Spin

Spin state $|s\rangle$ is a normalized vector in $\mathbb{C}^2$.

$$|s\rangle = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}, \quad |c_+|^2 + |c_-|^2 = 1$$

Spin measurement probabilities are the transition probabilities from $|s\rangle$ to an eigenstate.

For spin measurements in the z direction we have

$$\begin{aligned} \Pr\left(S_z = +\tfrac{\hbar}{2}\right) &= |\langle z_+|s\rangle|^2 = |c_+|^2 \\ \Pr\left(S_z = -\tfrac{\hbar}{2}\right) &= |\langle z_-|s\rangle|^2 = |c_-|^2 \end{aligned}$$

where the eigenstates are

$$|z_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |z_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

By definition of expectation value we have

$$\langle S_z\rangle = \tfrac{\hbar}{2}\Pr\left(S_z = +\tfrac{\hbar}{2}\right) - \tfrac{\hbar}{2}\Pr\left(S_z = -\tfrac{\hbar}{2}\right)$$

Rewrite as

$$\langle S_z\rangle = \tfrac{\hbar}{2}|\langle z_+|s\rangle|^2 - \tfrac{\hbar}{2}|\langle z_-|s\rangle|^2$$

Rewrite again as

$$\langle S_z\rangle = \tfrac{\hbar}{2}\langle s|z_+\rangle\langle z_+|s\rangle - \tfrac{\hbar}{2}\langle s|z_-\rangle\langle z_-|s\rangle$$

Then by

$$\langle S_z\rangle = \langle s|S_z|s\rangle$$

we have

$$S_z = \tfrac{\hbar}{2}|z_+\rangle\langle z_+| - \tfrac{\hbar}{2}|z_-\rangle\langle z_-| = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

From the commutator

$$S_+S_- - S_-S_+ = 2\hbar S_z$$

we have

$$S_+S_- - S_-S_+ = \hbar^2|z_+\rangle\langle z_+| - \hbar^2|z_-\rangle\langle z_-|$$

Rewrite as

$$S_+S_- - S_-S_+ = \hbar^2|z_+\rangle\langle z_-|z_-\rangle\langle z_+| - \hbar^2|z_-\rangle\langle z_+|z_+\rangle\langle z_-|$$

Hence

$$\begin{aligned} S_+ &= \hbar|z_+\rangle\langle z_-| = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ S_- &= \hbar|z_-\rangle\langle z_+| = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Then by

$$S_+ = S_x + iS_y$$
$$S_- = S_x - iS_y$$

we obtain

$$S_x = \frac{S_+ + S_-}{2} = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$
$$S_y = \frac{S_+ - S_-}{2i} = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

By solving for the eigenstates in

$$S_x|x_\pm\rangle = \pm\tfrac{\hbar}{2}|x_\pm\rangle$$
$$S_y|y_\pm\rangle = \pm\tfrac{\hbar}{2}|y_\pm\rangle$$

we obtain

$$|x_+\rangle = \frac{|z_+\rangle + |z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$$
$$|x_-\rangle = \frac{|z_+\rangle - |z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$$

and

$$|y_+\rangle = \frac{|z_+\rangle + i|z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$$
$$|y_-\rangle = \frac{|z_+\rangle - i|z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$$

The expected spin direction vector is

$$\langle \mathbf{S}\rangle = \begin{pmatrix}\langle s|S_x|s\rangle\\ \langle s|S_y|s\rangle\\ \langle s|S_z|s\rangle\end{pmatrix}, \quad |\langle \mathbf{S}\rangle| = \frac{\hbar}{2}$$

To convert a direction vector to a spin state, let θ and ϕ be the polar and azimuth angles such that

$$\langle \mathbf{S}\rangle = \frac{\hbar}{2}\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}$$

Then

$$|s\rangle = \begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)\exp(i\phi)\end{pmatrix}$$

By the identities

$$\cos^2(\theta/2) = \frac{\cos\theta + 1}{2}, \quad \sin^2(\theta/2) = \frac{1-\cos\theta}{2}$$

and noting that $0 \leq \theta \leq \pi$ we have

$$\cos(\theta/2) = \sqrt{\frac{\langle z\rangle + 1}{2}}, \quad \sin(\theta/2)\exp(i\phi) = \sqrt{\frac{1-\langle z\rangle}{2}}\frac{\langle x\rangle + i\langle y\rangle}{\sqrt{\langle x\rangle^2 + \langle y\rangle^2}}$$

where

$$\begin{aligned}
\langle x\rangle &= \tfrac{2}{\hbar}\langle S_x\rangle = \sin\theta\cos\phi \\
\langle y\rangle &= \tfrac{2}{\hbar}\langle S_y\rangle = \sin\theta\sin\phi \\
\langle z\rangle &= \tfrac{2}{\hbar}\langle S_z\rangle = \cos\theta
\end{aligned}$$

The following commutator was used to derive S_x and S_y.

$$S_+S_- - S_-S_+ = 2\hbar S_z$$

The commutator is a consequence of the following wave equation for spin.

$$\hat{\mathbf{S}}\psi = (\mathbf{r}\times\hat{\mathbf{p}})\psi, \quad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \hat{\mathbf{p}} = -i\hbar\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

Rewrite in component form.

