

Spontaneous emission rate

Find the spontaneous emission rate for hydrogen state $2p \rightarrow 1s$.

The wave function for hydrogen is

$$\psi_{nlm}(r, \theta, \phi) = R_{nl}(r)Y_{lm}(\theta, \phi)$$

where

$$R_{nl}(r) = \frac{2}{n^2} \sqrt{\frac{(n-l-1)!}{(n+l)!}} \left(\frac{2r}{na_0}\right)^l L_{n-l-1}^{2l+1} \left(\frac{2r}{na_0}\right) \exp\left(-\frac{r}{na_0}\right) a_0^{-3/2}$$

$$L_n^m(x) = (n+m)! \sum_{k=0}^n \frac{(-x)^k}{(n-k)!(m+k)!k!}$$

$$Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) \exp(im\phi)$$

$$P_l^m(\cos \theta) = \begin{cases} \left(\frac{\sin \theta}{2}\right)^m \sum_{k=0}^{l-m} (-1)^k \frac{(l+m+k)!}{(l-m-k)!(m+k)!k!} \left(\frac{1-\cos \theta}{2}\right)^k, & m \geq 0 \\ (-1)^m \frac{(l+m)!}{(l-m)!} P_l^{|m|}(\cos \theta), & m < 0 \end{cases}$$

State $2p$ is shorthand for $n = 2$ and $l = 1$. For $l = 1$ there are three ways to choose m hence all of the following processes correspond to the transition $2p \rightarrow 1s$. It turns out that all three processes have the same transition rate.

$$\left. \begin{array}{c} \psi_{2,1,1} \\ \psi_{2,1,0} \\ \psi_{2,1,-1} \end{array} \right\} \rightarrow \psi_{100} + \text{photon}$$

The spontaneous emission rate is

$$A_{21} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3} \omega_{21}^3 |r_{21}|^2$$

Noting that

$$e^2 = 4\pi\varepsilon_0\hbar c\alpha$$

we can also write

$$A_{21} = \frac{4\alpha}{3c^2} \omega_{21}^3 |r_{21}|^2 \quad (1)$$

Verify dimensions:

$$A_{21} \propto (\text{m/s})^{-2} \times \text{s}^{-3} \times \text{m}^2 = \text{s}^{-1} \text{ (or hertz)}$$

For angular frequency ω_{21} we have

$$\omega_{21} = \frac{E_2 - E_1}{\hbar}$$

and for hydrogen

$$E_n = -\frac{\alpha \hbar c}{2n^2 a_0}$$

Hence

$$\omega_{21} = \frac{3\alpha c}{8a_0}$$

For displacement r_{21} we have

$$|r_{21}|^2 = |x_{21}|^2 + |y_{21}|^2 + |z_{21}|^2$$

where

$$x_{21} = \int_0^\infty \int_0^\pi \int_0^{2\pi} x f_{21} dV, \quad y_{21} = \int_0^\infty \int_0^\pi \int_0^{2\pi} y f_{21} dV, \quad z_{21} = \int_0^\infty \int_0^\pi \int_0^{2\pi} z f_{21} dV$$

and

$$f_{21} = \psi_{100}^* \psi_{210}$$

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta$$

$$dV = r^2 \sin \theta dr d\theta d\phi$$

The integrals work out to be

$$x_{21} = 0, \quad y_{21} = 0, \quad z_{21} = \frac{2^7}{3^5} \sqrt{2} a_0$$

hence

$$|r_{21}|^2 = |z_{21}|^2 = \frac{2^{15}}{3^{10}} a_0^2$$

By equation (1) the spontaneous emission rate is

$$A_{21} = \frac{2^8}{3^8} \frac{\alpha^4 c}{a_0} = 6.26 \times 10^8 \text{ s}^{-1}$$

Noting that

$$a_0 = \frac{\hbar}{\alpha \mu c}$$

we can also write

$$A_{21} = \frac{2^8}{3^8} \frac{\alpha^5 \mu c^2}{\hbar} = 6.26 \times 10^8 \text{ s}^{-1}$$

where μ is reduced electron mass

$$\mu = \frac{m_e m_p}{m_e + m_p}$$