

Let  $|\Psi\rangle$  be the following “coherent” state where  $\bar{n}$  is the expected number of photons and  $|n\rangle$  is the state with exactly  $n$  photons.

$$|\Psi\rangle = \sum_{n=0}^{\infty} \sqrt{\frac{\bar{n}^n \exp(-\bar{n})}{n!}} \exp\left(-i\left(n + \frac{1}{2}\right)\omega t\right) |n\rangle$$

Let  $\hat{a}$  be the following “lowering” operator.

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

Apply operator  $\hat{a}$  to coherent state  $|\Psi\rangle$  to obtain (see derivation below)

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}} \exp(-i\omega t) |\Psi\rangle$$

and

$$\langle\Psi|\hat{a}^\dagger = (\hat{a}|\Psi\rangle)^\dagger = \sqrt{\bar{n}} \exp(i\omega t) \langle\Psi|$$

Let  $\hat{E}$  be the following electric field operator.

$$\hat{E} = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}(\hat{a} - \hat{a}^\dagger)$$

Note that  $\hat{E}$  is Hermitian.

$$\hat{E} = \hat{E}^\dagger$$

This is important because Hermitian operators have real eigenvalues. Hence we have confirmed that  $\hat{E}$  corresponds to an observable quantity.

The expected electric field is

$$\langle\hat{E}\rangle = \langle\Psi|\hat{E}|\Psi\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}} \langle\Psi|(\hat{a} - \hat{a}^\dagger)|\Psi\rangle$$

By distributive law

$$\langle\hat{E}\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}} (\langle\Psi|\hat{a}|\Psi\rangle - \langle\Psi|\hat{a}^\dagger|\Psi\rangle)$$

Substitute eigenvalues for operators.

$$\langle\hat{E}\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}} (\sqrt{\bar{n}} \exp(-i\omega t) \langle\Psi|\Psi\rangle - \sqrt{\bar{n}} \exp(i\omega t) \langle\Psi|\Psi\rangle)$$

By  $\langle \Psi | \Psi \rangle = 1$  we have

$$\langle \hat{E} \rangle = i \sqrt{\frac{\hbar \omega}{2 \epsilon_0}} (\sqrt{\bar{n}} \exp(-i \omega t) - \sqrt{\bar{n}} \exp(i \omega t))$$

Recalling that

$$2 \sin(\omega t) = i \exp(-i \omega t) - i \exp(i \omega t)$$

we have

$$\langle \hat{E} \rangle = \sqrt{\frac{2 \bar{n} \hbar \omega}{\epsilon_0}} \sin(\omega t)$$

Let  $\hat{B}$  be the following magnetic field operator.

$$\hat{B} = \sqrt{\frac{\hbar \omega \mu_0}{2}} (\hat{a} + \hat{a}^\dagger)$$

By deduction similar to that for  $\langle \hat{E} \rangle$  we obtain

$$\langle \hat{B} \rangle = \sqrt{2 \bar{n} \hbar \omega \mu_0} \cos(\omega t)$$

The energy of an electromagnetic wave is

$$U = \frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{1}{2 \mu_0} |\mathbf{B}|^2$$

For linear polarization and a suitable rotation matrix  $R$  we have

$$R \mathbf{E} = \begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix}, \quad R \mathbf{B} = \begin{pmatrix} 0 \\ B \\ 0 \end{pmatrix}$$

Hence in the rotated frame

$$U = \frac{\epsilon_0}{2} E^2 + \frac{1}{2 \mu_0} B^2$$

For a quantum field we have

$$U = \frac{\epsilon_0}{2} \langle \hat{E}^2 \rangle + \frac{1}{2 \mu_0} \langle \hat{B}^2 \rangle$$

where

$$\begin{aligned}\langle \hat{E}^2 \rangle &= \langle \Psi | \hat{E} \hat{E} | \Psi \rangle = -\frac{\hbar\omega}{2\epsilon_0} \langle \Psi | (\hat{a} - \hat{a}^\dagger)(\hat{a} - \hat{a}^\dagger) | \Psi \rangle \\ \langle \hat{B}^2 \rangle &= \langle \Psi | \hat{B} \hat{B} | \Psi \rangle = \frac{\hbar\omega\mu_0}{2} \langle \Psi | (\hat{a} + \hat{a}^\dagger)(\hat{a} + \hat{a}^\dagger) | \Psi \rangle\end{aligned}$$

