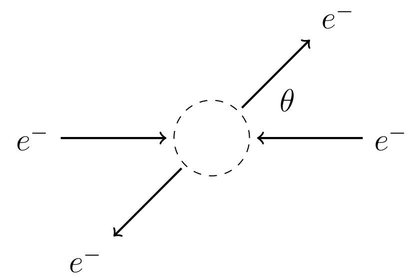
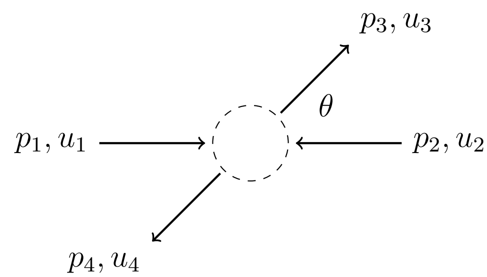
Moller scattering is the result of interactions between electrons. The following diagram represents a collider experiment with collinear electron beams.



Here is the same diagram with momentum and spinor labels.



In center of mass coordinates the momentum vectors are

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \qquad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} \qquad p_3 = \begin{pmatrix} E \\ p\sin\theta\cos\phi \\ p\sin\theta\sin\phi \\ p\cos\theta \end{pmatrix} \qquad p_4 = \begin{pmatrix} E \\ -p\sin\theta\cos\phi \\ -p\sin\theta\sin\phi \\ -p\cos\theta \end{pmatrix}$$

where $E = \sqrt{p^2 + m^2}$. The spinors are

$$u_{11} = \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix} \qquad u_{21} = \begin{pmatrix} E+m \\ 0 \\ -p \\ 0 \end{pmatrix} \qquad u_{31} = \begin{pmatrix} E+m \\ 0 \\ p_3^z \\ p_3^x + ip_3^y \end{pmatrix} \qquad u_{41} = \begin{pmatrix} E+m \\ 0 \\ p_4^z \\ p_4^x + ip_4^y \end{pmatrix}$$

$$u_{12} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix} \qquad u_{22} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ p \end{pmatrix} \qquad u_{32} = \begin{pmatrix} 0 \\ E+m \\ p_3^x - ip_3^y \\ -p_3^z \end{pmatrix} \qquad u_{42} = \begin{pmatrix} 0 \\ E+m \\ p_4^x - ip_4^y \\ -p_4^z \end{pmatrix}$$

The spinors shown above are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^4$ will be used.

The following formula computes a probability density $|\mathcal{M}_{abcd}|^2$ for Moller scattering where the subscripts $abcd$ are the spin states of the electrons.

$$|\mathcal{M}_{abcd}|^2 = \frac{e^4}{N} \left| \frac{1}{t} (\bar{u}_{3c}\gamma^\mu u_{1a})(\bar{u}_{4d}\gamma_\mu u_{2b}) - \frac{1}{u} (\bar{u}_{4d}\gamma^\nu u_{1a})(\bar{u}_{3c}\gamma_\nu u_{2b}) \right|^2$$

Symbol e is electron charge. Symbols t and u are Mandelstam variables $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$.

Let

$$a_1 = (\bar{u}_{3c}\gamma^\mu u_{1a})(\bar{u}_{4d}\gamma_\mu u_{2b}) \qquad a_2 = (\bar{u}_{4d}\gamma^\nu u_{1a})(\bar{u}_{3c}\gamma_\nu u_{2b})$$

Then

$$\begin{aligned}
|\mathcal{M}_{abcd}|^2 &= \frac{e^4}{N}\left|\frac{a_1}{t} - \frac{a_2}{u}\right|^2 \\
&= \frac{e^4}{N}\left(\frac{a_1}{t} - \frac{a_2}{u}\right)\left(\frac{a_1}{t} - \frac{a_2}{u}\right)^* \\
&= \frac{e^4}{N}\left(\frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{tu} - \frac{a_1^* a_2}{tu} + \frac{a_2 a_2^*}{u^2}\right)
\end{aligned}$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is computed by summing $|\mathcal{M}_{abcd}|^2$ over all spin states and dividing by the number of inbound states. There are four inbound states.

