0}^{N} a_i \\
b = 9n &= 9\left(\sum_{i=1}^{N} a_i \left(\sum_{j=0}^{i-1}(10)^j\right)\right) + \sum_{i=0}^{N} a_i \\
9n - 9\left(\sum_{i=1}^{N} a_i \left(\sum_{j=0}^{i-1}(10)^j\right)\right) &= \sum_{i=0}^{N} a_i
\end{aligned}$$

Thus the sum of the digits $\sum_{i=0}^{N} a_i = 9m$ is a multiple of nine, *e.g.* $41076 = 9 \times 4654$

$$\begin{aligned}
9 \times 4654 = 41076 &= 4 \times 10000 + 1 \times 1000 + 0 \times 100 + 7 \times 10 + 6 \\
&= (4 \times (10000 - 1) + 1 \times (1000 - 1) + 0 \times (100 - 1) + 7 \times (10 - 1)) + \\
&\quad (4 + 1 + 0 + 7 + 6) \\
&= (4 \times 9999 + 1 \times 999 + 0 \times 99 + 7 \times 9) + (4 + 1 + 0 + 7 + 6) \\
9 \times 4654 &= 9 \times (4 \times 1111 + 1 \times 111 + 0 \times 11 + 7 \times 1) + (4 + 1 + 0 + 7 + 6)
\end{aligned}$$

Hence $(4 + 1 + 0 + 7 + 6)$ must be a multiple of 9.

Note that the above argument is also valid for multiples of 3, *i.e.* digits of a number that is a multiple of 3 sum up to a multiple of 3.

-=-

What is the explicit formula for $\sum_{i=1}^{N} i^2$? I saw this on the bus on my way to work a long time ago, took me a day to finally figure it out. You can say that we can use induction to prove the formula however, induction only allows you to prove what you already know *not* to derive the formula itself in the first place! *inductio no ex nihilo* :)

The easiest way to derive the formula for $\sum_{i=1}^{N} i^2$ is to actually draw it, see Fig. 4. The series we want to sum is shown in the top red box, *i.e.* $\sum_{i=1}^{N=6} i^2 = 1^2+2^2+3^2+4^2+5^2+6^2$.

This sum $\sum_{i=1}^{N=6} i^2$ is also represented by the upper red triangle in Fig. 4 because that is the definition of squares, *e.g.* $2^2 = 2 + 2,\ 3^2 = 3 + 3 + 3$. However, Fig. 4 also shows

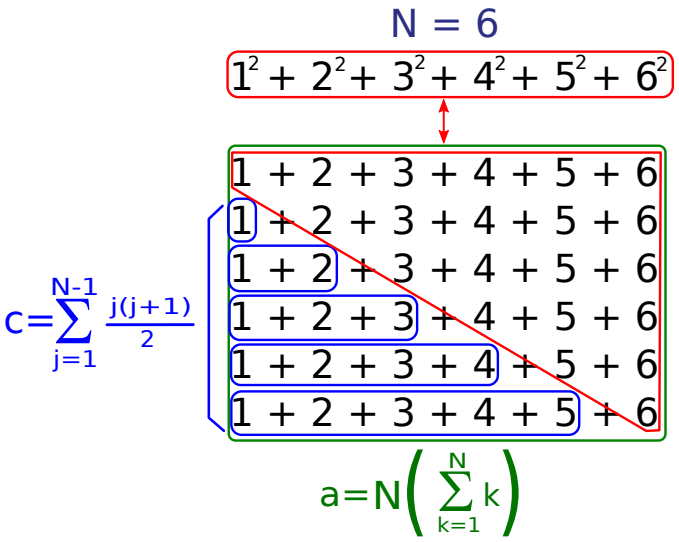


FIG. 4: An illustration on how to calculate sum of squares.

how we can calculate this upper red triangle, it is nothing but the sum in the green box a minus the sum in the lower blue triangle c.

The sum in the green box is given by $a = 6 \times (1+2+3+4+5+6) = N\sum_{k=1}^{N=6} k$, the lower blue triangle is given by $c = \sum_{j=1}^{N-1}\sum_{m=1}^{j} m = \sum_{j=1}^{N-1} j(j+1)/2$, where we have used the famous gauss formula for $\sum_{m=1}^{N} m = N(N+1)/2$.

The sum $\sum_{i=1}^{N} i^2$ is then given by

$$\begin{aligned}
\sum_{i=1}^{N} i^2 &= a - c \\
&= \left(N \sum_{k=1}^{N} k \right) - \left(\sum_{j=1}^{N-1} \frac{j(j+1)}{2} \right) \\
&= \left(N \frac{N(N+1)}{2} \right) - \left(\frac{1}{2} \sum_{j=1}^{N-1} j^2 + \frac{1}{2} \sum_{j=1}^{N-1} j \right) \\
\sum_{i=1}^{N} i^2 &= \left(N \frac{N(N+1)}{2} \right) - \left(\left\{ \frac{1}{2} \sum_{j=1}^{N} j^2 - \frac{1}{2} N^2 \right\} + \frac{1}{2} \sum_{j=1}^{N-1} j \right) \\
\sum_{i=1}^{N} i^2 + \frac{1}{2} \sum_{j=1}^{N} j^2 &= \left(\frac{N^3 + N^2}{2} \right) - \left(-\frac{1}{2} N^2 + \frac{1}{2} \frac{(N-1)N}{2} \right) \\
&= \frac{N^3 + N^2}{2} + \frac{1}{2} N^2 - \frac{(N-1)N}{4} \\
\frac{3}{2} \sum_{i=1}^{N} i^2 &= \frac{2N^3 + 2N^2 + 2N^2 - N^2 + N}{4} \\
&= \frac{2N^3 + 3N^2 + N}{4} = \frac{N(N+1)(2N+1)}{4} \\
\rightarrow \sum_{i=1}^{N} i^2 &= \frac{N(N+1)(2N+1)}{6}
\end{aligned}$$

And that is the formula we are after!

Bonus !!!

We can use the above result to calculate c explicitly in terms of N

$$\begin{aligned}
c \equiv \sum_{j=1}^{N} \frac{j(j+1)}{2} &= \frac{1}{2} \sum_{j=1}^{N} j^2 + \frac{1}{2} \sum_{j=1}^{N} j \\
&= \frac{1}{2} \frac{2N^3 + 3N^2 + N}{6} + \frac{1}{2} \frac{N^2 + N}{2} \\
&= \frac{2N^3 + 6N^2 + 4N}{12} = \frac{N^3 + 3N^2 + 2N}{6} \\
\sum_{j=1}^{N} \frac{j(j+1)}{2} &= \frac{N^3 + 3N^2 + 2N}{6}
\end{aligned}$$

Now, we can use this result to get something interesting. Note that the blue lower triangle

c in Fig. 4 can be seen as a sum in the vertical direction, *i.e.*

$$\begin{aligned} c &= 1(5) + 2(4) + 3(3) + 4(2) + 5(1) \\ &= 1(5-0) + 2(5-1) + 3(5-2) + 4(5-3) + 5(5-4) \\ &= 5(1+2+3+4+5) - (2(1) + 3(3) + 4(2) + 5(1)) \\ \frac{N^3+3N^2+2N}{6} &= \left(N\sum_{i=1}^{N} i\right) - \sum_{i=2}^{N} i(i-1), \quad N = 5 \end{aligned}$$

where in the last line we have used the explicit formula for c. Switching sides and generalizing to any N

$$\begin{aligned} \sum_{i=2}^{N} i(i-1) &= \left(N\sum_{i=1}^{N} i\right) - \frac{N^3+3N^2+2N}{6} \\ &= \frac{N^3+N^2}{2} - \frac{N^3+3N^2+2N}{6} \\ &= \frac{3N^3+3N^2-N^3-3N^2-2N}{6} \\ &= \frac{2N^3-2N}{6} \\ \sum_{i=2}^{N} i(i-1) &= \frac{N^3-N}{3} \end{aligned}$$

We can now in turn use this result to calculate $1 + 2(1) + 3(2) + 4(3) + \cdots + N(N-1)$, *i.e.*

$$1 + \sum_{i=2}^{N} i(i-1) = 1 + \sum_{i=2}^{N} \frac{i!}{(i-2)!} = \frac{N^3-N}{3} + 1$$

Looking back at how we calculated $\sum_{i=1}^{N} i^2$ using Fig. 4, we can easily generalize it to calculate $\sum_{i=1}^{N} i^k, k > 2$, see Fig. 5. To simplify things let's designate $f_k(N) \equiv \sum_{i=1}^{N} i^k$ as the sum of k-th powers.

From Fig. 5 it's immediately apparent that we can calculate $f_k(N)$ through

$$\begin{aligned} f_k(N) &= N f_{k-1}(N) - \sum_{m=1}^{N-1} f_{k-1}(m) \\ &= N f_{k-1}(N) - \left(\sum_{m=1}^{N} f_{k-1}(m) - f_{k-1}(N)\right) \\ f_k(N) &= (N+1) f_{k-1}(N) - \sum_{m=1}^{N} f_{k-1}(m) \end{aligned}$$

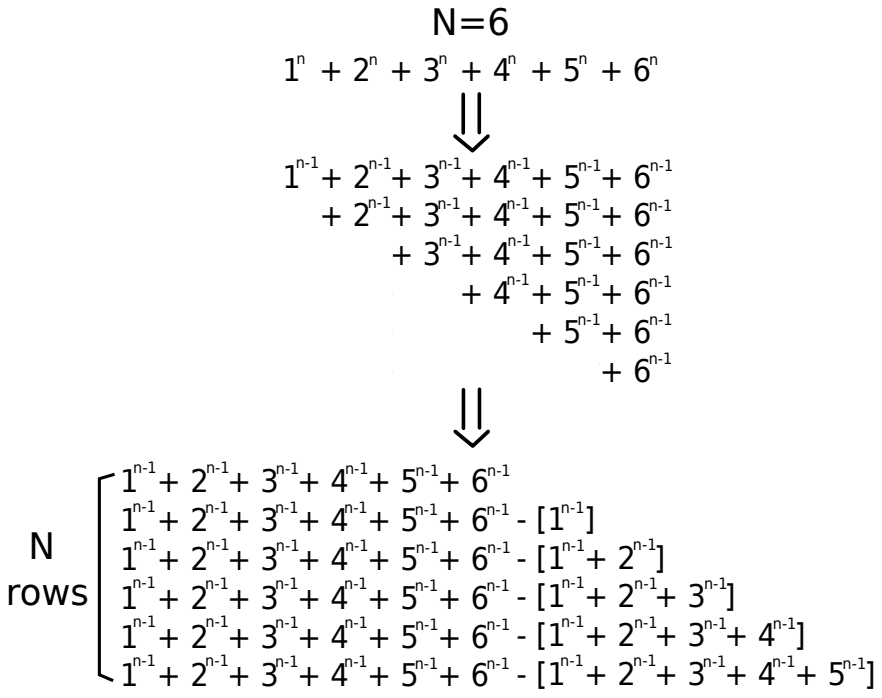


FIG. 5: An illustration on how to calculate sum of n-th powers.

which is a recursion relation for the sum of any k-th powers.

Let's see a concrete example by calculating $f_3(N) = \sum_{i=1}^{N} i^3$

$$
\begin{aligned}
f_3(N) &= (N+1)f_2(N) - \sum_{m=1}^{N} f_2(m) \\
&= (N+1)\frac{2N^3+3N^2+N}{6} - \sum_{m=1}^{N} \frac{2m^3+3m^2+m}{6} \\
\sum_{i=1}^{N} i^3 + \frac{2}{6}\sum_{m=1}^{N} m^3 &= \frac{2N^4+5N^3+4N^2+N}{6} - \frac{2N^3+3N^2+N}{12} - \frac{N^2+N}{12} \\
\frac{4}{3}\sum_{i=1}^{N} i^3 &= \frac{4N^4+8N^3+4N^2}{12} \\
&= \frac{N^4+2N^3+N^2}{3} = \frac{N^2(N+1)^2}{3} \\
\rightarrow \sum_{i=1}^{N} i^3 &= \frac{N^2(N+1)^2}{4}
\end{aligned}
$$

so *theoretically* we can compute the explicit formula for the sum of any k-th power using

$$f_k(N) = (N+1)f_{k-1}(N) - \sum_{m=1}^{N} f_{k-1}(m)$$

(with $f_0(N) = \sum_{i=1}^{N} 1 = N$) which is the gist of the algorithm. The following list shows $f_k(N)$ up to $k = 15$ when the expression starts becoming unwieldy, but there's a pattern, the even k starts with $N(N+1)(2N+1)$ while the odd k starts with $N^2(N+1)^2$.

$$\begin{aligned}
f_0 &= N \\
f_1 &= \frac{N^2+N}{2} &&= \frac{N(N+1)}{2} \\
f_2 &= \frac{2N^3+3N^2+N}{6} &&= \frac{N(N+1)(2N+1)}{6} \\
f_3 &= \frac{N^4+2N^2+N}{4} &&= \frac{N^2(N+1)^2}{4} \\
f_4 &= \frac{6N^5+15N^4+10N^3-N}{30} &&= \frac{N(N+1)(2N+1)(3N^2+3N-1)}{30} \\
f_5 &= \frac{2N^6+6N^5+5N^4-N^2}{12} &&= \frac{N^2(N+1)^2(2N^2+2N-1)}{12} \\
f_6 &= \frac{6N^7+21N^6+21N^5-7N^3+N}{42} &&= \frac{N(N+1)(2N+1)(3N^4+6N^3-3N+1)}{42} \\
f_7 &= \frac{3N^8+12N^7+14N^6-7N^4+2N^2}{24} &&= \frac{N^2(N+1)^2(3N^4+6N^3-N^2-4N+2)}{24} \\
f_8 &= \frac{10N^9+45N^8+60N^7-42N^5+20N^3-3N}{90} &&= \frac{N(N+1)(2N+1)(5N^6+15N^5+5N^4-15N^3-N^2+9N-3)}{90} \\
f_9 &= \frac{2N^{10}+10N^9+15N^8-14N^6+10N^4-3N^2}{20} &&= \frac{N^2(N+1)^2(N^2+N-1)(2N^4+4N^3-N^2-3N+3)}{20} \\
f_{10} &= \frac{6N^{11}+33N^{10}+55N^9-66N^7+66N^5-33N^3+5N}{66} &&= \frac{N(N+1)(2N+1)(N^2+N-1)(3N^6+9N^5+2N^4-11N^3+3N^2+10N-5)}{66} \\
f_{11} &= \frac{2N^{12}+12N^{11}+22N^{10}-33N^8+44N^6-33N^4+10N^2}{24} &&= \frac{N^2(N+1)^2(2N^8+8N^7+4N^6-16N^5-5N^4+26N^3-3N^2-20N+10)}{24} \\
f_{12} &= \frac{210N^{13}+1365N^{12}+2730N^{11}-5005N^9+8580N^7-9009N^5+4550N^3-691N}{2730} \\
&= \frac{N(N+1)(2N+1)(105N^{10}+525N^9-1050N^7-1190N^6+2310N^5+1420N^4-3285N^3-287N^2+2073N-691)}{2730} \\
f_{13} &= \frac{30N^{14}+210N^{13}+455N^{12}-1001N^{10}+2145N^8-3003N^6+2275N^4-691N^2}{420} \\
&= \frac{N^2(N+1)^2(30N^{10}+150N^9-400N^7-326N^6+1052N^5+367N^4-1786N^3-202N^2+1382N-691)}{420} \\
f_{14} &= \frac{6N^{15}+45N^{14}+105N^{13}-273N^{11}+715N^9-1287N^7+1365N^5-691N^3+105N}{90} \\
&= \frac{N(N+1)(2N+1)(3N^{12}+18N^{11}+24N^{10}-45N^9-81N^8+144N^7+182N^6-345N^5-217N^4+498N^3+44N^2-315N+105)}{90} \\
f_{15} &= \frac{3N^{16}+24N^{15}+60N^{14}-182N^{12}+572N^{10}-1287N^8+1820N^6-1382N^4+420N^2}{48} \\
&= \frac{N^2(N+1)^2(3N^{12}+18N^{11}+21N^{10}-60N^9-83N^8+226N^7+203N^6-632N^5-226N^4+1084N^3-122N^2-840N+420)}{48}
\end{aligned}$$

The above list was generated using maxima with a small C code to generate the maxima script. The algorithm is quite simple as can be seen from our calculation of $f_3(N)$ above. To compute f_k we need the coefficient of each N^p in $f_{k-1}(N)$, note that $f_m(N)$ is a

polynomial of order N^{m+1}. That is

$$f_{k-1}(N) = \sum_{p=0}^{k} C_p^{(k-1)} N^p$$

The superscript in $C_p^{(k-1)}$ indicates which f_{k-1} this coefficient belongs to. We then have

$$\begin{aligned}
f_k &= (N+1)f_{k-1} - \sum_{m=1}^{N} f_{k-1}(m) \\
&= (N+1)f_{k-1} - \sum_{m=1}^{N}\sum_{p=0}^{k} C_p^{(k-1)} m^p \\
&= (N+1)f_{k-1} - \sum_{p=0}^{k}\left(C_p^{(k-1)} \times \sum_{m=1}^{N} m^p\right) \\
&= (N+1)f_{k-1} - \sum_{p=0}^{k}\left(C_p^{(k-1)} f_p\right) \\
f_k &= (N+1)f_{k-1} - C_k^{(k-1)} f_k - \sum_{p=0}^{k-1} C_p^{(k-1)} f_p \\
f_k + C_k^{(k-1)} f_k &= (N+1)f_{k-1} - \sum_{p=0}^{k-1} C_p^{(k-1)} f_p \\
\rightarrow f_k &= \frac{1}{\left(1 + C_k^{(k-1)}\right)} \times \left((N+1)f_{k-1} - \sum_{p=0}^{k-1} C_p^{(k-1)} f_p\right)
\end{aligned}$$

The moral of the story here is that although Gauss's method (adding the first and last number and second and second last and so on) works for $\sum i$ it doesn't work for any other power of i^n. A blessing and a curse, I don't know what's worse :)

-=-

Now on to something fun :)

Here's the proof that 1 is equal to any number n

$$
\begin{aligned}
a &= b \\
ab^{n-1} &= b^n \\
a^{n-1}b &= b^n \\
a^{n-1}b - a^n &= b^n - a^n \\
a^{n-1}\cancel{(b-a)} &= \cancel{(b-a)}\sum_{i=1}^{n} b^{n-i}a^{i-1} \\
a^{n-1} &= \sum_{i=1}^{n} a^{n-i}a^{i-1} \\
\cancel{a^{n-1}} &= n\,\cancel{a^{n-1}} \\
1 &= n
\end{aligned}
$$

In case you forget how to factor $b^n - a^n = (b-a)\sum_{i=1}^{n} b^{n-i}a^{i-1}$, you start by pulling out the factors one at a time

$$
\begin{aligned}
b^n - a^n &= (b-a)(b^{n-1} + \dots \\
\rightarrow -a)(b^{n-1} &= -b^{n-1}a
\end{aligned}
$$

so you need to add $(b^{n-1} + b^{n-2}a \cdots \rightarrow (bb^{n-2}a)$ to cancel $-a)(b^{n-1} = -b^{n-1}a$ introduced by the first factor $(b^{n-1} + \dots$, but then again you generate $-a)(b^{n-2}a = -b^{n-2}a^2$ so you repeat the process until you get

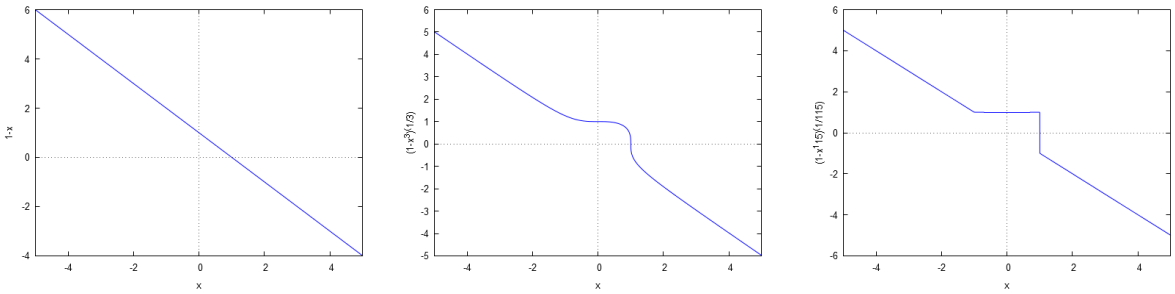
$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + b^{n-3}a^2 + \cdots + ba^{n-2} + a^{n-1})$$

-=-

A stroll with Monsieur Fermat

The most annoying thing about Fermat's last theorem is that it looks so simple. Proving it however, was almost impossible even for the greatest mathematical minds that ever walked this planet. So what am I doing here? Let's see.

One might be tempted to think that going from a pythogorean equation with power of two to a cubic one is trivial, after all $3 = 2 + O(1)$:) As an illustration, remember the quadratic formula? We can all derive it in five minutes or less, what it does is nothing



(a) Plot for $x^1 + y^1 = 1$ (b) Plot for $x^3 + y^3 = 1$ (c) Plot for $x^{115} + y^{115} = 1$

FIG. 6: The higher the power the sharper the bump. For all three plots, vertical axis is y and horizontal axis is x.

but completing the square. Not so with the cubic formula, we can't just complete the cube. The solution to a cubic equation is in general a cube root of a square root in the form of $(a + (b^{1/2}))^{1/3}$. Yup the roots are nested. It's not as intuitive as I thought.

