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The impulse-energy tensor of material particles

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PART I. MESONS AND ELECTRONS

The following is a direct and *general* construction, from the field quantities of the electrons and mesons and their derivatives, of a real, symmetrical and gauge-invariant tensor T_{kl} whose divergence to l is -1 times the four-force f_k experienced by the matter. Interpreting T_{kl} as the impulse-energy tensor of the material particles, for the case of no electromagnetic field the energy-momentum density obtained from T_{k4} is compared with that obtained from treating $(\hbar/i)\,(\partial/\partial x_k)$ as the energy-momentum operator. Further, in the general case the Hamiltonian of the matter is compared with $\int T_{44}\,dx\,dy\,dz$. Both times, the two quantities compared agree apart from small modifications.

Introduction

As is well known, the conservation of the combined energy and momentum of matter and the electromagnetic field follows if (i) there exists a tensor T_{kl} satisfying

$$\frac{\partial}{\partial x_l}T_{kl}+f_k = 0, \tag{1}$$

f_k being the four-force experienced by the matter and (ii) T_{kl} is interpreted as the impulse-energy tensor belonging to matter. For a Dirac electron, Tetrode (1928) has found a symmetrical T_{kl} satisfying (1), which was later expressed in spinor form by Laporte & Uhlenbeck (1932). The search for T_{kl} was nearly completed by Heisenberg & Pauli (1929) who found in a general way an expression for T_{kl} satisfying (1) for all sorts of particles whose wave equations are deducible from varying a Lagrangian, but their expression is not symmetrical between k and l.

In Part I of the present paper, an attempt is made to obtain the most *general* symmetrical T_{kl} belonging to scalar and vector mesons and electrons. § I deals with scalar mesons interacting with electrons and neutrinos and § II with vector mesons interacting with the same light particles. The cases in which we have a single kind of particle only are obviously special cases. Owing to the significance of T_{kl} as the impulse-energy tensor of matter, such an attempt is certainly worth while.

Two things are to be observed regarding this impulse-energy tensor of matter. Let us take the case of electrons for definiteness. First, in the case of no electromagnetic field, where a possible solution for the wave functions of the electrons contains x_r only through the factor $\exp(ic_r x_r)$ (c_r being a constant four-vector), the usual concept of treating $(\hbar/i)(\partial/\partial x_r)$ as the momentum operator leads one to expect $\rho c_r \hbar$ to be the energy-momentum density (ρ being the probability density of electrons) and thus to be equal to $(i/c)\,T_{k4}$. That this is nearly the case will be shown later. Secondly, one expects the integral of T_{44} over all space to be that of the Hamiltonian H for matter, which has the property that, from the rules of quantizing the field quantities ψ and from equations of the type

$$\frac{\partial \psi}{\partial t} = \frac{i}{\hbar}\left[\int H\,dV, \psi\right], \quad dV = dx\,dy\,dz,$$

one can obtain the wave equations for ψ. This is also found to be true apart from an unimportant modification. These results are certainly satisfactory.

I. Scalar mesons and electrons

Let x_r be the four-vector x, y, z, ict, ϕ_r be the electromagnetic four-potential A_x, A_y, A_z, $i\phi$ (A_x, A_y, A_z being the vector potential and ϕ the scalar potential), and Π_l, Π_l^*† be

$$-i\frac{\partial}{\partial x_l} - \frac{e}{\hbar c}\phi_l$$

and

$$i\frac{\partial}{\partial x_l} - \frac{e}{\hbar c}\phi_l \tag{2}$$

respectively. Before writing out the wave equations for A, A^* (the wave functions of scalar mesons), χ^E_μ, $\psi^E_{\dot\nu}$, $\chi^{E*}_{\dot\mu}$, ψ^{E*}_ν (those for electrons) and χ^N_μ, $\psi^N_{\dot\nu}$, $\chi^{N*}_{\dot\mu}$, ψ^{N*}_ν (those for neutrinos), their Lagrangians will be written out. With δ_l standing for $\partial/\partial x_l$ (which, it is assumed, does not operate through a bracket), $\delta_{\mu\dot\nu}$ for the spinor of δ_l, $\Pi_{\mu\dot\nu}$ for the spinor of Π_l, etc., and with κ^S, κ^E and κ^N as three constants connected

† In this paper, a tensor with a * symbol such as $T^*_{ik\ldots}$ is defined by

$$T^*_{ik\ldots} = (-1)^n\,(\overline{T_{ik\ldots}}),$$

where n is the number of suffices among i, k, ... which equal 4 and $\overline{\gamma}$ the complex conjugate of γ. (It is understood that by 'complex conjugate' is not meant numerically complex conjugate. This is evident since $\overline{T_{ik\ldots}}$ and $T_{ik\ldots}$ are different kinds of q-numbers.) For lack of better names, the tensor $T^*_{ik\ldots}$ is also called the complex conjugate of $T_{ik\ldots}$, and thus the complex conjugate of $T_{ik\ldots}$ may mean either $T^*_{ik\ldots}$ or $\overline{T_{ik\ldots}}$. For a spinor such as $t_{m\lambda\ldots\dot n\dot s\ldots}$, one can introduce a complex conjugate spinor $t^*_{m\lambda\ldots}$ defined by

$$t^*_{ns\ldots\dot m\dot\lambda\ldots} = \overline{(t_{m\lambda\ldots\dot n\dot s\ldots})},$$

where $\overline{\gamma}$ denotes the complex conjugate of γ as before. When t has an equal number of dotted and undotted suffices, the tensors corresponding to t and t^* are conjugate tensors. In the following the complex conjugate of a component $t_{m\lambda\ldots\dot n\dot s\ldots}$ of a spinor t with an equal number of dotted and undotted suffices may mean $t^*_{m\lambda\ldots\dot n\dot s\ldots}$ instead of $\overline{t_{m\lambda\ldots\dot n\dot s\ldots}}$. It is hoped that no confusion will arise from this double usage of the term 'complex conjugate'.

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to the mass of scalar mesons, electrons and neutrinos by $\kappa^S\hbar = m^S c$, $\kappa^E\hbar = m^E c$ and $\kappa^N\hbar = m^N c$, the Lagrangians are†

$$L^E = \hbar c[\kappa^E(\psi^{E\lambda}\chi^E_\lambda - \psi^E_{\dot\sigma}\chi^{E\dot\sigma}) + \tfrac{1}{2}\chi^{E\dot\mu}\Pi_{\lambda\dot\mu}\chi^{E\lambda} + \tfrac{1}{2}\chi^{E\lambda}\Pi^*_{\lambda\dot\mu}\chi^{E\dot\mu} + \tfrac{1}{2}\psi^{E\lambda}\Pi_{\lambda\dot\mu}\psi^{E\dot\mu} + \tfrac{1}{2}\psi^{E\dot\mu}\Pi^*_{\lambda\dot\mu}\psi^{E\lambda}], \tag{3·1}$$

$$L^N = \hbar c[\kappa^N(\psi^{N\lambda}\chi^N_\lambda - \psi^N_{\dot\sigma}\chi^{N\dot\sigma}) - \tfrac{1}{2}i\chi^{N\dot\mu}\delta_{\lambda\dot\mu}\chi^{N\lambda} + \tfrac{1}{2}i\chi^{N\lambda}\delta_{\lambda\dot\mu}\chi^{N\dot\mu} - \tfrac{1}{2}i\psi^{N\lambda}\delta_{\lambda\dot\mu}\psi^{N\dot\mu} + \tfrac{1}{2}i\psi^{N\dot\mu}\delta_{\lambda\dot\mu}\psi^{N\lambda}], \tag{3·2}$$

$$L^S = -\frac{\hbar c}{\kappa^S}\{(\kappa^S)^2(AA^* + F_l F_l^*) - g_2 g_2^* M_l M_l^* - \kappa^S(g_1 NA^* + g_1^* N^* A)\}, \tag{3·3}$$

where

$$\left.\begin{aligned} \kappa^S F_l &= i\Pi_l A + g_2 M_l,\\ M_l &= \sigma_{l\mu\dot\nu}(\psi^{N\mu}\psi^{E\dot\nu} + l_1\chi^{E\mu}\chi^{N\dot\nu}),\\ N &= l_2\psi^{N\mu}\chi^E_\mu + \chi^{N\dot\mu}\psi^E_{\dot\mu}, \end{aligned}\right\} \tag{4}$$

the $\sigma_{l\mu\dot\nu}$'s being the usual Pauli matrices and g_1, g_2, l_1, l_2 being arbitrary complex constants.‡ Varying $\int(L^S + L^E + L^N)\,dV\,dt$ to A^*, $\psi^{\lambda E}$, $\chi^{\sigma E}$, $\psi^{\sigma N}$, $\chi^{\dot\lambda N}$, one finds

$$-i\Pi_l F_l + \kappa^S A - g_1 N = 0, \tag{5·1}$$

$$\kappa^E\chi^E_\lambda + \Pi^*_{\mu\lambda}\psi^{E\mu} + \frac{g_2 i}{\kappa^S}(\Pi^*_{\mu\lambda}A^*)\psi^{N\mu} - g_1 A^*\chi^N_\lambda = 0, \tag{5·2}$$

$$\kappa^E\psi^E_\sigma - \Pi^*_{\sigma\dot\nu}\chi^{E\dot\nu} - \frac{g_2 l_1 i}{\kappa^S}(\Pi^*_{\sigma\dot\nu}A^*)\chi^{N\dot\nu} + g_1 l_2 A^*\psi^N_\sigma = 0, \tag{5·3}$$

$$\kappa^N\chi^N_\sigma - i\delta_{\sigma\dot\lambda}\psi^{N\dot\lambda} + \frac{g_2 i}{\kappa^S}(\Pi^*_{\sigma\dot\nu}A^*)\psi^{E\dot\nu} + g_1 l_2 A^*\chi^E_\sigma = 0, \tag{5·4}$$

$$\kappa^N\psi^N_\lambda + i\delta_{\mu\lambda}\chi^{N\mu} - \frac{g_2 l_1 i}{\kappa^S}(\Pi^*_{\mu\lambda}A^*)\chi^{N\mu} - g_1 A^*\psi^E_\lambda = 0. \tag{5·5}$$

For convenience of future description, the left-hand sides of the above equations will be referred to as $h(A)$, $h(\chi^E_\lambda)$, $h(\psi^E_\sigma)$, $h(\chi^N_\sigma)$ and $h(\psi^N_\lambda)$. If every term in the above equations is replaced by its complex conjugate, then equations whose left-hand sides will be referred to as $h(A^*)$, $h(\chi^E_{\dot\lambda})$, $h(\psi^E_{\dot\sigma})$, $h(\chi^N_{\dot\sigma})$ and $h(\psi^N_{\dot\lambda})$ are obtained. It is hoped that the common usage of the same notation 'h' to denote the left-hand sides of the different wave equations will not lead to misunderstandings.

