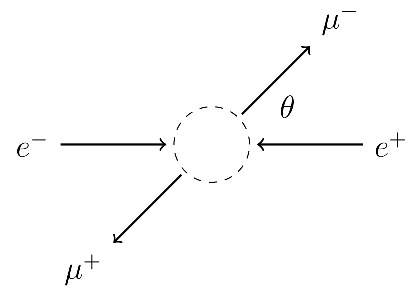
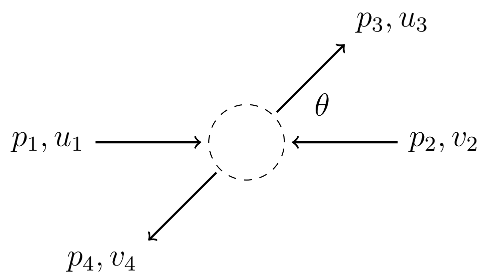
A high energy electron and positron collision can create two muons.



Here is the same diagram with momentum and spinor labels.



In a typical collider experiment the momentum vectors are

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \qquad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} \qquad p_3 = \begin{pmatrix} E \\ \rho \sin\theta \cos\phi \\ \rho \sin\theta \sin\phi \\ \rho \cos\theta \end{pmatrix} \qquad p_4 = \begin{pmatrix} E \\ -\rho \sin\theta \cos\phi \\ -\rho \sin\theta \sin\phi \\ -\rho \cos\theta \end{pmatrix}$$

where E is beam energy, $p = \sqrt{E^2 - m^2}$, $\rho = \sqrt{E^2 - M^2}$, m is electron mass $0.51\,\text{MeV}$, and M is muon mass $106\,\text{MeV}$. The spinors are

$$u_{11} = \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix} \quad v_{21} = \begin{pmatrix} -p \\ 0 \\ E+m \\ 0 \end{pmatrix} \quad u_{31} = \begin{pmatrix} E+M \\ 0 \\ p_3^z \\ p_3^x + ip_3^y \end{pmatrix} \quad v_{41} = \begin{pmatrix} p_4^z \\ p_4^x + ip_4^y \\ E+M \\ 0 \end{pmatrix}$$

$$u_{12} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix} \quad v_{22} = \begin{pmatrix} 0 \\ p \\ 0 \\ E+m \end{pmatrix} \quad u_{32} = \begin{pmatrix} 0 \\ E+M \\ p_3^x - ip_3^y \\ -p_3^z \end{pmatrix} \quad v_{42} = \begin{pmatrix} p_4^x - ip_4^y \\ -p_4^z \\ 0 \\ E+M \end{pmatrix}$$

The last digit in a spinor subscript is 1 for spin up and 2 for spin down. Note that the spinors are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^2(E+M)^2$ will be used where needed.

This is the probability density for muon production. Symbol $s = (p_1 + p_2)^2 = 4E^2$, symbol s_j selects the spin of spinor j, and e is electron charge.

$$|\mathcal{M}(s_1, s_2, s_3, s_4)|^2 = \frac{e^4}{s^2} \frac{1}{N} \left|(\bar{u}_3 \gamma_\mu v_4)(\bar{v}_2 \gamma^\mu u_1)\right|^2$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is computed by summing $|\mathcal{M}|^2$ over all spin states and dividing by the number of inbound states. There are four inbound states.

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{s_1=1}^{2}\sum_{s_2=1}^{2}\sum_{s_3=1}^{2}\sum_{s_4=1}^{2}|\mathcal{M}(s_1,s_2,s_3,s_4)|^2 \\ &= \frac{e^4}{4s^2}\sum_{s_1=1}^{2}\sum_{s_2=1}^{2}\sum_{s_3=1}^{2}\sum_{s_4=1}^{2}\frac{1}{N}\left|(\bar{u}_3\gamma_\mu v_4)(\bar{v}_2\gamma^\mu u_1)\right|^2\end{aligned}$$

Another way to compute $\langle|\mathcal{M}|^2\rangle$ is to use the Casimir trick.

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4s^2}\operatorname{Tr}\Big((\not{p}_3 + M)\gamma^\mu(\not{p}_4 - M)\gamma^\nu\Big)\operatorname{Tr}\Big((\not{p}_2 - m)\gamma_\mu(\not{p}_1 + m)\gamma_\nu\Big)$$

Here is a third way to compute $\langle|\mathcal{M}|^2\rangle$.

