Feynman and Hibbs problem 4-2

For a particle of charge e in a magnetic field the Lagrangian is

$$L(\dot{\mathbf{x}}, \mathbf{x}) = \frac{m}{2}\dot{\mathbf{x}}^2 + \frac{e}{c}\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}, t) - e\phi(\mathbf{x}, t)$$

where $\dot{\mathbf{x}}$ is the velocity vector, c is the velocity of light, and $\mathbf{A}$ and ϕ are the vector and scalar potentials. Show that the corresponding Schrodinger equation is

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar}\left(\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right) \cdot \left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right)\psi + e\phi\psi\right)$$

From equation (4.3)

$$\psi(\mathbf{x}, t + \epsilon) = \frac{1}{A}\int_{\mathbb{R}^3} \exp\left(\frac{i\epsilon}{\hbar} L\left(\frac{\mathbf{x} - \mathbf{y}}{\epsilon}, \frac{\mathbf{x} + \mathbf{y}}{2}\right)\right)\psi(\mathbf{y}, t)\, dy_1\, dy_2\, dy_3 \qquad (1)$$

By substitution of the given Lagrangian

$$\begin{aligned} L&\left(\frac{\mathbf{x} - \mathbf{y}}{\epsilon}, \frac{\mathbf{x} + \mathbf{y}}{2}\right) \\ &= \frac{m}{2\epsilon^2}(\mathbf{x} - \mathbf{y})^2 + \frac{e}{c\epsilon}(\mathbf{x} - \mathbf{y}) \cdot \mathbf{A}\left(\frac{\mathbf{x} + \mathbf{y}}{2}, t\right) - e\phi\left(\frac{\mathbf{x} + \mathbf{y}}{2}, t\right) \end{aligned}$$

Then from equation (1)

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) = \frac{1}{A}\int_{\mathbb{R}^3} \\ \exp\left(\frac{im}{2\hbar\epsilon}(\mathbf{x} - \mathbf{y})^2 + \frac{ie}{\hbar c}(\mathbf{x} - \mathbf{y}) \cdot \mathbf{A}\left(\frac{\mathbf{x} + \mathbf{y}}{2}, t\right) - \frac{ie\epsilon}{\hbar}\phi\left(\frac{\mathbf{x} + \mathbf{y}}{2}, t\right)\right) \\ \times \psi(\mathbf{y}, t)\, dy_1\, dy_2\, dy_3 \end{aligned}$$

Let

$$\mathbf{y} = \mathbf{x} + \boldsymbol{\eta}$$

Then

$$\mathbf{x} - \mathbf{y} = \boldsymbol{\eta}, \quad \frac{\mathbf{x} + \mathbf{y}}{2} = \mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, \quad dy_1\, dy_2\, dy_3 = d\eta_1\, d\eta_2\, d\eta_3$$

Hence

$$\psi(\mathbf{x}, t+\epsilon) = \frac{1}{A}\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2 + \frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right) - \frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right)\right) \times \psi(\mathbf{x}+\boldsymbol{\eta}, t)\, d\eta_1\, d\eta_2\, d\eta_3$$

Factor the exponential.

$$\psi(\mathbf{x}, t+\epsilon) = \frac{1}{A}\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2 + \frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right)\right) \exp\left(-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right)\right) \times \psi(\mathbf{x}+\boldsymbol{\eta}, t)\, d\eta_1\, d\eta_2\, d\eta_3 \tag{2}$$

From the identity $\exp(i\theta) = \cos(\theta) + i\sin(\theta)$ we have

$$\exp\left(-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right)\right) = \cos\left(-\frac{e\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right)\right) + i\sin\left(-\frac{e\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right)\right)$$

Then for small ϵ

$$\exp\left(-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right)\right) \approx 1 - \frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right) \tag{3}$$

Substitute (3) into (2).

