The following muon data is from Particle Data Group.

μ MEAN LIFE τ

Measurements with an error $> 0.001 \times 10^{-6}$ s have been omitted.

VALUE (10^{-6} s)	DOCUMENT ID		TECN	CHG	COMMENT
2.1969811±0.0000022 OUR AVERAGE					
2.1969803±0.0000021±0.0000007 [1]	TISHCHENKO	13	CNTR	+	Surface μ^+ at PSI
2.197083 ±0.000032 ±0.000015	BARCZYK	08	CNTR	+	Muons from π^+ decay at rest
2.197013 ±0.000021 ±0.000011	CHITWOOD	07	CNTR	+	Surface μ^+ at PSI
2.197078 ±0.000073	BARDIN	84	CNTR	+	
2.197025 ±0.000155	BARDIN	84	CNTR	−	
2.19695 ±0.00006	GIOVANETTI	84	CNTR	+	
2.19711 ±0.00008	BALANDIN	74	CNTR	+	
2.1973 ±0.0003	DUCLOS	73	CNTR	+	
• • • We do not use the following data for averages, fits, limits, etc. • • •					
2.1969803±0.0000022	WEBBER	11	CNTR	+	Surface μ^+ at PSI

[1] TISHCHENKO 13 uses 1.6×10^{12} μ^+ events and supersedes WEBBER 11.

From "V minus A" theory we have the following formula for muon lifetime τ.

$$\tau = \frac{96\pi^2 h}{G_F^2 \left(m_\mu c^2\right)^5}$$

Symbol G_F is Fermi coupling constant, m_μ is muon mass.

From NIST we have

$$\begin{aligned}
G_F &= 1.1663787 \times 10^{-5}\ \text{GeV}^{-2} \\
m_\mu &= 1.883531627 \times 10^{-28}\ \text{kilogram} \\
h &= 6.62607015 \times 10^{-34}\ \text{joule second (exact)} \\
c &= 299792458\ \text{meter second}^{-1}\ \text{(exact)} \\
1\,\text{eV} &= 1.602176634 \times 10^{-19}\ \text{joule (exact)}
\end{aligned}$$

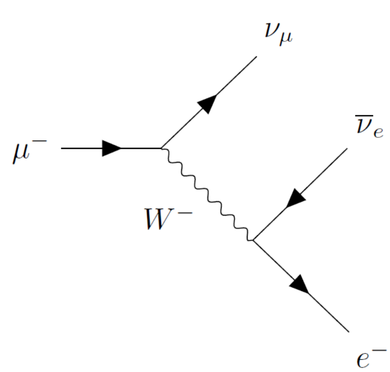
Hence

$$\tau = 2.18735 \times 10^{-6}\,\text{second}$$

The result is a bit smaller than the PDG value.

$$\frac{\tau}{2.1969811 \times 10^{-6}\ \text{second}} = 0.9956$$

A muon decays into a muon neutrino, an electron anti-neutrino, and an electron.



Particle	Symbol	Momentum	Spinor (up)	Spinor (down)
Muon	μ^-	p_1	u_{11}	u_{12}
Muon neutrino	ν_μ	p_2	u_{21}	u_{22}
Electron anti-neutrino	$\bar{\nu}_e$	p_3	v_{31}	v_{32}
Electron	e^-	p_4	u_{41}	u_{42}

We will use the following momentum vectors.

$$\underbrace{p_1 = \begin{pmatrix} E_1 \\ p_{1x} \\ p_{1y} \\ p_{1z} \end{pmatrix}}_{\mu^-} \qquad \underbrace{p_2 = \begin{pmatrix} E_2 \\ p_{2x} \\ p_{2y} \\ p_{2z} \end{pmatrix}}_{\nu_\mu} \qquad \underbrace{p_3 = \begin{pmatrix} E_3 \\ p_{3x} \\ p_{3y} \\ p_{3z} \end{pmatrix}}_{\bar{\nu}_e} \qquad \underbrace{p_4 = \begin{pmatrix} E_4 \\ p_{4x} \\ p_{4y} \\ p_{4z} \end{pmatrix}}_{e^-}$$

And we will use the following unnormalized Dirac spinors.

$$\underbrace{u_{11} = \begin{pmatrix} E_1 + m_\mu \\ 0 \\ p_{1z} \\ p_{1x} + ip_{1y} \end{pmatrix}}_{\mu^- \text{ up}} \qquad \underbrace{u_{21} = \begin{pmatrix} E_2 + m_2 \\ 0 \\ p_{2z} \\ p_{2x} + ip_{2y} \end{pmatrix}}_{\nu_\mu \text{ up}} \qquad \underbrace{v_{31} = \begin{pmatrix} p_{3z} \\ p_{3x} + ip_{3y} \\ E_3 + m_3 \\ 0 \end{pmatrix}}_{\bar{\nu}_e \text{ up}} \qquad \underbrace{u_{41} = \begin{pmatrix} E_4 + m_4 \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix}}_{e^- \text{ up}}$$

$$\underbrace{u_{12} = \begin{pmatrix} 0 \\ E_1 + m_\mu \\ p_{1x} - ip_{1y} \\ -p_{1z} \end{pmatrix}}_{\mu^- \text{ down}} \qquad \underbrace{u_{22} = \begin{pmatrix} 0 \\ E_2 + m_2 \\ p_{2x} - ip_{2y} \\ -p_{2z} \end{pmatrix}}_{\nu_\mu \text{ down}} \qquad \underbrace{v_{32} = \begin{pmatrix} p_{3x} - ip_{3y} \\ -p_{3z} \\ 0 \\ E_3 + m_3 \end{pmatrix}}_{\bar{\nu}_e \text{ down}} \qquad \underbrace{u_{42} = \begin{pmatrix} 0 \\ E_4 + m_4 \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}}_{e^- \text{ down}}$$

