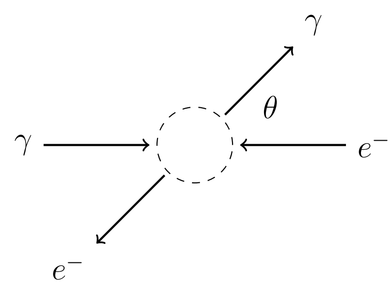
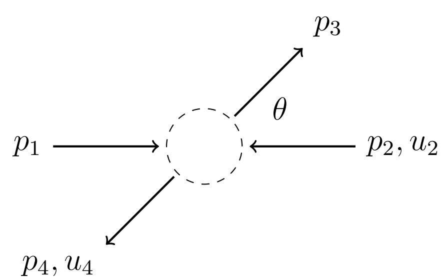
Compton scattering is the result of photons interacting with electrons. In a typical Compton scattering experiment the electron is at rest. However, it is easier to develop a theory using the center of mass frame in which the photon and the electron have equal and opposite momentum. The following diagram shows a photon and an electron scattered through angle θ in the center of mass frame.



Here is the same diagram with momentum and spinor labels.



In center of mass coordinates the momentum vectors are

$$p_1 = \underset{\text{inbound photon}}{\begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix}} \qquad p_2 = \underset{\text{inbound electron}}{\begin{pmatrix} E \\ 0 \\ 0 \\ -\omega \end{pmatrix}} \qquad p_3 = \underset{\text{outbound photon}}{\begin{pmatrix} \omega \\ \omega \sin\theta \cos\phi \\ \omega \sin\theta \sin\phi \\ \omega \cos\theta \end{pmatrix}} \qquad p_4 = \underset{\text{outbound electron}}{\begin{pmatrix} E \\ -\omega \sin\theta \cos\phi \\ -\omega \sin\theta \sin\phi \\ -\omega \cos\theta \end{pmatrix}}$$

Symbol ω is incident momentum, E is total energy $E = \sqrt{\omega^2 + m^2}$, and m is electron mass. Polar angle θ is the observed scattering angle. Azimuth angle ϕ cancels out in scattering calculations.

The spinors are

$$u_{21} = \underset{\substack{\text{inbound electron} \\ \text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ -\omega \\ 0 \end{pmatrix}} \qquad u_{22} = \underset{\substack{\text{inbound electron} \\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ \omega \end{pmatrix}} \qquad u_{41} = \underset{\substack{\text{outbound electron} \\ \text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix}} \qquad u_{42} = \underset{\substack{\text{outbound electron} \\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}}$$

The spinors shown above are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^2$ will be used.

The following formula computes a probability density $|\mathcal{M}_{ab}|^2$ for Compton scattering where a is the spin state of the inbound electron and b is the spin state of the outbound electron.

$$|\mathcal{M}_{ab}|^2 = \frac{e^4}{N}\left| -\frac{\bar{u}_{4b}\gamma^\mu(\not{q}_1 + m)\gamma^\nu u_{2a}}{s - m^2} - \frac{\bar{u}_{4b}\gamma^\nu(\not{q}_2 + m)\gamma^\mu u_{2a}}{u - m^2} \right|^2$$

Symbol e is electron charge. Symbols q_1 and q_2 are

$$\begin{aligned} q_1 &= p_1 + p_2 \\ q_2 &= p_4 - p_1 = p_2 - p_3 \end{aligned}$$

Symbols s and u are Mandelstam variables

$$\begin{aligned} s &= (p_1 + p_2)^2 \\ u &= (p_1 - p_4)^2 \end{aligned}$$

Let

$$a_1 = \bar{u}_{4b}\gamma^\mu(\not{q}_1 + m)\gamma^\nu u_{2a}, \quad a_2 = \bar{u}_{4b}\gamma^\nu(\not{q}_2 + m)\gamma^\mu u_{2a}$$

