Green's function

In this section we will find the Green's function $G(\mathbf{x})$ such that

$$(\nabla^2 + k^2)G(\mathbf{x}) = \delta^3(\mathbf{x}) \tag{1}$$

Let $g(\mathbf{y})$ be the Fourier transform of $G(\mathbf{x})$ such that

$$G(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \exp(i\mathbf{x}\cdot\mathbf{y}) g(\mathbf{y})\, d\mathbf{y} \tag{2}$$

Substitute (2) into (1) to obtain

$$(\nabla^2 + k^2)\left[\frac{1}{(2\pi)^{\frac{3}{2}}} \int \exp(i\mathbf{x}\cdot\mathbf{y}) g(\mathbf{y})\, d\mathbf{y}\right] = \delta(\mathbf{x})$$

By linearity of differentiation the $(\nabla^2 + k^2)$ can be moved inside the integral.

$$\frac{1}{(2\pi)^{\frac{3}{2}}} \int (\nabla^2 + k^2) \exp(i\mathbf{x}\cdot\mathbf{y}) g(\mathbf{y})\, d\mathbf{y} = \delta(\mathbf{x})$$

Noting that

$$\nabla^2 \exp(i\mathbf{x}\cdot\mathbf{y}) = -y^2 \exp(i\mathbf{x}\cdot\mathbf{y})$$

and

$$\delta(\mathbf{x}) = \frac{1}{(2\pi)^3} \int \exp(i\mathbf{x}\cdot\mathbf{y})\, d\mathbf{y}$$

we have

$$\frac{1}{(2\pi)^{\frac{3}{2}}} \int (-y^2 + k^2) \exp(i\mathbf{x}\cdot\mathbf{y}) g(\mathbf{y})\, d\mathbf{y} = \frac{1}{(2\pi)^3} \int \exp(i\mathbf{x}\cdot\mathbf{y})\, d\mathbf{y}$$

Hence

$$g(\mathbf{y}) = \frac{1}{(2\pi)^{\frac{3}{2}}(k^2 - y^2)}$$

Substitute for $g(\mathbf{y})$ in (2) to obtain

$$G(\mathbf{x}) = \frac{1}{(2\pi)^3} \int \frac{\exp(i\mathbf{x}\cdot\mathbf{y})}{k^2 - y^2}\, d\mathbf{y}$$

Change to polar coordinates where $x = |\mathbf{x}|$, $y = |\mathbf{y}|$, and θ and ϕ are the angular distance from $\mathbf{x}$ to $\mathbf{y}$.

$$G(\mathbf{x}) = \frac{1}{(2\pi)^3} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{\exp(ixy\cos\theta)}{k^2 - y^2}\, y^2 \sin\theta\, dy\, d\theta\, d\phi$$

For the integrals over θ and ϕ we have

$$\int_0^\pi \int_0^{2\pi} \exp(ixy\cos\theta) \sin\theta\, d\theta\, d\phi = \frac{4\pi\sin(xy)}{xy}$$

Hence

$$G(\mathbf{x}) = \frac{1}{2\pi^2 x} \int_0^\infty \frac{y\sin(xy)}{k^2 - y^2}\, dy$$

Noting that $y\sin(xy)$ is an even function of y we can change the integral limits as follows.

$$G(\mathbf{x}) = \frac{1}{4\pi^2 x} \int_{-\infty}^\infty \frac{y\sin(xy)}{k^2 - y^2}\, dy$$

Negate the denominator.

$$G(\mathbf{x}) = \frac{1}{4\pi^2 x} \int_{-\infty}^\infty -\frac{y\sin(xy)}{y^2 - k^2}\, dy$$

Change the sine function to exponential form and factor the denominator.

$$G(\mathbf{x}) = \frac{i}{8\pi^2 x} \left[\int_{-\infty}^\infty \frac{y\exp(ixy)}{(y-k)(y+k)}\, dy - \int_{-\infty}^\infty \frac{y\exp(-ixy)}{(y-k)(y+k)}\, dy \right]$$

By Cauchy's integral formula we have

$$\int_{-\infty}^\infty \frac{y\exp(ixy)}{y+k} \frac{1}{y-k}\, dy = i\pi\exp(ikx)$$

and

$$\int_{-\infty}^\infty \frac{y\exp(-ixy)}{y-k} \frac{1}{y+k}\, dy = -i\pi\exp(ikx)$$

Hence

$$\boxed{G(\mathbf{x}) = -\frac{\exp(ikx)}{4\pi x}} \tag{3}$$

where

$$x = |\mathbf{x}|$$

Verify that (3) satisfies (1).

