Unitary representations of the Lorentz group

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Certain quantities are introduced which are like tensors in space-time with an infinite enumerable number of components and with an invariant positive definite quadratic form for their squared length. Some of the main properties of these quantities are dealt with, and some applications to quantum mechanics are pointed out.

1. Introduction

Given any group, an important mathematical problem is to get a matrix representation of it, which means to make each element of the group correspond to a matrix in such a way that the matrix corresponding to the product of two elements is the product of the matrices corresponding to the factors. The matrices may be looked upon as linear transformations of the co-ordinates of a vector and then each element of the group corresponds to a linear transformation of a field of vectors. Of special interest are the *unitary* representations, in which the linear transformations leave invariant a positive definite quadratic form in the co-ordinates of a vector.

The Lorentz group is the group of linear transformations of four real variables $\xi_0, \xi_1, \xi_2, \xi_3$, such that $\xi_0^2-\xi_1^2-\xi_2^2-\xi_3^2$ is invariant. The finite representations of this group, i.e. those whose matrices have a finite number of rows and columns, are all well known, and are dealt with by the usual tensor analysis and its extension spinor analysis. None of them is unitary. The group has also some infinite representations which are unitary. These do not seem to have been studied much, in spite of their possible importance for physical applications.

The present paper gives a new method of attack on these representations, which was suggested by Fock's quantum theory of the harmonic oscillator. It leads to a new kind of tensor quantity in space-time, with an infinite number of components and a positive definite expression for its squared length.

2. Three-dimensional theory

This section will be devoted to some preliminary work applying to the rotation group of three-dimensional Euclidean space. Take an ascending power series

$$a_0+a_1\xi_1+a_2\xi_1^2+a_3\xi_1^3+\dots \tag{1}$$

in a real variable ξ_1 with real coefficients a_r. Consider these coefficients to be the co-ordinates of a vector in a certain space of an infinite number of dimensions, and define the squared length of the vector to be

$$\Sigma_0^\infty r!\,a_r^2. \tag{2}$$

The series (2) must converge for the vector to be a finite one.

Take two more similar power series $\Sigma_0^\infty b_s \xi_2^s$ and $\Sigma_0^\infty c_t \xi_3^t$ in the real variables ξ_2 and ξ_3 and consider their coefficients to be the co-ordinates of vectors in two more vector spaces, with squared lengths defined by the corresponding formula to (2). Now multiply the three vector spaces together. A general vector in the product space will be a sum of products of vectors of the three original vector spaces, and its co-ordinates A_{rst} can be represented as the coefficients in a power series

$$P = \Sigma_0^\infty A_{rst} \xi_1^r \xi_2^s \xi_3^t \tag{3}$$

in the three variables ξ_1, ξ_2, ξ_3. The squared length of such a vector is

$$\Sigma_0^\infty r!\,s!\,t!\,A_{rst}^2, \tag{4}$$

and two vectors with co-ordinates A_{rst} and B_{rst} have a scalar product

$$\Sigma_0^\infty r!\,s!\,t!\,A_{rst} B_{rst}. \tag{5}$$

If the variables ξ_1, ξ_2, ξ_3 are subjected to a linear transformation, going over into ξ_1', ξ_2', ξ_3', say, the power series (3) will go over into a power series in ξ_1', ξ_2', ξ_3',

$$P = \Sigma A'_{rst} \xi_1'^r \xi_2'^s \xi_3'^t,$$

in which the coefficients A' are linear functions of the previous coefficients A. Thus each linear transformation of the ξ's generates a linear transformation of the coefficients A.

The theorem will now be proved: *A linear transformation of ξ_1, ξ_2, ξ_3 which leaves $\xi_1^2+\xi_2^2+\xi_3^2$ invariant generates a linear transformation of the coefficients A which leaves the squared length* (4) *invariant.* Consider first the infinitesimal transformation

$$\xi_1 = \xi_1' + \epsilon \xi_2', \quad \xi_2 = \xi_2' - \epsilon \xi_1', \quad \xi_3 = \xi_3', \tag{6}$$

which leaves $\xi_1^2+\xi_2^2+\xi_3^2$ invariant, ϵ being a small quantity whose square is negligible. Substituting into (3), one gets

$$P = \Sigma A_{rst}(\xi_1'^r \xi_2'^s + r\epsilon \xi_1'^{r-1} \xi_2'^{s+1} - s\epsilon \xi_1'^{r+1} \xi_2'^{s-1})\, \xi_3'^t.$$

Hence $$A'_{rst} = A_{rst} + (r+1)\,\epsilon A_{r+1,s-1,t} - (s+1)\,\epsilon A_{r-1,s+1,t},$$

in which A_{rst} with a negative suffix is counted as zero. Thus

$$\Sigma r!\,s!\,t!\,A_{rst}'^2 = \Sigma r!\,s!\,t!\,[A_{rst}^2 + 2(r+1)\,\epsilon A_{rst} A_{r+1,s-1,t} - 2(s+1)\,\epsilon A_{r-1,s+1,t} A_{rst}].$$

The last two terms in the [ ] here cancel, as may be seen by substituting $r-1$ for r in the former and $s-1$ for s in the latter, and hence the squared length (4) is invariant for the transformation (6). Any linear transformation of ξ_1, ξ_2, ξ_3 which leaves $\xi_1^2+\xi_2^2+\xi_3^2$ invariant can be built up from the infinitesimal transformation (6) and similar infinitesimal transformations with ξ_1, ξ_2 and ξ_3 permuted, together with possibly a reflexion $\xi_1 = -\xi_1'$, $\xi_2 = \xi_2'$, $\xi_3 = \xi_3'$, which obviously leaves the squared length (4) invariant, and hence the theorem is proved.

