Dirac from boost

Let u_0 be the spinor for an electron at rest with spin up along the z axis.

$$u_0 = \begin{pmatrix} \sqrt{2m} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Matrix A boosts a spinor in the z direction where $E = \sqrt{p^2 + m^2}$.

$$A = \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix} E+m & 0 & -p & 0 \\ 0 & E+m & 0 & p \\ p & 0 & E+m & 0 \\ 0 & -p & 0 & E+m \end{pmatrix}$$

Hence

$$\begin{aligned} u &= Au_0 \\ &= \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix} E+m & 0 & p & 0 \\ 0 & E+m & 0 & -p \\ -p & 0 & E+m & 0 \\ 0 & p & 0 & E+m \end{pmatrix} \begin{pmatrix} \sqrt{2m} \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix} \end{aligned}$$

Noting that $E = m$ in the rest frame, spinor u has the same normalization as u_0.

$$u^\dagger u = 2E$$

Recall the Dirac equation for spinors.

$$\not{p} u = mu$$

Substitute Au_0 for u on the right-hand side.

$$\not{p} u = mAu_0$$

By the identity $\gamma^0 u_0 = u_0$, substitute $\gamma^0 u_0$ for u_0.

$$\not{p} u = mA\gamma^0 u_0$$

Substitute $A^{-1}u$ for u_0.

$$\not{p} u = mA\gamma^0 A^{-1} u$$

Cancel u to obtain

$$\not{p} = mA\gamma^0 A^{-1}$$

Divide both sides by m.

$$\frac{\not p}{m} = A\gamma^0 A^{-1}$$

Right multiply both sides by A.

$$\frac{\not p}{m} A = A\gamma^0 \tag{1}$$

To recover the Dirac equation from (1), start with the identity

$$\gamma^0 u_0 = u_0$$

Boost both sides of the equation.

$$A\gamma^0 u_0 = Au_0$$

By equation (1) substitute $(\not p/m)A$ for $A\gamma^0$.

$$\frac{\not p}{m} Au_0 = Au_0$$

Substitute u for Au_0.

$$\frac{\not p}{m} u = u$$

Multiply both sides by m to obtain the Dirac equation.

$$\not p u = mu \tag{2}$$

Eigenmath script