Hydrogen eigenfunctions

Verify

$$H\psi_{nlm}(r,\theta,\phi) = E_n\psi_{nlm}(r,\theta,\phi)$$

where H is the Hamiltonian operator

$$H\psi = -\frac{\hbar^2}{2\mu}\nabla^2\psi - \frac{\hbar^2}{\mu a_0 r}\psi$$

and E_n is the energy eigenvalue

$$E_n = -\frac{\hbar^2}{2\mu a_0^2 n^2} \approx -\frac{1}{n^2} \times 13.6\,\text{eV}$$

Symbol μ is the reduced electron mass

$$\mu = \frac{m_e m_p}{m_e + m_p}$$

Hydrogen eigenfunctions ψ_{nlm} are formed as

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi)$$

Radial eigenfunction R_{nl} is formed as

$$R_{nl}(r) = \frac{2}{n^2}\sqrt{\frac{(n-l-1)!}{(n+l)!}}\left(\frac{2r}{na_0}\right)^l L_{n-l-1}^{2l+1}\left(\frac{2r}{na_0}\right)\exp\left(-\frac{r}{na_0}\right)a_0^{-3/2}$$

Symbol L_n^m is the Laguerre polynomial

$$L_n^m(x) = (n+m)!\sum_{k=0}^{n}\frac{(-x)^k}{(n-k)!(m+k)!k!}$$

Symbol Y_{lm} is the spherical harmonic

$$Y_{lm}(\theta,\phi) = (-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\exp(im\phi)$$

Legendre polynomial $P_l^m(\cos\theta)$ is formed as (see arxiv.org/abs/1805.12125)

$$P_l^m(\cos\theta) = \begin{cases} \left(\dfrac{\sin\theta}{2}\right)^m \displaystyle\sum_{k=0}^{l-m}(-1)^k\frac{(l+m+k)!}{(l-m-k)!(m+k)!k!}\left(\frac{1-\cos\theta}{2}\right)^k, & m \ge 0 \\[2ex] (-1)^m\dfrac{(l+m)!}{(l-m)!}P_l^{|m|}(\cos\theta), & m < 0 \end{cases}$$

Symbol ∇^2 is the Laplacian operator in spherical coordinates.

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\psi\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\psi\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi$$

Noting that

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{e^2\mu}$$

we have

$$H\psi = -\frac{\hbar^2}{2\mu}\nabla^2\psi - \frac{e^2}{4\pi\varepsilon_0 r}\psi$$
$$E_n = -\frac{\mu}{2n^2}\left(\frac{e^2}{4\pi\varepsilon_0\hbar}\right)^2$$

Noting that

$$e^2 = 4\pi\epsilon_0\alpha\hbar c$$

we have

$$a_0 = \frac{\hbar}{\alpha\mu c}$$
$$H\psi = -\frac{\hbar^2}{2\mu}\nabla^2\psi - \frac{\alpha\hbar c}{r}\psi$$
$$E_n = -\frac{\mu\alpha^2c^2}{2n^2}$$