# Free particle propagator

A propagator is the amplitude for a particle to go from $x_a$ at time $t_a$ to $x_b$ at time $t_b$.

Let $K(b, a)$ be the propagator

$$K(b, a) = \langle x_b| \exp\left(-\frac{i}{\hbar}H(t_b - t_a)\right)|x_a\rangle$$

where $H$ is the free particle Hamiltonian

$$H = \frac{\hat{p}^2}{2m}$$

By the identity

$$\int |p\rangle\langle p|\,dp = 1$$

we can write

$$K(b, a) = \int \langle x_b| \exp\left(-\frac{i}{\hbar}H(t_b - t_a)\right)|p\rangle\langle p|x_a\rangle\,dp$$

Replace the momentum operator $\hat{p}$ with its eigenvalue $p$.

$$K(b, a) = \int \exp\left(-\frac{i}{2m\hbar}p^2(t_b - t_a)\right)\langle x_b|p\rangle\langle p|x_a\rangle\,dp$$

Recalling that

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ipx}{\hbar}\right)$$

we can write

$$K(b, a) = \frac{1}{2\pi\hbar}\int \exp\left(-\frac{i}{2m\hbar}p^2(t_b - t_a) + \frac{i}{\hbar}p(x_b - x_a)\right)\,dp$$

By the integral

$$\int_{-\infty}^{\infty} \exp(-ax^2 + bx)\,dx = \left(\frac{\pi}{a}\right)^{1/2}\exp\left(\frac{b^2}{4a}\right)$$

with

$$a = \frac{i}{2m\hbar}(t_b - t_a) \quad \text{and} \quad b = \frac{i}{\hbar}(x_b - x_a)$$

we have

$$K(b, a) = \frac{1}{2\pi\hbar}\left(\frac{2\pi m\hbar}{i(t_b - t_a)}\right)^{1/2}\exp\left(-\frac{m(x_b - x_a)^2}{2i\hbar(t_b - t_a)}\right)$$

Rewrite as

$$K(b, a) = \left(\frac{m}{2\pi i\hbar(t_b - t_a)}\right)^{1/2}\exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)}\right)$$