Varying the combined Lagrangian of the matter and the electromagnetic field subject to $\delta_l\phi_l = 0$, one finds

$$\delta_i\delta_i\phi_l = -(4\pi/c)j_l,$$

where

$$j_l = c(\partial/\partial\phi_l)(L^S + L^E + L^N) = ec[iAF_l^* - iA^*F_l - \sigma_{l\mu\dot\nu}(\chi^{E\mu}\chi^{E\dot\nu} + \psi^{E\mu}\psi^{E\dot\nu})] \tag{6}$$

† The *'s are dropped from $\chi^{E*}_{\dot\mu}$, $\psi^{E*}_{\dot\nu}$, $\chi^{N*}_{\dot\mu}$, $\psi^{N*}_{\dot\nu}$, since this will not cause confusion.

‡ In current literature g_1 and g_2 are real and l_1 and l_2 are unity, which is less general.

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and satisfies $\delta_l j_l = 0$ on account of equations (5·1–5·5). The problem now is to find T_{kl} satisfying

$$\delta_l T_{kl} + \frac{1}{c}(\delta_k \phi_l - \delta_l \phi_k) j_l = 0, \tag{7}$$

when the right-hand side of (6) is substituted for j_l.

Now impose the following conditions on T_{kl}:

(i) that it is gauge-invariant,

(ii) that the terms in T_{kl} involving A and A^* only do not contain symbolically more than two δ's and the terms involving χ^E_μ, $\psi^E_{\dot\nu}$, $\chi^E_{\dot\mu}$ and ψ^E_ν only (or χ^N_μ, $\psi^N_{\dot\nu}$, $\chi^N_{\dot\mu}$, ψ^N_ν only) do not contain symbolically more than one δ, and

(iii) that T_{kl} is real and symmetrical between k and l. T_{kl} can now be written down as a sum of terms such as

$$A^* \Pi_k \Pi_l A, \quad \delta_{kl} \Pi^*_s A^* \Pi_s A, \quad \sigma^{m\dot n}_k \sigma^{\lambda\dot s}_l \psi^E_{\dot n} \Pi^*_{\lambda\dot s} \psi^E_{m}, \tag{8}$$

etc., each with an unknown coefficient γ. Now split the coefficients of all terms containing A and A^* only into some power of κ^S times some quantities so that the latter are of the same dimensions as the coefficient of $A^* \Pi_k \Pi_l A$. It is at once found that the terms with some power of κ^S appearing in the coefficient can be eliminated with the help of the wave equations. Then assume that this elimination is carried out so that the coefficients of different terms containing A and A^* only contain no κ^S as a factor and are of the same dimensions. Let similar steps be taken toward the terms containing the wave functions of electrons (or neutrinos) only. After carrying out such manipulations, our assumed form for T_{kl} is substituted into (7), giving a long equation (8) which it is hoped can be made identically zero by properly choosing the value of the constants γ. The different terms in (8) are not all independent; relations between them will be given below. Following the method of unknown multipliers, multiply the relations by unknown constants λ, add them to (8), set the coefficients of different terms in the resulting equation to zero and solve concurrently for the λ's and the γ's. If in so doing no relation between terms in (8) has been left out, the result of solving for λ and γ gives the most general T_{kl} satisfying our conditions. All the algebra will be left out except for writing out for future reference the various relations between terms in (8) with their proper multiplying constants,

$$\tfrac{1}{2}\hbar c \quad \{-iA \; \Pi^*_k \; h(A^*) + i(\Pi^*_k \; A^*) \; h(A)\} = 0, \tag{9}$$

$$\tfrac{1}{8}\hbar c \sigma^{m\dot n}_k\{\; i\psi^{E\rho} \Pi_{\rho\dot n} \, h(\chi^E_m) - i(\Pi_{\rho\dot n} \chi^E_m) \; h(\psi^{E\rho})\} = 0, \tag{10·1}$$

$$\tfrac{3}{8}\hbar c \sigma^{m\dot n}_k\{\; i\psi^E_m \; \Pi^\rho{}_{\dot n} h(\chi^E_\rho) - i(\Pi^\rho{}_{\dot n} \chi^E_\rho) \; h(\psi^E_m)\} = 0, \tag{10·2}$$

$$\tfrac{3}{8}\hbar c \sigma^{m\dot n}_k\{-i\chi^E_m \; \Pi^*_{\rho\dot n} h(\psi^{E\rho}) + i(\Pi^*_{\rho\dot n} \psi^{E\rho}) \, h(\chi^E_m)\} = 0, \tag{10·3}$$

$$\tfrac{1}{8}\hbar c \sigma^{m\dot n}_k\{-i\chi^E_\rho \; \Pi^{*\rho}_{\dot n} h(\psi^E_m) + i(\Pi^{*\rho}_{\dot n} \psi^E_m) \; h(\chi^E_\rho)\} = 0, \tag{10·4}$$

$$\tfrac{1}{8}\hbar c \sigma^{m\dot n}_k\{\; \psi^{N\rho} \delta_{\rho\dot n} \; h(\chi^N_m) - (\delta_{\rho\dot n} \; \chi^N_m) \; h(\psi^{N\rho})\} = 0, \tag{11·1}$$

$$\tfrac{3}{8}\hbar c \sigma^{m\dot n}_k\{\; \psi^N_m \delta^\rho{}_{\dot n} \; h(\chi^N_\rho) - (\delta^\rho{}_{\dot n} \; \chi^N_\rho) \; h(\psi^N_m)\} = 0, \tag{11·2}$$

$$\tfrac{3}{8}\hbar c \sigma^{m\dot n}_k\{\; \chi^N_m \delta_{\rho\dot n} \; h(\psi^{N\rho}) - (\delta_{\rho\dot n} \; \psi^{N\rho}) \, h(\chi^N_m)\} = 0, \tag{11·3}$$

$$\tfrac{1}{8}\hbar c \sigma^{m\dot n}_k\{\; \chi^N{}_\rho \delta^\rho{}_{\dot n} \; h(\psi^N_m) - (\delta^\rho{}_{\dot n} \; \psi^N_m) \; h(\chi^N_\rho)\} = 0, \tag{11·4}$$

together with the conjugate complex equations.

The replacement of Π and Π^* by $-i\delta$ and $i\delta$ in equations (10) and the reverse replacement in (11) give in fact a number of other relations, but their multiplying constants are all zero.†

In giving the result for T_{kl}, it is convenient to put it as

$$Y_{kl}+Y_{lk}+Y^*_{kl}+Y^*_{lk}, \tag{12}$$

and write out Y_{kl}. It falls into four parts:

(i) The part that concerns A and A^* only,

$$\gamma^{(1)}\{\Pi^*_k A^*\Pi_l A - A^*\Pi_k\Pi_l A + \delta_{kl}A^*\Pi_s\Pi_s A - \delta_{kl}\Pi^*_s A^*\Pi_s A\} \\ -\tfrac{1}{4}(\hbar c/\kappa^S)\{\Pi^*_k A^*\Pi_l A + A^*\Pi_k\Pi_l A\}, \tag{13}$$

where $\gamma^{(1)}$ is an arbitrary real constant. This will be denoted by Y^S_{kl} and the corresponding T_{kl} by T^S_{kl}.

(ii) The part that concerns electrons only,

$$\tfrac{1}{4}\sigma_k^{m\dot{n}}\sigma_l^{\lambda\dot{s}}\{\gamma^{(2)}y^{(2)}_{m\lambda\dot{n}\dot{s}}+\gamma^{(3)}y^{(3)}_{m\lambda\dot{n}\dot{s}}+\ldots+\gamma^{(7)}y^{(7)}_{m\lambda\dot{n}\dot{s}}+y^{(8)}_{m\lambda\dot{n}\dot{s}}\}, \tag{14}$$

where $\gamma^{(2)}$, $\gamma^{(3)}$, ... are arbitrary real constants,

$$y^{(2)}_{m\lambda\dot{n}\dot{s}} = \psi^E_{\dot{n}}\Pi^*_{m\dot{s}}\psi^E_{\lambda} - \psi^E_{\dot{n}}\Pi^*_{\lambda\dot{s}}\psi^E_m + \tfrac{1}{2}\epsilon_{m\lambda}\epsilon_{\dot{n}\dot{s}}\psi^E_{\dot{\rho}}\Pi^{*\dot{\rho}\theta}\psi^E_{\theta}, \tag{15·1}$$

$$y^{(3)}_{m\lambda\dot{n}\dot{s}} = -\epsilon_{m\lambda}\psi^E_{\dot{s}}\Pi^{*\alpha}_{\dot{n}}\psi^E_{\alpha} + \tfrac{1}{2}\epsilon_{m\lambda}\epsilon_{\dot{n}\dot{s}}\psi^E_{\dot{\rho}}\Pi^{*\dot{\rho}\theta}\psi^E_{\theta}, \tag{15·2}$$

$$y^{(4)}_{m\lambda\dot{n}\dot{s}} = -\epsilon_{\dot{n}\dot{s}}\psi^E_{\dot{\rho}}\Pi^{*\dot{\rho}}_{m}\psi^E_{\lambda} + \tfrac{1}{2}\epsilon_{m\lambda}\epsilon_{\dot{n}\dot{s}}\psi^E_{\dot{\rho}}\Pi^{*\dot{\rho}\theta}\psi^E_{\theta}, \tag{15·3}$$

$$y^{(8)}_{m\lambda\dot{n}\dot{s}} = -\tfrac{1}{2}\hbar c(\psi^E_{\dot{n}}\Pi^*_{\lambda\dot{s}}\psi^E_m + \chi^E_{\dot{n}}\Pi_{\lambda\dot{s}}\chi^E_m), \tag{15·4}$$

($\epsilon_{m\lambda}$, $\epsilon_{\dot{n}\dot{s}}$ being the well-known antisymmetrical spinors), and the $y^{(5)}$, $y^{(6)}$, $y^{(7)}$ are spinors obtained from $y^{(2)}$, $y^{(3)}$, $y^{(4)}$ by replacing ψ by χ and Π^* by Π with all suffices remaining unchanged. This Y will be denoted by Y^E, and the corresponding T by T^E.

