Compton scattering
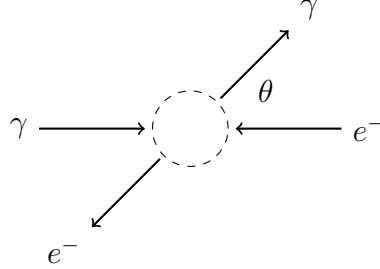
Compton scattering is the interaction $e^- + \gamma \to e^- + \gamma$.

In the center-of-mass frame we have the following momentum vectors where $E = \sqrt{\omega^2 + m^2}$.

$$
p_1 = \begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix}
\qquad
p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -\omega \end{pmatrix}
\qquad
p_3 = \begin{pmatrix} \omega \\ \omega \sin\theta \cos\phi \\ \omega \sin\theta \sin\phi \\ \omega \cos\theta \end{pmatrix}
\qquad
p_4 = \begin{pmatrix} E \\ -\omega \sin\theta \cos\phi \\ -\omega \sin\theta \sin\phi \\ -\omega \cos\theta \end{pmatrix}
$$

$$
\begin{array}{cccc}
\text{inbound} & \text{inbound} & \text{outbound} & \text{outbound} \\
\text{photon} & \text{electron} & \text{photon} & \text{electron}
\end{array}
$$

Spinors for the inbound electron.

$$
u_{21} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ -\omega \\ 0 \end{pmatrix}
\qquad
u_{22} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ 0 \\ \omega \end{pmatrix}
$$

$$
\begin{array}{cc}
\text{inbound electron} & \text{inbound electron} \\
\text{spin up} & \text{spin down}
\end{array}
$$

Spinors for the outbound electron.

$$
u_{41} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix}
\qquad
u_{42} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}
$$

$$
\begin{array}{cc}
\text{outbound electron} & \text{outbound electron} \\
\text{spin up} & \text{spin down}
\end{array}
$$

The probability amplitude $\mathcal{M}_{ab}$ for spin state ab is

$$
\mathcal{M}_{ba} = \mathcal{M}_{1ab} + \mathcal{M}_{2ab}
$$

where

$$
\mathcal{M}_{1ab} = \frac{\bar{u}_{4b}(-ie\gamma^\mu)(\slashed{q}_1 + m)(-ie\gamma^\nu)u_{2a}}{s - m^2}, \quad
\mathcal{M}_{2ab} = \frac{\bar{u}_{4b}(-ie\gamma^\nu)(\slashed{q}_2 + m)(-ie\gamma^\mu)u_{2a}}{u - m^2}
$$

Matrices $\slashed{q}_1$ and $\slashed{q}_2$ represent momentum transfer.

$$
\slashed{q}_1 = (p_1 + p_2)^\alpha g_{\alpha\beta}\gamma^\beta
$$

$$
\slashed{q}_2 = (p_4 - p_1)^\alpha g_{\alpha\beta}\gamma^\beta
$$

Scalars s and u are Mandelstam variables.

$$s = (p_1 + p_2)^2$$
$$u = (p_1 - p_4)^2$$

In component form (note that indices μ and ν are interchanged for $\mathcal{M}_{2ab}$)

$$(\mathcal{M}_{1ab})^{\mu\nu} = \frac{(\bar{u}_{4b})_\alpha(-ie\gamma^{\mu\alpha}{}_\beta)(\slashed{q}_1 + m)^\beta{}_\rho(-ie\gamma^{\nu\rho}{}_\sigma)(u_{2a})^\sigma}{s - m^2}$$

$$(\mathcal{M}_{2ab})^{\nu\nu} = \frac{(\bar{u}_{4b})_\alpha(-ie\gamma^{\nu\alpha}{}_\beta)(\slashed{q}_2 + m)^\beta{}_\rho(-ie\gamma^{\mu\rho}{}_\sigma)(u_{2a})^\sigma}{u - m^2}$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is the average of spin states.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}|\mathcal{M}_{ab}|^2$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\left(\mathcal{M}_{1ab}\mathcal{M}_{1ab}^* + \mathcal{M}_{1ab}\mathcal{M}_{2ab}^* + \mathcal{M}_{2ab}\mathcal{M}_{1ab}^* + \mathcal{M}_{2ab}\mathcal{M}_{2ab}^*\right)$$

Metric tensor $g_{\mu\nu}$ lowers indices.

