Some Derivation

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Abstract

Some ideas

I. DERIVING QED E.O.M FROM ITS HAMILTONIAN

The lagrangian (density) is given by

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu + i\overline{\psi}\gamma^\mu \partial_\mu \psi - \overline{\psi}\left(e\gamma^\mu A_\mu + m\right)\psi \\ &= -\frac{1}{2}\partial_\mu A_\nu F^{\mu\nu} + i\overline{\psi}\gamma^\mu \partial_\mu \psi - \overline{\psi}\left(e\gamma^\mu A_\mu + m\right)\psi\end{aligned}$$

We can simplyfy the first term by writing $\partial_\mu A_\nu$ in its symmetric and antisymmetric components

$$\begin{aligned}\partial_\mu A_\nu &= \left\{\frac{1}{2}(\partial_\mu A_\nu + \partial_\nu A_\mu) + \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\right\} \\ &= \partial_{\{\mu} A_{\nu\}} + \partial_{[\mu} A_{\nu]}\end{aligned}$$

but the symmetric component will vanish when contracted with $(-\partial^\mu A^\nu + \partial^\nu A^\mu)$ which is antisymmetric, $-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu$ then becomes

$$\begin{aligned}&= \frac{1}{2}\frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)(-\partial^\mu A^\nu + \partial^\nu A^\mu) \\ &= \frac{1}{4}F_{\mu\nu}(-F^{\mu\nu}) \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\end{aligned}$$

and the lagrangian becomes the usual one

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}\gamma^\mu \partial_\mu \psi - \overline{\psi}\left(e\gamma^\mu A_\mu + m\right)\psi$$

The (traditional) conjugate momenta are

$$\begin{aligned}p_A^{0\sigma} &= \frac{\delta\mathcal{L}}{\delta\partial_0\partial_\sigma} = -\frac{1}{2}\delta^0_\mu\delta^\sigma_\nu\partial^\mu A^\nu - \frac{1}{2}\partial_\mu A_\nu g^{0\mu}g^{\sigma\nu} + \frac{1}{2}\delta^0_\mu\delta^\sigma_\nu\partial^\nu A^\mu + \frac{1}{2}\partial_\mu A_\nu g^{0\nu}g^{\sigma\mu} \\ &= -\frac{1}{2}\partial^0 A^\sigma - \frac{1}{2}\partial^0 A^\sigma + \frac{1}{2}\partial^\sigma A^0 + \frac{1}{2}\partial^\sigma A^0 \\ &= -\partial^0 A^\sigma + \partial^\sigma A^0 \\ \rightarrow p_A^{0\sigma} &= -F^{0\sigma}\end{aligned}$$

$$\begin{aligned}\rightarrow p_\psi^0 &= \frac{\delta\mathcal{L}}{\delta\partial_0\psi} = i\overline{\psi}\gamma^0 \\ \rightarrow p_{\overline{\psi}}^0 &= \frac{\delta\mathcal{L}}{\delta\partial_0\overline{\psi}} = 0\end{aligned}$$

The hamiltonian (density) is then

$$\begin{aligned}\mathcal{H} &= p_A^{0\sigma}\partial_0 A_\sigma + p_\psi^0 \partial_0\psi - \mathcal{L} \\ &= -F^{0\sigma}\partial_0 A_\sigma + i\overline{\psi}\gamma^0\partial_0\psi + \frac{1}{2}\partial_\mu A_\nu F^{\mu\nu} - i\overline{\psi}\gamma^\mu\partial_\mu\psi + \overline{\psi}\left(e\gamma^\mu A_\mu + m\right)\psi\end{aligned}$$

Writing $\partial_0 A_\sigma$ in its symmetric and antisymmetric components

$$\partial_0 A_\sigma = \left\{\frac{1}{2}(\partial_0 A_\sigma + \partial_\sigma A_0) + \frac{1}{2}(\partial_0 A_\sigma - \partial_\sigma A_0)\right\}$$

and expecting to get rid of the symmetric part by contracting with $F^{0\sigma}$ might not work because

$$\begin{aligned}-F^{0\sigma}\partial_{\{0}A_{\sigma\}} &= F^{\sigma 0}\partial_{\{0}A_{\sigma\}} \\ &= F^{\sigma 0}\partial_{\{\sigma}A_{0\}}\end{aligned}$$

but now we can't swap $0 \leftrightarrow \sigma$ like what we do for the usual dummy indices, *i.e.*

$$\begin{aligned}-F^{\rho\sigma}\partial_{\{\rho}A_{\sigma\}} &= F^{\sigma\rho}\partial_{\{\rho}A_{\sigma\}} \\ &= F^{\sigma\rho}\partial_{\{\sigma}A_{\rho\}}, \quad \rho \leftrightarrow \sigma \\ -F^{\rho\sigma}\partial_{\{\rho}A_{\sigma\}} &= F^{\rho\sigma}\partial_{\{\rho}A_{\sigma\}}\end{aligned}$$

