

## Bell's theorem

Consider two observers  $A$  and  $B$ . Each observer has an apparatus for measuring spin. Each apparatus can be set in one of two orientations labeled 0 and 1. For independent random variables  $A$  and  $B$  we have

$$\langle A \rangle \langle B \rangle = \langle AB \rangle$$

Now consider all minimum and maximum expectation values along with a special formula.

$\langle A_0 \rangle$	$\langle A_1 \rangle$	$\langle B_0 \rangle$	$\langle B_1 \rangle$	$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle$
1	1	1	1	2
1	1	1	-1	2
1	1	-1	1	-2
1	1	-1	-1	-2
1	-1	1	1	2
1	-1	1	-1	-2
1	-1	-1	1	2
1	-1	-1	-1	-2
-1	1	1	1	-2
-1	1	1	-1	2
-1	1	-1	1	-2
-1	1	-1	-1	2
-1	-1	1	1	-2
-1	-1	1	-1	-2
-1	-1	-1	1	2
-1	-1	-1	-1	2

Since the table is for all minimum and maximum values we have by inspection the range

$$-2 \leq \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2 \quad (1)$$

Now suppose a third apparatus generates two spins in the following singlet state.

$$|s\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

One spin is sent to  $A$  and the other is sent to  $B$ .

Let

$$A_0 = \sigma_z, \quad A_1 = \sigma_x, \quad B_0 = -\frac{\sigma_x + \sigma_z}{\sqrt{2}}, \quad B_1 = \frac{\sigma_x - \sigma_z}{\sqrt{2}}$$

Then for the singlet state we have

$$\langle A_0 B_0 \rangle = \frac{1}{\sqrt{2}}, \quad \langle A_0 B_1 \rangle = \frac{1}{\sqrt{2}}, \quad \langle A_1 B_0 \rangle = \frac{1}{\sqrt{2}}, \quad \langle A_1 B_1 \rangle = -\frac{1}{\sqrt{2}}$$

Hence

$$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle = 2\sqrt{2} \quad (2)$$

The result in (2) conflicts with (1) because for the singlet state the random variables  $A$  and  $B$  are not independent. Any theory that asserts independence of  $A$  and  $B$  (for example, a hidden variable theory) is constrained by (1) and falsified by (2).

### Exercises

1. Verify equation (2).
2. Verify that for the singlet state

$$\langle A_0 \rangle = 0, \quad \langle A_1 \rangle = 0, \quad \langle B_0 \rangle = 0, \quad \langle B_1 \rangle = 0$$

Hence  $\langle A \rangle \langle B \rangle \neq \langle AB \rangle$  for the singlet state.