Klein-Nishina formula
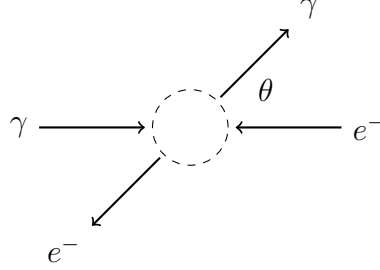
The Klein-Nishina formula is the lab frame cross section for $\gamma + e^- \to \gamma + e^-$.

It is easy to derive the Klein-Nishina formula from Dirac's equation by starting out in the center-of-mass frame and then boosting to the lab frame. The following center-of-mass momentum vectors have $E = \sqrt{\omega^2 + m^2}$.

$$
p_1 = \begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix} \quad
p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -\omega \end{pmatrix} \quad
p_3 = \begin{pmatrix} \omega \\ \omega \sin\theta \cos\phi \\ \omega \sin\theta \sin\phi \\ \omega \cos\theta \end{pmatrix} \quad
p_4 = \begin{pmatrix} E \\ -\omega \sin\theta \cos\phi \\ -\omega \sin\theta \sin\phi \\ -\omega \cos\theta \end{pmatrix}
$$
$$
\quad\; \gamma \longrightarrow \qquad\quad \longleftarrow e^- \qquad\qquad\quad\; \gamma \nearrow \qquad\qquad\qquad e^- \swarrow
$$

Spinors for p_2.

$$
u_{21} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ -\omega \\ 0 \end{pmatrix} \qquad
u_{22} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ 0 \\ \omega \end{pmatrix}
$$
$$
\qquad\qquad \text{spin up} \qquad\qquad\qquad\qquad\qquad \text{spin down}
$$

Spinors for p_4.

$$
u_{41} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix} \qquad
u_{42} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}
$$
$$
\qquad\qquad \text{spin up} \qquad\qquad\qquad\qquad\qquad \text{spin down}
$$

The scattering amplitude $\mathcal{M}_{ab}{}^{\mu\nu}$ for spin ab and polarization $\mu\nu$ is

$$
\mathcal{M}_{ab}{}^{\mu\nu} = \mathcal{M}_{1ab}{}^{\mu\nu} + \mathcal{M}_{2ab}{}^{\nu\mu}
$$

where

$$
\mathcal{M}_{1ab}{}^{\mu\nu} = \frac{\bar{u}_{4b}(-ie\gamma^\mu)(\slashed{q}_1 + m)(-ie\gamma^\nu)u_{2a}}{s - m^2}
$$
$$
\mathcal{M}_{2ab}{}^{\nu\mu} = \frac{\bar{u}_{4b}(-ie\gamma^\nu)(\slashed{q}_2 + m)(-ie\gamma^\mu)u_{2a}}{u - m^2}
$$

Matrices $\slashed{q}_1$ and $\slashed{q}_2$ represent momentum transfer.

$$\slashed{q}_1 = (p_1 + p_2)^\alpha g_{\alpha\beta} \gamma^\beta$$
$$\slashed{q}_2 = (p_4 - p_1)^\alpha g_{\alpha\beta} \gamma^\beta$$

Scalars s and u are Mandelstam variables.

$$s = (p_1 + p_2)^2$$
$$u = (p_1 - p_4)^2$$

In component form

$$\mathcal{M}_{1ab}{}^{\mu\nu} = \frac{(\bar{u}_{4b})_\alpha (-ie\gamma^{\mu\alpha}{}_\beta)(\slashed{q}_1 + m)^\beta{}_\rho (-ie\gamma^{\nu\rho}{}_\sigma)(u_{2a})^\sigma}{s - m^2}$$

$$\mathcal{M}_{2ab}{}^{\nu\mu} = \frac{(\bar{u}_{4b})_\alpha (-ie\gamma^{\nu\alpha}{}_\beta)(\slashed{q}_2 + m)^\beta{}_\rho (-ie\gamma^{\mu\rho}{}_\sigma)(u_{2a})^\sigma}{u - m^2}$$

Expected probability density $\langle |\mathcal{M}|^2 \rangle$ is the sum over squared amplitudes divided by the number of inbound states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{ab} \sum_{\mu\nu} |\mathcal{M}_{ab}{}^{\mu\nu}|^2$$

