Fun trick

Show that

$$[p^2, \mathbf{r}] = -2i\hbar\mathbf{p}$$

where

$$\mathbf{r} = \otimes(x, y, z), \quad \mathbf{p} = -i\hbar\nabla, \quad p^2 = \mathbf{p}\cdot\mathbf{p} = -\hbar^2\nabla^2$$

We have

$$\begin{aligned}
[p^2, \mathbf{r}] &= p^2\mathbf{r} - \mathbf{r}p^2 \\
&= \mathbf{p}\cdot\mathbf{p}\mathbf{r} - \mathbf{r}\mathbf{p}\cdot\mathbf{p} \\
&= \mathrm{Tr}[\mathbf{ppr} - \mathbf{prp} + \mathbf{prp} - \mathbf{rpp}] && \text{trick!} \\
&= \mathrm{Tr}[\mathbf{p}(\mathbf{pr} - \mathbf{rp}) + (\mathbf{pr} - \mathbf{rp})\mathbf{p}] \\
&= \mathbf{p}(-i\hbar) + (-i\hbar)\mathbf{p} \\
&= -2i\hbar\mathbf{p}
\end{aligned}$$

where Tr means trace (contraction of indices 1 and 2) and $\mathbf{I}$ is the 3×3 identity matrix.

Verify the following formulas.

$$[p^2, \mathbf{r}] = -2i\hbar\mathbf{p} \tag{1}$$

$$[p^2, \mathbf{r}] = \mathrm{Tr}[\mathbf{ppr} - \mathbf{prp} + \mathbf{prp} - \mathbf{rpp}] \tag{2}$$

$$\mathbf{pr} - \mathbf{rp} = -i\hbar\mathbf{I} \tag{3}$$

$$\mathbf{p}\cdot\mathbf{p} = \mathrm{Tr}[\mathbf{pp}] \tag{4}$$