The group of transformations of the ξ's which leave $\xi_1^2+\xi_2^2+\xi_3^2$ invariant is the rotation group in three-dimensional Euclidean space, so the transformations of the coefficients A provide a representation of this rotation group. One may restrict the function P to be homogeneous, of degree u say, and then the representation is a finite one. The coefficients A then form the components of a symmetrical tensor of rank u, the connexion with the usual tensor notation being effected by taking A_{rst} to be $u!/r!s!t!$ times the usual tensor component with the suffix 1 occurring r times, 2 occurring s times and 3 occurring t times, as may be seen from the invariance of expression (3) with (ξ_1, ξ_2, ξ_3) transforming like a vector.

One can make a straightforward generalization of the foregoing theory by introducing other triplets of variables, η_1, η_2, η_3 and $\zeta_1, \zeta_2, \zeta_3$ say, which transform together with ξ_1, ξ_2, ξ_3, and setting up a power series in all the variables. The transformations of the coefficients of these more general power series will provide further representations of the three-dimensional rotation group. If such a more general power series is restricted to be homogeneous, its coefficients will form the components of an unsymmetrical tensor.

3. Four-dimensional theory

Take a descending power series

$$k_0/\xi_0 + k_1/\xi_0^2 + k_2/\xi_0^3 + k_3/\xi_0^4 + \dots \tag{7}$$

in a real variable ξ_0 with real coefficients k_n. Consider these coefficients to be the co-ordinates of a vector in a certain space of an infinite number of dimensions, and define the squared length of the vector to be

$$\Sigma_0^\infty k_n^2/n!. \tag{8}$$

Multiply this vector space into the vector space of the preceding section. A general vector in the product space will have co-ordinates A_{nrst} which can be represented as the coefficients in a power series

$$Q = \Sigma_0^\infty A_{nrst}\xi_0^{-n-1}\xi_1^r\xi_2^s\xi_3^t \tag{9}$$

in the four variables $\xi_0, \xi_1, \xi_2, \xi_3$. The squared length of such a vector is

$$\Sigma_0^\infty n!^{-1} r!\,s!\,t!\,A_{nrst}^2, \tag{10}$$

and two vectors with co-ordinates A_{nrst} and B_{nrst} have a scalar product

$$\Sigma_0^\infty n!^{-1} r!\,s!\,t!\,A_{nrst}B_{nrst}. \tag{11}$$

The series (7) may be extended backwards to include some terms with non-negative powers of ξ_0, so that coefficients k_n occur with negative n-values, leading to coefficients A_{nrst} with negative n-values. Since $n!$ is infinite for n negative, these new coefficients do not contribute to the squared length of a vector or the scalar product of two vectors. Thus the terms with non-negative powers of ξ_0 should be counted as corresponding to the vector zero, and whether they are present in an expansion or not does not matter.

Now apply a Lorentz transformation to the ξ's,

$$\xi_\mu = \alpha_\mu^\nu \xi'_\nu, \tag{12}$$

the α's satisfying certain conditions so that $\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2$ is invariant. This makes $\xi_1^r \xi_2^s \xi_3^t$ go over into a finite polynomial in $\xi'_0, \xi'_1, \xi'_2, \xi'_3$, and ξ_0^{-n-1} go over into

$$(\alpha_0^0 \xi'_0 + \alpha_0^1 \xi'_1 + \alpha_0^2 \xi'_2 + \alpha_0^3 \xi'_3)^{-n-1}, \tag{13}$$

which may be expanded in ascending powers of ξ'_1, ξ'_2, ξ'_3 and descending powers of ξ'_0. (The question of the convergence of this expansion is left to the next section, so as not to break the main argument here.) The power series (9) then goes over into a series

$$Q = \Sigma A'_{nrst} \xi_0'^{-n-1} \xi_1'^r \xi_2'^s \xi_3'^t \tag{14}$$

in ascending powers of ξ_1, ξ_2, ξ_3 and descending powers of ξ_0, with coefficients A' which are linear functions of the previous coefficients A. There may be terms in (14) with non-negative powers of ξ_0, but on account of what was said above these can be discarded. If such terms are present in the original series (9), they will not affect any of the coefficients with non-negative n-values in (14).

