# Yukawa potential

Find $\sigma_{\text{total}}$ for the Yukawa potential

$$V(r) = -\frac{V_0 \exp(-\mu r)}{\mu r}$$

Let $f(\mathbf{k})$ be the scattering amplitude for $\mathbf{k} = \mathbf{k}_i - \mathbf{k}_f$. The following formula is the Born approximation for $f(\mathbf{k})$.

$$f(\mathbf{k}) = \frac{m}{2\pi\hbar^2} \int \exp(i\mathbf{k}\cdot\mathbf{r}) V(\mathbf{r})\, d\mathbf{r}$$

Convert to polar coordinates where $k = |\mathbf{k}|$.

$$f(\mathbf{k}) = \frac{m}{2\pi\hbar^2} \int_0^\infty \int_0^\pi \int_0^{2\pi} \exp(ikr\cos\theta) V(r,\theta,\phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Substitute the Yukawa potential

$$V(r,\theta,\phi) = -\frac{V_0 \exp(-\mu r)}{\mu r}$$

to obtain

$$f(\mathbf{k}) = -\frac{mV_0}{2\pi\hbar^2\mu} \int_0^\infty \int_0^\pi \int_0^{2\pi} \exp(ikr\cos\theta - \mu r)\, r \sin\theta\, dr\, d\theta\, d\phi$$

Integrate over $\phi$ (multiply by $2\pi$).

$$f(\mathbf{k}) = -\frac{mV_0}{\hbar^2\mu} \int_0^\infty \int_0^\pi \exp(ikr\cos\theta - \mu r)\, r \sin\theta\, dr\, d\theta$$

Let $y = \cos\theta$ and $dy = -\sin\theta\, d\theta$. The minus sign in $dy$ is canceled by interchanging integration limits $\cos(0) = 1$ and $\cos(\pi) = -1$.

$$f(\mathbf{k}) = -\frac{mV_0}{\hbar^2\mu} \int_0^\infty \int_{-1}^1 \exp(ikry - \mu r)\, r\, dr\, dy$$

Solve the integral over $y$.

$$f(\mathbf{k}) = -\frac{mV_0}{\hbar^2\mu} \int_0^\infty \left[ \frac{1}{ikr} \exp(ikry - \mu r) \right]_{y=-1}^{y=1} r\, dr$$

Cancel $r$ and evaluate the limits.

$$f(\mathbf{k}) = -\frac{mV_0}{\hbar^2\mu} \frac{1}{ik} \int_0^\infty \big[ \exp(ikr - \mu r) - \exp(-ikr - \mu r) \big]\, dr$$

Solve the integral over $r$.

$$f(\mathbf{k}) = -\frac{mV_0}{\hbar^2\mu} \frac{1}{ik} \left[ \frac{1}{ik - \mu} \exp(ikr - \mu r) + \frac{1}{ik + \mu} \exp(-ikr - \mu r) \right]_{r=0}^{r=\infty}$$

Evaluate the limits. The exponentials vanish at the upper limit.

$$f(\mathbf{k}) = -\frac{mV_0}{\hbar^2\mu}\frac{1}{ik}\left[-\frac{1}{ik-\mu} - \frac{1}{ik+\mu}\right] = -\frac{2mV_0}{\hbar^2\mu}\frac{1}{k^2+\mu^2} \tag{1}$$

Substitute

$$k^2 = \frac{|\mathbf{p}_i - \mathbf{p}_f|^2}{\hbar^2} = \frac{4mE(1-\cos\theta)}{\hbar^2}$$

to obtain

$$f(\theta) = -\frac{2mV_0}{\mu}\frac{1}{4mE(1-\cos\theta)+\mu^2\hbar^2} \tag{2}$$

Hence

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left(\frac{2mV_0}{\mu}\right)^2 \frac{1}{\left[4mE(1-\cos\theta)+\mu^2\hbar^2\right]^2}$$

For the total cross section we have

$$\sigma_{\text{total}} = \int \frac{d\sigma}{d\Omega}\,d\Omega = \left(\frac{2mV_0}{\mu}\right)^2 \int_0^\pi \int_0^{2\pi} \frac{1}{\left[4mE(1-\cos\theta)+\mu^2\hbar^2\right]^2}\sin\theta\,d\theta\,d\phi$$

Integrate over $\phi$ (multiply by $2\pi$).

$$\sigma_{\text{total}} = 2\pi\left(\frac{2mV_0}{\mu}\right)^2 \int_0^\pi \frac{1}{\left[4mE(1-\cos\theta)+\mu^2\hbar^2\right]^2}\sin\theta\,d\theta$$

Let $y = 1-\cos\theta$ and $dy = \sin\theta\,d\theta$. The limits transform as $1-\cos(0)=0$ and $1-\cos(\pi)=2$.

$$\sigma_{\text{total}} = 2\pi\left(\frac{2mV_0}{\mu}\right)^2 \int_0^2 \frac{1}{\left[4mEy+\mu^2\hbar^2\right]^2}\,dy$$

Solve the integral.

$$\sigma_{\text{total}} = 2\pi\left(\frac{2mV_0}{\mu}\right)^2 \left[-\frac{1}{4mE(4mEy+\mu^2\hbar^2)}\right]_{y=0}^{y=2}$$

Evaluate the limits.

$$\sigma_{\text{total}} = 2\pi\left(\frac{2mV_0}{\mu}\right)^2 \frac{2}{8mE\mu^2\hbar^2+\mu^4\hbar^4} \tag{3}$$

For the dimensions of $\sigma_{\text{total}}$ we have

$$\begin{aligned} mV_0 &\propto \text{mass} \times \text{energy} = \text{momentum}^2 \\ mE &\propto \text{mass} \times \text{energy} = \text{momentum}^2 \\ \mu\hbar &\propto \text{length}^{-1} \times \text{energy} \times \text{time} = \text{momentum} \end{aligned}$$

Hence

$$\begin{aligned} (mV_0)^2 &\propto \text{momentum}^4 \\ mE\mu^2\hbar^2 &\propto \text{momentum}^4 \\ \mu^4\hbar^4 &\propto \text{momentum}^4 \end{aligned}$$

The momentum dimensions cancel leaving units of area.

$$\sigma_{\text{total}} \propto \frac{1}{\mu^2} \propto \text{length}^2$$

See exercise 10.7 of *Quantum Mechanics* by Richard Fitzpatrick.