$$\mathcal{M}_{1ab}\mathcal{M}_{1ab}^* = (\mathcal{M}_{1ab})^{\mu\nu}(\mathcal{M}_{1ab}^*)_{\mu\nu} = (\mathcal{M}_{1ab})^{\mu\nu}\,g_{\mu\alpha}(\mathcal{M}_{1ab}^*)^{\alpha\beta}g_{\beta\nu}$$
$$\mathcal{M}_{1ab}\mathcal{M}_{2ab}^* = (\mathcal{M}_{1ab})^{\mu\nu}(\mathcal{M}_{2ab}^*)_{\nu\mu} = (\mathcal{M}_{1ab})^{\mu\nu}\,g_{\nu\alpha}(\mathcal{M}_{2ab}^*)^{\alpha\beta}g_{\beta\mu}$$
$$\mathcal{M}_{2ab}\mathcal{M}_{2ab}^* = (\mathcal{M}_{2ab})^{\nu\mu}(\mathcal{M}_{2ab}^*)_{\nu\mu} = (\mathcal{M}_{2ab})^{\nu\mu}\,g_{\nu\alpha}(\mathcal{M}_{2ab}^*)^{\alpha\beta}g_{\beta\mu}$$

The Casimir trick uses matrix arithmetic to sum over spin states.

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left(\frac{f_{11}}{(s-m^2)^2} + \frac{2f_{12}}{(s-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2}\right) \tag{1}$$

where

$$f_{11} = \text{Tr}\left((\slashed{p}_2 + m)\gamma^\mu(\slashed{q}_1 + m)\gamma^\nu(\slashed{p}_4 + m)\gamma_\nu(\slashed{q}_1 + m)\gamma_\mu\right)$$

$$f_{12} = \text{Tr}\left((\slashed{p}_2 + m)\gamma^\mu(\slashed{q}_2 + m)\gamma^\nu(\slashed{p}_4 + m)\gamma_\mu(\slashed{q}_1 + m)\gamma_\nu\right)$$

$$f_{22} = \text{Tr}\left((\slashed{p}_2 + m)\gamma^\mu(\slashed{q}_2 + m)\gamma^\nu(\slashed{p}_4 + m)\gamma_\nu(\slashed{q}_2 + m)\gamma_\mu\right)$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu}b^\nu$)

$$f_{11} = 32(p_1 \cdot p_2)(p_1 \cdot p_4) + 64m^2(p_1 \cdot p_2) - 32m^2(p_1 \cdot p_3) - 32m^2(p_1 \cdot p_4) + 32m^4$$
$$f_{12} = 16m^2(p_1 \cdot p_2) - 16m^2(p_1 \cdot p_4) + 32m^4$$
$$f_{22} = 32(p_1 \cdot p_2)(p_1 \cdot p_4) + 32m^2(p_1 \cdot p_2) - 32m^2(p_1 \cdot p_3) - 64m^2(p_1 \cdot p_4) + 32m^4$$

In Mandelstam variables

$$f_{11} = -8su + 24sm^2 + 8um^2 + 8m^4$$
$$f_{12} = 8sm^2 + 8um^2 + 16m^4 \tag{2}$$
$$f_{22} = -8su + 8sm^2 + 24um^2 + 8m^4$$

Compton scattering experiments are typically done in the lab frame where the electron is at rest. Define Lorentz boost Λ for transforming momentum vectors to the lab frame.

$$\Lambda = \begin{pmatrix} E/m & 0 & 0 & \omega/m \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega/m & 0 & 0 & E/m \end{pmatrix}$$

The electron is at rest in the lab frame.

$$\Lambda p_2 = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Mandelstam variables are invariant under a boost.

$$s = (p_1 + p_2)^2 = (\Lambda p_1 + \Lambda p_2)^2$$
$$t = (p_1 - p_3)^2 = (\Lambda p_1 - \Lambda p_3)^2$$
$$u = (p_1 - p_4)^2 = (\Lambda p_1 - \Lambda p_4)^2$$

In the lab frame, let ω_L be the angular frequency of the incident photon and let ω'_L be the angular frequency of the scattered photon.

$$\omega_L = \Lambda p_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2}{m} + \frac{\omega E}{m}$$

$$\omega'_L = \Lambda p_3 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2 \cos\theta}{m} + \frac{\omega E}{m}$$

It can be shown that

$$s = m^2 + 2m\omega_L$$
$$t = 2m(\omega'_L - \omega_L) \tag{3}$$
$$u = m^2 - 2m\omega'_L$$

Then by (1), (2), and (3) we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \left(\frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1 \right)^2 - 1 \right)$$

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Lab scattering angle θ_L is given by the Compton equation

$$\cos \theta_L = \frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \cos^2 \theta_L - 1 \right)$$
$$= 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} - \sin^2 \theta_L \right)$$

Cross section

Now that we have derived $\langle |\mathcal{M}|^2 \rangle$ we can investigate the angular distribution of scattered photons. For simplicity let us drop the L subscript from lab variables. From now on the symbols ω, ω', and θ will be lab frame variables.