Going back to the hamiltonian and grouping similar terms

$$\mathcal{H} = \left(-\partial_0 A_\nu F^{0\nu} + \frac{1}{2}\partial_\mu A_\nu F^{\mu\nu}\right) + \left(i\overline{\psi}\gamma^0\partial_0\psi - i\overline{\psi}\gamma^\mu\partial_\mu\psi\right) + \overline{\psi}\left(e\gamma^\mu A_\mu + m\right)\psi$$

focusing on the first bracket for now

$$\begin{aligned}-\partial_0 A_\nu F^{0\nu} + \frac{1}{2}\partial_\mu A_\nu F^{\mu\nu} &= -\partial_0 A_\nu F^{0\nu} + \frac{1}{2}\partial_0 A_\nu F^{0\nu} + \frac{1}{2}\partial_i A_\nu F^{i\nu} \\ &= -\frac{1}{2}\partial_0 A_\nu F^{0\nu} + \frac{1}{2}\partial_i A_\nu F^{i\nu}, \quad F^{00} = 0 \\ &= -\frac{1}{2}\partial_0 A_i F^{0i} + \frac{1}{2}\left(\partial_i A_0 F^{i0} + \partial_i A_j F^{ij}\right) \\ &= \frac{1}{2}\left(-\partial_0 A_i F^{0i} + \partial_i A_0 F^{i0}\right) + \frac{1}{4}F_{ij}F^{ij} \\ &= \frac{1}{2}\left(\partial_0 A_i F^{i0} + \partial_i A_0 F^{i0}\right) + \frac{1}{4}F_{ij}F^{ij} \\ &= \frac{1}{2}F^{i0}\left(\partial_0 A_i - \partial_i A_0 + 2\partial_i A_0\right) + \frac{1}{4}F_{ij}F^{ij} \\ &= \frac{1}{2}F^{i0}F_{0i} + F^{i0}\partial_i A_0 + \frac{1}{4}F_{ij}F^{ij}\end{aligned}$$

We can massage this to a more familiar form, to do this we must make sure we use the *correct metric*, otherwise we'll get all sorts of minus signs, the metric we have to use here is $(+---)$, *i.e.* $A^\mu = (\phi, \vec{A})$, $A_\mu = (\phi, -\vec{A})$

$$
\begin{aligned}
\vec{E} &= -\nabla\phi - \partial_0 \vec{A} \\
E^i &= \partial^i \phi - \partial_0 A^i, \quad \nabla = \partial_i = -\partial^i, \partial_0 = \partial^0, \\
&= \partial^i A^0 - \partial^0 A^i \\
E^i &= F^{i0}
\end{aligned}
$$

$$
\begin{aligned}
F_{0i} &= \partial_0 A_i - \partial_i A_0 \\
&= -\partial_0 A^i - \partial_i A^0 = E^i = -E_i \\
&= -\partial_0 (\vec{A})^i - (\nabla)_i \phi
\end{aligned}
$$

while for the magnetic field

$$
\begin{aligned}
\vec{B} &= \nabla \times \vec{A} \\
B^i &= \varepsilon^{ijk} \partial_j (-A_k) \\
\varepsilon_{ilm} B^i &= -\varepsilon_{ilm} \varepsilon^{ijk} \partial_j A_k \\
&= -(\delta_l^j \delta_m^k - \delta_l^k \delta_m^j) \partial_j A_k \\
&= -(\partial_l A_m - \partial_m A_l) \\
F_{lm} &= -\varepsilon_{ilm} B^i
\end{aligned}
$$

$$
F^{lm} = \varepsilon^{ilm} B_i
$$

The minus sign on $(-A_k)$ is due to the fact that the derivative was initially on the vector $\vec{A} \to A^k$ but since we're using the covariant version here we need to include a minus sign from the metric $(+---)$. Also there's no minus sign on $F^{lm} = \varepsilon^{ilm} B_i$ because $B_i = -B^i$ due to the metric $(+---)$, to check if this is correct let's do F_{12} which we know to be $F_{12} = F^{12} = -B^3$, $F_{12} = -\varepsilon_{312} B^3 = -B^3$ and $F^{12} = \varepsilon^{312} B_3 = -B^3$ which are correct.