The Lorentz transformation thus generates a linear transformation of the coefficients A_{nrst} with non-negative n-values. The theorem will now be proved: *The transformation of the coefficients A leaves the squared length* (10) *invariant.*

Consider first the infinitesimal Lorentz transformation

$$\xi_0 = \xi'_0 + \epsilon \xi'_1, \quad \xi_1 = \xi'_1 + \epsilon \xi'_0, \quad \xi_2 = \xi'_2, \quad \xi_3 = \xi'_3. \tag{15}$$

Substituting into (9), one gets

$$Q = \Sigma A_{nrst} [\xi_0'^{-n-1} \xi_1'^r - (n+1)\epsilon \xi_0'^{-n-2} \xi_1'^{r+1} + r\epsilon \xi_0'^{-n} \xi_1'^{r-1}] \xi_2'^s \xi_3'^t.$$

Hence
$$A'_{nrst} = A_{nrst} - n\epsilon A_{n-1,r-1,s,t} + (r+1)\epsilon A_{n+1,r+1,s,t},$$

in which A_{nrst} with a negative value for r, s or t is counted as zero. Thus

$$\begin{aligned} &\Sigma n!^{-1} r!\, s!\, t!\, A'^2_{nrst} \\ &\quad = \Sigma n!^{-1} r!\, s!\, t!\, [A^2_{nrst} - 2n\epsilon A_{nrst} A_{n-1,r-1,s,t} + 2(r+1)\epsilon A_{n+1,r+1,s,t} A_{nrst}]. \end{aligned}$$

The last two terms in the [ ] here cancel, as may be seen by substituting $n+1$ for n in the former and $r-1$ for r in the latter, and hence the squared length (10) is invariant for the transformation (15). Any Lorentz transformation can be built up from infinitesimal transformations like (15) and three-dimensional rotations like those considered in the preceding section, together with possibly a reflexion, which obviously leaves the squared length (10) invariant, and hence the theorem is proved.

The transformations of the coefficients A thus provide a unitary representation of the Lorentz group. The coefficients themselves form the components of a new kind of tensor quantity in space-time. I propose for it the name *expansor*, because of its connexion with binomial expansions. One may restrict the function (9) to be homogeneous and one then gets a simpler kind of expansor, which may be called a *homogeneous expansor*. The analogy with the three-dimensional case suggests that one should look upon a homogeneous expansor as a symmetrical tensor in space-time with the suffix 0 occurring in its components a negative number of times.

The foregoing theory can be generalized, like the three-dimensional theory, by the introduction of other quadruplets of variables, $\eta_0, \eta_1, \eta_2, \eta_3$ and $\zeta_0, \zeta_1, \zeta_2, \zeta_3$ say, which transform together with $\xi_0, \xi_1, \xi_2, \xi_3$. One can then set up a power series in all the variables, ascending in those variables with suffixes 1, 2 and 3, and descending in those variables with suffix 0. The transformations of the coefficients of these more general power series will provide further unitary representations of the Lorentz group, and the coefficients themselves will form the components of more general expansors.

There is another generalization which may readily be made in the theory, namely, to take the values of the index n in (9) to be not integers, but any set of real numbers $n_0, n_0+1, n_0+2, \ldots$ extending to infinity. In the formula (10) for the squared length $n!$ is then to be interpreted as $\Gamma(n+1)$. The terms with negative n-values can no longer be discarded. The expression for the squared length is still positive definite if the minimum value of n is greater than -1, which is the case if the function (9) is homogeneous and its degree is negative. The resulting representation is then still unitary. If, however, the function (9) is homogeneous and its degree is positive, there will be a finite number of negative terms in the expression for the squared length. The resulting kind of representation may be called *nearly unitary*.

4. Some theorems on convergence

If the series (2) is convergent, then (1) is convergent for all values of ξ_1. Similarly, if (4) is convergent, then (3) is convergent for all values of ξ_1, ξ_2, ξ_3. On the other hand, if (8) is convergent, (7) need not be convergent for any value of ξ_0, in which case, of course, it does not define a function of ξ_0. Similarly, if (10) is convergent, (9) need not be convergent for any values for the ξ's. Thus, corresponding to a general expansor A_{nrst}, there need not exist any function Q of the ξ's. However, it will now be proved that *if the series* (9) *is homogeneous and* (10) *is convergent, then* (9) *is absolutely convergent for all values of the ξ's satisfying*

$$\xi_0^2-\xi_1^2-\xi_2^2-\xi_3^2>0. \tag{16}$$

If the series (9) is homogeneous of degree $u-1$, it may be written

$$\Sigma_n \xi_0^{-n-1} \Sigma_S A_{nrst} \xi_1^r \xi_2^s \xi_3^t, \tag{17}$$

where Σ_S means a sum over all values of r, s and t satisfying

$$r+s+t=n+u. \tag{18}$$

With this notation the series (10) may be written

$$\Sigma_n n!^{-1} \Sigma_S r!\,s!\,t!\,A_{nrst}^2. \tag{19}$$

Now apply Cauchy's inequality

$$(x_1y_1+x_2y_2+x_3y_3+\dots)^2 \leqslant (x_1^2+x_2^2+x_3^2+\dots)(y_1^2+y_2^2+y_3^2+\dots) \tag{20}$$

in the following way. Take $(r+s+t)!/r!s!t!$ of the x's in (20) to be each equal to $\xi_1^r\xi_2^s\xi_3^t$ and the corresponding y's to be each equal to $A_{nrst}r!s!t!/(r+s+t)!$, and do this for all r, s, t subject to (18) for a fixed value of n. Then (20) becomes

$$(\Sigma_S A_{nrst}\xi_1^r\xi_2^s\xi_3^t)^2 \leqslant (\xi_1^2+\xi_2^2+\xi_3^2)^{n+u}(n+u)!^{-1}\Sigma_S r!s!t!A_{nrst}^2. \tag{21}$$