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4(4\pi\varepsilon_0)^2 s} \left(\frac{\omega'}{\omega} \right)^2 \langle |\mathcal{M}|^2 \rangle$$

where

$$s = m^2 + 2m\omega = (mc^2)^2 + 2(mc^2)(\hbar\omega)$$

and ω' is given by the Compton equation

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos \theta)}$$

For the lab frame we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

Hence in the lab frame

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\varepsilon_0)^2 s} \left(\frac{\omega'}{\omega} \right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

Noting that

$$e^2 = 4\pi\varepsilon_0 \alpha \hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 (\hbar c)^2}{2s} \left(\frac{\omega'}{\omega} \right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

Noting that

$$d\Omega = \sin \theta \, d\theta \, d\phi$$

we also have

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{2s} \left(\frac{\omega'}{\omega}\right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right) \sin\theta \, d\theta \, d\phi$$

Let $S(\theta_1, \theta_2)$ be the following integral of $d\sigma$.

$$S(\theta_1, \theta_2) = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{\pi\alpha^2(\hbar c)^2}{s}[I(\theta_2) - I(\theta_1)]$$

where

$$I(\theta) = -\frac{\cos\theta}{R^2} + \log\big(1 + R(1 - \cos\theta)\big)\left(\frac{1}{R} - \frac{2}{R^2} - \frac{2}{R^3}\right)$$

$$- \frac{1}{2R\big(1 + R(1 - \cos\theta)\big)^2} + \frac{1}{1 + R(1 - \cos\theta)}\left(-\frac{2}{R^2} - \frac{1}{R^3}\right)$$

and

$$R = \frac{\hbar\omega}{mc^2}$$

The cumulative distribution function is

$$F(\theta) = \frac{S(0, \theta)}{S(0, \pi)} = \frac{I(\theta) - I(0)}{I(\pi) - I(0)}, \quad 0 \le \theta \le \pi$$

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 < \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi) - I(0)} \left(\frac{\omega'}{\omega}\right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right) \sin\theta$$

Thomson scattering

For $\hbar\omega \ll mc^2$ we have

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos\theta)} \approx \omega$$

Hence we can use the approximations

$$\omega = \omega' \quad \text{and} \quad s = (mc^2)^2$$

to obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2\hbar^2}{2m^2c^2}(1 + \cos^2\theta)$$

which is the formula for Thomson scattering.

High energy approximation

For $\omega \gg m$ a useful approximation is to set $m = 0$ and obtain

$$f_{11} = -8su$$
$$f_{12} = 0$$
$$f_{22} = -8su$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left(\frac{-8su}{s^2} + \frac{-8su}{u^2}\right)$$
$$= 2e^4\left(-\frac{u}{s} - \frac{s}{u}\right)$$

The Mandelstam variables for $m = 0$ are

$$s = 4\omega^2$$
$$u = -2\omega^2(\cos\theta + 1)$$

Hence

$$\langle|\mathcal{M}|^2\rangle = 2e^4\left(\frac{\cos\theta + 1}{2} + \frac{2}{\cos\theta + 1}\right)$$

In the center of mass frame

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{4s(4\pi\epsilon_o)^2} = \frac{e^4}{2s(4\pi\epsilon_o)^2}\left(\frac{\cos\theta + 1}{2} + \frac{2}{\cos\theta + 1}\right)$$

Substitute $e^4 = (4\pi\varepsilon_0\alpha\hbar c)^2$ to obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\left(\frac{\cos\theta + 1}{2} + \frac{2}{\cos\theta + 1}\right) \times (\hbar c)^2$$

It follows that

$$\frac{d\sigma}{d\cos\theta} = 2\pi\frac{d\sigma}{d\Omega} = \frac{\pi\alpha^2}{s}\left(\frac{\cos\theta + 1}{2} + \frac{2}{\cos\theta + 1}\right) \times (\hbar c)^2$$

Cf. equation (1) of arxiv.org/pdf/hep-ex/0504012.