$$v(\breve{p}) = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \exp\left(\frac{i\breve{p}r\cos\theta}{\hbar}\right) V(r)\, r^2 \sin\theta \, dr\, d\theta\, d\phi$$

Substitute for $V(r)$.

$$v(\breve{p}) = -Ze^2 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \exp\left(\frac{i\breve{p}r\cos\theta}{\hbar}\right) \exp\left(-\frac{r}{a}\right) r \sin\theta \, dr\, d\theta\, d\phi$$

Integrate over ϕ.

$$v(\breve{p}) = -2\pi Ze^2 \int_0^{\pi}\int_0^{\infty} \exp\left(\frac{i\breve{p}r\cos\theta}{\hbar}\right) \exp\left(-\frac{r}{a}\right) r \sin\theta \, dr\, d\theta$$

Transform the integral over θ to an integral over y where $y = \cos\theta$, $dy = -\sin\theta\, d\theta$.

$$v(\breve{p}) = -2\pi Ze^2 \int_{-1}^{1}\int_0^{\infty} \exp\left(\frac{ipry}{\hbar}\right) \exp\left(-\frac{r}{a}\right) r \, dr\, dy$$

Solve the integral over y (note r in the integrand cancels).

$$v(\breve{p}) = -2\pi Ze^2 \int_0^{\infty} \frac{\hbar}{ip}\left[\exp\left(\frac{ipr}{\hbar}\right) - \exp\left(-\frac{ipr}{\hbar}\right)\right] \exp\left(-\frac{r}{a}\right) \, dr$$

Solve the integral over r.

$$v(\breve{p}) = -2\pi Ze^2 \frac{\hbar}{ip}\left[\frac{1}{ip/\hbar - 1/a}\exp\left(\frac{ipr}{\hbar} - \frac{r}{a}\right) + \frac{1}{ip/\hbar + 1/a}\exp\left(-\frac{ipr}{\hbar} - \frac{r}{a}\right)\right]_0^{\infty}$$

Evaluate the limits.

$$v(\breve{p}) = -2\pi Ze^2 \frac{\hbar}{ip}\left[-\frac{1}{ip/\hbar - 1/a} - \frac{1}{ip/\hbar + 1/a}\right] = -\frac{4\pi Ze^2}{(p/\hbar)^2 + (1/a)^2}$$

The cross section is

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |v(\breve{p})|^2 = \left(\frac{2mZe^2}{\breve{p}^2 + (\hbar/a)^2}\right)^2$$

Substitute $2mu\sin(\theta/2)$ for $\breve{p}$.

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mZe^2}{4m^2u^2\sin^2(\theta/2) + (\hbar/a)^2}\right)^2$$

Factor out $mu = p$ in the denominator.

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mZe^2}{m^2u^2\left[4\sin^2(\theta/2) + (\hbar/pa)^2\right]}\right)^2$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{Z^2e^4}{(mu^2/2)^2\left[4\sin^2(\theta/2) + (\hbar/pa)^2\right]^2}$$

The total cross section is

$$\sigma_T = \int_0^{2\pi} \int_0^{\pi} \frac{Z^2 e^4}{(mu^2/2)^2 \left[4\sin^2(\theta/2) + (\hbar/pa)^2\right]^2} \sin\theta \, d\theta \, d\phi$$

Integrate over ϕ.

$$\sigma_T = 2\pi \int_0^{\pi} \frac{Z^2 e^4}{(mu^2/2)^2 \left[4\sin^2(\theta/2) + (\hbar/pa)^2\right]^2} \sin\theta \, d\theta$$

Factor out 16 in the denominator and write as

$$\sigma_T = \frac{2\pi Z^2 e^4}{16(mu^2/2)^2} \int_0^{\pi} \frac{\sin\theta}{\left(\sin^2(\theta/2) + (\hbar/2pa)^2\right)^2} \, d\theta$$

By the definite integral

$$\int_0^{\pi} \frac{\sin\theta}{\left(\sin^2(\theta/2) + a\right)^2} \, d\theta = \frac{2}{a^2 + a}$$

we have

$$\sigma_T = \frac{2\pi Z^2 e^4}{16(mu^2/2)^2} \frac{2}{(\hbar/2pa)^4 + (\hbar/2pa)^2}$$

Rewrite as

$$\sigma_T = \frac{\pi Z^2 e^4}{2p^2 u^2} \left(\frac{2pa}{\hbar}\right)^2 \frac{2}{(\hbar/2pa)^2 + 1}$$

Hence

$$\sigma_T = \pi a^2 \frac{(2Ze^2/u\hbar)^2}{(\hbar/2pa)^2 + 1}$$