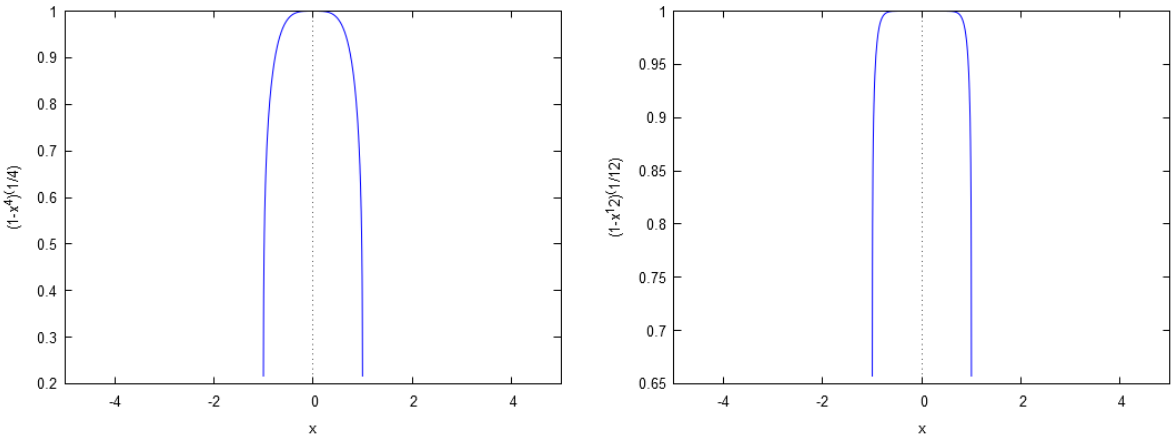
Prolegomenon: some greek sounding word for "Introduction"

So of course I'm not going to even try proving Fermat, what I'm doing here is to see if there's anything I can learn from it.

We start this discussion with something more obvious and lots of doodles :) The great divide for $n \leq 2$ and $n \geq 3$ can be seen immediately from the plot $x^n + y^n = 1$. Let's start with $x^1 + y^1 = 1$ which is a straight line, Fig. 6a, bumping it into $x^3 + y^3 = 1$ adds a little "bump" to that straight line, Fig. 6b, while increasing the power to 115 makes that bump a lot sharper, Fig. 6c.

But that's not the only great division, even powers have all the same shape (which is different from the odd one), Fig. 7a, except that the corners get sharper the higher the power, Fig. 7b. The only exception to this is the Pythagorean equation $x^2 + y^2 = 1$ which is a circle and the only plot with maximum symmetry among $x^n + y^n = 1$.

One interesting thing here (thanks to Wildberger) is whether those curves actually exist, *e.g.* for $x^3 + y^3 = 1$ the only rational solution is $(1, 0), (0, 1)$ this means that all other points on the curve are real numbers, however, the curves in Fig. 6b, 6c, 7 are calculated by a computer program and we all know that computers can *not* process real numbers.



(a) Plot for $x^4 + y^4 = 1$

(b) Plot for $x^{12} + y^{12} = 1$

FIG. 7: Sharper corners for sharper powers. For both plots, vertical axis is y and horizontal axis is x.

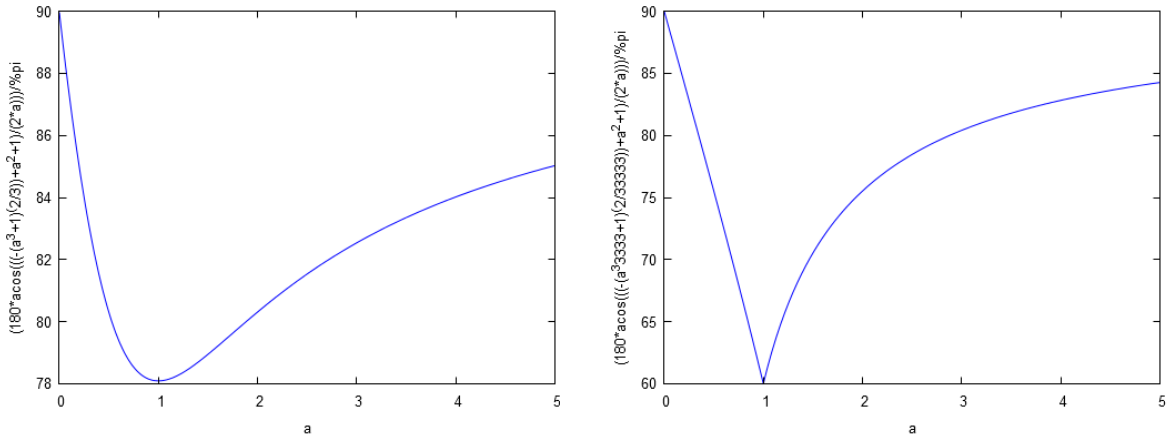
This seems to suggest that there is something special about Pythagoras, looks like "no. 2" is actually number 1 :)

Idea #1

Thanks to Wildberger's youtube videos the following ideas poped up in my head. The ancient greeks didn't use numbers like we do today, what we use are actually the arabic numerals. The greeks used line segments. To add one line segment on another you just join the two line segments to get a longer one. So I was curious as to what geometric meaning Fermat's last theorem presents us.

This prodded me to the idea that the angles between the two line segments (in the case where we are adding two numbers) actually means something. For example, in the case of $a^1 + b^1 = c^1$, we are adding two line segments a and b where the angles between them is $\pi/1$.

For the case of $a^2 + b^2 = c^2$, the angles between line segments a and b is actually $\pi/2$. Now how about the angle in $a^n + b^n = c^n$ with n any arbitrary number ≥ 3? is the angle between a and b given by π/n?



(a) Angle plot for triangle $(a, 1, \sqrt{a^3 + 1^3})$ (b) Angle plot for $(a, 1, \sqrt{a^{33333} + 1^{33333}})$

FIG. 8: For both plots, vertical axis is the angle between the 2 sides of triangle a and b with a fixed b normalized to 1

Using the cosine rule, the angle between line segments a and b is

$$\cos\theta = \frac{(a^n + b^n)^{2/n} - a^2 - b^2}{-2ab}$$

and its derivatives (if they mean anything) are given by

$$\frac{d\cos\theta}{dn} = (b^n + a^n)^{\frac{2}{n}} \cdot \left(\frac{2 \cdot (b^n \cdot \log(b) + a^n \cdot \log(a))}{(b^n + a^n) \cdot n} - \frac{2 \cdot \log(b^n + a^n)}{n^2} \right)$$

$$\frac{d\cos\theta}{da} = \frac{(b^n + a^n)^{\frac{2}{n}} - b^2 - a^2}{2 \cdot a^2 \cdot b} - \frac{2 \cdot a^{n-1} \cdot (b^n + a^n)^{\frac{2}{n}-1} - 2 \cdot a}{2 \cdot a \cdot b}$$

$$\frac{d\cos\theta}{db} = \frac{(b^n + a^n)^{\frac{2}{n}} - b^2 - a^2}{2 \cdot a \cdot b^2} - \frac{2 \cdot b^{n-1} \cdot (b^n + a^n)^{\frac{2}{n}-1} - 2 \cdot b}{2 \cdot a \cdot b}$$

The above exercise yields a provocative result. To simplify the problem, we can ask if $r_1^n + 1 = r_2^n$ has rational solutions $r_1 \equiv a/b$ and $r_2 \equiv c/b$. This normalization seems a bit odd at first. But it is actually easier to visualize things this way if one thinks of $r_1, 1$, and r_2 as sides of a triangle with r_2 the longest side among the three. This way one of the sides of the triangle is fixed to 1 while we vary the other two sides.

The angle θ between a and b is *not* fixed for $n \geq 3$. The angle always starts off at 90° when $r_1 = 1$ and will quickly plummet as r_1 increases, Fig. 8a. It will then climb back up *asymptotically* to 90°. As we go higher in n, the initial dip and rebound will become

more drastic but the overall behavior will remain the same, Fig. 8b. So there is a clear dichotomy of the behavior of the angle for $n = 1, 2$ and $n \geq 3$.

As I was thinking about what this might mean I was reminded of Galois theory (you might want to check Wildberger's youtube lecture on this as an introduction, see his history of mathematics series). The reason why quintic equations have no solution is because the symmetry structure does not cascade as it does for quartic and below. Can it be that a symmetry structure plays a role here as well? If so, what kind of symmetry?

Since the triangle formed by a, b, and c is a right triangle only for $n = 2$ I imagine it has something to do reflection symmetry with respect to the x and y axes in the cartesian coordinate. Say the triangle is given by these three points $(0, 0)$, $(a, 0)$, and $(0, b)$, by doing horizontal and vertical reflections (plus translations) we can cover the entire cartesian plane, *e.g.* a reflection w.r.t the x axis generates the triangle $(0, 0)(a, 0)(0, -b)$ while a subsequent relection w.r.t the y axis gives $(0, 0)(-a, 0)(0, -b)$. If we then translate this triangle using a vector $(a\hat{x}, b\hat{y})$ - and combining it with our original triangle - we will get a rectangle $(0, 0)(a, 0)(a, b)(0, b)$, repeating this process indefinitely produces a covering for the entire cartesian plane.

Not so for $n \geq 3$, since the angle can never reach 90° there is always some area that is not covered by the above symmetry operations. From this observation, the symmetry of interest is the discrete symmetry Z_2 (plus some translation symmetry).

Idea #2

At this point I'm not sure what this all means, or if it is actually profitable to pursue this line of thought since I can't find anything concrete to say about it.

My next idea is whether there is a higher dimensional object that represents $a^n + b^n = c^n$ for different n's. Since $n = 2$ is a triangle maybe $n \geq 3$ is some 3d object (after all $a^3 + b^3$ means that we are adding two volumes to get a third one) and $n = 4$ is an even higher dimensional object and so on. But again, I don't know what to make of it.

Idea #3

I then was reminded of quantum mechanics. Before learning quantum mechanics I wouldn't even imagine of something that have discrete and continuum spectrum in one

solution but there is (there are actually plenty examples in quantum mechanics). How is it related to Fermat? Because in Fermat, $a^n + b^n = c^n$ happens to just miss all of the integers a, b, c for $n \geq 3$ (while taking in real values just fine). So these seems like the inverse hydrogen atom, instead of having discrete spectrum it has an inverse discrete spectrum where all integers are skipped (is there such a physical object in the universe?).

So maybe there's a differential equations that yield a solution where all integers are skipped? can we just do a Fourier transform of $a^n + b^n = c^n$? Just a wild guess :)

Idea #4

Another strange idea I came up with was again inspired by symmetry. Say we move to the complex plane, does $a^n + b^n = c^n$ exhibit conformal symmetry? dilatation symmetry?

Why do they even relate to Fermat? It's all about angles! Back to idea #1, the angles between the two line segments a and b is always fixed for $n = 2$, this somewhat resembles conformal symmetry where the angle between two lines is always preserved. So maybe there's a conformal/dilatation symmetry preserved by $n = 2$ which is not preserved by $n \geq 3$. Again, just a wild guess, nothing concrete :)

Fun Excursions

Since the above ideas didn't get me anywhere I was thinking about other aspects of Fermat. We know we can't find solutions with just two numbers a and b, how many numbers do we need to get a solution? *i.e.* $a^n + b^n + d^n + \cdots = c^n$.

So I began with $n = 3$, and lo and behold, I found integer solutions for $a^3 + b^3 + d^3 = c^3$. Do these solutions have a pattern behind them? Euclid to the rescue ... (maybe)

Running along Euclid

Euclid (some people said it was the Babylonians) gave us an algorithm to find the solutions of $a^2 + b^2 = c^2$, which is as follows

$$(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$$

you simply pluck in whatever integers you fancy into m and n and Voila! you get a Pythagorean solution (one is usually interested in co-prime solutions, *i.e.* a, b, c have no common divisor). So maybe a similar algorithm exists for $a^3 + b^3 + d^3 = c^3$?

Well, there are actually plenty of algorithms for $a^3+b^3+d^3=c^3$. They are categorized by how many parameters are used and more imprtantly by what solutions they produce, it turns out that the different algorithms generate only selective solutions or families unlike the Babylonian algorithm above which generate every solution for the Pyhtagorean equation.

The algorithm above is a two parameter algorithm, there is actually a one parameter algorithm as well. But before we discuss all this let me show you my misadventures with these algorithms as I try to wildly guess what they might be like.

Euclidean roullette

What we really want to know is how those Babylonians or Euclid came up with the formula (or they might have just been really smart and saw the solution immediately, me no Euclid). At a glance it looked like just a good guess. So let say we start with a simplest of most guesses

$$a^2+(a+y)^2=(a+z)^2$$

we see that the solution for a has to be

$$\begin{aligned}(a+z)^2-\left\{(a+y)^2+a^2\right\}=z^2+2az-y^2-2ay-a^2=0\\ \rightarrow a=\pm\sqrt{2}\sqrt{z^2-yz}+z-y\end{aligned}$$

which in turn means that

$$z^2-yz=2w^2$$

for some integer w. One obvious solution is

$$\begin{aligned}z&=2w\\ y&=w\\ a&=2w+2w-w=3w\\ \rightarrow (3w)^2+(4w)^2&=(5w)^2\end{aligned}$$

And we get the $3,4,5$ solution and its integer multiples. There are actually many other solutions for $z^2-yz=2w^2$, this turns out to be a blessing in disguise, we shall see this in a little bit.

Now, what happens if we choose a different starting point?

$$a^2 + (a \cdot y)^2 = (a + z)^2$$
$$(a + z)^2 - \left\{(a \cdot y)^2 + a^2\right\} = z^2 + 2az - a^2y^2 = 0$$
$$\rightarrow a = \frac{z \pm \sqrt{y^2 + 1}}{y^2}$$

This time, we can only solve it if we change y into a rational number p/q but solving $p^2/q^2 + 1 = r^2/s^2$ is just solving the Pythagorean equation in the first place. However, the above equation also means

$$(ay^2 - z)^2 = y^2 + 1$$
$$a^2y^4 - 2ay^2z + z^2 = y^2 + 1$$
$$a^2p^4 - 2ap^2q^2z + z^2 = p^2q^2 + q^4$$

So why all this asinine substitutions?

Lagniappe

I wanted to say "small lagniappe" but lagniappe itself is already small :) The above two examples actually show us how to use Euclid's algorithm to solve other equations (instead of generating a new algorithm to solve the pyhtagorean equation). The first example

$$z^2 - yz = 2w^2$$

can be paraphrased into the Pythagorean equation, because it was a consequence of

$$a^2 + (a + y)^2 = (a + z)^2$$
$$(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$$

which means that

$$a = m^2 - n^2$$
$$a + y = 2mn$$
$$a + z = m^2 + n^2$$

By solving the above equations, the algorithm for the solutions of $z^2 - yz = 2w^2$ is then

$$z = 2n^2, \quad y = n^2 + 2mn - m^2, \quad w = n^2(n-m)^2$$

i.e. we can choose any two integers m and n and the integer solutions for $z^2 - yz = 2w^2$ are given above (note that we can also choose $a = -(m^2 - n^2)$ generating a slightly different final result).

It's the same with $a^2p^4 - 2ap^2q^2z + z^2 = p^2q^2 + q^4$, equating terms just like before gives

$$a = m^2 - n^2, \quad z = 2n^2, \quad y = \frac{p}{q} \rightarrow p = 2mn, q = m^2 - n^2$$

I called the above lagniappe because it was just a small boon. The boon is that there's actually a way to solve these seemingly random equations, *e.g.* $z^2 - yz = 2w^2$, $a^2p^4 - 2ap^2q^2z + z^2 = p^2q^2 + q^4$. However, those solutions we found above are not unique because Euclid's algorithm itself is not unique, as we shall see.

There's also another fun thing related to this, if we start with

$$(a+z)^2 + (a-z)^2 = (a \cdot y)^2$$
$$(a+z)^2 + (a-z)^2 - (a \cdot y)^2 = -2z^2 + a^2y^2 - 2a^2 = 0$$

we can now choose which one to solve

$$\text{solving } y: \ y = \pm\frac{\sqrt{2}\sqrt{z^2+a^2}}{a}$$
$$\text{solving } z: \ z = \pm\frac{a\sqrt{y^2-2}}{\sqrt{2}}$$
$$\text{solving } a: \ a = \pm\frac{\sqrt{2}z}{\sqrt{y^2-2}}$$

note that we are not solving y followed by z followed by a, we are only solving *one* of them but since we have three variables we can choose which one to solve. We then have the following constraints

$$\text{from solving } y \text{ we get}: \quad z^2 + a^2 = 2w^2$$
$$\text{from solving } z \text{ or } a \text{ we get}: \quad 2u^2 + 2 = y^2$$

matching terms $(a+z)^2+(a-z)^2=(a\cdot y)^2 \rightarrow (m^2-n^2)^2+(2mn)^2=(m^2+n^2)^2$

$$\begin{aligned} a &= m^2-n^2+2mn \\ z &= m^2-n^2-2mn \\ y &= \frac{2(m^2+n^2)}{m^2-n^2+2mn} \\ \rightarrow w &= m^2+n^2 \\ \rightarrow u &= \frac{n^2+2mn-m^2}{n^2-2mn-m^2} \end{aligned}$$

Note that we have inserted factor of 2 where convenient :) It is now obvious that we can play this game forever, just substitute any function you fancy into the Pythagorean equation and match terms, just be careful about factors of 2.

The moral of the story here is that the Babylonians actually gave us quite a powerful tool to solve deophantine equations (if we can massage them into the Pythagorean one), these ancient people were not just good at building towers :) But in a broader sense it means that we can map one Diophantine equation into a simpler one, which is good.

Euclidean bulldozer

We now come back to the subject of utilizing unsophisticated guesses in meting out algorithms such as Euclid's. The first attempt above $a^2+(a+y)^2=(a+z)^2$ wasn't fruitful and in a glance this should be the same as $(a+x)^2+(a+y)^2=(a+z)^2$ since we can always shift $a+x\rightarrow a$. However, it is not true. Adding an extra degree of freedom like x is a big deal, how big? let's see.

We can by brute force find an algorithm for $a^2+b^2=c^2$ from the most generic starting point

$$(a+z)^2=(a+x)^2+(a+y)^2$$
$$(a+z)^2-\{(a+x)^2+(a+y)^2\}=z^2+2az-y^2-2ay-x^2-2ax-a^2=0$$

Now we have four choices, we can solve for either, a, x, y, or z, which one we solve determines our fate. Solving for a gives

$$a=z-y-x\pm\sqrt{2}\sqrt{z^2+(-y-x)z+xy}$$

which means that

$$\begin{aligned} z^2+(-y-x)z+xy &= 2p^2 \\ \rightarrow x &= \frac{z^2-yz-2p^2}{z-y} \\ \rightarrow a &= -\frac{(y+2|p|)z-y^2-2|p|y-2p^2}{z-y} \end{aligned}$$

and there we have it, the full algorithm is

$$\begin{array}{lll} a+x &= z-y-2|p| & \rightarrow (z-y-2|p|)(z-y) \\ a+y &= -\frac{2|p|z-2|p|y-2p^2}{z-y} & \rightarrow 2|p|z-2|p|y-2p^2 \\ a+z &= \frac{z^2+(-2y-2|p|)z+y^2+2|p|y+2p^2}{z-y} & \rightarrow z^2+(-2y-2|p|)z+y^2+2|p|y+2p^2 \end{array}$$

we can multiply each term by $(z-y)$ if we want to and flip the minus sign. So apparently we can just go by brute force to bulldoze out an algorithm, in this case we have *three* final degrees of freedom.

However, we can also solve x first

$$x = -a \pm \sqrt{z^2+2az-y^2-2ay}$$

The next step is crucial, do we solve for a or y, z? Let's see, solving for a gives

$$a = -\frac{z^2-y^2-p^2}{2z-2y}$$

which is all well and good. Had we chosen to solve for y we'd have gotten

$$\begin{aligned} y &= -a \pm \sqrt{z^2+2az-p^2+a^2}, \quad \text{solving for } a \text{ next} \\ \rightarrow a &= -z \pm \sqrt{q^2+p^2} \quad \text{OR solving for } z \text{ next} \\ \rightarrow z &= -a \pm \sqrt{q^2+p^2} \end{aligned}$$

Solving either a or z produces no headway since now we have to solve for $q^2+p^2=r^2$ which is our original problem. Thus the order we choose to solve the degrees of freedom matters. Can we follow the same mechanical steps to arrive at algorithms for $a^3+b^3+c^3=d^3$?

Cubic conundrum

Applying the same guessing strategy to $a^3+b^3+c^3=d^3$ turns out to be tricky. My first attempt was $a^3+(a+y)^3+(a+z)^3=(a+w)^3$ but the cube root of a square root (generated by solving a in terms of y, z, w) was too hard to handle.