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}|\mathcal{M}_{abcd}|^2 \\
&= \frac{e^4}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}\frac{1}{N}\left(\frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{tu} - \frac{a_1^* a_2}{tu} + \frac{a_2 a_2^*}{u^2}\right)
\end{aligned}$$

Use the Casimir trick to replace sums over spins with matrix products.

$$\begin{aligned}
f_{11} &= \frac{1}{N}\sum_{\text{spins}} a_1 a_1^* = \operatorname{Tr}\left((\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu\right)\operatorname{Tr}\left((\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\right) \\
f_{12} &= \frac{1}{N}\sum_{\text{spins}} a_1 a_2^* = \operatorname{Tr}\left((\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\right) \\
f_{22} &= \frac{1}{N}\sum_{\text{spins}} a_2 a_2^* = \operatorname{Tr}\left((\not{p}_4 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu\right)\operatorname{Tr}\left((\not{p}_3 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\right)
\end{aligned}$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left(\frac{f_{11}}{t^2} - \frac{f_{12}}{tu} - \frac{f_{12}^*}{tu} + \frac{f_{22}}{u^2}\right)$$

Run “moller-scattering-1.txt” to verify the Casimir trick.

The following momentum formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$\begin{aligned}
f_{11} &= 32(p_1 \cdot p_2)(p_3 \cdot p_4) + 32(p_1 \cdot p_4)(p_2 \cdot p_3) - 32m^2(p_1 \cdot p_3) - 32m^2(p_2 \cdot p_4) + 64m^4 \\
f_{12} &= -32(p_1 \cdot p_2)(p_3 \cdot p_4) + 16m^2(p_1 \cdot p_2) + 16m^2(p_1 \cdot p_3) + 16m^2(p_1 \cdot p_4) \\
&\quad + 16m^2(p_2 \cdot p_3) + 16m^2(p_2 \cdot p_4) + 16m^2(p_3 \cdot p_4) - 32m^4 \\
f_{22} &= 32(p_1 \cdot p_2)(p_3 \cdot p_4) + 32(p_1 \cdot p_3)(p_2 \cdot p_4) - 32m^2(p_1 \cdot p_4) - 32m^2(p_2 \cdot p_3) + 64m^4
\end{aligned}$$

In Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, and $u = (p_1 - p_4)^2$ the formulas are

$$\begin{aligned}
f_{11} &= 8s^2 + 8u^2 - 64sm^2 - 64um^2 + 192m^4 \\
f_{12} &= -8s^2 + 64sm^2 - 96m^4 \\
f_{22} &= 8s^2 + 8t^2 - 64sm^2 - 64tm^2 + 192m^4
\end{aligned}$$

High energy approximation

When $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$\begin{aligned} f_{11} &= 8s^2 + 8u^2 \\ f_{12} &= -8s^2 \\ f_{22} &= 8s^2 + 8t^2 \end{aligned}$$

For $m = 0$ the Mandelstam variables are

$$\begin{aligned} s &= 4E^2 \\ t &= -2E^2(1 - \cos\theta) = -4E^2 \sin^2(\theta/2) \\ u &= -2E^2(1 + \cos\theta) = -4E^2 \cos^2(\theta/2) \end{aligned}$$

The corresponding expected probability density is

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left(\frac{8s^2 + 8u^2}{t^2} + \frac{16s^2}{tu} + \frac{8s^2 + 8t^2}{u^2} \right) \\ &= 2e^4 \left(\frac{s^2 + u^2}{t^2} + \frac{2s^2}{tu} + \frac{s^2 + t^2}{u^2} \right) \\ &= 2e^4 \left(\frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{8}{\sin^2\theta} + \frac{1 + \sin^4(\theta/2)}{\cos^4(\theta/2)} \right) \\ &= \frac{4e^4 \left(3 + \cos^2\theta\right)^2}{\sin^4\theta} \end{aligned}$$

Run “moller-scattering-2.txt” to verify the formulas on this page.

Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{64\pi^2 s} = \frac{e^4}{64\pi^2 E^2} \frac{(3 + \cos^2\theta)^2}{\sin^4\theta}$$

Substituting $e^2 = 4\pi\alpha$ yields

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2} \frac{(3 + \cos^2\theta)^2}{\sin^4\theta}$$

We can integrate $d\sigma$ to obtain a normalized probability density in spherical coordinates. Recall that

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

Hence

$$d\sigma = \frac{\alpha^2}{4E^2} \frac{(3 + \cos^2\theta)^2}{\sin^4\theta} \sin\theta \, d\theta \, d\phi$$

Let $I(\xi)$ be the following definite integral.

$$
\begin{aligned}
I(\xi) &= \frac{4E^2}{2\pi\alpha^2} \int_0^{2\pi} \int_a^{\xi} d\sigma \\
&= \int_a^{\xi} \frac{(3+\cos^2\theta)^2}{\sin^4\theta} \sin\theta \, d\theta \\
&= \left(-\cos\theta - \frac{8\cos\theta}{\sin^2\theta} \right) \Bigg|_a^{\xi} \\
&= \cos a + \frac{8\cos a}{\sin^2 a} - \cos\xi - \frac{8\cos\xi}{\sin^2\xi}, \qquad a \le \xi \le \pi - a
\end{aligned}
$$

Angular support is limited to $a > 0$ because $I(0)$ and $I(\pi)$ are undefined.

Let C be the normalization constant $C = I(\pi - a)$. Then the cumulative distribution function $F(\theta)$ is

$$F(\theta) = C^{-1} I(\theta), \qquad a \le \theta \le \pi - a$$

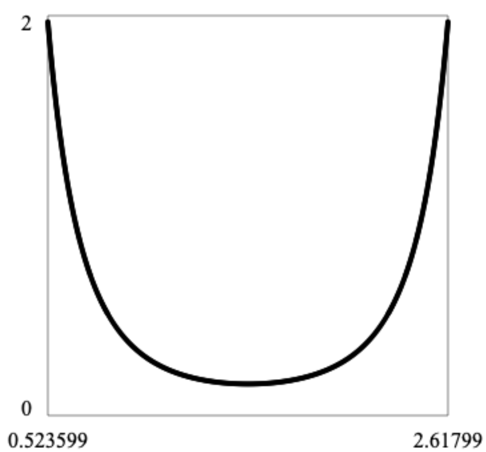
The probability of observing scattering events in the interval θ_1 to θ_2 can now be computed.

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

Probability density function $f(\theta)$ is the derivative of $F(\theta)$.

$$f(\theta) = \frac{dF(\theta)}{d\theta} = C^{-1} \frac{dI(\theta)}{d\theta} = C^{-1} \frac{(3+\cos^2\theta)^2}{\sin^3\theta}$$

Run “moller-scattering-3.txt” to draw a graph of $f(\theta)$ for $a = \pi/6 = 30°$.



The following table shows the probability distribution for 30° bins ($a = \pi/6 = 30°$).

θ_1	θ_2	$P(\theta_1 \le \theta \le \theta_2)$
0°	30°	–
30°	60°	0.40
60°	90°	0.10
90°	120°	0.10
120°	150°	0.40
150°	180°	–

Eigenmath code

Here are a few notes about how the scripts work.

In component notation, the trace operators of the Casimir trick become sums over a repeated index, in this case α.

$$
\begin{aligned}
f_{11} &= \Big((\not p_3+m)^\alpha{}_\beta\gamma^{\mu\beta}{}_\rho(\not p_1+m)^\rho{}_\sigma\gamma^{\nu\sigma}{}_\alpha\Big)\Big((\not p_4+m)^\alpha{}_\beta\gamma_\mu{}^\beta{}_\rho(\not p_2+m)^\rho{}_\sigma\gamma_\nu{}^\sigma{}_\alpha\Big)\\
f_{12} &= (\not p_3+m)^\alpha{}_\beta\gamma^{\mu\beta}{}_\rho(\not p_1+m)^\rho{}_\sigma\gamma^{\nu\sigma}{}_\tau(\not p_4+m)^\tau{}_\delta\gamma_\mu{}^\delta{}_\eta(\not p_2+m)^\eta{}_\xi\gamma_\nu{}^\xi{}_\alpha\\
f_{22} &= \Big((\not p_4+m)^\alpha{}_\beta\gamma^{\mu\beta}{}_\rho(\not p_1+m)^\rho{}_\sigma\gamma^{\nu\sigma}{}_\alpha\Big)\Big((\not p_3+m)^\alpha{}_\beta\gamma_\mu{}^\beta{}_\rho(\not p_2+m)^\rho{}_\sigma\gamma_\nu{}^\sigma{}_\alpha\Big)
\end{aligned}
$$