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4s^2}\big(32(p_1\cdot p_3)(p_2\cdot p_4) + 32(p_1\cdot p_4)(p_2\cdot p_3) + 32m^2(p_3\cdot p_4) + 32M^2(p_1\cdot p_2) + 64m^2M^2\big)$$

For the momentum vectors given above the result is

$$\langle|\mathcal{M}|^2\rangle = e^4\left(1+\cos^2\theta + \frac{m^2+M^2}{E^2}\sin^2\theta + \frac{m^2M^2}{E^4}\cos^2\theta\right)$$

The Stanford Linear Collider had a collision energy of $2E = 91$ GeV. For beam energies such as SLC where $E \gg M$ the above equation can be approximated as

$$\langle|\mathcal{M}|^2\rangle = e^4(1+\cos^2\theta)$$

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{64\pi^2 s} = \frac{e^4}{256\pi^2E^2}(1+\cos^2\theta)$$

Recall that $e^2 = 4\pi\alpha$ hence

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}(1+\cos^2\theta)$$

The total cross section calculation requires the following definite integral.

$$\int_\Omega (1+\cos^2\theta)\,d\Omega = \int_0^{2\pi}\int_0^{\pi}(1+\cos^2\theta)\sin\theta\,d\theta\,d\phi = \frac{8}{3}\int_0^{2\pi}d\phi = \frac{16\pi}{3}$$

Hence the total cross section is

$$\sigma = \int_\Omega d\sigma = \int_\Omega \frac{\alpha^2}{16E^2}(1+\cos^2\theta)\,d\Omega = \frac{\alpha^2}{16E^2}\frac{16\pi}{3} = \frac{\pi\alpha^2}{3E^2}$$

We can integrate the differential cross section to obtain a cumulative distribution function.

Let

$$I(\xi) = 2\pi \int_0^\xi \frac{d\sigma}{d\Omega} \sin\theta \, d\theta, \qquad 0 \le \xi \le \pi$$

The result is

$$I(\xi) = 2\pi \left(\frac{\alpha^2}{16E^2}\right)\left(-\frac{1}{3}\cos^3\xi - \cos\xi + \frac{4}{3}\right)$$

The cumulative distribution function is

$$F(\theta) = \frac{I(\theta)}{I(\pi)}, \qquad 0 \le \theta \le \pi$$

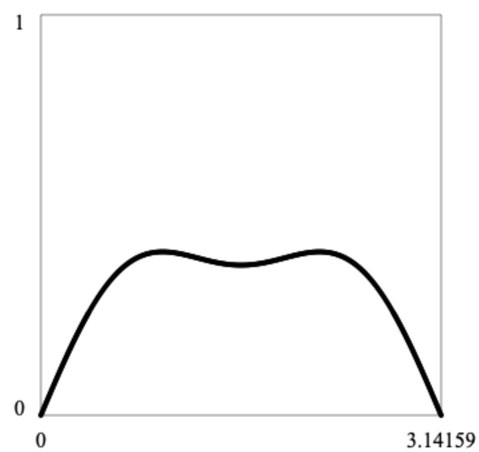
Hence

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

The normalized probability density is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{3}{8}(1 + \cos^2\theta)\sin\theta, \qquad 0 \le \theta \le \pi$$

Run "muon-production-5.txt" to draw the probability density function.



Run "muon-production-1.txt" to verify that

$$\frac{1}{N}\sum_{s_1=1}^{2}\sum_{s_2=1}^{2}\sum_{s_3=1}^{2}\sum_{s_4=1}^{2}|(\bar{u}_3\gamma_\mu v_4)(\bar{v}_2\gamma^\mu u_1)|^2 = \operatorname{Tr}\Big((\not{p}_3 + M)\gamma^\mu(\not{p}_4 - M)\gamma^\nu\Big)\operatorname{Tr}\Big((\not{p}_2 - m)\gamma_\mu(\not{p}_1 + m)\gamma_\nu\Big)$$

Run "muon-production-2.txt" to verify that

$$\frac{1}{64E^4}\operatorname{Tr}\Big((\not{p}_3 + M)\gamma^\mu(\not{p}_4 - M)\gamma^\nu\Big)\operatorname{Tr}\Big((\not{p}_2 - m)\gamma_\mu(\not{p}_1 + m)\gamma_\nu\Big)$$
$$= 1 + \cos^2\theta + \frac{m^2 + M^2}{E^2}\sin^2\theta + \frac{m^2M^2}{E^4}\cos^2\theta$$

and to verify that

$$\operatorname{Tr}\Big((\not{p}_3 + M)\gamma^\mu(\not{p}_4 - M)\gamma^\nu\Big)\operatorname{Tr}\Big((\not{p}_2 - m)\gamma_\mu(\not{p}_1 + m)\gamma_\nu\Big)$$
$$= 32(p_1\cdot p_3)(p_2\cdot p_4) + 32(p_1\cdot p_4)(p_2\cdot p_3) + 32m^2(p_3\cdot p_4) + 32M^2(p_1\cdot p_2) + 64m^2M^2$$

