Laplace transform example

Solve for $c_a(t)$ and $c_b(t)$ where $c_a(0) = 1$, $c_b(0) = 0$, and

$$\begin{aligned} i\hbar\dot{c}_a(t) &= Ae^{-i\omega t}c_b(t) \\ i\hbar\dot{c}_b(t) &= Be^{i\omega t}c_a(t) \end{aligned}$$

Start with Laplace transforms.

$$\begin{aligned} i\hbar[sC_a(s) - c_a(0)] &= AC_b(s + i\omega) \\ i\hbar[sC_b(s) - c_b(0)] &= BC_a(s - i\omega) \qquad (1) \end{aligned}$$

Solve for $C_a(s)$ with $c_a(0) = 1$.

$$C_a(s) = \frac{AC_b(s + i\omega) + i\hbar}{i\hbar s}$$

Hence

$$C_a(s - i\omega) = -\frac{iAC_b(s)}{\hbar(s - i\omega)} + \frac{1}{s - i\omega} \qquad (2)$$

Substitute (2) into (1) with $c_b(0) = 0$ to obtain

$$i\hbar sC_b(s) = -\frac{iABC_b(s)}{\hbar(s - i\omega)} + \frac{B}{s - i\omega}$$

Rearrange as

$$i\hbar sC_b(s) + \frac{iABC_b(s)}{\hbar(s - i\omega)} = \frac{B}{s - i\omega}$$

Rearrange again as

$$C_b(s)\left[i\hbar s + \frac{iAB}{\hbar(s - i\omega)}\right] = \frac{B}{s - i\omega}$$

Hence

$$C_b(s) = \frac{\dfrac{B}{s - i\omega}}{i\hbar s + \dfrac{iAB}{\hbar(s - i\omega)}} = \frac{-iB/\hbar}{s^2 - i\omega s + AB/\hbar^2}$$

Inverse Laplace transform:

$$\frac{1}{s^2 + as + b} \quad \Rightarrow \quad \frac{2}{k}\sin\left(\frac{kt}{2}\right)\exp\left(-\frac{at}{2}\right), \quad k = \sqrt{4b - a^2}$$

Hence for $a = -i\omega$ and $b = AB/\hbar^2$ we have

$$c_b(t) = -\frac{2iB}{\hbar k}\sin\left(\frac{kt}{2}\right)\exp\left(\frac{i\omega t}{2}\right), \quad k = \sqrt{\frac{4AB}{\hbar^2} + \omega^2}$$

Solve for $c_a(t)$.

$$c_a(t) = \frac{i\hbar\dot{c}_b(t)}{Be^{i\omega t}} = \cos\left(\frac{kt}{2}\right)\exp\left(-\frac{i\omega t}{2}\right) + \frac{i\omega}{k}\sin\left(\frac{kt}{2}\right)\exp\left(-\frac{i\omega t}{2}\right)$$