Schrodinger for charged particle

Derive the Schrodinger equation for a charged particle

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{q}{c}\mathbf{A}\right)^2\psi + q\phi\psi$$

from the Lagrangian

$$L(\mathbf{x},\dot{\mathbf{x}},t) = \frac{m\dot{\mathbf{x}}^2}{2} + \frac{q}{c}\dot{\mathbf{x}}\cdot\mathbf{A}(\mathbf{x},t) - q\phi(\mathbf{x},t)$$

Start with the path integral for an action S.

$$\psi(\mathbf{x}_b,t_b) = C\int_{\mathbb{R}^3}\exp\left(\frac{i}{\hbar}S(b,a)\right)\psi(\mathbf{x}_a,t_a)\,d\mathbf{x}_a,\quad \int_{\mathbb{R}^3} \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$$

For a small time interval $\epsilon = t_b - t_a$ we can use the approximation

$$S = \epsilon L$$

and write the path integral as

$$\psi(\mathbf{x}_b,t+\epsilon) = C\int_{\mathbb{R}^3}\exp\left[\frac{i}{\hbar}\epsilon L\left(\frac{\mathbf{x}_b-\mathbf{x}_a}{\epsilon},\frac{\mathbf{x}_b+\mathbf{x}_a}{2},t\right)\right]\psi(\mathbf{x}_a,t)\,d\mathbf{x}_a$$

Substitute for L.

$$\begin{aligned}\psi(\mathbf{x}_b,t+\epsilon) = C\int_{\mathbb{R}^3}\exp\Bigg[&\frac{im(\mathbf{x}_b-\mathbf{x}_a)^2}{2\hbar\epsilon} + \frac{iq}{\hbar c}(\mathbf{x}_b-\mathbf{x}_a)\cdot\mathbf{A}\left(\frac{\mathbf{x}_b+\mathbf{x}_a}{2},t\right)\\ &- \frac{iq\epsilon}{\hbar}\phi\left(\frac{\mathbf{x}_b+\mathbf{x}_a}{2},t\right)\Bigg]\psi(\mathbf{x}_a,t)\,d\mathbf{x}_a\end{aligned}$$

Let

$$\mathbf{x}_a = \mathbf{x}_b + \boldsymbol{\eta},\quad d\mathbf{x}_a = d\boldsymbol{\eta}$$

and write

$$\psi(\mathbf{x}_b,t+\epsilon) = C\int_{\mathbb{R}^3}\exp\left[\frac{im\boldsymbol{\eta}^2}{2\hbar\epsilon} - \frac{iq}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x}_b+\frac{\boldsymbol{\eta}}{2},t\right) - \frac{iq\epsilon}{\hbar}\phi\left(\mathbf{x}_b+\frac{\boldsymbol{\eta}}{2},t\right)\right]\psi(\mathbf{x}_b+\boldsymbol{\eta},t)\,d\boldsymbol{\eta}$$

Substitute $\mathbf{x}$ for $\mathbf{x}_b$.

$$\psi(\mathbf{x},t+\epsilon) = C\int_{\mathbb{R}^3}\exp\left[\frac{im\boldsymbol{\eta}^2}{2\hbar\epsilon} - \frac{iq}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x}+\frac{\boldsymbol{\eta}}{2},t\right) - \frac{iq\epsilon}{\hbar}\phi\left(\mathbf{x}+\frac{\boldsymbol{\eta}}{2},t\right)\right]\psi(\mathbf{x}+\boldsymbol{\eta},t)\,d\boldsymbol{\eta}$$

Because the exponential is highly oscillatory for large $|\boldsymbol{\eta}|$, most of the contribution to the integral is from small $|\boldsymbol{\eta}|$. Hence use the approximation $\mathbf{x}+\frac{1}{2}\boldsymbol{\eta}\approx\mathbf{x}$ for small $|\boldsymbol{\eta}|$.

$$\psi(\mathbf{x},t+\epsilon) = C\int_{\mathbb{R}^3}\exp\left(\frac{im\boldsymbol{\eta}^2}{2\hbar\epsilon} - \frac{iq}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}(\mathbf{x},t) - \frac{iq\epsilon}{\hbar}\phi\,(\mathbf{x},t)\right)\psi(\mathbf{x}+\boldsymbol{\eta},t)\,d\boldsymbol{\eta}$$

Use the approximation $\exp(y) \approx 1 + y$ for the exponential of ϕ.