Then

$$\begin{aligned} |\mathcal{M}_{ab}|^2 &= \frac{e^4}{N}\left| -\frac{a_1}{s - m^2} - \frac{a_2}{u - m^2} \right|^2 \\ &= \frac{e^4}{N}\left(-\frac{a_1}{s - m^2} - \frac{a_2}{u - m^2} \right)\left(-\frac{a_1}{s - m^2} - \frac{a_2}{u - m^2} \right)^* \\ &= \frac{e^4}{N}\left(\frac{a_1 a_1^*}{(s - m^2)^2} + \frac{a_1 a_2^*}{(s - m^2)(u - m^2)} + \frac{a_1^* a_2}{(s - m^2)(u - m^2)} + \frac{a_2 a_2^*}{(u - m^2)^2} \right) \end{aligned}$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is computed by summing $|\mathcal{M}_{ab}|^2$ over all spin and polarization states and then dividing by the number of inbound states. There are four inbound states. The sum over polarizations is already accomplished by contraction of aa^* over μ and ν.

$$\begin{aligned} \langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2} |\mathcal{M}_{ab}|^2 \\ &= \frac{e^4}{4N}\sum_{a=1}^{2}\sum_{b=1}^{2}\left(\frac{a_1 a_1^*}{(s - m^2)^2} + \frac{a_1 a_2^*}{(s - m^2)(u - m^2)} + \frac{a_1^* a_2}{(s - m^2)(u - m^2)} + \frac{a_2 a_2^*}{(u - m^2)^2} \right) \end{aligned}$$

Use the Casimir trick to replace sums over spins with matrix products.

$$\begin{aligned} f_{11} &= \frac{1}{N}\sum_{a=1}^{2}\sum_{b=1}^{2} a_1 a_1^* = \mathrm{Tr}\left((\not{p}_2 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu \right) \\ f_{12} &= \frac{1}{N}\sum_{a=1}^{2}\sum_{b=1}^{2} a_1 a_2^* = \mathrm{Tr}\left((\not{p}_2 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu \right) \\ f_{22} &= \frac{1}{N}\sum_{a=1}^{2}\sum_{b=1}^{2} a_2 a_2^* = \mathrm{Tr}\left((\not{p}_2 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\nu(\not{q}_2 + m)\gamma_\mu \right) \end{aligned}$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left(\frac{f_{11}}{(s-m^2)^2} + \frac{f_{12}}{(s-m^2)(u-m^2)} + \frac{f_{12}^*}{(s-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2}\right) \tag{1}$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$\begin{aligned}
f_{11} &= 32(p_1 \cdot p_2)(p_1 \cdot p_4) + 64m^2(p_1 \cdot p_2) - 32m^2(p_1 \cdot p_3) - 32m^2(p_1 \cdot p_4) + 32m^4 \\
f_{12} &= 16m^2(p_1 \cdot p_2) - 16m^2(p_1 \cdot p_4) + 32m^4 \\
f_{22} &= 32(p_1 \cdot p_2)(p_1 \cdot p_4) + 32m^2(p_1 \cdot p_2) - 32m^2(p_1 \cdot p_3) - 64m^2(p_1 \cdot p_4) + 32m^4
\end{aligned}$$

For Mandelstam variables

$$\begin{aligned}
s &= (p_1 + p_2)^2 \\
t &= (p_1 - p_3)^2 \\
u &= (p_1 - p_4)^2
\end{aligned}$$

the formulas are

$$\begin{aligned}
f_{11} &= -8su + 24sm^2 + 8um^2 + 8m^4 \\
f_{12} &= 8sm^2 + 8um^2 + 16m^4 \\
f_{22} &= -8su + 8sm^2 + 24um^2 + 8m^4
\end{aligned} \tag{2}$$

High energy approximation

For $\omega \gg m$ a useful approximation is to set $m = 0$ and obtain

$$\begin{aligned}
f_{11} &= -8su \\
f_{12} &= 0 \\
f_{22} &= -8su
\end{aligned}$$

Hence

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{e^4}{4}\left(\frac{-8su}{s^2} + \frac{-8su}{u^2}\right) \\
&= 2e^4\left(-\frac{u}{s} - \frac{s}{u}\right)
\end{aligned}$$

Also for $m = 0$ the Mandelstam variables s and u are

$$\begin{aligned}
s &= 4\omega^2 \\
u &= -2\omega^2(\cos\theta + 1)
\end{aligned}$$

Hence

$$\langle|\mathcal{M}|^2\rangle = 2e^4\left(\frac{\cos\theta + 1}{2} + \frac{2}{\cos\theta + 1}\right)$$

Lab frame

Compton scattering experiments are typically done in the "lab" frame where the electron is at rest. The following Lorentz boost Λ transforms momentum vectors from the center of mass frame to the lab frame.

$$
\Lambda = \begin{pmatrix} E/m & 0 & 0 & \omega/m \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega/m & 0 & 0 & E/m \end{pmatrix}, \quad \Lambda p_2 = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

Mandelstam variables are invariant under a boost.