This table shows SLAC-PEP muon pair production data obtained from HEP Data.[1]

x	y
-0.925	67.08
-0.85	58.67
-0.75	54.66
-0.65	51.72
-0.55	43.70
-0.45	41.12
-0.35	39.71
-0.25	35.34
-0.15	33.35
-0.05	34.69
0.05	34.05
0.15	34.48
0.25	34.66
0.35	35.23
0.45	35.60
0.55	40.13
0.65	42.56
0.75	46.37
0.85	49.28
0.925	55.70

Data x and y have the following relationship with cross section parameters.

$$x = \cos\theta \qquad y = (2E)^2 \frac{d\sigma}{d\cos\theta}$$

The differential cross section for muon production is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}(1 + \cos^2\theta)$$

Let us compute predicted values $\hat{y}$ from the cross section formula. Start by finding the relationship between $d\Omega$ and $d\cos\theta$. Since $1 + \cos^2\theta$ has no dependence on ϕ we have

$$\int_\Omega (1 + \cos^2\theta)\, d\Omega = \int_0^{2\pi} \int_0^{\pi} (1 + \cos^2\theta) \sin\theta \, d\theta \, d\phi = 2\pi \int_0^{\pi} (1 + \cos^2\theta) \sin\theta \, d\theta$$

Hence

$$d\Omega = 2\pi \sin\theta \, d\theta = -2\pi \, d\cos\theta$$

We want positive cross sections so drop the minus sign and set

$$\frac{d\sigma}{d\cos\theta} = 2\pi \frac{d\sigma}{d\Omega}$$

[1] `www.hepdata.net/record/ins216031` (Table 1, 29.0 GeV)

We can now write

$$
\begin{aligned}
y &= (2E)^2 \frac{d\sigma}{d\cos\theta} \\
&= (2E)^2 (2\pi) \frac{d\sigma}{d\Omega} \\
&= (2E)^2 (2\pi) \frac{\alpha^2}{16E^2} (1 + \cos^2\theta) \\
&= \frac{\pi\alpha^2}{2} (1 + \cos^2\theta)
\end{aligned}
$$

Multiply by $(\hbar c)^2$ to convert to SI and multiply by 10^{37} to convert square meters to nanobarns.

$$
y = \frac{\pi\alpha^2}{2} (1 + \cos^2\theta) \times (\hbar c)^2 \times 10^{37}
$$

Replace $\cos\theta$ with explanatory variable x to obtain $\hat{y}$.

$$
\hat{y} = \frac{\pi\alpha^2}{2} (1 + x^2) \times (\hbar c)^2 \times 10^{37}
$$

Here are the predicted values $\hat{y}$ based on the above formula.

x	y	$\hat{y}$
−0.925	67.08	60.44
−0.85	58.67	56.10
−0.75	54.66	50.89
−0.65	51.72	46.33
−0.55	43.70	42.42
−0.45	41.12	39.17
−0.35	39.71	36.56
−0.25	35.34	34.61
−0.15	33.35	33.30
−0.05	34.69	32.65
0.05	34.05	32.65
0.15	34.48	33.30
0.25	34.66	34.61
0.35	35.23	36.56
0.45	35.60	39.17
0.55	40.13	42.42
0.65	42.56	46.33
0.75	46.37	50.89
0.85	49.28	56.10
0.925	55.70	60.44

The coefficient of determination R^2 measures how well predicted values fit the real data.

$$
R^2 = 1 - \frac{\sum (y - \hat{y})^2}{\sum (y - \bar{y})^2} = 0.87
$$

The result indicates that the model $d\sigma$ explains 87% of the variance in the data.

Run "muon-production-3.txt" to compute the above results.

The following differential cross section formula from electroweak theory results in a better fit to the data.[2]

$$\frac{d\sigma}{d\Omega} = F(s)(1+\cos^2\theta) + G(s)\cos\theta$$

where

$$F(s) = \frac{\alpha^2}{4s}\left(1 + \frac{g_V^2}{\sqrt{2}\pi}\left(\frac{m_Z^2}{s-m_Z^2}\right)\left(\frac{sG}{\alpha}\right) + \frac{(g_A^2+g_V^2)^2}{8\pi^2}\left(\frac{m_Z^2}{s-m_Z^2}\right)^2\left(\frac{sG}{\alpha}\right)^2\right)$$

$$G(s) = \frac{\alpha^2}{4s}\left(\frac{\sqrt{2}g_A^2}{\pi}\left(\frac{m_Z^2}{s-m_Z^2}\right)\left(\frac{sG}{\alpha}\right) + \frac{g_A^2 g_V^2}{\pi^2}\left(\frac{m_Z^2}{s-m_Z^2}\right)^2\left(\frac{sG}{\alpha}\right)^2\right)$$

and

$$\begin{aligned} g_A &= -0.5 \\ g_V &= -0.0348 \\ m_Z &= 91.17\,\text{GeV} \\ G &= 1.166\times 10^{-5}\,\text{GeV}^{-2} \end{aligned}$$