If the sum with respect to n in (19) converges, there must be some number κ such that

$$n!^{-1}\Sigma_S r!s!t!A_{nrst}^2 < \kappa$$

for all n. Then from (21)

$$|\xi_0^{-n-1}\Sigma_S A_{nrst}\xi_1^r\xi_2^s\xi_3^t| < \kappa^{\frac{1}{2}}|\xi_0|^{u-1}\left\{\frac{n!}{(n+u)!}\right\}^{\frac{1}{2}}\left\{\frac{\xi_1^2+\xi_2^2+\xi_3^2}{\xi_0^2}\right\}^{\frac{1}{2}(n+u)}$$

which shows that (17) is convergent when (16) is satisfied. One may take ξ_0, ξ_1, ξ_2, ξ_3 and the A's to be all positive without disturbing the argument, so (17), considered as a quadruple series in n, r, s, t, is absolutely convergent.

Thus for any homogeneous expansor of finite length, there exists a function Q given by (9) defined within the light-cone (16). In transforming this function with a Lorentz transformation (12), it is legitimate to use the expansion of (13) in ascending powers of ξ_1', ξ_2', ξ_3' and descending powers of ξ_0', for the following reason. Suppose ξ_μ is within the light-cone and take for definiteness $\xi_0>0$. Then

$$\alpha_0^0\xi_0'+\alpha_0^1\xi_1'+\alpha_0^2\xi_2'+\alpha_0^3\xi_3'>0. \tag{22}$$

One can change the sign of any of the co-ordinates ξ_1', ξ_2', ξ_3', leaving ξ_0' unchanged, and ξ_μ will still lie within the light-cone with $\xi_0>0$, so (22) must still be satisfied. Hence

$$\alpha_0^0\xi_0'-|\alpha_0^1\xi_1'|-|\alpha_0^2\xi_2'|-|\alpha_0^3\xi_3'|>0,$$

which shows that the expansion of (13) in the required manner is absolutely convergent. The use of this expansion in the preceding section is thus justified for the case when (9) is homogeneous, the new coefficients A being determined by the transformed function Q within the light-cone. The justification for (9) not homogeneous then follows since terms of different degree do not interfere. It should be noted that all the foregoing arguments are valid also when the values of n are not integers.

There are some expansors which are invariant under all Lorentz transformations, namely those whose components are the coefficients of $(\xi_0^2-\xi_1^2-\xi_2^2-\xi_3^2)^{-l}$ expanded in ascending powers of ξ_1, ξ_2, ξ_3 and descending powers of ξ_0. For such an expansion to be possible, l must not be a negative integer or zero, but it can be any other real number. Again, there are expansors which transform like ordinary tensors under Lorentz transformations, namely those whose components are the coefficients in the expansion of

$$f(\xi_0,\xi_1,\xi_2,\xi_3)(\xi_0^2-\xi_1^2-\xi_2^2-\xi_3^2)^{-l}, \tag{23}$$

where f is any homogeneous integral polynomial in ξ_0, ξ_1, ξ_2, ξ_3, and l is restricted as before. (23) transforms like the polynomial f, and thus like a tensor of order equal to the degree of f. One would expect all these expansors to be of infinite length, as otherwise one could set up a positive definite form for the squared length of a tensor in space-time. A formal proof that they are is as follows.

Suppose the f in (23) is of degree u and express it as

$$f = g_u + \xi_0 g_{u-1} + (\xi_0^2 - \boldsymbol{\xi}^2)\, g_{u-2} + \xi_0(\xi_0^2 - \boldsymbol{\xi}^2)\, g_{u-3} + (\xi_0^2 - \boldsymbol{\xi}^2)^2\, g_{u-4} + \dots,$$

where $\boldsymbol{\xi}^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$ and each of the g's is a polynomial in ξ_1, ξ_2, ξ_3 only, of degree indicated by the suffix. The successive terms here contribute to (23) the amounts

$$g_u(\xi_0^2 - \boldsymbol{\xi}^2)^{-l} = \Sigma_{m=0}^{\infty} \frac{l(l+1)(l+2)\dots(l+m-1)}{m!} \frac{g_u \boldsymbol{\xi}^{2m}}{\xi_0^{2(m+l)}},$$

$$g_{u-1}\xi_0(\xi_0^2 - \boldsymbol{\xi}^2)^{-l} = \Sigma_{m=0}^{\infty} \frac{l(l+1)(l+2)\dots(l+m-1)}{m!} \frac{g_{u-1} \boldsymbol{\xi}^{2m}}{\xi_0^{2(m+l)-1}},$$

