

4-2. For a particle of charge e in a magnetic field the Lagrangian is

$$L(\dot{\mathbf{x}}, \mathbf{x}) = \frac{m}{2} \dot{\mathbf{x}}^2 + \frac{e}{c} \dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}, t) - e\phi(\mathbf{x}, t)$$

where $\dot{\mathbf{x}}$ is the velocity vector, c is the velocity of light, and \mathbf{A} and ϕ are the vector and scalar potentials. Show that the corresponding Schrodinger equation is

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} \left(\frac{1}{2m} \left(\frac{\hbar}{i} \nabla - \frac{e}{c} \mathbf{A} \right) \cdot \left(\frac{\hbar}{i} \nabla - \frac{e}{c} \mathbf{A} \right) \psi + e\phi \psi \right) \quad (4.18)$$

From equation (4.3) with a minor correction of $y - x$ instead of $x - y$.

$$\psi(\mathbf{x}, t + \epsilon) = \frac{1}{A} \int_{\mathbb{R}^3} \exp \left(\frac{i\epsilon}{\hbar} L \left(\frac{\mathbf{y} - \mathbf{x}}{\epsilon}, \frac{\mathbf{x} + \mathbf{y}}{2} \right) \right) \psi(\mathbf{y}, t) dy_1 dy_2 dy_3 \quad (1)$$

This is the Lagrangian with arguments from (1).

$$\begin{aligned} L \left(\frac{\mathbf{y} - \mathbf{x}}{\epsilon}, \frac{\mathbf{x} + \mathbf{y}}{2} \right) \\ = \frac{m}{2\epsilon^2} (\mathbf{x} - \mathbf{y})^2 - \frac{e}{c\epsilon} (\mathbf{x} - \mathbf{y}) \cdot \mathbf{A} \left(\frac{\mathbf{x} + \mathbf{y}}{2}, t \right) - e\phi \left(\frac{\mathbf{x} + \mathbf{y}}{2}, t \right) \end{aligned}$$

Hence

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) &= \frac{1}{A} \int_{\mathbb{R}^3} \\ &\exp \left(\frac{im}{2\hbar\epsilon} (\mathbf{x} - \mathbf{y})^2 - \frac{ie}{\hbar c} (\mathbf{x} - \mathbf{y}) \cdot \mathbf{A} \left(\frac{\mathbf{x} + \mathbf{y}}{2}, t \right) - \frac{ie\epsilon}{\hbar} \phi \left(\frac{\mathbf{x} + \mathbf{y}}{2}, t \right) \right) \\ &\times \psi(\mathbf{y}, t) dy_1 dy_2 dy_3 \end{aligned}$$

Let

$$\mathbf{y} = \mathbf{x} + \boldsymbol{\eta}$$

Then

$$\mathbf{x} - \mathbf{y} = \boldsymbol{\eta}, \quad \frac{\mathbf{x} + \mathbf{y}}{2} = \mathbf{x} + \frac{1}{2}\boldsymbol{\eta}, \quad dy_1 dy_2 dy_3 = d\eta_1 d\eta_2 d\eta_3$$

Hence

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) &= \frac{1}{A} \int_{\mathbb{R}^3} \exp \left(\frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 - \frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A} \left(\mathbf{x} + \frac{1}{2}\boldsymbol{\eta}, t \right) - \frac{ie\epsilon}{\hbar} \phi \left(\mathbf{x} + \frac{1}{2}\boldsymbol{\eta}, t \right) \right) \\ &\times \psi(\mathbf{x} + \boldsymbol{\eta}, t) d\eta_1 d\eta_2 d\eta_3 \end{aligned}$$

Factor the exponential.

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) &= \frac{1}{A} \int_{\mathbb{R}^3} \\ &\exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \exp\left(-\frac{ie}{\hbar c}\boldsymbol{\eta} \cdot \mathbf{A}\left(\mathbf{x} + \frac{1}{2}\boldsymbol{\eta}, t\right)\right) \exp\left(-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x} + \frac{1}{2}\boldsymbol{\eta}, t\right)\right) \\ &\times \psi(\mathbf{x} + \boldsymbol{\eta}, t) d\eta_1 d\eta_2 d\eta_3 \end{aligned} \quad (2)$$