The next natural progression is of course $(a+x)^3+(a+y)^3+(a+z)^3=(a+w)^3$ to allow more flexibility but just like the preceding attempt the resulting equations quickly became intractable. My next attempt was to simplify things by adding a minus sign $(a+x)^3+(-a+y)^3+(a+z)^3=(a+w)^3$ such that we no longer have a^3, however even with this modification the equations were still too long to be discernible

$$\begin{aligned}(a+w)^3-\{(a+x)^3+(-a+y)^3+(a+z)^3\}&=0\\ \implies a &= (6z+6y+6x-6w)^{-1}(-3z^2+3y^2-3x^2+3w^2\\ &\quad \pm\sqrt{3}(-z^4+(-4y-4x+4w)z^3+(-6y^2+6x^2-6w^2)z^2+\\ &\quad (-4y^3-4x^3+4w^3)z-y^4+(4w-4x)y^3+(6w^2-6x^2)y^2+\\ &\quad (4w^3-4x^3)y-x^4+4wx^3-6w^2x^2+4w^3x-w^4)^{1/2})\end{aligned}$$

here we have a super long square root (no more cube root since we have no a^3). The problem with this square root is that it contains fourth powers z^4, y^4, x^4. Solving for x, y or z inside the square root generates more problems, *i.e.* roots containing higher powers which in turn generates more complicated equations.

My final attempt was $(a+x)^3+(-b+y)^3+(b+z)^3=(a+w)^3$ however, solving a or b still generates ridiculously long square roots, solving for x however seems to be a lot simpler

$$x=(-z^3-3az^2-3a^2z-y^3-3by^2-3b^2y+w^3+3aw^2+3a^2w-b^3)^{1/3}+b$$

We are now faced with the same decision tree as discussed previously, who's next? Turns out that solving for y, z or w generates $p^3+q^3+r^3=s^3$ just like the Pythagorean case above. Seems like we need to solve b or a first but b gives a better outcome

$$b=(-z^3-3az^2-3a^2z+w^3+3aw^2+3a^2w-p^3)^{1/3}+y$$

It looks good as y decouples. Next we need to solve for a, solving any other variables generates $p^3+q^3+r^3=s^3$

$$a=\left(3z^2-3w^2\pm\sqrt{3}\sqrt{-z^4+4wz^3-6w^2z^2+(4w^3-4q^3-4p^3)z-w^4+(4q^3+4p^3)w}\right)\times (6z-6w)^{-1}$$

That long square root is a killer, solving it for z or w generates a root of a cube root of a root of a root, *four* nested roots! *Q.E.D* (*Q*.uickly *E*.nded *D*.reams)

In short, a brute force guessing game might be theoretically possible but not recommended :) in other words, it works well for certain obvious cases and thus it only has a very limited merit.

Euclidean apery

We now go back to guessing our own version of an algorithm for the Pythagorean equation (without a bulldozer). In our case, the goal here is to solve a in terms of two completely free parameters z and y. Let's try one with a quite generic structure

$$\begin{aligned}(a-z)^2+(a\cdot y)^2&=(a+z)^2\\(a+z)^2-\left\{(a\cdot y)^2+(a-z)^2\right\}&=4az-a^2y^2=0\\\rightarrow a&=\frac{4z}{y^2}\end{aligned}$$

Yay! It works! (the other solution is $a=0$). The full solution is then

$$\begin{aligned}\left(\frac{4z}{y^2}-z\right)^2+\left(\frac{4zy}{y^2}\right)^2&=\left(\frac{4z}{y^2}+z\right)^2\\\Rightarrow\left(4z-zy^2\right)^2+(4zy)^2&=\left(4z+zy^2\right)^2\end{aligned}$$

Although this is different from Euclid's, given any integer z and y we can generate a solution. Looks like these algorithms are not unique (as advertised) although I'm not sure (or too lazy to prove) that this version covers all solutions of the Pythagorean equation.

A more straighforward guess will be

$$(a+z)^2-(a-z)^2=4az$$

we can simply promote $a\rightarrow a^2, z\rightarrow z^2$ to get

$$(a^2+z^2)^2-(a^2-z^2)^2=4a^2z^2=(2az)^2$$

which is Euclid's (or the Babylonians').

So the trick in this guessing game is that you guess a term on the LHS and another on the RHS. You then want the difference between those two sides *not* to be a sum of terms, because then you're just going back to square one by needing to solve some equation in

the form of $x_1^{n_1} + x_2^{n_2} = a^m$. For example, if you're naively try to solve fermat's cubic last theorem the same way

$$(a-z)^3 + X^3 = (a+z)^3$$
$$\rightarrow X^3 = 2z^3 + 6a^2 z$$

You just get another deophantine equation to solve, back to square one.

One thing to note here is that we at first assume that all terms in our guess has the same power, *e.g.* a, z both have the power of one, we can for example promote things to a^n, z^m but our goal for now is to see how far these simple guesses take us.

Jabs in the dark

We now begin our journey in guessing the form of algorithm for $a^3 + b^3 + c^3 = d^3$.

1. I first tried the following solution (which looks simplest of all)

   $$a^3 + b^3 + K^3 = (a+b)^3$$

   with some unknown integer K (in Euclid's case K is just $2mn$). Let's see if we get lucky. Now K^3 is obviously $3ab(b+a)$. Sadly, the only solution is $a = 1$, $b = 8$, $K = 6$ and their integer multiples.

   We can try setting $a = 3^2 c^3$. First, why this choice? The goal here is to make $3a$ into $3a = P^3$ for some integer P. So the first thing to do is to complete the factor 3 into 3^3 by multiplying it by 3^2. What we need next is $b(b+a) = N^3$ which we will solve later. With this choice $3ab(b+a) = 3^3 c^3 N^3 = K^3 \rightarrow K = 3cN$ we would have found our algorithm. However, even with this scheme, the only solutions are integer multiples of $a = 1$, $b = 8$, $K = 6$.

2. From the list of solutions for $a^3 + b^3 + c^3 = d^3$ I saw a familiar face $3, 4, 5, 6$ (by the way, the tuplet $3, 4, 5, 6$ also represent the sides of a right triangle $3, 4, 5$ with an area of 6). Prompting me to guess a solution of the form

   $$a^3 + (a+1)^3 + (a+2)^3 = (a+3)^3$$
   $$(a+3)^3 - (a^3 + (a+1)^3 + (a+2)^3) = 2(3-a)(a^2 + 3a + 3) = 0$$

The last equation shows that this scheme will only work for $a = 3$, no other solution is sequential.

3. What if we mix products and additions of terms? Like this one

$$a^3 + (a \cdot x)^3 + (a \cdot y)^3 = (a + z)^3$$
$$(a + z)^3 - \left\{a^3 + (a \cdot x)^3 + (a \cdot y)^3\right\} = z^3 + 3az^2 + 3a^2z - a^3y^3 - a^3x^3$$

Now this doesn't look promising to some since we can always factor out a to get $y^3 + x^3 + 1 = w^3$, however, from our previous discussion about shifting $a + z \to a$, we know that adding a degree of freedom actually makes a difference. So let give this a shot. The solution is

$$a = \frac{z\left\{(y^3 + x^3 + 1)^{2/3} + (y^3 + x^3 + 1)^{1/3} + 1\right\}}{(y^3 + x^3)}$$

while the other two solutions give imaginary a. Which is rather disappointing since we now need to solve

$$y^3 + x^3 + 1 = w^3$$

so this time adding a degree of freedom doesn't help. If we insist in substituting the solution for a the factor z cancels from both LHS and RHS and the only free variables in $a^3 + (a \cdot x)^3 + (a \cdot y)^3 = (a + z)^3$ are just x and y which is the same free variables we have in $y^3 + x^3 + 1 = w^3$.

At this point I would like to list solutions of $y^3 + x^3 + 1 = w^3$, see Table I (sorted in ascending x). One can immediately see that $y = 138$ gives us *two* different sets of solutions (426 also gives two solutions but once as x and another as y). Also, from their prime factors we can see that the solutions are all co-prime to one another.

Another thing to note is that the difference $y - x$ and $w - y$ are all different for different solutions, showing that an algorithm, *i.e.* $a^3+(a+m)^3+(a+n)^3 = (a+p)^3$ where m, n, p are *fixed* will not work as each fixed difference m, n, p will only give a single solution.

In addition, this also means that it is a near miss for Fermat, $a^3 + b^3 = c^3$ has no solutions but $1 + a^3 + b^3 = c^3$ does :)

One last comment on this topic, we can arrange things differently $y^3 + x^3 + 1 = w^3 \to y^3 + x^3 = w^3 - 1$

$$\begin{aligned} y^3 + x^3 &= w^3 - 1 \\ &= (w-1)(w^2 + w + 1) \\ &= (w-1)((w-1)^2 + 3w) \\ &= (w-1)^3 + 3w(w-1) \end{aligned}$$

We can now add degrees of freedom as usual $y = b + z,\ x = b - z$

$$\begin{aligned} (b+z)^3 + (b-z)^3 &= (w-1)^3 + 3w(w-1) \\ 6bz^2 + 2b^3 &= (w-1)^3 + 3w(w-1) \end{aligned}$$

This looks promising as we have the same number of terms in both LHS and RHS. We know that $2b^3$ can't be u^3 for some integer u due to the factor of 2, the easiest algorithm for $y^3 + x^3 + 1 = w^3$ is to set $2b^3 = 3w(w-1),\ 6bz^2 = (w-1)^3$. However, from the $1, 6, 8, 9$ solution it is clear that we cannot make such identification.

We can try adding more variables, *e.g.* $1 = (b - z - a), y = (b + z)$, but this only complicates things with no clear benefit. In other words, only the combination, *e.g.* $6bz^2 + 2b^3$, constitutes $x_1^3 + x_2^3$, we cannot cast any one term $6bz^2$ or $2b^3$ as a pure power of 3, $6bz^2 = p^3$ or $2b^3 = q^3$.

4. Another potential solution is

$$(a-b)^3 + (b-c)^3 + (c-a)^3 = 3(a-b)(b-c)(c-a)$$

The nice thing is that the RHS is not a sum of terms, so we have a chance if we set

$$(a-b)(b-c)(c-a) = 3^2 w^3$$

To simplfy things let's denote $p = (a - b)$, $q = (b - c)$, $r = (c - a)$

$$p \cdot q \cdot r = 3^2 w^3$$
$$p + q + r = 0$$

We can solve the above equations by substituting $p = 3^2 w^3 / qr$ into $p + q + r = 0$

$$0 = 3^2 w^3 + q^2 r + q r^2$$
$$\rightarrow r = \frac{-q^2 \pm \sqrt{q^4 - 36 \cdot q \cdot w^3}}{2q}$$

To ensure that r is an integer, $q^4 - 36 q w^3 = v^2$ for some integer v. The interesting thing is that there are only two solutions I can find for this to happen

$$q \rightarrow q,\ w \rightarrow -2q\ :\ q^4 - 36q(-2q)^3 = 289q^4 = (17q^2)^2$$
$$q \rightarrow 4q,\ w \rightarrow -q\ :\ (4q)^4 - 36(4q)(-q)^3 = 400q^4 = (20q^2)^2$$

Regrettably this provision will only give one solution

$$(8q)^3 + (q)^3 + (-9q)^3 = (-6q)^3$$

which is just a multiple of the $1, 6, 8, 9$ solution.

We can do some minor rearrangement

$$p^3 + q^3 + 3pq(p + q) = (p + q)^3$$
$$\rightarrow (3p^3)^3 + (3q^3)^3 + (3pqz)^3 = (z^3)^3 \quad z^3 = 3p^3 + 3q^3 \quad \text{OR}$$
$$\rightarrow (3^2 p^3)^3 + (q^3)^3 + (3pqz)^3 = (z^3)^3 \quad z^3 = 3^2 p^3 + q^3 \quad \text{OR}$$
$$\rightarrow (3p^3)^3 + (q^3)^3 + (3pqz)^3 = (z^3)^3 \quad 3z^3 = 3p^3 + q^3 \quad \text{OR}$$
$$\rightarrow p^3 + q^3 + z^3 = (p + q)^3 \quad z^3 = 3pq(p + q)$$

we can either lump the factor of 3^2 into one of the variables or spread it onto any two of them or assume the whole thing $3pq(p + q)$ is a power of three z^3. I got

some interesting result from this exercise, *e.g.* the *largest* solution I found from this simple guessing game is

$$59707533^3 + 2268747144^3 + 981751158^3 = 2328454677^3$$

this one was from $p \to 3p^3,\ (p+q) \to 3u^3$.

5. Another promising solution is

$$\begin{aligned}(m-n)^3 + n^3 + A^3 &= (m+n)^3 \\ \to A^3 &= n^3 + 6m^2n\end{aligned}$$

We can solve for m or n or A, but m seems to yield the easiest solution, which is

$$m = \pm\sqrt{\frac{\frac{A^3}{n} - n^2}{6}}$$

Sadly, this only gives 2 integer solutions, they are

$$\begin{array}{cccc} m-n=17 & n=4 & A=22 & m+n=25 \\ m-n=-2 & n=9 & A=15 & m+n=16 \end{array}$$

I was hoping that these conditions, *e.g.* $m = \pm\sqrt{\frac{\frac{A^3}{n}-n^2}{6}}$ or $r = \frac{-q^2 \pm \sqrt{q^4 - 36 \cdot q \cdot w^3}}{2q}$ would yield a class of integer solutions instead of just one (or two).

So it looks like there's really no solution for $a^3 + b^3 + c^3 = d^3$ if we start we just simple guesses.

Expanding the horizon

Finding a generic algorithm for $a^3 + b^3 + c^3 = d^3$ just by guessing seems to be quite problematic as we have seen above. So what if we add another number? Say

$$a^3 + b^3 + c^3 + d^3 = e^3$$

TABLE I: (x) means prime factors of x

x	y	w	(x)	(y)	(w)
6	8	9	$(2,3)$	(2^3)	(3^2)
71	138	144	(71)	$(2,3,23)$	$(2^4,3^2)$
135	138	172	$(3^3,5)$	$(2,3,23)$	$(2^2,43)$
236	1207	1210	$(2^2,59)$	$(17,71)$	$(2,5,11^2)$
242	720	729	$(2,11^2)$	$(2^4,3^2,5)$	(3^6)
372	426	505	$(2^2,3,31)$	$(2,3,71)$	$(5,101)$
426	486	577	$(2,3,71)$	$(2,3^5)$	(577)
566	823	904	$(2,283)$	(823)	$(2^3,113)$
575	2292	2304	$(5^2,23)$	$(2^2,3,191)$	$(2^8,3^2)$
791	812	1010	$(7,113)$	$(2^2,7,29)$	$(2,5,101)$
1124	5610	5625	$(2^2,281)$	$(2,3,5,11,17)$	$(3^2,5^4)$
1938	2820	3097	$(2,3,17,19)$	$(2^2,3,5,47)$	$(19,163)$
1943	6702	6756	$(29,67)$	$(2,3,1117)$	$(2^2,3,563)$
2196	5984	6081	$(2^2,3^2,61)$	$(2^5,11,17)$	$(3,2027)$
2676	3230	3753	$(2^2,3,223)$	$(2,5,17,19)$	$(3^3,139)$

Going back to our symmetrical solution

$$\begin{aligned}(a-z)^3+\Delta&=(a+z)^3\\ \rightarrow (a+z)^3-(a-z)^3=\Delta&=2z^3+6a^2\\ \Delta&=z^3+z^3+6a^2z\end{aligned}$$

so as long as $a^2=6^2w^6$ and $z=v^3$ the above solution works since every term is a power of 3. The full solution is then

$$(a^3-6^2b^6)^3+(6^2b^6)^3+(6^2b^6)^3+(a^2\cdot 6b^2)^3=(a^3+6^2b^6)^3$$

But I don't like this solution since it contains a constant factor 2. There's yet another

solution, if we began with

$$(-a+b+c)^3+(a-b+c)^3+(a+b-c)^3+X=(a+b+c)^3$$
$$\rightarrow X=24abc$$

The above approach works as there is a so called "symmetry" where all complicated terms cancel for LHS and RHS. This was done by alternating the minus sign in each term to make sure there is only one a^3, b^3, c^3 on the LHS as there is on the RHS. Now, since X is just a product of terms the most obvious choice would be $a \rightarrow a^3, b \rightarrow b^3, c \rightarrow 3^2c^3$. The full solution is then

$$(-a^3+b^3+3^2c^3)^3+(a^3-b^3+3^2c^3)^3+(a^3+b^3-3^2c^3)^3+(2\cdot 3\cdot abc)^3=(a^3+b^3+3^2c^3)^3$$

There are other choices as well like $a \rightarrow a^3, b \rightarrow 3b^3, c \rightarrow 3c^3$. A more generic solution would be $a \rightarrow a^{3\cdot m_1}, b \rightarrow 3\cdot K_1^{3\cdot n_1}b^{3\cdot m_2}, c \rightarrow 3\cdot K_2^{3\cdot n_2}c^{3\cdot m_2}$ but this also allows for something else, namely $a^3+b^3+c^3+d^N=e^3$ by setting $a \rightarrow 2^{N-3}3^{N-1}a^N$, $2^3\cdot 3\ abc \rightarrow 2^3\cdot 3\ (2^{N-3}3^{N-1}a^N)b^Nc^N=(2\cdot 3\ abc)^N$, *i.e.*

$$(2^{N-3}3^{N-1}a^N+b^N+c^N)^3=(-2^{N-3}3^{N-1}a^N+b^N+c^N)^3+(2^{N-3}3^{N-1}a^N-b^N+c^N)^3+$$
$$(2^{N-3}3^{N-1}a^N+b^N-c^N)^3+(2\cdot 3abc)^N$$

The most general solution would be to distribute the prime factors of $P^3(\equiv 24abc)$ into a, b, and c,

$$P=\prod_{i=1}^{L}p_i^{n_i} \quad \longrightarrow \quad P^3=\prod_{i=1}^{L}p_i^{3n_i}$$

where each p_i is a prime number, n_i the multiplicity of p_i and L the number of different prime factors of P and the probability is infinite :)

One way we can use our newly found expanded horizon is to set one of the terms, $(-a+b+c)$ or $(a-b+c)$ or $(a+b-c)$, to zero, since we can't set the product term $24abc$ to zero without resulting in a trivial pursuit :)

By setting one of the terms above to zero we would get our prime objective $a^3+b^3+c^3=e^3$ from $a^3+b^3+c^3+\cancel{d^3}=e^3$. However, this exercise proves futile as the only solution generated is the infamous $1, 6, 8, 9$, *i.e.* $a=1, b=2, c=1 \rightarrow -9a^3+b^3+c^3=0$.

There is a possible (and potential) use of the above algorithm. Say we start with $(a+z)^3$ and $(a+y)^3$ on the LHS and RHS respectively, what we want is

$$\begin{aligned}
(a+z)^3 &= (a+y)^3 + C_1^3 + C_2^3 \\
(a+z)^3 - (a+y)^3 &= 3a^2z + 3az^2 + z^3 - (3a^2y + 3ay^2 + y^3) \\
&= 3a^2z + 3az^2 - y^3 - (3a^2y + 3ay^2 - z^3) \\
&= (a+z)^3 - a^3 - z^3 - y^3 - ((a+y)^3 - a^3 - y^3 - z^3) \\
&= C_1^3 - C_2^3 \\
\rightarrow C_1^3 &= (a+z)^3 - a^3 - z^3 - y^3 \\
\rightarrow C_2^3 &= (a+y)^3 - a^3 - y^3 - z^3
\end{aligned}$$

in the second line we need to swap $y \leftrightarrow z$ otherwise we'll get

$$\begin{aligned}
3a^2z + 3az^2 + z^3 &= (a+z)^3 - a^3 \\
\tilde{C}^3 &= (a+z)^3 - a^3
\end{aligned}$$

which is Fermat and we don't have a solution for it. However, we now have $a^3+b^3+c^3+d^3 = e^3$ for both C_1 and C_2, the bad news is that we might not be able to use the algorithm found above to solve them. The main problem is that C_2 has the same terms as C_1 on the RHS. Another problem is, even if we only consider C_1, matching terms to our algorithm above

$$\begin{aligned}
(a+z) &= -p^3 + q^3 + 3^2r^3 \\
a &= -p^3 + q^3 - 3^2r^3 \\
z &= -p^3 - q^3 + 3^2r^3 \\
y &= -2 \cdot 3\ pqr
\end{aligned}$$

the solution will be

$$\begin{aligned}
r &= \left(\frac{q^3+p^3}{9}\right)^{1/3} \\
a &= 2q^3 \\
y &= 2\ pq\ 3^{1/3}(q^3+p^3)^{1/3} \\
z &= -2(q^3+p^3)
\end{aligned}$$

TABLE II: Some sample of twins for $x^3 + y^3 + z^3 = w^3$, including one of the biggest one I found (see bottom row)

x	y	z	w
3	36	**37**	**46**
27	30	**37**	**46**
76	165	**192**	**229**
102	157	**192**	**229**
9	58	255	**229**
9	183	220	**229**
58	**201**	255	**292**
183	**201**	220	**292**
24	159	**382**	**391**
87	150	**382**	**391**
2678	10533	**43420**	**43629**
5157	10166	**43420**	**43629**

while p and q remain free parameters. The above requires that

$$q^3 + p^3 = \frac{w^3}{3}$$

and the only solution is $q = 1,\ p = 2,\ w = 3$ (this is just the 1,6,8,9 solution with $z = 0$). We could have chosen our other algorithm, *i.e.* $a + z = -p^3 + 3q^3 + 3r^3$ but this choice leaves us no way out. But why did we choose $(a + z) = -p^3 + q^3 + 3^2 r^3$ in the first place? because other choices generate complex solutions :)

Observation intermission

We now take a break from guessing and start staring at the numerical solutions of $a^3 + b^3 + c^3 = d^3$ we found and take stock of what we've found so far.