$$g_{u-2}(\xi_0^2 - \boldsymbol{\xi}^2)^{-l+1} = \Sigma_{m=-1}^{\infty} \frac{(l-1)\,l(l+1)\dots(l+m-1)}{(m+1)!} \frac{g_{u-2} \boldsymbol{\xi}^{2(m+1)}}{\xi_0^{2(m+l)}},$$

$$g_{u-3}\xi_0(\xi_0^2 - \boldsymbol{\xi}^2)^{-l+1} = \Sigma_{m=-1}^{\infty} \frac{(l-1)\,l(l+1)\dots(l+m-1)}{(m+1)!} \frac{g_{u-3} \boldsymbol{\xi}^{2(m+1)}}{\xi_0^{2(m+l)-1}},$$

and so on. These expansions show that, for large values of m, the terms arising from g_{u-2}, g_{u-4}, ... are of smaller order than the corresponding terms arising from g_u, and the terms arising from g_{u-3}, g_{u-5}, ... are of smaller order than the corresponding terms arising from g_{u-1}, so that in testing for the convergence of the series which gives the squared length, only the g_u and g_{u-1} terms need be taken into account. (It will be found that the convergence conditions are not sufficiently critical to be affected by this neglect.) Now express g_u and g_{u-1} as

$$g_u = S_u + \boldsymbol{\xi}^2 S_{u-2} + \boldsymbol{\xi}^4 S_{u-4} + \dots, \quad g_{u-1} = S_{u-1} + \boldsymbol{\xi}^2 S_{u-3} + \boldsymbol{\xi}^4 S_{u-5} + \dots,$$

where the S's are solid harmonic functions of ξ_1, ξ_2, ξ_3, of degrees indicated by the suffixes. Each of them gives a contribution to (23) of the form

$$S_{u-2r}\boldsymbol{\xi}^{2r}(\xi_0^2 - \boldsymbol{\xi}^2)^{-l} = S_{u-2r}\, \Sigma_{m=0}^{\infty} \frac{(m+l-1)!}{m!(l-1)!} \frac{\boldsymbol{\xi}^{2(m+r)}}{\xi_0^{2(m+l)}} \tag{24}$$

or

$$S_{u-2r-1}\xi_0\boldsymbol{\xi}^{2r}(\xi_0^2 - \boldsymbol{\xi}^2)^{-l} = S_{u-2r-1}\, \Sigma_{m=0}^{\infty} \frac{(m+l-1)!}{m!(l-1)!} \frac{\boldsymbol{\xi}^{2(m+r)}}{\xi_0^{2(m+l)-1}}. \tag{25}$$

Using the result (41) of the appendix, one finds for the squared lengths of the expansors whose components are the coefficients of (24) and (25), series of the form

$$c\,\Sigma_m \frac{(m+l-1)!^2}{m!^2(l-1)!^2} \frac{4^{m+r}(m+r)!\,(m+u-r+\frac{1}{2})!}{(2m+2l-1)!}$$

and

$$c\,\Sigma_m \frac{(m+l-1)!^2}{m!^2(l-1)!^2} \frac{4^{m+r}(m+r)!\,(m+u-r-\frac{1}{2})!}{(2m+2l-2)!},$$

respectively. In both these series the ratio of the $(m+1)$th term to the mth is $1+u/m$ for large m, and since u is necessarily positive or zero, the series diverge and the expansors are of infinite length. From the result (40) of the appendix, expansors associated with solid harmonics of different degrees are orthogonal to one another, and hence the total expansor with the coefficients of (23) as components must also be of infinite length.

5. A transformation of variables

Up to the present, expansors have always been considered in connexion with certain ξ-functions, the components of the expansor being the coefficients of the ξ-function. In the case of integral n-values, one can make a transformation of variables of the kind which is familiar in quantum mechanics, and get the expansors connected with some other functions, which serves to clarify some of their properties.

Introduce the four operators x_μ, defined by

$$2^{\frac{1}{2}}x_0 = \xi_0 - \partial/\partial\xi_0, \quad 2^{\frac{1}{2}}x_r = \xi_r + \partial/\partial\xi_r \quad (r = 1, 2, 3). \tag{26}$$

Under Lorentz transformations they transform like the components of a 4-vector. The operators $\partial/\partial x_\mu$ may be taken to be

$$2^{\frac{1}{2}}\, \partial/\partial x_0 = \xi_0 + \partial/\partial\xi_0, \quad 2^{\frac{1}{2}}\, \partial/\partial x_r = -\xi_r + \partial/\partial\xi_r, \tag{27}$$

as this gives the correct commutation relations between all the operators. One can now represent ξ-functions with integral n-values by functions of the x's according to the following scheme.

