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A new classical theory of electrons. II

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An analysis is made of the motion of a stream of electrons obeying Lorentz's equations of motion. It leads to a more general action principle than that of an earlier paper, containing two new field variables. The present theory allows vorticity in the electron stream, while still involving e and m only in the ratio e/m.

A Hamiltonian formulation of the equations is deduced.

1. Introduction

The fact that electric charges occur only in multiples of the electronic charge e is presumably a quantum phenomenon and therefore should not appear in a classical theory of electrodynamics. In order that one may have streams of electrons moving in accordance with Lorentz's equations of motion, it is necessary that the ratio e/m of the charge to the mass of the electron should appear, but it is not necessary that e and m should appear separately. It is desirable that e and m do not appear separately, because then one can hope that a method of quantization will eventually be discovered giving the simultaneous appearance of e and $\hbar$ with a connexion between them fixing the value of the fine structure constant $e^2/\hbar c$.

In a previous paper (Dirac 1951*b*) I proposed a classical theory of this kind, involving only the ratio $m/e = k$. It has been pointed out to me by D. Gabor that my theory allows only electron streams of a special kind, namely, the velocity 4-vector v_μ of the electron stream must be (see equations (14), (15) of that paper)

$$v_\mu = k^{-1}(\partial S/\partial x^\mu + A^*_\mu),$$

where the A^*_μ are the electromagnetic potentials with an arbitrary gauge. Thus the vector $kv_\mu - A^*_\mu$ must be irrotational. Now, one can easily obtain in practice an electron distribution for which this vector is vortical, for example, a cloud of electrons at rest in a magnetic field, with the electron density sufficiently small for the mutual repulsion of the electrons to be negligible. Such a distribution of electrons could not be described in my theory.

The present paper is concerned with generalizing the theory so as to allow vortical streams of electrons while keeping to the fundamental idea that only the ratio m/e appears in the theory. The equations are put in the form of an action principle and the Hamiltonian is deduced, as a preliminary to quantization.

2. Deductions from the Lorentz equations

Lorentz's equations of motion for an electron moving with velocity v_μ in an electromagnetic field $F^{\mu\nu}$ are

$$k\,\mathrm{d}v^\mu/\mathrm{d}s = v_\nu F^{\nu\mu}, \tag{1}$$

with $k = m/e$, the charge on the electron being $-e$. If there is a stream of electrons, we can look upon v^μ as a continuous function of the four x's describing a point in space-time, and equation (1) becomes

$$kv_\nu \partial v^\mu/\partial x_\nu = v_\nu F^{\nu\mu}. \tag{2}$$

From the equation $v_\nu v^\nu = 1$ it follows that

$$v_\nu \partial v^\nu/\partial x_\mu = 0,$$

so (2) may be written

$$v_\nu f^{\mu\nu} = 0, \tag{3}$$

where

$$f^{\mu\nu} = F^{\mu\nu} + k\left(\frac{\partial v^\mu}{\partial x_\nu} - \frac{\partial v^\nu}{\partial x_\mu}\right). \tag{4}$$

Equation (3) shows that there exists a 4-vector u_μ at each point of space-time such that

$$\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma} f^{\mu\nu} = v_\rho u_\sigma - v_\sigma u_\rho, \tag{5}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the antisymmetrical fourth-rank tensor density whose non-vanishing components are ± 1. (One can easily verify this result by taking a Lorentz frame of reference for which $v_0 = 1$, $v_1 = v_2 = v_3 = 0$.) From equation (4)

$$f^{\mu\nu} = \frac{\partial}{\partial x_\mu}(A^\nu - kv^\nu) - \frac{\partial}{\partial x_\nu}(A^\mu - kv^\mu), \tag{6}$$

where the A^μ are the potentials with any choice of gauge. Thus

$$\epsilon_{\mu\nu\rho\sigma}\partial f^{\mu\nu}/\partial x_\rho = 0, \tag{7}$$

and so from (5) $$\partial(v_\rho u_\sigma - v_\sigma u_\rho)/\partial x_\rho = 0$$

or $$v_\rho \frac{\partial u_\sigma}{\partial x_\rho} - u_\rho \frac{\partial v_\sigma}{\partial x_\rho} = v_\sigma \frac{\partial u_\rho}{\partial x_\rho} - u_\sigma \frac{\partial v_\rho}{\partial x_\rho}. \tag{8}$$

