

9-5. The momentum in the field is given by

$$\frac{1}{4\pi c} \int \mathbf{E} \times \mathbf{B} d^3\mathbf{r}$$

In the absence of matter (so $\phi_{\mathbf{k}} = 0$), show that this is

$$i \int \mathbf{k} (\mathbf{a}_{\mathbf{k}}^* \cdot \dot{\mathbf{a}}_{\mathbf{k}}) \frac{d^3\mathbf{k}}{(2\pi)^3}$$

Let

$$D = \exp(i\mathbf{k} \cdot \mathbf{r}) \frac{d^3\mathbf{k}}{(2\pi)^3}$$

By equation (9.14)

$$\mathbf{A} d^3\mathbf{r} = \sqrt{4\pi c} \mathbf{a}_{\mathbf{k}} D \quad (1)$$

By equation (1) and (9.9) with $\phi = 0$

$$\mathbf{E} d^3\mathbf{r} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = -\sqrt{4\pi} \dot{\mathbf{a}}_{\mathbf{k}} D$$

By equation (1) and (9.7)

$$\mathbf{B} d^3\mathbf{r} = \nabla \times \mathbf{A} d^3\mathbf{r} = \sqrt{4\pi ic} \mathbf{k} \times \mathbf{a}_{\mathbf{k}} D$$

Hence

$$\mathbf{E} \times \mathbf{B} = -4\pi ic \int \dot{\mathbf{a}}_{\mathbf{k}} \times (\mathbf{k} \times \mathbf{a}_{\mathbf{k}}) D$$

By the triple product formula

$$\mathbf{E} \times \mathbf{B} = -4\pi ic \int ((\dot{\mathbf{a}}_{\mathbf{k}} \cdot \mathbf{a}_{\mathbf{k}}) \mathbf{k} - (\dot{\mathbf{a}}_{\mathbf{k}} \cdot \mathbf{k}) \mathbf{a}_{\mathbf{k}}) D$$

By orthogonality of \mathbf{E} and \mathbf{k} (see problem 9-1), $\dot{\mathbf{a}}_{\mathbf{k}} \cdot \mathbf{k} = 0$ hence

$$\mathbf{E} \times \mathbf{B} = -4\pi ic \int (\dot{\mathbf{a}}_{\mathbf{k}} \cdot \mathbf{a}_{\mathbf{k}}) \mathbf{k} D$$

Integrate over all space.

$$\int \mathbf{E} \times \mathbf{B} d^3\mathbf{r} = -4\pi ic \int (\dot{\mathbf{a}}_{\mathbf{k}} \cdot \mathbf{a}_{\mathbf{k}}) \mathbf{k} \frac{d^3\mathbf{k}}{(2\pi)^3}$$

Note: Integrating over all \mathbf{r} is the same as integrating over all \mathbf{k} , hence the exponential in D is cancelled.