Harmonic oscillator propagator 2

Consider the harmonic oscillator wave function

$$\psi_n(x,t) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right) \exp\left(-\frac{m\omega x^2}{2\hbar} - i\left(n+\tfrac{1}{2}\right)\omega t\right)$$

and the harmonic oscillator propagator

$$K(x_b,t,x_a,0) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega t)}\right)^{\frac{1}{2}} \exp\left(\frac{im\omega}{2\hbar\sin(\omega t)}\left((x_b^2+x_a^2)\cos(\omega t) - 2x_b x_a\right)\right)$$

By definition of a propagator

$$\psi_n(x_b,t) = \int_{-\infty}^{\infty} K(x_b,t,x_a,0)\psi_n(x_a,0)\,dx_a$$

Prove for $n=1$. Let

$$I = \int_{-\infty}^{\infty} K(x_b,t,x_a,0)\psi_1(x_a,0)\,dx_a$$

Substitute for K and ψ_1.

$$\begin{aligned} I = \sqrt{2}\left(\frac{m^3\omega^3}{\pi\hbar^3}\right)^{\frac{1}{4}} &\left(\frac{m\omega}{2\pi i\hbar\sin(\omega t)}\right)^{\frac{1}{2}} \\ &\times \int_{-\infty}^{\infty} \exp\left(\frac{im\omega}{2\hbar\sin(\omega t)}\left((x_b^2+x_a^2)\cos(\omega t) - 2x_b x_a\right)\right) x_a \exp\left(-\frac{m\omega x_a^2}{2\hbar}\right) dx_a \end{aligned}$$

Simplify the integrand.

$$\begin{aligned} I = \sqrt{2}\left(\frac{m^3\omega^3}{\pi\hbar^3}\right)^{\frac{1}{4}} &\left(\frac{m\omega}{2\pi i\hbar\sin(\omega t)}\right)^{\frac{1}{2}} \\ &\times \int_{-\infty}^{\infty} x_a \exp\left(\frac{im\omega x_a^2\exp(i\omega t)}{2\hbar\sin(\omega t)} - \frac{im\omega x_a}{\hbar\sin(\omega t)} + \frac{im\omega x_b^2\cos(\omega t)}{2\hbar\sin(\omega t)}\right) dx_a \end{aligned}$$

Let

$$a = -\frac{im\omega\exp(i\omega t)}{2\hbar\sin(\omega t)}, \quad b = -\frac{im\omega}{\hbar\sin(\omega t)}, \quad c = \frac{im\omega x_b^2\cos(\omega t)}{2\hbar\sin(\omega t)}$$

so that

$$I = \sqrt{2}\left(\frac{m^3\omega^3}{\pi\hbar^3}\right)^{\frac{1}{4}} \left(\frac{m\omega}{2\pi i\hbar\sin(\omega t)}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} x_a \exp(-ax_a^2 + bx_a + c)\,dx_a$$

Solve the integral.

$$\begin{aligned} I &= \sqrt{2}\left(\frac{m^3\omega^3}{\pi\hbar^3}\right)^{\frac{1}{4}} \left(\frac{m\omega}{2\pi i\hbar\sin(\omega t)}\right)^{\frac{1}{2}} \frac{\sqrt{\pi}}{2}\frac{b}{a^{3/2}} \exp\left(\frac{b^2}{4a} + c\right) \\ &= \sqrt{2}\left(\frac{m^3\omega^3}{\pi\hbar^3}\right)^{\frac{1}{4}} x_b \exp\left(-\frac{m\omega x_b^2}{2\hbar} - \tfrac{3}{2}i\omega t\right) \\ &= \psi_1(x_b,t) \end{aligned}$$

Hence by solving the integral we have proven that

$$\psi_1(x_b, t) = \int_{-\infty}^{\infty} K(x_b, t, x_a, 0)\psi_1(x_a, 0)\, dx_a$$