This equation has a simple geometrical interpretation. It shows that, if ϵ is an infinitesimal number, the change in the vector ϵu_σ when one makes a displacement $\delta x_\rho = \epsilon v_\rho$ differs from the change in the vector ϵv_σ when one makes a displacement $\delta x_\rho = \epsilon u_\rho$ by ϵ^2 times the right-hand side of (8), which is a vector lying in the plane of the vectors u_σ and v_σ. It follows that *the vectors u_σ and v_σ lie in integrable two-dimensional surfaces in space-time.* The stream-lines of the electricity, i.e. the lines in space-time with the equations $\mathrm{d}x_\mu/\mathrm{d}s = v_\mu$, are contained in these surfaces.

Let ξ and η be two independent functions of the x's which are constant on these surfaces, so that

$$v_\sigma \frac{\partial \xi}{\partial x_\sigma} = u_\sigma \frac{\partial \xi}{\partial x_\sigma} = v_\sigma \frac{\partial \eta}{\partial x_\sigma} = u_\sigma \frac{\partial \eta}{\partial x_\sigma} = 0.$$

Then from (5) $$\epsilon_{\mu\nu\rho\sigma} f^{\mu\nu} \frac{\partial \xi}{\partial x_\sigma} = 0, \quad \epsilon_{\mu\nu\rho\sigma} f^{\mu\nu} \frac{\partial \eta}{\partial x_\sigma} = 0,$$

showing that the 6-vector $f^{\mu\nu}$ is of the form

$$f^{\mu\nu} = \alpha\left(\frac{\partial \xi}{\partial x_\mu}\frac{\partial \eta}{\partial x_\nu} - \frac{\partial \xi}{\partial x_\nu}\frac{\partial \eta}{\partial x_\mu}\right), \tag{9}$$

where α is some function of the x's. Substituting (9) in (7), we get

$$\epsilon_{\mu\nu\rho\sigma} \frac{\partial \xi}{\partial x_\mu}\frac{\partial \eta}{\partial x_\nu}\frac{\partial \alpha}{\partial x_\rho} = 0,$$

showing that the vectors $\partial\xi/\partial x_\mu$, $\partial\eta/\partial x_\nu$, $\partial\alpha/\partial x_\rho$ are coplanar, and thus α is a function of ξ and η only.

Let us introduce new variables ξ^* and η^* which are functions of the original ξ and η such that the Jacobian

$$\partial(\xi^*, \eta^*)/\partial(\xi, \eta) = \alpha.$$

Then $$f^{\mu\nu} = \frac{\partial \xi^*}{\partial x_\mu}\frac{\partial \eta^*}{\partial x_\nu} - \frac{\partial \xi^*}{\partial x_\nu}\frac{\partial \eta^*}{\partial x_\mu}. \tag{10}$$

We shall work only with these new variables in future, and drop the stars. Thus we have

$$f^{\mu\nu} = \frac{\partial}{\partial x_\mu}\left(\xi\frac{\partial \eta}{\partial x_\nu}\right) - \frac{\partial}{\partial x_\nu}\left(\xi\frac{\partial \eta}{\partial x_\mu}\right).$$

Comparing this with (6) we see that, with a suitable choice of gauge,

$$A^\mu = kv^\mu + \xi\partial\eta/\partial x_\mu. \tag{11}$$

The form (11) for the potentials is the foundation of the present theory. In my previous theory A^μ was simply equal to kv^μ. The extra term $\xi\partial\eta/\partial x_\mu$ appearing in (11) gives rise to vorticity in the electron stream. The variables ξ and η in (11) are not uniquely determined, but can be replaced by any two functions of them such that the Jacobian is unity.

3. The action principle

Let us take the action density

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\lambda(v_\mu v^\mu - 1), \tag{12}$$

with the $F^{\mu\nu}$ derived from the potentials (11), and take as independent field variables $v^\mu, \xi, \eta, \lambda$. Varying the action integral