(iii) The part that concerns neutrinos only; this can be obtained from (ii) by putting e to be zero. This Y will be denoted by Y^N, and its T by T^N.

(iv) The part that contains the g's,

$$\tfrac{1}{2}\hbar c\left\{\tfrac{1}{2}\delta_{kl}g_1 A^*N + \tfrac{1}{2}\delta_{kl}\frac{g_2 i}{\kappa^S}A^*\Pi_s M_s + \frac{g_2 i}{\kappa^S}M_l\Pi^*_k A^*\right\}. \tag{16}$$

This will be denoted by Y^I (the index 'I' signifying interaction), and its T by T^I. This completes the answer for T_{kl}.

† Strictly speaking, the whole set of relations between terms in (8) is not completed yet. By permuting in a proper way the spinor suffices m, $\dot{n}$, etc., in (10) and (11) (either with Π, Π^* replaced by $-i\delta$, $i\delta$ or not), new equations are derived, but these can also be obtained from the already existing ones by employing the rules

$$a_\lambda b^\lambda = -a^\lambda b_\lambda, \quad a_\lambda b^\lambda c_m + a_m b_\lambda c^\lambda + a^\lambda b_m c_\lambda = 0,$$

and are hence not independent equations. Further, there are equations with one side zero and the other side $(\delta_l\phi_l)$ times some expressions, which obviously may form part of the relations between terms in (8). Relations of such kinds will not be written out here.

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Now see what happens to T_{k4} for the special case in which $\phi = 0$, χ^E_μ and $\psi^E_{\dot{\nu}}$ contain x_r through the factor $\exp(ic^E_r x_r)$, χ^N_μ and $\psi^N_{\dot{\nu}}$ through the factor $\exp(ic^N_r x_r)$ and A through the factor $\exp(ic^S_r x_r)$. In the first place

$$c^S_r = c^E_r - c^N_r, \quad (\Pi_l A)\, M^*_l = A\Pi^*_l M^*_l,$$

etc. For electrons, on putting all the γ's in (14) zero, it is found that

$$T^E_{kl} = -\frac{\hbar c}{4}\sigma^{m\dot{n}}_k \sigma^{\lambda\dot{s}}_l \{c^E_{m\dot{n}}(\chi^E_\lambda \chi^E_{\dot{s}} + \psi^E_\lambda \psi^E_{\dot{s}}) + c^E_{\lambda\dot{s}}(\chi^E_m \chi^E_{\dot{n}} + \psi^E_m \psi^E_{\dot{n}})\}. \tag{17}$$

Now
$$\begin{aligned}\sigma^{\lambda\dot{s}}_l c^E_{m\dot{n}} \chi^E_\lambda \chi^E_{\dot{s}} &= -\sigma^{\lambda}_{l\dot{n}} c^E_{m\dot{s}} \chi^{E\dot{s}} \chi^E_\lambda - \sigma^{\lambda}_{l\dot{s}} c^{E\dot{s}}_m \chi^E_{\dot{n}} \chi^E_\lambda \\ &= -\sigma^{\lambda}_{l\dot{n}} c^E_{m\dot{s}} \chi^{E\dot{s}} \chi^E_\lambda + \sigma_{lm\dot{s}} c^{E\dot{s}}_\lambda \chi^E_{\dot{n}} \chi^{E\lambda} + \sigma_{l\lambda\dot{s}} c^{E\lambda\dot{s}} \chi^E_{\dot{n}} \chi^E_m.\end{aligned}$$

Hence T^E_{kl} is the sum of two parts, one with its T_{k4} equal to $-i\hbar c c^E_k \rho^E$ (ρ^E being the probability density of the electrons $\sum_{\lambda=1,2} (\chi^E_\lambda \chi^E_{\dot{\lambda}} + \psi^E_\lambda \psi^E_{\dot{\lambda}})$) and the other given by

$$-\frac{\hbar c}{4}\sigma^{m\dot{n}}_k \{\sigma^{\lambda}_{l\dot{n}} c^E_{m\dot{s}}(\chi^{E\dot{s}}\chi^E_\lambda + \psi^{E\dot{s}}\psi^E_\lambda) - \sigma_{lm\dot{s}} c^{E\dot{s}}_\lambda (\chi^E_{\dot{n}} \chi^{E\lambda} + \psi^E_{\dot{n}} \psi^{E\lambda})\}. \tag{18}$$

Similarly, T^N_{kl} is the sum of two parts, one with its T_{k4} equal to $-i\hbar c c^N_k \rho^N$ and the other given by something similar to (18). Eliminating from the latter and (18) $c^E_{m\dot{s}}\chi^{E\dot{s}}$, $c^E_{m\dot{s}}\psi^{E\dot{s}}$, $c^{E\dot{s}}_\lambda \chi^{E\lambda}$, $c^{E\dot{s}}_\lambda \psi^{E\lambda}$, etc., by means of the wave equations, so that the resulting expressions contain no other derivatives than that of A, and noting that each must be antisymmetrical between k and l, it is found that their sum reduces to

$$\frac{\hbar c}{2}\frac{g_2 i}{\kappa^S}\{-M_k \Pi^*_l A^* + M_l \Pi^*_k A^*\} + \text{comp. conj.}$$

Finally, for any value of $\gamma^{(1)}$ in (13), it is observed that

$$T^S_{k4} + \frac{\hbar c}{\kappa^S}(g_2 i M_4 \Pi^*_k A^* - g^*_2 i M^*_4 \Pi_k A) = -i\hbar c c^S_k \rho^S,$$

ρ^S being the probability density of the scalar mesons, and thus

$$T_{k4} = -i\hbar c(c^S_k \rho^S + c^E_k \rho^E + c^N_k \rho^N) - \tfrac{1}{2}\hbar c \triangle \delta_{k4}, \tag{19}$$

where $\triangle$ is short for

$$-g_1 A^* N - (g_2 i/\kappa^S)\, M_l \Pi^*_l A^* + \text{comp. conj.} \tag{20·1}$$

which in the present special case $\phi = 0$ reduces to

$$2\kappa^E(\chi^E_\lambda \psi^{E\lambda} + \chi^E_{\dot{s}} \psi^{E\dot{s}}) + 2c^E_{\lambda\dot{\mu}}(\chi^{E\lambda}\chi^{E\dot{\mu}} + \psi^{E\lambda}\psi^{E\dot{\mu}}) \tag{20·2}$$

or
$$2\kappa^N(\chi^N_\lambda \psi^{N\lambda} + \chi^N_{\dot{s}} \psi^{N\dot{s}}) + 2c^N_{\lambda\dot{\mu}}(\chi^{N\lambda}\chi^{N\dot{\mu}} + \psi^{N\lambda}\psi^{N\dot{\mu}}) \tag{20·3}$$

or
$$-2(\kappa^S)^{-1}[(\kappa^S)^2 + c^S_r c^S_r]\, AA^*. \tag{20·4}$$

Thus apart from the term $-\frac{1}{2}\hbar c \triangle \delta_{k4}$ in (19), $(i/c)\, T_{k4}$ gives the expected value of momentum density.

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It is impossible to compare T_{44} with the Hamiltonian H of matter, since the former is gauge-invariant while the latter is not. Let P be the canonical variable to A, P^* that to A^*, and let H' be

$$H+\frac{e}{\hbar}\phi_4(AP-A^*P^*)+ie\phi_4(\chi^{E\lambda}\chi^{E\dot\lambda}+\psi^{E\mu}\psi^{E\dot\mu}), \tag{21}$$

so that the wave equations take the gauge-invariant form

$$\left.\begin{aligned}
\Pi_4 A &= -\frac{i}{\hbar c}\left[\int H'dV, A\right], & \Pi_4^* P &= \frac{i}{\hbar c}\left[\int H'dV, P\right],\\
\Pi_4^* A^* &= \frac{i}{\hbar c}\left[\int H'dV, A^*\right], & \Pi_4 P^* &= -\frac{i}{\hbar c}\left[\int H'dV, P^*\right],\\
\Pi_4 \psi^{E\dot\mu} &= -\frac{i}{\hbar c}\left[\int H'dV, \psi^{E\dot\mu}\right], & \Pi_4^* \psi^{E\mu} &= \frac{i}{\hbar c}\left[\int H'dV, \psi^{E\mu}\right],\\
\frac{\partial}{\partial t}\psi^{N\dot\mu} &= \frac{i}{\hbar}\left[\int H'dV, \psi^{N\dot\mu}\right], & \frac{\partial}{\partial t}\psi^{N\mu} &= \frac{i}{\hbar}\left[\int H'dV, \psi^{N\mu}\right],
\end{aligned}\right\} \tag{22}$$

etc. Then, adding to the above T_{kl} the term $-\delta_{kl}$ times

$$L^E+L^N-\hbar c\triangle-\frac{\hbar c}{2\kappa^S}\{[(\kappa^S)^2AA^*+A^*(\Pi_l\Pi_lA-g_1\kappa^SN-g_2i\Pi_lM_l)]+\text{comp. conj.}\}, \tag{23}$$

which vanishes always, then by partial integrations over the space co-ordinates

$$\int H'dV=\int T_{44}dV, \tag{24}$$

showing that the modified Hamiltonian $\int H'dV$ is the total energy.

