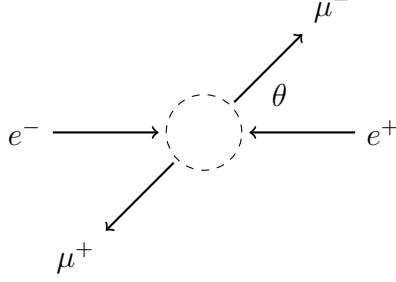
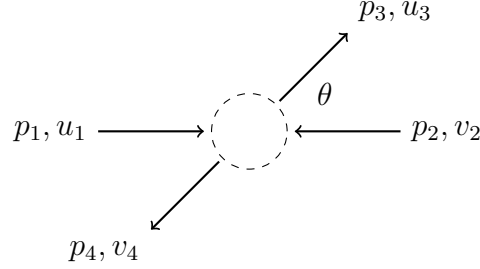


Muon pair production

A high energy electron and positron collision can create two muons.



Here is the same diagram with momentum and spinor labels.



In a typical collider experiment the momentum vectors are

$$\begin{aligned}
 p_1 &= \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} & p_2 &= \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} & p_3 &= \begin{pmatrix} E \\ \rho \sin \theta \cos \phi \\ \rho \sin \theta \sin \phi \\ \rho \cos \theta \end{pmatrix} & p_4 &= \begin{pmatrix} E \\ -\rho \sin \theta \cos \phi \\ -\rho \sin \theta \sin \phi \\ -\rho \cos \theta \end{pmatrix} \\
 &\text{inbound electron} & &\text{inbound positron} & &\text{outbound muon} & &\text{outbound anti-muon}
 \end{aligned}$$

Symbol E is beam energy, $p = \sqrt{E^2 - m^2}$, $\rho = \sqrt{E^2 - M^2}$, m is electron mass 0.51 MeV, and M is muon mass 106 MeV.

The spinors are

$$\begin{aligned}
 u_{11} &= \begin{pmatrix} E + m \\ 0 \\ p \\ 0 \end{pmatrix} & v_{21} &= \begin{pmatrix} -p \\ 0 \\ E + m \\ 0 \end{pmatrix} & u_{31} &= \begin{pmatrix} E + M \\ 0 \\ p_3^z \\ p_3^x + ip_3^y \end{pmatrix} & v_{41} &= \begin{pmatrix} p_4^z \\ p_4^x + ip_4^y \\ E + M \\ 0 \end{pmatrix} \\
 &\text{inbound electron} & &\text{inbound positron} & &\text{outbound muon} & &\text{outbound anti-muon} \\
 &\text{spin up} & &\text{spin up} & &\text{spin up} & &\text{spin up} \\
 u_{12} &= \begin{pmatrix} 0 \\ E + m \\ 0 \\ -p \end{pmatrix} & v_{22} &= \begin{pmatrix} 0 \\ p \\ E + m \\ 0 \end{pmatrix} & u_{32} &= \begin{pmatrix} 0 \\ E + M \\ p_3^x - ip_3^y \\ -p_3^z \end{pmatrix} & v_{42} &= \begin{pmatrix} p_4^x - ip_4^y \\ -p_4^z \\ 0 \\ E + M \end{pmatrix} \\
 &\text{inbound electron} & &\text{inbound positron} & &\text{outbound muon} & &\text{outbound anti-muon} \\
 &\text{spin down} & &\text{spin down} & &\text{spin down} & &\text{spin down}
 \end{aligned}$$

Spinor subscripts have 1 for spin up and 2 for spin down. The spinors are not individually normalized. Instead, a combined spinor normalization constant $N = (E + m)^2(E + M)^2$ will be used.