First, the solutions are not unique, we have pairs of numbers that appear in more than one solution, see Table. II. There are in total 1290 such twins from a total of 39607 solutions (around 3%, solutions are generated ascendingly by x, the smallest number in each

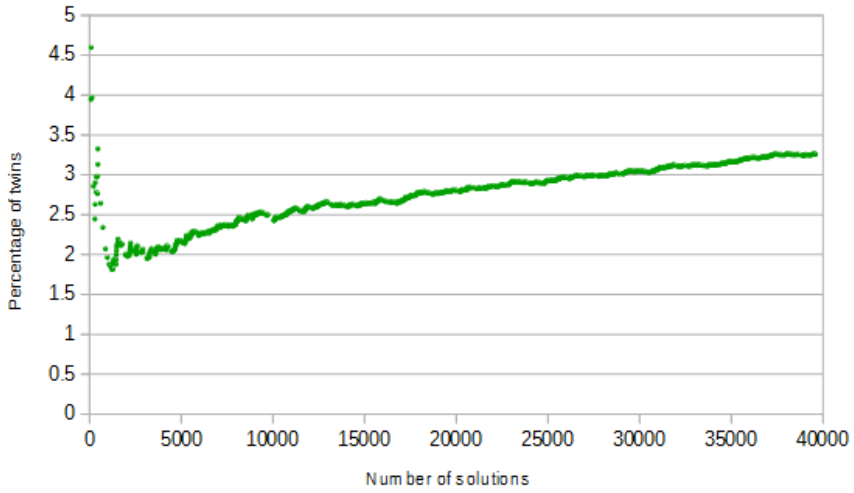


FIG. 9: Plot of the distribution of twins among solutions of $a^3 + b^3 + c^3 = d^3$. Each green dot indicates where each twin was born among the solutions. First data point is excluded as first twins occur as solutions #3 giving a distortedly high 33.3% value.

solution). However, this percentage depends on how we search the solution space. Twins are more likely to pop up when x, y, z are relatively close, see comments in "Intuition rumination" section.

Moreover, there are intriguing solutions like this one, where the difference between the two largerst terms is just one!

$$462^3 + 1981^3 + 51227^3 = 51228^3$$

An interesting question will be: the distribution of these twins, are they spread evenly? does their precentage keep fluctuating or does it stabilize at some point or does it increase without bound? For example see Fig. 9

Second, guessing for an algorithm is not the best strategy here because there are four terms in total. It is hard to cancel terms between LHS and RHS that will result only in a product of terms as we can see from my attempts above, *e.g.*

$$(a + z)^3 - (a - z)^3 = 2z^3 + 6a^2z$$

the factor of 2 in front of z^3 spoils everything because we need $2z^3 = w^3$ for some integer w. What if we add more variables? say $(x + y + z)^3$, the problem is that there are three

cubes on the RHS x^3, y^3, z^3 but we only have 3 terms on the LHS, this means that

$$(x+\Delta_1)^3 + (y+\Delta_2)^3 + (z+\Delta_3)^3 = (x+y+z)^3$$

We can cancel x^3, y^3, z^3 on both sides but we now have $\Delta_1^3, \Delta_2^3, \Delta_3^3$ on the LHS (plus cross terms on both sides) that we cannot cancel.

Contrast this situation to $a^3 + b^3 + c^3 + d^3 = e^3$. If we start with $(x+y+z)^3$ on the RHS, we can cycle a minus sign on the LHS $(-x+y+z)^3$, $(x-y+z)^3$, $(x+y-z)^3$ and we will have exactly one x^3, y^3, z^3 on both sides with an extra term d^3 to absorb the rest of the cross terms. No such luck with just four terms $a^3 + b^3 + c^3 = d^3$, even if we start with two terms $(x+y)^3$ on the RHS the best I could get was

$$(x+a)^3 + (y-a)^3 + 3(x+a)(y-a)(x+y) = (x+y)^3$$

which was attempt #4 with some minor rearrangement.

Speaking of attempt #4 and #5, we see something peculiar. Our usual observation says that we have one less constraint than the number of degrees of freedom and so we should have a multitude of solutions instead of just one. Let's start with #4, its constaints are

$$\begin{aligned} p^3 + q^3 + r^3 &= 3^3 w^3 \\ p \cdot q \cdot r &= 3^2 w^3 \\ p + q + r &= 0 \end{aligned}$$

so we have 4 degrees of freedom p, q, r, w with three constraints. For #5, its *two* constraints are

$$\begin{aligned} (m-n)^3 + n^3 + A^3 &= (m+n)^3 \\ n^3 + 6m^2 n &= A^3 \end{aligned}$$

Again one extra free term, two constraints with *three* degrees of freedom m, n, A. The odd thing here is that the above two examples yield only one solution each, despite the extra *free* degree of freedom.

How can this be? This is because we are so used to thinking about real numbers. When solving a system of equations, the equations are the only constraints there are.

Not so with integers/natural numbers. By requiring that the solutions are integers we are implicitly imposing another constraint, albeit a non traditional one. Going back to Fermat, $a^3 + b^3 = c^3$, we have only one constraint and three degrees of freedom and yet it has no solutions! It's because all the solutions are real numbers.

Let's see this in more detail through a concrete example

Attempt #6 must die!

This one was exhilirating because at a glance it gives you a lot of hope until you see it more closely. The starting point is

$$a^3 + (y^{1/3})^3 + (a - z)^3 = (a + 8z)^3$$

$$(a + 8z)^3 - \{a^3 + (y^{1/3})^3 + (a - z)^3\} = 513z^3 + 189az^2 + 27a^2z - y - a^3 = 0$$

The funny term $(y^{1/3})^3$ is due to hindsight :) and even though in itself it is a constraint, it's irrelevant to the point I'm trying to make here. On the onset it seems like we have three degrees of freedom with only one constraint. Implying

$$a = \left(\frac{\sqrt{1539648z^6 - 7344yz^3 + y^2}}{2} - \frac{y - 3672z^3}{2} \right)^{1/3}$$
$$+144z^2 \left(\frac{\sqrt{1539648z^6 - 7344yz^3 + y^2}}{2} - \frac{y - 3672z^3}{2} \right)^{-1/3} + 9z$$

To have any chance at all the inner square root has to produce an integer

$$1539648z^6 - 7344yz^3 + y^2 = u^2$$

but this is an additional constraint. One might object that it also introduces a new degree of freedom u. However, solving the above equation

$$y = 3672z^3 \pm \sqrt{11943936z^6 + u^2}$$

This in turn means we acquire another constraint $\sqrt{11943936z^6 + u^2}$ (which comes with another degree of freedom). This is a zero sum game, we are not gaining anything. We can continue with this forever!

Say we don't want to play this game, we can set

$$1539648z^6 - 7344yz^3 + y^2 = 0$$
$$\rightarrow y = 216z^3$$

which is good news because $216 = 6^3$ (another solution is $y = 7128z^3$). However, by this time we have utilized two constraints, the original equation and $1539648z^6 - 7344yz^3 + y^2 = 0$. This is no good. To see why, let's continue.

We have one more constraint to solve, $((y - 3672z^3)/2)^{1/3}$ has to be an integer, the solution is $z = -w/12$ for some integer w. This looks promising, we have a degree of freedom, but if you substitute this solution, $z = -w/12$, $y = 216z^3 = -w^3/8$, $a = -11w/4$, *i.e.* all terms in $a^3 + (y^{1/3})^3 + (a - z)^3 = (a + 8z)^3$ are just multiples of w

$$\left(\frac{-11w}{4}\right)^3 + \left(\frac{-w}{2}\right)^3 + \left(\frac{-8w}{3}\right)^3 = \left(\frac{-41w}{12}\right)^3$$
$$\rightarrow 33^3 + 6^3 + 32^3 = 41^3$$

Sadly we only get one final solution $6, 32, 33, 41$ because we have three degrees of freedom to begin with but utilize three constraints to get a solution, $a^3 + (y^{1/3})^3 + (a - z)^3 = (a+8z)^3$, $\sqrt{11943936z^6 + u^2}$, $((y-3672z^3)/2)^{1/3}$. Thus the requirement that the solutions are integers is itself a powerful constraint!

Another thing to note about the example above is that since the solution for a cubic polynomial is a cube root of a square root, $(\cdots + \sqrt{\ldots})^{1/3}$, we immediately incur two constraints, one for each root.

One quick side note, I initially wanted to treat the numbers in $a^3 + b^3 + c^3 = d^3$ as vectors much like in the case of $a^2 + b^2 = c^2 \rightarrow \vec{a} + \vec{b} = \vec{c}$ with a right angle between $\vec{a}$ and $\vec{b}$. What would become of $\vec{a} + \vec{b} + \vec{c} = \vec{d}$? maybe there's some sort of of new cosine rule? The problem is we now deal with a generic "parallelogram" and for such a four-sided object the angle between any two sides is not fixed as we can always squeeze the object and the length of all sides will still stay the same. This is instartk contrast to a triangle where changing the angle between any two sides would change the length of the other side.

Observation intermission part deux

The way I initially generated the solutions was through tripply nested loops $z_1 = 1 \rightarrow 1000, z_2 = 1 \rightarrow 1000, z_3 = 1 \rightarrow 1000$ where at each iteration I checked whether $z_1^3 + z_2^3 + z_3^3$ is equal to some w^3.

It was a symmetrical search among z_1, z_2, and z_3. It was very inefficient since it went rhrough things multiple times, *e.g.* 1,6,8,9 was rediscovered multiple times through 6,1,8,9 and 8,1,6,9.

I then modified it to remove any redundancy and also made it asymmetrical $z_1 = 1 \to 5200, z_2 = z_1 \to z_1 + 10000, z_3 = z_2 \to z_2 + 50000$ because I realized that z_1, z_2, z_3 are sometimes quite far apart.

There were in total 39607 solutions generated. They exhibit some interesting facts. First, the difference between the smallest and largest number can be quite large. Second, there is some sort of clustering going on, *e.g.* z_1 and z_2 are close together and z_3 and w are also close together but the gap between z_1, z_2 and z_3, w is quite big. Third, there's no two numbers are the same, *i.e.* $2z^3 + z_3^3 = w^3$ has no solution.

Finally, there are always an even number of odd numbers among z_1, z_2, z_3, w which means that the sum total of all four is always even. This is actually constructive because this means that we can do this

$$(a - z)^3 + (b + y)^3 + (b - y)^3 = (a + z)^3$$

we can do this because $(a + z) + (a - z) = 2a,\ (b + y) + (b - y) = 2b$ and since there is always an even number of odd numbers among z_1, z_2, z_3, w, this is always true. Continuing in this train of thought, the parameters a, z, b, y are not free, they have to satisfy

$$\begin{aligned}(a + z)^3 - \{(a - z)^3 + (b + y)^3 + (b - y)^3\} &= 0\\ z^3 + 3a^2z - 3by^2 - b^3 &= 0\\ \to z^3 + 3a^2z &= b^3 + 3y^2b\end{aligned}$$

Some words about convention. I sorted each solution set z_1, z_2, z_3, w in ascending order, *i.e.* $z_1 < z_2 < z_3 < w$. I then paired them up according their even/odd-ness in ascending order as well. I first search if z_1 and z_2 is of the same evenness, if so $z_1 + z_2 = 2b, z_2 - z_1 = 2y$. We cannot assign z_1, z_2 to a, z because z_1, z_2 are the smallest numbers of the lot, a, z are derived from z_3, w. If z_1 and z_2 is *not* of the same evenness, I tried z_1 and z_3, and if they're still not of the same evenness I'd try z_1, w and derive a, z, b, y accordingly.

The result was quite surprising, $b \geq z,\ b > y,\ b + y \leq a + z$, always! Also $b - z$ is always a multiple of 3, *i.e.* $b - z = 3k$ for some integer k. This is not at all surprising by

the way, let $b - z = q \;\rightarrow\; b = z + q$

$$\begin{aligned} b^3 - z^3 &= 3a^2z - 3y^2b \\ (z+q)^3 - z^3 &= 3(a^2z - y^2b) \\ \cancel{z^3} + 3z^2q + 3zq^2 + q^3 - \cancel{z^3} &= 3(a^2z - y^2b) \\ q^3 &= 3(a^2z - y^2b - z^2q - zq^2) \\ \Rightarrow q &= 3h \end{aligned}$$

This also sheds some light of the twin solutions. Let's take for example the first row in Table. II, the first twin is

$$\begin{aligned} 3^3 + 36^3 + 37^3 &= 46^3 \\ \rightarrow (b - y = 3)^3 + (a - z = 36)^3 + (b + y = 37)^3 &= (a + z = 46)^3 \end{aligned}$$

with $a = 41, z = 5, b = 20, y = 17$, while the second twin is given by

$$\begin{aligned} 27^3 + 30^3 + 37^3 &= 46^3 \\ \rightarrow (b - y = 27)^3 + (a - z = 30)^3 + (b + y = 37)^3 &= (a + z = 46)^3 \end{aligned}$$

with $a = 38, z = 8, b = 32, y = 5$.

In conclusion we can the above as a four-parameter algorithm

$$(a - z)^3 + (z + (3k + y))^3 + (z + (3k - y))^3 = (a + z)^3$$

with an extra constraint of

$$(3k)^3 = 3(a^2z - y^2(z + 3k) - z^2(3k) - z(3k)^2)$$

and since this constraint is cubic, this parameterization is not very useful.

Natural selection

Despite the limitation of simply guessing the algorithm for $a^3 + b^3 + c^3 = d^3$, other equations can benefit from it naturally, *e.g.* $a^4 + b^3 + c^3 = d^4$

$$\begin{aligned} (y+z)^4 - (y-z)^4 &= 8y^3z + 8yz^3 \\ \rightarrow (y^3 - z^3)^4 + (2y^3z)^3 + (2yz^3)^3 &= (y^3 + z^3)^4 \end{aligned}$$

and less natural for the 5^{th} power

$$(y+z)^5-(y-z)^5=2z^5+20y^2z^3+10y^4z$$

the factor of 2 (just like that of the 3^{rd} power) killed it. But, we can always do some substitutionary atonement $y\to 20y^3, z\to 2z^3$ and massage it into

$$(20y^3+2z^3)^5-(20y^3-2z^3)^5=(2z^2\cdot 2z^3)^3+(20y^2\cdot 2z^3)^3+(20y^3)^4(10\cdot 2z^3)$$

i.e. an algorithm for $a^3+b^3+c^4d+e^5=f^5$, except that this time we have a product of terms c^4d that the preceding ones didn't. Things get hairy when we go to power of 7

$$(y+z)^7-(y-z)^7=2z^7+42y^2z^5+70y^4z^3+14y^6z$$

Going with three terms doesn't get much better but it is now salvageable

$$\begin{aligned}
&(x+y+z)^7-(-x+y+z)^7-(x-y+z)^7-(x+y-z)^7\\
&\qquad=56xyz(3(x^2+y^2+z^2)^2+4(x^2y^2+x^2z^2+y^2z^2))\\
&\qquad=2\cdot 7\cdot 3xyz\cdot 2^2(x^2+y^2+z^2)^2+2^3x^3y^3z\cdot 7\cdot 4+2^3x^3yz^3\cdot 7\cdot 4+2^3xy^3z^3\cdot 7\cdot 4\\
&\qquad=(42xyz)(2p^2)^2+(2xy)^3(28z)+(2xz)^3(28y)+(2yz)^3(28x)
\end{aligned}$$

where $p^2=x^2+y^2+z^2$ whose algoritm will be discussed later. You really need to be creative to appease higher powers :)

Along the same line, $a^3+b^3+c^3=d^3$ can be post processed

$$(a+z)^3-(a-z)^3=2z^3+6a^2z$$

substituting $z\to 2z^3, a\to 2\cdot 6a^3$ we get

$$\begin{aligned}
(2\cdot 6a^3+2z^3)^3&=(2z^2)^3(2z^3)+(6a^2\cdot 2z)^3+(2\cdot 6a^3-2z^3)^3\\
\to(6a^3+z^3)^3&=(z^2)^3(2z^3)+(6a^2\cdot z)^3+(6a^3-z^3)^3
\end{aligned}$$

which is an algorithm for $a^3e+b^3+c^3=d^3$, not bad. Well, although this game gets really old really quickly, let's continue. The highest power that can still give some solace is eight

$$(a+z)^8-(a-z)^8=16az(z^2+a^2)^3+64a^3z^3(z^2+a^2)$$

It's tempting to conform (z^2+a^2) into $z^2+a^2=w^2$. To continue though, it requires $a \to 16^2a^6, z \to z^3$ to make $16az(z^2+a^2)^3 \to (16a^2z \cdot w^2)^3$. But this puts too many constraints. First, $z^2+a^2=w^2$ means that (at least) $m^2+n^2=z^3$, which might not be solvable. Unless we find a solution, we will get products of terms $(16az)w^6$ for $16az(z^2+a^2)^3$ and $(4a)3w^2$ for $64a^3z^3(z^2+a^2)$.

A better choice will be $a \to 4a^3, z \to z^3$, this way we'll get less products of terms

$$(4a^3+z^3)^8-(4a^3-z^3)^8=(4az\cdot(z^6+16a^6))^3+(4\cdot 4a^3\cdot z)^3(z^6+16a^6)$$

which is an algorithm for $a^8+b^3+c^3d=e^8$. One last example but this time with three variables

$$(x+y+z)^5-(-x+y+z)^5-(x-y+z)^5-(x+y-z)^5=80xyz(x^2+y^2+z^2)$$

We can easily find an algorithm for $x^2+y^2+z^2=w^2$, which we will discuss in more details later. The solution is

$$(a^2+b^2+c^2)^2=(2ab)^2+(2ac)^2+(-a^2+b^2+c^2)^2$$

Even with this proviso, we still have difficulty recasting $80xyz(x^2+y^2+z^2) \to u^n$. For example, since we can transmute $x^2+y^2+z^2=w^2$, we might want to upgrade $x \to 80x^2, y \to y^2, z \to z^2$ to transform $80xyz(x^2+y^2+z^2) \to 80^2x^2y^2z^2w^2$.

However, this puts a constraint on $-a^2+b^2+c^2=80x^2$ which might not be solvable; $x^2+y^2+z^2=w^2$ requires that $x^2=-a^2+b^2+c^2$ which we will discuss later. Thus it is more than likely that we have have a product of terms for $80xyz(x^2+y^2+z^2)$.

Rhythm of the rain

The case of power of two (*i.e.* the Pythagorean equation) is special because we can extend the formula for any number of terms. For example, let's go back to $x^2+y^2+z^2=w^2$, if we expand

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$$
$$(-a+b+c)^2=a^2+b^2+c^2-2ab-2ac+2bc$$

the difference between those two quantities will only contain terms multiplying a

$$(a+b+c)^2-(-a+b+c)^2=4ab+4ac$$

And here you see the pattern, if we extend to any number of terms $z_1^2+z_2^2+\cdots+z_n^2=w^2$ we can use the same result above

$$(a_1+a_2+\cdots+a_n)^2-(-a_1+a_2+\cdots+a_n)^2=4a_1a_2+4a_1a_3+\cdots+4a_1a_n$$
$$\to(a_1^2+a_2^2+\cdots+a_n^2)^2-(-a_1^2+a_2^2+\cdots+a_n^2)^2=(2a_1a_2)^2+(2a_1a_3)^2+\cdots+(2a_1a_n)^2$$

where we have done the usual trick $a_i\to a_i^2$ going to the second line. Thus the generic algorithm for $z_1^2+z_2^2+\cdots+z_n^2=w^2$ is

$$z_i=2a_1a_{i+1},\quad 1\le i<n$$
$$z_n=-a_1^2+\sum_{j=2}^{n}a_j^2$$
$$w=\sum_{k=1}^{n}a_k^2$$

well of course you can choose any other a's beside a_1 as the one having the minus sign, also n is good from 2 onwards.

We can easily extend (or maybe distend) this even further to $z_1^{N_1}+\cdots+z_{n-1}^{N_{n-1}}+z_n^2=w^2$ with $N_1,\ldots,N_{n-1}$ any natural numbers. We simply do the following

$$z_i=2a_1^{M/N_i}a_{i+1},\quad 1\le i<n,\quad M=\prod_{j=1}^{n-1}N_j$$
$$z_n=-2^{M-2}a_1^M+\sum_{j=1}^{n-1}a_{j+1}^{N_j}$$
$$w=2^{M-2}a_1^M+\sum_{k=1}^{n-1}a_{k+1}^{N_k}$$

i.e. we merely substitute $4a_1a_{j+1}\to 4\cdot 2^{M-2}a_1^Ma_{j+1}^{N_j}=(2a_1^{M/N_j}a_{j+1})^{N_j}$. We will discuss more about extending things to generic power in the next section, but this shows that Pythagoras is one flexible fellow, it can be effortlessly expanded to numerous directions from just a simple guess.

