

Fermi golden rule step by step

Consider the transition rate $\Gamma_{1 \rightarrow 2}$ where $\psi_1(x)$ is the initial eigenstate and $\psi_2(x)$ is the final eigenstate.

- ① Let $\Psi(x, t)$ be the following linear combination of eigenstates.

$$\Psi(x, t) = c_1(t)\psi_1(x) \exp(-i\omega_1 t) + c_2(t)\psi_2(x) \exp(-i\omega_2 t)$$

We need to solve for $c_2(t)$ to find the transition rate.

- ② Let the perturbing Hamiltonian be

$$H'(x, t) = 2V(x) \cos(\omega t + \phi)$$

- ③ From the time-dependent Schrodinger equation we have

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = H_0(x)\Psi(x, t) + H'(x, t)\Psi(x, t)$$

- ④ This reduces to

$$i\hbar \frac{\partial c_1(t)}{\partial t} \psi_1(x) \exp(-i\omega_1 t) + i\hbar \frac{\partial c_2(t)}{\partial t} \psi_2(x) \exp(-i\omega_2 t) = H'(x, t)\Psi(x, t)$$

- ⑤ Take the inner product of $\psi_2^*(x)$ with the above equation to obtain

$$i\hbar \frac{\partial c_2(t)}{\partial t} \exp(-i\omega_2 t) = 2 \cos(\omega t + \phi) \left(c_1(t) M_{21} \exp(-i\omega_1 t) + c_2(t) M_{22} \exp(-i\omega_2 t) \right)$$

where M_{21} and M_{22} are the matrix elements

$$M_{21} = \int \psi_2^*(x) V(x) \psi_1(x) dx$$
$$M_{22} = \int \psi_2^*(x) V(x) \psi_2(x) dx$$

- ⑥ Rewrite as

$$i\hbar \frac{\partial c_2(t)}{\partial t} = 2 \cos(\omega t + \phi) \left(c_1(t) M_{21} \exp(i(\omega_2 - \omega_1)t) + c_2(t) M_{22} \right)$$

- ⑦ Let the initial state be $\Psi(x, 0) = \psi_1(x)$ hence the initial conditions are

$$c_1(0) = 1, \quad c_2(0) = 0$$

For small t we use the approximations $c_1(t) = 1$ and $c_2(t) = 0$ to obtain

$$i\hbar \frac{\partial c_2(t)}{\partial t} = 2 \cos(\omega t + \phi) M_{21} \exp(i(\omega_2 - \omega_1)t)$$

⑧ Solve for $c_2(t)$ by integrating.

$$c_2(t) = \frac{2M_{21}}{i\hbar} \int_0^t \cos(\omega t' + \phi) \exp(i(\omega_2 - \omega_1)t') dt'$$

The solution is

$$c_2(t) = -\frac{M_{21}}{\hbar} \left(\frac{\exp(i(\omega_2 - \omega_1 - \omega)t) - 1}{\omega_2 - \omega_1 - \omega} \right) \exp(-i\phi) - \frac{M_{21}}{\hbar} \left(\frac{\exp(i(\omega_2 - \omega_1 + \omega)t) - 1}{\omega_2 - \omega_1 + \omega} \right) \exp(i\phi)$$

⑨ For ω such that $\omega \approx \omega_2 - \omega_1$ the first term dominates so discard the second term and write

$$c_2(t) = -\frac{M_{21}}{\hbar} \left(\frac{\exp(i(\omega_2 - \omega_1 - \omega)t) - 1}{\omega_2 - \omega_1 - \omega} \right) \exp(-i\phi)$$

⑩ Rewrite using a sinc function.

$$c_2(t) = -\frac{it}{\hbar} M_{21} \exp\left(i \frac{\omega_2 - \omega_1 - \omega}{2} t - i\phi\right) \text{sinc}\left(\frac{\omega_2 - \omega_1 - \omega}{2} t\right)$$

⑪ Hence the transition probability is

$$P(1 \rightarrow 2) = |c_2(t)|^2 = \frac{t^2}{\hbar^2} |M_{21}|^2 \text{sinc}^2\left(\frac{\omega_2 - \omega_1 - \omega}{2} t\right)$$

⑫ Integrate $P(1 \rightarrow 2)$ to obtain the total transition probability $P_{tot}(1 \rightarrow 2)$.

$$P_{tot}(1 \rightarrow 2) = \frac{t^2}{\hbar^2} |M_{21}|^2 \int_{E-\epsilon}^{E+\epsilon} \text{sinc}^2\left(\frac{E'/\hbar - \omega}{2} t\right) g(E') dE'$$

where

$$E = \hbar(\omega_2 - \omega_1)$$

and $g(E')$ is the density of photon states for energy E' .

⑬ Use the approximation $g(E') \approx g(\hbar\omega)$ to obtain

$$P_{tot}(1 \rightarrow 2) = \frac{t^2}{\hbar^2} |M_{21}|^2 g(\hbar\omega) \int_{E-\epsilon}^{E+\epsilon} \text{sinc}^2\left(\frac{E'/\hbar - \omega}{2} t\right) dE'$$

⑭ Let

$$y = \frac{E'/\hbar - \omega}{2} t$$

It follows that

$$E' = \frac{2\hbar y}{t} + \hbar\omega$$

and

$$dE' = \frac{2\hbar}{t} dy$$

The integration limits transform as

$$E \pm \epsilon \rightarrow \frac{(E \pm \epsilon)/\hbar - \omega}{2} t = \frac{Et}{2\hbar} - \frac{\omega t}{2} \pm \frac{\epsilon t}{2\hbar} \approx \pm \frac{\epsilon t}{2\hbar}$$

Hence the integral is transformed as

$$P_{tot}(1 \rightarrow 2) = \frac{2t}{\hbar} |M_{21}|^2 g(\hbar\omega) \int_{-\epsilon t/2\hbar}^{\epsilon t/2\hbar} \text{sinc}^2 y \, dy$$

⑮ The sinc squared function is very narrow so use the approximation

$$\int_{-\epsilon t/2\hbar}^{\epsilon t/2\hbar} \text{sinc}^2 y \, dy \approx \int_{-\infty}^{\infty} \text{sinc}^2 y \, dy = \pi$$

to obtain

$$P_{tot}(1 \rightarrow 2) = \frac{2\pi t}{\hbar} |M_{21}|^2 g(\hbar\omega)$$

⑯ The transition rate is the derivative of $P_{tot}(1 \rightarrow 2)$.

$$\Gamma_{1 \rightarrow 2} = \frac{d}{dt} P_{tot}(1 \rightarrow 2) = \frac{2\pi}{\hbar} |M_{21}|^2 g(\hbar\omega)$$