$$\hat{S}_x\psi = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\psi$$

$$\hat{S}_y\psi = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\psi$$

$$\hat{S}_z\psi = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\psi$$

By computer algebra we have

$$\begin{aligned}
(\hat{S}_y\hat{S}_z - \hat{S}_z\hat{S}_y)\psi &= i\hbar\hat{S}_x\psi \\
(\hat{S}_z\hat{S}_x - \hat{S}_x\hat{S}_z)\psi &= i\hbar\hat{S}_y\psi \\
(\hat{S}_x\hat{S}_y - \hat{S}_y\hat{S}_x)\psi &= i\hbar\hat{S}_z\psi
\end{aligned}$$

Let

$$\begin{aligned}
\hat{S}_+ &= \hat{S}_x + i\hat{S}_y \\
\hat{S}_- &= \hat{S}_x - i\hat{S}_y
\end{aligned}$$

By computer algebra

$$(\hat{S}_+\hat{S}_- - \hat{S}_-\hat{S}_+)\psi = 2\hbar\hat{S}_z\psi$$

Exercises

1. Verify that

$$\begin{aligned}
S_x &= \tfrac{\hbar}{2}(|x_+\rangle\langle x_+| - |x_-\rangle\langle x_-|)\\
S_y &= \tfrac{\hbar}{2}(|y_+\rangle\langle y_+| - |y_-\rangle\langle y_-|)\\
S_z &= \tfrac{\hbar}{2}(|z_+\rangle\langle z_+| - |z_-\rangle\langle z_-|)
\end{aligned}$$

2. Let $|s\rangle$ be the following spin state.

$$|s\rangle = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$$

Verify that $|s\rangle$ is normalized and that

$$\langle \mathbf{S} \rangle = \langle s|\mathbf{S}|s\rangle = \frac{\hbar}{2}\begin{pmatrix} \frac{4}{9} \\ \frac{8}{9} \\ \frac{1}{9} \end{pmatrix}$$

where

$$\mathbf{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$$

Note: In component form we have

$$\langle s|\mathbf{S}|s\rangle = s^*_\beta S^{\alpha\beta}{}_\gamma s^\gamma$$

Eigenmath requires a transpose so that the β indices are adjacent.

$$\langle s|\mathbf{S}|s\rangle = s^*_\beta S^{\beta\alpha}{}_\gamma s^\gamma$$

3. Let $|s\rangle$ be the following spin state.

$$|s\rangle = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$$

Verify the following measurement probabilities for $|s\rangle$.

$$\begin{aligned}
\Pr\left(S_x = +\tfrac{\hbar}{2}\right) &= |\langle x_+|s\rangle|^2 = \tfrac{13}{18}\\
\Pr\left(S_x = -\tfrac{\hbar}{2}\right) &= |\langle x_-|s\rangle|^2 = \tfrac{5}{18}
\end{aligned}$$

$$\begin{aligned}
\Pr\left(S_y = +\tfrac{\hbar}{2}\right) &= |\langle y_+|s\rangle|^2 = \tfrac{17}{18}\\
\Pr\left(S_y = -\tfrac{\hbar}{2}\right) &= |\langle y_-|s\rangle|^2 = \tfrac{1}{18}
\end{aligned}$$

$$\begin{aligned}
\Pr\left(S_z = +\tfrac{\hbar}{2}\right) &= |\langle z_+|s\rangle|^2 = \tfrac{5}{9}\\
\Pr\left(S_z = -\tfrac{\hbar}{2}\right) &= |\langle z_-|s\rangle|^2 = \tfrac{4}{9}
\end{aligned}$$

4. Let $|s\rangle$ be the following spin state.

$$|s\rangle = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$$

Verify that the following spin state $|\chi\rangle$ is indistinguishable from $|s\rangle$.

$$|\chi\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\exp(i\phi) \end{pmatrix}$$

where

$$\cos(\theta/2) = \sqrt{\frac{\langle z\rangle + 1}{2}} = \frac{\sqrt{5}}{3}$$

and

$$\sin(\theta/2)\exp(i\phi) = \sqrt{\frac{1-\langle z\rangle}{2}}\,\frac{\langle x\rangle + i\langle y\rangle}{\sqrt{\langle x\rangle^2 + \langle y\rangle^2}} = \frac{2+4i}{3\sqrt{5}}$$

with

$$\begin{aligned} \langle x\rangle &= \tfrac{2}{\hbar}\langle S_x\rangle \\ \langle y\rangle &= \tfrac{2}{\hbar}\langle S_y\rangle \\ \langle z\rangle &= \tfrac{2}{\hbar}\langle S_z\rangle \end{aligned}$$

5. Verify the following commutators for $\mathbf{S}\psi = (\mathbf{r}\times\mathbf{p})\psi$.

$$\begin{aligned} [S_y, S_z] &= i\hbar S_x \\ [S_z, S_x] &= i\hbar S_y \\ [S_x, S_y] &= i\hbar S_z \end{aligned}$$

$$\begin{aligned} [S^2, S_x] &= 0 \\ [S^2, S_y] &= 0 \\ [S^2, S_z] &= 0 \end{aligned}$$

$$[S_+, S_-] = 2\hbar S_z$$

where

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

and

$$\begin{aligned} S_+ &= S_x + iS_y \\ S_- &= S_x - iS_y \end{aligned}$$