This is the probability density for spin state $abcd$. Symbol e is electron charge and $s = (p_1 + p_2)^2 = 4E^2$. The formula is derived from Feynman diagrams for muon pair production.

$$|\mathcal{M}_{abcd}|^2 = \frac{e^4}{N s^2} |(\bar{u}_{3c} \gamma_\mu v_{4d})(\bar{v}_{2b} \gamma^\mu u_{1a})|^2$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is computed by summing $|\mathcal{M}_{abcd}|^2$ over all spin states and dividing by the number of inbound states. There are four inbound states.

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{1}{4} \sum_{a=1}^2 \sum_{b=1}^2 \sum_{c=1}^2 \sum_{d=1}^2 |\mathcal{M}_{abcd}|^2 \\ &= \frac{e^4}{4N s^2} \sum_{a=1}^2 \sum_{b=1}^2 \sum_{c=1}^2 \sum_{d=1}^2 |(\bar{u}_{3c} \gamma_\mu v_{4d})(\bar{v}_{2b} \gamma^\mu u_{1a})|^2 \end{aligned}$$

The Casimir trick uses matrix arithmetic to compute sums.

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4s^2} \text{Tr} \left((\not{p}_3 + M) \gamma^\mu (\not{p}_4 - M) \gamma^\nu \right) \text{Tr} \left((\not{p}_2 - m) \gamma_\mu (\not{p}_1 + m) \gamma_\nu \right)$$

The following formula is equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4s^2} \left(32(p_1 \cdot p_3)(p_2 \cdot p_4) + 32(p_1 \cdot p_4)(p_2 \cdot p_3) \right. \\ &\quad \left. + 32m^2(p_3 \cdot p_4) + 32M^2(p_1 \cdot p_2) + 64m^2M^2 \right) \end{aligned}$$

For the momentum vectors given above the result is

$$\langle |\mathcal{M}|^2 \rangle = e^4 \left(1 + \cos^2 \theta + \frac{m^2 + M^2}{E^2} \sin^2 \theta + \frac{m^2 M^2}{E^4} \cos^2 \theta \right)$$

Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{4(4\pi\epsilon_0)^2 s}$$

For high energy experiments $E \gg M$ a useful approximation is

$$\langle |\mathcal{M}|^2 \rangle = e^4 (1 + \cos^2 \theta) \quad \text{and} \quad s = 4E^2$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{4(4\pi\epsilon_0)^2 s} (1 + \cos^2 \theta)$$

Noting that

$$e^2 = 4\pi\epsilon_0\alpha\hbar c$$

we can also write

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{4s} (1 + \cos^2 \theta)$$

We can integrate $d\sigma$ to obtain a cumulative distribution function. Recall that

$$d\Omega = \sin \theta d\theta d\phi$$

Hence

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{4s} (1 + \cos^2 \theta) \sin \theta d\theta d\phi$$

Let $I(\theta)$ be the following integral of $d\sigma$.

$$I(\theta) = \int (1 + \cos^2 \theta) \sin \theta d\theta$$

The result is

$$I(\theta) = -\frac{\cos^3 \theta}{3} - \cos \theta$$

The cumulative distribution function is

$$F(\theta) = \frac{I(\theta) - I(0)}{I(\pi) - I(0)} = -\frac{\cos^3 \theta}{8} - \frac{3 \cos \theta}{8} + \frac{1}{2}, \quad 0 \leq \theta \leq \pi$$

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 \leq \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

Data from SLAC PEP experiment

See www.hepdata.net/record/ins216031, Table 1, 29.0 GeV.

x	y
-0.925	67.08
-0.85	58.67
-0.75	54.66
-0.65	51.72
-0.55	43.70
-0.45	41.12
-0.35	39.71
-0.25	35.34
-0.15	33.35
-0.05	34.69
0.05	34.05
0.15	34.48
0.25	34.66
0.35	35.23
0.45	35.60
0.55	40.13
0.65	42.56
0.75	46.37
0.85	49.28
0.925	55.70

Data x and y have the following relationship with cross section parameters.

