

9-7. Show, for the vacuum state, the expectation value of  $\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{q}}$  is  $(\hbar/2kc)\delta_{\mathbf{k},\mathbf{q}}$  and that of  $\bar{a}_{1,\mathbf{k}} \bar{a}_{1,\mathbf{q}}$  is  $(\hbar/2kc)\delta_{-\mathbf{k},\mathbf{q}}$ .

We will use the following table of integrals.

$$\int_{-\infty}^{\infty} \exp(-ax^2 + b) dx = \sqrt{\frac{\pi}{a}} \exp(b) \quad (1)$$

$$\int_{-\infty}^{\infty} x \exp(-ax^2 + b) dx = 0 \quad (2)$$

$$\int_{-\infty}^{\infty} x^2 \exp(-ax^2 + b) dx = \frac{1}{2a} \sqrt{\frac{\pi}{a}} \exp(b) \quad (3)$$

For simplicity of notation, let

$$A = \bar{a}_{1,\mathbf{k}}^c \quad B = \bar{a}_{1,\mathbf{k}}^s \quad C = \bar{a}_{1,\mathbf{q}}^c \quad D = \bar{a}_{1,\mathbf{q}}^s$$

From problem 9-6

$$\begin{aligned} \bar{a}_{1,\mathbf{k}} &= \frac{1}{\sqrt{2}}(A - iB) \\ \bar{a}_{1,\mathbf{q}} &= \frac{1}{\sqrt{2}}(C - iD) \end{aligned} \quad (4)$$

Adapted from equation (8.84).

$$\langle \Phi_0 | f | \Phi_0 \rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_0^* f \Phi_0 dA dB dC dD$$

Adapted from problem 9-6 with  $q$  as a mode number (not electric charge).

$$\Phi_0 = \exp \left( -\frac{kc}{4\hbar} A^2 - \frac{kc}{4\hbar} B^2 - \frac{qc}{4\hbar} C^2 - \frac{qc}{4\hbar} D^2 \right)$$

It follows that

$$\Phi_0^* \Phi_0 = \exp \left( -\frac{kc}{2\hbar} A^2 - \frac{kc}{2\hbar} B^2 - \frac{qc}{2\hbar} C^2 - \frac{qc}{2\hbar} D^2 \right)$$

Compute the normalization constant  $K$ .

$$K = \langle \Phi_0 | 1 | \Phi_0 \rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_0^* \Phi_0 dA dB dC dD$$

By integral (1) for each factor in the measure (see problem 8-5)

$$K = \left(\frac{2\pi\hbar}{kc}\right)^{1/2} \left(\frac{2\pi\hbar}{kc}\right)^{1/2} \left(\frac{2\pi\hbar}{qc}\right)^{1/2} \left(\frac{2\pi\hbar}{qc}\right)^{1/2}$$

Compute the expectation of  $\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}}$ . From (4) we have

$$\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} = \frac{A^2 + B^2}{2}$$

Hence

$$\langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle = \frac{1}{K} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_0^* \Phi_0 \frac{A^2 + B^2}{2} dA dB dC dD$$

Rewrite as

$$\begin{aligned} \langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle &= \frac{1}{2K} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_0^* \Phi_0 A^2 dA dB dC dD \\ &\quad + \frac{1}{2K} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_0^* \Phi_0 B^2 dA dB dC dD \end{aligned}$$

By integrals (1) and (3) we have

$$\langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle = \frac{1}{K} \frac{\hbar}{kc} \left(\frac{2\pi\hbar}{kc}\right)^{1/2} \left(\frac{2\pi\hbar}{kc}\right)^{1/2} \left(\frac{2\pi\hbar}{qc}\right)^{1/2} \left(\frac{2\pi\hbar}{qc}\right)^{1/2}$$

Hence

$$\langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle = \frac{\hbar}{kc} \tag{5}$$

Compute the expectation of  $\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{q}}$ .

$$\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{q}} = \frac{AC + BD - iAD + iBC}{2}$$

Hence

$$\begin{aligned} \langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{q}} | \Phi_0 \rangle &= \\ \frac{1}{K} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_0^* \Phi_0 \frac{AC + BD - iAD + iBC}{2} dA dB dC dD \end{aligned}$$

By integral (2) all terms are zero, hence

$$\langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{q}} | \Phi_0 \rangle = 0 \quad (6)$$

Combine (5) and (6) to obtain

$$\langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{q}} | \Phi_0 \rangle = \frac{\hbar}{kc} \delta_{\mathbf{k},\mathbf{q}}$$

Note that by equation (8.77)

$$\bar{a}_{1,\mathbf{k}}^* = \bar{a}_{1,-\mathbf{k}}$$

Hence

$$\langle \Phi_0 | \bar{a}_{1,-\mathbf{k}} \bar{a}_{1,\mathbf{q}} | \Phi_0 \rangle = \frac{\hbar}{kc} \delta_{\mathbf{k},\mathbf{q}}$$

(9-7 cont'd) Develop a formula for the expectation of  $(\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}})^r$  for integral  $r$  and explain thereby how the expectation of such quantities as  $(\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}})^r (\bar{a}_{1,\mathbf{q}}^* \bar{a}_{1,\mathbf{q}})^s$  can be got for  $\mathbf{q} \neq \mathbf{k}$ .