$$I = \int \mathscr{L}\, d^4x, \tag{13}$$

taken over the whole of space-time, we get

$$\begin{aligned}
\delta I &= \int \left\{ F_{\mu\nu}\frac{\partial \delta A^\mu}{\partial x_\nu} + \lambda v_\mu \delta v^\mu + \tfrac{1}{2}(v_\mu v^\mu - 1)\delta\lambda \right\} d^4x \\
&= \int \left\{ -\frac{\partial F_{\mu\nu}}{\partial x_\nu}\left(k\,\delta v^\mu + \frac{\partial \eta}{\partial x_\mu}\delta\xi + \xi\frac{\partial \delta\eta}{\partial x_\mu}\right) + \lambda v_\mu \delta v^\mu + \tfrac{1}{2}(v_\mu v^\mu - 1)\delta\lambda \right\} d^4x \\
&= \int \left\{ -\frac{\partial F_{\mu\nu}}{\partial x_\nu}\left(k\,\delta v^\mu + \frac{\partial \eta}{\partial x_\mu}\delta\xi - \frac{\partial \xi}{\partial x_\mu}\delta\eta\right) + \lambda v_\mu \delta v^\mu + \tfrac{1}{2}(v_\mu v^\mu - 1)\delta\lambda \right\} d^4x.
\end{aligned}$$

The condition that I shall be stationary gives

$$v_\mu v^\mu = 1, \tag{14}$$

$$k\,\partial F_{\mu\nu}/\partial x_\nu = \lambda v_\mu, \tag{15}$$

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu}\frac{\partial \eta}{\partial x_\mu} = 0, \quad \frac{\partial F_{\mu\nu}}{\partial x_\nu}\frac{\partial \xi}{\partial x_\mu} = 0. \tag{16}$$

Equation (15) becomes Maxwell's field equations if we identify the charge-current density as

$$j_\mu = -k^{-1}\lambda v_\mu. \tag{17}$$

Equations (16) may be replaced, for $\lambda \neq 0$, by

$$v_\mu \partial\eta/\partial x_\mu = 0, \quad v_\mu \partial\xi/\partial x_\mu = 0,$$

and give, with the help of (11),

$$v_\mu\left\{ \frac{\partial}{\partial x_\mu}(A^\nu - kv^\nu) - \frac{\partial}{\partial x_\nu}(A^\mu - kv^\mu)\right\} = 0,$$

which leads back to Lorentz's equation of motion (2). Thus the action principle gives all the equations of motion correctly.

It should be noted that the action density involves only the field variables $v^\mu, \xi, \eta, \lambda$ and their first derivatives, the second derivatives of η cancelling out. Thus the action integral (13) taken over a finite region of space-time will be stationary provided the equations (14), (15), (16) hold and the values of v^μ, ξ, η, λ at the boundary are kept fixed.

4. An alternative action principle

One could take several field variables $\xi_a (a = 1, 2, \ldots)$, each associated with a field variable η_a, and put

$$A^\mu = kv^\mu + \Sigma_a \xi_a \partial \eta_a / \partial x_\mu. \tag{18}$$

On carrying through the variation procedure with this more general expression for A^μ instead of (11) in the action density (12), one finds that equations (14) and (15) hold unchanged and equations (16) hold for each ξ_a and η_a. One easily sees that the equations are still all in agreement with the Maxwell-Lorentz theory. They cannot describe a physically more general motion because, as we saw in § 2, the potentials, (11) are adequate to describe a general electron stream, so they must be merely a more complicated description of the same physical motion. This result is a stepping-stone to a more interesting change in the action principle.

Each of the quantities ξ_a, η_a is constant along a stream-line. The action principle based on (12) and (18) requires, in order that the action integral over a finite region shall be stationary, that each of the quantities ξ_a, η_a shall be kept fixed at the boundary. Thus, provided at least three of these quantities are independent, *the end-points of the stream-lines must be kept fixed.* One might think that it is now unnecessary to keep the end values of the velocity v^μ fixed, and on investigation this proves to be the case. One can thus have an action principle in which the independent field variables determine the positions of the stream-lines but not (unless one takes their derivatives) the velocities.

A convenient way of describing all the stream-lines is by expressing the co-ordinates of each point of space-time as a function of four parameters u_1, u_2, u_3, τ, say

$$x_\mu = y_\mu(u, \tau),$$

where the u's are constants along a stream-line. Then the u's are labels specifying a particular stream-line and τ is a parameter describing a point on it. We shall consider τ as the time variable and denote derivatives with respect to it by a dot.