Symbol E_n is total energy of particle n.

$$E_1 = \sqrt{(p_{1x})^2 + (p_{1y})^2 + (p_{1z})^2 + m_\mu^2}$$
$$E_2 = \sqrt{(p_{2x})^2 + (p_{2y})^2 + (p_{2z})^2 + m_2^2}$$
$$E_3 = \sqrt{(p_{3x})^2 + (p_{3y})^2 + (p_{3z})^2 + m_3^2}$$
$$E_4 = \sqrt{(p_{4x})^2 + (p_{4y})^2 + (p_{4z})^2 + m_4^2}$$

From the Feynman diagram above we have the following amplitude $\mathcal{M}_{abcd}$ where each letter in $abcd$ can be either 1 (spin up) or 2 (spin down).

$$\mathcal{M}_{abcd} = \frac{G_F}{\sqrt{2}\sqrt{N}} \big(\underset{e^-}{\bar{u}_{4d}}\ \gamma^\mu(1-\gamma^5)\ \underset{\bar{\nu}_e}{v_{3c}}\big)\big(\underset{\nu_\mu}{\bar{u}_{2b}}\ \gamma_\mu(1-\gamma^5)\ \underset{\mu^-}{u_{1a}}\big)$$

Symbol N is the following spinor normalization constant.

$$N = (E_1 + m_\mu)(E_2 + m_2)(E_3 + m_3)(E_4 + m_4)$$

Recall that the magnitude squared of an amplitude is a probability density and also an observable.

$$|\mathcal{M}_{abcd}|^2 = \mathcal{M}^*_{abcd}\mathcal{M}_{abcd}$$

In a typical muon decay experiment the spins are not observed. Consequently, the experimental result is an average of spin states. The average is computed by summing over all spin states and dividing by the number of initial spin states. The muon has two spin states hence the divisor is two.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{2}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}|\mathcal{M}_{abcd}|^2$$

The result is a simple formula.

$$\langle|\mathcal{M}|^2\rangle = 64G_F^2(p_1\cdot p_3)(p_2\cdot p_4) \tag{1}$$

In component notation we have

$$\langle|\mathcal{M}|^2\rangle = 64G_F^2\Big((p_1)^\mu\; g_{\mu\nu}\;(p_3)^\nu\Big)\Big((p_2)^\mu\; g_{\mu\nu}\;(p_4)^\nu\Big)$$

where

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Muon decay rate Γ is the integral of $\langle|\mathcal{M}|^2\rangle$ over all possible values of momentum. In the muon rest frame $p_1 = (m_\mu, 0, 0, 0)$ we have by Fermi's golden rule the following formula.

$$\Gamma = \frac{(2\pi)^4}{2m_\mu}\int\langle|\mathcal{M}|^2\rangle\,\delta(p_1-p_2-p_3-p_4)\,\frac{d^3p_2}{2(2\pi)^3E_2}\,\frac{d^3p_3}{2(2\pi)^3E_3}\,\frac{d^3p_4}{2(2\pi)^3E_4}$$

It can be shown that

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

Muon lifetime τ is the inverse of decay rate.

$$\tau = \frac{1}{\Gamma} = \frac{192\pi^3}{G_F^2 m_\mu^5}$$

Converting from natural units to physical values for h and c yields

$$\tau = \frac{96\pi^2 h}{G_F^2\,(m_\mu c^2)^5}$$

Probability density $\langle|\mathcal{M}|^2\rangle$ can also be computed using the following Casimir trick.

$$\langle|\mathcal{M}|^2\rangle = \frac{G_F^2}{4}\operatorname{Tr}\Big(\not{p}_4\gamma^\mu(1-\gamma^5)\not{p}_3\gamma^\nu(1-\gamma^5)\Big)\operatorname{Tr}\Big(\not{p}_2\gamma_\mu(1-\gamma^5)\not{p}_1\gamma_\nu(1-\gamma^5)\Big)$$

The slashed symbols are 4×4 matrices computed as

$$\not{p} = p\cdot\gamma = p^0\gamma^0 - p^1\gamma^1 - p^2\gamma^2 - p^3\gamma^3$$

For example,

$$
\not{p}_1 = \begin{pmatrix} E_1 & 0 & -p_{1z} & -p_{1x} + ip_{1y} \\ 0 & E_1 & -p_{1x} - ip_{1y} & p_{1z} \\ p_{1z} & p_{1x} - ip_{1y} & -E_1 & 0 \\ p_{1x} + ip_{1y} & -p_{1z} & 0 & -E_1 \end{pmatrix}
$$