II. Vector mesons and electrons

The Lagrangian of the vector mesons is

$$-\frac{\hbar c}{\kappa^V}\{(\kappa^V)^2A_iA_i^*+\tfrac{1}{2}(\kappa^V)^2F_{ij}F_{ij}^*-\tfrac{1}{2}g_2g_2^*\Gamma_{ij}\Gamma_{ij}^*-\kappa^V(g_1M_iA_i^*+g_1^*M_i^*A_i)\}, \tag{25}$$

where
$$\kappa^VF_{ij}=i(\Pi_iA_j-\Pi_jA_i)+g_2\Gamma_{ij}, \tag{26·1}$$

$$M_l=\sigma_{l\mu\dot\nu}(\psi^{N\mu}\psi^{E\dot\nu}+l_1\chi^{N\dot\nu}\chi^{E\mu}), \tag{26·2}$$

$$\Gamma_{ij}=\sigma_{i,n\dot\alpha}\sigma_{j,m\dot\beta}[l_3\epsilon^{\dot\alpha\dot\beta}(\psi^{Nn}\chi^{Em}+\psi^{Nm}\chi^{En})+\epsilon^{nm}(\psi^{E\dot\alpha}\chi^{N\dot\beta}+\psi^{E\dot\beta}\chi^{N\dot\alpha})], \tag{26·3}$$

g_1, g_2, l_1, l_3 being again arbitrary real or complex constants. With L^E and L^N given in the previous section, it is easy to see by performing variations that

$$\kappa^VA_j-i\Pi_iF_{ij}-g_1M_j=0, \tag{27·1}$$

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$$\kappa^E \chi^E_{\dot\lambda} + \Pi^*_{\mu\dot\lambda} \psi^{E\mu} + g_1 A^*_{\nu\dot\lambda} \psi^{N\nu} + \frac{g_2 i}{\kappa^V} (\Pi^*_{\alpha\dot\lambda} A^{*\alpha}_{\dot\mu} - \Pi^{*\alpha}_{\dot\mu} A^*_{\alpha\dot\lambda}) \chi^{N\dot\mu} = 0, \tag{27·2}$$

$$\kappa^E \psi^E_\sigma - \Pi^*_{\sigma\dot\lambda} \chi^{E\dot\lambda} - g_1 l_1 A^*_{\sigma\dot\mu} \chi^{N\dot\mu} - \frac{g_2 l_3 i}{\kappa^V} (\Pi^*_{\mu\dot\alpha} A^{*\dot\alpha}_{\sigma} - \Pi^{*\dot\alpha}_{\sigma} A^*_{\mu\dot\alpha}) \psi^{N\mu} = 0, \tag{27·3}$$

$$\kappa^N \chi^N_\sigma - i\delta_{\sigma\dot\lambda} \psi^{N\dot\lambda} + g_1 A^*_{\sigma\dot\mu} \psi^{E\dot\mu} + \frac{g_2 l_3 i}{\kappa^V} (\Pi^*_{\sigma\dot\alpha} A^{*\dot\alpha}_{\mu} - \Pi^{*\dot\alpha}_{\mu} A^*_{\sigma\dot\alpha}) \chi^{E\mu} = 0, \tag{27·4}$$

$$\kappa^N \psi^N_{\dot\lambda} + i\delta_{\mu\dot\lambda} \chi^{N\mu} - g_1 l_1 A^*_{\sigma\dot\lambda} \chi^{E\sigma} - \frac{g_2 i}{\kappa^V} (\Pi^*_{\alpha\dot\nu} A^{*\alpha}_{\dot\lambda} - \Pi^{*\alpha}_{\dot\lambda} A^*_{\alpha\dot\nu}) \psi^{E\dot\nu} = 0. \tag{27·5}$$

As before, the left-hand sides of the above equations and their conjugates will be referred to as $h(A_j)$ or $h(\chi^E_{\dot\lambda})$, $h(\psi^E_\sigma)$, $h(\chi^N_\sigma)$, etc. Varying the combined Lagrangian of matter and field, one finds

$$\delta_i \delta_i \phi_l = -\frac{4\pi}{c} j_l,$$

where j_l is given by

$$c(\partial/\partial\phi_l)(L^V + L^E + L^N) = ec\{iA_i F^*_{li} - iA^*_i F_{li} - \sigma_{l,\mu\dot\nu}(\psi^{E\mu}\psi^{E\dot\nu} + \chi^{E\mu}\chi^{E\dot\nu})\}, \tag{28}$$

and satisfies $\delta_l j_l = 0$ on account of the wave equations.

The task is to substitute this j_l into

$$\delta_l T_{kl} + \frac{1}{c}(\delta_k \phi_l - \delta_l \phi_k) j_l = 0$$

and find the corresponding T_{kl}. The same set of conditions will be imposed on T_{kl} as before and the same procedure employed in finding it. The only difference is that equation (9) is to be replaced by

$$\left.\begin{aligned}
&\tfrac{1}{2}\hbar c\{-iA^*_k \Pi_i h(A_i) + i(\Pi_i A_i) h(A^*_k)\} = 0,\\
&\tfrac{1}{2}\hbar c\{-iA^*_i \Pi_k h(A_i) + i(\Pi_k A_i) h(A^*_i)\} = 0,\\
&\tfrac{1}{2}\hbar c\{\ \ iA^*_i \Pi_i h(A_k) - i(\Pi_i A_k) h(A^*_i)\} = 0,
\end{aligned}\right\} \tag{29}$$

and that the symbols h now denote entirely different functions. Writing T_{kl} as $Y_{kl} + Y_{lk} + Y^*_{kl} + Y^*_{lk}$, the result for Y_{kl} falls into four parts, of which that concerning electrons and that concerning neutrinos are exactly those given in the previous section. The other two parts are:

(i) That concerning vector mesons only,

$$\begin{aligned}
&\gamma^{(8)}\{\Pi^*_s A^*_k \Pi_s A_l - \Pi^*_k A^*_s \Pi_s A_l - \Pi^*_k A^*_l \Pi_s A_s - A^*_k \Pi_s \Pi_s A_l\\
&\qquad + A^*_k \Pi_l \Pi_s A_s + A^*_s \Pi_k \Pi_s A + \tfrac{1}{2}\delta_{kl} \Pi^*_r A^*_s \Pi_s A_r\\
&\qquad + \tfrac{1}{2}\delta_{kl} \Pi^*_r A^*_r \Pi_s A_s - \tfrac{1}{2}\delta_{kl} A^*_r \Pi_r \Pi_s A_s - \tfrac{1}{2}\delta_{kl} A^*_r \Pi_s \Pi_r A_s\}\\
&+ \gamma^{(9)}\{\Pi^*_k A^*_s \Pi_l A_s - A^*_s \Pi_k \Pi_l A_s + \delta_{kl} A^*_r \Pi_s \Pi_s A_r - \delta_{kl} \Pi^*_r A^*_s \Pi_r A_s\}\\
&- \frac{\hbar c}{2\kappa^V}\{\Pi^*_k A^*_l \Pi_s A_s + \Pi^*_k A^*_s \Pi_l A_s - \Pi^*_k A^*_s \Pi_s A_l\\
&\qquad - A^*_s \Pi_s \Pi_l A_k - \tfrac{1}{2}\delta_{kl} \Pi^*_r A^*_s \Pi_r A_s - \tfrac{1}{2}\delta_{kl} \Pi^*_r A^*_r \Pi_s A_s\\
&\qquad + \tfrac{1}{2}\delta_{kl} A^*_r \Pi_r \Pi_s A_s + \tfrac{1}{2}\delta_{kl} A^*_r \Pi_s \Pi_s A_r\},
\end{aligned} \tag{30}$$

where $\gamma^{(8)}$ and $\gamma^{(9)}$ are arbitrary real constants. This is denoted by Y^V and its T by T^V.

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(ii) That containing either g_1 or g_2,

$$-\tfrac{1}{2}\hbar c\frac{g_2 i}{\kappa^V}\{\Gamma_{sk}(\Pi_l^* A_s^* - \Pi_s^* A_l^*) + A_l^* \Pi_s \Gamma_{sk}\} + \tfrac{1}{4}\hbar c\,\delta_{kl}\left\{g_1 A_i^* M_i + \frac{g_2 i}{\kappa^V} A_i^* \Pi_s \Gamma_{st}\right\}. \tag{31}$$

This is denoted by Y^I and its T by T^I.

For the special case in which $\phi = 0$ and the solution for χ_μ^E, χ_μ^N and A_i contains x_r through the factors $\exp(ic_r^E x_r)$, $\exp(ic_r^N x_r)$ and $\exp(ic_r^V x_r)$, it is noted in the same way that after setting all the γ's in (14) zero, T_{kl}^E can be reduced to two parts, one with its T_{k4} equal to $-i\hbar c c_k^E \rho^E$ and the other given by (18). (18) and a similar expression arising from T_{kl}^N combine to give

$$\tfrac{1}{2}\hbar c\frac{g_2 i}{\kappa^V}[\Gamma_{ki}(\Pi_i^* A_l^* - \Pi_l^* A_i^*) + \Gamma_{li}(\Pi_k^* A_i^* - \Pi_i^* A_k^*)] + \tfrac{1}{2}\hbar c g_1(-A_l^* M_k + A_k^* M_l) + \text{comp. conj.}$$

Noting that

$$-\frac{g_2 i}{\kappa^V}(\Gamma_{li}\Pi_i^* A_k^* + \Gamma_{sk}\Pi_s^* A_l^*) + g_1(-A_l^* M_k + A_k^* M_l) - \frac{1}{\kappa^V}(\Pi_k A_s \Pi_s^* A_l^* - \Pi_l A_s \Pi_s^* A_k^*) + \text{comp. conj.} = 0$$

and that

$$-\frac{1}{\kappa^V}(c_4^V A_s A_s^* - c_s^V A_s A_4^*) + \frac{g_2 i}{\kappa^V}\Gamma_{4i} A_i^* + \text{comp. conj.} = -i\rho^V,$$

where ρ^V is the probability density of the vector mesons, one obtains finally for any value of $\gamma^{(8)}$ and $\gamma^{(9)}$

$$T_{k4} = -i\hbar c(c_k^V \rho^V + c_k^E \rho^E + c_k^N \rho^N) - \tfrac{1}{2}\hbar c \triangle \delta_{k4}, \tag{32}$$

where
$$\triangle = -g_1 A_i^* M_i - \frac{g_2 i}{\kappa} A_i^* \Pi_s \Gamma_{st} + \text{comp. conj.}, \tag{33·1}$$

which, expressed in the wave functions of electrons (or neutrinos) for the present special case, reduces exactly to the expression (20·2) or (20·3), and expressed in the wave functions of the vector mesons, to

$$-\frac{1}{\kappa^V}[(\kappa^V)^2 A_i A_i^* + c_i^V c_i^V A_j A_j^* - c_i^V c_j^V A_i A_j^*] + \text{comp. conj.} \tag{33·2}$$

