Free particle propagator

A propagator is the amplitude for a particle to go from x_a at time t_a to x_b at time t_b.

Let $K(b,a)$ be the propagator

$$K(b,a) = \langle x_b | \exp\left(-\frac{i}{\hbar} H (t_b - t_a)\right) | x_a \rangle$$

where H is the free particle Hamiltonian

$$H = \frac{\hat{p}^2}{2m}$$

By the identity

$$\int |p\rangle\langle p| \, dp = 1$$

we can write

$$K(b,a) = \int \langle x_b | \exp\left(-\frac{i}{\hbar} H (t_b - t_a)\right) |p\rangle\langle p | x_a \rangle \, dp$$

Replace the momentum operator $\hat{p}$ with its eigenvalue p.

$$K(b,a) = \int \exp\left(-\frac{i}{2m\hbar} p^2 (t_b - t_a)\right) \langle x_b | p \rangle \langle p | x_a \rangle \, dp$$

Recalling that

$$\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right)$$

we can write

$$K(b,a) = \frac{1}{2\pi\hbar} \int \exp\left(-\frac{i}{2m\hbar} p^2 (t_b - t_a) + \frac{i}{\hbar} p (x_b - x_a)\right) \, dp$$

By the integral

$$\int_{-\infty}^{\infty} \exp(-ax^2 + bx) \, dx = \left(\frac{\pi}{a}\right)^{1/2} \exp\left(\frac{b^2}{4a}\right)$$

with

$$a = \frac{i}{2m\hbar}(t_b - t_a) \quad \text{and} \quad b = \frac{i}{\hbar}(x_b - x_a)$$

we have

$$K(b,a) = \frac{1}{2\pi\hbar} \left(\frac{2\pi m\hbar}{i(t_b - t_a)}\right)^{1/2} \exp\left(-\frac{m(x_b - x_a)^2}{2i\hbar(t_b - t_a)}\right)$$

Rewrite as

$$K(b,a) = \left(\frac{m}{2\pi i\hbar(t_b - t_a)}\right)^{1/2} \exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)}\right)$$