The corresponding formula for $\hat{y}$ is

$$\hat{y} = 2\pi\left[F(s)(1+x^2) + G(s)x\right] \times (\hbar c)^2 \times 10^{37}$$

where $\sqrt{s} = 29\,\text{GeV}$ is the center of mass collision energy. Here are the predicted values $\hat{y}$ based on the above formula.

x	y	$\hat{y}$
−0.925	67.08	65.59
−0.85	58.67	60.84
−0.75	54.66	55.07
−0.65	51.72	49.96
−0.55	43.70	45.49
−0.45	41.12	41.69
−0.35	39.71	38.53
−0.25	35.34	36.02
−0.15	33.35	34.17
−0.05	34.69	32.97
0.05	34.05	32.42
0.15	34.48	32.53
0.25	34.66	33.28
0.35	35.23	34.69
0.45	35.60	36.75
0.55	40.13	39.47
0.65	42.56	42.83
0.75	46.37	46.85
0.85	49.28	51.52
0.925	55.70	55.45

[2] F. Mandl and G. Shaw, *Quantum Field Theory Revised Edition*, 316.

The coefficient of determination R^2 is

$$R^2 = 1 - \frac{\sum(y - \hat{y})^2}{\sum(y - \bar{y})^2} = 0.98$$

The result indicates that electroweak theory explains 98% of the variance in the data.

Run “muon-production-4.txt” to verify.

Here are a few notes about how the scripts work.

In component notation the traces become sums over the repeated index α.

$$\operatorname{Tr}\Big((\not{p}_3 + M)\gamma^\mu(\not{p}_4 - M)\gamma^\nu\Big) = (\not{p}_3 + M)^\alpha{}_\beta\gamma^{\mu\beta}{}_\rho(\not{p}_4 - M)^\rho{}_\sigma\gamma^{\nu\sigma}{}_\alpha$$
$$\operatorname{Tr}\Big((\not{p}_2 - m)\gamma_\mu(\not{p}_1 + m)\gamma_\nu\Big) = (\not{p}_2 - m)^\alpha{}_\beta\gamma_\mu{}^\beta{}_\rho(\not{p}_1 + m)^\rho{}_\sigma\gamma_\nu{}^\sigma{}_\alpha$$

To convert the above formulas to Eigenmath code, the γ tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply γ^μ by the metric tensor to lower the index.

$\gamma^{\beta\mu}{}_\rho$	$\rightarrow$	`gammaT = transpose(gamma)`
$\gamma^\beta{}_{\mu\rho}$	$\rightarrow$	`gammaL = transpose(dot(gmunu,gamma))`

Define the following 4×4 matrices.

$(\not{p}_1 + m)$	$\rightarrow$	`X1 = pslash1 + m I`
$(\not{p}_2 - m)$	$\rightarrow$	`X2 = pslash2 - m I`
$(\not{p}_3 + M)$	$\rightarrow$	`X3 = pslash3 + M I`
$(\not{p}_4 - M)$	$\rightarrow$	`X4 = pslash4 - M I`

Then

$(\not{p}_3 + M)^\alpha{}_\beta\gamma^{\mu\beta}{}_\rho(\not{p}_4 - M)^\rho{}_\sigma\gamma^{\nu\sigma}{}_\alpha$	$\rightarrow$	`T1 = contract(dot(X3,gammaT,X4,gammaT),1,4)`
$(\not{p}_2 - m)^\alpha{}_\beta\gamma_\mu{}^\beta{}_\rho(\not{p}_1 + m)^\rho{}_\sigma\gamma_\nu{}^\sigma{}_\alpha$	$\rightarrow$	`T2 = contract(dot(X2,gammaL,X1,gammaL),1,4)`

Next, multiply matrices and sum over repeated indices. The dot function sums over ν then the contract function sums over μ. The transpose makes the ν indices adjacent as required by the dot function.

$\operatorname{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu)\operatorname{Tr}(\cdots\gamma_\mu\cdots\gamma_\nu)$	$\rightarrow$	`contract(dot(T1,transpose(T2)))`