Thus

$$
\begin{aligned}
F^{i0}F_{0i} &= E^i(-E_i) = -g_{ij}E^iE^j, \quad g_{ij} = (---) \\
&= E^iE^i \\
\rightarrow \frac{1}{2}F^{i0}F_{0i} &= \frac{1}{2}\vec{E}\cdot\vec{E}
\end{aligned}
$$

$$
\begin{aligned}
F_{ij}F^{ij} &= -\varepsilon_{ijk}B^k\varepsilon^{ijl}B_l = \varepsilon_{ijk}\varepsilon^{ijl}B^kB_l \\
&= -2\delta^l_kB^kB_l = -2B^kB_k = 2B^kB^k \\
\rightarrow \frac{1}{4}F_{ij}F^{ij} &= \frac{1}{2}\vec{B}\cdot\vec{B}
\end{aligned}
$$

$$
\begin{aligned}
F^{i0}\partial_iA_0 &= -\partial_iF^{i0}A_0 = \partial_iE^iA_0 \\
\rightarrow F^{i0}\partial_iA_0 &= -A_0(\nabla\cdot\vec{E})
\end{aligned}
$$

And the photon part of the hamiltonian is

$$
\mathcal{H}_{\text{ph}} = \frac{1}{2}\vec{E}\cdot\vec{E} + \frac{1}{2}\vec{B}\cdot\vec{B} - A_0(\nabla\cdot\vec{E})
$$

Going back to our original hamiltonian

$$
\mathcal{H} = \frac{1}{2}F^{i0}F_{0i} + F^{i0}\partial_iA_0 + \frac{1}{4}F_{ij}F^{ij} - i\overline{\psi}\gamma^i\partial_i\psi + \overline{\psi}\left(e\gamma^\mu A_\mu + m\right)\psi
$$

Let's rewrite the first term

$$
\begin{aligned}
F_{0i} &= g_{0\mu}g_{i\nu}F^{\mu\nu} = g_{00}g_{ij}F^{0j}, g_{00} = 1, g_{ij} = -\delta_{ij} \\
&= (-F^{0i}) = F^{i0} \\
\rightarrow F_{0i} &= p_A^{0i} \\
\rightarrow F^{i0}F_{0i} &= p_A^{0i}p_A^{0i}
\end{aligned}
$$

We now want to derive the equations of motion from this hamiltonian. Let's start with the photons, the first is

$$
\begin{aligned}
\frac{\delta\mathcal{H}}{\delta p_A^{0i}} &= \partial_0A_i \\
\frac{\delta\mathcal{H}}{\delta p_A^{0i}} &= \frac{1}{2}p_A^{0i} + \frac{1}{2}p_A^{0i} + \partial_iA_0 \\
&= F_{0i} + \partial_iA_0 = \partial_0A_i - \partial_iA_0 + \partial_iA_0 \\
\partial_0A_i &= \partial_0A_i
\end{aligned}
$$

Thus this equation only gives us a trivial identity. The next e.o.m is

$$\frac{\delta\mathcal{H}}{\delta A_\rho} = -\partial_0 p_A^{0\rho}$$

Note that $p_A^{00} = F^{00} = 0 \rightarrow \delta\mathcal{H}/\delta A_0 = 0$, so let's do that first since that seems harmless enough, to do this we need to do integration by parts on $F^{i0}\partial_i A_0 \rightarrow -\partial_i F^{i0} A_0$, we can do this because the hamiltonian is the integral of the density, $H = \int d^3x\mathcal{H}$, note that integration by parts can only be done on spatial derivatives, H is not integrated in time!

$$\begin{aligned}
\frac{\delta\mathcal{H}}{\delta A_0} &= -\partial_i F^{i0} + e\overline{\psi}\gamma^0\psi \\
0 &= -\partial_i F^{i0} + e\overline{\psi}\gamma^0\psi \\
\partial_i F^{i0} &= e\overline{\psi}\gamma^0\psi \\
\rightarrow \nabla\cdot\vec{E} &= \rho
\end{aligned}$$

Where $J^\mu = e\overline{\psi}\gamma^\mu\psi = (\rho, \vec{J})$, so we obtain the first of the Maxwell's equations. Next is $\delta\mathcal{H}/\delta A_k$, we will do it slowly :) First, the terms we need are $\frac{1}{4}F_{ij}F^{ij} + e\overline{\psi}\gamma^\mu A_\mu\psi$, we do not include $\frac{1}{2}F^{i0}F_{0i} + F^{i0}\partial_i A_0$ because they are actually terms of p_A^{0i} and in the hamiltonian formalism the conjugate momenta are independent of the position variables A_μ.

