Rutherford scattering 2

Find the scattering cross section for the screened Coulomb potential

$$V(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r}\exp\left(-\frac{r}{a}\right)$$

Let $f(\mathbf{p})$ be the scattering amplitude for momentum transfer vector $\mathbf{p} = \mathbf{p}_i - \mathbf{p}_f$. The following formula is the Born approximation for $f(\mathbf{p})$.

$$f(\mathbf{p}) = \frac{m}{2\pi\hbar^2}\int \exp\left(\frac{i\mathbf{p}\cdot\mathbf{r}}{\hbar}\right) V(\mathbf{r})\,d\mathbf{r}$$

Convert to polar coordinates where $p = |\mathbf{p}|$.

$$f(\mathbf{p}) = \frac{m}{2\pi\hbar^2}\int_0^\infty\int_0^\pi\int_0^{2\pi}\exp\left(\frac{ipr\cos\theta}{\hbar}\right) V(r,\theta,\phi)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

Substitute the screened Coulomb potential

$$V(r,\theta,\phi) = -\frac{Ze^2}{4\pi\varepsilon_0 r}\exp\left(-\frac{r}{a}\right)$$

to obtain

$$f(\mathbf{p}) = -\frac{mZe^2}{8\pi^2\varepsilon_0\hbar^2}\int_0^\infty\int_0^\pi\int_0^{2\pi}\exp\left(\frac{ipr\cos\theta}{\hbar}-\frac{r}{a}\right) r\sin\theta\,dr\,d\theta\,d\phi$$

Integrate over ϕ (multiply by 2π).

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\varepsilon_0\hbar^2}\int_0^\infty\int_0^\pi\exp\left(\frac{ipr\cos\theta}{\hbar}-\frac{r}{a}\right) r\sin\theta\,dr\,d\theta$$

Transform the integral over θ into an integral over u where $u = \cos\theta$ and $du = -\sin\theta\,d\theta$. The minus sign in du is canceled by interchanging integration limits $\cos 0 = 1$ and $\cos\pi = -1$.

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\varepsilon_0\hbar^2}\int_0^\infty\int_{-1}^1\exp\left(\frac{ipru}{\hbar}-\frac{r}{a}\right) r\,dr\,du$$

Solve the integral over u.

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\varepsilon_0\hbar^2}\int_0^\infty\left[\frac{\hbar}{ipr}\exp\left(\frac{ipru}{\hbar}-\frac{r}{a}\right)\right]_{u=-1}^{u=1} r\,dr$$

Cancel r and evaluate the limits.

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\varepsilon_0\hbar^2}\int_0^\infty\frac{\hbar}{ip}\left[\exp\left(\frac{ipr}{\hbar}-\frac{r}{a}\right)-\exp\left(-\frac{ipr}{\hbar}-\frac{r}{a}\right)\right]dr$$

Solve the integral over r.

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\varepsilon_0\hbar^2}\frac{\hbar}{ip}\left[\frac{1}{ip/\hbar - 1/a}\exp\left(\frac{ipr}{\hbar} - \frac{r}{a}\right) + \frac{1}{ip/\hbar + 1/a}\exp\left(-\frac{ipr}{\hbar} - \frac{r}{a}\right)\right]_{r=0}^{r=\infty}$$

Evaluate the limits. The exponentials vanish at the upper limit.

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\varepsilon_0\hbar^2}\frac{\hbar}{ip}\left[-\frac{1}{ip/\hbar - 1/a} - \frac{1}{ip/\hbar + 1/a}\right] = -\frac{mZe^2}{2\pi\varepsilon_0\left[p^2 + (\hbar/a)^2\right]} \tag{1}$$

Substitute $e^2 = 4\pi\varepsilon_0\alpha\hbar c$.

$$f(\mathbf{p}) = -\frac{2mZ\alpha\hbar c}{p^2 + (\hbar/a)^2}$$

Note that $\mathbf{p}$ is momentum transfer such that

$$p^2 = |\mathbf{p}|^2 = 4mE(1 - \cos\theta)$$

Hence

$$f(\theta) = -\frac{2mZ\alpha\hbar c}{4mE(1 - \cos\theta) + (\hbar/a)^2} \tag{2}$$

Cancel m in the numerator.

$$f(\theta) = -\frac{2Z\alpha\hbar c}{4E(1 - \cos\theta) + \frac{1}{m}(\hbar/a)^2}$$

The cross section is

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{4Z^2\alpha^2(\hbar c)^2}{\left[4E(1 - \cos\theta) + \frac{1}{m}(\hbar/a)^2\right]^2}$$

Let $a \to \infty$ to obtain the ordinary Rutherford cross section.

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2(\hbar c)^2}{4E^2(1 - \cos\theta)^2}$$