Fun trick

Show that

$$[p^2, \mathbf{r}] = -2i\hbar\mathbf{p}$$

where

$$\mathbf{r} = \otimes(x, y, z), \quad \mathbf{p} = -i\hbar\nabla, \quad p^2 = \mathbf{p}\cdot\mathbf{p} = -\hbar^2\nabla^2$$

We have

$$\begin{aligned}
[p^2, \mathbf{r}] &= p^2\mathbf{r} - \mathbf{r}p^2 \\
&= \mathbf{p}\cdot\mathbf{p}\mathbf{r} - \mathbf{r}\mathbf{p}\cdot\mathbf{p} \\
&= \mathrm{Tr}[\mathbf{ppr} - \mathbf{rpp}] \\
&= \mathrm{Tr}[\mathbf{ppr} - \mathbf{prp} + \mathbf{prp} - \mathbf{rpp}] && \text{trick!} \\
&= \mathrm{Tr}[\mathbf{p}(\mathbf{pr} - \mathbf{rp}) + (\mathbf{pr} - \mathbf{rp})\mathbf{p}] \\
&= \mathbf{p}(-i\hbar) + (-i\hbar)\mathbf{p} \\
&= -2i\hbar\mathbf{p}
\end{aligned}$$

The trick is adding null term $-\mathbf{prp} + \mathbf{prp}$ so that the operators can be factored. Trace operator Tr contracts on indices 1 and 2.

Verify the following formulas.

$$[p^2, \mathbf{r}] = -2i\hbar\mathbf{p} \tag{1}$$

$$[p^2, \mathbf{r}] = \mathrm{Tr}[\mathbf{ppr} - \mathbf{rpp}] \tag{2}$$

$$[p^2, \mathbf{r}] = \mathrm{Tr}[\mathbf{ppr} - \mathbf{prp} + \mathbf{prp} - \mathbf{rpp}] \tag{3}$$

$$\mathbf{pr} - \mathbf{rp} = -i\hbar\mathbf{I} \tag{4}$$

$$\mathbf{p}\cdot\mathbf{p} = \mathrm{Tr}[\mathbf{pp}] \tag{5}$$