# Sine perturbation

Let $\Psi(\mathbf{r},t)$ be the following linear combination of two wave functions where $c_a(t)$ and $c_b(t)$ are dimensionless time-dependent coefficients such that $|c_a(t)|^2 + |c_b(t)|^2 = 1$ for all time $t$.

$$\Psi(\mathbf{r},t) = c_a(t)\psi_a(\mathbf{r})\exp\left(-\tfrac{i}{\hbar}E_a t\right) + c_b(t)\psi_b(\mathbf{r})\exp\left(-\tfrac{i}{\hbar}E_b t\right)$$

Let $\hat{H}(\mathbf{r},t)$ be the Hamiltonian

$$\hat{H}(\mathbf{r},t) = \hat{H}_0(\mathbf{r}) + \hat{H}_1(\mathbf{r},t)$$

where

$$\hat{H}_0\psi_a = E_a\psi_a, \quad \hat{H}_0\psi_b = E_b\psi_b$$

Let $E_b > E_a$. The first-order perturbation solution for $c_b(t)$ is

$$c_b(t) = -\frac{i}{\hbar}\int_0^t \langle\psi_b|\hat{H}_1|\psi_a\rangle \exp(i\omega_0 t')\,dt', \quad \omega_0 = \frac{E_b - E_a}{\hbar}$$

Let $\hat{H}_1(\mathbf{r},t)$ be the perturbation

$$\hat{H}_1(\mathbf{r},t) = \hat{V}(\mathbf{r})\cos(\omega t)$$

Then by substitution

$$c_b(t) = -\frac{i}{\hbar}\langle\psi_b|\hat{V}|\psi_a\rangle \int_0^t \cos(\omega t')\exp(i\omega_0 t')\,dt'$$

Solve the integral.

$$\int_0^t \cos(\omega t')\exp(i\omega_0 t')\,dt' = -\frac{i}{2}\left(\frac{\exp\big(i(\omega_0-\omega)t\big)-1}{\omega_0-\omega} + \frac{\exp\big(i(\omega_0+\omega)t\big)-1}{\omega_0+\omega}\right) \tag{1}$$

Hence

$$c_b(t) = -\frac{\langle\psi_b|\hat{V}|\psi_a\rangle}{2\hbar}\left(\frac{\exp\big(i(\omega_0-\omega)t\big)-1}{\omega_0-\omega} + \frac{\exp\big(i(\omega_0+\omega)t\big)-1}{\omega_0+\omega}\right) \tag{2}$$

As an approximation, discard the second term since the first term dominates for $\omega \approx \omega_0$.

$$c_b(t) = -\frac{\langle\psi_b|\hat{V}|\psi_a\rangle}{2\hbar}\frac{\exp\big(i(\omega_0-\omega)t\big)-1}{\omega_0-\omega}$$

Rewrite $c_b(t)$ using a sine function.

$$c_b(t) = -\frac{i}{\hbar}\langle\psi_b|\hat{V}|\psi_a\rangle\frac{\sin\left(\frac{1}{2}(\omega_0-\omega)t\right)}{\omega_0-\omega}\exp\left(\tfrac{i}{2}(\omega_0-\omega)t\right) \tag{3}$$

The transition probability is

$$\Pr_{a\to b}(t) = |c_b(t)|^2 = \frac{\left|\langle\psi_b|\hat{V}|\psi_a\rangle\right|^2}{\hbar^2}\frac{\sin^2\left(\frac{1}{2}(\omega_0-\omega)t\right)}{(\omega_0-\omega)^2} \tag{4}$$