$$\psi(\mathbf{x}, t+\epsilon) = \frac{1}{A}\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2 + \frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right)\right) \left(1 - \frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right)\right) \times \psi(\mathbf{x}+\boldsymbol{\eta}, t)\, d\eta_1\, d\eta_2\, d\eta_3 \tag{4}$$

Next we will use the following Taylor series approximations.

$$\begin{aligned} \psi(\mathbf{x}, t+\epsilon) &\approx \psi(\mathbf{x}, t) + \epsilon\frac{\partial\psi}{\partial t} \\ \psi(\mathbf{x}+\boldsymbol{\eta}, t) &\approx \psi(\mathbf{x}, t) + \boldsymbol{\eta}\cdot\nabla\psi + \tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi) \end{aligned} \tag{5}$$

Note: In component notation

$$\boldsymbol{\eta}\cdot\nabla\psi = \eta_1\frac{\partial\psi}{\partial x_1} + \eta_2\frac{\partial\psi}{\partial x_2} + \eta_2\frac{\partial\psi}{\partial x_2}$$

and

$$\begin{aligned}\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi) = \eta_1^2\frac{\partial^2\psi}{\partial x_1^2} + \eta_2^2\frac{\partial^2\psi}{\partial x_2^2} + \eta_3^2\frac{\partial^2\psi}{\partial x_3^2}& \\ + 2\eta_1\eta_2\frac{\partial^2\psi}{\partial x_1\partial x_2} + 2\eta_1\eta_3\frac{\partial^2\psi}{\partial x_1\partial x_3} + 2\eta_2\eta_3\frac{\partial^2\psi}{\partial x_2\partial x_3}&\end{aligned}$$

Substitute the approximations (5) into (4).

$$\begin{aligned}&\psi(\mathbf{x},t) + \epsilon\frac{\partial\psi}{\partial t} = \frac{1}{A}\int_{\mathbb{R}^3} \\ &\quad\exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2 + \frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)\left(1-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right) \\ &\quad\times\left(\psi(\mathbf{x},t) + \boldsymbol{\eta}\cdot\nabla\psi + \tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi)\right)\,d\eta_1\,d\eta_2\,d\eta_3\end{aligned}$$

Without any justification, drop the $\boldsymbol{\eta}$ term in $\mathbf{A}$ and ϕ.

$$\begin{aligned}&\psi(\mathbf{x},t) + \epsilon\frac{\partial\psi}{\partial t} = \frac{1}{A}\int_{\mathbb{R}^3} \\ &\quad\exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2 + \frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x},t\right)\right)\left(1-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x},t\right)\right) \\ &\quad\times\left(\psi(\mathbf{x},t) + \boldsymbol{\eta}\cdot\nabla\psi + \tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi)\right)\,d\eta_1\,d\eta_2\,d\eta_3\end{aligned}\tag{6}$$

Let a_k be the exponential argument in component notation.

$$a_k = \frac{im}{2\hbar\epsilon}\eta_k^2 + \frac{ie}{\hbar c}\eta_k A_k(\mathbf{x},t)$$

Expand the right-hand side of (6) using $a = a_1 + a_2 + a_3$.

$$\psi(\mathbf{x},t) + \epsilon\frac{\partial\psi}{\partial t} = \frac{\psi(\mathbf{x},t)}{A}\int_{\mathbb{R}^3} \exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 \tag{7}$$

$$+ \frac{1}{A}\int_{\mathbb{R}^3} (\nabla\psi\cdot\boldsymbol{\eta})\exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 \tag{8}$$

$$+ \frac{1}{2A}\int_{\mathbb{R}^3} (\nabla(\nabla\psi\cdot\boldsymbol{\eta})\cdot\boldsymbol{\eta})\exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 \tag{9}$$

$$- \frac{ie\epsilon}{A\hbar}\phi\,(\mathbf{x},t)\,\psi(\mathbf{x},t)\int_{\mathbb{R}^3} \exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 \tag{10}$$

$$- \frac{ie\epsilon}{A\hbar}\phi\,(\mathbf{x},t)\int_{\mathbb{R}^3} (\nabla\psi\cdot\boldsymbol{\eta})\exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 \tag{11}$$

$$- \frac{ie\epsilon}{2A\hbar}\phi\,(\mathbf{x},t)\int_{\mathbb{R}^3} (\nabla(\nabla\psi\cdot\boldsymbol{\eta})\cdot\boldsymbol{\eta})\exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 \tag{12}$$

