9-8. For the state for which there is just one photon present in level $1, \mathbf{k}$, all of the factors in the wave function are ϕ_0 except one, which is ϕ_1. But for an oscillator $\phi_1(x) = \sqrt{2}x\phi_0(x)$. The wave function representing an excited running wave is a linear superposition of the state with the cosine mode excited and i times the state with the sine wave excited, so show that the unnormalized wave function for just one photon present in $1, \mathbf{k}$ is $\bar{a}^*_{1,\mathbf{k}}\Phi_0$. The normalization is $\int \Phi^*_0 \bar{a}_{1,\mathbf{k}} \bar{a}^*_{1,\mathbf{k}} \Phi_0 \, d\bar{a}$, or the expectation of $\bar{a}_{1,\mathbf{k}}\bar{a}^*_{1,\mathbf{k}}$ for the vacuum, which we have seen in the preceding problem is $\hbar/2kc$. Hence the normalized one-photon state is $\sqrt{2kc/\hbar}\bar{a}^*_{1,\mathbf{k}}\Phi_0$.

This is a state with the cosine mode excited.

$$\bar{a}^c_{1,\mathbf{k}}\Phi_0$$

This is a state with the sine mode excited.

$$\bar{a}^s_{1,\mathbf{k}}\Phi_0$$

This is a superposition of the cosine state and i times the sine state. The factor $1/\sqrt{2}$ is for normalization, i.e., $|1 + i| = \sqrt{2}$.

$$\frac{1}{\sqrt{2}}\left(\bar{a}^c_{1,\mathbf{k}}\Phi_0 + i\bar{a}^s_{1,\mathbf{k}}\Phi_0\right) = \bar{a}^*_{1,\mathbf{k}}\Phi_0$$

Here are some additional results.

From equation (9.43)

$$|\Phi_0|^2 = \Phi^*_0\Phi_0 = \exp\left(-\frac{kc}{\hbar}(\bar{a}^c_{1,\mathbf{k}})^2 - \frac{kc}{\hbar}(\bar{a}^s_{1,\mathbf{k}})^2 - \frac{kc}{\hbar}(\bar{a}^c_{2,\mathbf{k}})^2 - \frac{kc}{\hbar}(\bar{a}^s_{2,\mathbf{k}})^2\right)$$

For simplicity of notation, let

$$d\bar{a} = d\bar{a}^c_{1,\mathbf{k}}\, d\bar{a}^s_{1,\mathbf{k}}\, d\bar{a}^c_{2,\mathbf{k}}\, d\bar{a}^s_{2,\mathbf{k}}$$

The expectation of Φ_0 is

$$\langle\Phi_0\rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\Phi_0|^2 \, d\bar{a} = \left(\frac{\pi\hbar}{kc}\right)^2 \tag{1}$$

Let

$$\Phi_1 = \bar{a}^*_{1,\mathbf{k}} \Phi_0$$

Then

$$|\Phi_1|^2 = \Phi_1^* \Phi_1 = \left(\frac{(\bar{a}^c_{1,\mathbf{k}})^2 + (\bar{a}^s_{1,\mathbf{k}})^2}{2} \right) |\Phi_0|^2$$

The expectation of Φ_1 is

$$\langle \Phi_1 \rangle = \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} \left(\frac{(\bar{a}^c_{1,\mathbf{k}})^2 + (\bar{a}^s_{1,\mathbf{k}})^2}{2} \right) |\Phi_0|^2 \, d\bar{a} = \frac{\hbar}{2kc} \left(\frac{\pi \hbar}{kc} \right)^2 \tag{2}$$

The expectation for n photons is

$$\langle \Phi_n \rangle = \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} \left(\frac{(\bar{a}^c_{1,\mathbf{k}})^2 + (\bar{a}^s_{1,\mathbf{k}})^2}{2} \right)^n |\Phi_0|^2 \, d\bar{a}$$

By a result from problem 9-7

$$\langle \Phi_n \rangle = n! \left(\frac{\hbar}{2kc} \right)^n \left(\frac{\pi \hbar}{kc} \right)^2 = n! \left(\frac{\hbar}{2kc} \right)^n \langle \Phi_0 \rangle \tag{3}$$