From the identity $\exp(i\theta) = \cos(\theta) + i\sin(\theta)$ we have

$$\begin{aligned} \exp\left(-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x} + \frac{1}{2}\boldsymbol{\eta}, t\right)\right) \\ = \cos\left(-\frac{e\epsilon}{\hbar}\phi\left(\mathbf{x} + \frac{1}{2}\boldsymbol{\eta}, t\right)\right) + i\sin\left(-\frac{e\epsilon}{\hbar}\phi\left(\mathbf{x} + \frac{1}{2}\boldsymbol{\eta}, t\right)\right) \end{aligned}$$

Then for small ϵ

$$\exp\left(-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x} + \frac{1}{2}\boldsymbol{\eta}, t\right)\right) \approx 1 - \frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x} + \frac{1}{2}\boldsymbol{\eta}, t\right)$$

The $\boldsymbol{\eta}$ term can be discarded because the integral is Gaussian. (Contributions to the integral are small for $\boldsymbol{\eta}^2 > 2\hbar\epsilon/m$.)

$$\exp\left(-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x} + \frac{1}{2}\boldsymbol{\eta}, t\right)\right) \approx 1 - \frac{ie\epsilon}{\hbar}\phi(\mathbf{x}, t) \quad (3)$$

Substitute (3) into (2).

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) &= \frac{1}{A} \left(1 - \frac{ie\epsilon}{\hbar}\phi\right) \int_{\mathbb{R}^3} \\ &\exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \exp\left(-\frac{ie}{\hbar c}\boldsymbol{\eta} \cdot \mathbf{A}\left(\mathbf{x} + \frac{1}{2}\boldsymbol{\eta}, t\right)\right) \psi(\mathbf{x} + \boldsymbol{\eta}, t) d\eta_1 d\eta_2 d\eta_3 \end{aligned}$$

Approximate the exponential involving \mathbf{A} with a Taylor series.

$$\exp\left(-\frac{ie}{\hbar c}\boldsymbol{\eta} \cdot \mathbf{A}\left(\mathbf{x} + \frac{1}{2}\boldsymbol{\eta}, t\right)\right) \approx \exp\left(-\frac{ie}{\hbar c}\boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{x}) - \frac{ie}{2\hbar c}\boldsymbol{\eta} \cdot (\boldsymbol{\eta} \cdot \nabla \mathbf{A}(\mathbf{x}))\right)$$

Expand the right-hand side as a power series.

$$\exp\left(-\frac{ie}{\hbar c}\boldsymbol{\eta} \cdot \mathbf{A}\left(\mathbf{x} + \frac{1}{2}\boldsymbol{\eta}, t\right)\right) \approx (1 + T + \frac{1}{2}T^2)$$

where

$$T = -\frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{x}) - \frac{ie}{2\hbar c} \boldsymbol{\eta} \cdot (\boldsymbol{\eta} \cdot \nabla \mathbf{A}(\mathbf{x}))$$

Discard high order terms.

$$\begin{aligned} & \exp \left(-\frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A} \left(\mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \\ & \approx 1 - \frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{x}) - \frac{ie}{2\hbar c} \boldsymbol{\eta} \cdot (\boldsymbol{\eta} \cdot \nabla \mathbf{A}(\mathbf{x})) + \frac{1}{2} \left(-\frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{x}) \right)^2 \end{aligned}$$

Hence

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) &= \frac{1}{A} \left(1 - \frac{ie\epsilon}{\hbar} \phi \right) \int_{\mathbb{R}^3} \exp \left(\frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 \right) \\ &\times \left(1 - \frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A} - \frac{ie}{2\hbar c} \boldsymbol{\eta} \cdot (\boldsymbol{\eta} \cdot \nabla \mathbf{A}) + \frac{1}{2} \left(-\frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A} \right)^2 \right) \\ &\times \psi(\mathbf{x} + \boldsymbol{\eta}, t) d\eta_1 d\eta_2 d\eta_3 \end{aligned} \quad (4)$$

Next we will use the following Taylor series approximations.