The first of those terms we want to tackle is

$$\begin{aligned}
F_{ij}F^{ij} &= (\partial_i A_j - \partial_j A_i)F^{ij} \\
F_{ij}F^{ij} &= -A_j\partial_i F^{ij} + A_i\partial_j F^{ij}
\end{aligned}$$

$$\begin{aligned}
F_{ij}F^{ij} &= F_{ij}(\partial^i A^j - \partial^j A^i) \\
F_{ij}F^{ij} &= -\partial^i F_{ij}A^j + \partial^j F_{ij}A^i
\end{aligned}$$

And we have done plenty of integration by parts (for spatial derivatives only), again since the hamiltonian density is integrated $H = \int d^3x\mathcal{H}$, thus

$$\begin{aligned}
\frac{\delta\left(\frac{1}{4}F_{ij}F^{ij}\right)}{\delta A_k} &= \frac{1}{4}\left(-\partial_i F^{ik} + \partial_j F^{kj} - \partial^i F_i^{\ k} + \partial^j F^k_{\ j}\right) \\
&= \frac{1}{4}\left(-\partial_i F^{ik} + \partial_j F^{kj} - \partial_i F^{ik} + \partial_j F^{kj}\right) \\
&= \frac{1}{4}\left(-2\partial_i F^{ik} + 2\partial_j F^{kj}\right) = -\partial_i F^{ik}
\end{aligned}$$

while the fermionic part gives

$$\frac{\delta(e\overline{\psi}\gamma^\mu A_\mu\psi)}{\delta A_k} = e\overline{\psi}\gamma^k\psi$$

Combining both we get

$$\begin{aligned}\frac{\delta\mathcal{H}}{\delta A_k} &= -\partial_i F^{ik} + e\overline{\psi}\gamma^k\psi \\ -\partial_0 p_A^{0k} &= -\partial_i F^{ik} + e\overline{\psi}\gamma^k\psi\end{aligned}$$

We can write it in a more familiar form by remembering that $p_A^{0k} = E^k$, $J^k = e\overline{\psi}\gamma^k\psi$ and

$$\begin{aligned}-\partial_i F^{ik} &= -\partial_i\varepsilon^{ikm}B_m = -\varepsilon^{ikm}\partial_i B_m \\ &= \varepsilon^{kim}\partial_i B_m = -(\nabla\times\vec{B})^k\end{aligned}$$

the extra minus sign is again due to the fact that the vector $\vec{B} \to B^k = -B_k$ thanks to the choice of metric $(+---)$, the e.o.m can then be written as

$$\begin{aligned}-\partial_0 E^k &= -(\nabla\times\vec{B})^k + J^k \\ \to \nabla\times\vec{B} &= \partial_0\vec{E} + \vec{J}\end{aligned}$$

which is the other familiar Maxwell's equation. For a more modern representation we can add both e.o.m's we get earlier, *i.e.* $\partial_i F^{i0} = e\overline{\psi}\gamma^0\psi$ and $\partial_0 F^{0k} + \partial_i F^{ik} = e\overline{\psi}\gamma^k\psi$

$$\begin{aligned}\partial_i F^{i0} + \partial_0 F^{0k} + \partial_i F^{ik} &= e\overline{\psi}\gamma^0\psi + e\overline{\psi}\gamma^k\psi \\ (\partial_0 F^{00} + \partial_0 F^{0k}) + (\partial_i F^{i0} + \partial_i F^{ik}) &= e\overline{\psi}\gamma^\mu\psi, \quad F^{00} = 0 \\ \partial_0 F^{0\mu} + \partial_i F^{i\mu} &= J^\mu \\ \to \partial_\nu F^{\nu\mu} &= J^\mu\end{aligned}$$

We now do the variation of the fermionic parts of the hamiltonian given by

$$\mathcal{H}_{\text{fm}} = -i\overline{\psi}\gamma^i\partial_i\psi + \overline{\psi}\left(e\gamma^\mu A_\mu + m\right)\psi$$

notice that it does *not* contain the time derivative of the field, either $\dot{\psi}$ or $\dot{\overline{\psi}}$. What this means is that we have a constrained system, another anomaly is that $p_{\overline{\psi}}^0 = 0$, yet another constraint, we have to treat this hamiltonian according to the Dirac-Bergmann algorithm.