$$\psi(\mathbf{x}, t+\epsilon) = C\left(1 - \frac{iq\epsilon}{\hbar}\phi(\mathbf{x}, t)\right)\int_{\mathbb{R}^3} \exp\left(\frac{im\boldsymbol{\eta}^2}{2\hbar\epsilon} - \frac{iq}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}(\mathbf{x}, t)\right)\psi(\mathbf{x}+\boldsymbol{\eta}, t)\,d\boldsymbol{\eta}$$

Expand $\psi(\mathbf{x}+\boldsymbol{\eta}, t)$ as the power series

$$\psi(\mathbf{x}+\boldsymbol{\eta}, t) \approx \psi(\mathbf{x}, t) + \boldsymbol{\eta}\cdot\nabla\psi + \frac{1}{2}\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi)$$

to obtain

$$\begin{aligned}\psi(\mathbf{x}, t+\epsilon) = C\left(1 - \frac{iq\epsilon}{\hbar}\phi(\mathbf{x}, t)\right)\int_{\mathbb{R}^3} \exp\left(\frac{im\boldsymbol{\eta}^2}{2\hbar\epsilon} - \frac{iq}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}(\mathbf{x}, t)\right)& \\ \times\left(\psi(\mathbf{x}, t) + \boldsymbol{\eta}\cdot\nabla\psi + \frac{1}{2}\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi)\right)&\,d\boldsymbol{\eta}\end{aligned}$$

Rewrite as

$$\psi(\mathbf{x}, t+\epsilon) = C\left(1 - \frac{iq\epsilon}{\hbar}\phi(\mathbf{x}, t)\right)\sum_k I_k$$

where

$$\begin{aligned}I_1 &= \int_{\mathbb{R}^3} \exp\left(\frac{im\boldsymbol{\eta}^2}{2\hbar\epsilon} - \frac{iq}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}(\mathbf{x}, t)\right)\psi\,d\boldsymbol{\eta}\\ I_2 &= \int_{\mathbb{R}^3} \exp\left(\frac{im\boldsymbol{\eta}^2}{2\hbar\epsilon} - \frac{iq}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}(\mathbf{x}, t)\right)\boldsymbol{\eta}\cdot\nabla\psi\,d\boldsymbol{\eta}\\ I_3 &= \int_{\mathbb{R}^3} \exp\left(\frac{im\boldsymbol{\eta}^2}{2\hbar\epsilon} - \frac{iq}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}(\mathbf{x}, t)\right)\frac{1}{2}\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi)\,d\boldsymbol{\eta}\end{aligned}$$

The solutions are

$$\begin{aligned}I_1 &= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{\frac{3}{2}}\exp\left(-\frac{iq^2\epsilon}{2\hbar mc^2}\mathbf{A}^2\right)\psi\\ I_2 &= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{\frac{3}{2}}\exp\left(-\frac{iq^2\epsilon}{2\hbar mc^2}\mathbf{A}^2\right)\frac{q\epsilon}{mc}\mathbf{A}\cdot\nabla\psi\\ I_3 &= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{\frac{3}{2}}\exp\left(-\frac{iq^2\epsilon}{2\hbar mc^2}\mathbf{A}^2\right)\left(\frac{i\hbar\epsilon}{2m}\nabla^2\psi + \frac{q\epsilon}{2mc}\mathbf{A}\cdot\nabla\psi + O(\epsilon^2)\right)\end{aligned}$$

Use the approximation $\exp(y) \approx 1 + y$ to write the integrals this way.

$$\begin{aligned}I_1 &= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{\frac{3}{2}}\left(1-\frac{iq^2\epsilon}{2\hbar mc^2}\mathbf{A}^2\right)\psi\\ I_2 &= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{\frac{3}{2}}\left(1-\frac{iq^2\epsilon}{2\hbar mc^2}\mathbf{A}^2\right)\frac{q\epsilon}{mc}\mathbf{A}\cdot\nabla\psi\\ I_3 &= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{\frac{3}{2}}\left(1-\frac{iq^2\epsilon}{2\hbar mc^2}\mathbf{A}^2\right)\left(\frac{i\hbar\epsilon}{2m}\nabla^2\psi + \frac{q\epsilon}{2mc}\mathbf{A}\cdot\nabla\psi + O(\epsilon^2)\right)\end{aligned}$$

Discarding terms of order ϵ^2 we have

$$\sum_k I_k = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{\frac{3}{2}} \left(\psi - \frac{iq^2\epsilon}{2\hbar mc^2}\mathbf{A}^2\psi + \frac{q\epsilon}{mc}\mathbf{A}\cdot\nabla\psi + \frac{i\hbar\epsilon}{2m}\nabla^2\psi + \frac{q\epsilon}{2mc}\mathbf{A}\cdot\nabla\psi\right)$$