8-3. Show that Q^c_α, Q^s_α are normal coordinates corresponding to standing wave normal modes $\cos(2\pi\alpha j/N)$ and $\sin(2\pi\alpha j/N)$, in the sense that (for N odd)

$$q_j(t) = \sqrt{\frac{2}{N}}\left(\frac{1}{2}Q^c_0(t) + \sum_{\alpha=1}^{(N-1)/2}\left(Q^c_\alpha(t)\cos\frac{2\pi\alpha j}{N} + Q^s_\alpha(t)\sin\frac{2\pi\alpha j}{N}\right)\right) \quad (8.82)$$

Consider the following equations.

$$Q_\alpha(t) = \frac{1}{\sqrt{N}}\sum_{k=1}^{N} q_k(t)\left(\cos\frac{2\pi\alpha k}{N} - i\sin\frac{2\pi\alpha k}{N}\right) \quad (8.77)$$

$$Q^c_\alpha = \frac{1}{\sqrt{2}}(Q_\alpha + Q^*_\alpha) \quad (8.79)$$

$$Q^s_\alpha = \frac{i}{\sqrt{2}}(Q_\alpha - Q^*_\alpha) \quad (8.80)$$

Let

$$\begin{aligned}
T_1 &= \cos\frac{2\pi\alpha k}{N}\cos\frac{2\pi\alpha j}{N} - i\sin\frac{2\pi\alpha k}{N}\cos\frac{2\pi\alpha j}{N} \\
T_2 &= \cos\frac{2\pi\alpha k}{N}\cos\frac{2\pi\alpha j}{N} + i\sin\frac{2\pi\alpha k}{N}\cos\frac{2\pi\alpha j}{N} \\
T_3 &= i\cos\frac{2\pi\alpha k}{N}\sin\frac{2\pi\alpha j}{N} + \sin\frac{2\pi\alpha k}{N}\sin\frac{2\pi\alpha j}{N} \\
T_4 &= i\cos\frac{2\pi\alpha k}{N}\sin\frac{2\pi\alpha j}{N} - \sin\frac{2\pi\alpha k}{N}\sin\frac{2\pi\alpha j}{N}
\end{aligned}$$

Hence

$$T = T_1 + T_2 + T_3 - T_4 = 2\cos\frac{2\pi\alpha k}{N}\cos\frac{2\pi\alpha j}{N} + 2\sin\frac{2\pi\alpha k}{N}\sin\frac{2\pi\alpha j}{N}$$

By trigonometric identities

$$T = 2\cos\left(\frac{2\pi\alpha}{N}(j-k)\right)$$

It follows that

$$\sum_{\alpha=1}^{(N-1)/2}\left(Q_\alpha^c\cos\frac{2\pi\alpha j}{N}+Q_\alpha^s\sin\frac{2\pi\alpha j}{N}\right)=\sum_{\alpha=1}^{(N-1)/2}\frac{1}{\sqrt{N}}\sum_{k=1}^{N}q_k\frac{T}{\sqrt{2}}$$
$$=\sqrt{\frac{2}{N}}\sum_{\alpha=1}^{(N-1)/2}\sum_{k=1}^{N}q_k\cos\left(\frac{2\pi\alpha}{N}(j-k)\right)$$