8-3. Show that Q_α^c, Q_α^s are normal coordinates corresponding to standing wave normal modes $\cos(2\pi\alpha j/N)$ and $\sin(2\pi\alpha j/N)$, in the sense that (for N odd)

$$q_j(t) = \sqrt{\frac{2}{N}} \left(\frac{1}{2} Q_0^c(t) + \sum_{\alpha=1}^{(N-1)/2} \left(Q_\alpha^c(t) \cos \frac{2\pi\alpha j}{N} + Q_\alpha^s(t) \sin \frac{2\pi\alpha j}{N} \right) \right)$$

$$(8.82)$$

Consider the following equations.

$$Q_\alpha(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} q_k(t) \left(\cos \frac{2\pi\alpha k}{N} - i \sin \frac{2\pi\alpha k}{N} \right) \tag{8.77}$$

$$Q_\alpha^c = \frac{1}{\sqrt{2}} (Q_\alpha + Q_\alpha^*) \tag{8.79}$$

$$Q_\alpha^s = \frac{i}{\sqrt{2}} (Q_\alpha - Q_\alpha^*) \tag{8.80}$$

Substitute (8.77) into (8.82).

$$q_j = \frac{1}{\sqrt{2N}} Q_0^c + \frac{1}{N} \sum_{\alpha=1}^{(N-1)/2} \sum_{k=1}^{N} q_k (T_1 + T_2 + T_3 + T_4) \tag{1}$$

where

$$T_1 = \cos \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N} - i \sin \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N}$$

$$T_2 = \cos \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N} + i \sin \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N}$$

$$T_3 = i \cos \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N} + \sin \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N}$$

$$T_4 = -i \cos \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N} + \sin \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N}$$

It follows that

$$T_1 + T_2 + T_3 + T_4 = 2 \cos \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N} + 2 \sin \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N}$$

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By trigonometric identities

$$T_1 + T_2 + T_3 + T_4 = 2\cos\left(\frac{2\pi\alpha}{N}(j-k)\right) \tag{2}$$

Substitute (2) into (1) to obtain

$$q_j = \frac{1}{\sqrt{2N}}Q_0^c + \frac{2}{N}\sum_{\alpha=1}^{(N-1)/2}\sum_{k=1}^{N} q_k \cos\left(\frac{2\pi\alpha}{N}(j-k)\right) \tag{3}$$

By equations (8.77) and (8.79) with $\alpha = 0$ we have

$$Q_0^c = \sqrt{\frac{2}{N}}\sum_{k=1}^{N} q_k \tag{4}$$

Substitute (4) into (3).

$$q_j = \frac{1}{N}\sum_{k=1}^{N} q_k + \frac{2}{N}\sum_{\alpha=1}^{(N-1)/2}\sum_{k=1}^{N} q_k \cos\left(\frac{2\pi\alpha}{N}(j-k)\right)$$

Rewrite as

$$q_j = \sum_{k=1}^{N} q_k \left(\frac{1}{N} + \frac{2}{N}\sum_{\alpha=1}^{(N-1)/2}\cos\left(\frac{2\pi\alpha}{N}(j-k)\right)\right) \tag{5}$$

For the sum over α in (5) we have

$$\sum_{\alpha=1}^{(N-1)/2}\cos\left(\frac{2\pi\alpha}{N}(j-k)\right) = \begin{cases}(N-1)/2 & j = k \\ -1/2 & j \neq k\end{cases}$$

Hence for $j = k$

$$\frac{1}{N} + \frac{2}{N}\sum_{\alpha=1}^{(N-1)/2}\cos\left(\frac{2\pi\alpha}{N}(j-k)\right) = 1$$

and for $j \neq k$

$$\frac{1}{N} + \frac{2}{N}\sum_{\alpha=1}^{(N-1)/2}\cos\left(\frac{2\pi\alpha}{N}(j-k)\right) = 0$$

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It follows that (5) reduces to the tautology

$$q_j = \sum_{k=1}^{N} q_k \delta(j - k) = q_j$$

Hence (8.82) is proven to be correct.