If we insist of using the traditional way we will immediately face an ambiguity as what should we substitute for p_ψ. We can choose to substitute all $\overline{\psi}$ for p^0_ψ but then we will still have the problem of

$$\begin{aligned}\frac{\delta\mathcal{H}}{\delta p^0_{\overline{\psi}}} &= \partial_0\overline{\psi} \\ \to 0 &= \partial_0\overline{\psi}\end{aligned}$$

which is wrong. Ignoring this issue for now, the fastest way to get the fermionic e.o.m's is actually to substitute all $\overline{\psi}$ for p^0_ψ, $p^0_\psi = i\overline{\psi}\gamma^0 \to \overline{\psi} = -ip^0_\psi\gamma^0$

$$\begin{aligned}\mathcal{H}_{\text{fm}} &= -p^0_\psi\gamma^0\gamma^i\partial_i\psi - iep^0_\psi A_0\psi - eip^0_\psi\gamma^0\gamma^i A_i\psi - imp^0_\psi\gamma^0\psi \\ &= p^0_\psi\gamma^i\gamma^0\partial_i\psi - iep^0_\psi A_0\psi + eip^0_\psi\gamma^i\gamma^0 A_i\psi - imp^0_\psi\gamma^0\psi\end{aligned}$$

where $\gamma^0\gamma^i = -\gamma^i\gamma^0$ from the anticommutation relation of the gamma matrices. Moving on with this hamiltonian, we can recover all of Dirac's e.o.m's

$$\begin{aligned}\frac{\delta\mathcal{H}}{\delta p^0_\psi} &= \partial_0\psi \\ \frac{\delta\mathcal{H}}{\delta p^0_\psi} &= -\gamma^0\gamma^i\partial_i\psi - ieA_0\psi - ei\gamma^0\gamma^i A_i\psi - im\gamma^0\psi \\ \partial_0\psi &= -\gamma^0\gamma^i\partial_i\psi - ieA_0\psi - ei\gamma^0\gamma^i A_i\psi - im\gamma^0\psi \\ i\gamma^0\partial_0\psi &= -i\gamma^i\partial_i\psi + e\gamma^0 A_0\psi + e\gamma^i A_i\psi + m\psi \\ i\gamma^\mu\partial_\mu\psi &= e\gamma^\mu A_\mu\psi + m\psi\end{aligned}$$

The next e.o.m is easier to derive if we use the second line (with γ^i on the left of γ^0) of the hamiltonian above because we want to multiply by γ^0 from the right.

$$\begin{aligned}\frac{\delta\mathcal{H}}{\delta\psi} &= -\partial_0 p^0_\psi \\ \frac{\delta\mathcal{H}}{\delta\psi} &= -\partial_i p^0_\psi\gamma^i\gamma^0 - iep^0_\psi A_0 + eip^0_\psi\gamma^i\gamma^0 A_i - imp^0_\psi\gamma^0 \\ -\partial_0 p^0_\psi\gamma^0 &= -\partial_i p^0_\psi\gamma^i - iep^0_\psi\gamma^0 A_0 + eip^0_\psi\gamma^i A_i - imp^0_\psi \\ -i\partial_0\overline{\psi}\gamma^0\gamma^0 &= -i\partial_i\overline{\psi}\gamma^0\gamma^i + e\overline{\psi}\gamma^0\gamma^0 A_0 - e\overline{\psi}\gamma^0\gamma^i A_i + m\overline{\psi}\gamma^0 \\ -i\partial_0\overline{\psi}\gamma^0\gamma^0 &= i\partial_i\overline{\psi}\gamma^i\gamma^0 + e\overline{\psi}\gamma^0\gamma^0 A_0 + e\overline{\psi}\gamma^i\gamma^0 A_i + m\overline{\psi}\gamma^0 \\ -i\partial_0\overline{\psi}\gamma^0 &= i\partial_i\overline{\psi}\gamma^i + e\overline{\psi}\gamma^0 A_0 + e\overline{\psi}\gamma^i A_i + m\overline{\psi} \\ -i\partial_\mu\overline{\psi}\gamma^\mu &= e\overline{\psi}\gamma^\mu A_\mu + m\overline{\psi}\end{aligned}$$

from third to fourth line we have substitued p^0_ψ for $i\overline{\psi}\gamma^0$ and from fourth to fifth line we have swap the order of $\gamma^0\gamma^i \to \gamma^i\gamma^0$ incurring a minus sign. Again, although the above derivation yields the correct e.o.m's they are fundamentally wrong! We need to employ Dirac-Bergmann algorithm to do it correctly. As this algorithm is quite an extensive subject, I will deal with it in a separate article.

