

In his 1917 paper, “On the Quantum Theory of Radiation,” Einstein uses the following argument to derive Planck’s law. The argument requires induced emission, a process which was a theoretical discovery by Einstein. Prior to Einstein, no one was aware that induced emission existed.

Consider a gas at temperature  $T$ . Let  $N$  be the number of molecules in the gas and let  $N_n$  be the number of molecules with internal energy  $E_n$ . By the Maxwell-Boltzmann distribution we have

$$\frac{N_n}{N} = p_n \exp\left(-\frac{E_n}{kT}\right) \quad (1)$$

where  $k$  is Boltzmann’s constant. The coefficient  $p_n$  is a statistical weighting factor that does not depend on  $T$ .

Let us now consider the processes by which an atom or molecule transitions between energy levels. The processes are absorption, induced emission, and spontaneous emission. Let  $E_m$  and  $E_n$  be energy levels such that  $E_m > E_n$ . Let  $N_{m \rightarrow n}/\Delta t$  be the number of atoms or molecules that transition from energy level  $E_m$  to  $E_n$  in time  $\Delta t$ . Finally, let  $B_{nm}$ ,  $B_{mn}$ , and  $A_{mn}$  be coefficients with dimensions such that

$$\frac{N_{n \rightarrow m}}{\Delta t} = \underbrace{B_{nm}N_n\rho(\nu, T)}_{\text{absorption}}, \quad \frac{N_{m \rightarrow n}}{\Delta t} = \underbrace{B_{mn}N_m\rho(\nu, T)}_{\text{induced emission}} + \underbrace{A_{mn}N_m}_{\text{spontaneous emission}}$$

Absorption and induced emission are proportional to  $\rho(\nu, T)$  which is the radiant energy density of the gas as a function of radiant frequency  $\nu$  and temperature  $T$ . The  $A$  and  $B$  coefficients do not depend on  $T$ .

At equilibrium, the transition rates are equal.

$$\frac{N_{n \rightarrow m}}{\Delta t} = \frac{N_{m \rightarrow n}}{\Delta t}$$

Hence

$$\underbrace{B_{nm}N_n\rho(\nu, T)}_{\text{absorption}} = \underbrace{B_{mn}N_m\rho(\nu, T)}_{\text{induced emission}} + \underbrace{A_{mn}N_m}_{\text{spontaneous emission}}$$

Divide through by  $N$ .

$$\underbrace{B_{nm}\frac{N_n}{N}\rho(\nu, T)}_{\text{absorption}} = \underbrace{B_{mn}\frac{N_m}{N}\rho(\nu, T)}_{\text{induced emission}} + \underbrace{A_{mn}\frac{N_m}{N}}_{\text{spontaneous emission}}$$

Then by the Maxwell-Boltzmann distribution (1) we have

$$\underbrace{B_{nm}p_n\rho(\nu, T)\exp\left(-\frac{E_n}{kT}\right)}_{\text{absorption}} = \underbrace{B_{mn}p_m\rho(\nu, T)\exp\left(-\frac{E_m}{kT}\right)}_{\text{induced emission}} + \underbrace{A_{mn}p_m\exp\left(-\frac{E_m}{kT}\right)}_{\text{spontaneous emission}} \quad (2)$$

Multiply both sides by  $\exp(E_m/kT)$ .

$$\underbrace{B_{nm}p_n\rho(\nu, T)\exp\left(\frac{E_m - E_n}{kT}\right)}_{\text{absorption}} = \underbrace{B_{mn}p_m\rho(\nu, T)}_{\text{induced emission}} + \underbrace{A_{mn}p_m}_{\text{spontaneous emission}}$$

Note that for increasing  $T$  we have

$$\lim_{T \rightarrow \infty} \exp\left(\frac{E_m - E_n}{kT}\right) = 1$$

It follows that for  $T \rightarrow \infty$  the equilibrium formula is

$$B_{nm}p_n\rho(\nu, T) = B_{mn}p_m\rho(\nu, T) + A_{mn}p_m$$

Divide through by  $\rho(\nu, T)$ .