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) &\approx \psi + \epsilon \frac{\partial \psi}{\partial t} \\ \psi(\mathbf{x} + \boldsymbol{\eta}, t) &\approx \psi + \boldsymbol{\eta} \cdot \nabla \psi + \frac{1}{2} \boldsymbol{\eta} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \psi) \end{aligned} \quad (5)$$

Substitute the approximations (5) into (4).

$$\begin{aligned} \psi + \epsilon \frac{\partial \psi}{\partial t} &= \frac{1}{A} \left(1 - \frac{ie\epsilon}{\hbar} \phi \right) \int_{\mathbb{R}^3} \exp \left(\frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 \right) \\ &\times \left(1 - \frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A} - \frac{ie}{2\hbar c} \boldsymbol{\eta} \cdot (\boldsymbol{\eta} \cdot \nabla \mathbf{A}) + \frac{1}{2} \left(-\frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A} \right)^2 \right) \\ &\times \left(\psi + \boldsymbol{\eta} \cdot \nabla \psi + \frac{1}{2} \boldsymbol{\eta} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \psi) \right) d\eta_1 d\eta_2 d\eta_3 \end{aligned} \quad (6)$$

To solve the above integral, we will use the following formulas provided by the authors.

$$\int_{-\infty}^{\infty} \exp \left(\frac{im\eta_k^2}{2\hbar\epsilon} \right) d\eta_k = \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{1/2} \quad (4.7)$$

$$\int_{-\infty}^{\infty} \eta_k \exp \left(\frac{im\eta_k^2}{2\hbar\epsilon} \right) d\eta_k = 0 \quad (4.9)$$

$$\int_{-\infty}^{\infty} \eta_k^2 \exp \left(\frac{im\eta_k^2}{2\hbar\epsilon} \right) d\eta_k = \frac{i\hbar\epsilon}{m} \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{1/2} \quad (4.10)$$

The integrand in (6) has twelve terms. The following table summarizes the integrals $\int uv$ for each factor pair uv . Integrals I_9 and I_{12} are proportional to ϵ^2 due to (4.10) and are discarded.

	$v = \psi$	$v = \boldsymbol{\eta} \cdot \nabla \psi$	$v = \frac{1}{2} \boldsymbol{\eta} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \psi)$
$u = 1$	I_1	0	I_3
$u = -\frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A}$	0	I_5	0
$u = -\frac{ie}{2\hbar c} \boldsymbol{\eta} \cdot (\boldsymbol{\eta} \cdot \nabla \mathbf{A})$	I_7	0	$I_9 \propto \epsilon^2$
$u = \frac{1}{2} \left(-\frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A} \right)^2$	I_{10}	0	$I_{12} \propto \epsilon^2$

$$\begin{aligned}
I_1 &= \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{3/2} \psi \\
I_3 &= \frac{i \hbar \epsilon}{2m} \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{3/2} \nabla^2 \psi \\
I_5 &= \frac{i \hbar \epsilon}{m} \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{3/2} \frac{-ie}{\hbar c} \mathbf{A} \cdot \nabla \psi \\
I_7 &= \frac{i \hbar \epsilon}{m} \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{3/2} \frac{-ie}{2\hbar c} \nabla \cdot \mathbf{A} \psi \\
I_{10} &= \frac{i \hbar \epsilon}{m} \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{3/2} \frac{1}{2} \left(\frac{-ie}{\hbar c} \right)^2 \mathbf{A}^2 \psi
\end{aligned}$$

Substitute the solved integrals into (6) to obtain

$$\begin{aligned} \psi + \epsilon \frac{\partial \psi}{\partial t} &= \frac{1}{A} \left(1 - \frac{ie\epsilon}{\hbar} \phi \right) \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{3/2} \\ &\times \left(\psi + \frac{i\hbar\epsilon}{2m} \nabla^2 \psi + \frac{e\epsilon}{mc} \mathbf{A} \cdot \nabla \psi + \frac{e\epsilon}{2mc} \nabla \cdot \mathbf{A} \psi - \frac{ie^2\epsilon}{2m\hbar c^2} \mathbf{A}^2 \psi \right) \end{aligned}$$

In the limit as $\epsilon \rightarrow 0$ we have

$$\psi = \frac{1}{A} \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{3/2} \psi$$

hence

$$A = \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{3/2}$$

Cancel A.

$$\begin{aligned} \psi + \epsilon \frac{\partial \psi}{\partial t} &= \left(1 - \frac{ie\epsilon}{\hbar} \phi \right) \\ &\times \left(\psi + \frac{i\hbar\epsilon}{2m} \nabla^2 \psi + \frac{e\epsilon}{mc} \mathbf{A} \cdot \nabla \psi + \frac{e\epsilon}{2mc} \nabla \cdot \mathbf{A} \psi - \frac{ie^2\epsilon}{2m\hbar c^2} \mathbf{A}^2 \psi \right) \end{aligned}$$

Expand the product and discard terms of order ϵ^2 .

$$\begin{aligned} \psi + \epsilon \frac{\partial \psi}{\partial t} &= \psi + \frac{i\hbar\epsilon}{2m} \nabla^2 \psi + \frac{e\epsilon}{mc} \mathbf{A} \cdot \nabla \psi + \frac{e\epsilon}{2mc} \nabla \cdot \mathbf{A} \psi - \frac{ie^2\epsilon}{2m\hbar c^2} \mathbf{A}^2 \psi - \frac{ie\epsilon}{\hbar} \phi \psi \end{aligned}$$

Cancel leading terms ψ and divide through by ϵ .

