Problem 1.3. Verify (1.33).

This is equation (1.33).

$$\begin{aligned}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,\boldsymbol{\sigma}\cdot\boldsymbol{\pi} &= \boldsymbol{\pi}^2 + i\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\times\boldsymbol{\pi}\\ &= \boldsymbol{\pi}^2 - \frac{e\hbar}{c}\boldsymbol{\sigma}\cdot\mathbf{B} \qquad (1.33)\end{aligned}$$

The first equality is an identity hence the important part is to show that

$$i\boldsymbol{\pi}\times\boldsymbol{\pi} = -\frac{e\hbar}{c}\mathbf{B}$$

We are given

$$\boldsymbol{\pi} = \mathbf{p} - (e/c)\mathbf{A}, \quad \mathbf{p} = -i\hbar\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}, \quad \mathbf{B} = \operatorname{curl}\mathbf{A}$$

The cross product is distributive hence

$$\boldsymbol{\pi}\times\boldsymbol{\pi} = \mathbf{p}\times\mathbf{p} - \mathbf{p}\times(e/c)\mathbf{A} - (e/c)\mathbf{A}\times\mathbf{p} + (e/c)^2\mathbf{A}\times\mathbf{A}$$

The terms $\mathbf{p}\times\mathbf{p}$ and $\mathbf{A}\times\mathbf{A}$ vanish leaving

$$\boldsymbol{\pi}\times\boldsymbol{\pi} = -\frac{e}{c}(\mathbf{p}\times\mathbf{A} + \mathbf{A}\times\mathbf{p})$$

It is useful at this point to apply the operators to a wave function and write

$$(\boldsymbol{\pi}\times\boldsymbol{\pi})\psi = -\frac{e}{c}(\mathbf{p}\times(\mathbf{A}\psi) + \mathbf{A}\times\mathbf{p}\psi)$$

It can be shown that

$$\mathbf{p}\times(\mathbf{A}\psi) = (\mathbf{p}\times\mathbf{A})\psi + \mathbf{p}\psi\times\mathbf{A} \qquad (1)$$

Hence

$$(\boldsymbol{\pi}\times\boldsymbol{\pi})\psi = -\frac{e}{c}((\mathbf{p}\times\mathbf{A})\psi + \mathbf{p}\psi\times\mathbf{A} + \mathbf{A}\times\mathbf{p}\psi)$$

The last two terms cancel by antisymmetry hence

$$(\boldsymbol{\pi}\times\boldsymbol{\pi})\psi = -\frac{e}{c}(\mathbf{p}\times\mathbf{A})\psi$$

Note that

$$\mathbf{p} \times \mathbf{A} = -i\hbar \begin{pmatrix} \frac{\partial}{\partial_y} A_z - \frac{\partial}{\partial_z} A_y \\ \frac{\partial}{\partial_z} A_x - \frac{\partial}{\partial_x} A_z \\ \frac{\partial}{\partial_x} A_y - \frac{\partial}{\partial_y} A_x \end{pmatrix} = -i\hbar \operatorname{curl} \mathbf{A}$$

Hence

$$\boldsymbol{\pi} \times \boldsymbol{\pi} = i\frac{e\hbar}{c} \operatorname{curl} \mathbf{A}$$

Multiply both sides by i to complete the proof.

$$i\boldsymbol{\pi} \times \boldsymbol{\pi} = -\frac{e\hbar}{c} \operatorname{curl} \mathbf{A}$$