

Hydrogen transition 1

Start with the perturbing Hamiltonian

$$H_1(\mathbf{r}, t) = -\frac{eA_0\boldsymbol{\epsilon} \cdot \mathbf{p}}{m_e} \cos(-\mathbf{k} \cdot \mathbf{r} + \omega t)$$

In exponential form

$$H_1(\mathbf{r}, t) = -\frac{eA_0\boldsymbol{\epsilon} \cdot \mathbf{p}}{m_e} \left(\frac{1}{2} \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t) + \frac{1}{2} \exp(-i\mathbf{k} \cdot \mathbf{r} + i\omega t) \right)$$

Given the initial condition $c_b(0) = 0$ the first-order approximation for $c_b(t)$ is

$$c_b(t) = -\frac{i}{\hbar} \int_0^t \langle \psi_b | H_1 | \psi_a \rangle \exp(i\omega_0 t') dt', \quad \omega_0 = \frac{E_b - E_a}{\hbar}$$

Factor the integrand.

$$\begin{aligned} c_b(t) = & \frac{ieA_0}{2\hbar m_e} \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle \int_0^t \exp(-i\omega t) \exp(i\omega_0 t') dt' \\ & + \frac{ieA_0}{2\hbar m_e} \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(-i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle \int_0^t \exp(i\omega t) \exp(i\omega_0 t') dt' \end{aligned}$$

Solve the integral to obtain

$$\begin{aligned} c_b(t) = & \frac{eA_0}{2\hbar m_e} \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle \frac{\exp(i(\omega_0 - \omega)t) - 1}{\omega_0 - \omega} \\ & + \frac{eA_0}{2\hbar m_e} \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(-i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle \frac{\exp(i(\omega_0 + \omega)t) - 1}{\omega_0 + \omega} \quad (1) \end{aligned}$$

As an approximation, discard the second term since the first term dominates for $\omega \approx \omega_0$.

$$c_b(t) = \frac{eA_0}{2\hbar m_e} \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle \frac{\exp(i(\omega_0 - \omega)t) - 1}{\omega_0 - \omega}$$

Rewrite $c_b(t)$ in the form of a sine function.

$$c_b(t) = \frac{ieA_0}{\hbar m_e} \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle \frac{\sin\left(\frac{1}{2}(\omega_0 - \omega)t\right)}{\omega_0 - \omega} \exp\left(\frac{i}{2}(\omega_0 - \omega)t\right) \quad (2)$$

The probability to go from state a to b is

$$\Pr_{a \rightarrow b}(t) = |c_b(t)|^2$$