

# Feynman and Hibbs problem 4-2

For a particle of charge  $e$  in a magnetic field the Lagrangian is

$$L(\dot{\mathbf{x}}, \mathbf{x}) = \frac{m}{2} \dot{\mathbf{x}}^2 + \frac{e}{c} \dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}, t) - e\phi(\mathbf{x}, t)$$

where  $\dot{\mathbf{x}}$  is the velocity vector,  $c$  is the velocity of light, and  $\mathbf{A}$  and  $\phi$  are the vector and scalar potentials. Show that the corresponding Schrodinger equation is

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} \left( \frac{1}{2m} \left( \frac{\hbar}{i} \nabla - \frac{e}{c} \mathbf{A} \right) \cdot \left( \frac{\hbar}{i} \nabla - \frac{e}{c} \mathbf{A} \right) \psi + e\phi \psi \right)$$

From equation (4.3)

$$\psi(\mathbf{x}, t + \epsilon) = \frac{1}{A} \int_{\mathbb{R}^3} \exp \left( \frac{i\epsilon}{\hbar} L \left( \frac{\mathbf{x} - \mathbf{y}}{\epsilon}, \frac{\mathbf{x} + \mathbf{y}}{2} \right) \right) \psi(\mathbf{y}, t) dy_1 dy_2 dy_3 \quad (1)$$

This is the Lagrangian with arguments from (1).

$$\begin{aligned} L \left( \frac{\mathbf{x} - \mathbf{y}}{\epsilon}, \frac{\mathbf{x} + \mathbf{y}}{2} \right) \\ = \frac{m}{2\epsilon^2} (\mathbf{x} - \mathbf{y})^2 + \frac{e}{c\epsilon} (\mathbf{x} - \mathbf{y}) \cdot \mathbf{A} \left( \frac{\mathbf{x} + \mathbf{y}}{2}, t \right) - e\phi \left( \frac{\mathbf{x} + \mathbf{y}}{2}, t \right) \end{aligned}$$

Hence

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) &= \frac{1}{A} \int_{\mathbb{R}^3} \\ &\exp \left( \frac{im}{2\hbar\epsilon} (\mathbf{x} - \mathbf{y})^2 + \frac{ie}{\hbar c} (\mathbf{x} - \mathbf{y}) \cdot \mathbf{A} \left( \frac{\mathbf{x} + \mathbf{y}}{2}, t \right) - \frac{ie\epsilon}{\hbar} \phi \left( \frac{\mathbf{x} + \mathbf{y}}{2}, t \right) \right) \\ &\times \psi(\mathbf{y}, t) dy_1 dy_2 dy_3 \end{aligned}$$

Let

$$\mathbf{y} = \mathbf{x} + \boldsymbol{\eta}$$

Then

$$\mathbf{x} - \mathbf{y} = -\boldsymbol{\eta}, \quad \frac{\mathbf{x} + \mathbf{y}}{2} = \mathbf{x} + \frac{1}{2}\boldsymbol{\eta}, \quad dy_1 dy_2 dy_3 = d\eta_1 d\eta_2 d\eta_3$$

Hence

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) = \frac{1}{A} \int_{\mathbb{R}^3} \exp \left( \frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 + \frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A} \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) - \frac{ie\epsilon}{\hbar} \phi \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \\ \times \psi(\mathbf{x} + \boldsymbol{\eta}, t) d\eta_1 d\eta_2 d\eta_3 \end{aligned}$$

Factor the exponential.

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) = \frac{1}{A} \int_{\mathbb{R}^3} \exp \left( \frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 \right) \exp \left( \frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A} \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \exp \left( -\frac{ie\epsilon}{\hbar} \phi \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \\ \times \psi(\mathbf{x} + \boldsymbol{\eta}, t) d\eta_1 d\eta_2 d\eta_3 \end{aligned} \quad (2)$$

From the identity  $\exp(i\theta) = \cos(\theta) + i \sin(\theta)$  we have

$$\begin{aligned} \exp \left( -\frac{ie\epsilon}{\hbar} \phi \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \\ = \cos \left( -\frac{e\epsilon}{\hbar} \phi \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) + i \sin \left( -\frac{e\epsilon}{\hbar} \phi \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \end{aligned}$$

Then for small  $\epsilon$

$$\exp \left( -\frac{ie\epsilon}{\hbar} \phi \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \approx 1 - \frac{ie\epsilon}{\hbar} \phi \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right)$$

The authors write that the  $\boldsymbol{\eta}$  term can be dropped “because the error is of higher order than  $\epsilon$ .” Hence

$$\exp \left( -\frac{ie\epsilon}{\hbar} \phi \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \approx 1 - \frac{ie\epsilon}{\hbar} \phi(\mathbf{x}, t) \quad (3)$$

Substitute (3) into (2).