Take as starting point the ξ-function ξ_0^{-1}. It vanishes when operated on by $\partial/\partial\xi_1$, $\partial/\partial\xi_2$, or $\partial/\partial\xi_3$, and also effectively vanishes when multiplied by ξ_0, since the result of the multiplication can be discarded. It must therefore be represented by a function of the x's which vanishes when operated on by $x_r+\partial/\partial x_r$ with $r = 1, 2$ or 3, or by $x_0+\partial/\partial x_0$, and thus by a multiple of $e^{-\frac{1}{2}(x_0^2+\mathbf{x}^2)}$, where $\mathbf{x}^2 = x_1^2+x_2^2+x_3^2$. Take

$$\xi_0^{-1} \equiv \pi^{-1}e^{-\frac{1}{2}(x_0^2+\mathbf{x}^2)}, \tag{28}$$

where the sign $\equiv$ means 'represented by'. Multiply the left-hand side of (28) by the operator $(-\partial/\partial\xi_0)^n\, \xi_1^r\xi_2^s\xi_3^t$ and the right-hand side by the operator equal to it according to (26) and (27). The result is, after dividing through by $n!$,

$$\begin{aligned}\xi_0^{-n-1}\xi_1^r\xi_2^s\xi_3^t &\equiv \pi^{-1}n!^{-1}\,2^{-\frac{1}{2}(n+r+s+t)}\left(x_0-\frac{\partial}{\partial x_0}\right)^n\left(x_1-\frac{\partial}{\partial x_1}\right)^r\left(x_2-\frac{\partial}{\partial x_2}\right)^s\left(x_3-\frac{\partial}{\partial x_3}\right)^t e^{-\frac{1}{2}(x_0^2+\mathbf{x}^2)}\\ &= F_{nrst}(x_0, x_1, x_2, x_3),\end{aligned} \tag{29}$$

say. This gives the function F of the x's which represents $\xi_0^{-n-1}\xi_1^r\xi_2^s\xi_3^t$. A general ξ-function is now represented thus,

$$\Sigma A_{nrst}\,\xi_0^{-n-1}\xi_1^r\xi_2^s\xi_3^t \equiv \Sigma A_{nrst}F_{nrst}(x_0, x_1, x_2, x_3). \tag{30}$$

In this way a general expansor A_{nrst} gets connected with a function of the x's.

The chief interest of this connexion is that the law for the scalar product of two expansors becomes very simple when expressed in terms of the functions of the x's. *The scalar product is the integral of the product of the two functions of the x's over all* x_0, x_1, x_2 *and* x_3. To prove this, first evaluate the integral

$$\int_{-\infty}^{\infty}\left(z-\frac{d}{dz}\right)^m e^{-\frac{1}{2}z^2}.\left(z-\frac{d}{dz}\right)^{m'} e^{-\frac{1}{2}z^2}\,dz, \tag{31}$$

where the dot has the meaning that operators to the left of it do not operate on functions to the right of it. For $m>0$, (31) goes over by partial integration into

$$\int_{-\infty}^{\infty}\left(z-\frac{d}{dz}\right)^{m-1} e^{-\frac{1}{2}z^2}.\left(z+\frac{d}{dz}\right)\left(z-\frac{d}{dz}\right)^{m'} e^{-\frac{1}{2}z^2}\,dz$$
$$=\int_{-\infty}^{\infty}\left(z-\frac{d}{dz}\right)^{m-1} e^{-\frac{1}{2}z^2}.\left\{\left(z-\frac{d}{dz}\right)^{m'}\left(z+\frac{d}{dz}\right)+2m'\left(z-\frac{d}{dz}\right)^{m'-1}\right\} e^{-\frac{1}{2}z^2}\,dz$$
$$=2m'\int_{-\infty}^{\infty}\left(z-\frac{d}{dz}\right)^{m-1} e^{-\frac{1}{2}z^2}.\left(z-\frac{d}{dz}\right)^{m'-1} e^{-\frac{1}{2}z^2}\,dz.$$

Applying this procedure m times, one gets zero if $m'<m$ and $2^m m!\,\pi^{\frac{1}{2}}$ if $m'=m$. Since there is symmetry between m and m', the result is

$$\int_{-\infty}^{\infty}\left(z-\frac{d}{dz}\right)^m e^{-\frac{1}{2}z^2}.\left(z-\frac{d}{dz}\right)^{m'} e^{-\frac{1}{2}z^2}\,dz = 2^m m!\,\pi^{\frac{1}{2}}\delta_{mm'}.$$

Now substitute for z each of the variables x_0, x_1, x_2, x_3 in turn, with m equal to n, r, s, t and m' equal to n', r', s', t' respectively, and multiply the four equations so obtained. The result, after dividing through by $\pi^2 n!^2\, 2^{n+r+s+t}$, is

$$\iiiint F_{nrst}F_{n'r's't'}\,dx_0\,dx_1\,dx_2\,dx_3 = n!^{-1}r!\,s!\,t!\,\delta_{nn'}\delta_{rr'}\delta_{ss'}\delta_{tt'}.$$

This proves the theorem for the case when the expansors each have only one non-vanishing component, which is sufficient to prove it generally.

One can obviously get a unitary representation of the Lorentz group by considering the transformations of a set of vectors in a space of an infinite number of dimensions, where each vector corresponds to a function of four Lorentz variables x_0, x_1, x_2, x_3, and where the squared length of a vector is defined as the integral of the square of the function over all x_0, x_1, x_2, x_3. The above work shows how this obvious unitary representation is connected with the expansor theory.