We can say there's sorta rhythm to this. No such swing with the power of 3, just the rain. Well maybe there's a bit of rhythm but it is ugly. We have to start with 3 terms

$$(a+b+c)^3-\sum[a,b,c]^3=24abc$$

where $\sum[a,b,c]^3$ is the sum of

$$(-a+b+c)^3+(a-b+c)^3+(a+b-c)^3$$

i.e. we cycle the minus sign through each term. If we had 4 terms we would have

$$2(a+b+c+d)^3-\sum[a,b,c,d]^3=24(abc+abd+acd+bcd)$$

First we need a factor of 2 in front of $(a+b+c+d)^3$ because each parameter in $\sum[a,b,c,d]^3$, *e.g.* a^3, is multiplied by 2, $-a^3+a^3+a^3+a^3=2a^3$, the same with b^3,c^3,d^3. The RHS is just the combination of 4 choose 3. The general formula is thus

$$(n-2)(a_1+a_2+\cdots+a_n)^3-\sum[a_1,\ldots,a_n]^3=24C[a_1,\ldots,a_n]$$

where $C[a_1,\ldots,a_n]$ is the combination of n choose three $a_1a_2a_3+a_1a_2a_4+\cdots+a_{n-2}a_{n-1}a_n$ and here is the source of the problem.

For simplicity let $n=4$, so we have $24C[a,b,c,d]=24(abc+abd+acd+bcd)$. We can promote $a\to 3^2a^3, b\to b^3, c\to c^3, d\to d^3$. The first three terms become $(2\cdot 3abc)^3+(2\cdot 3abd)^3+(2\cdot 3acd)^3$, however, this ruins the last term $24bcd\to 24(bcd)^3$.

No holy tri(3)nity ¯\(°_o)/¯

If $n\geq 5$, we will start having trouble at $24a_2a_3a_4$ because we have promoted $a_2,\ldots,a_n\to a_2^3,\ldots,a_n^3$ to deal with $24a_1a_ja_k\to 8\cdot 27a_1^3a_j^3a_k^3,\quad j,k>1$.

Thus for a power of 3 it seems that only $a^3+b^3+c^3+d^3=e^3$ is bueno. Another significant difference is that the number of a's is *not* the same as the number of z's, *e.g.* 3 a's generate $3+1=4$ z's while 4 a's generate $4+4=8$ z's. The next difference from the power of 2 is that the above generic formula is only good for $n\geq 4$ compared to $n\geq 2$ for power of 2. Also for the power of three, the number of terms $z_1^3+\cdots+z_n^3=w^3$ increases factorially and not to mention that there's a constant factor $(n-2)$ in front of $(a_1+a_2+\cdots+a_n)^3$.

There is still something we can do for power of 3, we can always massage

$$24a_i^3a_j^3a_k^3=(2a_i)^3(3a_j^3a_k^3)$$

giving the generic formula for power of 3, before we state the generic formula there are some bookkeeping to settle

- the number of a's is not the same as the number of z's. The number of z's, denote it m is

$$m = n + \binom{n}{3}$$

n represents $\sum[a_1, \ldots, a_n]$, *i.e.* the sum of $(-a_1 + \cdots + a_n) + \cdots + (a_1 + \cdots - a_n)$ with the negative sign cycled through each term. $\binom{n}{3}$ is for $24(a_1a_2a_3 + \cdots + a_{n-2}a_{n-1}a_n)$. Note that

$$\binom{n}{3} = \sum_{j}^{n-2} \binom{n-j}{2}$$

i.e. the terms that contain a_1 as the first factor can be thought of as removing a_1 from the pool as choosing 2 from the rest, and we can repeat the process for terms with a_2 as first factor and so on.

- Some of the z's have an extra factor multiplying it, *i.e.* $(2a_i)^3(3a_j^3a_k^3) = z^3y$, thus some z's have an extra factor multiplying it.

- The equation we want to generate an algorithm for is

$$\left(z_1^3 + \cdots + z_n^3\right) + \left(z_{n+1}^3 + \cdots + z_{n+\binom{n-1}{2}}^3\right) + \left(y_1 z_{n+\binom{n-1}{2}+1}^3 + \cdots + y_l z_m^3\right) = (n-2)w^3$$

where $l = \binom{n}{3} - \binom{n-1}{2}$

The generic formula is then

$$\begin{aligned}
&z_i = [3^2a_1^3, \ldots, a_n^3], \quad 1 \le i \le n \\
&z_j = 2 \cdot 3a_1a_pa_q, \quad n+1 \le j \le n + \binom{n-1}{2}, \quad 2 \le p, q \le n \\
&z_k = 2a_r, \qquad\qquad n + \binom{n-1}{2} + 1 \le k \le m, \quad 2 \le r \le n \\
&y_q = 3a_s^3a_t^3, \qquad\quad 1 \le q \le \binom{n}{3} - \binom{n-1}{2}, \qquad 2 \le s, t(\neq r) \le n \\
&w = (3^2a_1^3 + \cdots + a_n^3)
\end{aligned}$$

and as we can see it is a lot more complicated than that of the power of 2.

To infinity and beyond

In the above examples we were stopped in our tracks by constants

$$(a+z)^3-(a-z)^3=6a^2z+2z^3$$

Since 2 is prime we can't massage $2z^3\to u^3$.There is a way though to cast it another way $2z^3\to u^N$ for some integers u and N (we have actually done this for the generic power of 2 earlier). We know that 2 is prime so

$$z=2^m g^n$$
$$\to 2z^3=2^{3m+1}g^{3n}$$

What we want now is

$$3m+1=NK$$
$$3n=NQ$$
$$\to 2z^3=2^{3m+1}g^{3n}=(2^K g^Q)^N$$

However, we have another obstacle, $6a^2z$ has to be recast into $6a^2z\to v^q$, in this case we want v^3 since we are dealing with power of 3. We can apply the same process to a and indeed we have, $a\to 6a^3$, since $6a^2=(6a^2)^3$ is already a power of 3

$$z=2^m g^n=d^3\to m=3S,\ n=3T$$

m and n have to be multiples of 3. This in turn means that $3m+1=9S+1=NK\Longrightarrow N$ is any factor of $9S+1$ and that's it, we have arrived, we can now set $2z^3$ into u^N with N any factor of $9S+1$. How about $3n=9T=NQ$? That can be easily dealt with since both R and Q are free parameters, the more than obvious choice is

$$Q=9\quad T=N$$

Four our beloved cubic friend $(a-z)^3+6a^2z+2z^3=(a+z)^3$, we can conveniently choose $S=1,\ 9S+1=10,\ T=N=2,\ K=5,\ Q=9,\ m=3S=3,\ n=NQ/3=6,\ z\to 2^3z^6$

$$(6a^3-2^3z^6)^3+6^3a^62^3z^6+2\cdot 2^9z^{18}=(6a^3+2^3z^6)^3$$
$$(6a^3-2^3z^6)^3+(6a^22z^2)^3+(2^5z^{14})^2=(6a^3+2^3z^6)^3$$
$$\to a^3+b^3+c^2=d^3$$

The only ironic thing about this is that N can never be 3 (or a multiple of 3) since if $N = 3 \rightarrow 3(K - m) = 1$ which is impossible.

The generic algorithm to massage $C^q z^p$ into u^N (with C some constant factor) is

$$\begin{aligned} z &= C^m g^n \\ C^q z^p &= C^{pm+q} g^{pn} = u^N \\ \Rightarrow pm + q &= NK \\ \Rightarrow pn &= NQ \end{aligned}$$

$N = \{pm + q\}$ with $n = N,\ Q = p$ where I have used the notation $\{A\}$ as "any factor of A".

With this we can extend known algorithms to "infinity and beyond" and our first guinnea pig is gonna be Euclid's algorithm

$$\begin{aligned} (a^2 + b^2)^2 - (a^2 - b^2)^2 &= 4a^2b^2 \\ a &\rightarrow 4^m g^n \\ 4a^2 &= 4^{2m+1} g^{2n} \\ 2m + 1 &= NK \\ 2n &= NQ \\ n = N \qquad Q &= 2 \end{aligned}$$

$N = \{2m+1\}$ means N is any factor of any odd number, we can also cast this differently

$$\begin{aligned} a &\rightarrow 2^m g^n \\ 4a^2 = 2^2 a^2 &= 2^{2m+2} g^{2n} \\ 2m + 2 &= NK \\ 2n &= NQ \\ n = N \qquad Q &= 2 \end{aligned}$$

which means that N is any factor of any even number, a factor that is any factor of any odd number and any even number is any number (actually we only need it to be a factor of any even number for it to be any number since even numbers are $2m$ with m

any integer), *i.e.* we can recast $4a^2b^2$ into u^N with N any number :) the only thing to do is $b \to b^N,\ (b^N)^2 = (b^2)^N$. This is actually interesting, Pythagorean equation is extra special, it not only works with $a^2 + b^2 = c^2$ but also $a^2 + b^N = c^2$ for *any* N.

Pythagoreans surely are a lucky lot, not only that they can be easily extended indefinitely in terms of the number of terms in $z_1^2 + \cdots + z_n^2 = w^2$ but also in the exponent $z_1^2 + z_2^N = w^2$. Remember, we started this whole business from nothing but a simple guess!

Not so with the power of 3, a simple guess won't do. Looking at the example above, $a^3 + b^3 + c^N = d^3$, N is limited to the factors of $9S+1$, for example for $S = 2, 4, 8, 12, 14, 18, 20$ the combination $9S + 1$ is actually a prime number, they are $19, 37, 73, 109, 127, 163, 181$ respectively.

There's a much easier way to establish $z_1^2 + z_2^N = w^2 \longrightarrow z_1 = r - q,\ w = r + q$

$$\begin{aligned} w^2 - z_1^2 &= (r+q)^2 - (r-q)^2 \\ \to z_2^N &= 4qr \end{aligned}$$

we can then conveniently set $q = 2^{N-2}d^N,\ r = h^N$ such that $z_2 = 2dh$, *i.e.*

$$(h^N - 2^{N-2}d^N)^2 + (2dh)^N = (h^N + 2^{N-2}d^N)^2$$

Note: going to $z_1^M + z_2^N = w^2$ is much harder because even though casting $r - q = y^M$ is straightforward, satisfying it while requiring $z_2^N = 4qr$ at the same time is challenging.

Silly tally

Now comes the time to gather what I've found. In this section I'll list all the algorithms I discovered, sorted in ascending power (some might not have been previously discussed because I just stumbled upon them). Recall that I started this whole business from a silly guess, $(a + z)^n - (a - z)^n = \Delta$, thus this tally is indeed rather silly :)

- power of 2

1. $z_1^2 + z_2^2 = w^2$ (the original Pythagorean) with two and *three* degrees of freedom

Two degrees of freedom

$$z_1 = 4a - ab^2,\ z_2 = 4ab,\ w = 4a + ab^2$$

Three degrees of freedom

$$z_1 = (a - b - 2|c|)(a - b),\ z_2 = 2|c|(a - b) - 2c^2$$
$$w = a^2 - 2(b + |c|)a + b^2 + 2b|c| + 2c^2$$

2. $z^2 - zy = 2w^2$ (A.I.P: An Incarnation of the Pythagorean)

$$z = 2a^2,\ y = a^2 + 2ab - b^2,\ w = a^2(a - b)^2$$

3. $z^2p^4 - 2zp^2q^2y + y^2 = p^2q^2 + q^4$ (Y.A.I.P: Yet Another Incarnation of the Pythagorean)

$$z = a^2 - b^2,\ y = 2b^2,\ p = 2ab,\ q = a^2 - b^2$$

4. $z^2 + y^2 = 2w^2$ (A.Y.A.I.P: And Yet Another Incarnation of the Pythagorean)

$$z = a^2 - b^2 + 2ab,\ y = a^2 - b^2 - 2ab,\ w = a^2 + b^2$$

5. $2y^2 + 2 = z^2$ (S.Y.A.I.P: Stil Yet Another Incarnation of the Pythagorean)

$$z = \frac{2(a^2 + b^2)}{a^2 - b^2 + 2ab},\ y = \frac{b^2 + 2ab - a^2}{b^2 - 2ab - a^2}$$

6. $z_1^2 + \cdots + z_n^2 = w^2$ (crowd funded Pythagorean)

$$z_i = 2a_1a_{i+1}, \quad 1 \le i < n$$
$$z_n = -a_1^2 + \sum_{j=2}^{n} a_j^2$$
$$w = \sum_{k=1}^{n} a_k^2$$

7. $z_1^2 + z_2^N = w^2$ (Buzzlightyear meet Pythagoras)

Choosing factors from odd numbers

$$z_1 = (4^m a^{NQ/2})^2 - b^{2N},\ z_2 = 4^K a^Q b^2,\ w = (4^m a^{NQ/2})^2 + b^{2N}$$

$$N = \{2m+1\},\ NK = 2m+1,\ Q = 2$$

Choosing factors from even numbers

$$z_1 = (2^m a^{NQ/2})^2 - b^{2N},\ z_2 = 2^K a^Q b^2,\ w = (2^m a^{NQ/2})^2 + b^{2N}$$

$$N = \{2m+2\},\ NK = 2m+2,\ Q = 2$$

A much easier choice

$$z_1 = a^N - 2^{N-2} b^N,\ z_2 = 2ab,\ w = a^N + 2^{N-2} b^N$$

8. $z_1^{N_1} + \cdots + z_{n-1}^{N_{n-1}} + z_n^2 = w^2$ (Phytagoras on steroids)

$$z_i = 2a_1^{M/N_i} a_{i+1}, \quad 1 \le i < n, \quad M = \prod_{j=1}^{n-1} N_j$$

$$z_n = -2^{M-2} a_1^M + \sum_{j=1}^{n-1} a_{j+1}^{N_j}$$

$$w = 2^{M-2} a_1^M + \sum_{k=1}^{n-1} a_{k+1}^{N_k}$$

- power of 3

1. $z_1^3 + z_2^3 + z_3^3 = w^3 \rightarrow$ no generic algo, more than just a silly guess required :)

$$\begin{aligned}
&z_1 = a,\ z_2 = b,\ z_3^3 = 3ab(b+a),\ w = a+b && \longrightarrow\ 1, 6, 8, 9\\
&z_1 = a,\ z_2 = a+1,\ z_3 = a+2,\ w = a+3 && \longrightarrow\ a = 3\\
&z_1 = a,\ z_2 = a \cdot x,\ z_3 = a \cdot y,\ w = a+z && \longrightarrow\ 1 + z_1^3 + z_2^3 = w^3\\
&z_1 = a-b,\ z_2 = b-c,\ z_3 = c-a,\ w = 3(a-b)(b-c)(c-a) && \longrightarrow\ 1, 6, 8, 9\\
&z_1 = a-b,\ z_2 = b,\ z_3^3 = b^3 + 6a^2 b,\ w = a+b && \longrightarrow\ 17, 4, 22, 25\\
&z_1 = a,\ z_2 = b^{1/3},\ z_3 = a-c,\ w = a+8c && \longrightarrow\ 6, 32, 33, 41
\end{aligned}$$

2. $1 + z_1^3 + z_2^3 = w^3$, see Table. I

3. $1 + z_1^3 + z_2^3 + z_3^3 + z_4^3 = w^3$, this was an accident fromtrying to solve no. 2 above

$$z_1 = a,\ z_2 = 2a,\ z_3 = 3a^2,\ z_4 = 3a^3,\ w = (3a^3 + 1)$$

4. $z_1^3 + z_2^{2m+1} + 2 \cdot z_3^{2m+1} = w^3$

$$z_1 = 6^m a^{2m+1} - b^{2m+1},\ z_2 = 6a^2 \cdot b,\ z_3 = b^3,\ w = 6^m a^{2m+1} + b^{2m+1}$$

5. $z_1^3 + z_2^{2m+1} + y \cdot z_3^N = w^3,$ $\quad N$ is any factor of $2(2m+1)$

$$z_1 = 6^m a^{2m+1} - b^{2m+1},\ z_2 = 6a^2 \cdot b,\ y = 2b^{2m+1}$$
$$z_3 = b^{2(2m+1)/N},\ w = 6^m a^{2m+1} + b^{2m+1}$$

6. $z_1^3 + z_2^3 + z_3^3 + z_4^3 = w^3$

the simplest choice

$$z_1 = -a^3 + b^3 + 3^2 c^3,\ z_2 = a^3 - b^3 + 3^2 c^3,\ z_3 = a^3 + b^3 - 3^2 c^3$$
$$z_4 = 2 \cdot 3 \cdot abc,\ w = a^3 + b^3 + 3^2 c^3$$

another simple choice

$$z_1 = -a^3 + 3b^3 + 3c^3,\ z_2 = a^3 - 3b^3 + 3c^3,\ z_3 = a^3 + 3b^3 - 3c^3$$
$$z_4 = 2 \cdot 3 \cdot abc,\ w = a^3 + 3b^3 + 3c^3$$

there are many other bloated solutions as discussed previously.

7. $z_1^3 + z_2^3 + z_3^3 + z_4^N = w^3$

the simplest choice

$$z_1 = -a^N + b^N + 2^{N-3} 3^{N-1} c^N,\ z_2 = a^N - b^N + 2^{N-3} 3^{N-1} c^N$$
$$z_3 = a^N + b^N - 2^{N-3} 3^{N-1} c^N,\ z_4 = 2 \cdot 3 \cdot abc$$
$$w = a^N + b^N + 2^{N-3} 3^{N-1} c^N$$

8. *Generic power of 3*

$$\left(z_1^3 + \cdots + z_n^3\right) + \left(z_{n+1}^3 + \cdots + z_{n+\binom{n-1}{2}}^3\right) + \left(y_1 z_{n+\binom{n-1}{2}+1}^3 + \cdots + y_l z_m^3\right) = (n-2)w^3$$

where $n \geq 3,\ l = \binom{n}{3} - \binom{n-1}{2}$

$$z_i = [3^2 a_1^3, \ldots, a_n^3], \quad 1 \leq i \leq n$$
$$z_j = 2 \cdot 3 a_1 a_p a_q, \quad n+1 \leq j \leq n + \binom{n-1}{2}, \quad 2 \leq p, q \leq n$$
$$z_k = 2a_r, \qquad n + \binom{n-1}{2} + 1 \leq k \leq m, \quad 2 \leq r \leq n$$
$$y_q = 3a_s^3 a_t^3, \qquad 1 \leq q \leq \binom{n}{3} - \binom{n-1}{2}, \quad 2 \leq s, t(\neq r) \leq n$$
$$w = (3^2 a_1^3 + \cdots + a_n^3)$$

9. $z_1^3 + z_2^3 + z_3^N = w^3$ (to cube-finity and beyond)

$$z_1 = 6a^3 - 2^{3S} b^{NQ/3},\ z_2 = 6a^2 2^S b^{NQ/9},\ z_3 = 2^K b^Q,\ w = 6a^3 + 2^{3S} b^{NQ/3}$$
$$N = \{9S+1\},\ NK = 9S+1,\ Q = 9$$

- power of 4

1. $z_1^4 + z_2^3 + z_3^3 = w^4$

$$z_1 = a^3 - b^3,\ z_2 = 2a^3 b,\ z_3 = 2ab^3,\ w = a^3 + b^3$$

- power of 5

1. $z_1^3 + z_2^3 + y \cdot z_3^4 + z_4^5 = w^5$

$$z_1 = 4b^5,\ z_2 = 40a^2 b^3,\ y = 20b^3,\ z_3 = 20a^3,\ z_4 = 20a^3 - 2b^3,\ w = 20a^3 + 2b^3$$

2. $z_1^5 + z_2^5 + z_3^5 + y_1 \cdot z_4^3 + y_2 \cdot z_5^3 + y_3 \cdot z_6^3 = w^5$

$$z_1 = -a + b + c,\ z_2 = a - b + c,\ z_3 = a + b - c$$
$$y_1 = 80bc,\ y_2 = 80ac,\ y_3 = 80ab$$
$$z_4 = a,\ z_5 = b,\ z_6 = c$$
$$w = a + b + c$$

- power of 6

1. $z_1^6 + y \cdot z_2^N = w^6$

$$z_1 = 2^{N-2} a^N - b^N,\ y = (3 \cdot 2^{2(N-2)} a^{2N} + b^{2N})(2^{2(N-2)} a^{2N} + 3b^{2N})$$
$$z_2 = 2ab,\ w = 2^{N-2} a^N + b^N$$

- power of 7

  1. $z_1^7 + z_2^7 + z_3^7 + y_1 \cdot z_4^2 + y_2 \cdot z_5^3 + y_3 \cdot z_6^3 + y_4 \cdot z_7^3 = w^7$

$$z_1 = -a + b + c,\ z_2 = a - b + c,\ z_3 = a + b - c$$
$$y_1 = 42abc,\ y_2 = 28c,\ y_3 = 28b,\ y_4 = 28a$$
$$z_4 = 2p^2,\ z_5 = 2ab,\ z_6 = 2ac,\ z_7 = 2bc$$
$$w = a + b + c,\ p^2 = a^2 + b^2 + c^2$$

  here we need the formula for $a^2 + b^2 + c^2 = p^2$ above, *i.e.* $a = 2rs, b = 2rt, c = (-r^2 + s^2 + t^2), p = (r^2 + s^2 + t^2)$ but to avoid clutter I didn't substitute them into the above formula. This is the first time we have a cascade of algorithms

- power of 8

  1. $z_1^8 + z_2^3 + y \cdot z_3^3 = w^8$

$$z_1 = 4a^3 - b^3,\ z_2 = 4ab(b^6 + 16a^6),\ y = b^6 + 16a^6,\ z_3 = 16a^3b,\ w = 4a^3 + b^3$$

Note that variables used above are either integers or natural numbers which can be clearly deduced from their associated context.