Before concluding this section, consider the relation between the Hamiltonian of matter H and T_{44}. Let P_i be the canonical variable to A_i ($i = 1, 2, 3$) and P_i^* that to A_i^* ($i = 1, 2, 3$), and let

$$H' = H + \frac{e}{\hbar}\phi_4 \sum_{i=1}^{3}(P_i A_i - P_i^* A_i^*) + ie\phi_4(\chi^{E\lambda}\chi^{E\dot\lambda} + \psi^{E\mu}\psi^{E\dot\mu}). \tag{34}$$

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Then $$\left.\begin{aligned} \Pi_4 A_i &= -\frac{i}{\hbar c}\left[\int H'dV, A_i\right], \quad \Pi_4^* P_i = \frac{i}{\hbar c}\left[\int H'dV, P_i\right], \quad (i = 1, 2, 3), \\ \Pi_4^* A_i^* &= \frac{i}{\hbar c}\left[\int H'dV, A_i^*\right], \quad \Pi_4 P_i^* = -\frac{i}{\hbar c}\left[\int H'dV., P_i^*\right], \end{aligned}\right\} \tag{35}$$

equivalent to the wave equations for A_i, A_i^*, P_i and P_i^* ($i = 1, 2, 3$), and

$$\Pi_4 \psi^{E\mu} = -\frac{i}{\hbar c}\left[\int H'dV, \psi^{E\mu}\right],$$

etc., equivalent to the wave equations for $\psi^{E\mu}$, etc. It is satisfactory to see that if to the above T_{kl} is added $-\delta_{kl}$ times

$$L^E + L^N - \hbar c\triangle - \frac{\hbar c}{2\kappa^V}\{[(\kappa^V)^2 A_i A_i^* + A_i^*(\Pi_j \Pi_j A_i - \Pi_j \Pi_i A_j \\ - g_2 i \Pi_j \Gamma_{ji} - g_1 \kappa^V M_i)] + \text{comp. conj.}\}, \tag{36}$$

which vanishes always, by partial integrations over the space coordinates

$$\int H'dV = \int T_{44} dV$$

is obtained. Thus the Hamiltonian is again the total energy.

It is perhaps needless to add that the above calculations and conclusions can be extended to include protons and neutrons. So long as there is no direct interaction between the heavy and the light particles, the extension can be easily made. For this reason, it will be left out here.

PART II. PARTICLES OF SPIN 2 OR 3/2

Calculations of a general impulse-energy tensor T_{kl} for mesons and electrons are here extended to particles of spin 2 or 3/2, whose wave equations were given by Fierz & Pauli. Results are generally similar, though expressions for T_{kl} are now much more complicated.

Introduction

In part I of this paper, it is pointed out that for vector or scalar mesons and electrons, there exists a real and symmetrical tensor T_{kl} satisfying

$$\delta_l T_{kl} + f_k = 0, \tag{1}$$

where f_k is the four-force and δ_l denotes differentiation to x_l, (x_l being the vector x, y, z, ict). On interpreting T_{k4} ($k = 1, 2, 3$) as $-ic$ times the momentum density of matter, T_{44} as the energy density of matter, the conservation law for the total energy and momentum of the matter and the electromagnetic field in any region of space is obtained, account being taken for their flow across the surface bounding the region. Needless to say, in constructing expressions for the energy or momentum of matter, such a conservation must be constantly kept in sight and thus the correct construction must start from determining T_{kl} from (1), obtaining the density expressions from T_{kl} and performing a final integration over space.

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In this part of the paper the investigation of T_{kl} will be extended to particles of spin 2 and 3/2. Expressions for T_{kl} for such particles in a force-free field have been given by Fierz (1939), and thus here attention will be concentrated on T_{kl} for particles in a general electromagnetic field. Throughout the entire calculations, the wave equations given by Fierz & Pauli (1939) for such particles will be employed.

I. Particles of spin 2

Following Fierz, let the wave functions be a symmetrical tensor A_{kl} together with a scalar C and obtain the wave equations by varying a Lagrangian $L^{(2)}$. They are

$$2\kappa^2 A_{ik} + 2\Pi_l \Pi_l A_{ik} - 2\{\Pi_i \Pi_s A_{sk} + \Pi_k \Pi_s A_{si}\} - \frac{ei}{\hbar c}(\phi_{li} A_{lk} + \phi_{lk} A_{li})$$
$$+ \tfrac{1}{2}(\Pi_i \Pi_k + \Pi_k \Pi_i) C - \tfrac{1}{2}\delta_{ik}(-2\Pi_r \Pi_s A_{rs} + \tfrac{1}{2}\Pi_l \Pi_l C) = 0, \quad (2\cdot1)$$

$$2\kappa^2 C + \Pi_l \Pi_l C - \tfrac{4}{3}\Pi_r \Pi_s A_{rs} = 0, \quad (2\cdot2)$$

where ϕ_{li} denotes $\delta_l \phi_i - \delta_i \phi_l$ and all the other symbols have the same meaning as in part I. As before, the left-hand sides of the above equations will be called $h(A_{ik})$ and $h(C)$ and those of the conjugate equations $h(A^*_{ik})$ and $h(C^*)$.

Fierz has not given an expression for the current, but one can be obtained from varying the Lagrangian to ϕ_l. Varying the combined Lagrangian of field and matter subject to $\delta_l \phi_l = 0$, one finds

$$\frac{1}{4\pi}\delta_i \delta_i \phi_l - \delta_l \Phi = -\frac{1}{c} j_l,$$

where Φ is a scalar to be determined later by $\delta_l \phi_l = 0$, and j_l the vector

$$c\frac{\partial L}{\partial \phi_l} = \frac{ec}{\kappa}\{A^*_{ik}\Pi_l A_{ik} - A^*_{ik}\Pi_i A_{lk} - A^*_{lk}\Pi_s A_{sk}$$
$$+ \tfrac{1}{4}A^*_{kl}\Pi_k C + \tfrac{1}{2}C^* \Pi_k A_{kl} - \tfrac{3}{8}C^* \Pi_l C + \text{comp. conj.}\}. \quad (3)$$

It is easy to verify from the wave equations that $\delta_l j_l = 0$, thus we take it as the current vector. At the same time, since $\delta_l j_l = 0$, then $\delta_l \delta_l \Phi = 0$ or $\Phi = 0$, and thus

$$\delta_i \delta_i \phi_l = -\frac{4\pi}{c} j_l, \quad (4)$$

as usual. Incidentally, it may be pointed out that the condition $\delta_l j_l = 0$ is not sufficient to determine j_l. An example is provided by the vector

$$A^*_{ik}\Pi_i A_{lk} - A^*_{lk}\Pi_i A_{ik} + \text{comp. conj.},$$

whose divergence is zero under all conditions.

As before the method is to substitute the right side of (3) for j_l into

$$\delta_l T_{kl} + \frac{1}{c}(\delta_k \phi_l - \delta_l \phi_k) j_l = 0$$

and solve for T_{kl}. The procedure of finding T_{kl} is exactly the same as before, and its

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description will therefore be omitted here. The equations playing the part of (9), (10) and (11) in part I are

$$\left.\begin{aligned}
\frac{1}{4}\frac{\hbar c}{\kappa}\{-iA_{il}\Pi_k^*\,h(A_{il}^*)+i(\Pi_k^*A_{il}^*)\,h(A_{il})\} &= 0,\\
-\frac{1}{2}\frac{\hbar c}{\kappa}\{-iA_{il}\Pi_l^*\,h(A_{ik}^*)+i(\Pi_l^*A_{ik}^*)\,h(A_{il})\} &= 0,\\
\frac{1}{2}\frac{\hbar c}{\kappa}\{-iA_{ik}\Pi_l^*\,h(A_{il}^*)+i(\Pi_l^*A_{il}^*)\,h(A_{ik})\} &= 0,\\
0\{-iA_{ik}\Pi_i^*\,h(C^*)+i(\Pi_i^*C^*)\,h(A_{ik})\} &= 0,\\
0\{-iC\Pi_l^*\,h(A_{lk}^*)+i(\Pi_l^*A_{lk}^*)\,h(C)\} &= 0,\\
-\frac{3}{16}\frac{\hbar c}{\kappa}\{-iC\Pi_k^*\,h(C^*)+i(\Pi_k^*C^*)\,h(C)\} &= 0,
\end{aligned}\right\} \tag{5}$$

and their conjugates. The result for T_{kl} is that, if it is written as

$$Y_{kl}+Y_{lk}+Y_{kl}^*+Y_{lk}^*,$$

Y_{kl} is given by

$$\gamma^{(1)}Y_{kl}^{(1)}+\gamma^{(2)}Y_{kl}^{(2)}+\ldots+\gamma^{(5)}Y^{(5)}+Y^{(6)}, \tag{6}$$

where $\gamma^{(1)}$, $\gamma^{(2)}$, ... are arbitrary real constants and

$$\begin{aligned}
Y_{kl}^{(1)} = {}& \Pi_l^*A_{lr}^*\Pi_rA_{kl}+\Pi_s^*A_{ls}^*\Pi_rA_{kr}-2\Pi_r^*A_{rs}^*\Pi_sA_{kl}\\
&-A_{kr}^*\Pi_s\Pi_rA_{ls}-A_{kr}^*\Pi_r\Pi_sA_{ls}+A_{rs}^*\Pi_r\Pi_sA_{kl}+A_{kl}^*\Pi_r\Pi_sA_{rs},
\end{aligned}\tag{7·1}$$

