# Rutherford scattering 2

Find the cross section for Rutherford scattering with the following potential.

$$V(r) = -\frac{Ze^2}{r}\exp\left(-\frac{r}{a}\right)$$

Start with

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2\epsilon_0^2}\left(\frac{mQ}{2\pi\hbar^2}\right)^2, \quad Q = \int \exp\left(\frac{i\mathbf{p}\cdot\mathbf{r}}{\hbar}\right)V(\mathbf{r})\,d^3\mathbf{r}$$

Convert $Q$ to polar coordinates.

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\exp\left(\frac{ipr\cos\theta}{\hbar}\right)V(r,\theta,\phi)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

Substitute the shielded Coulomb potential for $V(r,\theta,\phi)$ and note $r^2$ becomes $r$.

$$Q = -Ze^2\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\exp\left(\frac{ipr\cos\theta}{\hbar}\right)\exp\left(-\frac{r}{a}\right)r\sin\theta\,dr\,d\theta\,d\phi$$

Integrate over $\phi$.

$$Q = -2\pi Ze^2\int_0^{\pi}\int_0^{\infty}\exp\left(\frac{ipr\cos\theta}{\hbar}\right)\exp\left(-\frac{r}{a}\right)r\sin\theta\,dr\,d\theta$$

Transform the integral over $\theta$ to an integral over $y$ where $y=\cos\theta$, $dy = -\sin\theta\,d\theta$.

$$Q = -2\pi Ze^2\int_{-1}^{1}\int_0^{\infty}\exp\left(\frac{ipry}{\hbar}\right)\exp\left(-\frac{r}{a}\right)r\,dr\,dy$$

Solve the integral over $y$ (note $r$ in the integrand cancels).

$$Q = -2\pi Ze^2\int_0^{\infty}\frac{\hbar}{ip}\left[\exp\left(\frac{ipr}{\hbar}\right) - \exp\left(-\frac{ipr}{\hbar}\right)\right]\exp\left(-\frac{r}{a}\right)\,dr$$

Solve the integral over $r$.

$$Q = -2\pi Ze^2\frac{\hbar}{ip}\left[\frac{1}{ip/\hbar - 1/a}\exp\left(\frac{ipr}{\hbar}-\frac{r}{a}\right) + \frac{1}{ip/\hbar + 1/a}\exp\left(-\frac{ipr}{\hbar}-\frac{r}{a}\right)\right]_0^{\infty}$$

Evaluate the limits.

$$Q = -2\pi Ze^2\frac{\hbar}{ip}\left[-\frac{1}{ip/\hbar - 1/a} - \frac{1}{ip/\hbar + 1/a}\right] = -\frac{4\pi Ze^2}{(p/\hbar)^2 + (1/a)^2} \tag{1}$$

The cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2\epsilon_0^2}\left(\frac{mQ}{2\pi\hbar^2}\right)^2 = \frac{1}{64\pi^2\epsilon_0^2}\frac{4m^2Z^2e^4}{\left[p^2 + (\hbar/a)^2\right]^2}$$

Substitute $(4\pi\epsilon_0\alpha\hbar c)^2$ for $e^4$.

$$\frac{d\sigma}{d\Omega} = \frac{m^2 Z^2 \alpha^2 (\hbar c)^2}{\left[p^2 + (\hbar/a)^2\right]^2}$$

Symbol $p$ is momentum transfer $|\mathbf{p}_i| - |\mathbf{p}_f|$ such that

$$p^2 = 2mE(1 - \cos\theta)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{m^2 Z^2 \alpha^2 (\hbar c)^2}{\left[2mE(1 - \cos\theta) + (\hbar/a)^2\right]^2}$$

Cancel $m^2$ in the numerator.

$$\frac{d\sigma}{d\Omega} = \frac{Z^2 \alpha^2 (\hbar c)^2}{\left[2E(1 - \cos\theta) + \frac{1}{m}(\hbar/a)^2\right]^2} \tag{2}$$

Let $a \to \infty$ to obtain the ordinary Rutherford cross section.