$$x = \cos \theta, \quad y = (2E)^2 \frac{d\sigma}{d\cos \theta} = 2\pi(2E)^2 \frac{d\sigma}{d\Omega}$$

The differential cross section for muon production is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2} (1 + \cos^2 \theta) \times (\hbar c)^2$$

Let us compute predicted values \hat{y} from the cross section formula. We have

$$\begin{aligned} y &= 2\pi(2E)^2 \frac{d\sigma}{d\Omega} \\ &= 2\pi(2E)^2 \frac{\alpha^2}{16E^2} (1 + \cos^2 \theta) \times (\hbar c)^2 \\ &= \frac{\pi\alpha^2}{2} (1 + \cos^2 \theta) \times (\hbar c)^2 \end{aligned}$$

Multiply by 10^{37} to convert square meters to nanobarns.

$$y = \frac{\pi\alpha^2}{2} (1 + \cos^2 \theta) \times (\hbar c)^2 \times 10^{37}$$

Replace $\cos \theta$ with explanatory variable x to obtain predicted value \hat{y} .

$$\hat{y} = \frac{\pi\alpha^2}{2} (1 + x^2) \times (\hbar c)^2 \times 10^{37}$$

Here are the predicted values \hat{y} based on the above formula.

x	y	\hat{y}
-0.925	67.08	60.44
-0.85	58.67	56.10
-0.75	54.66	50.89
-0.65	51.72	46.33
-0.55	43.70	42.42
-0.45	41.12	39.17
-0.35	39.71	36.56
-0.25	35.34	34.61
-0.15	33.35	33.30
-0.05	34.69	32.65
0.05	34.05	32.65
0.15	34.48	33.30
0.25	34.66	34.61
0.35	35.23	36.56
0.45	35.60	39.17
0.55	40.13	42.42
0.65	42.56	46.33
0.75	46.37	50.89
0.85	49.28	56.10
0.925	55.70	60.44

The coefficient of determination R^2 measures how well predicted values fit the real data.

$$R^2 = 1 - \frac{\sum(y - \hat{y})^2}{\sum(y - \bar{y})^2} = 0.87$$

The result indicates that the model $d\sigma$ explains 87% of the variance in the data.

Electroweak model

The following differential cross section formula from electroweak theory results in a better fit to the data.¹

$$\frac{d\sigma}{d\Omega} = F(s)(1 + \cos^2 \theta) + G(s) \cos \theta$$

where

$$F(s) = \frac{\alpha^2}{4s} \left(1 + \frac{g_V^2}{\sqrt{2}\pi} \left(\frac{m_Z^2}{s - m_Z^2} \right) \left(\frac{sG}{\alpha} \right) + \frac{(g_A^2 + g_V^2)^2}{8\pi^2} \left(\frac{m_Z^2}{s - m_Z^2} \right)^2 \left(\frac{sG}{\alpha} \right)^2 \right)$$

$$G(s) = \frac{\alpha^2}{4s} \left(\frac{\sqrt{2}g_A^2}{\pi} \left(\frac{m_Z^2}{s - m_Z^2} \right) \left(\frac{sG}{\alpha} \right) + \frac{g_A^2 g_V^2}{\pi^2} \left(\frac{m_Z^2}{s - m_Z^2} \right)^2 \left(\frac{sG}{\alpha} \right)^2 \right)$$

¹F. Mandl and G. Shaw, *Quantum Field Theory Revised Edition*, 316.

and

$$\begin{aligned}
g_A &= -0.5 \\
g_V &= -0.0348 \\
m_Z &= 91.17 \text{ GeV} \\
G &= 1.166 \times 10^{-5} \text{ GeV}^{-2}
\end{aligned}$$

The corresponding formula for \hat{y} is

$$\hat{y} = 2\pi [F(s)(1 + x^2) + G(s)x] \times (\hbar c)^2 \times 10^{37}$$

where $\sqrt{s} = 29 \text{ GeV}$ is the center of mass collision energy. Here are the predicted values \hat{y} based on the above formula.

