Apostol's Analytic Number Theory

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(Dated: December 26, 2016)

Abstract

Just for fun :)

Chapter 2

Problem 2.1 Find all integers n such that

(a) $\varphi(n) = n/2$ (b) $\varphi(n) = \varphi(2n)$ (c) $\varphi(n) = 12$

For (a), using the definition of $\varphi(n)$, $\varphi(n) = n\prod_{p|n}\left(1-\frac{1}{p}\right)$

$$\begin{aligned}
\frac{\cancel{n}}{2} &= \cancel{n}\prod_{p|n}\left(1-\frac{1}{p}\right) \\
\frac{1}{2} &= \frac{\prod_{p|n}(p-1)}{\prod_{p|n}p} \\
\prod_{p|n}p &= 2\prod_{p|n}(p-1)
\end{aligned}$$

if n is odd the LHS is odd while the RHS is even, so it can't be. If n is even the LHS only has one factor of 2 while the RHS has many so it will only work if $n = 2$.

For (b)

$$\cancel{n}\prod_{p|n}\left(1-\frac{1}{p}\right) = 2\cancel{n}\prod_{p|2n}\left(1-\frac{1}{p}\right)$$

If n is even then

$$\prod_{p|2n}\left(1-\frac{1}{p}\right) = \prod_{p|n}\left(1-\frac{1}{p}\right)$$

and so

$$\begin{aligned}
\prod_{p|n}\left(1-\frac{1}{p}\right) &= 2\prod_{p|n}\left(1-\frac{1}{p}\right) \\
&\to 1 = 2
\end{aligned}$$

which is impossible, so n has to be odd, in that case

$$\begin{aligned}
\prod_{p|2n}\left(1-\frac{1}{p}\right) &= \left(1-\frac{1}{2}\right)\prod_{p|n}\left(1-\frac{1}{p}\right) \\
&= \frac{1}{2}\prod_{p|n}\left(1-\frac{1}{p}\right)
\end{aligned}$$

and therefore

$$\begin{aligned}\prod_{p|n}\left(1-\frac{1}{p}\right) &= 2\frac{1}{2}\prod_{p|n}\left(1-\frac{1}{p}\right)\\ &\to 1 = 1\end{aligned}$$

and therefore $\varphi(n) = \varphi(2n)$ for all odd n.

For (c)

$$\begin{aligned}\varphi(n) = 12 &= 2\cdot 2\cdot 3\\ &= \prod_{p|n} p^{\alpha_p} - p^{\alpha_p - 1}\\ \varphi\left(\prod_{p|n} p^{\alpha_p}\right) &= \prod_{p|n} p^{\alpha_p-1}(p-1)\end{aligned}$$

the only possible solution is $n = 13$

Problem 2.2. For each of the following statements either give a proof or exhibit a counter example.

(a) If $(m, n) = 1$ then $(\varphi(m), \varphi(n)) = 1$

(b) If n is composite, then $(n, \varphi(n)) > 1$

(c) If the same primes divide m and n, then $n\varphi(m) = m\varphi(n)$

For (a) a counter example will be $(3, 4) = 1$, while $\varphi(3) = 2,\ \varphi(4) = 2$

For (b) a counter example would be $n = 15$ which means that $\varphi(15) = 8$ and $(15, 8) = 1$

For (c) I think what it means by "the same primes divide m and n" is that $m = \prod p^{\alpha_p}$ and $n = \prod p^{\beta_p}$, so they both have the same primes but they might have different exponents for each prime, in this case $\prod_{p|n} = \prod_{p|m}$

$$\begin{aligned}n\varphi(m) &= n\left(m\prod_{p|m}\left(1-\frac{1}{p}\right)\right)\\ &= m\left(n\prod_{p|n}\left(1-\frac{1}{p}\right)\right)\\ n\varphi(m) &= m\varphi(n)\end{aligned}$$

Problem 2.3. Prove that

$$\frac{n}{\varphi(n)} = \sum_{d|n}\frac{\mu^2(d)}{\varphi(d)}$$

Since $\mu(n)$ and $\varphi(n)$ are both multiplicative so is μ^2/φ, in that case $g(n) = \sum_{d|n} \frac{\mu^2(d)}{\varphi(d)}$ is also multiplicative. To determine $g(n)$ we need only compute $g(p^\alpha)$ for prime powers

$$\begin{aligned}
g(p^\alpha) &= \sum_{d|p^\alpha} \frac{\mu^2(d)}{\varphi(d)} \\
&= \frac{\mu^2(1)}{\varphi(1)} + \frac{\mu^2(p)}{\varphi(p)} + \ldots + \frac{\mu^2(p^\alpha)}{\varphi(p^\alpha)} \\
&= 1 + \frac{1}{p-1} \\
&= \frac{p}{p-1} \\
&= p^\alpha \cdot \frac{p}{p^\alpha(p-1)} \\
\to \sum_{d|p^\alpha} \frac{\mu^2(d)}{\varphi(d)} &= \frac{p^\alpha}{\varphi(p^\alpha)}
\end{aligned}$$

We can also prove it the other way around by assuming the LHS, to do this it is easiest to use the Mobius inversion formula

$$\frac{n}{\varphi(n)} = \sum_{d|n} g(d)$$

and we want to find out what this $g(d)$ is, which is

$$g(n) = \sum_{d|n} \frac{d}{\varphi(d)} \mu\left(\frac{n}{d}\right)$$

The RHS is multiplicative so like above we just need to evaluate $g(p^\alpha)$ for prime powers

$$\begin{aligned}
g(p^\alpha) &= \sum_{d|p^\alpha} \frac{d}{\varphi(d)} \mu\left(\frac{p^\alpha}{d}\right) \\
&= \frac{p^{\alpha-1}}{\varphi(p^{\alpha-1})} \mu\left(\frac{p^\alpha}{p^{\alpha-1}}\right) + \frac{p^\alpha}{\varphi(p^\alpha)} \mu\left(\frac{p^\alpha}{p^\alpha}\right) \\
&= -\frac{p^{\alpha-1}}{\varphi(p^{\alpha-1})} + \frac{p^\alpha}{\varphi(p^\alpha)} \\
&= -\frac{p^\alpha}{\varphi(p^\alpha)} + \frac{p^\alpha}{\varphi(p^\alpha)} \\
&= 0
\end{aligned}$$

if $\alpha > 1$ and if $\alpha = 1$ we get

$$\begin{aligned}
g(p) &= \sum_{d|p} \frac{d}{\varphi(d)} \mu\left(\frac{p}{d}\right) \\
&= \frac{1}{\varphi(1)} \mu\left(\frac{p}{1}\right) + \frac{p}{\varphi(p)} \mu\left(\frac{p}{p}\right) \\
&= -1 + \frac{p}{\varphi(p)} \\
&= -1 + \frac{p}{p-1} \\
&= \frac{1}{p-1} \\
g(p) &= \frac{1}{\varphi(p)}
\end{aligned}$$

This means that $g(p^\alpha) = 1/\varphi(p^\alpha)$ is $\alpha = 1$ and $g(p^\alpha) = 0$ if $\alpha > 1$, in other words $g(p^\alpha) = \mu^2(p^\alpha)/\varphi(p^\alpha)$