

# Jumbled up thoughts

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## Abstract

Stray thoughts of a stray man. This is a collection of things that I thought was fun to do and also things that I always come back to after forgetting it :) There's no order or structure here just whatever my mind facies at the moment.

The second book on number theory that I use is ntb v2 by Victor Shoup, I'm reading it concurrently with William Stein's ent.pdf.

**Theorem 1.5.** He said that by “Applying Theorem 1.4 with  $a - \lceil x \rceil$  in place of  $a$  we obtain”

Theorem 1.4 is the statement that for  $a, b \in \mathbb{Z}$ ,  $a = bq + r$ . Although it is somewhat obvious there are a few things to consider.

First the range of  $x$  is  $[x, x + b)$ , here  $b$  is not included because otherwise we will have too many integers in the range. This only happens if  $x$  coincides with an integer, *e.g.*  $x = 1$ ,  $b = 1$ , if the range is  $[x, x + b]$  then there will be two integers in the range, 1 and 2. By excluding  $x + b$  we only have one integer in the range, which is  $x = 1$ .

Second, we need to take care some subtleties, *e.g.* say we want  $a = 5$ ,  $b = 3$ , and  $x = 7$ . If we blindly calculate it

$$\begin{aligned} 5 - 7 &= 3q + r \\ -2 &= 3q + r \\ -2 &= 3 \cdot 0 + -2 \\ 5 &= 3 \cdot 0 + 5 \end{aligned}$$

which is wrong since 5 is not between  $[7, 7 + 3)$ . Note that Theorem 1.4 is not the usual division statement. We might think that  $-2/3 = 0$  (at least that's what you'll get using a computer when both numbers are integers since  $|-2| < |3|$ ). However, Theorem 1.4 requires that the **remainder** be positive, also the quotient is defined as  $q = \lfloor a/b \rfloor \rightarrow \lfloor -2/3 \rfloor = -1$ . Thus the right way is

$$\begin{aligned} -2 &= 3q + r \\ -2 &= 3(-1) + 1 \\ 5 &= 3(-1) + 8 \end{aligned}$$

which is correct since now 8 is in the range of  $[7, 7 + 3)$ .

**Exercise 1.7.** Show that Theorem 1.5 also holds for the interval  $(x, x + b]$ . Does it hold in general for the intervals  $[x, x + b]$  or  $(x, x + b)$ ?

The key here is the observation that  $(x, x + b]$  contains  $b$  integers  $\lfloor x \rfloor + 1, \lfloor x \rfloor + 2, \dots, \lfloor x \rfloor + b$ . Since there are  $b$  integers we can apply the same trick, applying Theorem 1.4 with  $a - (\lfloor x \rfloor + 1)$  we obtain our goal.

The intervals  $[x, x + b]$  or  $(x, x + b)$  don't work because  $[x, x + b]$  can contain  $b + 1$  integers and  $(x, x + b)$  can contain  $b - 1$  integers, those happen when  $x$  itself is an integer. We need  $b$  integers in the range since  $r$  can contain  $b$  different values  $0, 1, \dots, b - 1$ .

**Exercise 1.2.** Let  $n$  be a composite integer. Show that there exists a prime  $p$  dividing  $n$ , with  $p \leq n^{1/2}$ .

First, we utilize the fundamental theorem that  $n$  must have prime factors. Next, we use contradiction. If all primes factors  $p_i$  of  $n$  are bigger than  $n^{1/2}$  this means that every product  $p_i p_j > n$  which is a contradiction. Thus there must be a factor  $p$  that is  $\leq n^{1/2}$ .

**Theorem 1.6** This is one cool theorem, it also brings up the dilemma on whether we need some intuitive understanding before attacking a problem or do we just blindly follows the mathematical consequence from one statement to another to get to where we want to go. I came from a physics background where you need to have (physical) intuitions on the theories you have, not just some mathematical tricks to get you somewhere (something that Feynman himself corroborated).

However, this approach might not be profitable in mathematics. The proof of this theorem just utilizes contradiction and some keen observation as always. Mechanically going from one deduction reasoning to another seems to be the fastest route for this particular problem. Note that the proof doesn't give us an intuition as to how we can compute it or what it is.

