White dwarf star

The size of a white dwarf star can be estimated using the electron gas model of a solid.

The total electron energy of a spherical electron gas with radius r is

$$E = \left(\frac{3\pi^2}{2}\right)^{\frac{1}{3}} \frac{9\hbar^2 n^{\frac{5}{3}}}{20 m_e r^2}$$

where n is the number of free electrons and m_e is electron mass.

The gravitational energy of a sphere with mass M and uniform density is

$$U = -\frac{3GM^2}{5r}$$

Minimize the total energy by finding r such that

$$\frac{d}{dr}(E + U) = 0$$

Hence

$$-\left(\frac{3\pi^2}{2}\right)^{\frac{1}{3}} \frac{9\hbar^2 n^{\frac{5}{3}}}{10 m_e r^3} + \frac{3GM^2}{5r^2} = 0$$

Multiply both sides by r^3.

$$-\left(\frac{3\pi^2}{2}\right)^{\frac{1}{3}} \frac{9\hbar^2 n^{\frac{5}{3}}}{10 m_e} + \frac{3GM^2}{5} r = 0$$

Hence

$$r = \left(\frac{3\pi^2}{2}\right)^{\frac{1}{3}} \frac{9\hbar^2 n^{\frac{5}{3}}}{10 m_e} \frac{5}{3GM^2} = \left(\frac{3\pi^2}{2}\right)^{\frac{1}{3}} \frac{3\hbar^2 n^{\frac{5}{3}}}{2 m_e G M^2} \tag{1}$$

The number of free electrons is estimated to be one-half the number of nucleons. For one solar mass we have

$$n = \frac{M_\odot}{2m_p} = 6 \times 10^{56}$$

For $M = M_\odot$ the radius is

$$r = 7146\,\mathrm{km}$$