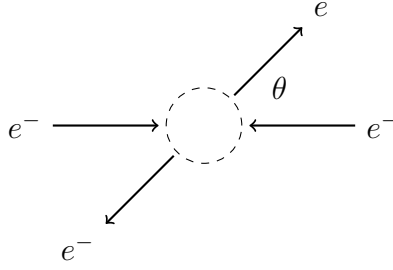


Moller scattering

Moller scattering is the process $e^- + e^- \rightarrow e^- + e^-$.



The following center-of-mass momentum vectors have $E = \sqrt{p^2 + m^2}$.

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}_{e^- \rightarrow} \quad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix}_{\leftarrow e^-} \quad p_3 = \begin{pmatrix} E \\ p \sin \theta \cos \phi \\ p \sin \theta \sin \phi \\ p \cos \theta \end{pmatrix}_{e^- \nearrow} \quad p_4 = \begin{pmatrix} E \\ -p \sin \theta \cos \phi \\ -p \sin \theta \sin \phi \\ -p \cos \theta \end{pmatrix}_{\nwarrow e^-}$$

Spinors for p_1 .

$$u_{11} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix}_{\text{spin up}} \quad u_{12} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix}_{\text{spin down}}$$

Spinors for p_2 .

$$u_{21} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ -p \\ 0 \end{pmatrix}_{\text{spin up}} \quad u_{22} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ 0 \\ p \end{pmatrix}_{\text{spin down}}$$

Spinors for p_3 .

$$u_{31} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p_{3z} \\ p_{3x} + ip_{3y} \end{pmatrix}_{\text{spin up}} \quad u_{32} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ p_{3x} - ip_{3y} \\ -p_{3z} \end{pmatrix}_{\text{spin down}}$$

Spinors for p_4 .

$$u_{41} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix}_{\text{spin up}} \quad u_{42} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}_{\text{spin down}}$$

The scattering amplitude \mathcal{M}_{abcd} for spin state $abcd$ is

$$\mathcal{M}_{abcd} = \mathcal{M}_{1abcd} + \mathcal{M}_{2abcd}$$

where

$$\mathcal{M}_{1abcd} = \frac{e^2}{t} (\bar{u}_{3c} \gamma^\mu u_{1a}) (\bar{u}_{4d} \gamma_\mu u_{2b}), \quad \mathcal{M}_{2abcd} = -\frac{e^2}{u} (\bar{u}_{4d} \gamma^\nu u_{1a}) (\bar{u}_{3c} \gamma_\nu u_{2b})$$

no electron interchange electron interchange

Symbols t and u are Mandelstam variables.

$$t = (p_1 - p_3)^2$$

$$u = (p_1 - p_4)^2$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is the sum of squared amplitudes divided by the number of inbound states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{abcd} |\mathcal{M}_{abcd}|^2$$

Expand the summand and label the terms. By positivity $\boxed{2} = \boxed{3}$.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{abcd} (\underbrace{\mathcal{M}_{1abcd} \mathcal{M}_{1abcd}^*}_{\boxed{1}} + \underbrace{\mathcal{M}_{1abcd} \mathcal{M}_{2abcd}^*}_{\boxed{2}} + \underbrace{\mathcal{M}_{2abcd} \mathcal{M}_{1abcd}^*}_{\boxed{3}} + \underbrace{\mathcal{M}_{2abcd} \mathcal{M}_{2abcd}^*}_{\boxed{4}})$$

The following Casimir trick uses matrix arithmetic to sum over spin states.

$$\begin{aligned} \sum_{abcd} \boxed{1} &= \frac{e^4}{t^2} \text{Tr} \left[(\not{p}_3 + m) \gamma^\mu (\not{p}_1 + m) \gamma^\nu \right] \text{Tr} \left[(\not{p}_4 + m) \gamma_\mu (\not{p}_2 + m) \gamma_\nu \right] \\ \sum_{abcd} \boxed{2} &= -\frac{e^4}{tu} \text{Tr} \left[(\not{p}_3 + m) \gamma^\mu (\not{p}_1 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\mu (\not{p}_2 + m) \gamma_\nu \right] \\ \sum_{abcd} \boxed{4} &= \frac{e^4}{u^2} \text{Tr} \left[(\not{p}_4 + m) \gamma^\mu (\not{p}_1 + m) \gamma^\nu \right] \text{Tr} \left[(\not{p}_3 + m) \gamma_\mu (\not{p}_2 + m) \gamma_\nu \right] \end{aligned}$$