The vector v_μ lies in the same direction as $\dot{y}_\mu$, so

$$v_\mu = \dot{y}_\mu (\dot{y}_\nu \dot{y}^\nu)^{-\frac{1}{2}}.$$

Equation (11) shows that

$$A^\mu v_\mu = k, \tag{19}$$

or

$$A^\mu \dot{y}_\mu = k(\dot{y}_\nu \dot{y}^\nu)^{\frac{1}{2}}. \tag{20}$$

With given directions for the stream-lines, the potentials are arbitrary except for the condition (20).

The variables A^μ, y_μ will be taken as the independent field quantities in the new action principle. In order to ensure that the condition (20) holds between them, we take as the action density

$$\mathscr{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \lambda \{A^\mu \dot{y}_\mu - k(\dot{y}_\nu \dot{y}^\nu)^{\frac{1}{2}}\}. \tag{21}$$

On displacing all the stream-lines by δy_μ (a function of the x's), we get

$$\delta \dot{y}_\mu = \frac{\mathrm{d}}{\mathrm{d}\tau} \delta y_\mu = \dot{y}_\nu \frac{\partial \delta y_\mu}{\partial x_\nu}. \tag{22}$$

The $\delta\dot{y}_\mu$ in (22) denotes the change in $\dot{y}_\mu$ for given values of the u's and τ. If we let $\delta\dot{y}_\mu$ denote instead the change in $\dot{y}_\mu$ for given values of the x's (to correspond to the other δ's in the variation procedure), we must replace (22) by

$$\delta\dot{y}_\mu + \frac{\partial \dot{y}_\mu}{\partial x_\nu}\delta y_\nu = \dot{y}_\nu \frac{\partial \delta y_\mu}{\partial x_\nu}. \tag{23}$$

We have now

$$\begin{aligned}
\delta I &= \delta \int \mathscr{L}\, d^4x \\
&= \int \Big\{ F_{\mu\nu}\frac{\partial \delta A^\mu}{\partial x_\nu} + [A^\mu \dot{y}_\mu - k(\dot{y}_\nu \dot{y}^\nu)^{\frac{1}{2}}]\,\delta\lambda + \lambda \dot{y}_\mu \delta A^\mu \\
&\qquad + \lambda[A^\mu - k(\dot{y}_\nu \dot{y}^\nu)^{-\frac{1}{2}}\dot{y}_\mu]\left[\dot{y}_\nu \frac{\partial \delta y_\mu}{\partial x_\nu} - \frac{\partial \dot{y}_\mu}{\partial x_\nu}\delta y_\nu\right]\Big\} d^4x \\
&= \int \Big\{ -\frac{\partial F_{\mu\nu}}{\partial x_\nu}\delta A^\mu + [A^\mu \dot{y}_\mu - k(\dot{y}_\nu \dot{y}^\nu)^{\frac{1}{2}}]\,\delta\lambda + \lambda \dot{y}_\mu \delta A^\mu \\
&\qquad - \frac{\partial}{\partial x_\nu}[\lambda(A^\mu - kv^\mu)\dot{y}_\nu]\,\delta y_\mu - \lambda(A^\mu - kv^\mu)\frac{\partial \dot{y}_\mu}{\partial x_\nu}\delta y_\nu \Big\} d^4x,
\end{aligned}$$

provided δA^μ and δy_μ vanish on the boundary. Thus the conditions for the action integral to be stationary are (20),

$$\partial F_{\mu\nu}/\partial x_\nu = \lambda \dot{y}_\mu \tag{24}$$

and

$$\frac{\partial}{\partial x_\nu}[\lambda(A^\mu - kv^\mu)\dot{y}_\nu] + \lambda(A^\nu - kv^\nu)\frac{\partial \dot{y}_\nu}{\partial x_\mu} = 0. \tag{25}$$

Equation (24) gives Maxwell's equations with the charge-current density $j_\mu = -\lambda \dot{y}_\mu$. From (24)

$$\partial(\lambda \dot{y}_\nu)/\partial x_\nu = 0, \tag{26}$$

so (25) reduces, when $\lambda \neq 0$, to

$$\dot{y}_\nu \frac{\partial}{\partial x_\nu}(A^\mu - kv^\mu) + (A^\nu - kv^\nu)\frac{\partial \dot{y}_\nu}{\partial x_\mu} = 0.$$

Since $(A^\nu - kv^\nu)\dot{y}_\nu = 0$, this may be written

$$\dot{y}_\nu \frac{\partial}{\partial x_\nu}(A^\mu - kv^\mu) - \dot{y}_\nu \frac{\partial}{\partial x_\mu}(A^\nu - kv^\nu) = 0,$$

which leads to Lorentz's equation (2). Thus the new action principle gives all the equations of motion correctly.

