

2-2. For a harmonic oscillator $L = (m/2)(\dot{x}^2 - \omega^2 x^2)$. With T equal to $t_b - t_a$, show that the classical action is

$$S_{cl} = \frac{m\omega}{2 \sin \omega T} ((x_b^2 + x_a^2) \cos \omega T - 2x_b x_a) \quad (2.9)$$

We will need equation (2.7) to determine the classical motion $x(t)$.

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \quad (2.7)$$

For the Lagrangian L given in problem 2-2 we have

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = m\ddot{x} \quad \frac{\partial L}{\partial x} = -m\omega^2 x$$

By equation (2.7) we have for the classical acceleration $\ddot{x}(t)$

$$\ddot{x}(t) = -\omega^2 x \quad (1)$$

The well-known solution to (1) is

$$x(t) = A \sin(\omega t) + B \cos(\omega t)$$

We have the following boundary conditions.

$$x(0) = x_a \quad x(T) = x_b$$

Solve for B .

$$B = x(0) = x_a$$

For $x(T)$ we have

$$x(T) = A \sin(\omega T) + B \cos(\omega T)$$

Solve for A .

$$A = \frac{x(T) - B \cos(\omega T)}{\sin(\omega T)} = \frac{x_b - x_a \cos(\omega T)}{\sin(\omega T)}$$

Hence the equation of motion is

$$x(t) = \frac{x_b - x_a \cos(\omega T)}{\sin(\omega T)} \sin(\omega t) + x_a \cos(\omega t) \quad (2)$$

Differentiate $x(t)$ to obtain velocity $\dot{x}(t)$.

$$\dot{x}(t) = \frac{d}{dt}x(t) = \omega \left(\frac{x_b - x_a \cos(\omega T)}{\sin(\omega T)} \cos(\omega t) - x_a \sin(\omega t) \right) \quad (3)$$

Using the action integral

$$S = \int_0^T L dt$$

we have for the classical action

$$\begin{aligned} S_{cl} &= \frac{m}{2} \int_0^T (\dot{x}^2 - \omega^2 x^2) dt \\ &= \frac{m}{2} \left(\int_0^T \dot{x}^2 dt - \int_0^T \omega^2 x^2 dt \right) \end{aligned}$$

Let $u = v = \dot{x}$ and note that

$$\begin{aligned} \dot{u} &= \ddot{x} \\ \int v dt &= x \end{aligned}$$

Apply integration by parts to the integral of \dot{x}^2 .

$$\begin{aligned} \int_0^T \dot{x}^2 dt &= \int_0^T uv dt \\ &= \left(u \int v dt \right)_0^T - \int_0^T \dot{u} \left(\int v dt \right) dt \\ &= \dot{x}x \Big|_0^T - \int_0^T \ddot{x}x dt \end{aligned}$$

Hence

$$S_{cl} = \frac{m}{2} \left(\dot{x}x \Big|_0^T - \int_0^T \ddot{x}x dt - \int_0^T \omega^2 x^2 dt \right)$$

The two integrals on the right cancel by $\ddot{x} = -\omega^2 x$ from equation (1).

We now have

$$S_{cl} = \frac{m}{2} \dot{x}x \Big|_0^T = \frac{m}{2} \dot{x}(T)x(T) - \frac{m}{2} \dot{x}(0)x(0) \quad (4)$$

By evaluation of $x(t)$ and $\dot{x}(t)$ at $t = T$ and $t = 0$ we have

$$S_{cl} = \frac{m\omega}{2\sin(\omega T)} \left((x_b^2 + x_a^2) \cos(\omega T) - 2x_b x_a \right) \quad (5)$$