Most of the formulae above can be easily modified into a more general power, *e.g.* $z_1^4 + z_2^3 + z_3^3 = w^4 \longrightarrow z_1^4 + z_2^{3N} + y \cdot z_3^N = w^4$. The starting point was

$$(a + b)^4 - (a - b)^4 = 8ab^3 + 8a^3b$$

We can of course substitute $a \to 8^{3N-1}a^{3N},\ b \to b^N$ such that

$$\begin{aligned} 8ab^3 + 8a^3b \Rightarrow 8^{3N}a^{3N}b^{3N} + 8^{9N-2}a^{9N}b^N \\ = (8ab)^{3N} + (8^{9N-2}a^{8N})(ab)^N \\ = z_2^{3N} + y \cdot z_3^N \end{aligned}$$

But I feel that this is too contrived, $8 = 2^3$ and I went for the most natural adaptation. This was my philosophy in building the table above although I went for the generalization if it is warranted. You can play this game endlessly so the rest is left as an exercise for the readers (beside myself, if any).

Hitting the books

So that's all I've got for fooling around with a "simple guess", the phrase I kept repeating in the previous several sections. There's no big discovery obviously, except for a few interesting things that I have mentioned above.

I then checked the literature about the "Cubic Pyhtagorean Equation", $a^3+b^3+c^3 = d^3$. There are many algorithms! There are even many one parameter algorithms. The really interesting point is that these algorithms usually only cover only certain solutions not all of them.

This explains why it was hard getting the overall vibe from the solutions because we need to group them, for example, in one of the one-parameter algorithm, the $1, 6, 8, 9$ is on its own, it is the only solution the algorithm produces that contains 1.

Getting one parameter algorithms is actually not that hard but it surely is tedious. Let's start with Pythagoras equation. What we want to do here is cast each number into a poluynomial $\sum a_n x^n$. The only question is how high a degree of a polynomial we want. We can even extend this to two or more parameters

$$\sum_{n=0,m=0}^{N_1,N_2} a_{mn} x^n y^m$$

where a_{mn} is some sort of a coefficient matrix. This formulation is more generic than $(\sum a_n x^n)(\sum b_m y^m)$ because in the case of product of sums we cannot have $a_{N_1} x^{N_1} + b_{N_2} y^{N_2}$ without having $a_{N_1} b_{N_2} x^{N_1} y^{N_2}$ as well. On the other hand, a_{mn} allows anything :) In fact, Euler's formula for the cubic Pythagorean is exactly like it, it contains y^4 and x^4 but it doesn't contain the product $x^4 \cdot y^4$. The only problem is that solving these coefficients, a_{mn}, might be quite laborious.

Let's start with Pythagoras, $z_1^2 + z_2^2 = w^2$

$$w = b_1 x + a_1$$

$$z_1 = b_2 x + a_2$$

$$z_2 = b_3 x + a_3$$

Substituting this into the Pythagorean equation $z_1^2 + z_2^2 - w^2 = 0$ and grouping coefficients

of different powers of x

$$x^2 : b_3^2 + b2^2 - b_1^2 = 0$$
$$x^1 : 2a_3b_3 + 2a_2b_2 - 2a_1b_1 = 0$$
$$x^0 : a_3^2 + a_2^2 - a_1^2 = 0$$

Two things to note here. Coefficients of the highest and lowest powers are themselves Pythagoreas equations, this is still true with cubic and higher Diophantines and is still obviously true no matter how high the polynomial we use for x. This can be good and bad.

Bad if we don't already have solutions but good that we might be able to generate new solutions from a known one. Going the bad news route, we have to set one of the b's and a's to zero, say $b_3 = 0 \rightarrow b_1 = b_2$. Now for a we don't want to set a_3 to zero as well, this is generally a bad strategy as $z_3 = b_3x + a_3 = 0$, this principle still applies for the cubic Pythagorean and higher.

So we need to set one of the b's and one of the a's to zero but we want to stagger them so that they don't belong to the same z. However, in the case of Pythagoras with just x this strategy doesn't work, it will only generate a trivial solution.

So now let's go the good news route which is not so good in this case. For example, setting $b_1 = 13, b_2 = 12, b_3 = 5$ generates $a_2 = 12a_1/13, a_3 = 5a_1/13$ which means that we are only generating integer multiples of $5, 12, 13$.

This means that we have to start with at least x^2 and indeed we do, starting with at least x^2 (and choosing the bad route) we have the following parameterizations

$z_1 = 2x + 2$	$z_2 = x^2 + 2x$	$w = x^2 + 2x + 2$
$z_1 = -x^2 + x + 2$	$z_2 = 2x - \frac{x^3}{2}$	$w = -\frac{x^3}{2} + x + 2$
$z_1 = \frac{x^4}{4} - x^3 + 2x$	$z_2 = -x^2 + 2x + 2$	$w = \frac{x^4}{4} - x^3 + 2x + 2$

Things on the good side is quite promising too, say you have a solution $A^2+B^2 = C^2,\ A <$

$B < C$ you can create new solutions $z_1^2 + z_2^2 = w^2,\ z_1 < z_2 < w$ using

$$w = A \cdot x^2 + b_1 x + a_1$$
$$z_2 = B \cdot x^2 + b_2 x + a_2$$
$$z_1 = C \cdot x^2 + b_3 x + a_3$$

$$a1 = \frac{C \cdot b_2^2 - 2B \cdot b_1 b_2 + C \cdot b_1^2}{(2A)^2}$$
$$a2 = -\frac{B \cdot b_2^2 - 2C \cdot b_1 b_2 + B \cdot b_1^2}{(2A)^2}$$
$$a_3 = -\frac{b_2^2 - b_1^2}{4A}$$
$$b_3 = -\frac{B \cdot b_2 - C \cdot b_1}{A}$$

$b_1,\ b_2$ remain free parameters. We can also subtitute the known solutions A, B, C into a_1, a_2, a_3 instead with a similar outcome. Since $b_1,\ b_2$, and x are free parameters, we immediately have a three-parameter solution. If we substitute the Babylonian algorithm for A, B, C we will then have a *five*-parameter algorithm.

For the cubic Pythagorean, $z_1^3 + z_2^3 + z_3^3 = w^3, z_1 < z_2 < z_3 < w$, the minimum would be x^2, no solution with just x. The solution I found was a generalization of a known solution, the starting point is

$$w = d_1 x^3 + c_1 x^2 + b_1 x + a_1$$
$$z_1 = d_2 x^3 + c_2 x^2 + b_2 x + a_2$$
$$z_2 = d_3 x^3 + c_3 x^2 + b_3 x + a_3$$
$$z_3 = d_4 x^3 + c_4 x^2 + b_4 x + a_4$$

To prep the solution we need to set $d_1 = d_4 = 1,\ d_2 = d_3 = 0,\ a_3 = a_1,\ a_2 = a_4 = 0$ and then $c_3 = c_4 = c_1$ followed by $b_3 = b_4 = b_1,\ b_2 = 0,$ *i.e.*

$$w = x^3 + c_1 x^2 + b_1 x + a_1$$
$$z_1 = \quad c_2 x^2$$
$$z_2 = \quad c_1 x^2 + b_1 x + a_1$$
$$z_3 = x^3 + c_1 x^2 + b_1 x$$

All this might seem random but if you do it you'll see the progression. First you'll need to set up a's and d's because they themselves have to satisfy the cubic Pythagorean. Once you do this, you'll see that other variables have to take certain values.

For our current assignment, the coefficients of the various powers of x are

$$\begin{aligned} x^6 &: c_2^3 + c_1^3 - 3a_1 = 0 \\ x^5 &: 3b_1c_1^2 - 6a_1c_1 = 0 \\ x^4 &: 3b_1^2c_1 - 6a_1b_1 = 0 \\ x^3 &: b_1^3 - 3a_1^2 = 0 \end{aligned}$$

Solving them altogether generates the following one parameter algorithm for the cubic Pythagorean

$$a_1 = 3\left(\frac{c_1}{2}\right)^3, \quad b_1 = 3\left(\frac{c_1}{2}\right)^2, \quad c_2 = \frac{c_1}{2}$$

$$\begin{aligned} w &= x^3 + c_1x^2 + 3\left(\frac{c_1}{2}\right)^2 x + 3\left(\frac{c_1}{2}\right)^3 \\ z_1 &= \left(\frac{c_1}{2}\right) x^2 \\ z_2 &= c_1x^2 + 3\left(\frac{c_1}{2}\right)^2 x + 3\left(\frac{c_1}{2}\right)^3 \\ z_3 &= x^3 + c_1x^2 + 3\left(\frac{c_1}{2}\right)^2 x \end{aligned}$$

while c_1 remains a free parameter. This means that we effectively have a two-parameter solution, x and c_1.

We can now see how we have multiple algorithms. For example, choosing a parameterization with up to x^3 means that we have $4 \times 4 = 16$ coefficients, $a_i, b_i, c_i, d_i,\ 1 \leq i \leq 4$. Since we are dealing with cubes we then have $x^j,\ 0 \leq j \leq 12$ which means that we have 13 constraints for 16 coefficients, some freedom is warranted. We must be careful however, as the requirement that all coefficients must be integers is a very strong condition.

One interesting thing to note here is that solving the coefficients of the polynomials x^n in itself is actually solving a set of Diophantine equations although in this case they are **coupled** Diophantine equations. The good thing is that we *just* need to find *one* solution not all of them, it may sometimes even be a trivial solution, *e.g.* in the cubic Pythagorean

example above we set $d_1 = d_4 = 1,\ d_2 = d_3 = 0$ which means that $-d_1^3 + d_2^3 + d_3^3 + d_4^3 = 0 \to 1^3 + 0 + 0 + 1^3 = 0$, a trivial solution. *One solution to rule them all* :)

Another interesting thing is that we can generate two-parameter algorithms from one-parameter algorithms. In Chamberland's paper there is an interesting formula

$$(ax^2 + cx - c)^3 + (bx^2 - bx + d)^3 + (cx^2 - cx - a)^3 = (dx^2 - bx + b)^3$$

where a, b, c, d are the solution of cubic Pythagorean $a^3 + b^3 + c^3 = d^3$, however, there's an extra constraint on a, b, c, d

$$c(c^3 - a^3) = b(d^2 - b^2)$$

Now, if we use the parameterization of a, b, c, d that obey the above constraint, we can generate a two-parameter algorithm by substituting it into the above algorithm. This process can *potentially* be repeated indefinitely if the new parameterization utilizes solutions of the cubic Pythagorean. Chamberland actually provides the *only* parameterization of a, b, c, d obeying the above constraint.

Intuition rumination

This short section is dedicated to the reflection of my numerical digression above. My biggest mistake was sticking with that stupid guess mechanism for too long. I was hoping that by starting from $(a + z)^n - (a - z)^n$, a and z will sort themselves out, *i.e.* they will transform into some polynomial automatically.

Sadly, that was not the case. This is actually my personal weakness, I always want to know the deeper reason behind anything. And so I was sucked into trying to understand why those silly guesses wouldn't work. I was also preoccupied with finding some sort of first principles that would allow me to universally generate parametrized algorithms for Diophantine's solutions.

In this kind of problems, there's no use in trying to get a generic principle, speed is all that matters. We just need to try as many potential candidates as fast as possible. Something that I do not like since in solving textbook physics problems you always start with some underlying principle and then work with it.

Now, let's see why this simple guess trick won't do but before that let's why it works for the power of *2*. The answer is **l.u.c.k**, $(a + z)^2 - (a - z)^2 = 4az$, 4 is a square and

we have only product of terms, gold mine! This is why we can easily extend this to any number of terms. Other powers are more complicated.

Recall that the formula for a cubic equation is a cube root of a square root $(a+(b^{1/2}))^{1/3}$ and not just a cube root from "completing the cube"?

Looking back, this is actually expected because $(a+b)^n$ contains all polynomials of degree $\leq n$ or in our current lingo, there's some "Exponent-Inception" going on, *e.g.*

$$(a+b)^5 = ((a+b)^2)^3 = (a^2+2ab+b^2)^3$$

Again, we see why $n=2$ is the lucky one, $(a+b)^2$ only contains squares and power of 1, on the other hand $(a+b)^5$ contains all powers of 5 and below, and from the equation above we can say that in a sense the powers are nested, and as such it is natural that in solving a polynomial of degree 3 we get a cube root of a square root.

This is somewhat reminiscent of Galois theory where the symmetry groups are nested although as to why and how they are precisely related, I do not know.

One more pertinent point, the constant coefficients in expanding a polynomial like this one $(a+b)^n$ are computed from Pascal triangle. And we all know that Pascal triangle is a nested (recursive) structure; you need its previous result before computing the next one. Hold on you say, we can use Newton's binomial expansion to compute it, we are modern people after all.

True, but binomial expansion requires factorials and a factorial itself is a recursive structure, there's no explicit formula for a factorial, you have to compute it one step at a time, which is the very definition of recursion. In other words, when we were trying to find algorithms for $z_1^n + \cdots + z_m^n = w^m$ what we were trying to do was recast a factorial into a polynomial, *e.g.*

$$\begin{aligned} (a+z)^3 &= \binom{3}{0}a^3 + \binom{3}{1}a^2z + \binom{3}{2}az^2 + \binom{3}{3}z^3 \\ \rightarrow (a+z)^3 - (a-z)^3 &= 6a^2z + 2z^3 \end{aligned}$$

The constant factors, 6 and 2, are (2 times) the binomial expansions (factorials) from $(a+z)^3$ and we are trying to somehow cast it into some polynomial $6 \rightarrow u^3$. It is of no surprise then that this will not always succeed, except for a few serendipitous cases.

If we trace it back even further, this recursivity is derived directly from the definition of exponentiation, *e.g.* 5^n is $5 \times 5 \times 5 \times 5 \times 5$. There is no explicit formula for this, we recurse some multiplications and happily denote it 5^n, in the words of a supermodel "image is powerful but image is also superficial". (*Note: addition is the basic operation in arithmetic, multiplication is a derived operation, i.e. repeated additions. Taking a number to some power is repeated multiplications, a derived operation of a derived operation, continuing this trend we can define something like* $a[n] \equiv a^{a^{a^{\cdots}}}$*, i.e. recursively exponentiating* n *times and see what kind of number theoretical implications we'll get from it but this might be taking it too far*)

This recursive business might also be the culprit behind the indomitability of Fermat's last theorem. Increasing the power by one is not just going one more above the previous one, the recursive nature requires that it is at least "one order of magnitude" more complicated so to say (although the dicothomy between $n = 2$ and $n \geq 3$ is still a mystery to me).

On a related but distant note, the formula for $\sum_{i=1}^{N} i$ is misleadingly simple. Misleading because it gives us the impression that we can avoid doing N additions by merely calculating $(N^2 + N)/2$. However, this is a pipe dream, sorry Gauss.

We are so used to multiplying things that we don't even realize it. To multiply things we need to add repeatedly, there's no other way. Even when we use our familiar algorithm of multiplying each digit what we are really doing is using the multiplication table. And that's all we do, we memorize the multiplication table for small numbers and we use it to calculate larger multiplications. The notation for multiplication is just that, a notation, a shorthand for $a \times b \equiv \sum_{i=1}^{b} a$.

The formula for $\sum_i i^n, n \geq 2$ is even more involved as it now concerns recursions. In fact, as far as I know, the formulae for $n \geq 3$ can only be derived recursively from $\sum_i i^2$, see my other article on this.

In short, unrolling a recursion into something straightforward is difficile (if not impossible) and that's what I was inflexibly trying to do using the simple guess tactic.

Now, why does polynomial work then? Simple, taylor expansion. Remember that any (well behaved) function can be Taylor expanded including factorials. But more than that,

we now have a lot more flexibility. However, the point here is not about the underlying principle behind all this numerical discourse, the lesson is that different problems require different approaches no one single approach is supreme, sorry string theory.

Farewell to Fermat

The main reason Fermat (or number theory for that matter) is alluring is because the subject matter is widely accessible. But accessibility does not equal simplicity. It is in fact extremely convoluted.

The biggest disappointment is that I was too stubborn to change my strategy but it was fun. Although I must say that this wasn't my best work, I wasn't feeling to well emotionally either, I was kinda depressed. This was mainly my escape; those bursts of excitements when you solved/realized something.

Well, that's all for now with me on this subject, Au Revoir Monsieur Fermat!

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What other operations can we add to our beloved arithmetic that might yield useful results?

We already have

1. additions
2. multiplications $\Leftarrow$ repeated additions
3. exponentiations $\Leftarrow$ repeated multiplications
4. factorial (and double factorial) $\Leftarrow$ modified repeated multiplications
5. higher op 1 $\Leftarrow$ repeated exponentiations ???
6. random op 1 $\Leftarrow$ anything we fancy ???

Substraction and division are inverse operators of addition and multiplication respectively while logarithm is the inverse of exponentiation. The interesting question would be what the inverse of factorial is. Denote $!^{-1}$ the inverse of factorial then

$$!^{-1}120 = 5$$

$$!^{-1}15 = \text{undefined?}$$

Although $!^{-1}15$ is undefined, can we still assign some real number value to it? in analogy to the square root of two, $\sqrt{2} = 1.414\ldots$

Some potential higher/random ops

1. $[a * b]_{\pm} \equiv a^b \pm b^a$

2. $[a, b]_{\pm} \equiv a \cdot b \pm b \cdot a$, proven useful in quantum mechanics

3. $a \; !{+}! \; b \equiv a + (b-1) + (a-2) + (b-3) + (a-4) + \cdots + 1$

4. $a!b \equiv a \cdot (a+1) \cdot (a+2) \cdot \cdots \cdot (a+b)$

=-

For $a, b \in \mathbb{Z}$ there's a unique $q, r \in \mathbb{Z}$ such that $a = bq + r$.

Proof. For simplicity let's assume that $a, b \geq 0$, the argument for negative integers is similar. We assume $b \neq a$ otherwise it is trivial, *i.e.* $q = 1,\ r = 0$. If $b > a$ then $q = 0,\ r = a$. If $b < a$ then $r < b$. If $r > b$ we can keep substracting r until it is less than b say

$$\begin{aligned} a &= bq + r \\ &= bq + \underbrace{b + b + b + b + \ldots}_{n \text{ number of } b\text{'s}} + (r - \underbrace{b - b - b - b - \ldots}_{n \text{ number of } b\text{'s}}) \\ a &= b(q+n) + r' \\ a &= bq' + r' \end{aligned}$$

where $q' = q + bn$ and $r' = r - bn$. Suppose there are two sets of q, r that satisfy $a = bq + r$, let's denote them $a = bq_1 + r_1$ and $a = bq_2 + r_2$ then

$$\begin{aligned} bq_1 + r_1 &= bq_2 + r_2 \\ b(q_1 - q_2) &= (r_2 - r_1) \end{aligned}$$

this means that b divides $(r_2 - r_1)$, *i.e.* $r_2 - r_1 = bm$ for some $m \in \mathbb{Z}$ but since $r_1, r_2 < b$ this is only possible if $m = r_2 - r_1 = 0$ thus

$$\begin{aligned} b(q_1 - q_2) &= 0 \\ &\rightarrow q_1 = q_2 \end{aligned}$$

since $b \neq 0$.

=-

This is my (baby step) attempt to prove the fundamental theorem of arithmetic that states that every integer is a unique product of primes, we start with a product of two primes.

The product of primes p_1p_2 will be equal to the product of other primes p_3, p_4, *i.e.* $p_1p_2 = p_3p_4$, if and only if p_1, p_2 and p_3, p_4 are the same, *i.e.* $p_1 = p_3,\ p_2 = p_4$ or $p_1 = p_4,\ p_2 = p_3$.

I tried different things and I only got circular arguments :(this is the first circular argument.

Proof? I'm not sure about this proof but here it goes. Assume the opposite, $p_1p_2 = p_3p_4$ but $\{p_1, p_2\} \neq \{p_3, p_4\}$.

First observation, say $p_1 > p_3$, this means that $p_4 > p_2$ such that the equality has a chance to hold, this also means that

$$p_1 = qp_3 + r$$
$$p_4 = \tilde{q}p_2 + \tilde{r}$$

for some $q, r, \tilde{q}, \tilde{r} \in \mathbb{Z}$

$$\begin{aligned} p_1p_2 &= p_2(qp_3 + r) \\ p_3p_4 &= p_3(\tilde{q}p_2 + \tilde{r}) \\ qp_2p_3 + p_2r &= \tilde{q}p_2p_3 + p_3\tilde{r} \\ p_2p_3(q - \tilde{q}) &= p_3\tilde{r} - p_2r \\ \cancel{p_2p_3}(q - \tilde{q}) &= \cancel{p_2p_3}\left(\frac{\tilde{r}}{p_2} - \frac{r}{p_3}\right) \end{aligned}$$

since $r < p_3,\ \tilde{r} < p_2 \to \frac{r}{p_3} < 1,\ \frac{\tilde{r}}{p_2} < 1$. Now the minimum of $q - \tilde{q}$ is 1 while the maximum of $\frac{\tilde{r}}{p_2} - \frac{r}{p_3}$ is < 1. The only way this can be consistent is if $\frac{\tilde{r}}{p_2} - \frac{r}{p_3} = q - \tilde{q} = 0$.