The new action principle is possibly more fundamental than that of § 3, since the new one refers directly to the stream-lines of the electric charges and makes the v_μ appear as velocities. Also the new action integral can be looked upon as a natural generalization of the usual action integral for a single electron, namely

$$I = -\tfrac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\, d^4x + \int eA^\mu v_\mu \, ds - \int m\, ds. \tag{27}$$

The terms $\lambda A^\mu \dot{y}_\mu$ and $-k\lambda(\dot{y}_\nu \dot{y}^\nu)^{\frac{1}{2}}$ in (21) correspond to the second and third terms on the right-hand side of (27) respectively, and are just what the latter terms should become when the single electron is replaced by a stream.

5. The Hamiltonian

Before one can consider the quantization of the theory one must obtain the equations of motion in the Hamiltonian form. The standard methods of passing from the Lagrangian to the Hamiltonian may be applied to both the action principles of §§3 and 4, and lead to two Hamiltonians which look rather different but must, of course, be equivalent. It is not clear which of the two will be more convenient for quantization. The one from the action principle of §3 will be worked out here, to show the method.

To get a covariant formulation of the Hamiltonian one must work in terms of physical quantities on a general space-like three-dimensional surface in space-time, defining a physical state on that surface. The general mathematical technique for dealing with these surfaces has been given (Dirac 1951*a*), and the rules there set up will be followed here, with the same notation. Thus the surface will be specified by four functions $y_\mu(u)$ of three parameters u_r.

The Lagrangian is

$$L \equiv -\tfrac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\dot{y}_l \Gamma\, d^3u, \tag{28}$$

with the $F^{\mu\nu}$ derived from the potentials (11). The second term of the action density (12) has been omitted, because it is now convenient to consider (14) as a strong equation and to look upon the normal component v_l of v_μ as identically equal to a function of the tangential components

$$v_l \equiv (1 - v_r v^r)^{\frac{1}{2}}. \tag{29}$$

Thus L is a function of the independent dynamical co-ordinates y_μ, v^r, ξ, η for all values of the u's, and of their τ derivatives, the velocities.

It is preferable to take the A^r as basic dynamical co-ordinates instead of the v^r. This leads to more convenient expressions for the momenta. The change cannot affect the equations of motion that follow from the action principle, because the equation

$$A^r \equiv kv^r + \xi\eta^r \tag{30}$$

connecting A^r and v^r involves only dynamical co-ordinates and no velocities.

To obtain the momenta we must vary the velocities in (28), keeping the dynamical co-ordinates unchanged. By expressing $F^{\mu\nu}$ in terms of its normal and tangential components, we get

$$L \equiv -\int \{\tfrac{1}{4}F_{rs}F^{rs} + \tfrac{1}{2}F_{rl}F^{r}{}_{l}\}\dot{y}_l \Gamma\, d^3u. \tag{31}$$

Here

$$F^{rs} \equiv A^{sr} - A^{rs}$$

and does not involve any velocities, so

$$\delta F^{rs} = 0. \tag{32}$$

Also

$$\delta\dot{y}_l = l^\sigma \delta\dot{y}_\sigma.$$

The easiest way to evaluate $\delta F^r{}_l$ is first to form

$$\begin{aligned}\dot{y}_\sigma F^{r\sigma} &\equiv \dot{y}_\sigma y_\mu{}^r (A^{\sigma\mu} - A^{\mu\sigma}) \equiv \dot{y}_\sigma A^{\sigma r} - y_\mu{}^r A^{\mu\tau} \\ &\equiv (\dot{y}_\sigma A^\sigma)^r - (y_\mu{}^r A^\mu)^\tau \\ &\equiv (\dot{y}_\sigma kv^\sigma + \xi\eta^\tau)^r - A^{r\tau},\end{aligned} \tag{33}$$

where the upper suffix τ added to a quantity denotes its τ derivative, an alternative to the dot notation. Varying both sides of (33) we get, with the help of (32),

$$\dot{y}_l \delta F^r{}_l + F^{r\sigma} \delta \dot{y}_\sigma = (k v^\sigma \delta \dot{y}_\sigma + \xi \delta \eta^\tau)^r - \delta A^{r\tau}.$$