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m} \nabla^2 \psi + \frac{e}{mc} \mathbf{A} \cdot \nabla \psi + \frac{e}{2mc} \nabla \cdot \mathbf{A} \psi - \frac{ie^2}{2m\hbar c^2} \mathbf{A}^2 \psi - \frac{ie}{\hbar} \phi \psi \quad (7)$$

To show that (7) is the same as (4.18), expand (4.18) step by step. First we have

$$\left(\frac{\hbar}{i} \nabla - \frac{e}{c} \mathbf{A} \right) \psi = -i\hbar \nabla \psi - \frac{e}{c} \mathbf{A} \psi$$

Next

$$\begin{aligned}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right)^2 \psi &= \left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right) \left(-i\hbar\nabla\psi - \frac{e}{c}\mathbf{A}\psi\right) \\ &= -\hbar^2\nabla(\nabla\psi) + \frac{ie\hbar}{c}\nabla(\mathbf{A}\psi) + \frac{ie\hbar}{c}\mathbf{A} \cdot \nabla\psi + \frac{e^2}{c^2}\mathbf{A} \cdot \mathbf{A}\psi\end{aligned}$$

Because $\nabla\psi$ is a vector we have

$$\nabla(\nabla\psi) = \nabla \cdot \nabla\psi = \nabla^2\psi$$

and because \mathbf{A} is a vector and ψ is a scalar we have

$$\nabla(\mathbf{A}\psi) = \nabla \cdot \mathbf{A}\psi + \mathbf{A} \cdot \nabla\psi$$

Hence

$$\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right)^2 \psi = -\hbar^2\nabla^2\psi + \frac{2ie\hbar}{c}\mathbf{A} \cdot \nabla\psi + \frac{ie\hbar}{c}\nabla \cdot \mathbf{A}\psi + \frac{e^2}{c^2}\mathbf{A}^2\psi$$

Divide through by $2m$.

$$\begin{aligned}\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right)^2 \psi \\ = -\frac{\hbar^2}{2m}\nabla^2\psi + \frac{ie\hbar}{mc}\mathbf{A} \cdot \nabla\psi + \frac{ie\hbar}{2mc}\nabla \cdot \mathbf{A}\psi + \frac{e^2}{2mc^2}\mathbf{A}^2\psi\end{aligned}$$

Add the scalar potential.

$$\begin{aligned}\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right)^2 \psi + e\phi\psi \\ = -\frac{\hbar^2}{2m}\nabla^2\psi + \frac{ie\hbar}{mc}\mathbf{A} \cdot \nabla\psi + \frac{ie\hbar}{2mc}\nabla \cdot \mathbf{A}\psi + \frac{e^2}{2mc^2}\mathbf{A}^2\psi + e\phi\psi\end{aligned}$$

Finally, multiply by $-i/\hbar$.

$$\begin{aligned}-\frac{i}{\hbar}\left(\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right)^2 \psi + e\phi\psi\right) \\ = \frac{i\hbar}{2m}\nabla^2\psi + \frac{e}{mc}\mathbf{A} \cdot \nabla\psi + \frac{e}{2mc}\nabla \cdot \mathbf{A}\psi - \frac{ie^2}{2m\hbar c^2}\mathbf{A}^2\psi - \frac{ie}{\hbar}\phi\psi\end{aligned}$$

The result is identical to (7).

$$\frac{\partial\psi}{\partial t} = \frac{i\hbar}{2m}\nabla^2\psi + \frac{e}{mc}\mathbf{A} \cdot \nabla\psi + \frac{e}{2mc}\nabla \cdot \mathbf{A}\psi - \frac{ie^2}{2m\hbar c^2}\mathbf{A}^2\psi - \frac{ie}{\hbar}\phi\psi \quad (7)$$