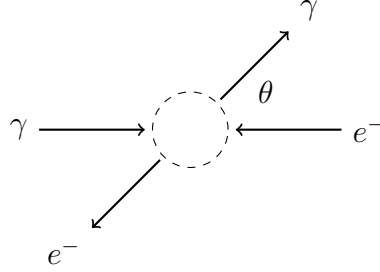


# Compton scattering

Compton scattering is the interaction  $e^- + \gamma \rightarrow e^- + \gamma$ .



Define the following momentum vectors and spinors. Symbol  $\omega$  is incident energy. Symbol  $E$  is total energy  $E = \sqrt{\omega^2 + m^2}$  where  $m$  is electron mass. Polar angle  $\theta$  is the observed scattering angle. Azimuth angle  $\phi$  cancels out in scattering calculations.

$$\begin{aligned}
 p_1 &= \begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix} & p_2 &= \begin{pmatrix} E \\ 0 \\ 0 \\ -\omega \end{pmatrix} & p_3 &= \begin{pmatrix} \omega \\ \omega \sin \theta \cos \phi \\ \omega \sin \theta \sin \phi \\ \omega \cos \theta \end{pmatrix} & p_4 &= \begin{pmatrix} E \\ -\omega \sin \theta \cos \phi \\ -\omega \sin \theta \sin \phi \\ -\omega \cos \theta \end{pmatrix} \\
 &\text{inbound photon} & &\text{inbound electron} & &\text{outbound photon} & &\text{outbound electron} \\
 u_{21} &= \begin{pmatrix} E + m \\ 0 \\ -\omega \\ 0 \end{pmatrix} & & & & u_{41} &= \begin{pmatrix} E + m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix} \\
 &\text{spin up} & & & & &\text{spin up} \\
 u_{22} &= \begin{pmatrix} 0 \\ E + m \\ 0 \\ \omega \end{pmatrix} & & & & u_{42} &= \begin{pmatrix} 0 \\ E + m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix} \\
 &\text{spin down} & & & & &\text{spin down}
 \end{aligned}$$

The spinors are not individually normalized. Instead, a combined spinor normalization constant  $N = (E + m)^2$  will be used.

This is the probability density for spin state  $ab$ . The formula is derived from Feynman diagrams for Compton scattering.

$$|\mathcal{M}_{ab}|^2 = \frac{e^4}{N} \left| -\frac{\bar{u}_{4b} \gamma^\mu (\not{q}_1 + m) \gamma^\nu u_{2a}}{s - m^2} - \frac{\bar{u}_{4b} \gamma^\nu (\not{q}_2 + m) \gamma^\mu u_{2a}}{u - m^2} \right|^2$$

Symbol  $e$  is electron charge and

$$\begin{aligned}
 \not{q}_1 &= (p_1 + p_2)^\mu g_{\mu\nu} \gamma^\nu \\
 \not{q}_2 &= (p_4 - p_1)^\mu g_{\mu\nu} \gamma^\nu
 \end{aligned}$$

Symbols  $s$  and  $u$  are Mandelstam variables

$$s = (p_1 + p_2)^2 = (E + \omega)^2$$

$$u = (p_1 - p_4)^2 = (p_1 - p_4)^\mu g_{\mu\nu} (p_1 - p_4)^\nu$$

Let

$$a_1 = \bar{u}_{4b} \gamma^\mu (\not{p}_1 + m) \gamma^\nu u_{2a}, \quad a_2 = \bar{u}_{4b} \gamma^\nu (\not{p}_2 + m) \gamma^\mu u_{2a}$$

Then

$$\begin{aligned} |\mathcal{M}_{ab}|^2 &= \frac{e^4}{N} \left| -\frac{a_1}{s-m^2} - \frac{a_2}{u-m^2} \right|^2 \\ &= \frac{e^4}{N} \left( -\frac{a_1}{s-m^2} - \frac{a_2}{u-m^2} \right) \left( -\frac{a_1}{s-m^2} - \frac{a_2}{u-m^2} \right)^* \\ &= \frac{e^4}{N} \left( \frac{a_1 a_1^*}{(s-m^2)^2} + \frac{a_1 a_2^*}{(s-m^2)(u-m^2)} + \frac{a_1^* a_2}{(s-m^2)(u-m^2)} + \frac{a_2 a_2^*}{(u-m^2)^2} \right) \end{aligned}$$

The expected probability density  $\langle |\mathcal{M}|^2 \rangle$  is computed by summing  $|\mathcal{M}_{ab}|^2$  over all spin and polarization states and then dividing by the number of inbound states. There are four inbound states. The sum over polarizations is already accomplished by contraction of  $aa^*$  over  $\mu$  and  $\nu$ .

