6-3. For a free particle, equation (4.29) reduces to

$$\frac{\partial}{\partial t_b}K_0(b,a) + \frac{i}{\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2}K_0(b,a)\right) = \delta(t_b - t_a)\delta(x_b - x_a) \tag{6.20}$$

Show, from this result and equation (6.19), that the kernel K_V satisfies the differential equation

$$\frac{\partial}{\partial t_b}K_V(b,a) + \frac{i}{\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2}K_V(b,a) + V(b)K_V(b,a)\right) = \delta(t_b - t_a)\delta(x_b - x_a) \tag{6.21}$$

From equation (6.19)

$$K_0(b,a) = K_V(b,a) + \frac{i}{\hbar}\int K_0(b,c)V(c)K_V(c,a)\,d\tau_c \tag{1}$$

Apply $\partial/\partial t_b$ and $\partial^2/\partial x_b^2$ to (1).

$$\frac{\partial}{\partial t_b}K_0(b,a) = \frac{\partial}{\partial t_b}K_V(b,a) + \frac{i}{\hbar}\int \frac{\partial}{\partial t_b}K_0(b,c)V(c)K_V(c,a)\,d\tau_c \tag{2}$$

$$\frac{\partial^2}{\partial x_b^2}K_0(b,a) = \frac{\partial^2}{\partial x_b^2}K_V(b,a) + \frac{i}{\hbar}\int \frac{\partial^2}{\partial x_b^2}K_0(b,c)V(c)K_V(c,a)\,d\tau_c \tag{3}$$

Multiply (3) by $-\hbar^2/2m$.

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2}K_0(b,a) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2}K_V(b,a) + \frac{i}{\hbar}\int\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2}K_0(b,c)\right)V(c)K_V(c,a)\,d\tau_c \tag{4}$$

From equation (6.20)

$$\frac{i}{\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2}K_0(b,a)\right) = -\frac{\partial}{\partial t_b}K_0(b,a) + \delta(t_b - t_a)\delta(x_b - x_a)$$

It follows that

$$\frac{i}{\hbar}\int\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2}K_0(b,c)\right)V(c)K_V(c,a)\,d\tau_c = -\int\frac{\partial}{\partial t_b}K_0(b,c)V(c)K_V(c,a)\,d\tau_c + V(b)K_V(b,a) \tag{5}$$

where we have used the identity

$$\int \delta(t_b - t_a)\delta(x_b - x_a)V(c)K_V(c,a)\,d\tau_c = V(b)K_V(b,a)$$

Substitute (5) into (4) to obtain

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2}K_0(b,a) = &-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2}K_V(b,a) \\ &- \int \frac{\partial}{\partial t_b}K_0(b,c)V(c)K_V(c,a)\,d\tau_c + V(b)K_V(b,a) \quad (6) \end{aligned}$$

Substitute (2) and (6) into (6.20) to obtain (6.21).