To convert the above formulas to Eigenmath code, the γ tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply γ^μ by the metric tensor to lower the index.

$$
\begin{aligned}
\gamma^{\beta\mu}{}_\rho &\quad\rightarrow\quad \texttt{gammaT = transpose(gamma)}\\
\gamma^\beta{}_{\mu\rho} &\quad\rightarrow\quad \texttt{gammaL = transpose(dot(gmunu,gamma))}
\end{aligned}
$$

Define the following 4×4 matrices.

$$
\begin{aligned}
(\not p_1+m) &\quad\rightarrow\quad \texttt{X1 = pslash1 + m I}\\
(\not p_2+m) &\quad\rightarrow\quad \texttt{X2 = pslash2 + m I}\\
(\not p_3+m) &\quad\rightarrow\quad \texttt{X3 = pslash3 + m I}\\
(\not p_4+m) &\quad\rightarrow\quad \texttt{X4 = pslash4 + m I}
\end{aligned}
$$

Then for f_{11} we have the following Eigenmath code. The contract function sums over α.

$$
\begin{aligned}
(\not p_3+m)^\alpha{}_\beta\gamma^{\mu\beta}{}_\rho(\not p_1+m)^\rho{}_\sigma\gamma^{\nu\sigma}{}_\alpha &\quad\rightarrow\quad \texttt{T1 = contract(dot(X3,gammaT,X1,gammaT),1,4)}\\
(\not p_4+m)^\alpha{}_\beta\gamma_\mu{}^\beta{}_\rho(\not p_2+m)^\rho{}_\sigma\gamma_\nu{}^\sigma{}_\alpha &\quad\rightarrow\quad \texttt{T2 = contract(dot(X4,gammaL,X2,gammaL),1,4)}
\end{aligned}
$$

Next, multiply then sum over repeated indices. The dot function sums over ν then the contract function sums over μ. The transpose makes the ν indices adjacent as required by the dot function.

$$
f_{11} = \mathrm{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu)\,\mathrm{Tr}(\cdots\gamma_\mu\cdots\gamma_\nu) \quad\rightarrow\quad \texttt{contract(dot(T1,transpose(T2)))}
$$

Follow suit for f_{22}.

$$
\begin{aligned}
(\not p_4+m)^\alpha{}_\beta\gamma^{\mu\beta}{}_\rho(\not p_1+m)^\rho{}_\sigma\gamma^{\nu\sigma}{}_\alpha &\quad\rightarrow\quad \texttt{T1 = contract(dot(X4,gammaT,X1,gammaT),1,4)}\\
(\not p_3+m)^\alpha{}_\beta\gamma_\mu{}^\beta{}_\rho(\not p_2+m)^\rho{}_\sigma\gamma_\nu{}^\sigma{}_\alpha &\quad\rightarrow\quad \texttt{T2 = contract(dot(X3,gammaL,X2,gammaL),1,4)}
\end{aligned}
$$

Then

$$
f_{22} = \mathrm{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu)\,\mathrm{Tr}(\cdots\gamma_\mu\cdots\gamma_\nu) \quad\rightarrow\quad \texttt{contract(dot(T1,transpose(T2)))}
$$

The calculation of f_{12} begins with

$$
\begin{aligned}
&(\not p_3+m)^\alpha{}_\beta\gamma^{\mu\beta}{}_\rho(\not p_1+m)^\rho{}_\sigma\gamma^{\nu\sigma}{}_\tau(\not p_4+m)^\tau{}_\delta\gamma_\mu{}^\delta{}_\eta(\not p_2+m)^\eta{}_\xi\gamma_\nu{}^\xi{}_\alpha\\
&\qquad\rightarrow\quad \texttt{T = contract(dot(X3,gammaT,X1,gammaT,X4,gammaL,X2,gammaL),1,6)}
\end{aligned}
$$

Then sum over repeated indices μ and ν.

$$
f_{12} = \mathrm{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu\cdots\gamma_\mu\cdots\gamma_\nu) \quad\rightarrow\quad \texttt{contract(contract(T,1,3))}
$$