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) = \frac{1}{A} \int_{\mathbb{R}^3} \exp \left( \frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 \right) \exp \left( \frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A} \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \left( 1 - \frac{ie\epsilon}{\hbar} \phi(\mathbf{x}, t) \right) \\ \times \psi(\mathbf{x} + \boldsymbol{\eta}, t) d\eta_1 d\eta_2 d\eta_3 \end{aligned}$$

Factor out the term involving  $\phi$ .

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) &= \frac{1}{A} \left( 1 - \frac{ie\epsilon}{\hbar} \phi(\mathbf{x}, t) \right) \int_{\mathbb{R}^3} \\ &\quad \exp \left( \frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 \right) \exp \left( \frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A} \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \psi(\mathbf{x} + \boldsymbol{\eta}, t) d\eta_1 d\eta_2 d\eta_3 \end{aligned}$$

Let

$$T = \frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{x}, t)$$

and use the approximation

$$\exp \left( \frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A} \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \approx 1 + T + \frac{1}{2} T^2$$

Hence

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) &= \frac{1}{A} \left( 1 - \frac{ie\epsilon}{\hbar} \phi(\mathbf{x}, t) \right) \\ &\quad \times \int_{\mathbb{R}^3} \exp \left( \frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 \right) (1 + T + \frac{1}{2} T^2) \psi(\mathbf{x} + \boldsymbol{\eta}, t) d\eta_1 d\eta_2 d\eta_3 \quad (4) \end{aligned}$$

Next we will use the following Taylor series approximations.

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) &\approx \psi(\mathbf{x}, t) + \epsilon \frac{\partial \psi}{\partial t} \\ \psi(\mathbf{x} + \boldsymbol{\eta}, t) &\approx \psi(\mathbf{x}, t) + \boldsymbol{\eta} \cdot \nabla \psi + \frac{1}{2} \boldsymbol{\eta} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \psi) \end{aligned} \quad (5)$$

Note: In component notation

$$\boldsymbol{\eta} \cdot \nabla \psi = \eta_1 \frac{\partial \psi}{\partial x_1} + \eta_2 \frac{\partial \psi}{\partial x_2} + \eta_3 \frac{\partial \psi}{\partial x_3}$$

and

$$\begin{aligned} \boldsymbol{\eta} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \psi) &= \eta_1^2 \frac{\partial^2 \psi}{\partial x_1^2} + \eta_2^2 \frac{\partial^2 \psi}{\partial x_2^2} + \eta_3^2 \frac{\partial^2 \psi}{\partial x_3^2} \\ &\quad + 2\eta_1 \eta_2 \frac{\partial^2 \psi}{\partial x_1 \partial x_2} + 2\eta_1 \eta_3 \frac{\partial^2 \psi}{\partial x_1 \partial x_3} + 2\eta_2 \eta_3 \frac{\partial^2 \psi}{\partial x_2 \partial x_3} \end{aligned}$$

Substitute the approximations (5) into (4).

$$\begin{aligned} \psi(\mathbf{x}, t) + \epsilon \frac{\partial \psi}{\partial t} &= \frac{1}{A} \left( 1 - \frac{ie\epsilon}{\hbar} \phi(\mathbf{x}, t) \right) \int_{\mathbb{R}^3} \exp \left( \frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 \right) \\ &\times \left( 1 + T + \frac{1}{2} T^2 \right) \left( \psi(\mathbf{x}, t) + \boldsymbol{\eta} \cdot \nabla \psi + \frac{1}{2} \boldsymbol{\eta} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \psi) \right) d\eta_1 d\eta_2 d\eta_3 \quad (6) \end{aligned}$$

To solve the above integral, we will use the following formulas provided by the authors.

$$I_k = \int_{-\infty}^{\infty} \exp \left( \frac{im\eta_k^2}{2\hbar\epsilon} \right) d\eta_k = \left( \frac{2\pi i\hbar\epsilon}{m} \right)^{1/2} \quad \text{from F\&H (4.7)}$$

$$J_k = \int_{-\infty}^{\infty} \eta_k \exp \left( \frac{im\eta_k^2}{2\hbar\epsilon} \right) d\eta_k = 0 \quad \text{from F\&H (4.9)}$$

$$K_k = \int_{-\infty}^{\infty} \eta_k^2 \exp \left( \frac{im\eta_k^2}{2\hbar\epsilon} \right) d\eta_k = \frac{i\hbar\epsilon}{m} \left( \frac{2\pi i\hbar\epsilon}{m} \right)^{1/2} \quad \text{from F\&H (4.10)}$$