6. Applications to quantum mechanics

The four x's of the preceding section may be looked upon as the co-ordinates of a four-dimensional harmonic oscillator, the four operators $i\partial/\partial x_\mu$ being the conjugate momenta p^μ, and the energy of the oscillator may be taken to be the Lorentz invariant

$$\tfrac{1}{2}[x_1^2+x_2^2+x_3^2-x_0^2+(p^1)^2+(p^2)^2+(p^3)^2-(p^0)^2]. \tag{32}$$

The 1, 2, 3 components of the oscillator thus have positive energies and the 0 component negative energy.

The state of the oscillator for which the 0, 1, 2, 3 components are in the nth, rth, sth, tth quantum states respectively is then represented by the function F_{nrst} defined by (29), with a suitable normalizing factor. This representative may be transformed to the ξ-representation and becomes $\xi_0^{-n-1}\xi_1^r\xi_2^s\xi_3^t$. Thus a state of the oscillator for which each of its components is in a quantum state corresponds to an expansor with one non-vanishing component. A general state of the oscillator therefore corresponds to a general expansor with integral n-values. A stationary state of the oscillator corresponds to a homogeneous expansor, the degree of the expansor giving the energy of the state with neglect of zero-point energy.

Four-dimensional harmonic oscillators of the above type occur in the theory of the electromagnetic field. Each Fourier component of the field, specified by a particular frequency and a particular direction of motion of the waves, provides one such oscillator, its four components coming from the four electromagnetic potentials. Thus a state of the electromagnetic field in quantum mechanics is described by a number of expansors, one for each Fourier component. By using the electromagnetic equation which gives the value of the divergence of the potentials, one can eliminate in a non-relativistic way the 0 component and one other component of each of the four-dimensional oscillators, so that only two-dimensional oscillators are left. This circumstance has made it possible for people to develop quantum electrodynamics without using expansors.

Another possible application of expansors is to the spins of particles. The wave function describing a particle may be a function of the four co-ordinates x_μ of the particle in space-time and also of the four variables ξ_μ whose coefficients are the components of an expansor. As a simple example of relativistic wave equations for such a particle in the absence of external forces, one may consider

$$\left(\hbar^2\frac{\partial^2}{\partial x_\mu \partial x^\mu}+m^2\right)\psi = 0, \quad \xi_\mu\frac{\partial}{\partial \xi_\mu}\psi = -\psi, \quad \xi_\mu\frac{\partial}{\partial x_\mu}\psi = 0. \tag{33}$$

The first of these is the usual equation for the motion of the particle as a whole. The second shows that at each point in space-time the wave function ψ is homogeneous in the ξ's of degree -1. The third shows that the state for which the momentum-energy four-vector of the particle has the value p_μ is represented by the wave function

$$\psi = \frac{m}{\xi_0 p_0 - (\boldsymbol{\xi}\mathbf{p})} e^{-i[p_0x_0-(\mathbf{p}\mathbf{x})]/\hbar}$$

in three-dimensional vector notation. This may be expanded as

$$\psi = me^{-i[p_0x_0-(\mathbf{p}\mathbf{x})]/\hbar}\textstyle\sum_{n=0}^{\infty}\dfrac{(\boldsymbol{\xi}\mathbf{p})^n}{\xi_0^{n+1}p_0^{n+1}}. \tag{34}$$

For the state for which the particle is at rest, $p_1 = p_2 = p_3 = 0$, $p_0 = m$, and ψ reduces to

$$\psi = e^{-imx_0/\hbar}\,\xi_0^{-1}.$$

This ψ is spherically symmetrical, showing that when the particle is at rest it has no spin. But when the particle is moving, it is represented by the general ψ (34) and has a finite probability of a non-zero spin. In fact, taking for simplicity $p_2 = p_3 = 0$, the particle has a probability $(mp_1^n/p_0^{n+1})^2$ of being in a state of spin corresponding to the transformations of ξ_1^n under three-dimensional rotations.

This example shows there is a possibility of a particle having no spin when at rest but acquiring a spin when moving, a state of affairs which was not allowed by previous theory. It is desirable that the new spin possibilities opened up by the present theory should be investigated to see whether they could in some cases give an improved description of Nature. The present theory of expansors applies, of course, only to integral spins, but probably it will be possible to set up a corresponding theory of two-valued representations of the Lorentz group, which will apply to half odd integral spins.

Appendix

The rules (5), (11) for forming scalar products are not always convenient for direct use. There are various ways of transforming them and making them more suitable for practical application. One such way has been given (Dirac 1942, equation 3·22) for the case of a single ξ with ascending power series. Another way, applicable to the case of homogeneous functions of ξ_1, ξ_2, ξ_3, is provided by the following.