Note that the goal of is to massage any ideal into  $d\mathbb{Z}$ , *i.e.*  $d$  must be the smallest positive element  $\in I$  that divides any element in  $I$ . So, in some sense, given any two elements  $a, b \in I$ , we can find another number  $d$  that divides both  $a$  and  $b$ . However, what we want is **not** the smallest number that divides  $a$  and  $b$ , that would be 1, we want the largest number  $d$  that divides  $a$  and  $b$  since that will be sufficient for  $d\mathbb{Z}$  to contain both  $a$  and  $b$  or in other words since  $as + bt \in I$  and  $g \in I$  is a must

$$as + bt = g$$

where  $s, t$  are some integers and  $g = \gcd(a, b)$ . This statement more naturally flows as a

consequence of the “ideal” formalism rather than its precedence since it is not immediately clear how we can systematically compute  $s$  and  $t$ , “idealism” is indeed a powerful tool.

To intuitively prove this theorem we can use Euclid’s gcd algorithm

$$\begin{aligned} a &= bq_1 + r_1 \\ b &= r_1q_2 + r_2 \\ &\vdots \\ r_{n-1} &= r_nq + g \end{aligned}$$

Since  $b \in I \rightarrow q_1b \in I$  as well which means that  $r_1 = a - q_1b \in I$  which in turn means that  $r_2$  is also  $\in I$  and so on. Hence  $g \in I$ .

We now need to repeat the process, we calculate the pairwise gcd of all the positive numbers in  $I$  and then the pairwise gcd of all the gcd’s and so on. The smallest gcd of gcd of gcd of  $\dots$  of the numbers will be  $d$  since it obviously divides all elements of  $I$ .

The “astute reader” would have also noticed that we can use Euclid’s gcd algorithm to get  $s$  and  $t$  in  $as + bt = g$  by simply keeping track the residue of each step

$$\begin{aligned} a &= bq_0 + r_0 \\ a &= r_0q_1 + r_1, \quad r_0 = a - bq_0 \\ a &= r_1q_2 + r_2, \quad r_1 = a - r_0q_1 = a - (a - bq_0)q_1 \\ &\vdots \\ a &= r_{n-1}q_n + g, \quad r_{n-1} = a - q_{n-1}(a - q_{n-1}(a - \dots q_1(a - bq_0))) \end{aligned}$$

where  $g \equiv \gcd(a, b)$ , by substituting each  $r_i$  we will get the values for  $s$  and  $t$ . Let’s see an example

$$\begin{aligned} 7 &= 5 \cdot 1 + 2, \quad 2 = 7 - 5 \\ 7 &= 2 \cdot 3 + 1 = (7 - 5) \cdot 3 + 1 \\ \rightarrow 1 &= (-2)7 + (3)5 \end{aligned}$$

In conclusion, everytime you do the gcd calculation ala Euclid, you practically generate other elements in  $I$  (due to the closure property of an ideal) and because of that  $g$  is included as well, that’s the reasoning behind this theorem.

So which way reigns supreme? Do we need some sort of intuition of how things work or just heartlessly traverse through logical conclusions to get to our main goal? Intuition takes time to develop (although it gives you a deeper understanding) while mechanical derivation is faster, so  $\neg \setminus (^{\circ} \_ o) / \neg$

**Exercise 1.8.** Let  $I$  be a non-empty set of integers that is closed under addition (i.e.,  $a + b \in I$  for all  $a, b \in I$ ). Show that  $I$  is an ideal if and only if  $-a \in I$  for all  $a \in I$ .

If  $I$  is an ideal and  $a \in I$  then  $(-1)a = -a \in I$ ,  $-1 \in \mathbb{Z}$  per the definition of an ideal.

Going the other direction requires more work. Since  $I$  is closed under addition this means that  $I$  is also closed under multiplication by a positive integer  $za \in I, a \in I, z \in \mathbb{Z}, z > 0$ . Since a multiplication is nothing but repeated additions

$$za = \underbrace{a + a + a + a + \cdots + a}_{z \text{ numbers of } a}$$

thus closed under addition means closed under positive multiplication. If for every  $a \in I$ ,  $-a \in I$  is also true this means that  $-a + a = 0 \in I$  and  $I$  is closed under negative multiplication as

$$za = \underbrace{-a + -a + -a + -a + \cdots + -a}_{z \text{ numbers of } -a}$$

$z < 0$ . Thus  $I$  is closed under  $za \in I, z \in \mathbb{Z}$  and this completes the proof.