$$\begin{aligned}
Y_{kl}^{(2)} = {}& \Pi_l^*A_{lr}^*\Pi_lA_{kr}-\Pi_l^*A_{lr}^*\Pi_kA_{lr}-\Pi_s^*A_{is}^*\Pi_lA_{ik}\\
&-A_{kr}^*\Pi_s\Pi_sA_{lr}+A_{kr}^*\Pi_l\Pi_sA_{rs}+A_{rs}^*\Pi_l\Pi_sA_{kr}\\
&+\tfrac{1}{2}\delta_{kl}\Pi_s^*A_{sl}^*\Pi_rA_{rl}+\tfrac{1}{2}\delta_{kl}\Pi_s^*A_{rl}^*\Pi_rA_{sl}\\
&-\tfrac{1}{2}\delta_{kl}A_{rs}^*\Pi_r\Pi_lA_{sl}-\tfrac{1}{2}\delta_{kl}A_{rs}^*\Pi_l\Pi_rA_{sl},
\end{aligned}\tag{7·2}$$

$$Y_{kl}^{(3)} = -\Pi_l^*A_{tr}^*\Pi_kA_{tr}+A_{rs}^*\Pi_k\Pi_lA_{rs}+\delta_{kl}\Pi_r^*A_{st}^*\Pi_rA_{st}-\delta_{kl}A_{rs}^*\Pi_t\Pi_tA_{rs}, \tag{7·3}$$

$$Y_{kl}^{(4)} = \Pi_k^*C^*\Pi_lC-C^*\Pi_k\Pi_lC-\delta_{kl}\Pi_s^*C^*\Pi_sC+\delta_{kl}C^*\Pi_s\Pi_sC, \tag{7·4}$$

$$\begin{aligned}
Y_{kl}^{(5)} = {}& \Pi_s^*C^*\Pi_sA_{kl}-\Pi_l^*C^*\Pi_sA_{ks}-\Pi_s^*C^*\Pi_lA_{ks}+C^*\Pi_k\Pi_rA_{lr}\\
&-\tfrac{1}{2}C^*\Pi_r\Pi_rA_{kl}+A_{rl}^*\Pi_k\Pi_rC-\tfrac{1}{2}A_{kl}^*\Pi_s\Pi_sC\\
&+\delta_{kl}\Pi_s^*C^*\Pi_rA_{rs}-\tfrac{1}{2}\delta_{kl}A_{rs}^*\Pi_r\Pi_sC-\tfrac{1}{2}\delta_{kl}C^*\Pi_r\Pi_sA_{rs},
\end{aligned}\tag{7·5}$$

$$\begin{aligned}
-\frac{2\kappa}{\hbar c}Y_{kl}^{(6)} = {}& \tfrac{1}{2}\Pi_k^*A_{tr}^*\Pi_lA_{tr}-\Pi_r^*A_{kr}^*\Pi_sA_{ls}-2\Pi_k^*A_{tr}^*\Pi_lA_{lr}\\
&+2\Pi_i^*A_{ik}^*\Pi_sA_{is}+\Pi_l^*A_{kr}^*\Pi_rA_{ll}+\tfrac{1}{2}A_{rs}^*\Pi_k\Pi_lA_{rs}\\
&+A_{rs}^*\Pi_r\Pi_sA_{kl}-A_{kl}^*\Pi_r\Pi_sA_{rs}+A_{rs}^*\Pi_l\Pi_rA_{ks}\\
&-3A_{rs}^*\Pi_r\Pi_lA_{ks}-\delta_{kl}\Pi_r^*A_{rt}^*\Pi_sA_{st}+\delta_{kl}A_{rs}^*\Pi_r\Pi_tA_{st}\\
&+\Pi_l^*C^*\Pi_sA_{ks}-\tfrac{1}{2}\Pi_s^*C^*\Pi_sA_{kl}+\tfrac{1}{2}A_{kr}^*\Pi_l\Pi_rC\\
&-\tfrac{1}{2}A_{kr}^*\Pi_r\Pi_lC+\tfrac{1}{2}A_{kl}^*\Pi_r\Pi_rC+\tfrac{1}{4}\delta_{kl}C^*\Pi_r\Pi_sA_{rs}\\
&-\tfrac{1}{4}\delta_{kl}A_{rs}^*\Pi_r\Pi_sC-\tfrac{3}{16}\Pi_k^*C^*\Pi_lC-\tfrac{3}{16}C^*\Pi_k\Pi_lC.
\end{aligned}\tag{7·6}$$

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Examining $-e^{-1}j_4 c_k \hbar$ and T_{k4} for the special case in which $\phi = 0$, and A_{ik} and C contain x_r through the factor $\exp(ic_r x_r)$, it is satisfactory to find that they are equal for any values of the constants γ in (6). In proving the equality, the equations

$$C = C^* = c_r A_{rs} = c_r A^*_{rs} = 0,$$

are utilized, which can be proved from the wave equations for this special case.

With regard to the Hamiltonian of such particles and T_{44}, no attempt will be made to compare them in the present paper, for a theory of second quantization for such particles in an electromagnetic field is still lacking. For second quantization of such particles in a force-free field, reference should be made to Fierz's paper (1939).

II. Particles of spin 3/2

Following Fierz & Pauli (1939) let the wave functions be c_α, $d_{\dot\beta}$; $a^{\alpha\beta}_{\dot\gamma}$ and $b^{\gamma}_{\dot\alpha\dot\beta}$, of which the last two are symmetrical in α, β and $\dot\alpha, \dot\beta$ respectively. The conjugate wave functions are $c^*_{\dot\alpha}$, d^*_{β}, $a^{*\dot\alpha\dot\beta}_{\gamma}$ and $b^{*\dot\gamma}_{\alpha\beta}$. Whenever they have their suffices explicitly written out, the *'s are omitted, since this will not cause confusion.

The wave equations can be deduced as usual from varying a Lagrangian. In the present case, the Lagrangian $L^{(3/2)}$ is

$$\begin{aligned} -\hbar c\{\kappa[a^{\gamma}_{\dot\alpha\dot\beta} b^{\dot\alpha\dot\beta}_{\gamma} + b^{\alpha\beta}_{\dot\gamma} a^{\dot\gamma}_{\alpha\beta}] - [a^{\gamma}_{\dot\alpha\dot\beta} \Pi^{\dot\beta\rho} a^{\dot\alpha}_{\gamma\rho} + b^{\alpha\beta}_{\dot\gamma} \Pi_{\alpha\dot\rho} b^{\dot\gamma\dot\rho}_{\beta}] + [a^{\gamma}_{\dot\alpha\dot\beta} \Pi^{\dot\beta}_{\gamma} d^{\dot\alpha} + b^{\alpha\beta}_{\dot\gamma} \Pi^{\dot\gamma}_{\beta} c_\alpha + \text{c.c.}] \\ -3(d^{\alpha} \Pi_{\alpha\dot\beta} d^{\dot\beta} + c_{\dot\alpha} \Pi^{\dot\alpha\beta} c_\beta) - 6\kappa(d^\alpha c_\alpha + d^{\dot\alpha} c_{\dot\alpha})\}. \end{aligned} \tag{8}$$

The wave equations are therefore

$$2\kappa b^{\dot\alpha\dot\beta}_{\gamma} - (\Pi^{\dot\beta\rho} a^{\dot\alpha}_{\gamma\rho} + \Pi^{\dot\alpha\rho} a^{\dot\beta}_{\gamma\rho}) + (\Pi^{\dot\beta}_{\gamma} d^{\dot\alpha} + \Pi^{\dot\alpha}_{\gamma} d^{\dot\beta}) = 0, \tag{9·1}$$

$$2\kappa a^{\dot\gamma}_{\alpha\beta} - (\Pi_{\alpha\dot\rho} b^{\dot\gamma\dot\rho}_{\beta} + \Pi_{\beta\dot\rho} b^{\dot\gamma\dot\rho}_{\alpha}) + (\Pi^{\dot\gamma}_{\beta} c_\alpha + \Pi^{\dot\gamma}_{\alpha} c_\beta) = 0, \tag{9·2}$$

$$2\kappa c_\alpha - \tfrac{1}{3}\Pi^{\beta}_{\dot\gamma} a^{\dot\gamma}_{\alpha\beta} + \Pi_{\alpha\dot\beta} d^{\dot\beta} = 0, \tag{9·3}$$

$$2\kappa d^{\dot\alpha} - \tfrac{1}{3}\Pi^{\gamma}_{\dot\beta} b^{\dot\alpha\dot\beta}_{\gamma} + \Pi^{\dot\alpha\beta} c_\beta = 0. \tag{9·4}$$

According to the way the notation 'h' has been used (see part I), the conjugate equations to the above are simply

$$h(b^{\alpha\beta}_{\dot\gamma}) = h(a^{\gamma}_{\dot\alpha\dot\beta}) = h(c_{\dot\alpha}) = h(d^\alpha) = 0.$$

At this juncture it may be pointed out that the numbers -3 and -6 in the Lagrangian were given by Fierz & Pauli as 3 and 6 which are wrong. Using the numbers 3 and 6 instead of -3 and -6, the coefficients in (9·3) and (9·4) are all positive, thus $c = d = 0$ for the case $\phi = 0$ cannot be deduced.

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To get the current, vary the combined Lagrangian of the particles and the field to $\phi_{\mu\dot{\nu}}$ subject to $\delta^{\mu\dot{\nu}}\phi_{\mu\dot{\nu}} = 0$. In this way

$$\frac{1}{16\pi}\delta_{\rho\dot{\theta}}\delta^{\rho\dot{\theta}}\phi_{\mu\dot{\nu}} - \delta_{\mu\dot{\nu}}\Phi = \frac{1}{2c}j_{\mu\dot{\nu}}$$

is obtained, where Φ is a scalar to be determined by $\delta^{\mu\dot{\nu}}\phi_{\mu\dot{\nu}} = 0$ and $j_{\mu\dot{\nu}}$ is the spinor

$$-2c\frac{\partial L^{(3/2)}}{\partial\phi^{\mu\dot{\nu}}} = 2ce\{a^{\gamma}_{\ \dot{\rho}\dot{\nu}}a^{\dot{\rho}}_{\ \mu\gamma} + b^{\gamma}_{\ \dot{\rho}\dot{\nu}}b^{\dot{\rho}}_{\ \mu\gamma} - a^{\dot{\rho}}_{\ \mu\dot{\nu}}d_{\dot{\rho}} - a^{\rho}_{\ \mu\dot{\nu}}d_{\rho} + b^{\dot{\rho}}_{\ \mu\dot{\nu}}c_{\dot{\rho}} + b^{\rho}_{\ \mu\dot{\nu}}c_{\rho} + 3d_{\mu}d_{\dot{\nu}} + 3c_{\mu}c_{\dot{\nu}}\}. \tag{10}$$

It is easy to verify from the wave equations that $\delta^{\mu\dot{\nu}}j_{\mu\dot{\nu}} = 0$, so that $j_{\mu\dot{\nu}}$ can be taken as the current spinor. At the same time from $\delta^{\mu\dot{\nu}}j_{\mu\dot{\nu}} = 0$ the equation $\delta_{\mu\dot{\nu}}\delta^{\mu\dot{\nu}}\Phi = 0$ or $\Phi = 0$ is derived, and thus

$$-\tfrac{1}{2}\delta_{\rho\dot{\theta}}\delta^{\rho\dot{\theta}}\phi_{\mu\dot{\nu}} = -\frac{4\pi}{c}j_{\mu\dot{\nu}}, \tag{11}$$

as usual.