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{1}{4} \sum_{a=1}^2 \sum_{b=1}^2 |\mathcal{M}_{ab}|^2 \\ &= \frac{e^4}{4N} \sum_{a=1}^2 \sum_{b=1}^2 \left( \frac{a_1 a_1^*}{(s-m^2)^2} + \frac{a_1 a_2^*}{(s-m^2)(u-m^2)} + \frac{a_1^* a_2}{(s-m^2)(u-m^2)} + \frac{a_2 a_2^*}{(u-m^2)^2} \right) \end{aligned}$$

The Casimir trick uses matrix arithmetic to compute sums.

$$\begin{aligned} f_{11} &= \frac{1}{N} \sum_{a=1}^2 \sum_{b=1}^2 a_1 a_1^* = \text{Tr} \left( (\not{p}_2 + m) \gamma^\mu (\not{p}_1 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\nu (\not{p}_1 + m) \gamma_\mu \right) \\ f_{12} &= \frac{1}{N} \sum_{a=1}^2 \sum_{b=1}^2 a_1 a_2^* = \text{Tr} \left( (\not{p}_2 + m) \gamma^\mu (\not{p}_2 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\mu (\not{p}_1 + m) \gamma_\nu \right) \\ f_{22} &= \frac{1}{N} \sum_{a=1}^2 \sum_{b=1}^2 a_2 a_2^* = \text{Tr} \left( (\not{p}_2 + m) \gamma^\mu (\not{p}_2 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\nu (\not{p}_2 + m) \gamma_\mu \right) \end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left( \frac{f_{11}}{(s-m^2)^2} + \frac{f_{12}}{(s-m^2)(u-m^2)} + \frac{f_{12}^*}{(s-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2} \right) \quad (1)$$

The following formulas are equivalent to the Casimir trick. (Recall that  $a \cdot b = a^\mu g_{\mu\nu} b^\nu$ )

$$\begin{aligned} f_{11} &= 32(p_1 \cdot p_2)(p_1 \cdot p_4) + 64m^2(p_1 \cdot p_2) - 32m^2(p_1 \cdot p_3) - 32m^2(p_1 \cdot p_4) + 32m^4 \\ f_{12} &= 16m^2(p_1 \cdot p_2) - 16m^2(p_1 \cdot p_4) + 32m^4 \\ f_{22} &= 32(p_1 \cdot p_2)(p_1 \cdot p_4) + 32m^2(p_1 \cdot p_2) - 32m^2(p_1 \cdot p_3) - 64m^2(p_1 \cdot p_4) + 32m^4 \end{aligned}$$

For Mandelstam variables

$$\begin{aligned}s &= (p_1 + p_2)^2 \\ t &= (p_1 - p_3)^2 \\ u &= (p_1 - p_4)^2\end{aligned}$$

the formulas are

$$\begin{aligned}f_{11} &= -8su + 24sm^2 + 8um^2 + 8m^4 \\ f_{12} &= 8sm^2 + 8um^2 + 16m^4 \\ f_{22} &= -8su + 8sm^2 + 24um^2 + 8m^4\end{aligned}\tag{2}$$

Compton scattering experiments are typically done in the lab frame where the electron is at rest. Apply Lorentz boost  $\Lambda$  to transform momentum vectors to the lab frame.

$$\Lambda = \begin{pmatrix} E/m & 0 & 0 & \omega/m \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega/m & 0 & 0 & E/m \end{pmatrix}, \quad \Lambda p_2 = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Mandelstam variables are invariant under a boost.