To solve the above integrals, we will use the following formulas obtained from online resources.

$$\int_{-\infty}^{\infty} \exp(a_k)\, d\eta_k$$

$$= -\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{1/2} \exp\left(-\frac{ie^2\epsilon A_k(\mathbf{x},t)^2}{2m\hbar c^2}\right) \tag{13}$$

$$\int_{-\infty}^{\infty} \eta_k \exp(a_k)\, d\eta_k$$

$$= -\frac{e\epsilon A_k(\mathbf{x},t)}{mc}\left(\frac{2\pi i\hbar}{m}\right)^{1/2} \exp\left(-\frac{ie^2\epsilon A_k(\mathbf{x},t)^2}{2m\hbar c^2}\right) \tag{14}$$

$$\int_{-\infty}^{\infty} \eta_k^2 \exp(a_k)\, d\eta_k$$

$$= \left(\frac{e^2\epsilon^2 A_k(\mathbf{x},t)^2}{m^2c^2} + \frac{i\hbar\epsilon}{m}\right)\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{1/2} \exp\left(-\frac{ie^2\epsilon A_k(\mathbf{x},t)^2}{2m\hbar c^2}\right) \tag{15}$$

By equation (13)

$$\int_{\mathbb{R}^3} \exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 = -\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \exp\left(-\frac{ie^2\epsilon\mathbf{A}^2}{2m\hbar c^2}\right) \tag{16}$$

where

$$\mathbf{A}^2 = \mathbf{A}\cdot\mathbf{A} = A_1(\mathbf{x},t)^2 + A_2(\mathbf{x},t)^2 + A_3(\mathbf{x},t)^2$$

Rewrite the integral in (8) and (11) in component notation.

$$\int_{\mathbb{R}^3} (\nabla\psi\cdot\boldsymbol{\eta})\exp(a)\,d\eta_1\,d\eta_2\,d\eta_3 = \int_{\mathbb{R}^3} \frac{\partial\psi}{\partial x_1}\eta_1\exp(a)\,d\eta_1\,d\eta_2\,d\eta_3 \\ + \int_{\mathbb{R}^3} \frac{\partial\psi}{\partial x_2}\eta_2\exp(a)\,d\eta_1\,d\eta_2\,d\eta_3 + \int_{\mathbb{R}^3} \frac{\partial\psi}{\partial x_3}\eta_3\exp(a)\,d\eta_1\,d\eta_2\,d\eta_3$$

Then by equations (13) and (14)

$$\int_{\mathbb{R}^3} \frac{\partial\psi}{\partial x_1}\eta_1\exp(a)\,d\eta_1\,d\eta_2\,d\eta_3 \\ = -\frac{e\epsilon A_1(\mathbf{x},t)}{mc}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\exp\left(-\frac{ie^2\epsilon\mathbf{A}^2}{2m\hbar c^2}\right)\frac{\partial\psi}{\partial x_1}$$

$$\int_{\mathbb{R}^3} \frac{\partial\psi}{\partial x_2}\eta_2\exp(a)\,d\eta_1\,d\eta_2\,d\eta_3 \\ = -\frac{e\epsilon A_2(\mathbf{x},t)}{mc}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\exp\left(-\frac{ie^2\epsilon\mathbf{A}^2}{2m\hbar c^2}\right)\frac{\partial\psi}{\partial x_2}$$

$$\int_{\mathbb{R}^3} \frac{\partial\psi}{\partial x_3}\eta_3\exp(a)\,d\eta_1\,d\eta_2\,d\eta_3 \\ = -\frac{e\epsilon A_3(\mathbf{x},t)}{mc}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\exp\left(-\frac{ie^2\epsilon\mathbf{A}^2}{2m\hbar c^2}\right)\frac{\partial\psi}{\partial x_3}$$

