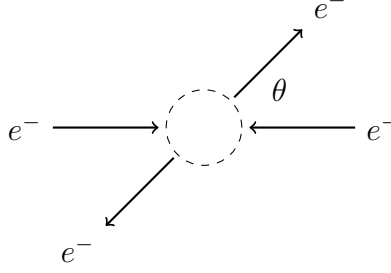
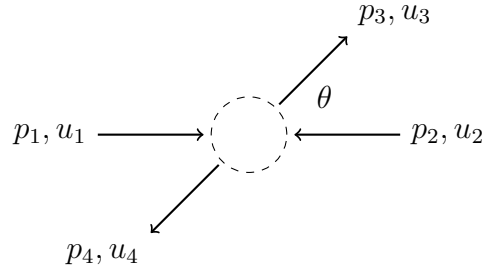


The following diagram shows the geometry of a typical collider experiment that generates electron scattering data.



Here is the same diagram with momentum labels p and Dirac spinor labels u .



In center of mass coordinates the momentum vectors are

$$\begin{aligned}
 p_1 &= \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} & p_2 &= \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} & p_3 &= \begin{pmatrix} E \\ p \sin \theta \cos \phi \\ p \sin \theta \sin \phi \\ p \cos \theta \end{pmatrix} & p_4 &= \begin{pmatrix} E \\ -p \sin \theta \cos \phi \\ -p \sin \theta \sin \phi \\ -p \cos \theta \end{pmatrix} \\
 \text{particle 1} & & \text{particle 2} & & \text{particle 3} & & \text{particle 4}
 \end{aligned}$$

Symbol p is incident momentum, E is total energy $E = \sqrt{p^2 + m^2}$, and m is electron mass. Polar angle θ is the observed scattering angle. Azimuth angle ϕ cancels out in scattering calculations.

The spinors are

$$\begin{aligned}
 u_{11} &= \begin{pmatrix} E + m \\ 0 \\ p \\ 0 \end{pmatrix} & u_{12} &= \begin{pmatrix} 0 \\ E + m \\ 0 \\ -p \end{pmatrix} & u_{21} &= \begin{pmatrix} E + m \\ 0 \\ -p \\ 0 \end{pmatrix} & u_{22} &= \begin{pmatrix} 0 \\ E + m \\ 0 \\ p \end{pmatrix} \\
 \text{particle 1} & & \text{particle 1} & & \text{particle 2} & & \text{particle 2} \\
 \text{spin up} & & \text{spin down} & & \text{spin up} & & \text{spin down} \\
 \\
 u_{31} &= \begin{pmatrix} E + m \\ 0 \\ p_{3z} \\ p_{3x} + ip_{3y} \end{pmatrix} & u_{32} &= \begin{pmatrix} 0 \\ E + m \\ p_{3x} - ip_{3y} \\ -p_{3z} \end{pmatrix} & u_{41} &= \begin{pmatrix} E + m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix} & u_{42} &= \begin{pmatrix} 0 \\ E + m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix} \\
 \text{particle 3} & & \text{particle 3} & & \text{particle 4} & & \text{particle 4} \\
 \text{spin up} & & \text{spin down} & & \text{spin up} & & \text{spin down}
 \end{aligned}$$

The spinors shown above are not individually normalized. Instead, a combined spinor normalization constant $N = (E + m)^4$ will be used.

The following formula computes a probability density $|\mathcal{M}_{abcd}|^2$ for electron scattering where the subscripts $abcd$ are the spin states of the electrons.