$$\begin{aligned}s &= (p_1 + p_2)^2 = (\Lambda p_1 + \Lambda p_2)^2 \\ t &= (p_1 - p_3)^2 = (\Lambda p_1 - \Lambda p_3)^2 \\ u &= (p_1 - p_4)^2 = (\Lambda p_1 - \Lambda p_4)^2\end{aligned}$$

In the lab frame, let  $\omega_L$  be the angular frequency of the incident photon and let  $\omega'_L$  be the angular frequency of the scattered photon.

$$\begin{aligned}\omega_L &= \Lambda p_1 \cdot (1, 0, 0, 0) = \frac{\omega^2}{m} + \frac{\omega E}{m} \\ \omega'_L &= \Lambda p_3 \cdot (1, 0, 0, 0) = \frac{\omega^2 \cos \theta}{m} + \frac{\omega E}{m}\end{aligned}$$

It follows that

$$\begin{aligned}s &= (p_1 + p_2)^2 = m^2 + 2m\omega_L \\ t &= (p_1 - p_3)^2 = 2m(\omega'_L - \omega_L) \\ u &= (p_1 - p_4)^2 = m^2 - 2m\omega'_L\end{aligned}$$

Then from equations (1) and (2)

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left( \frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \left( \frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1 \right)^2 - 1 \right)$$

Lab scattering angle  $\theta_L$  is given by the Compton formula.

$$\cos \theta_L = \frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1$$

Hence

$$\begin{aligned}\langle |\mathcal{M}|^2 \rangle &= 2e^4 \left( \frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \cos^2 \theta_L - 1 \right) \\ &= 2e^4 \left( \frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} - \sin^2 \theta_L \right)\end{aligned}$$

## Cross section

Now that we have derived  $\langle |\mathcal{M}|^2 \rangle$  we can investigate the angular distribution of scattered photons. For simplicity let us drop the  $L$  subscript from lab variables. From now on the symbols  $\omega$ ,  $\omega'$ , and  $\theta$  will be lab frame variables.

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4(4\pi\epsilon_0)^2 s} \left( \frac{\omega'}{\omega} \right)^2 \langle |\mathcal{M}|^2 \rangle$$

where

$$s = m^2 + 2m\omega = (mc^2)^2 + 2(mc^2)(\hbar\omega)$$

For the lab frame we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

Substitute for  $\langle |\mathcal{M}|^2 \rangle$ .

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\epsilon_0)^2 s} \left( \frac{\omega'}{\omega} \right)^2 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

Noting that

$$e^2 = 4\pi\epsilon_0\alpha\hbar c$$

we can also write

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s} \left( \frac{\omega'}{\omega} \right)^2 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

The scattered photon frequency  $\omega'$  is computed from the Compton equation.

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos \theta)}$$

We can integrate  $d\sigma$  to obtain a cumulative distribution function. Let  $I(\theta)$  be the following integral of  $d\sigma$ . (The  $\sin \theta$  is due to  $d\Omega = \sin \theta d\theta d\phi$ .)

$$I(\theta) = \int \left( \frac{\omega'}{\omega} \right)^2 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right) \sin \theta d\theta$$

The result is

$$\begin{aligned} I(\theta) = & -\frac{\cos \theta}{R^2} + \log(1 + R(1 - \cos \theta)) \left( \frac{1}{R} - \frac{2}{R^2} - \frac{2}{R^3} \right) \\ & - \frac{1}{2R(1 + R(1 - \cos \theta))^2} + \frac{1}{1 + R(1 - \cos \theta)} \left( -\frac{2}{R^2} - \frac{1}{R^3} \right) \end{aligned}$$

where

$$R = \frac{\hbar\omega}{mc^2}$$

The cumulative distribution function is

$$F(\theta) = \frac{I(\theta) - I(0)}{I(\pi) - I(0)}, \quad 0 \leq \theta \leq \pi$$

The probability of observing scattering events in the interval  $\theta_1$  to  $\theta_2$  is

$$P(\theta_1 \leq \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

Let  $N$  be the number of scattering events from an experiment. Then the number of scattering events in the interval  $\theta_1$  to  $\theta_2$  is predicted to be

$$N (F(\theta_2) - F(\theta_1))$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi) - I(0)} \left( \frac{\omega'}{\omega} \right)^2 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right) \sin \theta$$

Note that if we had carried through the  $\alpha^2(\hbar c)^2/2s$  in  $I(\theta)$ , it would have cancelled out in  $F(\theta)$ .

