

# Mott problem

Consider a system of two hydrogen atoms ionized by an  $\alpha$  particle. The Hamiltonian is

$$\hat{H} = \hat{K}_\alpha + \hat{K}_1 + \hat{K}_2 + V_1 + V_2 + U_1 + U_2$$

where

$\hat{K}_\alpha = -\frac{\hbar^2}{2M}\nabla_\alpha^2$	kinetic energy of $\alpha$ particle
$\hat{K}_1 = -\frac{\hbar^2}{2m}\nabla_1^2$	kinetic energy of 1st electron
$\hat{K}_2 = -\frac{\hbar^2}{2m}\nabla_2^2$	kinetic energy of 2nd electron
$V_1 = -\frac{e^2}{ \mathbf{r}_1 - \mathbf{a}_1 }$	potential energy of 1st electron
$V_2 = -\frac{e^2}{ \mathbf{r}_2 - \mathbf{a}_2 }$	potential energy of 2nd electron
$U_1 = -\frac{2e^2}{ \mathbf{R} - \mathbf{r}_1 }$	potential energy of $\alpha$ and 1st electron
$U_2 = -\frac{2e^2}{ \mathbf{R} - \mathbf{r}_2 }$	potential energy of $\alpha$ and 2nd electron

and

$\mathbf{R}$	coordinate of $\alpha$ particle
$\mathbf{r}_1, \mathbf{r}_2$	coordinates of electrons
$\mathbf{a}_1, \mathbf{a}_2$	coordinates of hydrogen atoms

Let  $\psi_1$  and  $\psi_2$  be hydrogen atom wavefunctions such that

$$\hat{H}\psi_1 = E_1\psi_1, \quad \hat{H}\psi_2 = E_2\psi_2$$

We want to find a wavefunction  $F(\mathbf{R}, \mathbf{r}_1, \mathbf{r}_2)$  such that

$$\hat{H}F = EF$$

Let

$$F = F_0 + F_1 + F_2 + \dots$$

and let

$$\hat{H}_0 = \hat{K}_\alpha + E_1 + E_2$$

Start by finding an  $F_0$  such that

$$\hat{H}_0 F_0 = E F_0$$

The solution is

$$F_0(\mathbf{R}, \mathbf{r}_1, \mathbf{r}_2) = f_0(\mathbf{R})\psi_1(\mathbf{r}_1 - \mathbf{a}_1)\psi_2(\mathbf{r}_2 - \mathbf{a}_2) \quad (1)$$

where

$$f_0(\mathbf{R}) = \frac{1}{|\mathbf{R}|} \exp\left(\frac{ik|\mathbf{R}|}{\hbar}\right), \quad k = \sqrt{2M(E - E_1 - E_2)}$$

It follows that for the full Hamiltonian

$$\hat{H}F_0 = EF_0 + (U_1 + U_2)F_0$$

To cancel  $(U_1 + U_2)F_0$  from the full Hamiltonian, find an  $F_1$  such that

$$\hat{H}_0F_1 = EF_1 - (U_1 + U_2)F_0$$

Rewrite as

$$(\hat{H}_0 - E)F_1 = -(U_1 + U_2)F_0$$

Expand  $F_1$  and  $F_0$ .

$$(\hat{H}_0 - E)f_1(\mathbf{R})\psi_1(\mathbf{r}_1 - \mathbf{a}_1)\psi_2(\mathbf{r}_2 - \mathbf{a}_2) = -(U_1 + U_2)f_0(\mathbf{R})\psi_1(\mathbf{r}_1 - \mathbf{a}_1)\psi_2(\mathbf{r}_2 - \mathbf{a}_2)$$

To solve for  $f_1(\mathbf{R})$  multiply both sides by

$$\psi_1^*(\mathbf{r}_1 - \mathbf{a}_1)\psi_2^*(\mathbf{r}_2 - \mathbf{a}_2)$$

and integrate over  $\mathbf{r}_1$  and  $\mathbf{r}_2$  to obtain

$$(\hat{H}_0 - E)f_1(\mathbf{R}) = V(\mathbf{R})f_0(\mathbf{R}) \quad (2)$$

where

$$V(\mathbf{R}) = 2e^2 \int \frac{|\psi_1(\mathbf{r}_1 - \mathbf{a}_1)|^2 |\psi_2(\mathbf{r}_2 - \mathbf{a}_2)|^2}{|\mathbf{R} - \mathbf{r}_1|} d\mathbf{r}_1 d\mathbf{r}_2 + 2e^2 \int \frac{|\psi_1(\mathbf{r}_1 - \mathbf{a}_1)|^2 |\psi_2(\mathbf{r}_2 - \mathbf{a}_2)|^2}{|\mathbf{R} - \mathbf{r}_2|} d\mathbf{r}_1 d\mathbf{r}_2$$