$$|\mathcal{M}_{abcd}|^2 = \frac{e^4}{N} \left| \frac{1}{t} (\bar{u}_{3c} \gamma^\mu u_{1a}) (\bar{u}_{4d} \gamma_\mu u_{2b}) - \frac{1}{u} (\bar{u}_{4d} \gamma^\nu u_{1a}) (\bar{u}_{3c} \gamma_\nu u_{2b}) \right|^2$$

from Feynman diagram with
photon exchange
no electron interchange

from Feynman diagram with
photon exchange
electron interchange

Symbol e is electron charge. Symbols t and u are Mandelstam variables

$$t = (p_1 - p_3)^2$$

$$u = (p_1 - p_4)^2$$

Let

$$a_1 = (\bar{u}_{3c} \gamma^\mu u_{1a}) (\bar{u}_{4d} \gamma_\mu u_{2b}) \quad a_2 = (\bar{u}_{4d} \gamma^\nu u_{1a}) (\bar{u}_{3c} \gamma_\nu u_{2b})$$

Then

$$\begin{aligned} |\mathcal{M}_{abcd}|^2 &= \frac{e^4}{N} \left| \frac{a_1}{t} - \frac{a_2}{u} \right|^2 \\ &= \frac{e^4}{N} \left(\frac{a_1}{t} - \frac{a_2}{u} \right) \left(\frac{a_1}{t} - \frac{a_2}{u} \right)^* \\ &= \frac{e^4}{N} \left(\frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{tu} - \frac{a_1^* a_2}{tu} + \frac{a_2 a_2^*}{u^2} \right) \end{aligned}$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is computed by summing $|\mathcal{M}_{abcd}|^2$ over all spin states and dividing by the number of inbound states. There are four inbound states.

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{1}{4} \sum_{a=1}^2 \sum_{b=1}^2 \sum_{c=1}^2 \sum_{d=1}^2 |\mathcal{M}_{abcd}|^2 \\ &= \frac{e^4}{4N} \sum_{a=1}^2 \sum_{b=1}^2 \sum_{c=1}^2 \sum_{d=1}^2 \left(\frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{tu} - \frac{a_1^* a_2}{tu} + \frac{a_2 a_2^*}{u^2} \right) \end{aligned}$$

Use the Casimir trick to replace sums over spins with matrix products.

$$\begin{aligned} f_{11} &= \frac{1}{N} \sum_{abcd} a_1 a_1^* = \text{Tr} \left((\not{p}_3 + m) \gamma^\mu (\not{p}_1 + m) \gamma^\nu \right) \text{Tr} \left((\not{p}_4 + m) \gamma_\mu (\not{p}_2 + m) \gamma_\nu \right) \\ f_{12} &= \frac{1}{N} \sum_{abcd} a_1 a_2^* = \text{Tr} \left((\not{p}_3 + m) \gamma^\mu (\not{p}_1 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\mu (\not{p}_2 + m) \gamma_\nu \right) \\ f_{22} &= \frac{1}{N} \sum_{abcd} a_2 a_2^* = \text{Tr} \left((\not{p}_4 + m) \gamma^\mu (\not{p}_1 + m) \gamma^\nu \right) \text{Tr} \left((\not{p}_3 + m) \gamma_\mu (\not{p}_2 + m) \gamma_\nu \right) \end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left(\frac{f_{11}}{t^2} - \frac{f_{12}}{tu} - \frac{f_{12}^*}{tu} + \frac{f_{22}}{u^2} \right)$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$\begin{aligned} f_{11} &= 32(p_1 \cdot p_2)^2 + 32(p_1 \cdot p_4)^2 - 64m^2(p_1 \cdot p_3) + 64m^4 \\ f_{12} &= -32(p_1 \cdot p_2)^2 + 32m^2(p_1 \cdot p_2) + 32m^2(p_1 \cdot p_3) + 32m^2(p_1 \cdot p_4) - 32m^4 \\ f_{22} &= 32(p_1 \cdot p_2)^2 + 32(p_1 \cdot p_3)^2 - 64m^2(p_1 \cdot p_4) + 64m^4 \end{aligned}$$

In Mandelstam variables

$$\begin{aligned} s &= (p_1 + p_2)^2 \\ t &= (p_1 - p_3)^2 \\ u &= (p_1 - p_4)^2 \end{aligned}$$

the formulas are

$$\begin{aligned} f_{11} &= 8s^2 + 8u^2 - 64sm^2 - 64um^2 + 192m^4 \\ f_{12} &= -8s^2 + 64sm^2 - 96m^4 \\ f_{22} &= 8s^2 + 8t^2 - 64sm^2 - 64tm^2 + 192m^4 \end{aligned}$$