## Thomson scattering

For  $\hbar\omega \ll mc^2$  we have

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2} (1 - \cos \theta)} \approx \omega$$

Hence we can use the approximations

$$\omega = \omega' \quad \text{and} \quad s = (mc^2)^2$$

to obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \hbar^2}{2m^2 c^2} (1 + \cos^2 \theta)$$

which is the formula for Thomson scattering.

## High energy approximation

For  $\omega \gg m$  a useful approximation is to set  $m = 0$  and obtain

$$\begin{aligned} f_{11} &= -8su \\ f_{12} &= 0 \\ f_{22} &= -8su \end{aligned}$$

Hence

$$\begin{aligned}\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left( \frac{-8su}{s^2} + \frac{-8su}{u^2} \right) \\ &= 2e^4 \left( -\frac{u}{s} - \frac{s}{u} \right)\end{aligned}$$

Also for  $m = 0$  the Mandelstam variables  $s$  and  $u$  are

$$\begin{aligned}s &= 4\omega^2 \\ u &= -2\omega^2(\cos\theta + 1)\end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left( \frac{\cos\theta + 1}{2} + \frac{2}{\cos\theta + 1} \right)$$

## Notes

Here are a few notes regarding the Eigenmath scripts.

Start by writing out  $a_1$  and  $a_2$  in full component form.

$$a_1^{\mu\nu} = \bar{u}_{4\alpha} \gamma^{\mu\alpha}{}_{\beta} (\not{q}_1 + m)^{\beta}{}_{\rho} \gamma^{\nu\rho}{}_{\sigma} u_2^{\sigma}, \quad a_2^{\nu\mu} = \bar{u}_{4\alpha} \gamma^{\nu\alpha}{}_{\beta} (\not{q}_2 + m)^{\beta}{}_{\rho} \gamma^{\mu\rho}{}_{\sigma} u_2^{\sigma}$$

Transpose  $\gamma$  tensors to form inner products over  $\alpha$  and  $\rho$ .

$$a_1^{\mu\nu} = \bar{u}_{4\alpha} \gamma^{\alpha\mu}{}_{\beta} (\not{q}_1 + m)^{\beta}{}_{\rho} \gamma^{\rho\nu}{}_{\sigma} u_2^{\sigma}, \quad a_2^{\nu\mu} = \bar{u}_{4\alpha} \gamma^{\alpha\nu}{}_{\beta} (\not{q}_2 + m)^{\beta}{}_{\rho} \gamma^{\rho\mu}{}_{\sigma} u_2^{\sigma}$$

Convert transposed  $\gamma$  to Eigenmath code.

$$\gamma^{\alpha\mu}{}_{\beta} \quad \rightarrow \quad \text{gammaT} = \text{transpose}(\text{gamma})$$

Then to compute  $a_1$  we have

$$\begin{aligned}a_1 &= \bar{u}_{4\alpha} \gamma^{\alpha\mu}{}_{\beta} (\not{q}_1 + m)^{\beta}{}_{\rho} \gamma^{\rho\nu}{}_{\sigma} u_2^{\sigma} \\ &\rightarrow \quad \text{a1} = \text{dot}(\text{u4bar}[\text{s4}], \text{gammaT}, \text{qslash1} + \text{m I}, \text{gammaT}, \text{u2}[\text{s2}])\end{aligned}$$

where  $s_2$  and  $s_4$  are spin indices. Similarly for  $a_2$  we have

$$\begin{aligned}a_2 &= \bar{u}_{4\alpha} \gamma^{\alpha\nu}{}_{\beta} (\not{q}_2 + m)^{\beta}{}_{\rho} \gamma^{\rho\mu}{}_{\sigma} u_2^{\sigma} \\ &\rightarrow \quad \text{a2} = \text{dot}(\text{u4bar}[\text{s4}], \text{gammaT}, \text{qslash2} + \text{m I}, \text{gammaT}, \text{u2}[\text{s2}])\end{aligned}$$