x	y	\hat{y}
-0.925	67.08	65.59
-0.85	58.67	60.84
-0.75	54.66	55.07
-0.65	51.72	49.96
-0.55	43.70	45.49
-0.45	41.12	41.69
-0.35	39.71	38.53
-0.25	35.34	36.02
-0.15	33.35	34.17
-0.05	34.69	32.97
0.05	34.05	32.42
0.15	34.48	32.53
0.25	34.66	33.28
0.35	35.23	34.69
0.45	35.60	36.75
0.55	40.13	39.47
0.65	42.56	42.83
0.75	46.37	46.85
0.85	49.28	51.52
0.925	55.70	55.45

The coefficient of determination R^2 is

$$R^2 = 1 - \frac{\sum (y - \hat{y})^2}{\sum (y - \bar{y})^2} = 0.98$$

The result indicates that electroweak theory explains 98% of the variance in the data.

Notes

Here are a few notes about how the demo script works.

In component notation, traces are sums over a repeated index, in this case α .

$$\begin{aligned}\text{Tr} \left((\not{p}_3 + M) \gamma^\mu (\not{p}_4 - M) \gamma^\nu \right) &= (\not{p}_3 + M)^\alpha_\beta \gamma^{\mu\beta}_\rho (\not{p}_4 - M)^\rho_\sigma \gamma^{\nu\sigma}_\alpha \\ \text{Tr} \left((\not{p}_2 - m) \gamma_\mu (\not{p}_1 + m) \gamma_\nu \right) &= (\not{p}_2 - m)^\alpha_\beta \gamma^\beta_{\mu\rho} (\not{p}_1 + m)^\rho_\sigma \gamma^\sigma_{\nu\alpha}\end{aligned}$$

To convert the above formulas to Eigenmath code, the γ tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply γ^μ by the metric tensor to lower the index.

$$\begin{aligned}\gamma^{\beta\mu}_\rho &\rightarrow \text{gammaT} = \text{transpose}(\text{gamma}) \\ \gamma^\beta_{\mu\rho} &\rightarrow \text{gammaL} = \text{transpose}(\text{dot}(\text{gmunu}, \text{gamma}))\end{aligned}$$

Define the following 4×4 matrices.

$$\begin{aligned}(\not{p}_1 + m) &\rightarrow \text{X1} = \text{pslash1} + \text{m I} \\ (\not{p}_2 - m) &\rightarrow \text{X2} = \text{pslash2} - \text{m I} \\ (\not{p}_3 + M) &\rightarrow \text{X3} = \text{pslash3} + \text{M I} \\ (\not{p}_4 - M) &\rightarrow \text{X4} = \text{pslash4} - \text{M I}\end{aligned}$$

Then

$$\begin{aligned}(\not{p}_3 + M)^\alpha_\beta \gamma^{\mu\beta}_\rho (\not{p}_4 - M)^\rho_\sigma \gamma^{\nu\sigma}_\alpha &\rightarrow \text{T1} = \text{contract}(\text{dot}(\text{X3}, \text{gammaT}, \text{X4}, \text{gammaT}), 1, 4) \\ (\not{p}_2 - m)^\alpha_\beta \gamma^\beta_{\mu\rho} (\not{p}_1 + m)^\rho_\sigma \gamma^\sigma_{\nu\alpha} &\rightarrow \text{T2} = \text{contract}(\text{dot}(\text{X2}, \text{gammaL}, \text{X1}, \text{gammaL}), 1, 4)\end{aligned}$$

Next, multiply matrices and sum over repeated indices. The dot function sums over ν then the contract function sums over μ . The transpose makes the ν indices adjacent as required by the dot function.

$$\text{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu) \text{Tr}(\cdots \gamma_\mu \cdots \gamma_\nu) \rightarrow \text{contract}(\text{dot}(\text{T1}, \text{transpose}(\text{T2})))$$