Hence

$$\begin{aligned} &\int_{\mathbb{R}^3} \exp \left( \frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 \right) \psi(\mathbf{x}, t) d\eta_1 d\eta_2 d\eta_3 \\ &= I_1 I_2 I_3 \psi(\mathbf{x}, t) \\ &= \left( \frac{2\pi i\hbar\epsilon}{m} \right)^{3/2} \psi(\mathbf{x}, t) \quad (7) \end{aligned}$$

$$\begin{aligned} &\int_{\mathbb{R}^3} \exp \left( \frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 \right) \boldsymbol{\eta} \cdot \nabla \psi(\mathbf{x}, t) d\eta_1 d\eta_2 d\eta_3 \\ &= J_1 I_2 I_3 \frac{\partial \psi}{\partial x_1} + I_1 J_2 I_3 \frac{\partial \psi}{\partial x_2} + I_1 I_2 J_3 \frac{\partial \psi}{\partial x_3} \\ &= 0 \quad (8) \end{aligned}$$

$$\begin{aligned}
& \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \frac{1}{2}\boldsymbol{\eta} \cdot \nabla(\boldsymbol{\eta} \cdot \nabla\psi) d\eta_1 d\eta_2 d\eta_3 \\
&= \frac{1}{2}K_1I_2I_3\frac{\partial^2\psi}{\partial x_1^2} + \frac{1}{2}I_1K_2I_3\frac{\partial^2\psi}{\partial x_2^2} + \frac{1}{2}I_1I_2K_3\frac{\partial^2\psi}{\partial x_3^2} \\
&+ J_1J_2I_1\frac{\partial^2\psi}{\partial x_1x_2} + J_1I_2J_3\frac{\partial^2\psi}{\partial x_1x_3} + I_1J_2J_3\frac{\partial^2\psi}{\partial x_2x_3} \\
&= \frac{i\hbar\epsilon}{2m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\nabla^2\psi
\end{aligned} \tag{9}$$

$$\begin{aligned}
& \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \left(\frac{ie}{\hbar c}\boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{x}, t)\right) \psi(\mathbf{x}, t) d\eta_1 d\eta_2 d\eta_3 \\
&= J_1I_2I_3\frac{ie}{\hbar c}A_1\psi + I_1J_2I_3\frac{ie}{\hbar c}A_2\psi + I_1I_2J_3\frac{ie}{\hbar c}A_3\psi \\
&= 0
\end{aligned} \tag{10}$$

$$\begin{aligned}
& \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \left(\frac{ie}{\hbar c}\boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{x}, t)\right) (\boldsymbol{\eta} \cdot \nabla\psi) d\eta_1 d\eta_2 d\eta_3 \\
&= \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \\
&\times \frac{ie}{\hbar c}(\eta_1A_1 + \eta_2A_2 + \eta_3A_3) \left(\eta_1\frac{\partial\psi}{\partial x_1} + \eta_2\frac{\partial\psi}{\partial x_2} + \eta_3\frac{\partial\psi}{\partial x_3}\right) d\eta_1 d\eta_2 d\eta_3 \\
&= K_1I_2I_3\frac{ieA_1}{\hbar c}\frac{\partial\psi}{\partial x_1} + J_1J_2I_3\frac{ieA_1}{\hbar c}\frac{\partial\psi}{\partial x_2} + J_1I_2J_3\frac{ieA_1}{\hbar c}\frac{\partial\psi}{\partial x_3} \\
&+ J_1J_2I_3\frac{ieA_2}{\hbar c}\frac{\partial\psi}{\partial x_1} + I_1K_2I_3\frac{ieA_2}{\hbar c}\frac{\partial\psi}{\partial x_2} + I_1J_2J_3\frac{ieA_2}{\hbar c}\frac{\partial\psi}{\partial x_3} \\
&+ J_1I_2J_3\frac{ieA_3}{\hbar c}\frac{\partial\psi}{\partial x_1} + I_1J_2J_3\frac{ieA_3}{\hbar c}\frac{\partial\psi}{\partial x_2} + I_1I_2K_3\frac{ieA_3}{\hbar c}\frac{\partial\psi}{\partial x_3} \\
&= \frac{i\hbar\epsilon}{m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \frac{ie}{\hbar c} \left(A_1\frac{\partial\psi}{\partial x_1} + A_2\frac{\partial\psi}{\partial x_2} + A_3\frac{\partial\psi}{\partial x_3}\right) \\
&= \frac{i\hbar\epsilon}{m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \frac{ie}{\hbar c} \mathbf{A} \cdot \nabla\psi
\end{aligned} \tag{11}$$