In component notation the product  $a_1 a_1^*$  is

$$a_1 a_1^* = a_1^{\mu\nu} a_1^{*\mu\nu}$$

To sum over  $\mu$  and  $\nu$  it is necessary to lower indices with the metric tensor. Also, transpose  $a_1^*$  to form an inner product with  $\nu$ .

$$a_1 a_1^* = a_1^{\mu\nu} a_{1\nu\mu}^*$$

Convert to Eigenmath code. The dot function sums over  $\nu$  and the contract function sums over  $\mu$ .

$$a_1 a_1^* \rightarrow \text{a11} = \text{contract}(\text{dot}(\text{a1}, \text{gmunu}, \text{transpose}(\text{conj}(\text{a1}))), \text{gmunu})$$

Similarly for  $a_2 a_2^*$  we have

$$a_2 a_2^* \rightarrow \text{a22} = \text{contract}(\text{dot}(\text{a2}, \text{gmunu}, \text{transpose}(\text{conj}(\text{a2}))), \text{gmunu})$$

The product  $a_1 a_2^*$  does not require a transpose because  $a_1 a_2^* = a_1^{\mu\nu} a_2^{*\nu\mu}$ .

$$a_1 a_2^* \rightarrow \text{a12} = \text{contract}(\text{dot}(\text{a1}, \text{gmunu}, \text{conj}(\text{a2})), \text{gmunu})$$

In component notation, a trace operator becomes a sum over an index, in this case  $\alpha$ .

$$\begin{aligned} f_{11} &= \text{Tr} \left( (\not{p}_2 + m) \gamma^\mu (\not{q}_1 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\nu (\not{q}_1 + m) \gamma_\mu \right) \\ &= (\not{p}_2 + m)^\alpha_\beta \gamma^{\mu\beta}_\rho (\not{q}_1 + m)^\rho_\sigma \gamma^{\nu\sigma}_\tau (\not{p}_4 + m)^\tau_\delta \gamma^\delta_{\nu\eta} (\not{q}_1 + m)^\eta_\xi \gamma^\xi_{\mu\alpha} \end{aligned}$$

As before, transpose  $\gamma$  tensors to form inner products.

$$f_{11} = (\not{p}_2 + m)^\alpha_\beta \gamma^{\beta\mu}_\rho (\not{q}_1 + m)^\rho_\sigma \gamma^{\sigma\nu}_\tau (\not{p}_4 + m)^\tau_\delta \gamma^\delta_{\nu\eta} (\not{q}_1 + m)^\eta_\xi \gamma^\xi_{\mu\alpha}$$

To convert to Eigenmath code, use an intermediate variable for the inner product.

$$T^{\alpha\mu\nu}_{\nu\mu\alpha} \rightarrow \text{T} = \text{dot}(\text{P2}, \text{gammaT}, \text{Q1}, \text{gammaT}, \text{P4}, \text{gammaL}, \text{Q1}, \text{gammaL})$$

Now sum over the indices of  $T$ . The innermost contract sums over  $\nu$  then the next contract sums over  $\mu$ . Finally the outermost contract sums over  $\alpha$ .

$$f_{11} \rightarrow \text{f11} = \text{contract}(\text{contract}(\text{contract}(\text{T}, 3, 4), 2, 3))$$

Follow suit for  $f_{22}$ . For  $f_{12}$  the order of the rightmost  $\mu$  and  $\nu$  is reversed.

$$f_{12} = \text{Tr} \left( (\not{p}_2 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\mu (\not{q}_1 + m) \gamma_\nu \right)$$

The resulting inner product is  $T^{\alpha\mu\nu}_{\mu\nu\alpha}$  so the contraction is different.

$$f_{12} \rightarrow \text{f12} = \text{contract}(\text{contract}(\text{contract}(\text{T}, 3, 5), 2, 3))$$

The innermost contract sums over  $\nu$  followed by sum over  $\mu$  then sum over  $\alpha$ .