High energy approximation

When $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$\begin{aligned} f_{11} &= 8s^2 + 8u^2 \\ f_{12} &= -8s^2 \\ f_{22} &= 8s^2 + 8t^2 \end{aligned}$$

Hence

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left(\frac{f_{11}}{t^2} - \frac{f_{12}}{tu} - \frac{f_{12}^*}{tu} + \frac{f_{22}}{u^2} \right) \\ &= \frac{e^4}{4} \left(\frac{8s^2 + 8u^2}{t^2} - \frac{-8s^2}{tu} - \frac{-8s^2}{tu} + \frac{8s^2 + 8t^2}{u^2} \right) \\ &= 2e^4 \left(\frac{s^2 + u^2}{t^2} + \frac{2s^2}{tu} + \frac{s^2 + t^2}{u^2} \right) \end{aligned}$$

Combine terms so $\langle |\mathcal{M}|^2 \rangle$ has a common denominator.

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{u^2(s^2 + u^2) + 2s^2tu + t^2(s^2 + t^2)}{t^2u^2} \right)$$

For $m = 0$ the Mandelstam variables are

$$\begin{aligned} s &= 4E^2 \\ t &= 2E^2(\cos \theta - 1) \\ u &= -2E^2(\cos \theta + 1) \end{aligned}$$

Hence

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= 2e^4 \left(\frac{32E^8 \cos^4 \theta + 192E^8 \cos^2 \theta + 288E^8}{16E^8(\cos \theta - 1)^2(\cos \theta + 1)^2} \right) \\
&= 4e^4 \frac{(\cos^2 \theta + 3)^2}{(\cos \theta - 1)^2(\cos \theta + 1)^2} \\
&= 4e^4 \frac{(\cos^2 \theta + 3)^2}{\sin^4 \theta}
\end{aligned}$$

The following equivalent formula can also be used.

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= 2e^4 \left(\frac{s^2 + u^2}{t^2} + \frac{2s^2}{tu} + \frac{s^2 + t^2}{u^2} \right) \\
&= 2e^4 \left(\underbrace{\frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)}}_{\substack{\text{from Feynman diagram with} \\ \text{photon exchange} \\ \text{no electron interchange}}} + \underbrace{\frac{2}{\sin^2(\theta/2) \cos^2(\theta/2)}}_{\text{interaction term}} + \underbrace{\frac{1 + \sin^4(\theta/2)}{\cos^4(\theta/2)}}_{\substack{\text{from Feynman diagram with} \\ \text{photon exchange} \\ \text{electron interchange}}} \right)
\end{aligned}$$

For example, see A. Zee, p. 134.

Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{64\pi^2 s} = \frac{e^4}{16\pi^2 s} \frac{(\cos^2 \theta + 3)^2}{\sin^4 \theta}, \quad s \gg m$$

Substituting $e^4 = 16\pi^2 \alpha^2$ yields

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s} \frac{(\cos^2 \theta + 3)^2}{\sin^4 \theta}$$

We can integrate $d\sigma$ to obtain a cumulative distribution function. Recall that

$$d\Omega = \sin \theta d\theta d\phi$$

Hence

$$d\sigma = \frac{\alpha^2}{s} \frac{(\cos^2 \theta + 3)^2}{\sin^3 \theta} d\theta d\phi$$

Let $I(\theta)$ be the integral of $d\sigma$.

$$\begin{aligned}
I(\theta) &= \int_0^{2\pi} \int d\sigma \\
&= \frac{2\pi\alpha^2}{s} \int \frac{(\cos^2 \theta + 3)^2}{\sin^3 \theta} d\theta \\
&= \frac{2\pi\alpha^2}{s} \left(\frac{8 \cos \theta}{\cos^2 \theta - 1} - \cos \theta \right), \quad a \leq \theta \leq \pi - a
\end{aligned}$$

Angular support is limited to an arbitrary $a > 0$ because $I(0)$ and $I(\pi)$ are undefined.