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = (\Lambda p_1 + \Lambda p_2)^2 \\
t &= (p_1 - p_3)^2 = (\Lambda p_1 - \Lambda p_3)^2 \\
u &= (p_1 - p_4)^2 = (\Lambda p_1 - \Lambda p_4)^2
\end{aligned}
$$

In the lab frame, let ω_L be the angular frequency of the incident photon and let ω'_L be the angular frequency of the scattered photon.

$$
\begin{aligned}
\omega_L &= \Lambda p_1 \cdot (1, 0, 0, 0) = \frac{\omega^2}{m} + \frac{\omega E}{m} \\
\omega'_L &= \Lambda p_3 \cdot (1, 0, 0, 0) = \frac{\omega^2 \cos\theta}{m} + \frac{\omega E}{m}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = 2m\omega_L + m^2 \\
t &= (p_1 - p_3)^2 = 2m(\omega'_L - \omega_L) \\
u &= (p_1 - p_4)^2 = -2m\omega'_L + m^2
\end{aligned}
$$

Then from equations (1) and (2)

$$
\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \left(\frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1 \right)^2 - 1 \right)
$$

From the Compton formula

$$
\frac{1}{\omega'_L} - \frac{1}{\omega_L} = \frac{1 - \cos\theta_L}{m}
$$

we have

$$
\cos\theta_L = \frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1
$$

Hence

$$
\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \cos^2\theta_L - 1 \right) \\
&= 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} - \sin^2\theta_L \right)
\end{aligned}
$$

Cross section

Now that we have derived $\langle|\mathcal{M}|^2\rangle$ we can investigate the angular distribution of scattered photons. For simplicity let us drop the L subscript from lab variables. From now on the symbols ω, ω', and θ will be lab frame variables.

The differential cross section for Compton scattering is

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 m^2}\left(\frac{\omega'}{\omega}\right)^2\langle|\mathcal{M}|^2\rangle \\ &= \frac{e^4}{32\pi^2 m^2}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta\right)\end{aligned}$$

Substitute $e^4 = 16\pi^2\alpha^2$ to obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta\right)$$

The scattered photon frequency ω' is computed from the Compton equation.

$$\omega' = \frac{m\omega}{m+\omega\,(1-\cos\theta)}$$

We can integrate $d\sigma$ to obtain a cumulative distribution function. Recall that

$$d\Omega = \sin\theta\,d\theta\,d\phi$$

Hence

$$d\sigma = \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta\right)\sin\theta\,d\theta\,d\phi$$

Let $I(\theta)$ be the following integral of $d\sigma$.

$$I(\theta) = \int_0^{2\pi}\int\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta\right)\sin\theta\,d\theta\,d\phi$$

Here is a computer solution for $I(\theta)$ with $\mathtt{w}\equiv\omega$.

```
I = m^4 / (-m w^3 - w^4 + w^4 cos(theta)) +
    2 m^3 w / (-m w^3 - w^4 + w^4 cos(theta)) -
    m^3 / (2 w (m + w (-cos(theta) + 1))^2) -
    2 m^3 log(m + w - w cos(theta)) / w^3 -
    m^2 w / (-m w^2 - w^3 + w^3 cos(theta)) +
    m^2 / (w (-m + w (cos(theta) - 1))) -
    m^2 cos(theta) / w^2 -
    2 m^2 log(m + w - w cos(theta)) / w^2 -
    m w^2 / (-m w^2 - w^3 + w^3 cos(theta)) +
    m log(m + w - w cos(theta)) / w -
    m / (m + w (-cos(theta) + 1))
```

The cumulative distribution function is

$$F(\theta) = \frac{I(\theta) - I(0)}{I(\pi) - I(0)}, \quad 0 \le \theta \le \pi$$

The probability of observing scattered photons in the interval θ_1 to θ_2 is

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

Thomson scattering

For $\omega \ll m$ we have

$$\frac{m}{m + \omega\,(1 - \cos\theta)} \approx 1$$

Hence we can use the approximation

$$\omega = \omega'$$

It follows that

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2}\left(1 + \cos^2\theta\right)$$

which is the formula for Thomson scattering.

