

Harmonic oscillator propagator 2

Consider the harmonic oscillator wave function

$$\psi_n(x, t) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi \hbar} \right)^{\frac{1}{4}} H_n \left(\sqrt{\frac{m\omega}{\hbar}} x \right) \exp \left(-\frac{m\omega x^2}{2\hbar} - i \left(n + \frac{1}{2} \right) \omega t \right)$$

and the harmonic oscillator propagator

$$K(x_b, t_b, x_a, t_a) = \left(\frac{m\omega}{2\pi i \hbar \sin(\omega T)} \right)^{\frac{1}{2}} \exp \left[\frac{im\omega}{2\hbar \sin(\omega T)} ((x_b^2 + x_a^2) \cos(\omega T) - 2x_b x_a) \right]$$

where $T = t_b - t_a$.

We should have

$$\psi_n(x_b, T) = \int_{-\infty}^{\infty} K(x_b, T, x_a, 0) \psi_n(x_a, 0) dx_a$$

Try for $n = 1$. Let

$$I = \int_{-\infty}^{\infty} K(x_b, T, x_a, 0) \psi_1(x_a, 0) dx_a$$

Substitute for K and ψ_1 .

$$\begin{aligned} I &= \sqrt{2} \left(\frac{m^3 \omega^3}{\pi \hbar^3} \right)^{\frac{1}{4}} \left(\frac{m\omega}{2\pi i \hbar \sin(\omega T)} \right)^{\frac{1}{2}} \\ &\quad \times \int_{-\infty}^{\infty} \exp \left[\frac{im\omega}{2\hbar \sin(\omega T)} ((x_b^2 + x_a^2) \cos(\omega T) - 2x_b x_a) \right] x_a \exp \left(-\frac{m\omega x_a^2}{2\hbar} \right) dx_a \end{aligned}$$

Simplify the integrand.

$$\begin{aligned} I &= \sqrt{2} \left(\frac{m^3 \omega^3}{\pi \hbar^3} \right)^{\frac{1}{4}} \left(\frac{m\omega}{2\pi i \hbar \sin(\omega T)} \right)^{\frac{1}{2}} \\ &\quad \times \int_{-\infty}^{\infty} x_a \exp \left(\frac{im\omega x_a^2 \exp(i\omega T)}{2\hbar \sin(\omega T)} - \frac{im\omega x_a}{\hbar \sin(\omega T)} + \frac{im\omega x_b^2 \cos(\omega T)}{2\hbar \sin(\omega T)} \right) dx_a \end{aligned}$$

Let

$$a = -\frac{im\omega \exp(i\omega T)}{2\hbar \sin(\omega T)}, \quad b = -\frac{im\omega}{\hbar \sin(\omega T)}, \quad c = \frac{im\omega x_b^2 \cos(\omega T)}{2\hbar \sin(\omega T)}$$

so that

$$I = \sqrt{2} \left(\frac{m^3 \omega^3}{\pi \hbar^3} \right)^{\frac{1}{4}} \left(\frac{m\omega}{2\pi i \hbar \sin(\omega T)} \right)^{\frac{1}{2}} \int_{-\infty}^{\infty} x_a \exp(-ax_a^2 + bx_a + c) dx_a$$

Solve the integral.

$$\begin{aligned} I &= \sqrt{2} \left(\frac{m^3 \omega^3}{\pi \hbar^3} \right)^{\frac{1}{4}} \left(\frac{m\omega}{2\pi i \hbar \sin(\omega T)} \right)^{\frac{1}{2}} \frac{\sqrt{\pi}}{2} \frac{b}{a^{3/2}} \exp \left(\frac{b^2}{4a} + c \right) \\ &= \sqrt{2} \left(\frac{m^3 \omega^3}{\pi \hbar^3} \right)^{\frac{1}{4}} x_b \exp \left(-\frac{m\omega x_b^2}{2\hbar} - \frac{3}{2} i\omega T \right) \\ &= \psi_1(x_b, T) \end{aligned}$$

Hence solving the integral shows that

$$\psi_1(x_b, T) = \int_{-\infty}^{\infty} K(x_b, T, x_a, 0) \psi_1(x_a, 0) dx_a$$