The cumulative distribution function $F(\theta)$ is

$$F(\theta) = \frac{I(\theta) - I(a)}{I(\pi) - I(a)}, \quad a \leq \theta \leq \pi - a$$

The probability of observing scattering events in the interval θ_1 to θ_2 can now be computed.

$$P(\theta_1 \leq \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

Notes

In component notation, the trace operators of the Casimir trick become sums over a repeated index, in this case α .

$$\begin{aligned} f_{11} &= \left((\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \right) \left((\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \right) \\ f_{12} &= (\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_2 + m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha \\ f_{22} &= \left((\not{p}_4 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \right) \left((\not{p}_3 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \right) \end{aligned}$$

To convert the above formulas to Eigenmath code, the γ tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply γ^μ by the metric tensor to lower the index.

$$\begin{aligned} \gamma^{\beta\mu}{}_\rho &\rightarrow \text{gammaT} = \text{transpose}(\text{gamma}) \\ \gamma^\beta{}_{\mu\rho} &\rightarrow \text{gammaL} = \text{transpose}(\text{dot}(\text{gmunu}, \text{gamma})) \end{aligned}$$

Define the following 4×4 matrices.

$$\begin{aligned} (\not{p}_1 + m) &\rightarrow \text{X1} = \text{pslash1} + \text{m I} \\ (\not{p}_2 + m) &\rightarrow \text{X2} = \text{pslash2} + \text{m I} \\ (\not{p}_3 + m) &\rightarrow \text{X3} = \text{pslash3} + \text{m I} \\ (\not{p}_4 + m) &\rightarrow \text{X4} = \text{pslash4} + \text{m I} \end{aligned}$$

Then for f_{11} we have the following Eigenmath code. The contract function sums over α .

$$\begin{aligned} (\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha &\rightarrow \text{T1} = \text{contract}(\text{dot}(\text{X3}, \text{gammaT}, \text{X1}, \text{gammaT}), 1, 4) \\ (\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha &\rightarrow \text{T2} = \text{contract}(\text{dot}(\text{X4}, \text{gammaL}, \text{X2}, \text{gammaL}), 1, 4) \end{aligned}$$

Next, multiply then sum over repeated indices. The dot function sums over ν then the contract function sums over μ . The transpose makes the ν indices adjacent as required by the dot function.

$$f_{11} = \text{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu) \text{Tr}(\cdots \gamma_\mu \cdots \gamma_\nu) \rightarrow \text{contract}(\text{dot}(\text{T1}, \text{transpose}(\text{T2})))$$

Follow suit for f_{22} .

$$\begin{aligned} (\not{p}_4 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha &\rightarrow \text{T1} = \text{contract}(\text{dot}(\text{X4}, \text{gammaT}, \text{X1}, \text{gammaT}), 1, 4) \\ (\not{p}_3 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha &\rightarrow \text{T2} = \text{contract}(\text{dot}(\text{X3}, \text{gammaL}, \text{X2}, \text{gammaL}), 1, 4) \end{aligned}$$

Then

$$f_{22} = \text{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu) \text{Tr}(\cdots \gamma_\mu \cdots \gamma_\nu) \rightarrow \text{contract}(\text{dot}(\text{T1}, \text{transpose}(\text{T2})))$$

The calculation of f_{12} begins with

$$\begin{aligned} & (\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_2 + m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha \\ & \rightarrow \text{T} = \text{contract}(\text{dot}(\text{X3}, \text{gammaT}, \text{X1}, \text{gammaT}, \text{X4}, \text{gammaL}, \text{X2}, \text{gammaL}), 1, 6) \end{aligned}$$

Then sum over repeated indices μ and ν .

$$f_{12} = \text{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu \cdots \gamma_\mu \cdots \gamma_\nu) \rightarrow \text{contract}(\text{contract}(\text{T}, 1, 3))$$