By the binomial theorem

$$\left( \frac{A^2 + B^2}{2} \right)^r = \frac{1}{2^r} \sum_{j=0}^r \binom{r}{j} A^{2j} B^{2(r-j)}$$

Hence

$$\langle \Phi_0^* | (\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}})^r | \Phi_0 \rangle = \frac{1}{2^r} \sum_{j=0}^r \binom{r}{j} \langle \Phi_0^* | A^{2j} B^{2(r-j)} | \Phi_0 \rangle \quad (7)$$

To evaluate (7) we need the following integral.

$$\begin{aligned} \int_{-\infty}^{\infty} x^{2n} \exp(-ax^2 + b) dx &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n a^n} \sqrt{\frac{\pi}{a}} \exp(b) \\ &= (2n-1)!! \frac{1}{2^n a^n} \sqrt{\frac{\pi}{a}} \exp(b) \end{aligned}$$

Based on the above integral, define the following function  $F$ . (The  $\sqrt{\pi/a}$  factor is left out because it gets cancelled by the normalization constant  $K$ .)

$$F(n) = (2n-1)!! \left( \frac{\hbar}{kc} \right)^n$$

Note that

$$F(j)F(r-j) = (2j-1)!! (2r-2j-1)!! \left(\frac{\hbar}{kc}\right)^j \left(\frac{\hbar}{kc}\right)^{r-j}$$

It turns out that

$$\frac{1}{2^r} \sum_{j=0}^r \binom{r}{j} (2j-1)!! (2r-2j-1)!! = r!$$

Hence equation (7) can be written as

$$\langle \Phi_0^* | (\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}})^r | \Phi_0 \rangle = r! \left(\frac{\hbar}{kc}\right)^r$$

Regarding the  $\mathbf{q} \neq \mathbf{k}$  part of the problem, we have

$$\begin{aligned} & \left(\frac{A^2 + B^2}{2}\right)^r \left(\frac{C^2 + D^2}{2}\right)^s \\ &= \left(\frac{1}{2^r} \sum_{j=0}^r \binom{r}{j} A^{2j} B^{2(r-j)}\right) \left(\frac{1}{2^s} \sum_{k=0}^s \binom{s}{k} C^{2k} D^{2(s-k)}\right) \\ &= \frac{1}{2^{r+s}} \sum_{j=0}^r \sum_{k=0}^s \binom{r}{j} \binom{s}{k} A^{2j} B^{2(r-j)} C^{2k} D^{2(s-k)} \end{aligned}$$

Hence

$$\begin{aligned} \langle \Phi_0^* | (\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}})^r (\bar{a}_{1,\mathbf{q}}^* \bar{a}_{1,\mathbf{q}})^s | \Phi_0 \rangle &= \\ \frac{1}{2^{r+s}} \sum_{j=0}^r \sum_{k=0}^s \binom{r}{j} \binom{s}{k} \langle \Phi_0^* | A^{2j} B^{2(r-j)} C^{2k} D^{2(s-k)} | \Phi_0 \rangle \end{aligned}$$

(9-7 cont'd) Show that the expectation of  $(\bar{a}_{1,\mathbf{k}})^2$  or  $(\bar{a}_{1,\mathbf{k}}^*)^2$  vanishes.

We have

$$(\bar{a}_{1,\mathbf{k}})^2 = \frac{A^2 - B^2}{2} - iAB \quad (\bar{a}_{1,\mathbf{k}}^*)^2 = \frac{A^2 - B^2}{2} + iAB$$

The integrals of  $A^2$  and  $-B^2$  cancel each other. The integral of  $AB$  vanishes by integral (2).

(9-7 cont'd) Show that the expectation of the product of any odd number of  $\bar{a}$ 's is zero and that you can compute the expectation value of any product of  $\bar{a}$ 's or  $\bar{a}^*$ 's for the vacuum state.

Isn't the expectation of *any* number of  $\bar{a}$ 's zero? It is shown above that the expectation of  $(\bar{a}_{1,\mathbf{k}})^2$  vanishes.

FIXME: The result  $\hbar/kc$  is wrong by a factor of 2 everywhere it appears. Should be  $\hbar/2kc$ .