The equation for the spinor $t_{\dot{n}\dot{s}m\lambda}$ of T_{kl} is

$$\delta^{\lambda\dot{s}}t_{\dot{n}\dot{s}m\lambda} + \frac{1}{c}(\delta_{m\dot{n}}\phi_{\lambda\dot{s}} - \delta_{\lambda\dot{s}}\phi_{m\dot{n}})j^{\lambda\dot{s}} = 0, \tag{12}$$

and the procedure of solving for $t_{\dot{n}\dot{s}m\lambda}$ is exactly the same as that for T_{kl} in the previous section or that in part I. The equations playing the part of (9), (10) and (11) in part I consist of

(i) equations of the type

$$\left.\begin{aligned} a^{\dot{\gamma}}_{\ \alpha\beta}\Pi^{*}_{\lambda\dot{s}}h(b^{\phi\psi}_{\dot{\theta}}) - (\Pi^{*}_{\lambda\dot{s}}b^{\phi\psi}_{\dot{\theta}})\,h(a^{\dot{\gamma}}_{\ \alpha\beta}) &= 0,\\ b^{\dot{\gamma}}_{\alpha\beta}\Pi_{\lambda\dot{s}}h(a^{\phi\psi}_{\dot{\theta}}) - (\Pi_{\lambda\dot{s}}a^{\phi\psi}_{\dot{\theta}})\,h(b^{\dot{\gamma}}_{\ \alpha\beta}) &= 0, \end{aligned}\right\} \tag{13}$$

(ii) equations of the type

$$\left.\begin{aligned} d^{\dot{\alpha}}\Pi^{*}_{\lambda\dot{s}}h(b^{\phi\psi}_{\dot{\theta}}) - (\Pi^{*}_{\lambda\dot{s}}b^{\phi\psi}_{\dot{\theta}})\,h(d^{\dot{\alpha}}) &= 0,\\ -c^{\dot{\alpha}}\Pi_{\lambda\dot{s}}h(a^{\phi\psi}_{\dot{\theta}}) + (\Pi_{\lambda\dot{s}}a^{\phi\psi}_{\dot{\theta}})\,h(c^{\dot{\alpha}}) &= 0, \end{aligned}\right\} \tag{14}$$

(iii) equations of the type

$$\left.\begin{aligned} a^{\phi\psi}_{\dot{\theta}}\Pi^{*}_{\lambda\dot{s}}h(c^{\dot{\alpha}}) - (\Pi^{*}_{\lambda\dot{s}}c^{\dot{\alpha}})\,h(a^{\phi\psi}_{\dot{\theta}}) &= 0,\\ -b^{\phi\psi}_{\dot{\theta}}\Pi_{\lambda\dot{s}}h(d^{\dot{\alpha}}) + (\Pi_{\lambda\dot{s}}d^{\dot{\alpha}})\,h(b^{\phi\psi}_{\dot{\theta}}) &= 0, \end{aligned}\right\} \tag{15}$$

(iv) equations of the type

$$\left.\begin{aligned} d_{\dot{\theta}}\Pi^{*}_{\lambda\dot{s}}h(c_{\dot{\alpha}}) - (\Pi^{*}_{\lambda\dot{s}}c_{\dot{\alpha}})\,h(d_{\dot{\theta}}) &= 0,\\ c_{\dot{\theta}}\Pi_{\lambda\dot{s}}h(d_{\dot{\alpha}}) - (\Pi_{\lambda\dot{s}}d_{\dot{\alpha}})\,h(c_{\dot{\theta}}) &= 0, \end{aligned}\right\} \tag{16}$$

and their conjugates. By putting one dotted and one undotted suffix to be $\dot{n}$ and m, raising or lowering some of the suffices and performing some contractions, each of their left-hand sides can be made the $\dot{n}m$ component of a spinor. In this way, (13) and their conjugates provide sixteen independent equations,† (14) and their conjugates eight equations, (15) and the conjugates eight equations, and (16) and their conjugates eight equations. Needless to say, there are equations obtained from the above by replacing Π, Π^* by $-i\delta$, $i\delta$, and also equations with all terms containing $\delta_{\mu\dot{\nu}}\phi^{\mu\dot{\nu}}$ as a factor. Multiplying them by unknown quantities λ_1, λ_2, ..., adding them to the equation obtained by substituting in (12) an assumed form for $t_{\dot{n}\dot{s}m\lambda}$, setting the coefficients of independent terms of the resulting equation to be zero and solving for the λ's and the constants in the assumed form of t, the required result is obtained.

Putting $t_{\dot{n}\dot{s}m\lambda}$ as

$$y_{\dot{n}\dot{s}m\lambda}+y_{\dot{s}\dot{n}\lambda m}+y^*_{\dot{n}\dot{s}m\lambda}+y^*_{\dot{s}\dot{n}\lambda m}, \tag{17}$$

the result is

$$y_{\dot{n}\dot{s}m\lambda}=\sum_{i=1}^{24}\gamma^{(i)}y^{(i)}_{\dot{n}\dot{s}m\lambda}+y^{(25)}_{\dot{n}\dot{s}m\lambda}, \tag{18}$$

where $\gamma^{(1)}$, $\gamma^{(2)}$, ... are arbitrary real constants and

$$y^{(1)}=\quad a^{\alpha}_{\lambda\dot{s}}\Pi^*_{\alpha\dot{\beta}}a^{\dot{\beta}}_{m\dot{n}}-a^{\alpha}_{m\dot{s}}\Pi^*_{\alpha\dot{\beta}}a^{\dot{\beta}}_{\lambda\dot{n}}+\tfrac{1}{2}\epsilon_{m\lambda}\epsilon_{\dot{n}\dot{s}}a^{\alpha\rho}_{\dot{\theta}}\Pi^*_{\alpha\dot{\beta}}a^{\dot{\beta}\dot{\theta}}_{\rho}, \tag{19·1}$$

$$y^{(2)}=-\epsilon_{m\lambda}a^{\alpha\rho}_{\dot{s}}\Pi^*_{\alpha\dot{\beta}}a^{\dot{\beta}}_{\dot{n}\rho}+\tfrac{1}{2}\epsilon_{m\lambda}\epsilon_{\dot{n}\dot{s}}a^{\alpha\rho}_{\dot{\theta}}\Pi^*_{\alpha\dot{\beta}}a^{\dot{\beta}\dot{\theta}}_{\rho}, \tag{19·2}$$

$$y^{(3)}=-\epsilon_{\dot{n}\dot{s}}a^{\alpha}_{m\dot{\theta}}\Pi^*_{\alpha\dot{\beta}}a^{\dot{\beta}\dot{\theta}}_{\lambda}+\tfrac{1}{2}\epsilon_{m\lambda}\epsilon_{\dot{n}\dot{s}}a^{\alpha\rho}_{\dot{\theta}}\Pi^*_{\alpha\dot{\beta}}a^{\dot{\beta}\dot{\theta}}_{\rho}, \tag{19·3}$$

$$y^{(4)}=\quad d_{\dot{n}}\Pi^*_{m\dot{s}}d_{\lambda}-d_{\dot{n}}\Pi^*_{\lambda\dot{s}}d_m+\tfrac{1}{2}\epsilon_{m\lambda}\epsilon_{\dot{n}\dot{s}}d^{\dot{\beta}}\Pi^*_{\alpha\dot{\beta}}d^{\alpha}, \tag{19·4}$$

$$y^{(5)}=-\epsilon_{m\lambda}d_{\dot{s}}\Pi^{*\alpha}_{\dot{n}}d_{\alpha}+\tfrac{1}{2}\epsilon_{m\lambda}\epsilon_{\dot{n}\dot{s}}d^{\dot{\beta}}\Pi^*_{\alpha\dot{\beta}}d^{\alpha}, \tag{19·5}$$

$$y^{(6)}=-\epsilon_{\dot{n}\dot{s}}d_{\dot{\rho}}\Pi^{*\dot{\rho}}_{m}d_{\lambda}+\tfrac{1}{2}\epsilon_{m\lambda}\epsilon_{\dot{n}\dot{s}}d^{\dot{\beta}}\Pi^*_{\alpha\dot{\beta}}d^{\alpha}, \tag{19·6}$$

$$y^{(7)}=\quad d_{\dot{n}}\Pi^*_{\dot{\beta}\lambda}a^{\dot{\beta}}_{\dot{s}m}-d_{\dot{n}}\Pi^*_{m\dot{\beta}}a^{\dot{\beta}}_{\dot{s}\lambda}+\tfrac{1}{2}\epsilon_{m\lambda}\epsilon_{\dot{n}\dot{s}}d_{\dot{\theta}}\Pi^*_{\rho\dot{\beta}}a^{\rho\dot{\theta}\dot{\beta}}, \tag{19·7}$$

$$y^{(8)}=\quad \epsilon_{m\lambda}d_{\dot{n}}\Pi^*_{\alpha\dot{\beta}}a^{\alpha\dot{\beta}}_{\dot{s}}+\tfrac{1}{2}\epsilon_{m\lambda}\epsilon_{\dot{n}\dot{s}}d_{\dot{\theta}}\Pi^*_{\rho\dot{\beta}}a^{\rho\dot{\theta}\dot{\beta}}, \tag{19·8}$$