Let

$$\begin{aligned} f_{11} &= \text{Tr} \left[(\not{p}_3 + m) \gamma^\mu (\not{p}_1 + m) \gamma^\nu \right] \text{Tr} \left[(\not{p}_4 + m) \gamma_\mu (\not{p}_2 + m) \gamma_\nu \right] \\ f_{12} &= -\text{Tr} \left[(\not{p}_3 + m) \gamma^\mu (\not{p}_1 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\mu (\not{p}_2 + m) \gamma_\nu \right] \\ f_{22} &= \text{Tr} \left[(\not{p}_4 + m) \gamma^\mu (\not{p}_1 + m) \gamma^\nu \right] \text{Tr} \left[(\not{p}_3 + m) \gamma_\mu (\not{p}_2 + m) \gamma_\nu \right] \end{aligned}$$

so that

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left(\frac{f_{11}}{t^2} + \frac{2f_{12}}{tu} + \frac{f_{22}}{u^2} \right)$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$\begin{aligned} f_{11} &= 32(p_1 \cdot p_2)^2 + 32(p_1 \cdot p_4)^2 - 64(p_1 \cdot p_2)m^2 + 64(p_1 \cdot p_4)m^2 \\ f_{12} &= 32(p_1 \cdot p_2)^2 - 64(p_1 \cdot p_2)m^2 \\ f_{22} &= 32(p_1 \cdot p_2)^2 + 32(p_1 \cdot p_3)^2 - 64(p_1 \cdot p_2)m^2 + 64(p_1 \cdot p_3)m^2 \end{aligned}$$

In Mandelstam variables

$$\begin{aligned}f_{11} &= 8s^2 + 8u^2 - 64sm^2 - 64um^2 + 192m^4 \\f_{12} &= 8s^2 - 64sm^2 + 96m^4 \\f_{22} &= 8s^2 + 8t^2 - 64sm^2 - 64tm^2 + 192m^4\end{aligned}$$

For $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$\begin{aligned}f_{11} &= 8s^2 + 8u^2 \\f_{12} &= 8s^2 \\f_{22} &= 8s^2 + 8t^2\end{aligned}$$

For $m = 0$ the Mandelstam variables are

$$\begin{aligned}s &= 4E^2 \\t &= -2E^2(1 - \cos \theta) \\u &= -2E^2(1 + \cos \theta)\end{aligned}$$

Hence

$$\begin{aligned}\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left(\frac{f_{11}}{t^2} + \frac{2f_{12}}{tu} + \frac{f_{22}}{u^2} \right) \\&= 2e^4 \left(\frac{s^2 + u^2}{t^2} + \frac{2s^2}{tu} + \frac{s^2 + t^2}{u^2} \right) \\&= 2e^4 \left(\underbrace{\frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)}}_{\text{no electron interchange}} + \underbrace{\frac{2}{\sin^2(\theta/2) \cos^2(\theta/2)}}_{\text{interaction term}} + \underbrace{\frac{1 + \sin^4(\theta/2)}{\cos^4(\theta/2)}}_{\text{electron interchange}} \right)\end{aligned}$$

The expected probability density can be written more compactly as

$$\langle |\mathcal{M}|^2 \rangle = 4e^4 \frac{(\cos^2 \theta + 3)^2}{\sin^4 \theta}$$

Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{4(4\pi\epsilon_0)^2 s}$$

where

$$s = (p_1 + p_2)^2 = 4E^2$$

For high energy experiments we have

$$\langle |\mathcal{M}|^2 \rangle = 4e^4 \frac{(\cos^2 \theta + 3)^2}{\sin^4 \theta}$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{(4\pi\epsilon_0)^2 s} \frac{(\cos^2 \theta + 3)^2}{\sin^4 \theta}$$

Noting that

$$e^2 = 4\pi\epsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{s} \frac{(\cos^2 \theta + 3)^2}{\sin^4 \theta}$$

Noting that

$$d\Omega = \sin \theta d\theta d\phi$$

we also have

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{s} \frac{(\cos^2 \theta + 3)^2}{\sin^4 \theta} \sin \theta d\theta d\phi$$

Let $S(\theta_1, \theta_2)$ be the following integral of $d\sigma$.

$$S(\theta_1, \theta_2) = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{2\pi\alpha^2(\hbar c)^2}{s} [I(\theta_2) - I(\theta_1)]$$

where

$$I(\theta) = -\frac{8 \cos \theta}{\sin^2 \theta} - \cos \theta$$

The cumulative distribution function is

$$F(\theta) = \frac{S(a, \theta)}{S(a, \pi - a)} = \frac{I(\theta) - I(a)}{I(\pi - a) - I(a)}, \quad a \leq \theta \leq \pi - a$$

Angular support is reduced by an arbitrary angle $a > 0$ because $I(0)$ and $I(\pi)$ are undefined.

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 < \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi - a) - I(a)} \frac{(\cos^2 \theta + 3)^2}{\sin^4 \theta} \sin \theta$$

Note

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$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4\pi} \right)^2 \frac{1}{8E^2} f(\theta)$$

where $f(\theta)$ is the probability density function

$$f(\theta) = \frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{2}{\sin^2(\theta/2) \cos^2(\theta/2)} + \frac{1 + \sin^4(\theta/2)}{\cos^4(\theta/2)}$$

The probability density function is equivalent to

$$f(\theta) = \frac{2(\cos^2 \theta + 3)^2}{\sin^4 \theta}$$

Hence for natural units $\varepsilon_0 = \hbar = c = 1$ and $e^2 = 4\pi\alpha$ the above cross section is equivalent to

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{4E^2} \frac{(\cos^2 \theta + 3)^2}{\sin^4 \theta}$$