In summary the equations of motion we get from the hamiltonian (density) are:

For the photons

$$\begin{aligned} \partial_i F^{i0} &= e\overline{\psi}\gamma^0\psi \\ \partial_0 F^{0k} + \partial_i F^{ik} &= e\overline{\psi}\gamma^k\psi \\ \to \partial_\nu F^{\nu\mu} &= e\overline{\psi}\gamma^\mu\psi \end{aligned}$$

and for the fermions

$$\begin{aligned} \to -i\partial_\mu\overline{\psi}\gamma^\mu &= e\overline{\psi}(\gamma^\mu A_\mu + m) \\ \to i\gamma^\mu\partial_\mu\psi &= (e\gamma^\mu A_\mu + m)\psi \end{aligned}$$

One might ask as to whatever happened to the other two Maxwell's equations, the sourceless ones? They are actually not equations of motion, they are just a consequence of Bianchi identity, which is

$$\begin{aligned} \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} &= \partial_\lambda\partial_\mu A_\nu - \partial_\lambda\partial_\nu A_\mu + \partial_\mu\partial_\nu A_\lambda - \partial_\mu\partial_\lambda A_\nu + \partial_\nu\partial_\lambda A_\mu - \partial_\nu\partial_\mu A_\lambda \\ &= (\partial_\lambda\partial_\mu A_\nu - \partial_\mu\partial_\lambda A_\nu) + (-\partial_\lambda\partial_\nu A_\mu + \partial_\nu\partial_\lambda A_\mu) + (\partial_\mu\partial_\nu A_\lambda - \partial_\nu\partial_\mu A_\lambda) \\ \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} &= 0 \end{aligned}$$

note that none of the indices are summed. Let's start by setting $_{\lambda=0,\mu=i,\nu=j}$ and using our usual dictionary $F_{0i} = E^i = -E_i,\ F_{lm} = -\varepsilon_{ilm}B^i,\ F^{lm} = \varepsilon^{ilm}B_i$

$$\begin{aligned} 0 &= \partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} \\ &= -\partial_0\varepsilon_{kij}B^k + \partial_i E_j - \partial_j E_i \\ &= -\varepsilon^{lij}\varepsilon_{kij}\partial_0 B^k + \varepsilon^{lij}(\partial_i E_j - \partial_j E_i) \\ &= -2\delta^l_k\partial_0 B^k + \varepsilon^{lij}\partial_i E_j - \varepsilon^{lij}\partial_j E_i \\ &= -2\partial_0 B^l + 2\varepsilon^{lij}\partial_i E_j \\ \to 0 &= -\partial_0 B^l + \varepsilon^{lij}\partial_i E_j \end{aligned}$$

note that if $i = j$ line 2 will become $0 = 0$ thus $i \neq j \neq k$, we can rewrite this result as

$$\begin{aligned} 0 &= -\partial_0 \vec{B} - \nabla \times \vec{E} \\ \to -\partial_0 \vec{B} &= \nabla \times \vec{E} \end{aligned}$$

the extra minus sign in front of $\nabla \times \vec{E}$ is because $\vec{E} \to E^i = -E_i$, thus we have recovered one of the sourceless Maxwell's equations.

Next we set ${}_{\lambda=i,\mu=j,\nu=k}$ with $i \neq j \neq k$

$$\begin{aligned} 0 &= \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} \\ &= -\partial_i \varepsilon_{ljk} B^l - \partial_j \varepsilon_{lki} B^l - \partial_k \varepsilon_{lij} B^l \\ &= -\varepsilon^{ijk} (\partial_i \varepsilon_{ljk} B^l + \partial_j \varepsilon_{lki} B^l + \partial_k \varepsilon_{lij} B^l) \\ &= -\delta^i_l \partial_i B^l - \delta^j_l \partial_j B^l - \delta^k_l \partial_k B^l \\ &= -3 \ \partial_l B^l \\ 0 &= \partial_l B^l \end{aligned}$$

or in the usual notation

$$\to 0 = \nabla \cdot \vec{B}$$

Thus we recover the other sourceless Maxwell's equation, Voila! and since these are just Bianchi identity, they are always true regardless of the presence or absence of sources.