Hence

$$\int_{\mathbb{R}^3} (\nabla\psi\cdot\boldsymbol{\eta})\exp(a)\,d\eta_1\,d\eta_2\,d\eta_3 \\ = -\frac{e\epsilon}{mc}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\exp\left(-\frac{ie^2\epsilon\mathbf{A}^2}{2m\hbar c^2}\right)\nabla\psi\cdot\mathbf{A}(\mathbf{x},t) \tag{17}$$

Rewrite the integral in (9) and (12) in component notation.

$$
\begin{aligned}
&\int_{\mathbb{R}^3} (\nabla(\nabla\psi\cdot\boldsymbol{\eta})\cdot\boldsymbol{\eta}) \exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad = \int_{\mathbb{R}^3} \eta_1^2 \frac{\partial^2\psi}{\partial x_1^2} \exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad\quad + \int_{\mathbb{R}^3} \eta_2^2 \frac{\partial^2\psi}{\partial x_2^2} \exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad\quad + \int_{\mathbb{R}^3} \eta_3^2 \frac{\partial^2\psi}{\partial x_3^2} \exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad\quad + \int_{\mathbb{R}^3} 2\eta_1\eta_2 \frac{\partial^2\psi}{\partial x_1\partial x_2} \exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad\quad + \int_{\mathbb{R}^3} 2\eta_1\eta_3 \frac{\partial^2\psi}{\partial x_1\partial x_3} \exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad\quad + \int_{\mathbb{R}^3} 2\eta_2\eta_3 \frac{\partial^2\psi}{\partial x_2\partial x_3} \exp(a)\, d\eta_1\, d\eta_2\, d\eta_3
\end{aligned}
$$

By equations (13) and (15)

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \eta_1^2 \frac{\partial^2\psi}{\partial x_1^2} \exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad = \left(\frac{e^2\epsilon^2 A_1(\mathbf{x},t)^2}{m^2c^2} + \frac{i\hbar\epsilon}{m}\right)\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \exp\left(-\frac{ie^2\epsilon\mathbf{A}^2}{2m\hbar c^2}\right)\frac{\partial^2\psi}{\partial x_1^2} \\
&\int_{\mathbb{R}^3} \eta_2^2 \frac{\partial^2\psi}{\partial x_2^2} \exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad = \left(\frac{e^2\epsilon^2 A_2(\mathbf{x},t)^2}{m^2c^2} + \frac{i\hbar\epsilon}{m}\right)\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \exp\left(-\frac{ie^2\epsilon\mathbf{A}^2}{2m\hbar c^2}\right)\frac{\partial^2\psi}{\partial x_2^2} \\
&\int_{\mathbb{R}^3} \eta_3^2 \frac{\partial^2\psi}{\partial x_3^2} \exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad = \left(\frac{e^2\epsilon^2 A_3(\mathbf{x},t)^2}{m^2c^2} + \frac{i\hbar\epsilon}{m}\right)\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \exp\left(-\frac{ie^2\epsilon\mathbf{A}^2}{2m\hbar c^2}\right)\frac{\partial^2\psi}{\partial x_3^2}
\end{aligned}
$$

By equations (13) and (14)

