Tunneling probability

Consider the following potential energy function.

$$V(x) = \begin{cases} 0, & x < 0 \\ V_0, & 0 \le x \le L \\ 0, & x > L \end{cases}$$

Suppose a particle with mass m and energy $E < V_0$ is traveling from left to right along the x axis. The particle has a Schrodinger equation for each region of $V(x)$.

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_1 &= E\psi_1, && x < 0 \\ -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_2 + V_0\psi_2 &= E\psi_2, && 0 \le x \le L \\ -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_3 &= E\psi_3, && x > L \end{aligned}$$

Let ψ_1 and ψ_3 have the most general free-particle solutions.

$$\begin{aligned} \psi_1(x) &= A\exp\left(i\sqrt{\frac{2mE}{\hbar^2}}x\right) + B\exp\left(-i\sqrt{\frac{2mE}{\hbar^2}}x\right) \\ \psi_3(x) &= F\exp\left(i\sqrt{\frac{2mE}{\hbar^2}}x\right) + G\exp\left(-i\sqrt{\frac{2mE}{\hbar^2}}x\right) \end{aligned}$$

Use the WKB approximation to solve for ψ_2.

$$\psi_2(x) \approx C\exp\left(i\int\sqrt{\frac{2m(E-V_0)}{\hbar^2}}\,dx\right) + D\exp\left(-i\int\sqrt{\frac{2m(E-V_0)}{\hbar^2}}\,dx\right)$$

Cancel i by swapping E and V_0.

$$\psi_2(x) \approx C\exp\left(\int\sqrt{\frac{2m(V_0-E)}{\hbar^2}}\,dx\right) + D\exp\left(-\int\sqrt{\frac{2m(V_0-E)}{\hbar^2}}\,dx\right)$$

Substitute x for $\int dx$.

$$\psi_2(x) \approx C\exp\left(\sqrt{\frac{2m(V_0-E)}{\hbar^2}}x\right) + D\exp\left(-\sqrt{\frac{2m(V_0-E)}{\hbar^2}}x\right)$$

To simplify the formulas let

$$k = \sqrt{\frac{2mE}{\hbar^2}}, \quad \beta = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

and write

$$\begin{aligned}
\psi_1(x) &= A\exp(ikx) + B\exp(-ikx) \\
\psi_2(x) &= C\exp(\beta x) + D\exp(-\beta x) \\
\psi_3(x) &= F\exp(ikx) + G\exp(-ikx)
\end{aligned}$$

Exponentials of $-i$ represent particles moving from right to left. The B exponential represents a particle reflected from the boundary at $x = 0$. There is no particle moving right to left at $x > L$ hence $G = 0$.

Let us now solve for the coefficients using boundary conditions. Four boundary conditions are needed to ensure continuity at $x = 0$ and $x = L$.

$$\begin{aligned}
\psi_1(0) &= \psi_2(0) \\
\psi_1'(0) &= \psi_2'(0) \\
\psi_2(L) &= \psi_3(L) \\
\psi_2'(L) &= \psi_3'(L)
\end{aligned}$$

From the boundary condition $\psi_2(L) = \psi_3(L)$ we have

$$C\exp(\beta L) + D\exp(-\beta L) = F\exp(ikL) \tag{1}$$

From the boundary condition $\psi_2'(L) = \psi_3'(L)$ we have

$$\beta C\exp(\beta L) - \beta D\exp(-\beta L) = ikF\exp(ikL) \tag{2}$$

Add β times (1) to (2) to obtain

$$2\beta C\exp(\beta L) = (\beta + ik)F\exp(ikL)$$

Hence

$$C = \frac{(\beta + ik)F\exp(ikL - \beta L)}{2\beta} \tag{3}$$

Add minus β times (1) to (2) to obtain

$$-2\beta D\exp(-\beta L) = (-\beta + ik)F\exp(ikL)$$

Hence

$$D = \frac{(\beta - ik)F\exp(ikL + \beta L)}{2\beta} \tag{4}$$

From the boundary condition $\psi_1(0) = \psi_2(0)$ we have

$$A + B = C + D \tag{5}$$

From the boundary condition $\psi_1'(0) = \psi_2'(0)$ we have

$$ik(A - B) = \beta(C - D) \tag{6}$$

Add ik times (5) to (6) to obtain

$$2ikA = \beta(C - D) + ik(C + D)$$

Hence

$$A = \frac{\beta(C - D)}{2ik} + \frac{C + D}{2}$$

Substitute (3) and (4) for C and D to obtain

$$A = F \exp(ikL)\big(\cosh(\beta L) + i\gamma \sinh(\beta L)\big) \tag{7}$$

where

$$\gamma = \frac{1}{2}\left(\frac{\beta}{k} - \frac{k}{\beta}\right)$$

The tunneling probability T is the magnitude of the transmitted wave divided by the magnitude of the inbound wave.

$$T = \frac{|F|^2}{|A|^2} = \left|\frac{1}{\exp(ikL)\big(\cosh(\beta L) + i\gamma \sinh(\beta L)\big)}\right|^2$$

Hence

$$T = \frac{1}{\cosh^2(\beta L) + \gamma^2 \sinh^2(\beta L)} \tag{8}$$

The result can also be written as

$$T = \left(1 + \frac{V_0^2 \sinh^2(\beta L)}{4E(V_0 - E)}\right)^{-1} \tag{9}$$

For small values of T the following approximation can be used.

$$T \approx \left(\frac{V_0^2 \frac{1}{4} \exp(2\beta L)}{4E(V_0 - E)}\right)^{-1} = \frac{16E(V_0 - E)}{V_0^2} \exp(-2\beta L)$$

Example. For an electron with

$$\begin{aligned} E &= 1\,\mathrm{eV} \\ V_0 &= 1.1\,\mathrm{eV} \\ L &= 10^{-9}\,\mathrm{meter} \end{aligned}$$

the tunneling probability is

$$T = 0.053$$

The approximate value is

$$T = 0.052$$

(Ref. “Quantum Tunneling of Particles through Potential Barriers” at phys.libretexts.org)