(37.3) *A one-line derivation of the Dirac equation.*

(a) Given that the left- and right-handed parts of the Dirac spinor for a fermion at rest are identical, explain why we may write $(\gamma^0 - 1)u(p^0) = 0$.

(b) Prove that $e^{i\mathbf{K}\cdot\boldsymbol{\phi}}\gamma^0 e^{-i\mathbf{K}\cdot\boldsymbol{\phi}} = \not{p}/m$.

(c) Use the result in (b) to boost

$$(\gamma^0 - 1)u(p^0) = 0$$

and show that you recover the Dirac equation.

(a) Consider equation (36.29).

$$u(p^0) \equiv \begin{pmatrix} u_L(p^0) \\ u_R(p^0) \end{pmatrix} = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix} \tag{36.29}$$

It follows that

$$\gamma^0 u(p^0) = \sqrt{m}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \xi \end{pmatrix} = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix} = u(p^0)$$

Hence

$$(\gamma^0 - 1)u(p^0) = u(p^0) - u(p^0) = 0$$

(b) Let $\exp(-i\mathbf{K}\cdot\boldsymbol{\sigma})$ be the negative boost such that

$$\exp(-i\mathbf{K}\cdot\boldsymbol{\sigma})u(p) = u(p^0) \tag{1}$$

Trivially we have

$$\exp(i\mathbf{K}\cdot\boldsymbol{\sigma})\exp(-i\mathbf{K}\cdot\boldsymbol{\sigma})u(p) = u(p) \tag{2}$$

Combine (1) and (2) with the result from part (a) to obtain

$$\exp(i\mathbf{K}\cdot\boldsymbol{\sigma})\gamma^0\exp(-i\mathbf{K}\cdot\boldsymbol{\sigma})u(p) = u(p) \tag{3}$$

Substitute (3) into the Dirac equation $\not{p}u(p) = mu(p)$ to obtain

$$\not{p}u(p) = m\exp(i\mathbf{K}\cdot\boldsymbol{\sigma})\gamma^0\exp(-i\mathbf{K}\cdot\boldsymbol{\sigma})u(p)$$

Divide through by m and cancel $u(p)$ to obtain

$$\not{p}/m = \exp(i\mathbf{K}\cdot\boldsymbol{\sigma})\gamma^0 \exp(-i\mathbf{K}\cdot\boldsymbol{\sigma})$$

(c) Applying the boost we have

$$\exp(i\mathbf{K}\cdot\boldsymbol{\phi})\gamma^0 u(p^0) - u(p) = 0 \tag{4}$$

Substitute (1) into (4) to obtain

$$\exp(i\mathbf{K}\cdot\boldsymbol{\phi})\gamma^0 \exp(-i\mathbf{K}\cdot\boldsymbol{\phi})u(p) - u(p) = 0$$

Then by the result from part (b) we have

$$m^{-1}\not{p}u(p) - u(p) = 0$$

Multiply through by m to obtain the Dirac equation.

$$(\not{p} - m)u(p) = 0$$