But this means

$$\frac{\tilde{r}}{p_2} = \frac{r}{p_3}$$
$$p_3\tilde{r} = p_2r$$

This means that p_2 divides $p_3\tilde{r}$, since p_3 is prime p_2 can only divide $\tilde{r}$ but since $\tilde{r} < p_2$, p_2 doesn't divide $\tilde{r}$ either so there's a contradiction.

The above statement is a circular argument, why? There's no theorem about $a|bc \to a|c$ if and only if a and b are primes, unless you already know the fundamental theorem of arithmetic. I tried proving it in various ways but it always comes down to a circular argument.

My first attempt was that $a|bc$ means that a is the product of the prime factors of b and c, now since a is prime, it is not a product of anything. We also have $a \neq b$, the only conclusion is that a is one of the prime factor of c. But this is only true if bc can be decomposed into a *unique* product of primes which is a circular argument (if the product is not unique bc might be cast into $bc = az$ and $a|bc$ is just a tautology, *i.e.* something like $0 = 0$, something that is obviously true).

Another argument I tried was using rational numbers

$$p_3\tilde{r} = p_2 r$$
$$\frac{p_3}{p_2} = \frac{r}{\tilde{r}}$$

but this is a contradiction as $p_2,\ p_3$ are primes and we cannot simplify the ratio p_3/p_2, but since $r/\tilde{r}$ has $r < p_3,\ \tilde{r} < p_2$ we have a "smaller" ratio which in turn means that p_3/p_2 can be simplified, a contradiction indeed. However, the canonical form of rationals is a consequence of the fundamental theorem. We always know that we can simplify ratios because we know that every number is a unique product of primes.

I then saw that $p_1p_2 = p_3p_4$ is a common multiple of p_2 and p_3. I then proceeded to prove that the least common multiple of p_2 and p_3 is p_2p_3, the proof can be found below. However, a common multiple is not necessarily the least common multiple, so even though the least common multiple is the product p_2p_3 we might have other common multiples that are larger, *e.g.* $p_1p_2 = p_3p_4,\ p_1 > p_3,\ p_4 > p_2$.

++

Say we have two relatively prime numbers p_2 and p_3, what is their least common multiple? Say their least common multiple is $mp_2 = p_3n$ since this is the **least** common

multiple m and n are the smallest (relatively prime) numbers that satisfy this equation, relatively prime such that we cannot cancel common factors and create an even lesser common multiple.

My goal here is to show that $m = p_3,\ n = p_2$.

Say that $m > p_3$, this means that $n > p_2$ such that the equality has a chance to hold, this also means that

$$m = qp_3 + r$$
$$n = \tilde{q}p_2 + \tilde{r}$$

for some $q, r, \tilde{q}, \tilde{r} \in \mathbb{Z}$

$$\begin{aligned} mp_2 &= p_2(qp_3 + r) \\ p_3 n &= p_3(\tilde{q}p_2 + \tilde{r}) \\ qp_2p_3 + p_2r &= \tilde{q}p_2p_3 + p_3\tilde{r} \\ p_2p_3(q - \tilde{q}) &= p_3\tilde{r} - p_2r \\ \cancel{p_2}\cancel{p_3}(q - \tilde{q}) &= \cancel{p_2}\cancel{p_3}\left(\frac{\tilde{r}}{p_2} - \frac{r}{p_3}\right) \end{aligned}$$

since $r < p_3,\ \tilde{r} < p_2 \rightarrow \frac{r}{p_3} < 1,\ \frac{\tilde{r}}{p_2} < 1$. Now the minimum nonzero value of $q - \tilde{q}$ is 1 while the maximum of $\frac{\tilde{r}}{p_2} - \frac{r}{p_3}$ is < 1. The only way this can be consistent is if $\frac{\tilde{r}}{p_2} - \frac{r}{p_3} = q - \tilde{q} = 0$ such that

$$\begin{aligned} \frac{r}{p_3} &= \frac{\tilde{r}}{p_2} \\ rp_2 &= p_3\tilde{r} \end{aligned}$$

but since $r < m$, $\tilde{r} < n$, this is a contradiction, *i.e.* m can't be larger than p_3!

Next option is $m < p_3$, this means that $n < p_2$ such that the equality has a chance to hold, this also means that

$$p_3 = qm + r$$
$$p_2 = \tilde{q}n + \tilde{r}$$

for some $q, r, \tilde{q}, \tilde{r} \in \mathbb{Z}$

$$
\begin{aligned}
mp_2 &= m(\tilde{q}n + \tilde{r}) \\
p_3 n &= n(qm + r) \\
\cancel{mn}\left(\tilde{q} + \frac{\tilde{r}}{n}\right) &= \cancel{nm}\left(q + \frac{r}{m}\right) \\
\tilde{q} - q &= \frac{r}{m} - \frac{\tilde{r}}{n}
\end{aligned}
$$

since $r < m,\ \tilde{r} < n \rightarrow \frac{r}{m} < 1,\ \frac{\tilde{r}}{n} < 1$. Now the minimum nonzero value of $q - \tilde{q}$ is 1 while the maximum of $\frac{r}{m} - \frac{\tilde{r}}{n}$ is < 1. The only way this can be consistent is if $\frac{r}{m} - \frac{\tilde{r}}{n} = \tilde{q} - q = 0$. But this means

$$
\begin{aligned}
\frac{\tilde{r}}{n} &= \frac{r}{m} \\
m\tilde{r} &= rn \\
\rightarrow \frac{\cancel{m}\,\tilde{r}}{\cancel{m}p_2} &= \frac{r\,\cancel{n}}{p_3\cancel{n}} \\
rp_2 &= p_3\tilde{r}
\end{aligned}
$$

since $r < m,\ \tilde{r} < n$, this is again a contradiction as we claim that $m,\ n$ are the smallest numbers that satisfy $mp_2 = p_3 n$. This means that $m < p_3$ is wrong but we have shown that $m > p_3$ is also wrong, the only conclusion is that $m = p_3$ and therefore $n = p_2$.

++

Algorithm to find least common multiple, an accidental byproduct of the above mishap.

You want to find the least common multiple of m and n, say $m > n$ (and of course they are co prime).

$$
\begin{aligned}
m &= nq + r \\
am &= nb \\
n(aq) + ar &= nb \\
ar &= n(b - aq)
\end{aligned}
$$

Now we have the common multiple of r (the remainder of m) and n. So the question of finding the least common multiple of m and n becomes the question of finding the

common multiple of r and n. This is reminiscent of the gcd algorithm of Euclid. The next step is repeating the process, since $r < n,\ n = rq' + r'$ and we now search for the common multiple r and r' and so on.

++

$m|a$, m is biggest divisor of $a,\ m < a$, and $n|b$ and $n < b$ is the biggest divisor of b, what is the biggest divisor of ab ?

$$\begin{aligned} ab &= (mc)(nd) \\ &= (mn)(cd) \\ ab &= kw \end{aligned}$$

Say $k > mn$

$$\begin{aligned} (mn)(cd) &= (mnq + r)w \\ \cancel{(mn)}(cd) &= \cancel{(mn)}w\left(q + \frac{r}{mn}\right) \\ cd &= wq + \frac{wr}{mn} \\ (mn)(cd - wq) &= wr \\ \cancel{w}(mn)\left(\frac{cd}{w} - q\right) &= \cancel{w}r \end{aligned}$$

Now $r > 0,\ r < mn$, so this is only consistent if $r = 0,\ cd/w = q$ but this means that $k = mn,\ w = cd$.

++

The pdf book I've been reading about elementary number theory (ent) is ent.pdf. It is written by William Stein, there are multiple versions and revisions of the book. One is called "*Primes, Congruences, and Secrets*" dated Nov 16, 2011. Another version is called "*A Computational Approach*" dated March 2007. As far as I can tell, they are pretty much the same but I'll use "*Primes, Congruences, and Secrets*" in this article.

The proofs in these books are not the easiest to understand but they are really cool. So I'll list here explanations about the proofs that hopefully clarify them.

Lemma 1.1.9 This is a really cool way to prove something. He was using the concept of (sub)sets, something seemingly unrelated to prove properties of gcd.

The goal here is to show that

$$\gcd(a,b) = \gcd(b,a) = \gcd(\pm a, \pm b) = \gcd(a, b-a) = \gcd(a, b+a)$$

The first thing he showed is that $\gcd(a,b) \leq \gcd(a,b-a)$ followed by $\gcd(a,b-a) \leq \gcd(a,b)$ which means that $\gcd(a,b) = \gcd(a,b-a)$.

To show that $\gcd(a,b) \leq \gcd(a,b-a)$ he uses the idea of a subset. He shows that every common divisor of a and b is also a common divisor of $b-a$, *i.e.* $a = dc_1, b = dc_2 \rightarrow b - a = d(c_2 - c_1)$. Now since every common divisor of a and b is also a divisor of $b-a$ this means that the common divisors of a and b is a subset of the divisor of $b-a$.

Since they are a subset $\gcd(a,b) \leq \gcd(a,b-a)$ as the max element of the subset is either the same as the max element of the superset or lower, *e.g.* consider a set $\{1,4,8,9\}$ if you form a subset of this, the maximum of this subset is 9 or lower, depending what elements you choose for the subset but it can't be higher than 9.

The next step in the proof is to show that $\gcd(a,b-a) \leq \gcd(a,b)$. He did this by replacing $a \rightarrow -a,\ b \rightarrow b-a$ such that $\gcd(a,b) \rightarrow \gcd(-a,b-a)$. He then repeats the reasoning above, every common divisor of $-a$ and $b-a$ is also a divisor of $(b-a)-(-a) = b$ this means that $\gcd(-a,b-a) \leq \gcd(-a,b)$ but $\gcd(-a,b-a) = \gcd(a,b-a),\ \gcd(-a,b) = \gcd(a,b)$. This completes the proof.

Lemma 1.1.17 This is another cool one. He uses (strong) induction in a way I'd neven seen before. Usually you use induction by proofing the link between n and $n+1$ but he doesn't. Instead he retrospects, let's see what I mean.

The goal here is to prove $\gcd(an,bn) = \gcd(a,b) \cdot |n|$ something very obvious.

He started by (implicitly) assuming that all numbers a', b' smaller than a, b, *i.e.* $a'+b' < a+b$, have this property. Note that he uses the sum of the numbers for induction instead the numbers themselves. As the base case he uses $a+b=2$.

The proof now continues through several bridges.

1. Show that $\gcd(an,bn) = \gcd(bn,rn)$, this is achieved through Euclid's algorithm as $an = bnq + rn$.

2. Next, show that $\gcd(bn, rn) = \gcd(b, r) \cdot |n|$, how? by showing that $b + r < a + b$, why? Because if $b + r < a + b$ then the case of $\gcd(bn, rn)$ is already covered by the induction. Recall that induction assumes all pairs $b + r$ leading up to $a + b$ already have this property, so if $b + r < a + b$ then b, r is already in the induction assumption. This is why I said that he's using a strong induction.

As we have seen, the induction proof was not a typical one, we're not proving the link between n and $n+1$, he assumes that all elements less than $n+1$ are true and then show that our target is equal to one of those earlier elements. Very clever indeed.

Lemma 1.1.17, "Gangnam Style". Despite being a very clever proof I would like a simpler one :) A much simpler proof will utilize Euclid's algo

$$\begin{aligned} a &= bq_1 + r_1 \\ b &= r_1 q_2 + r_2 \\ &\vdots \\ r_{m-2} &= r_{m-1} q_m + (1?) \end{aligned}$$

where if the last remainder is 1 the gcd between a and b is 1, otherwise it is r_{m-1}. If we multiply a and b by n, the multiplication will cascade down all the way

$$\begin{aligned} na &= nb \cdot q_1 + nr_1 \\ nb &= nr_1 \cdot q_2 + nr_2 \\ &\vdots \\ nr_{m-2} &= nr_{m-1} \cdot q_m + (n \cdot 1?) \end{aligned}$$

if the last remainder is n then the gcd is n since $n | nr_{m-1}$ otherwise the remainder is nr_{m-1}. Thus $\gcd(na, nb) = |n| \cdot \gcd(a, b)$.

Theorem 1.1.19 Another clever proof, another indirect one. The goal is to show that if p is prime and $p|ab$ then either $p|a$ or $p|b$.

He approaches this by showing that $p | \gcd(pb, ab)$ why? because $\gcd(pb, ab) = b \gcd(p, a) = b$. So his approach was to find "how can I represent b? are there any other forms of b that are divisible by p?" The answer of course is that $p | \gcd(pb, ab)$ and $\gcd(pb, ab)$ happens to be b itself. Nice trick.

Theorem 1.1.19, "Gangnam Style". I, on the other hand, will personally do it the other way. Starting with there was a prime p and if $p \nmid a$ and $p \nmid b$ then $p \nmid ab$, *i.e.* we turn it around, from "'if A then B" into "if not B then not A".

Say $p \nmid a$ and $p \nmid b$ and we then assume a contradiction $p|ab$. Now $p|pa$ and $p|ab$, from Lemma 1.1.17 $p|\gcd(pa, ab) \to p|a\gcd(p, b)$. Since p is prime $\gcd(p, b)$ is either p or 1. If $\gcd(p, b) = 1$ then $p|a\gcd(p, b) = p|a \cdot 1$ which is a contradiction, if $\gcd(p, b) = p$ then $p|b$ which is also a contradiction, which means that our assumption $p|ab$ is wrong, *i.e.* $p \nmid ab$.

Since $p \nmid a$ and $p \nmid b \longrightarrow p \nmid ab$ this means that $p|ab \longrightarrow p|a$ or $p|b$ with the understanding that p is prime.

Theorem 1.1.6, the proof is actually on page 10. I just want to recap the strategy he used to prove this theorem. The key is Theorem 1.1.19 but to prove 1.1.19 you need to know about gcd and its properties, they are

1. Lemma 1.1.9, for any integers a and b we have

$$\gcd(a, b) = \gcd(b, a) = \gcd(\pm a, \pm b) = \gcd(a, b - a) = \gcd(a, b + a)$$

2. Lemma 1.1.10, suppose $a, b, n \in \mathbb{Z}$. Then $\gcd(a, b) = \gcd(a, b - an)$.

3. Proposition 1.1.11, suppose that a and b are integers with $b \neq 0$. Then there exists unique integers q and r such that $0 \leq r < |b|$ and $a = bq + r$

4. Algorithm 1.1.12 (Division Algorithm) Suppose a and b are integers with $b \neq 0$. This algorithm computes integers q and r such that $0 \leq r < |b|$ and $a = bq + r$

5. Algorithm 1.1.13 (Greatest Common Division), this is Euclid's algorithm

6. Lemma 1.1.17, for any integers a, b, n we have

$$\gcd(an, bn) = \gcd(a, b) \cdot |n|$$

7. Lemma 1.1.18, suppose $a, b, n \in \mathbb{Z}$ are such that $n|a$ and $n|b$. Then $n|\gcd(a, b)$

8. Theorem 1.1.19, let p be a prime and $a, b \in \mathbb{N}$. If $p|ab$ then $p|a$ or $p|b$

So it's nowhere near straightforward, trying to prove this theorem straightforwardly is an oxymoron.

Algorithm 1.2.3 This is actually about proof of step 2, the stopping condition. The proof of this step doesn't give any intuition as to how this is true, it basically boils down to "false assumption leads to contradictions".

There's actually a better explanation which is very intuitive. Once you come to a number that is roughly $\sim \sqrt{n}$ you know you can stop, why? The current number in question $\sqrt{n}$ is of course a prime since the non primes have been crossed off by the previous primes and their multiples. But some of the multiples of $\sqrt{n}$, *i.e.* $p\sqrt{n},\ p < \sqrt{n}$ have been crossed off by multiples of the previous lesser primes. The next thing to cross is of course $\sqrt{n} \cdot \sqrt{n}$, but this is already bigger than n, so we can stop.

The above explanation might still be confusing. Let's see it with a concrete example, for simplicity let's use the same example as in the book $n = 40$ but let's jump straight to after we've crossed off $2, 3, 5$ and their multiples. The set X is now

$$X = [7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37]$$

We now need to cross off 7 and its multiples, however,

- 7×2 was crossed off by 2×7
- 7×3 was crossed off by 3×7
- 7×4 was crossed off by 2×14
- 7×5 was crossed off by 5×7.

The next thing to cross off is then 7×7 but this is larger than n so we can immediately stop. For any other number > 7 the situation is even more dire since some of their multiples have been crossed off by the lesser primes and their square is definitely bigger than n.

Proposition 1.2.5 The goal here is to show that there are infinitely many primes in the form $4x - 1$. However the proof was a bit confusing. He started like Euclid, constructing a new number in the form

$$N = 4p_1p_2 \cdots p_n - 1$$

Notice that he began by assuming that $p_1 p_2 \cdots p_n$ are primes of the form $4x-1$. This is required as we want to show that no known prime of the form $4x-1$ divides N, if p_i is just any prime we would only show that the new prime that divides N is not the same as any of p_i but since p_i is not of the form $4x-1$ there might be some other primes of the form $4x-1$ that wasn't included in constructing N. Thus the new prime factor of N is not really new, it just wan't included in constructing N.

As to why $p_i \nmid N$ is because if $p_i | N \rightarrow N = p_i m$ then

$$\begin{aligned} N = p_i m &= p_i \cdot (4 \prod_{j \neq i} p_j) - 1 \\ p_i (m - 4 \prod_{j \neq i} p_j) &= -1 \\ \rightarrow p_i &\mid -1 \end{aligned}$$

which is a contradiction (this is the reasoning used in Euclid's famous "there are infinite prime numbers" proof).

He then argues that if all prime factors of N is of the form $4x+1$ then N will be of the form $4x+1$ as well which is true. The mysterious phrase is then (the phrase in the "*A Computational Approach*" is even worse) "since N is odd, each prime divisor p_i is odd so there is a $p|N$ that is of the form $4x-1$". Why not $2x+1$? which is the most generic form of an odd number. The thing is that N is of the form $4x-1$ so we want the product of its factors to conform to $4x-1$. However, his requirement is actually not stringent enough (or he didn't explain it clearly enough), say that we have an odd number whose factors are

$$(2x_1+1)(4x_2-1) = 8x_1x_2 - 2x_1 + 4x_2 - 1$$

just because one of the factor is $4x-1$ it doesn't mean that we are guaranteed to have $N \sim 4x-1$. What we really need (as seen from the example above) is that all other factors are of the form $4x+1$ while at least one of them (or an odd number of them) is of the form $4x-1$.

This is because the product of of the of x's with nothing but the 1's need to be in the form of $4x$, thus we need to have all the x's multiplied by 4 and we need an odd number of $4x-1$ because the product of all the 1's needs to be -1.

How about the last sentence "We can repeat this process indefinitely"? Remember that the primes that construct N, *i.e.* p_i, were all of the form $4x-1$. Since we find another one of the form $4x-1$ (as one of the factor of N) we can construct a new number $N' = 4p_1p_2\cdots p_np_{\text{new}} - 1$ which in turn produces another prime of the form $4x-1$ which we can use to construct another number and so on.

Problem 1.1 Warm up! :) Compute the greatest common divisor $\gcd(455, 1235)$ by hand.

I did it by hand no calculator involved :)

$$\begin{aligned} 1235 &= 455 \times 2 + 325 \\ 455 &= 325 \times 1 + 130 \\ 325 &= 130 \times 2 + 65 \\ 130 &= 65 \times 2 \end{aligned}$$

gcd is 65!

Problem 1.3 Prove that there are infinitely many primes of the form $6x-1$.

This closely follows the proof of Proposition 1.2.5. Let

$$N = 6p_1p_2\cdots p_n - 1$$

where p_i is of the form $6x-1$. We know that $p_i \nmid N$. To get the form $6x-1$ the prime factors of N has to be of the form $6x \pm 1$ and an odd number of them of them has to be $6x-1$. Thus we have a new prime of the form $6x-1$ which we can use to construct a new Number $N' = \prod 6p_i - 1$ and the whole process repeats.

Problem 1.4 Use Theorem 1.2.10 to deduce that $\lim_{x\to\infty} \pi(x)/x = 0$.

Theorem 1.2.10 is

$$\begin{aligned} \lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)} &= 1 \\ \lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)} - 1 &= 0 \end{aligned}$$

The thing I'm not sure about is whether we can do the usual algebraic manipulations with the lim involved, for example, is the following legit?

$$\lim_{x\to\infty} \left(\frac{\pi(x)}{x/\log(x)} - 1 \right) \times \frac{1}{\log(x)} = 0 \times \frac{1}{\log(x)}$$

I'm not sure if we can do the above but

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)} = 1$$
$$\lim_{x\to\infty} \pi(x) = \lim_{x\to\infty} \frac{x}{\log(x)}$$
$$\lim_{x\to\infty} \frac{1}{x} \times \pi(x) = \lim_{x\to\infty} \frac{1}{\cancel{x}} \times \frac{\cancel{x}}{\log(x)} = 0$$

I think that was legal as the first line says that $\lim_{x\to\infty} \pi(x) = \lim_{x\to\infty} x/\log(x)$, *i.e.* as x goes to ∞ the two approach the same value. This means that if we divide both sides by the same value we will still get the same limit for both sides.