Data from a CERN LEP experiment

See "Compton Scattering of Quasi-Real Virtual Photons at LEP," arxiv.org/abs/hep-ex/0504012.

x	y
-0.74	13380
-0.60	7720
-0.47	6360
-0.34	4600
-0.20	4310
-0.07	3700
0.06	3640
0.20	3340
0.33	3500
0.46	3010
0.60	3310
0.73	3330

The data are for the center of mass frame and have the following relationship with the differential cross section formula.

$$x = \cos\theta, \quad y = \frac{d\sigma}{d\cos\theta} = 2\pi\frac{d\sigma}{d\Omega}$$

From equation (3) we have for the center of mass frame

$$\langle|\mathcal{M}|^2\rangle = 2e^4\left(\frac{\cos\theta + 1}{2} + \frac{2}{\cos\theta + 1}\right)$$

The corresponding cross section formula is

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{64\pi^2 s} = \frac{e^4}{32\pi^2 s}\left(\frac{\cos\theta + 1}{2} + \frac{2}{\cos\theta + 1}\right), \qquad s \gg m$$

Substituting $e^4 = 16\pi^2\alpha^2$ yields

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\left(\frac{\cos\theta + 1}{2} + \frac{2}{\cos\theta + 1}\right)$$

Multiply by 2π to obtain

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{s}\left(\frac{\cos\theta + 1}{2} + \frac{2}{\cos\theta + 1}\right)$$

To compute predicted values $\hat{y}$ from the above formula, multiply by $(\hbar c)^2$ to convert to SI and multiply by 10^{40} to convert square meters to picobarns.

$$\hat{y} = \frac{\pi\alpha^2}{s}\left(\frac{x + 1}{2} + \frac{2}{x + 1}\right) \times (\hbar c)^2 \times 10^{40}$$

The following table shows $\hat{y}$ for $s = 40\,\mathrm{GeV}^2$ (i.e., $\omega = 100\,\mathrm{MeV}$).

x	y	$\hat{y}$
−0.74	13380	12739
−0.60	7720	8468
−0.47	6360	6577
−0.34	4600	5472
−0.20	4310	4723
−0.07	3700	4259
0.06	3640	3936
0.20	3340	3691
0.33	3500	3532
0.46	3010	3420
0.60	3310	3338
0.73	3330	3291

The coefficient of determination R^2 measures how well predicted values fit the real data.

$$R^2 = 1 - \frac{\sum(y - \hat{y})^2}{\sum(y - \bar{y})^2} = 0.97$$

The result indicates that the model $d\sigma$ explains 97% of the variance in the data.

Notes

Here are a few notes regarding the Eigenmath scripts.

Start by writing out a_1 and a_2 in full component form.

$$a_1^{\mu\nu} = \bar{u}_{4\alpha}\gamma^{\mu\alpha}{}_\beta(\not{q}_1 + m)^\beta{}_\rho\gamma^{\nu\rho}{}_\sigma u_2^\sigma, \quad a_2^{\nu\mu} = \bar{u}_{4\alpha}\gamma^{\nu\alpha}{}_\beta(\not{q}_2 + m)^\beta{}_\rho\gamma^{\mu\rho}{}_\sigma u_2^\sigma$$

Transpose γ tensors to form inner products over α and ρ.

$$a_1^{\mu\nu} = \bar{u}_{4\alpha}\gamma^{\alpha\mu}{}_\beta(\not{q}_1 + m)^\beta{}_\rho\gamma^{\rho\nu}{}_\sigma u_2^\sigma, \quad a_2^{\nu\mu} = \bar{u}_{4\alpha}\gamma^{\alpha\nu}{}_\beta(\not{q}_2 + m)^\beta{}_\rho\gamma^{\rho\mu}{}_\sigma u_2^\sigma$$

Convert transposed γ to Eigenmath code.

