

What is the mean lifetime of 2p hydrogen?

Let us begin by writing down the wave function ψ for hydrogen.

$$\psi_{nlm}(r, \theta, \phi) = R_{nl}(r)Y_{lm}(\theta, \phi)$$

where

$$R_{nl}(r) = \frac{2}{n^2} \left(\frac{(n-l-1)!}{(n+l)!} \right)^{1/2} \left(\frac{2r}{na_0} \right)^l L_{n-l-1}^{2l+1} \left(\frac{2r}{na_0} \right) \exp \left(-\frac{r}{na_0} \right) a_0^{-3/2}$$

$$L_n^m(x) = (n+m)! \sum_{k=0}^n \frac{(-x)^k}{(n-k)!(m+k)!k!}$$

$$Y_{lm}(\theta, \phi) = (-1)^m \left(\frac{2l+1}{4\pi} \right)^{1/2} \left(\frac{(l-m)!}{(l+m)!} \right)^{1/2} P_l^m(\cos \theta) \exp(im\phi)$$

$$P_n^m(x) = \frac{1}{2^n n!} (1-x^2)^{m/2} \frac{d^{n+m}}{dx^{n+m}} (x^2-1)^n$$

$$a_0 = \frac{4\pi\epsilon_0 \hbar^2}{e^2 \mu} \approx 0.529 \times 10^{-10} \text{ meter}$$

The state 2p means that $n = 2$ and $l = 1$. That leaves three ways to choose m hence all of the following processes correspond to the transition $2p \rightarrow 1s$.

$$\left. \begin{array}{l} \psi_{2,1,1} \\ \psi_{2,1,0} \\ \psi_{2,1,-1} \end{array} \right\} \rightarrow \psi_{100} + \text{photon}$$

By Fermi's Golden Rule and the dipole approximation we have for the transition rate A_{21}

$$A_{21} = \frac{e^2}{3\pi\epsilon_0 \hbar c^3} \omega_{21}^3 |r_{21}|^2 \quad (1)$$

where

$$\omega_{21} = \frac{E_2 - E_1}{\hbar}$$

$$|r_{21}|^2 = |x_{21}|^2 + |y_{21}|^2 + |z_{21}|^2$$

$$x_{21} = \int_0^{2\pi} \int_0^\pi \int_0^\infty x f_{21} dV \quad y_{21} = \int_0^{2\pi} \int_0^\pi \int_0^\infty y f_{21} dV \quad z_{21} = \int_0^{2\pi} \int_0^\pi \int_0^\infty z f_{21} dV$$

$$f_{21} = \psi_{100}^* \psi_{210}$$

$$dV = r^2 \sin \theta dr d\theta d\phi$$

For $|r_{21}|^2$ we have

$$|r_{21}|^2 = \frac{32768}{59049} a_0^2$$

By equation (1) the transition rate is

$$A_{21} = 6.26 \times 10^8 \text{ second}^{-1}$$

Hence the mean lifetime is

$$\frac{1}{A_{21}} = 1.60 \times 10^{-9} \text{ second}$$