Problem 1.8 (a) if $a|n$ and $b|n$ with $\gcd(a,b) = 1$, then $ab|n$

The easiest way is of course to use the fundamental theorem of arithmetic but here I try not to do that if at all possible. First $a|n \to n = aa',\ b|n \to n = bb'$, suppose $a < b$ (we know $a \neq b$), it doesn't matter which we choose to be smaller, the argument will be the same

$$aa' = bb'$$
$$aa' = (aq + r)b'$$
$$a(a' - qb') = rb'$$

which means that $a|rb'$. We also know from Euclid's algorithm that $\gcd(b,a) = \gcd(a,r)$ but since $\gcd(a,b) = 1 \to \gcd(a,r) = 1$. We know that $a|rb'$ and obviously $a|ab'$ then from Lemma 1.1.18 $\to a|\gcd(rb', ab')$, following the proof of Theorem 1.1.19

$$a \mid \gcd(rb', ab')$$
$$\gcd(ar, ab') = |b'| \cdot \gcd(r,a) = |b'| \cdot 1$$
$$\to a \mid b'$$

so $b' = aw \to n = bb' = (ba)w \to ab|n$ which completes the proof.

Problem 1.8 (b) if $a|bc$ and $\gcd(a,b) = 1$, then $a|c$

Again, let's not use the fundamental theorem. Since we know that $a \neq b$, the first

possibility is $a < b \rightarrow b = aq + r$

$$\begin{aligned}(aq + r)c &= an \\ rc &= a(n - q) \\ \rightarrow a &\mid rc\end{aligned}$$

Following the footsteps of Problem 1.8 (a) above, $\gcd(b, a) = 1 = \gcd(a, r)$. Next, $a|rc$ and $a|ac$, so according to Lemma 1.1.18 $\rightarrow a|\gcd(rc, ac) = c \cdot \gcd(r, a) = c \cdot 1 = c \rightarrow a|c$.

The only other possibility is $b < a \rightarrow b = a - \Delta$

$$\begin{aligned}(a - \Delta)c &= an \\ -\Delta c &= a(n - c) \\ \rightarrow a &\mid \Delta c\end{aligned}$$

To proceed we need to show that

$$\begin{aligned}\Delta &= -b + a \\ \rightarrow \gcd(a, \Delta) &= \gcd(a, -b + a) = \gcd(a, -b) = \gcd(a, b) \\ &= 1\end{aligned}$$

where we have used Lemma 1.1.9 in the second line, thus $\gcd(a, \Delta) = 1$, since $a|\Delta c$ and $a|ac$ using Lemma 1.1.18 $\rightarrow a|\gcd(\Delta c, ac) = c \cdot \gcd(\Delta, a) = c \cdot 1 = c \rightarrow a|c$.

Problem 1.9 (a) if $a|b$ and $b|c$ then $a|c$.

$$\begin{aligned}a|b \rightarrow b &= am \\ b|c \rightarrow c &= bn = amn \\ c = a(mn) &\rightarrow a|c\end{aligned}$$

Problem 1.9 (b) if $a|b$ and $c|d$ then $ac|bd$.

$$\begin{aligned}a|b \rightarrow b &= am \\ c|d \rightarrow d &= cn \\ ac = (ac)(mn) &\rightarrow ac|bd\end{aligned}$$

Problem 1.9 (c) if $m \neq 0$, then $a|b$ if and only if $ma|mb$.

$$a|b \to b = aw$$
$$mb = maw \to ma|mb$$

The other way

$$ma|mb \to mb = maw \to b = aw$$
$$b = aw \to a|b$$

Problem 1.9 (c) if $d|a$ and $a \neq 0$, then $|d| \leq |a|$.

$$d|a \to a = dw \to |a| = |dw| = |d||w|$$
$$|a| = |d||w| \to |d| \leq |a|$$

Problem 1.11 (b) Fun facts! The two numbers are actually the first 27 digits of π and e and their gcd is 13. Their factors are

$$314159265358979323846264338 = 2 \times 3 \times 7 \times \mathbf{13} \times 17 \times 23 \times 53 \times 27765442739383319011$$
$$271828182845904523536028747 = \mathbf{13} \times 31 \times 14419 \times 48287851 \times 968760484921$$

Problem 1.12 (a) Suppose $a,\ b$ and n are positive integers. Prove that if $a^n|b^n$ then $a|b$.

Let's try if we can do it without the fundamental theorem, since it'll be too easy otherwise ¯\(°_o)/¯

The problem is an "if A then B" type but it also means "if not B then not A". We'll go by induction but before that, from $a^n|b^n$ we know that $a < b$ (which can be easily proven by induction). We also know that $a|b^n$ either through Lemma 1.1.10, because $a^n|b^n \to b^n = a^n m$ and

$$\begin{aligned}\gcd(a, b^n) &= \gcd(a, a^n m) \\ &= \gcd(a, a^n m - ad), \quad d = a^{n-1}m - 1 \\ \gcd(a, b^n) &= \gcd(a, a) = a \\ &\to a \mid b^n\end{aligned}$$

or through $b^n = a^n m = a(a^{n-1}m) = aw \to a|b^n$ or through Problem 1.9 (a), $a|a^n$ and $a^n|b^n$ then $a|b^n$.

We now want to show that if $a \nmid b$ then $a \nmid b^n$ by induction. We start by proving the base case $a \nmid b^2$. Before we start we want to divide both a and b by $m \equiv \gcd(a, b)$ and denote them $a' = a/m$ and $b' = b/m$ where $\gcd(a', b') = 1$.

From Problem 1.9 (c) we have $a \nmid b \to a' \nmid b'$. Now what about $a'|b'^2$? Say $a'|b'^2$ then

$$\begin{aligned} &a'|b'^2 = a'|b'b', \quad \gcd(a', b') = 1 \\ &\to a' \mid b' \end{aligned}$$

where we have used Problem 1.8 (b). But this is a contradiction since $a' \nmid b'$, thus $a' \nmid b'^2$. Now the induction step, suppose $a' \nmid b'^n$ we want to show $a' \nmid b'^{n+1}$, assume otherwise

$$\begin{aligned} &a'|b'^{n+1} = a'|b'b'^n, \quad \gcd(a', b') = 1 \\ &\to a' \mid b'^n \end{aligned}$$

where we have used Problem 1.8 (b). This is a contradiction as $a' \nmid b'^n$ thus we have proved that if $a' \nmid b'$ then $a' \nmid b'^n$ where $\gcd(a', b') = 1$. In other words if $a'|b'^n$ then $a'|b'$.

Our problem states that $a^n|b^n = \cancel{m^n}a'^n|\cancel{m^n}b'^n$ where $m = \gcd(a, b)$ (we have used Problem 1.9 (c)). But $a'^n|b'^n$ means that $a'|b'^n$ (as shown above) and by the above result $a'|b'^n \to a'|b' = ma'|mb' = a|b$ using Problem 1.9 (c). This completes the proof.

I initially solved it this way

$$\begin{aligned} b^n &= a^n m \\ m &= \left(\frac{b}{a}\right)^n \end{aligned}$$

Since $m \in \mathbb{Z} \to b/a \in \mathbb{Z} \to b = an,\ n \in \mathbb{Z} \to a|b$. But the dodgy step is the requirement that $b/a \in \mathbb{Z}$, this smells like we are implicitly assuming the fundamental theorem as we are trying to simplify the ratio into the canonical form.

I tried many other more straightforward ways but they didn't come to anything, one of the ways was to assume $a \nmid b \to b = aq + r$ with not much luck.

Problem 1.12 (b) Suppose p is a prime and a and k are positive integers. Prove that if $p|a^k$, then $p^k|a^k$.

Again, let's try not to use the fundamental theorem. Start with $p|a^k \rightarrow p|aa^{k-1}$, we also have $p|pa^{k-1}$. From Lemma 1.1.18 $\rightarrow p|\gcd(aa^{k-1}, pa^{k-1}) = p|a^{k-1}\gcd(a,p)$. From Theorem 1.1.19 it is either $p|a^{k-1}$ or $p|\gcd(p,a)$.

Now $\gcd(p,a)$ is either 1 or p since p is prime. If $\gcd(p,a) = p$ then we are done since $p|a$ and $a|a^k$ means $p|a^k$ (see Problem 1.9 (a)). If $\gcd(p,a) = 1$ then $p|a^{k-1}$.

We now repeat the process, $p|\gcd(aa^{k-2}, pa^{k-2}) \rightarrow p|a^{k-2}$ since $\gcd(p,a) = 1$. Continuing in this path to $p|a$ we get a contradiction because we have assumed that $\gcd(p,a) = 1$, thus $\gcd(p,a) = p$ and according to Problem 1.9 (a), $p|a$ and $a|a^k$ means $p|a^k$.

Problem 1.13 (a) Prove that if a positive integer n is a perfect square, then n cannot be written in the form $4k+3$.

Since $4k+3$ is odd n has to be odd. In that case $n = (2x+1)^2$

$$\begin{aligned}(2x+1)^2 &= 4x^2 + 4x + 1 \\ &= 4(x^2+x) + 1 \\ &= 2\{2x(x+1)\} + 1\end{aligned}$$

while

$$\begin{aligned}4k+3 &= 4k+2+1 \\ &= 2(2k+1)+1\end{aligned}$$

This means that $2x(x+1) = 2k+1$ which is impossible since the LHS is even while the RHS is odd.

Problem 1.13 (b) Prove that no integer in the sequence

$$11, 111, 1111, 11111, 111111, \ldots$$

is a perfect square. (Hint: $111\cdots111 = 111\cdots108 + 3 = 4k+3$)

We just need to follow the hint :)

$$\begin{aligned}
\underbrace{111\cdots111}_{N \text{ number of 1's}} &= \sum_{i=0}^{N-1} 10^i \\
&= 11 + \sum_{i=2}^{N-1} 10^i \\
&= 11 + \sum_{i=1}^{N-2} 100^i \\
&= 4 \cdot 2 + 3 + \sum_{i=1}^{N-2} (4 \cdot 25)^i \\
&= 4 \left(2 + \sum_{i=1}^{N-2} 4^{i-1} 25^i \right) + 3 \\
&= 4k + 3
\end{aligned}$$

in the above derivation $N > 2$ for $N = 2 \rightarrow 11 = 4 \cdot 2 + 3$. From Problem 1.13 (a) we know that no perfect square is of the form $4k + 3$ and this completes the proof.

Problem 1.14 Prove that a positive integer n is prime if and only if n is not divisible by any prime p with $1 < p \leq \sqrt{n}$.

The first part is easy, if n is prime then its only factors are 1 and n thus it is not divisible by p with $1 < p \leq \sqrt{n}$.

The other way "if n is not divisible by any prime p with $1 < p \leq \sqrt{n}$ then n is prime" is a bit harder.

The proof proceeds like the field sieve algorithm. Say we list all N primes $p_i \leq \sqrt{n}$ that do not divide n sorted by $p_1 < p_2 < \cdots < p_N$. We know from the fundamental theorem that n contains prime factors that are smaller than n.

Since $p_1, p_2, \ldots, p_N$ do not divide n the next prime factor candidate will be $p_{N+1} > p_n > \sqrt{n}$. But for $n = p_{N+1} m$, m has to be $< \sqrt{n}$ and contain one of the $p_1, p_2, \ldots, p_N$, otherwise the only other option is $p_{N+1} p_{N+1} > \sqrt{n}\sqrt{n} > n$ but none of $p_1, p_2, \ldots, p_N$ divides n thus no prime divides n and n itself must be prime.

Chapter 1 Conclusion

- On the way through proving the fundamental theorem we see how important $\gcd(na, nb) = n \cdot \gcd(a, b)$ is.

- We don't need to use the fundamental theorem as I have shown in solving the various problems above. This reiterates the idea that math is an interconnected web of theorems as mentioned by Feynman, if we don't have a theorem we can use other theorems as bridges to the one we need.
- Keen observations are a must, we need all we can get on those numbers.

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Example 2.1.5.

$$\mathbb{Z}/3\mathbb{Z} = \{\{\underbrace{\ldots,-3,0,3,\ldots}_{\text{“0”}}\},\{\underbrace{\ldots,-2,1,4,\ldots}_{\text{“1”}}\},\{\underbrace{\ldots,-1,2,5,\ldots}_{\text{“2”}}\}\}$$

Explanation: the set $\mathbb{Z}/3\mathbb{Z}$ only has *three* elements but each element is a set. Please do not confuse each element with an ideal, we know that the ideal

$$3\mathbb{Z} = \{\ldots,-12,-9,-6,-3,0,3,6,9,12,\ldots\}$$

because if a is an element of an ideal then $-a$ is also an element. Therefore $\{\ldots,-2,1,2\ldots\}$ and $\{\ldots,-1,2,5\ldots\}$ are **not** ideals.

What happens to $\mathbb{Z}/3\mathbb{Z}$ is it is a quotient of the ring $\mathbb{Z}$ by some "ideal" $n\mathbb{Z}$ (notice the quatation marks) because as we have seen that two of the sets are not ideals. What we have is shifted ideals

$$\begin{aligned}
\mathbb{Z}/3\mathbb{Z} &= \{0+3\mathbb{Z}, 1+3\mathbb{Z}, 2+3\mathbb{Z}\} \\
0+3\mathbb{Z} &= \{\ldots,\underbrace{\mathbf{0}+-3}_{\text{-3}},\underbrace{\mathbf{0}+0}_{0},\underbrace{\mathbf{0}+3}_{3},\ldots\} \\
1+3\mathbb{Z} &= \{\ldots,\underbrace{\mathbf{1}+-3}_{\text{-2}},\underbrace{\mathbf{1}+0}_{1},\underbrace{\mathbf{1}+3}_{4},\ldots\} \\
2+3\mathbb{Z} &= \{\ldots,\underbrace{\mathbf{2}+-3}_{\text{-1}},\underbrace{\mathbf{2}+0}_{2},\underbrace{\mathbf{2}+3}_{5},\ldots\}
\end{aligned}$$

Or in a physicist friendly way (in talking about quotient of a group), we identify each element of $\mathbb{Z}$ using $n\mathbb{Z}$, thus $0 \to 0+n\mathbb{Z}$, etc. How about n itself? n has been covered by $0+n\mathbb{Z}$. So in this way $n\mathbb{Z}$ acts as a gauge transformation and each $k+n\mathbb{Z}$ is a gauge orbit.

Proposition 2.1.9 I proved this before in this document, "*A number $n \in \mathbb{Z}$ is divisible by 3 if and only if the sum of the digits of n is divisible by 3.*"

His proof boils down to

$$n = a + 10b + 100c + \cdots \equiv a + b + c + \cdots (\text{mod } 3)$$

Now why does this proof work? It says that $n = (a + b + c)(\text{mod } 3)$. The meaning of (mod 3) is that $a = b(\text{mod } 3) \rightarrow 3|a - b$ so in this case $3|n - (a + b + c)$ but since $3|n$ therefore $3|(a + b + c)$.

Definition 2.1.11 "(Complete Set of Residues). ... pairwise distinct". The name is apt since the set is the set of residues of n, *e.g.* $1, 2, 3, \ldots, n - 1$. The "pairwise distinct" phrase, I think, means that every two elements of R are not related by n, *i.e.* every two elements x_1 and x_2 of R, $n \nmid (x_1 - x_2)$.

Lemma 2.1.12,"Gangnam Style". The way I'll prove it is to show it through a contradiction. Assume that aR is not a complete set of residues modulo n, which means that there are two elements of aR, say $ax_1 - ax_2$ such that

$$n|a(x_1 - x_2)$$

since $\gcd(n, a) = 1$ from Problem 1.8 (b) we know that $n|(x_1 - x_2)$ but this is a contradiction since the elements of R are pairwise distinct. Thus aR is a complete set of residues modulo n.

Problem 2.12 Let p be prime. Prove that $\mathbb{Z}/p\mathbb{Z}$ is a field.

What is the reasoning behind this? Remember that a field is a ring where each non-zero element has an "inverse" under multiplication, *i.e.* for each element $a \neq 0$ we can find another element b such that $ab = 1$.

If q is not prime we can show that $\mathbb{Z}/q\mathbb{Z}$ is **not** a field. Here is why. For simplicity let's denote $\mathbb{Z}/q\mathbb{Z}$

$$\mathbb{Z}/q\mathbb{Z} = \{0, 1, 2, 3, \ldots, q - 1\}$$

where each number is actually $k + q\mathbb{Z}$. So, if q is not prime, at least one of the numbers in $\mathbb{Z}/q\mathbb{Z}$ less than q has to be a factor of q, let's denote this number w, $w < q$. Since w is a factor of $q \rightarrow q = wm$.

The identity element, 1, in the ring $\mathbb{Z}/q\mathbb{Z}$ is actually $qn + 1 = w(mn) + 1$. Now we want to find another element u such that $uw = 1 = qn + 1 = w(mn) + 1$, but this is a contradiction since $w(u - mn) = 1 \rightarrow w|1$. Thus $\mathbb{Z}/q\mathbb{Z}$ for non prime q is **not** a field.

Back to our original problem, we need to show the opposite, that for a prime p, $\mathbb{Z}/p\mathbb{Z}$ is always a field. As shown above, since none of the elements of $\mathbb{Z}/p\mathbb{Z}$ is a factor of p, this should be possible, the hard part is showing **how** this can definitely be done (answer: induction).

The trick here is to utilize the circular structure of $\mathbb{Z}/p\mathbb{Z}$ in the induction step. The base case is obviously $a \in \mathbb{Z}/p\mathbb{Z},\ a = 1$ with the inverse being itself $1 \cdot 1 = 1$. Now suppose that we have found the inverse for each non-zero element of $\mathbb{Z}/p\mathbb{Z}$ up to n, how about $v = n + 1$?

Notice that each non-zero element of $\mathbb{Z}/p\mathbb{Z}$ is less than p. This means that we can write

$$p = vq + r$$

What we want is to just exceed p, if we add one more $v \rightarrow v(q + 1)$ and substracting p from it we get $v(q+1) - p = v - r$. We know that $r < v$ therefore $v - r < v$ but the case of $< v$, *i.e.* all elements up to n, already has an inverse and we can use that inverse to get the inverse of v. For example $v - r = d$ and the inverse of d is y such that $dy = ps + 1 = 1$. This means that since $p = vq + r$ and $v - r = d$

$$\begin{aligned} v(q + 1) &= p + d \\ v(q + 1)y &= py + dy = py + (ps + 1) \\ v(q + 1)y &= p(y + s) + 1 \end{aligned}$$

i.e. $v(q + 1)y = 1$ and the inverse of v is $(q + 1)y$. The primality of p was needed to guarantee that $r \neq 0,\ v \nmid p$ otherwise the proof doesn't work.

$(q + 1)y$ is generally not the "smallest" inverse of v but we can of course bring it back to be $< p$ by substracting it by $pt \rightarrow v((q + 1)y - pt) = p(y + s - tv) + 1 = 1$ for some t such that $0 < (q + 1)y - pt < p$.

Now, we are dealing with $k + n\mathbb{Z}$. If the element we are inspecting is $> p$ or < 0 we

can still repeat the same argument but this time instead of just exceeding p we want to exceed $|np|$ with $|n|$ the smallest number such that $|np| > |v|$.

Notice that in the above argument, we can identify $v(q+1) = d$ because of the circular structure of $\mathbb{Z}/p\mathbb{Z}$ which also allows us to utilize previously discovered inverses. Let's see this with a concrete example using $\mathbb{Z}/7\mathbb{Z}$ and pick from it the element $v = 3$ (lucky seven meet holy trinity).

$$7 = 3 \times 2 + 1$$

Adding one more 3 and substracting 7

$$\begin{aligned}((3 \times 2) + 3) - 7 &= 3 - 1 \\ &= 2\end{aligned}$$

But in the induction process leading up to 3 we have found the inverse for 2 which is 4, $2 \cdot 4 = 8 = 1$. Thus the inverse of 3 is $(2+1) \cdot 4 = 12 \rightarrow 3 \times 12 = 36 = 7 \cdot 5 + 1$.

Some sidenote, we can of course use the above method to get the inverse of 2 from the inverse of 1 but it can actually be computed a lot easier. In the case of 2 the inverse is y such that $2y = pn + 1$. We want $pn + 1$ to be even and unless p is 2, p is always odd. Therefore any odd n will do ($n = 1$ is a very good choice) and y is just $(p+1)/2$ while the case of $p = 2$ is trivial since the only members of $\mathbb{Z}/2\mathbb{Z}$ are $\{0, 1\}$.

In conclusion we have done a 2-in-1 combo deal. We have proven that if p is not prime then $\mathbb{Z}/p\mathbb{Z}$ is a not field, in other words if $\mathbb{Z}/p\mathbb{Z}$ is a field then p is prime and we have proven that if p is prime then $\mathbb{Z}/p\mathbb{Z}$ is a field. Therefore we have proven a much stronger statement, *p is prime if and only if $\mathbb{Z}/p\mathbb{Z}$ is a field.*