$$y^{(9)}=\quad \epsilon_{\dot{n}\dot{s}}d_{\dot{\theta}}\Pi^*_{m\dot{\beta}}a^{\dot{\theta}\dot{\beta}}_{\lambda}+\tfrac{1}{2}\epsilon_{m\lambda}\epsilon_{\dot{n}\dot{s}}d_{\dot{\theta}}\Pi^*_{\rho\dot{\beta}}a^{\rho\dot{\theta}\dot{\beta}}, \tag{19·9}$$

$$y^{(10)}=\quad a^{\alpha}_{\lambda\dot{n}}\Pi^*_{\alpha\dot{s}}d_m-a^{\alpha}_{\lambda\dot{s}}\Pi^*_{\alpha\dot{n}}d_m+\tfrac{1}{2}\epsilon_{m\lambda}\epsilon_{\dot{n}\dot{s}}a^{\rho\alpha\dot{\beta}}\Pi^*_{\alpha\dot{\beta}}d_{\rho}, \tag{19·10}$$

$$y^{(11)}=\quad \epsilon_{m\lambda}a^{\alpha\rho}_{\dot{s}}\Pi^*_{\dot{n}\alpha}d_{\rho}+\tfrac{1}{2}\epsilon_{m\lambda}\epsilon_{\dot{n}\dot{s}}a^{\rho\alpha\dot{\beta}}\Pi^*_{\alpha\dot{\beta}}d_{\rho}, \tag{19·11}$$

$$y^{(12)}=\quad \epsilon_{\dot{n}\dot{s}}a^{\alpha\dot{\beta}}_{\lambda}\Pi^*_{\alpha\dot{\beta}}d_m+\tfrac{1}{2}\epsilon_{m\lambda}\epsilon_{\dot{n}\dot{s}}a^{\rho\alpha\dot{\beta}}\Pi^*_{\alpha\dot{\beta}}d_{\rho}, \tag{19·12}$$

† By independent equations are meant those which do not have a linear dependence between them owing to relations of the type

$$a_{\lambda}b^{\lambda}=-a^{\lambda}b_{\lambda},\quad a_{\lambda}b^{\lambda}c_m+a_m b_{\lambda}c^{\lambda}+a^{\lambda}b_m c_{\lambda}=0.$$

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$$
\begin{aligned}
y^{(25)} = \quad & \hbar c\{-\tfrac{1}{2}a^{\dot\beta}_{m\lambda}\Pi^*_{\alpha\dot\beta}a^{\alpha}_{\dot n\dot s}-\tfrac{1}{2}a^{\alpha\dot\beta}_{\lambda}\Pi^*_{\alpha\dot\beta}a_{m\dot n\dot s} \\
& +\tfrac{4}{3}a_{m\lambda\dot s}\Pi^*_{\alpha\dot\beta}a^{\alpha\dot\beta}_{\dot n}-\tfrac{1}{2}a^{\alpha}_{m\dot s}\Pi^*_{\alpha\dot\beta}a^{\dot\beta}_{\dot n\lambda}-\tfrac{1}{2}d_{\dot s}\Pi^*_{\lambda\dot n}d_m \\
& -\tfrac{1}{2}\epsilon_{m\lambda}\epsilon_{\dot n\dot s}d^{\dot\beta}\Pi^*_{\alpha\dot\beta}d^{\alpha}+\tfrac{1}{2}d^{\dot\beta}\Pi^*_{\lambda\dot\beta}a_{m\dot n\dot s} \\
& +\tfrac{3}{2}d_{\dot n}\Pi^*_{\lambda\dot\beta}a^{\dot\beta}_{\dot s m}+\epsilon_{\dot n\dot s}\epsilon_{m\lambda}d_{\dot\theta}\Pi^*_{\rho\dot\beta}a^{\rho\dot\theta\dot\beta} \\
& -2a_{m\dot n\lambda}\Pi^*_{\alpha\dot s}d^{\alpha}+\tfrac{1}{2}a^{\alpha}_{\lambda\dot n}\Pi^*_{\alpha\dot s}d_m+\tfrac{1}{4}\epsilon_{\dot n\dot s}\epsilon_{m\lambda}a^{\alpha\dot\beta}_{\rho}\Pi^*_{\alpha\dot\beta}d^{\rho}\}+\dots,
\end{aligned}
\tag{20}
$$

$y^{(13)}, y^{(14)}, \dots$ are $y^{(1)}, y^{(2)}, \dots$ with a replaced by b, d by $-c$, and Π^* by Π, and the dots in the right-hand side of (20) denote the preceding terms treated with the same replacement.

In the case of no field, compare $-e^{-1}j_{s\dot s}c_{m\dot n}\hbar$ and $t_{\dot n\dot s m s}$, where $c_{m\dot n}$ is the spinor appearing in the solution $\exp\left(-\frac{i}{2}c_{m\dot n}x^{m\dot n}\right)$ for the unstarred wave functions. For this special case

$$c_{\alpha\dot\beta}a^{\alpha\dot\beta}_{\rho} = c_{\alpha\dot\beta}b^{\alpha\dot\beta}_{\rho} = c_{\rho} = d_{\rho} = 0. \tag{21}$$

So $-e^{-1}j_{s\dot s}c_{m\dot n}\hbar$ is simply

$$-2c\hbar c_{m\dot n}(a^{\dot\beta}_{s\alpha}a^{\alpha}_{\dot s\dot\beta}+b^{\dot\beta}_{s\alpha}b^{\alpha}_{\dot s\dot\beta}). \tag{22}$$

If all the γ's in (18) are zero, then

$$t_{\dot n\dot s m s} = -\hbar c\{2c_{\alpha\dot\beta}a^{\dot\beta}_{ms}a^{\alpha}_{\dot n\dot s}+c_{\alpha\dot\beta}a^{\alpha}_{m\dot s}a^{\dot\beta}_{\dot n s}+c_{\alpha\dot\beta}a^{\alpha}_{\dot n s}a^{\dot\beta}_{m\dot s}\}, \tag{23}$$

plus exactly the same terms with a replaced by b. Making use of (21), one has

$$c_{\alpha\dot\beta}a^{\dot\beta}_{ms}a^{\alpha}_{\dot n\dot s} = -c_{m\dot\beta}a^{\alpha\dot\beta}_{s}a_{\dot n\dot s\alpha} = c_{m\dot n}a^{\alpha}_{s\dot\beta}a^{\dot\beta}_{\alpha\dot s}+c^{\dot\beta}_{m}a^{\alpha}_{\dot n s}a_{\dot\beta\dot s\alpha},$$

$$c_{\alpha\dot\beta}a^{\dot\beta}_{ms}a^{\alpha}_{\dot n\dot s} = -c_{\alpha\dot n}a_{\dot\beta ms}a^{\alpha\dot\beta}_{\dot s} = c_{m\dot n}a^{\alpha}_{s\dot\beta}a^{\dot\beta}_{\alpha\dot s}+c^{\alpha}_{\dot n}a_{\dot\beta\alpha\dot s}a^{\dot\beta}_{m s}.$$

Similarly $\quad c_{\alpha\dot\beta}a^{\alpha}_{m\dot s}a^{\dot\beta}_{\dot n s} = -c_{s\dot\beta}a_{m\dot s\alpha}a^{\alpha\dot\beta}_{\dot n} = \tfrac{1}{2}a^{\alpha}_{m\dot s}(c_{\alpha\dot\beta}a^{\dot\beta}_{\dot n s}+c_{s\dot\beta}a^{\dot\beta}_{\dot n\alpha}) = -\kappa a^{\alpha}_{m\dot s}b_{\alpha\dot n s},$

the last equality being the result of applying the wave equation. In the same way it is found that

$$
\begin{aligned}
c_{\alpha\dot\beta}a^{\alpha}_{s\dot n}a^{\dot\beta}_{\dot s m} &= -\kappa a^{\dot\beta}_{m\dot s}b_{\dot n\dot\beta s}, \\
c^{\dot\beta}_{m}a^{\alpha}_{s\dot n}a_{\dot\beta\dot s\alpha} &= \quad \kappa a^{\alpha}_{s\dot n}b_{\dot s\alpha m}, \\
c^{\alpha}_{\dot n}a^{\dot\beta}_{m\dot s}a_{\dot\beta\alpha s} &= \quad \kappa a^{\dot\beta}_{m\dot s}b_{\dot n\dot\beta s}.
\end{aligned}
$$

Hence the right-hand side of (23) reduces to

$$-\hbar c\{2c_{m\dot n}a^{\alpha}_{s\dot\beta}a^{\dot\beta}_{\dot s\alpha}-\kappa a^{\alpha}_{m\dot s}b_{\dot n s\alpha}+\kappa a^{\alpha}_{s\dot n}b_{\dot s\alpha m}\}.$$

Similarly, the right-hand side of (23) with a replaced by b reduces to

$$-\hbar c\{2c_{m\dot n}b^{\alpha}_{s\dot\beta}b^{\dot\beta}_{\dot s\alpha}+\kappa b^{\alpha}_{m\dot s}a_{\dot n s\alpha}-\kappa b^{\alpha}_{s\dot n}a_{\dot s\alpha m}\}.$$

Adding, it follows that

$$t_{\dot n\dot s m s} = -2\hbar c\, c_{m\dot n}(a^{\alpha}_{s\dot\beta}a^{\dot\beta}_{\dot s\alpha}+b^{\alpha}_{s\dot\beta}b^{\dot\beta}_{\dot s\alpha}),$$

which is exactly $-e^{-1}j_{s\dot s}c_{m\dot n}\hbar$.

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The comparison of the Hamiltonian of such particles and their T_{44} is not attempted here, for a theory of second quantization for such particles in an electromagnetic field is still lacking. A theory of second quantization for such particles has been given only for the case $\phi = 0$, which can be found in Fierz's paper.

Thus the search for a symmetrical T_{kl} for elementary particles with spin ranging from 0 to 2 has been completed. The only unpleasantness in the result is that T_{kl} is not unique. Also, only the lowest possible derivatives have been considered, i.e. introduction into T_{kl} of higher derivatives of the wave functions than those contemplated here could have been made, but this has not been done.

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