We will need the following formula from Griffiths and Schroeter problem 10.8.

$$\nabla^2(1/r) = -4\pi\delta^3(\mathbf{r}) \tag{4}$$

Recall that $\nabla^2 = \nabla\cdot\nabla$ and for a scalar function f and a vector function $\mathbf{F}$ we have

$$\nabla\cdot(f\mathbf{F}) = \frac{\partial}{\partial x}(fF_x) + \frac{\partial}{\partial y}(fF_y) + \frac{\partial}{\partial z}(fF_z) = \nabla f\cdot\mathbf{F} + f\nabla\cdot\mathbf{F}$$

Substituting $\mathbf{r}$ for $\mathbf{x}$ in (3) we have for the Laplacian of $G(\mathbf{r})$

$$\begin{aligned}
\nabla^2 G(\mathbf{r}) &= -\frac{1}{4\pi}\nabla\cdot\nabla\left(\frac{e^{ikr}}{r}\right) \\
&= -\frac{1}{4\pi}\nabla\cdot\left(\frac{1}{r}\nabla e^{ikr} + e^{ikr}\nabla\frac{1}{r}\right) \\
&= -\frac{1}{4\pi}\Bigg(\underbrace{\nabla\frac{1}{r}\cdot\nabla e^{ikr}}_{\text{subst. (6)}} + \underbrace{\frac{1}{r}\nabla^2 e^{ikr}}_{\text{subst. (7)}} + \underbrace{\nabla e^{ikr}\cdot\nabla\frac{1}{r}}_{\text{subst. (6)}} + \underbrace{e^{ikr}\nabla^2\frac{1}{r}}_{\text{subst. (4)}}\Bigg)
\end{aligned} \tag{5}$$

In spherical coordinates

$$\nabla\frac{1}{r}\cdot\nabla e^{ikr}=\left(-\frac{1}{r^2}\mathbf{e}_r+0\mathbf{e}_\theta+0\mathbf{e}_\phi\right)\cdot\left(ike^{ikr}\mathbf{e}_r+0\mathbf{e}_\theta+0\mathbf{e}_\phi\right)=-\frac{ike^{ikr}}{r^2}\tag{6}$$

and

$$\begin{aligned}\frac{1}{r}\nabla^2 e^{ikr}&=\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(re^{ikr})\\&=\frac{1}{r^2}\frac{\partial}{\partial r}\left(e^{ikr}+ikre^{ikr}\right)\\&=\frac{1}{r^2}\left(2ike^{ikr}-k^2re^{ikr}\right)\\&=\frac{2ike^{ikr}}{r^2}-\frac{k^2e^{ikr}}{r}\end{aligned}\tag{7}$$

Substitute (4), (6), and (7) into (5) to obtain

$$\nabla^2 G(\mathbf{r})=\frac{k^2e^{ikr}}{4\pi r}+\delta^3(\mathbf{r})e^{ikr}$$

Then by equation (3)

$$\nabla^2 G(\mathbf{r})=-k^2G(\mathbf{r})+\delta^3(\mathbf{r})e^{ikr}$$

Noting that $e^{ikr}=1$ at $r=0$, the e^{ikr} term can be discarded leaving

$$\nabla^2 G(\mathbf{r})=-k^2G(\mathbf{r})+\delta^3(\mathbf{r})\tag{8}$$

Hence

$$(\nabla^2+k^2)G(\mathbf{r})=\delta^3(\mathbf{r})$$

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