$$\begin{aligned}
& \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \left(\frac{ie}{\hbar c}\boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{x}, t)\right) \left(\frac{1}{2}\boldsymbol{\eta} \cdot (\boldsymbol{\eta} \cdot \nabla\psi)\right) d\eta_1 d\eta_2 d\eta_3 \\
&= \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \frac{ie}{\hbar c}(\eta_1 A_1 + \eta_2 A_2 + \eta_3 A_3) \\
&\times \left(\frac{1}{2}\eta_1^2 \frac{\partial^2 \psi}{\partial x_1^2} + \frac{1}{2}\eta_2^2 \frac{\partial^2 \psi}{\partial x_2^2} + \frac{1}{2}\eta_3^2 \frac{\partial^2 \psi}{\partial x_3^2} + \eta_1 \eta_2 \frac{\partial^2 \psi}{\partial x_1 \partial x_2} + \eta_1 \eta_3 \frac{\partial^2 \psi}{\partial x_1 \partial x_3} + \eta_2 \eta_3 \frac{\partial^2 \psi}{\partial x_2 \partial x_3}\right) \\
&= 0
\end{aligned} \tag{12}$$

$$\begin{aligned}
& \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \frac{1}{2} \left(\frac{ie}{\hbar c}\boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{x}, t)\right)^2 \psi(\mathbf{x}, t) d\eta_1 d\eta_2 d\eta_3 \\
&= \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \left(-\frac{e^2}{2\hbar^2 c^2}\right) (\eta_1 A_1 + \eta_2 A_2 + \eta_3 A_3)^2 \psi(\mathbf{x}, t) d\eta_1 d\eta_2 d\eta_3 \\
&= \frac{i\hbar\epsilon}{m} \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \left(-\frac{e^2}{2\hbar^2 c^2}\right) (A_1^2 + A_2^2 + A_3^2) \psi(\mathbf{x}, t) \\
&= \frac{i\hbar\epsilon}{m} \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \left(-\frac{e^2}{2\hbar^2 c^2}\right) \mathbf{A}^2 \psi
\end{aligned} \tag{13}$$

$$\begin{aligned}
& \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \frac{1}{2} \left(\frac{ie}{\hbar c}\boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{x}, t)\right)^2 (\boldsymbol{\eta} \cdot \nabla\psi) d\eta_1 d\eta_2 d\eta_3 \\
&= \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \left(-\frac{e^2}{2\hbar^2 c^2}\right) \\
&\times (\eta_1 A_1 + \eta_2 A_2 + \eta_3 A_3)^2 \left(\eta_1 \frac{\partial \psi}{\partial x_1} + \eta_2 \frac{\partial \psi}{\partial x_2} + \eta_3 \frac{\partial \psi}{\partial x_3}\right) d\eta_1 d\eta_2 d\eta_3 \\
&= 0
\end{aligned} \tag{14}$$

$$\begin{aligned}
& \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \frac{1}{2} \left(\frac{ie}{\hbar c}\boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{x}, t)\right)^2 \left(\frac{1}{2}\boldsymbol{\eta} \cdot (\boldsymbol{\eta} \cdot \nabla\psi)\right) d\eta_1 d\eta_2 d\eta_3 \\
&= \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \left(-\frac{e^2}{2\hbar^2 c^2}\right) (\eta_1 A_1 + \eta_2 A_2 + \eta_3 A_3)^2 \\
&\times \left(\frac{1}{2}\eta_1^2 \frac{\partial^2 \psi}{\partial x_1^2} + \frac{1}{2}\eta_2^2 \frac{\partial^2 \psi}{\partial x_2^2} + \frac{1}{2}\eta_3^2 \frac{\partial^2 \psi}{\partial x_3^2} + \eta_1 \eta_2 \frac{\partial^2 \psi}{\partial x_1 \partial x_2} + \eta_1 \eta_3 \frac{\partial^2 \psi}{\partial x_1 \partial x_3} + \eta_2 \eta_3 \frac{\partial^2 \psi}{\partial x_2 \partial x_3}\right) \\
&= \frac{e^2 \epsilon^2}{2m^2 c^2} \left(\frac{2\pi i \hbar \epsilon}{m}\right)^2 \\
&\times \left(\frac{1}{2}\mathbf{A}^2 \nabla^2 \psi + \mathbf{A} \nabla \nabla \psi \mathbf{A} - \frac{3}{2} \left(A_1^2 \frac{\partial^2}{\partial x_1^2} + A_2^2 \frac{\partial^2}{\partial x_2^2} + A_3^2 \frac{\partial^2}{\partial x_3^2}\right)\right) \quad (15)
\end{aligned}$$