Because  $|\psi|^2$  is a normalized probability density function we have

$$V(\mathbf{R}) = 2e^2 \int \frac{|\psi_1(\mathbf{r}_1 - \mathbf{a}_1)|^2}{|\mathbf{R} - \mathbf{r}_1|} d\mathbf{r}_1 + 2e^2 \int \frac{|\psi_2(\mathbf{r}_2 - \mathbf{a}_2)|^2}{|\mathbf{R} - \mathbf{r}_2|} d\mathbf{r}_2$$

Per Mott the solution to (2) is

$$f_1(\mathbf{R}) = \frac{M}{2\pi\hbar^2} \int \frac{V(\mathbf{r})f_0(\mathbf{r})}{|\mathbf{R} - \mathbf{r}|} \exp\left(\frac{ik|\mathbf{R} - \mathbf{r}|}{\hbar}\right) d\mathbf{r}, \quad k = \sqrt{2M(E - E_1 - E_2)}$$

Substitute for  $f_0(\mathbf{r})$  to obtain

$$f_1(\mathbf{R}) = \frac{M}{2\pi\hbar^2} \int \frac{V(\mathbf{r})}{|\mathbf{R} - \mathbf{r}||\mathbf{r}|} \exp\left(\frac{ik|\mathbf{R} - \mathbf{r}|}{\hbar} + \frac{ik|\mathbf{r}|}{\hbar}\right) d\mathbf{r}$$

Change of variable  $\mathbf{r} \rightarrow \mathbf{y} + \mathbf{a}_1$ .

$$f_1(\mathbf{R}) = \frac{M}{2\pi\hbar^2} \int \frac{V(\mathbf{y} + \mathbf{a}_1)}{|\mathbf{R} - \mathbf{y} - \mathbf{a}_1||\mathbf{y} + \mathbf{a}_1|} \exp\left(\frac{ik|\mathbf{R} - \mathbf{y} - \mathbf{a}_1|}{\hbar} + \frac{ik|\mathbf{y} + \mathbf{a}_1|}{\hbar}\right) d\mathbf{y}$$

Per Mott (see also Figari and Teta)

$$f_1(\mathbf{R}) \approx \frac{\exp(ik|\mathbf{R} - \mathbf{a}_1|)}{|\mathbf{R} - \mathbf{a}_1|} \frac{M}{2\pi\hbar^2} \int \frac{V(\mathbf{y} + \mathbf{a}_1)}{|\mathbf{y} + \mathbf{a}_1|} \exp\left(-\frac{ik\mathbf{u} \cdot \mathbf{y}}{\hbar} + \frac{ik|\mathbf{y}|}{\hbar}\right) d\mathbf{y} \quad (3)$$

where

$$\mathbf{u} = \frac{\mathbf{R} - \mathbf{a}_1}{|\mathbf{R} - \mathbf{a}_1|}$$

By the method of stationary phase the integral vanishes except for

$$\frac{d}{d\mathbf{y}} (-\mathbf{u} \cdot \mathbf{y} + |\mathbf{y} + \mathbf{a}_1|) = -\mathbf{u} + \frac{\mathbf{y} + \mathbf{a}_1}{|\mathbf{y} + \mathbf{a}_1|} = 0$$

Because  $V(\mathbf{y} + \mathbf{a}_1) \rightarrow 0$  as  $|\mathbf{y}|$  increases we only require stationarity for small  $|\mathbf{y}|$  hence

$$\mathbf{u} \approx \frac{\mathbf{a}_1}{|\mathbf{a}_1|}, \quad |\mathbf{y}| \ll |\mathbf{a}_1|$$

It follows that the  $\alpha$  particle trajectory is along the line  $\mathbf{a}_1$ .