For the coherent state

$$\begin{aligned}\langle \Psi | \hat{a} \hat{a} | \Psi \rangle &= (\sqrt{\bar{n}} \exp(-i\omega t))^2 &= \bar{n} \exp(-2i\omega t) \\ \langle \Psi | \hat{a}^\dagger \hat{a} | \Psi \rangle &= (\sqrt{\bar{n}} \exp(i\omega t)) (\sqrt{\bar{n}} \exp(-i\omega t)) &= \bar{n} \\ \langle \Psi | \hat{a} \hat{a}^\dagger | \Psi \rangle &= \langle \Psi | (\hat{a}^\dagger \hat{a} + 1) | \Psi \rangle = \langle \Psi | \hat{a}^\dagger \hat{a} | \Psi \rangle + \langle \Psi | \Psi \rangle &= \bar{n} + 1 \\ \langle \Psi | \hat{a}^\dagger \hat{a}^\dagger | \Psi \rangle &= (\sqrt{\bar{n}} \exp(i\omega t))^2 &= \bar{n} \exp(2i\omega t)\end{aligned}$$

The expectation  $\bar{n} + 1$  for  $\hat{a} \hat{a}^\dagger$  is from the commutator

$$\hat{a} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} = 1$$

Using the expectation values derived above we now have

$$\begin{aligned}\langle \hat{E}^2 \rangle &= -\frac{\hbar\omega}{2\epsilon_0} (\bar{n} \exp(-2i\omega t) + \bar{n} \exp(2i\omega t) - 2\bar{n} - 1) \\ \langle \hat{B}^2 \rangle &= \frac{\hbar\omega\mu_0}{2} (\bar{n} \exp(-2i\omega t) + \bar{n} \exp(2i\omega t) + 2\bar{n} + 1)\end{aligned}$$

Noting that

$$\begin{aligned}-4 \sin(\omega t)^2 &= \exp(-2i\omega t) + \exp(2i\omega t) - 2 \\ 4 \cos(\omega t)^2 &= \exp(-2i\omega t) + \exp(2i\omega t) + 2\end{aligned}$$

we have

$$\begin{aligned}\langle \hat{E}^2 \rangle &= -\frac{\hbar\omega}{2\epsilon_0} (-4\bar{n} \sin(\omega t)^2 - 1) \\ \langle \hat{B}^2 \rangle &= \frac{\hbar\omega\mu_0}{2} (4\bar{n} \cos(\omega t)^2 + 1)\end{aligned}$$

Rewrite as

$$\frac{\epsilon_0}{2} \langle \hat{E}^2 \rangle = \hbar\omega \left( \bar{n} \sin(\omega t)^2 + \frac{1}{4} \right) \quad (1)$$

$$\frac{1}{2\mu_0} \langle \hat{B}^2 \rangle = \hbar\omega \left( \bar{n} \cos(\omega t)^2 + \frac{1}{4} \right) \quad (2)$$

The total energy per unit volume is the sum of (1) and (2).

$$U = \frac{\epsilon_0}{2} \langle \hat{E}^2 \rangle + \frac{1}{2\mu_0} \langle \hat{B}^2 \rangle = \hbar\omega \left( \bar{n} + \frac{1}{2} \right)$$

Check units.

$$\hbar\omega = h\nu \propto \text{joule second} \times \frac{1}{\text{second}} = \text{joule}$$

We will now show that

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}} \exp(-i\omega t) |\Psi\rangle$$

Apply operator  $\hat{a}$  to coherent state  $|\Psi\rangle$  to obtain

$$\hat{a}|\Psi\rangle = \sum_{n=0}^{\infty} \sqrt{\frac{\bar{n}^n \exp(-\bar{n})}{n!}} \exp\left(-i\left(n + \frac{1}{2}\right)\omega t\right) \sqrt{n} |n-1\rangle$$

The  $n=0$  term vanishes hence the sum can start from  $n=1$ .

$$\hat{a}|\Psi\rangle = \sum_{n=1}^{\infty} \sqrt{\frac{\bar{n}^n \exp(-\bar{n})}{n!}} \exp\left(-i\left(n + \frac{1}{2}\right)\omega t\right) \sqrt{n} |n-1\rangle$$

The  $\sqrt{n}$  cancels with  $n$  factorial.

$$\hat{a}|\Psi\rangle = \sum_{n=1}^{\infty} \sqrt{\frac{\bar{n}^n \exp(-\bar{n})}{(n-1)!}} \exp\left(-i\left(n + \frac{1}{2}\right)\omega t\right) |n-1\rangle$$

Factor out  $\sqrt{\bar{n}} \exp(-i\omega t)$ .

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}} \exp(-i\omega t) \sum_{n=1}^{\infty} \sqrt{\frac{\bar{n}^{n-1} \exp(-\bar{n})}{(n-1)!}} \exp\left(-i\left(n - \frac{1}{2}\right)\omega t\right) |n-1\rangle$$

Substitute  $n+1$  for index  $n$ .

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}} \exp(-i\omega t) \sum_{n=0}^{\infty} \sqrt{\frac{\bar{n}^n \exp(-\bar{n})}{n!}} \exp\left(-i\left(n + \frac{1}{2}\right)\omega t\right) |n\rangle$$

Hence

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}} \exp(-i\omega t) |\Psi\rangle$$