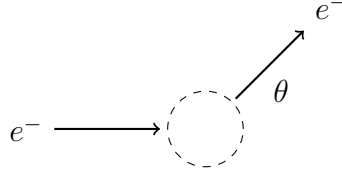
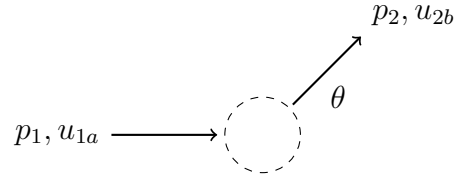


## Rutherford scattering

Consider an electron scattered by an atomic nucleus.



Here is the same diagram with momentum and spinor labels.



The path of the incident electron can be modeled as the  $z$  axis, resulting in the following momentum vectors.

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \quad p_2 = \begin{pmatrix} E \\ p \sin \theta \cos \phi \\ p \sin \theta \sin \phi \\ p \cos \theta \end{pmatrix}$$

inbound electron outbound electron

Symbol  $p$  is incident momentum. Symbol  $E$  is total energy  $E = \sqrt{p^2 + m^2}$  where  $m$  is electron mass. Polar angle  $\theta$  is the observed scattering angle. Azimuth angle  $\phi$  cancels out in scattering calculations.

The spinors are

$$u_{11} = \begin{pmatrix} E + m \\ 0 \\ p \\ 0 \end{pmatrix} \quad u_{21} = \begin{pmatrix} E + m \\ 0 \\ p_{2z} \\ p_{2x} + ip_{2y} \end{pmatrix}$$

inbound electron spin up outbound electron spin up

$$u_{12} = \begin{pmatrix} 0 \\ E + m \\ 0 \\ -p \end{pmatrix} \quad u_{22} = \begin{pmatrix} 0 \\ E + m \\ p_{2x} - ip_{2y} \\ -p_{2z} \end{pmatrix}$$

inbound electron spin down outbound electron spin down

Spinor subscripts have 1 for spin up and 2 for spin down. The spinors are not individually normalized. Instead, a combined spinor normalization constant  $N = (E + m)^2$  will be used.

This is the probability density for spin state  $ab$ . Symbol  $Z$  is the atomic number of the nucleus,  $e$  is electron charge, and  $q = p_1 - p_2$  is momentum transfer. The formula is derived from Feynman diagrams for Rutherford scattering.

$$|\mathcal{M}_{ab}|^2 = \frac{Z^2 e^4}{q^4} \frac{1}{N} |\bar{u}_{2b} \gamma^0 u_{1a}|^2$$

The expected probability density  $\langle |\mathcal{M}|^2 \rangle$  is computed by summing  $|\mathcal{M}_{ab}|^2$  over all spin states and then dividing by the number of inbound states. There are two inbound states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{2} \sum_{a=1}^2 \sum_{b=1}^2 |\mathcal{M}_{ab}|^2$$

The Casimir trick uses matrix arithmetic to compute the sum.

$$\langle |\mathcal{M}|^2 \rangle = \frac{Z^2 e^4}{2q^4} \text{Tr} \left( (\not{p}_1 + m) \gamma^0 (\not{p}_2 + m) \gamma^0 \right)$$

The result is

$$\langle |\mathcal{M}|^2 \rangle = \frac{2Z^2 e^4}{q^4} (E^2 + m^2 + p^2 \cos \theta)$$

## Low energy approximation

For low energy electrons such that  $p \ll m$  we can use the following approximation.

$$E^2 + m^2 + p^2 \cos \theta \approx 2m^2$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{4m^2 Z^2 e^4}{q^4}$$

Substituting  $e^4 = 16\pi^2 \alpha^2$  and  $q^4 = 4p^4 (\cos \theta - 1)^2$  we have

$$\langle |\mathcal{M}|^2 \rangle = \frac{16\pi^2 \alpha^2 m^2 Z^2}{p^4 (\cos \theta - 1)^2}$$

## Cross section

The differential cross section for Rutherford scattering is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{16\pi^2}$$

For the low energy approximation we have

$$\langle |\mathcal{M}|^2 \rangle = \frac{16\pi^2 \alpha^2 m^2 Z^2}{p^4 (\cos \theta - 1)^2}$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 m^2 Z^2}{p^4 (\cos \theta - 1)^2}$$

We can integrate  $d\sigma$  to obtain a cumulative distribution function. Let  $I(\theta)$  be the following integral of  $d\sigma$ . (The  $\sin \theta$  is from  $d\Omega = \sin \theta d\theta d\phi$ .)

$$I(\theta) = \int \frac{1}{(\cos \theta - 1)^2} \sin \theta d\theta$$

The result is

$$I(\theta) = \frac{1}{\cos \theta - 1}$$

The cumulative distribution function is

$$F(\theta) = \frac{I(\theta) - I(a)}{I(\pi) - I(a)} = \frac{2(\cos a - \cos \theta)}{(1 + \cos a)(1 - \cos \theta)}, \quad a \leq \theta \leq \pi$$

Angular support is reduced by an arbitrary angle  $a > 0$  because  $I(0)$  is undefined.

The probability of observing scattering events in the interval  $\theta_1$  to  $\theta_2$  is

$$P(\theta_1 \leq \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

Let  $N$  be the number of scattering events from an experiment. Then the number of scattering events in the interval  $\theta_1$  to  $\theta_2$  is predicted to be

$$N (F(\theta_2) - F(\theta_1))$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi) - I(a)} \frac{1}{(\cos \theta - 1)^2} \sin \theta$$

Note that if we had carried through the  $\alpha^2 m^2 Z^2 / p^4$  in  $I(\theta)$ , it would have cancelled out in  $F(\theta)$ .

## Notes

1. The original Rutherford scattering experiment in 1911 used alpha particles, not electrons. However, scattering of any charged particle by Coulomb interaction is now known as Rutherford scattering. The first Rutherford scattering experiment using electrons appears to have been done by F. L. Arnot, then a student of Rutherford, in 1929.

2. Lancaster and Blundell page 356 has

$$\frac{d\sigma}{d\Omega} = \frac{Z^2 \alpha^2}{4m^2 \mathbf{v}^4 \sin^4(\theta/2)}$$

Noting that

$$\frac{1}{m^2 \mathbf{v}^4} = \frac{m^2}{m^4 \mathbf{v}^4} = \frac{m^2}{p^4}$$

and

$$4 \sin^4(\theta/2) = (\cos \theta - 1)^2$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{Z^2 \alpha^2}{4m^2 \mathbf{v}^4 \sin^4(\theta/2)} = \frac{m^2 Z^2 \alpha^2}{p^4 (\cos \theta - 1)^2}$$