Wien's displacement law

For absolute temperature T, Wien's displacement law gives the peak EM radiation wavelength as

$$\lambda_{\text{peak}} = \frac{b}{T}$$

where b is Wien's displacement constant

$$b = 2.898 \times 10^{-3}\,\text{kelvin meter}$$

Wien estimated the value of b empirically. However, b has an exact value that can be derived from Planck's law

$$u = \frac{2hc^2}{\lambda^5}\frac{1}{\exp\left(\frac{hc}{kT\lambda}\right) - 1}$$

Take the derivative of u with respect to λ.

$$\frac{du}{d\lambda} = \frac{2h^2c^3\exp\left(\frac{hc}{kT\lambda}\right)}{kT\lambda^7\left(\exp\left(\frac{hc}{kT\lambda}\right) - 1\right)^2} - \frac{10hc^2}{\lambda^6\left(\exp\left(\frac{hc}{kT\lambda}\right) - 1\right)}$$

Peak intensity is at $du/d\lambda = 0$.

$$\frac{2h^2c^3\exp\left(\frac{hc}{kT\lambda}\right)}{kT\lambda^7\left(\exp\left(\frac{hc}{kT\lambda}\right) - 1\right)^2} - \frac{10hc^2}{\lambda^6\left(\exp\left(\frac{hc}{kT\lambda}\right) - 1\right)} = 0$$

Multiply both sides by

$$\frac{\lambda^6}{2hc^2}\left(\exp\left(\frac{hc}{kT\lambda}\right) - 1\right)$$

to obtain

$$\frac{hc\exp\left(\frac{hc}{kT\lambda}\right)}{kT\lambda\left(\exp\left(\frac{hc}{kT\lambda}\right) - 1\right)} - 5 = 0$$

Define

$$x = \frac{hc}{kT\lambda}$$

and write

$$\frac{xe^x}{e^x - 1} - 5 = 0$$

Multiply both sides by $e^x - 1$ to obtain

$$xe^x - 5e^x + 5 = 0 \tag{1}$$

The solution to (1) is

$$x = W_0(-5e^{-5}) + 5 \approx 4.96511$$

where W_0 is the Lambert W function

$$W_0(z) = \sum_{n=1}^{\infty}\frac{(-n)^{n-1}z^n}{n!}$$

Hence

$$b = \lambda T = \frac{hc}{kx} = 0.00289777\,\text{kelvin meter}$$