

7-3. If

$$F(j(\mathbf{r}, t)) = \exp \left(\frac{1}{2} \int \int \int \int j(\mathbf{r}_1, t_1) j(\mathbf{r}_2, t_2) R(\mathbf{r}_2 - \mathbf{r}_1, t_2 - t_1) d^3 \mathbf{r}_2 dt_2 d^3 \mathbf{r}_1 dt_1 \right)$$

where the integrals extend over all space and time, show that

$$\frac{\delta F}{\delta j(\mathbf{x}, s)} = F \int \int j(\mathbf{r}, t) \frac{1}{2} (R(\mathbf{r} - \mathbf{x}, t - s) + R(\mathbf{x} - \mathbf{r}, s - t)) d^3 \mathbf{r} dt \quad (7.27)$$

By equation (7.20) to first order we have

$$F(j(\mathbf{r}, t) + \eta(\mathbf{r}, t)) - F(j(\mathbf{r}, t)) = \int \int \frac{\delta F}{\delta j(\mathbf{x}, s)} \eta(\mathbf{x}, s) d^3 \mathbf{x} ds$$

Let

$$f(f_1, f_2) = \frac{1}{2} \int \int \int \int f_1(\mathbf{r}_1, t_1) f_2(\mathbf{r}_2, t_2) R(\mathbf{r}_2 - \mathbf{r}_1, t_2 - t_1) d^3 \mathbf{r}_2 dt_2 d^3 \mathbf{r}_1 dt_1$$

Then

$$F(j(\mathbf{r}, t) + \eta(\mathbf{r}, t)) = F(j(\mathbf{r}, t)) \exp(f(j, \eta)) \exp(f(\eta, j)) \exp(f(\eta, \eta))$$

Use the approximation $\exp(x) = 1 + x$ to obtain to first order in η

$$F(j(\mathbf{r}, t) + \eta(\mathbf{r}, t)) = F(j(\mathbf{r}, t)) (1 + f(j, \eta) + f(\eta, j))$$

Then by equation (7.20)

$$F(j(\mathbf{r}, t)) (f(j, \eta) + f(\eta, j)) = \int \int \frac{\delta F}{\delta j(\mathbf{x}, s)} \eta(\mathbf{x}, s) d^3 \mathbf{x} ds \quad (1)$$

Substitute (7.27) into (1). The $F(j(\mathbf{r}, t))$ cancel leaving

$$\begin{aligned} f(j, \eta) + f(\eta, j) &= \\ &= \frac{1}{2} \int \int \int \int j(\mathbf{r}, t) R(\mathbf{r} - \mathbf{x}, t - s) \eta(\mathbf{x}, s) d^3 \mathbf{r} dt d^3 \mathbf{x} ds \\ &+ \frac{1}{2} \int \int \int \int j(\mathbf{r}, t) R(\mathbf{x} - \mathbf{r}, s - t) \eta(\mathbf{x}, s) d^3 \mathbf{r} dt d^3 \mathbf{x} ds \end{aligned}$$

which is easily verified.