Let $|\Psi\rangle$ be a coherent state where $\bar{n}$ is the expected number of photons.

$$|\Psi\rangle = \sum_{n=0}^{\infty} \sqrt{\frac{\bar{n}^n \exp(-\bar{n})}{n!}} \exp\left(-i\left(n+\tfrac{1}{2}\right)\omega t\right)|n\rangle$$

It can be shown that

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}}\exp(-i\omega t)|\Psi\rangle$$

It follows that

$$\langle\Psi|\hat{a}^\dagger = (\hat{a}|\Psi\rangle)^\dagger = \sqrt{\bar{n}}\exp(i\omega t)\langle\Psi|$$

Let $\hat{E}$ be the electric field operator

$$\hat{E} = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}(\hat{a}-\hat{a}^\dagger)$$

The expected electric field is

$$\langle\hat{E}\rangle = \langle\Psi|\hat{E}|\Psi\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\langle\Psi|(\hat{a}-\hat{a}^\dagger)|\Psi\rangle$$

By distributive law

$$\langle\hat{E}\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\left(\langle\Psi|\hat{a}|\Psi\rangle - \langle\Psi|\hat{a}^\dagger|\Psi\rangle\right)$$

Substitute eigenvalues for operators.

$$\langle\hat{E}\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\left(\sqrt{\bar{n}}\exp(-i\omega t)\langle\Psi|\Psi\rangle - \sqrt{\bar{n}}\exp(i\omega t)\langle\Psi|\Psi\rangle\right)$$

By $\langle\Psi|\Psi\rangle = 1$ we have

$$\langle\hat{E}\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\left(\sqrt{\bar{n}}\exp(-i\omega t) - \sqrt{\bar{n}}\exp(i\omega t)\right)$$

Recalling that

$$2\sin(\omega t) = i\exp(-i\omega t) - i\exp(i\omega t)$$

we have

$$\langle\hat{E}\rangle = \sqrt{\frac{2\bar{n}\hbar\omega}{\epsilon_0}}\sin(\omega t)$$

Hence the peak amplitude is proportional to $\sqrt{\bar{n}}$.

The total energy of the electromagnetic field per unit volume is

$$U = \frac{\epsilon_0}{2}|\mathbf{E}|^2 + \frac{1}{2\mu_0}|\mathbf{B}|^2$$

For linearly polarized light and a suitable rotation matrix R we have

$$R\mathbf{E} = \begin{pmatrix} E_x \\ 0 \\ 0 \end{pmatrix}, \quad R\mathbf{B} = \begin{pmatrix} 0 \\ B_y \\ 0 \end{pmatrix}$$

Hence in the rotated frame

$$U = \frac{\epsilon_0}{2}E_x^2 + \frac{1}{2\mu_0}B_y^2$$

For the quantum field we have

$$U = \frac{\epsilon_0}{2}\langle \hat{E}_x^2 \rangle + \frac{1}{2\mu_0}\langle \hat{B}_y^2 \rangle$$

where

$$\hat{E}_x = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}(\hat{a} - \hat{a}^\dagger), \quad \hat{B}_y = \sqrt{\frac{\hbar\omega\mu_0}{2}}(\hat{a} + \hat{a}^\dagger)$$

For the coherent state we have

$$\begin{aligned}
\langle\Psi|\hat{a}\hat{a}|\Psi\rangle &= \left(\sqrt{\bar{n}}\exp(-i\omega t)\right)^2 &&= \bar{n}\exp(-2i\omega t) \\
\langle\Psi|\hat{a}\hat{a}^\dagger|\Psi\rangle &= \langle\Psi|(\hat{a}^\dagger\hat{a} + 1)|\Psi\rangle &&= \bar{n} + 1 \\
\langle\Psi|\hat{a}^\dagger\hat{a}|\Psi\rangle &= \left(\sqrt{\bar{n}}\exp(i\omega t)\right)\left(\sqrt{\bar{n}}\exp(-i\omega t)\right) &&= \bar{n} \\
\langle\Psi|\hat{a}^\dagger\hat{a}^\dagger|\Psi\rangle &= \left(\sqrt{\bar{n}}\exp(i\omega t)\right)^2 &&= \bar{n}\exp(2i\omega t)
\end{aligned}$$