$\gamma^{\alpha\mu}{}_\beta \quad \rightarrow$ `gammaT = transpose(gamma)`

Then to compute a_1 we have

$a_1 = \bar{u}_{4\alpha}\gamma^{\alpha\mu}{}_\beta(\not{q}_1 + m)^\beta{}_\rho\gamma^{\rho\nu}{}_\sigma u_2^\sigma$

$\rightarrow$ `a1 = dot(u4bar[s4],gammaT,qslash1 + m I,gammaT,u2[s2])`

where s_2 and s_4 are spin indices. Similarly for a_2 we have

$a_2 = \bar{u}_{4\alpha}\gamma^{\alpha\nu}{}_\beta(\not{q}_2 + m)^\beta{}_\rho\gamma^{\rho\mu}{}_\sigma u_2^\sigma$

$\rightarrow$ `a2 = dot(u4bar[s4],gammaT,qslash2 + m I,gammaT,u2[s2])`

In component notation the product $a_1 a_1^*$ is

$$a_1 a_1^* = a_1^{\mu\nu} a_1^{*\mu\nu}$$

To sum over μ and ν it is necessary to lower indices with the metric tensor. Also, transpose a_1^* to form an inner product with ν.

$$a_1 a_1^* = a_1^{\mu\nu} a_{1\nu\mu}^*$$

Convert to Eigenmath code. The dot function sums over ν and the contract function sums over μ.

$a_1 a_1^* \quad \rightarrow$ `a11 = contract(dot(a1,gmunu,transpose(conj(a1)),gmunu))`

Similarly for $a_2 a_2^*$ we have

$a_2 a_2^* \quad \rightarrow$ `a22 = contract(dot(a2,gmunu,transpose(conj(a2)),gmunu))`

The product $a_1 a_2^*$ does not require a transpose because $a_1 a_2^* = a_1^{\mu\nu} a_2^{*\nu\mu}$.

$a_1 a_2^* \quad \rightarrow$ `a12 = contract(dot(a1,gmunu,conj(a2),gmunu))`

In component notation, a trace operator becomes a sum over an index, in this case α.

$$
\begin{aligned}
f_{11} &= \operatorname{Tr}\left((\not{p}_2 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\right)\\
&= (\not{p}_2 + m)^\alpha{}_\beta\gamma^{\mu\beta}{}_\rho(\not{q}_1 + m)^\rho{}_\sigma\gamma^{\nu\sigma}{}_\tau(\not{p}_4 + m)^\tau{}_\delta\gamma_\nu{}^\delta{}_\eta(\not{q}_1 + m)^\eta{}_\xi\gamma_\mu{}^\xi{}_\alpha
\end{aligned}
$$

As before, transpose γ tensors to form inner products.

$$
f_{11} = (\not{p}_2 + m)^\alpha{}_\beta\gamma^{\beta\mu}{}_\rho(\not{q}_1 + m)^\rho{}_\sigma\gamma^{\sigma\nu}{}_\tau(\not{p}_4 + m)^\tau{}_\delta\gamma^\delta{}_{\nu\eta}(\not{q}_1 + m)^\eta{}_\xi\gamma^\xi{}_{\mu\alpha}
$$

To convert to Eigenmath code, use an intermediate variable for the inner product.

$T^{\alpha\mu\nu}{}_{\nu\mu\alpha}$ $\rightarrow$ `T = dot(P2,gammaT,Q1,gammaT,P4,gammaL,Q1,gammaL)`

Now sum over the indices of T. The innermost contract sums over ν then the next contract sums over μ. Finally the outermost contract sums over α.

f_{11} $\rightarrow$ `f11 = contract(contract(contract(T,3,4),2,3))`

Follow suit for f_{22}. For f_{12} the order of the rightmost μ and ν is reversed.

$$
f_{12} = \operatorname{Tr}\left((\not{p}_2 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\right)
$$

The resulting inner product is $T^{\alpha\mu\nu}{}_{\mu\nu\alpha}$ so the contraction is different.

f_{12} $\rightarrow$ `f12 = contract(contract(contract(T,3,5),2,3))`

The innermost contract sums over ν followed by sum over μ then sum over α.