$$B_{nm}p_n = B_{mn}p_m + \frac{A_{mn}p_m}{\rho(\nu, T)}$$

Energy density  $\rho(\nu, T)$  increases with temperature  $T$  hence  $A_{mn}p_m/\rho(\nu, T)$  vanishes for  $T \rightarrow \infty$  leaving

$$B_{nm}p_n = B_{mn}p_m \quad (3)$$

Einstein reasoned that equation (3) is true in general based on the assumption that the factors involved do not depend on  $T$ . By substitution in the absorption term we can now eliminate  $B_{nm}p_n$  and obtain

$$\underbrace{B_{mn}p_m\rho(\nu, T) \exp\left(\frac{E_m - E_n}{kT}\right)}_{\text{absorption}} = \underbrace{B_{mn}p_m\rho(\nu, T)}_{\text{induced emission}} + \underbrace{A_{mn}p_m}_{\text{spontaneous emission}}$$

Divide both sides by  $B_{mn}p_m$ .

$$\underbrace{\rho(\nu, T) \exp\left(\frac{E_m - E_n}{kT}\right)}_{\text{absorption}} = \underbrace{\rho(\nu, T)}_{\text{induced emission}} + \underbrace{\frac{A_{mn}}{B_{mn}}}_{\text{spontaneous emission}}$$

Rearrange terms.

$$\underbrace{\rho(\nu, T) \exp\left(\frac{E_m - E_n}{kT}\right)}_{\text{absorption}} - \underbrace{\rho(\nu, T)}_{\text{induced emission}} = \underbrace{\frac{A_{mn}}{B_{mn}}}_{\text{spontaneous emission}}$$

Factor out  $\rho(\nu, T)$ .

$$\rho(\nu, T) \left( \exp\left(\frac{E_m - E_n}{kT}\right) - 1 \right) = \frac{A_{mn}}{B_{mn}}$$

Solve for  $\rho(\nu, T)$ .

$$\rho(\nu, T) = \frac{A_{mn}}{B_{mn}} \frac{1}{\exp\left(\frac{E_m - E_n}{kT}\right) - 1}$$

We now consider the limit of  $\rho(\nu, T)$  as  $E_m - E_n \rightarrow \infty$ .

$$\lim_{E_m - E_n \rightarrow \infty} \rho(\nu, T) = \frac{A_{mn}}{B_{mn}} \exp\left(-\frac{E_m - E_n}{kT}\right)$$

Then by equivalence with Wien's law (which is accurate for large  $\nu$ )

$$\rho_{\text{wien}}(\nu, T) = \frac{2h\nu^3}{c^2} \exp\left(-\frac{h\nu}{kT}\right)$$

we have

$$\frac{A_{mn}}{B_{mn}} = \frac{2h\nu^3}{c^2} \quad (4)$$

and

$$E_m - E_n = h\nu$$

Then by substitution we obtain Planck's law.

$$\begin{aligned} \rho(\nu, T) &= \frac{A_{mn}}{B_{mn}} \frac{1}{\exp\left(\frac{E_m - E_n}{kT}\right) - 1} \\ &= \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1} \end{aligned}$$

Let us now consider the values of the  $A$  and  $B$  coefficients. The coefficient for spontaneous emission can be computed from quantum mechanics. For example, for hydrogen we have

$$A_{21} = \frac{e^{10} m_e}{26244 \pi^5 \epsilon_0^5 \hbar^6 c^3} = 6.27 \times 10^8 \text{ second}^{-1}$$

The coefficient for induced emission can be obtained from equation (4).

$$B_{mn} = \frac{c^2}{2h\nu^3} A_{mn}$$

The coefficient for absorption can be computed from equation (3).

$$B_{nm} = \frac{p_m}{p_n} B_{mn}$$

The ratio  $p_m/p_n$  is equal to  $g_m/g_n$  where  $g_m$  is the multiplicity associated with energy level  $m$ .

$$g = (2s + 1)(2\ell + 1)$$