4-12. Carry out the integral in equation (4.64) by completing the square. Show that the correct free particle kernel (i.e., the three dimensional version of equation 3.3) results.

$$K_0(\mathbf{x}_b, t_b, \mathbf{x}_a, t_a) = \int \exp\left(\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{x}_b - \mathbf{x}_a)\right)\exp\left(-\frac{i}{\hbar}\frac{p^2}{2m}(t_b - t_a)\right)\frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \tag{4.64}$$

Let

$$T = t_b - t_a, \qquad (X_1, X_2, X_3) = \mathbf{x}_b - \mathbf{x}_a$$

In component notation

$$\mathbf{p}\cdot(\mathbf{x}_b - \mathbf{x}_a) = p_1X_1 + p_2X_2 + p_3X_3, \qquad p^2 = \mathbf{p}\cdot\mathbf{p} = p_1^2 + p_2^2 + p_3^2$$

Hence from (4.64)

$$K_0(\mathbf{x}_b, t_b, \mathbf{x}_a, t_a) = \frac{1}{(2\pi\hbar)^3} \times \int_{\mathbb{R}^3} \exp\left[\frac{i}{\hbar}\left(p_1X_1 + p_2X_2 + p_3X_3 - \frac{p_1^2 + p_2^2 + p_3^2}{2m}T\right)\right] dp_1\,dp_2\,dp_3$$

Rewrite as

$$K_0(\mathbf{x}_b, t_b, \mathbf{x}_a, t_a) = \frac{1}{(2\pi\hbar)^3}\int_{\mathbb{R}^3}\exp\left(\frac{i}{\hbar}p_1X_1 - \frac{i}{\hbar}\frac{p_1^2}{2m}T\right) \times \exp\left(\frac{i}{\hbar}p_2X_2 - \frac{i}{\hbar}\frac{p_2^2}{2m}T\right)\exp\left(\frac{i}{\hbar}p_3X_3 - \frac{i}{\hbar}\frac{p_3^2}{2m}T\right) dp_1\,dp_2\,dp_3$$

Let

$$A = \frac{i}{2m\hbar}T \qquad B = \frac{i}{\hbar}X_k$$

Then

$$\int_{-\infty}^{\infty}\exp(-Ap_k^2 + Bp_k)\,dp_k = \left(\frac{\pi}{A}\right)^{1/2}\exp\left(\frac{B^2}{4A}\right) = \left(\frac{2\pi m\hbar}{iT}\right)^{1/2}\exp\left(\frac{imX_k^2}{2\hbar T}\right) \tag{1}$$

Hence

$$K_0(\mathbf{x}_b, t_b, \mathbf{x}_a, t_a) = \frac{1}{(2\pi\hbar)^3} \prod_{k=1}^{3} \left(\frac{2\pi m\hbar}{iT}\right)^{1/2} \exp\left(\frac{imX_k^2}{2\hbar T}\right)$$

Combine coefficients and convert to vector notation.

$$K_0(\mathbf{x}_b, t_b, \mathbf{x}_a, t_a) = \left(\frac{m}{2\pi i\hbar(t_b - t_a)}\right)^{3/2} \exp\left(\frac{im(\mathbf{x}_b - \mathbf{x}_a)^2}{2\hbar(t_b - t_a)}\right) \qquad (2)$$