

Fun trick

Show that

$$[p^2, \mathbf{r}] = -2i\hbar\mathbf{p}$$

where

$$\mathbf{r} = \otimes(x, y, z), \quad \mathbf{p} = -i\hbar\nabla, \quad p^2 = \mathbf{p} \cdot \mathbf{p} = -\hbar^2\nabla^2$$

We have

$$\begin{aligned} [p^2, \mathbf{r}] &= p^2\mathbf{r} - \mathbf{r}p^2 \\ &= \mathbf{p} \cdot \mathbf{p}\mathbf{r} - \mathbf{r}\mathbf{p} \cdot \mathbf{p} \\ &= \text{Tr}[\mathbf{p}\mathbf{p}\mathbf{r} - \mathbf{r}\mathbf{p}\mathbf{p}] \\ &= \text{Tr}[\mathbf{p}\mathbf{p}\mathbf{r} - \mathbf{p}\mathbf{r}\mathbf{p} + \mathbf{p}\mathbf{r}\mathbf{p} - \mathbf{r}\mathbf{p}\mathbf{p}] && \text{trick!} \\ &= \text{Tr}[\mathbf{p}(\mathbf{p}\mathbf{r} - \mathbf{r}\mathbf{p}) + (\mathbf{p}\mathbf{r} - \mathbf{r}\mathbf{p})\mathbf{p}] \\ &= (-i\hbar)\mathbf{p} + (-i\hbar)\mathbf{p} \\ &= -2i\hbar\mathbf{p} \end{aligned}$$

The trick is introducing null term $\mathbf{p}\mathbf{r}\mathbf{p} - \mathbf{p}\mathbf{r}\mathbf{p}$ so that the operators can be factored. Trace operator Tr contracts on the first and second indices.

Verify the following formulas.

$$[p^2, \mathbf{r}] = -2i\hbar\mathbf{p} \tag{1}$$

$$[p^2, \mathbf{r}] = \text{Tr}[\mathbf{p}\mathbf{p}\mathbf{r} - \mathbf{r}\mathbf{p}\mathbf{p}] \tag{2}$$

$$[p^2, \mathbf{r}] = \text{Tr}[\mathbf{p}\mathbf{p}\mathbf{r} - \mathbf{p}\mathbf{r}\mathbf{p} + \mathbf{p}\mathbf{r}\mathbf{p} - \mathbf{r}\mathbf{p}\mathbf{p}] \tag{3}$$

$$\mathbf{p}\mathbf{r} - \mathbf{r}\mathbf{p} = -i\hbar\mathbf{I} \tag{4}$$

$$\mathbf{p} \cdot \mathbf{p} = \text{Tr}[\mathbf{p}\mathbf{p}] \tag{5}$$