

# Harmonic oscillator propagator

Given

$$K(b, a) = \langle x_b | \exp \left( -\frac{iHt}{\hbar} \right) | x_a \rangle \quad (1)$$

where

$$H = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2}$$

show that

$$K(b, a) = \left( \frac{m\omega}{2\pi i \hbar \sin(\omega t)} \right)^{\frac{1}{2}} \exp \left( \frac{im\omega(x_b^2 + x_a^2) \cot(\omega t)}{2\hbar} - \frac{im\omega x_b x_a}{\hbar \sin(\omega t)} \right)$$

This is van Kortryk's identity.

$$\exp \left( -\frac{iHt}{\hbar} \right) = \exp \left( -\frac{im\omega \hat{x}^2}{2\hbar} \tan \frac{\omega t}{2} \right) \exp \left( -\frac{i\hat{p}^2 \sin(\omega t)}{2m\hbar\omega} \right) \exp \left( -\frac{im\omega \hat{x}^2}{2\hbar} \tan \frac{\omega t}{2} \right) \quad (2)$$

Substitute (2) into (1) to obtain

$$K(b, a) = \langle x_b | \exp \left( -\frac{im\omega \hat{x}^2}{2\hbar} \tan \frac{\omega t}{2} \right) \exp \left( -\frac{i\hat{p}^2 \sin(\omega t)}{2m\hbar\omega} \right) \exp \left( -\frac{im\omega \hat{x}^2}{2\hbar} \tan \frac{\omega t}{2} \right) | x_a \rangle$$

Replace operators  $\hat{x}^2$  with eigenvalues  $x_b^2$  and  $x_a^2$ .

$$K(b, a) = \langle x_b | \exp \left( -\frac{im\omega x_b^2}{2\hbar} \tan \frac{\omega t}{2} \right) \exp \left( -\frac{i\hat{p}^2 \sin(\omega t)}{2m\hbar\omega} \right) \exp \left( -\frac{im\omega x_a^2}{2\hbar} \tan \frac{\omega t}{2} \right) | x_a \rangle$$

With the  $\hat{x}^2$  operators eliminated, states  $\langle x_b |$  and  $| x_a \rangle$  can be moved inwards.

$$K(b, a) = \exp \left( -\frac{im\omega x_b^2}{2\hbar} \tan \frac{\omega t}{2} \right) \langle x_b | \exp \left( -\frac{i\hat{p}^2 \sin(\omega t)}{2m\hbar\omega} \right) | x_a \rangle \exp \left( -\frac{im\omega x_a^2}{2\hbar} \tan \frac{\omega t}{2} \right)$$

Combine exponentials.

$$K(b, a) = \exp \left( -\frac{im\omega(x_b^2 + x_a^2)}{2\hbar} \tan \frac{\omega t}{2} \right) \langle x_b | \exp \left( -\frac{i\hat{p}^2 \sin(\omega t)}{2m\hbar\omega} \right) | x_a \rangle$$

Rewrite as

$$K(b, a) = \exp \left( -\frac{im\omega(x_b^2 + x_a^2)}{2\hbar} \tan \frac{\omega t}{2} \right) K_0(b, a) \quad (3)$$

where

$$K_0(b, a) = \langle x_b | \exp \left( -\frac{i\hat{p}^2 \sin(\omega t)}{2m\hbar\omega} \right) | x_a \rangle$$

Noting that  $K_0$  is a free particle propagator with  $t_b - t_a = \sin(\omega t)/\omega$  we have

$$K_0(b, a) = \left( \frac{m\omega}{2\pi i \hbar \sin(\omega t)} \right)^{\frac{1}{2}} \exp \left( \frac{im\omega(x_b - x_a)^2}{2\hbar \sin(\omega t)} \right) \quad (4)$$

Substitute (4) into (3) to obtain

$$K(b, a) = \left( \frac{m\omega}{2\pi i \hbar \sin(\omega t)} \right)^{\frac{1}{2}} \exp \left( -\frac{im\omega(x_b^2 + x_a^2)}{2\hbar} \tan \frac{\omega t}{2} + \frac{im\omega(x_b - x_a)^2}{2\hbar \sin(\omega t)} \right) \quad (5)$$

Using the identity

$$\cot \alpha = -\tan \frac{\alpha}{2} + \frac{1}{\sin \alpha}$$

rewrite (5) as

$$K(b, a) = \left( \frac{m\omega}{2\pi i \hbar \sin(\omega t)} \right)^{\frac{1}{2}} \exp \left( \frac{im\omega(x_b^2 + x_a^2) \cot(\omega t)}{2\hbar} - \frac{im\omega x_b x_a}{\hbar \sin(\omega t)} \right) \quad (6)$$