

3-4. Suppose a free particle has a definite momentum at the time $t = 0$ (that is, the wave function is $Ce^{ipx/\hbar}$). With the help of equations (3.3) and (3.42), show that at some later time the particle has the same definite momentum (i.e., the wave function depends on x through the function $e^{ipx/\hbar}$) and varies in time as $e^{-ip^2t/2m\hbar}$. This means that the particle has definite energy $p^2/2m$.

We are given

$$\psi(x, 0) = C \exp\left(\frac{ipx}{\hbar}\right)$$

This is equation (3.3).

$$K_0(b, a) = \left(\frac{m}{2\pi i\hbar(t_b - t_a)}\right)^{1/2} \exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)}\right) \quad (3.3)$$

Rewrite (3.3) as follows.

$$\begin{aligned} K_0(b, a) &= \left(\frac{m}{2\pi i\hbar(t_b - t_a)}\right)^{1/2} \\ &\quad \times \exp\left(\frac{imx_b^2}{2\hbar(t_b - t_a)}\right) \exp\left(\frac{imx_a^2}{2\hbar(t_b - t_a)} - \frac{imx_bx_a}{\hbar(t_b - t_a)}\right) \end{aligned}$$

Then for $t_a = 0$

$$K_0(x_b, t_b, x_a, 0) = \left(\frac{m}{2\pi i\hbar t_b}\right)^{1/2} \exp\left(\frac{imx_b^2}{2\hbar t_b}\right) \exp\left(\frac{imx_a^2}{2\hbar t_b} - \frac{imx_bx_a}{\hbar t_b}\right)$$

By equation (3.42) we have

$$\psi(x_b, t_b) = \int_{-\infty}^{\infty} K_0(x_b, t_b, x_a, 0) \psi(x_a, 0) dx_a$$

Hence

$$\begin{aligned} \psi(x_b, t_b) &= C \left(\frac{m}{2\pi i\hbar t_b}\right)^{1/2} \exp\left(\frac{imx_b^2}{2\hbar t_b}\right) \\ &\quad \times \int_{-\infty}^{\infty} \exp\left(\frac{imx_a^2}{2\hbar t_b} - \frac{imx_bx_a}{\hbar t_b}\right) \exp\left(\frac{ipx_a}{\hbar}\right) dx_a \quad (1) \end{aligned}$$

Rewrite the integral as

$$\int_{-\infty}^{\infty} \exp(Ax_a^2 + Bx_a) dx_a$$

where

$$A = \frac{im}{2\hbar t_b}$$

$$B = -\frac{imx_b}{\hbar t_b} + \frac{ip}{\hbar}$$

Solve the integral.

$$\begin{aligned} & \int_{-\infty}^{\infty} \exp\left(\frac{imx_a^2}{2\hbar t_b} - \frac{imx_b x_a}{\hbar t_b}\right) dx_a \\ &= \left(-\frac{\pi}{A}\right)^{1/2} \exp\left(-\frac{B^2}{4A}\right) \\ &= \left(\frac{2\pi i\hbar t_b}{m}\right)^{1/2} \exp\left(-\frac{imx_b^2}{2\hbar t_b} - \frac{ip^2 t_b}{2m\hbar} + \frac{ipx_b}{\hbar}\right) \end{aligned} \quad (2)$$

Substitute (2) into (1) to obtain

$$\psi(x_b, t_b) = C \exp\left(-\frac{ip^2 t_b}{2m\hbar} + \frac{ipx_b}{\hbar}\right)$$

Substitute x for x_b and t for t_b .

$$\psi(x, t) = C \exp\left(-\frac{ip^2 t}{2m\hbar} + \frac{ipx}{\hbar}\right)$$

Hence the wave function $\psi(x, t)$ depends on x through the function $e^{ipx/\hbar}$ and varies in time as $e^{-ip^2 t/2m\hbar}$.