

## Spin state

Fermion spin state  $|s\rangle$  is a unit vector in  $\mathbb{C}^2$ .

$$|s\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad c_1^* c_1 + c_2^* c_2 = 1$$

Here is spin state  $|s\rangle$  as a linear combination of basis states “up” and “down.”

$$|s\rangle = c_1|u\rangle + c_2|d\rangle, \quad |u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These are the spin operators.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Expectation of spin operators is a projection of  $|s\rangle$  onto Euclidean space.

$$\langle x \rangle = \langle s | \sigma_x | s \rangle, \quad \langle y \rangle = \langle s | \sigma_y | s \rangle, \quad \langle z \rangle = \langle s | \sigma_z | s \rangle$$

Let  $\mathbf{u}$  be the following spin direction vector determined by  $|s\rangle$ .

$$\mathbf{u} = \begin{pmatrix} \langle x \rangle \\ \langle y \rangle \\ \langle z \rangle \end{pmatrix} = \langle s | \boldsymbol{\sigma} | s \rangle, \quad \boldsymbol{\sigma} = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}$$

Let  $\theta$  and  $\phi$  be polar and azimuth angles such that

$$\mathbf{u} = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix}$$

Then

$$|s\rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \exp(i\phi) \end{pmatrix} = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

where

$$c_+ = \sqrt{\frac{\langle z \rangle + 1}{2}}, \quad c_- = \sqrt{\frac{1 - \langle z \rangle}{2}} \frac{\langle x \rangle + i\langle y \rangle}{\sqrt{\langle x \rangle^2 + \langle y \rangle^2}}$$

In component notation  $\boldsymbol{\sigma} = \sigma^{\alpha\beta}{}_{\gamma}$  hence

$$u^\alpha = s_\beta^* \sigma^{\alpha\beta}{}_{\gamma} s^\gamma$$

A transpose swaps  $\alpha$  and  $\beta$  so that summed-over indices are adjacent.

$$u^\alpha = s_\beta^* \sigma^{\beta\alpha}{}_{\gamma} s^\gamma$$

Hence the Eigenmath code is

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sigma = (sigmax,sigmay,sigmaz)
u = dot(conj(s),transpose(sigma),s)
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