By partial integration with respect to ξ', one gets, for $m > 0$,

$$\iint_{-\infty}^{\infty} \xi^m \xi'^n e^{i\xi\xi'} d\xi d\xi' = \int_{-\infty}^{\infty} \xi^{m-1} d\xi \left\{ \left[-i\xi'^n e^{i\xi\xi'} \right]_{\xi'=-\infty}^{\xi'=\infty} + in \int_{-\infty}^{\infty} \xi'^{n-1} e^{i\xi\xi'} d\xi' \right\}.$$

If the integrals are made precise in the sense of Cesaro, which means neglecting oscillating terms like $\xi'^n e^{i\xi\xi'}$ for ξ' infinite, this gives

$$\iint_{-\infty}^{\infty} \xi^m \xi'^n e^{i\xi\xi'} d\xi d\xi' = in \iint_{-\infty}^{\infty} \xi^{m-1} \xi'^{n-1} e^{i\xi\xi'} d\xi d\xi'.$$

Taking $m \geqslant n$ and applying the partial integration process n times, one gets

$$\begin{aligned} \iint_{-\infty}^{\infty} \xi^m \xi'^n e^{i\xi\xi'} d\xi d\xi' &= i^n n! \iint_{-\infty}^{\infty} \xi^{m-n} e^{i\xi\xi'} d\xi d\xi' \\ &= 2\pi i^n n! \int_{-\infty}^{\infty} \xi^{m-n} \delta(\xi) d\xi \\ &= 2\pi i^n n! \delta_{mn}. \end{aligned} \tag{35}$$

It follows that if A and B are homogeneous functions of ξ_1, ξ_2, ξ_3 of degree u, their scalar product according to (5) is

$$(AB) = (2\pi)^{-3} i^{-u} \iint_{\cdots\cdots} A(\xi_1 \xi_2 \xi_3)\, B(\xi_1' \xi_2' \xi_3')\, e^{i(\xi_1\xi_1' + \xi_2\xi_2' + \xi_3\xi_3')} d\xi_1 d\xi_1' d\xi_2 d\xi_2' d\xi_3 d\xi_3'. \tag{36}$$

As an application of this rule, take

$$A = (\xi_1^2+\xi_2^2+\xi_3^2)^r S_{u-2r}, \quad B = (\xi_1^2+\xi_2^2+\xi_3^2)^s S_{u-2s},$$

where the S's are solid harmonic functions. Then, using three-dimensional vector notation, (36) gives

$$(AB) = (2\pi)^{-3} i^{-u} \iint_{\dots\dots} \boldsymbol{\xi}^{2r} \boldsymbol{\xi}'^{2s} S_{u-2r}(\xi)\, S_{u-2s}(\xi')\, e^{i(\boldsymbol{\xi}\boldsymbol{\xi}')}\, d\xi_1 \dots d\xi_3'. \tag{37}$$

From Green's theorem

$$\iiint \left[e^{i(\boldsymbol{\xi}\boldsymbol{\xi}')} \left(\frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} + \frac{\partial^2}{\partial \xi_3^2} \right) \left\{ \boldsymbol{\xi}^{2r} S_{u-2r}(\xi) \right\} - \boldsymbol{\xi}^{2r} S_{u-2r}(\xi) \left(\frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} + \frac{\partial^2}{\partial \xi_3^2} \right) e^{i(\boldsymbol{\xi}\boldsymbol{\xi}')} \right] d\xi_1 d\xi_2 d\xi_3$$

equals a surface integral of an oscillating kind which is to be counted as vanishing at infinity. This result reduces to

$$4r(u-r+\tfrac{1}{2}) \iiint_{-\infty}^{\infty} \boldsymbol{\xi}^{2(r-1)} S_{u-2r}(\xi)\, e^{i(\boldsymbol{\xi}\boldsymbol{\xi}')}\, d\xi_1 d\xi_2 d\xi_3 + \boldsymbol{\xi}'^2 \iiint_{-\infty}^{\infty} \boldsymbol{\xi}^{2r} S_{u-2r}(\xi)\, e^{i(\boldsymbol{\xi}\boldsymbol{\xi}')}\, d\xi_1 d\xi_2 d\xi_3 = 0.$$

so (37) becomes, for $r, s > 0$,

$$(AB) = (2\pi)^{-3} i^{-u+2}\, 4r(u-r+\tfrac{1}{2}) \iint_{\dots\dots} \boldsymbol{\xi}^{2(r-1)} \boldsymbol{\xi}'^{2(s-1)} S_{u-2r}(\xi)\, S_{u-2s}(\xi')\, e^{i(\boldsymbol{\xi}\boldsymbol{\xi}')}\, d\xi_1 \dots d\xi_3'. \tag{38}$$

Now suppose $s \geqslant r$ and apply the procedure by which (37) was changed to (38) r times. The result is

$$(AB) = (2\pi)^{-3} i^{-u+2r}\, 4^r r!\,(u-r+\tfrac{1}{2})!\,(u-2r+\tfrac{1}{2})!^{-1} \times \iint_{\dots\dots} \boldsymbol{\xi}'^{2(s-r)} S_{u-2r}(\xi)\, S_{u-2s}(\xi')\, e^{i(\boldsymbol{\xi}\boldsymbol{\xi}')}\, d\xi_1 \dots d\xi_3'. \tag{39}$$

where $n!$ means $\Gamma(n+1)$ for n not an integer. If $s > r$, the procedure can be applied once more, and then shows that

$$(AB) = 0 \quad \text{for} \quad r \neq s. \tag{40}$$

If $s = r$, (39) shows that

$$(AB) = c 4^r r!(u-r+\tfrac{1}{2})!, \tag{41}$$

where c depends only on $u-2r$ and on the two S functions.

Reference

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