Thus the variation of (31) gives

$$\delta L = -\int \{F_{rl} \delta F^r{}_l \dot{y}_l + (\tfrac{1}{4} F_{rs} F^{rs} + \tfrac{1}{2} F_{rl} F^r{}_l)\, \delta \dot{y}_l\}\, \Gamma d^3u$$

$$= \int \{F_{rl}(F^{r\sigma} \delta \dot{y}_\sigma + \delta A^{r\tau})\, \Gamma + (F_{rl} \Gamma)^r\, (k v^\sigma \delta \dot{y}_\sigma + \xi \delta \eta^\tau)$$

$$- (\tfrac{1}{4} F_{rs} F^{rs} + \tfrac{1}{2} F_{rl} F^r{}_l)\, \Gamma l^\sigma \delta \dot{y}_\sigma\}\, d^3u.$$

The momenta $B_r, \alpha, \beta, w^\sigma$ conjugate to A^r, ξ, η, y_σ respectively are defined to make

$$\delta L = \int \{B_r \delta A^{r\tau} + \dot{\alpha} \delta \xi^\tau + \beta \delta \eta^\tau + w^\sigma \delta \dot{y}_\sigma\}\, \mathrm{d}^3u.$$

Comparing coefficients, we get

$$B_r = F_{rl} \Gamma, \tag{34}$$

$$\alpha = 0, \tag{35}$$

$$\beta = (F_{rl} \Gamma)^r \xi, \tag{36}$$

$$w^\sigma = F_{rl} F^{r\sigma} \Gamma + k (F_{rl} \Gamma)^r v^\sigma - (\tfrac{1}{4} F_{rs} F^{rs} + \tfrac{1}{2} F_{rl} F^r{}_l)\, \Gamma l^\sigma. \tag{37}$$

From these equations connecting the momenta with the co-ordinates and velocities we must eliminate the velocities to obtain the ϕ equations. Equation (35) as it stands is a ϕ equation. From (34) and (36)

$$\beta - B_r{}^r \xi = 0, \tag{38}$$

another ϕ equation. The tangential part of (37) gives

$$w^s - B_r F^{rs} - B_r{}^r (A^s - \xi \eta^s) = 0, \tag{39}$$

which is a further ϕ equation. Finally, the normal part of (37) gives

$$w_l + \tfrac{1}{4} F_{rs} F^{rs} \Gamma - \tfrac{1}{2} B_r B^r \Gamma^{-1} - k B_r{}^r v_l = 0.$$

Eliminating v_l with the help of (29) and (30), we obtain the ϕ equation

$$\{w_l + \tfrac{1}{4} F_{rs} F^{rs} \Gamma - \tfrac{1}{2} B_r B^r \Gamma^{-1}\}^2 - (B_r{}^r)^2 \{k^2 - (A_s - \xi \eta_s)\, (A^s - \xi \eta^s)\} = 0. \tag{40}$$

These are all the ϕ equations, and it is easily seen that there are no χ equations.

Except in the special case $B_r{}^r = 0$, the ϕ equations (35) and (38) are second class, since the P.b. of their left-hand sides does not vanish. The remaining ϕ equations (39) and (40) must be first class, or at any rate we must be able to obtain first class ϕ equations from them with the help of (35) and (38), because they must give rise to arbitrary changes of parametrization of the surface and an arbitrary motion of the surface normal to itself respectively.

To obtain a Hamiltonian suitable for quantization one must convert the second class ϕ equations into strong equations and introduce new P.b.'s consistent with them as strong equations. A general method of doing this has been given (Dirac 1950, see §8), but the present case is so simple that the result is obvious, namely, we must consider (35) and (38) as strong equations and suppose that the degrees of freedom associated with the α, ξ variables do not contribute to the P.b.'s. We can use (35) and (38) to eliminate the α, ξ variables from the other ϕ equations and then ignore the α, ξ variables altogether. We are left with

$$w^s + \beta\eta^s + B_r A^{rs} - (B_r A^s)^r = 0, \tag{41}$$

$$\{w_l + \tfrac{1}{4}F_{rs}F^{rs}\Gamma - \tfrac{1}{2}B_r B^r\Gamma^{-1}\}^2 + \{B_r{}^r A_s - \beta\eta_s\}\{B_p{}^p A^s - \beta\eta^s\} - (B_r{}^r)^2 k^2 = 0. \tag{42}$$

Equation (41) is just the ϕ equation one would expect, to give rise to changes of parametrization of the surface when the independent field functions are a scalar η and a 3-vector A^r in the surface. Equation (42) is the Hamiltonian of the present theory, and governs the motion of all the dynamical variables as the surface moves in space-time.