**Exercise 1.9 (c).**  $\gcd(a, 0) = \gcd(a, a) = |a|$  and  $\gcd(a, 1) = 1$

Using Theorem 1.8

$$as + 0 \cdot t = \gcd(a, 0)$$

$$as = \gcd(a, 0)$$

Since  $\gcd$  is the smallest of  $|as + bt|$ , the only solution above is  $s = \pm 1$ . Also

$$as + at = \gcd(a, a)$$

$$a(s + t) = \gcd(a, a)$$

$\gcd$  is the smallest of  $|a(s + t)|$ , the only solution above is  $s + t = \pm 1$ .

**Exercise 1.9 (d).**  $\gcd(ca, cb) = |c| \gcd(a, b)$

In ent.pdf a sick induction proof was employed, we can avoid it by using ideals. First  $g \equiv \gcd(a, b)$ , by Theorem 1.8 we have

$$\begin{aligned} as + bt &= g \\ c \times (as + bt) &= c \times g \\ (ca)s + (cb)t &= cg \end{aligned}$$

Thus  $g' \equiv \gcd(ca, cb) | cg$ . The question is now whether  $g' < cg$  or  $g' = cg$ ? Applying Theorem 1.8 to  $g'$

$$\begin{aligned} cas' + cbt' &= g' \\ c(as' + bt') &= g' \end{aligned}$$

Thus  $c|g' \rightarrow g' = ck$ , now suppose  $g' < cg$

$$\begin{aligned} \mathfrak{C}(as' + bt') &= \mathfrak{C}k \\ ck &< cg \rightarrow k < g \end{aligned}$$

which is a contradiction as  $g$  is the smallest  $as + bt$  possible.

**Exercise 1.10.** Show that for all integers  $a, b$  with  $d := \gcd(a, b) \neq 0$ , we have  $\gcd(a/d, b/d) = 1$ .

Again, using Theorem 1.8

$$\begin{aligned} as + bt &= d \\ \frac{a}{d}s + \frac{b}{d}t &= 1 \end{aligned}$$

**Exercise 1.11.** Let  $n$  be an integer. Show that if  $a, b$  are relatively prime integers, each of which divides  $n$ , then  $ab$  divides  $n$ .

$a|n \rightarrow n = aa'$ ,  $b|n \rightarrow n = bb'$ ,  $aa' = bb'$  which means that  $a|bb'$  but since  $\gcd(a, b) = 1$  according to Theorem 1.9  $a|b' \rightarrow b' = am$ . Thus  $n = bb' = (ab)m \rightarrow ab|n$ .

**Exercise 1.14.** Let  $p$  be a prime and  $k$  an integer, with  $0 < k < p$ . Show that the binomial coecient

$$\binom{p}{k} = \frac{p!}{k!(p-k)!},$$

which is an integer (see §A2), is divisible by  $p$ .

I'm more interested in the fact that  $\binom{p}{k}$  is an integer. If you check (§A2), you need two identities to prove this

$$\binom{p}{p} = \binom{p}{0} = 1$$

and **Pascal identity**

$$\binom{p}{k} = \binom{p-1}{k-1} + \binom{p-1}{k}$$

So the goal is to reduce  $\binom{p}{k}$  into a sum of  $\binom{p}{p}$  and  $\binom{p}{0}$

$$\begin{aligned} \binom{p}{k} &= \binom{p-1}{k-1} + \binom{p-1}{k} \\ &= \left\{ \binom{p-2}{k-2} + \binom{p-2}{k-1} \right\} + \binom{p-1}{k} \\ &\vdots \\ &= \binom{p-k}{k-k} + \binom{p-k}{k-k+1} + \binom{p-k+1}{2} + \cdots + \binom{p-2}{k-1} + \binom{p-1}{k} \\ &= 1 + (p-k) + \binom{p-k+1}{2} + \cdots + \binom{p-2}{k-1} + \binom{p-1}{k} \end{aligned}$$

So we substitute  $A$  at each step with a lower  $p$  and  $k$ , we now need to substitute each term again using Pascal identity, very laborious indeed but it does the job.

Note that for this particular problem  $p$  needs to be prime, otherwise it won't work, *e.g.*  $\binom{18}{6} = 18564$  and  $18 \nmid 18564$ .

**Exercise 1.17.** Let  $a, b, c$  be positive integers satisfying  $\gcd(a, b) = 1$  and  $c \geq (a-1)(b-1)$ . Show that there exist non-negative integers  $s, t$  such that  $c = as + bt$ .

$$(a-1)(b-1) = ab - a - b + 1$$

$$\rightarrow c > ab - a - b + 1$$

$$(as + bt) + a + b > ab + 1$$

$$a(s+1) + b(t+1) > ab + 1$$

$s \geq b, t \geq 0$  will do.