

Matrix mechanics 2

Show that for hydrogen¹

$$L_1 = \begin{matrix} & \psi_{1,0,0} & \psi_{2,1,-1} & \psi_{2,1,0} & \psi_{2,1,1} \\ \begin{matrix} \psi_{1,0,0} \\ \psi_{2,1,-1} \\ \psi_{2,1,0} \\ \psi_{2,1,1} \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{pmatrix} \end{matrix}$$

$$L_2 = \begin{matrix} & \psi_{1,0,0} & \psi_{2,1,-1} & \psi_{2,1,0} & \psi_{2,1,1} \\ \begin{matrix} \psi_{1,0,0} \\ \psi_{2,1,-1} \\ \psi_{2,1,0} \\ \psi_{2,1,1} \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i\hbar}{\sqrt{2}} & 0 \\ 0 & -\frac{i\hbar}{\sqrt{2}} & 0 & \frac{i\hbar}{\sqrt{2}} \\ 0 & 0 & -\frac{i\hbar}{\sqrt{2}} & 0 \end{pmatrix} \end{matrix}$$

$$L_3 = \begin{matrix} & \psi_{1,0,0} & \psi_{2,1,-1} & \psi_{2,1,0} & \psi_{2,1,1} \\ \begin{matrix} \psi_{1,0,0} \\ \psi_{2,1,-1} \\ \psi_{2,1,0} \\ \psi_{2,1,1} \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\hbar & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hbar \end{pmatrix} \end{matrix}$$

and

$$L^2 = \begin{matrix} & \psi_{1,0,0} & \psi_{2,1,-1} & \psi_{2,1,0} & \psi_{2,1,1} \\ \begin{matrix} \psi_{1,0,0} \\ \psi_{2,1,-1} \\ \psi_{2,1,0} \\ \psi_{2,1,1} \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\hbar^2 & 0 & 0 \\ 0 & 0 & 2\hbar^2 & 0 \\ 0 & 0 & 0 & 2\hbar^2 \end{pmatrix} \end{matrix}$$

Verify

$$L^2 = L_1^2 + L_2^2 + L_3^2$$

and

$$L_2 L_3 - L_3 L_2 = i\hbar L_1$$

$$L_3 L_1 - L_1 L_3 = i\hbar L_2$$

$$L_1 L_2 - L_2 L_1 = i\hbar L_3$$

¹See *Quantum Mechanics in Matrix Form* by Günter Ludyk, p. 73.

In spherical coordinates the angular momentum operators are

$$\begin{aligned}\hat{L}_1 &= i\hbar \left(\sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right) \\ \hat{L}_2 &= i\hbar \left(-\cos \phi \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right) \\ \hat{L}_3 &= -i\hbar \frac{\partial}{\partial \phi}\end{aligned}$$

and

$$\hat{L}^2 = -\hbar^2 \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right)$$