6. Conclusion

The main problem with a classical theory of the above type is to get quantization of electricity to appear when one goes over to the quantum theory. This requires presumably that the dynamical variables conjugate to the electric charge at all points of the surface should be of a cyclic character. There is nothing in the present theory to make them cyclic, so some further development is necessary. The introduction of magnetic poles may answer the purpose, but it would not fix the value of the fine-structure constant $e^2/\hbar c$, so it would not be a complete solution of the problem.

An important feature of the new theory, distinguishing it from the usual field theories of atomic physics, is that it involves a velocity field v_μ satisfying $v_\mu v^\mu = 1$, so that one cannot take all the field quantities to be zero, even in a perfect vacuum. Where there is electric charge, v_μ is its velocity. Let us discuss the significance of v_μ where there is no charge.

For a region of space-time where there is no charge, the equations of motion that follow from the action principle of §3 leave v_μ indeterminate, since equations (16) are automatically satisfied. The equations of motion that follow from the action principle of §4 also leave v_μ indeterminate, since λ is zero, so that (25) is automatically satisfied. Thus it would seem that v_μ is of no importance where there is no charge.

However, it is more reasonable to take the Hamiltonian (42) as the foundation of the equations of motion, because the Hamiltonian is the all-important thing from the quantum point of view. The Hamiltonian (42) is not quite equivalent to the action principles, because the method of deriving it fails for $B_r{}^r = 0$, as equations (35) and (38) are then first class. From the Hamiltonian standpoint, there can be no arbitrariness in the motion of the Hamiltonian dynamical variables except that associated with first class ϕ equations, which in the present case means the arbitrary motion of the surface and its parametrization. Where $B_r{}^r$ vanishes, equation (38)

shows that β also vanishes, and in order that there may be no arbitrariness in the motion of η as given by the Hamiltonian (42), it is necessary that the ratio β/B_r^r should be definite. Then, from the equation

$$A^r = kv^r + \beta\eta^r/B_s^s, \tag{43}$$

the velocity v^r is definite. Thus the Hamiltonian formulation requires a definite v_μ throughout space-time.

One may picture v_μ as the velocity of an aether. There is no contradiction with relativity, because all the equations are Lorentz invariant. A perfect vacuum, in the theory after quantization, should be a region in which all directions for v_μ are equally probable, so that there is no preferred time-axis.

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High-polymer solutions

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In describing the configurations of a polymer molecule in terms of the 'equivalent chain' of N elements, each of length l, it has been usual to simplify the problem by assuming the equivalent chain to have position only and zero volume. The weights of the various configurations of such a 'random flight' chain are different from those of a real chain in which there exists an interaction potential between any pair of chain elements. These differences are particularly important in the theory of solutions of chain molecules, since they are responsible for the deviation of the osmotic pressure from van't Hoff's law.

In this paper the average dimensions of a chain with interactions are calculated by a statistical method. For $\langle s^2\rangle$, the average square distance of the elements from the centre of gravity, the result is $\langle s^2\rangle = (Nl^2/6)\,[1-0{\cdot}857(\beta_1/l^3)\,N^{-\frac{1}{2}}]$, (i)

where β_1 is the 'excluded volume' integral for free chain elements. For large N this reduces to the well-known result for a random flight chain. Similar results are obtained for other average dimensions.

The possibility of checking (i) from experimental determinations of $\langle s^2\rangle$ for chain molecules using the light-scattering technique is examined, and it is shown that a very accurate knowledge of the chain-length distribution in the fractions used will be required if the influence of the second term in (i) is to be detected in this way.

A natural extension of the statistical method is used to calculate the pair distribution function $F_2(X_{12})$ governing the probability of occurrence of the centres of gravity of two chains in equal volume elements separated by the distance X_{12}. This function is needed to calculate the second coefficient A_2 in the osmotic pressure expansion

$$\pi = RT[M^{-1}c + A_2c^2 + \ldots].$$

Here M is the molecular weight of the solute and c the concentration. For random flight chains F_2 is unity for all values of X_{12}; A_2 is zero and the osmotic pressure follows van't Hoff's