

Atomic transitions 4

From the previous section

$$R_{a \rightarrow b} = \frac{\pi e^2 \rho(\omega_0)}{\varepsilon_0 m^2 \hbar^2 \omega_0^2} \left| \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle \right|^2$$

Use the dipole approximation

$$\exp(i\mathbf{k} \cdot \mathbf{r}) \approx 1$$

to obtain

$$R_{a \rightarrow b} = \frac{\pi e^2 \rho(\omega_0)}{\varepsilon_0 m^2 \hbar^2 \omega_0^2} \left| \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \right|^2$$

By the identity

$$\mathbf{p} = \frac{im}{\hbar} [H_0, \mathbf{r}] \tag{1}$$

we have

$$\begin{aligned} \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle &= \frac{im}{\hbar} \langle \psi_b | \boldsymbol{\epsilon} \cdot [H_0, \mathbf{r}] | \psi_a \rangle \\ &= \frac{im}{\hbar} \langle \psi_b | \boldsymbol{\epsilon} \cdot H_0 \mathbf{r} | \psi_a \rangle - \frac{im}{\hbar} \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} H_0 | \psi_a \rangle \\ &= \frac{im}{\hbar} E_b \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle - \frac{im}{\hbar} E_a \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle \\ &= \frac{im}{\hbar} (E_b - E_a) \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle \\ &= im\omega_0 \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle \end{aligned}$$

Hence

$$R_{a \rightarrow b} = \frac{\pi e^2 \rho(\omega_0)}{\varepsilon_0 \hbar^2} \left| \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle \right|^2$$

Verify dimensions.

$$R_{a \rightarrow b} \propto \frac{\frac{e^2}{\text{C}^2} \frac{\rho(\omega)}{\text{J m}^{-3} \text{s}}}{\frac{\varepsilon_0}{\text{C}^2 \text{J}^{-1} \text{m}^{-1}} \frac{\hbar^2}{\text{J}^2 \text{s}^2}} \times \frac{\left| \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle \right|^2}{\text{m}^2} = \text{s}^{-1}$$