Substitute the solved integrals into (6) to obtain

$$\psi(\mathbf{x}, t) + \epsilon \frac{\partial \psi}{\partial t} = \frac{1}{A} \left(1 - \frac{ie\epsilon}{\hbar} \phi(\mathbf{x}, t)\right) I$$

where

$$\begin{aligned}
I &= \left(\frac{2\pi i \hbar \epsilon}{m}\right)^{3/2} \psi(\mathbf{x}, t) && \text{from (7)} \\
&+ \frac{i\hbar\epsilon}{2m} \left(\frac{2\pi i \hbar \epsilon}{m}\right)^{3/2} \nabla^2 \psi && \text{from (9)} \\
&+ \frac{i\hbar\epsilon}{m} \left(\frac{2\pi i \hbar \epsilon}{m}\right)^{3/2} \frac{ie}{\hbar c} \mathbf{A} \nabla \psi && \text{from (11)} \\
&+ \frac{i\hbar\epsilon}{m} \left(\frac{2\pi i \hbar \epsilon}{m}\right)^{3/2} \left(-\frac{e^2}{2\hbar^2 c^2}\right) \mathbf{A}^2 \psi && \text{from (13)} \\
&+ 0 && \text{from (15)}
\end{aligned}$$

The result from (15) is discarded because it is proportional to  $\epsilon^2$ .

Simplify  $I$  as follows.

$$I = \left(\frac{2\pi i \hbar \epsilon}{m}\right)^{3/2} \left(\psi(\mathbf{x}, t) + \frac{i\hbar\epsilon}{2m} \nabla^2 \psi - \frac{e\epsilon}{mc} \mathbf{A} \nabla \psi - \frac{ie^2 \epsilon}{2m \hbar c^2} \mathbf{A}^2 \psi\right) \quad (16)$$

In the limit as  $\epsilon \rightarrow 0$  we have

$$\psi(\mathbf{x}, t) = \frac{1}{A} \left( \frac{2\pi i \hbar \epsilon}{m} \right)^{3/2} \psi(\mathbf{x}, t)$$

hence

$$A = \left( \frac{2\pi i \hbar \epsilon}{m} \right)^{3/2}$$

Cancel  $A$  with the coefficient in  $I$  to obtain

$$\begin{aligned} \psi(\mathbf{x}, t) + \epsilon \frac{\partial \psi}{\partial t} &= \left( 1 - \frac{ie\epsilon}{\hbar} \phi(\mathbf{x}, t) \right) \\ &\times \left( \psi(\mathbf{x}, t) + \frac{i\hbar\epsilon}{2m} \nabla^2 \psi - \frac{e\epsilon}{mc} \mathbf{A} \nabla \psi - \frac{ie^2\epsilon}{2m\hbar c^2} \mathbf{A}^2 \psi \right) \end{aligned}$$

Expand the product and discard terms of order greater than  $\epsilon$ .

$$\begin{aligned} \psi(\mathbf{x}, t) + \epsilon \frac{\partial \psi}{\partial t} &= \psi(\mathbf{x}, t) + \frac{i\hbar\epsilon}{2m} \nabla^2 \psi - \frac{e\epsilon}{mc} \mathbf{A} \nabla \psi - \frac{ie^2\epsilon}{2m\hbar c^2} \mathbf{A}^2 \psi - \frac{ie\epsilon}{\hbar} \phi(\mathbf{x}, t) \psi \\ &= \psi(\mathbf{x}, t) + \frac{i\hbar\epsilon}{2m} \nabla^2 \psi - \frac{e\epsilon}{mc} \mathbf{A} \nabla \psi - \frac{ie^2\epsilon}{2m\hbar c^2} \mathbf{A}^2 \psi - \frac{ie\epsilon}{\hbar} \phi(\mathbf{x}, t) \psi \end{aligned}$$

Cancel leading terms  $\psi(\mathbf{x}, t)$  and divide through by  $\epsilon$ .

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m} \nabla^2 \psi - \frac{e}{mc} \mathbf{A} \nabla \psi - \frac{ie^2}{2m\hbar c^2} \mathbf{A}^2 \psi - \frac{ie}{\hbar} \phi(\mathbf{x}, t) \psi \quad (17)$$