The expectation $\bar{n} + 1$ for $\hat{a}\hat{a}^\dagger$ is from the commutator

$$\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$$

Hence

$$\begin{aligned}
\langle \hat{E}_x^2 \rangle = \langle\Psi|\hat{E}_x\hat{E}_x|\Psi\rangle &= -\frac{\hbar\omega}{2\epsilon_0}\langle\Psi|(\hat{a} - \hat{a}^\dagger)(\hat{a} - \hat{a}^\dagger)|\Psi\rangle \\
&= -\frac{\hbar\omega}{2\epsilon_0}\left(\bar{n}\exp(-2i\omega t) + \bar{n}\exp(2i\omega t) - 2\bar{n} - 1\right)
\end{aligned}$$

and

$$\begin{aligned}\langle \hat{B}_y^2 \rangle = \langle \Psi | \hat{B}_y \hat{B}_y | \Psi \rangle &= \frac{\hbar \omega \mu_0}{2} \langle \Psi | (\hat{a} + \hat{a}^\dagger)(\hat{a} + \hat{a}^\dagger) | \Psi \rangle \\ &= \frac{\hbar \omega \mu_0}{2} \left(\bar{n} \exp(-2 i \omega t) + \bar{n} \exp(2 i \omega t) + 2\bar{n} + 1 \right)\end{aligned}$$

Noting that

$$\begin{aligned}4 \sin(\omega t)^2 &= -\exp(-2 i \omega t) - \exp(2 i \omega t) + 2 \\ 4 \cos(\omega t)^2 &= \exp(-2 i \omega t) + \exp(2 i \omega t) + 2\end{aligned}$$

we have

$$\begin{aligned}\langle \hat{E}_x^2 \rangle &= \frac{2 \bar{n} \hbar \omega}{\epsilon_0} \sin(\omega t)^2 + \frac{\hbar \omega}{2 \epsilon_0} \\ \langle \hat{B}_y^2 \rangle &= 2 \bar{n} \hbar \omega \mu_0 \cos(\omega t)^2 + \frac{\hbar \omega \mu_0}{2}\end{aligned}$$

Rewrite as

$$\begin{aligned}\frac{\epsilon_0}{2} \langle \hat{E}_x^2 \rangle &= \bar{n} \hbar \omega \sin(\omega t)^2 + \frac{\hbar \omega}{4} \\ \frac{1}{2\mu_0} \langle \hat{B}_y^2 \rangle &= \bar{n} \hbar \omega \cos(\omega t)^2 + \frac{\hbar \omega}{4}\end{aligned}$$

Hence the total energy per unit volume is

$$U = \frac{\epsilon_0}{2} \langle \hat{E}_x^2 \rangle + \frac{1}{2\mu_0} \langle \hat{B}_y^2 \rangle = \left(\bar{n} + \tfrac{1}{2} \right) \hbar \omega$$

Checking physical dimensions we have

$$\hbar \omega = h \nu \propto \text{joule second} \times \frac{1}{\text{second}} = \text{joule}$$

We will now show that

$$\hat{a} | \Psi \rangle = \sqrt{\bar{n}} \exp(-i \omega t) | \Psi \rangle$$

Apply operator $\hat{a}$ to coherent state $|\Psi\rangle$ to obtain

$$\hat{a} | \Psi \rangle = \sum_{n=0}^{\infty} \sqrt{\frac{\bar{n}^n \exp(-\bar{n})}{n!}} \exp\left(-i \left(n + \tfrac{1}{2} \right) \omega t \right) \sqrt{n} | n - 1 \rangle$$

The $n = 0$ term vanishes hence the sum can start from $n = 1$.

$$\hat{a}|\Psi\rangle = \sum_{n=1}^{\infty} \sqrt{\frac{\bar{n}^n \exp(-\bar{n})}{n!}} \exp\left(-i\left(n + \tfrac{1}{2}\right)\omega t\right) \sqrt{n}|n-1\rangle$$

The $\sqrt{n}$ cancels with n factorial.

$$\hat{a}|\Psi\rangle = \sum_{n=1}^{\infty} \sqrt{\frac{\bar{n}^n \exp(-\bar{n})}{(n-1)!}} \exp\left(-i\left(n + \tfrac{1}{2}\right)\omega t\right) |n-1\rangle$$

Factor out $\sqrt{\bar{n}} \exp(-i\omega t)$.

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}} \exp(-i\omega t) \sum_{n=1}^{\infty} \sqrt{\frac{\bar{n}^{n-1} \exp(-\bar{n})}{(n-1)!}} \exp\left(-i\left(n - \tfrac{1}{2}\right)\omega t\right) |n-1\rangle$$

Substitute $n + 1$ for index n.

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}} \exp(-i\omega t) \sum_{n=0}^{\infty} \sqrt{\frac{\bar{n}^n \exp(-\bar{n})}{n!}} \exp\left(-i\left(n + \tfrac{1}{2}\right)\omega t\right) |n\rangle$$

Hence

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}} \exp(-i\omega t)|\Psi\rangle$$