$$\begin{aligned}
&\int_{\mathbb{R}^3} 2\eta_1\eta_2 \frac{\partial^2\psi}{\partial x_1 \partial x_2} \exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad = \left(\frac{e^2\epsilon^2 A_1(\mathbf{x},t)^2}{m^2c^2} + \frac{i\hbar\epsilon}{m}\right)\left(\frac{e^2\epsilon^2 A_2(\mathbf{x},t)^2}{m^2c^2} + \frac{i\hbar\epsilon}{m}\right) \\
&\qquad\qquad \times \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \exp\left(-\frac{ie^2\epsilon\mathbf{A}^2}{2m\hbar c^2}\right)\frac{\partial^2\psi}{\partial x_1 x_2} \\
&\int_{\mathbb{R}^3} 2\eta_1\eta_2 \frac{\partial^2\psi}{\partial x_1 \partial x_3} \exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad = \left(\frac{e^2\epsilon^2 A_1(\mathbf{x},t)^2}{m^2c^2} + \frac{i\hbar\epsilon}{m}\right)\left(\frac{e^2\epsilon^2 A_3(\mathbf{x},t)^2}{m^2c^2} + \frac{i\hbar\epsilon}{m}\right) \\
&\qquad\qquad \times \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \exp\left(-\frac{ie^2\epsilon\mathbf{A}^2}{2m\hbar c^2}\right)\frac{\partial^2\psi}{\partial x_1 x_3} \\
&\int_{\mathbb{R}^3} 2\eta_1\eta_2 \frac{\partial^2\psi}{\partial x_1 \partial x_2} \exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\quad = \left(\frac{e^2\epsilon^2 A_2(\mathbf{x},t)^2}{m^2c^2} + \frac{i\hbar\epsilon}{m}\right)\left(\frac{e^2\epsilon^2 A_3(\mathbf{x},t)^2}{m^2c^2} + \frac{i\hbar\epsilon}{m}\right)\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \\
&\qquad \times \exp\left(-\frac{ie^2\epsilon\mathbf{A}^2}{2m\hbar c^2}\right)\frac{\partial^2\psi}{\partial x_2 x_3}
\end{aligned}$$

Discard terms involving powers of ϵ to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\nabla(\nabla\psi\cdot\boldsymbol{\eta})\cdot\boldsymbol{\eta}) \exp(a)\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad\qquad = \frac{i\hbar\epsilon}{m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \exp\left(-\frac{ie^2\epsilon\mathbf{A}^2}{2m\hbar c^2}\right)\nabla^2\psi(\mathbf{x},t) \qquad (18)
\end{aligned}$$

Substitute the solved integrals into (6) to obtain

$$
\begin{aligned}
&\psi(\mathbf{x},t) + \epsilon\frac{\partial\psi}{\partial t} \\
&\quad = \frac{1}{A}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \exp\left(-\frac{ie^2\epsilon\mathbf{A}^2}{2m\hbar c^2}\right)\psi(\mathbf{x},t) && \text{from (7) and (16)} \\
&\qquad - \frac{e\epsilon}{Amc}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \exp\left(-\frac{ie^2\epsilon\mathbf{A}^2}{2m\hbar c^2}\right)\nabla\psi\cdot\mathbf{A}(\mathbf{x},t) && \text{from (8) and (17)} \\
&\qquad + \frac{i\hbar\epsilon}{2Am}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \exp\left(-\frac{ie^2\epsilon\mathbf{A}^2}{2m\hbar c^2}\right)\nabla^2\psi(\mathbf{x},t) && \text{from (9) and (18)} \\
&\qquad - \phi(\mathbf{x},t)\frac{ie\epsilon}{A\hbar}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \exp\left(-\frac{ie^2\epsilon\mathbf{A}^2}{2m\hbar c^2}\right)\psi(\mathbf{x},t) && \text{from (10) and (16)} \\
&\qquad + \phi(\mathbf{x},t)\frac{ie^2\epsilon^2}{Am\hbar c}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \exp\left(-\frac{ie^2\epsilon\mathbf{A}^2}{2m\hbar c^2}\right)\nabla\psi\cdot\mathbf{A}(\mathbf{x},t) \\
&&& \text{from (11) and (17)} \\
&\qquad - \phi(\mathbf{x},t)\frac{ie\epsilon^2}{2Am}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \exp\left(-\frac{ie^2\epsilon\mathbf{A}^2}{2m\hbar c^2}\right)\nabla^2\psi(\mathbf{x},t) \\
&&& \text{from (12) and (18)}
\end{aligned}
$$