Summing over $\mu\nu$ requires $g_{\mu\nu}$ to lower indices.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{ab} \mathcal{M}_{ab}{}^{\mu\nu} \left(g_{\mu\alpha} \mathcal{M}_{ab}{}^{\alpha\beta} g_{\beta\nu} \right)^*$$

Expand the summand and label the terms. By positivity $\boxed{2} = \boxed{3}$.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{ab} \Big[\underbrace{\mathcal{M}_{1ab}{}^{\mu\nu} \left(g_{\mu\alpha} \mathcal{M}_{1ab}{}^{\alpha\beta} g_{\beta\nu} \right)^*}_{\boxed{1}} + \underbrace{\mathcal{M}_{1ab}{}^{\mu\nu} \left(g_{\nu\alpha} \mathcal{M}_{2ab}{}^{\alpha\beta} g_{\beta\mu} \right)^*}_{\boxed{2}}$$

$$+ \underbrace{\mathcal{M}_{2ab}{}^{\nu\mu} \left(g_{\mu\alpha} \mathcal{M}_{1ab}{}^{\alpha\beta} g_{\beta\nu} \right)^*}_{\boxed{3}} + \underbrace{\mathcal{M}_{2ab}{}^{\nu\mu} \left(g_{\nu\alpha} \mathcal{M}_{2ab}{}^{\alpha\beta} g_{\beta\mu} \right)^*}_{\boxed{4}} \Big]$$

The following Casimir trick uses matrix arithmetic to sum over spin and polarization states.

$$\sum_{ab} \boxed{1} = \frac{e^4}{(s - m^2)^2} \, \text{Tr} \left[(\slashed{p}_2 + m)\gamma^\mu (\slashed{q}_1 + m)\gamma^\nu (\slashed{p}_4 + m)\gamma_\nu (\slashed{q}_1 + m)\gamma_\mu \right]$$

$$\sum_{ab} \boxed{2} = \frac{e^4}{(s - m^2)(u - m^2)} \, \text{Tr} \left[(\slashed{p}_2 + m)\gamma^\mu (\slashed{q}_2 + m)\gamma^\nu (\slashed{p}_4 + m)\gamma_\mu (\slashed{q}_1 + m)\gamma_\nu \right]$$

$$\sum_{ab} \boxed{4} = \frac{e^4}{(u - m^2)^2} \, \text{Tr} \left[(\slashed{p}_2 + m)\gamma^\mu (\slashed{q}_2 + m)\gamma^\nu (\slashed{p}_4 + m)\gamma_\nu (\slashed{q}_2 + m)\gamma_\mu \right]$$

Probability density $\langle |\mathcal{M}|^2 \rangle$ can be reformulated as

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left[\frac{f_{11}}{(s - m^2)^2} + \frac{2f_{12}}{(s - m^2)(u - m^2)} + \frac{f_{22}}{(u - m^2)^2} \right]$$

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with Casimir trick terms

$$f_{11} = \text{Tr}\left[(\not p_2 + m)\gamma^\mu(\not q_1 + m)\gamma^\nu(\not p_4 + m)\gamma_\nu(\not q_1 + m)\gamma_\mu\right]$$

$$f_{12} = \text{Tr}\left[(\not p_2 + m)\gamma^\mu(\not q_2 + m)\gamma^\nu(\not p_4 + m)\gamma_\mu(\not q_1 + m)\gamma_\nu\right]$$

$$f_{22} = \text{Tr}\left[(\not p_2 + m)\gamma^\mu(\not q_2 + m)\gamma^\nu(\not p_4 + m)\gamma_\nu(\not q_2 + m)\gamma_\mu\right]$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$.)

$$\begin{aligned}
f_{11} &= 32(p_1 \cdot p_2)(p_1 \cdot p_4) + 32(p_1 \cdot p_2)m^2 + 32m^4 \\
f_{12} &= 16(p_1 \cdot p_2)m^2 - 16(p_1 \cdot p_4)m^2 + 32m^4 \\
f_{22} &= 32(p_1 \cdot p_2)(p_1 \cdot p_4) - 32(p_1 \cdot p_4)m^2 + 32m^4
\end{aligned} \tag{2}$$

In Mandelstam variables

$$\begin{aligned}
f_{11} &= -8su + 24sm^2 + 8um^2 + 8m^4 \\
f_{12} &= 8sm^2 + 8um^2 + 16m^4 \\
f_{22} &= -8su + 8sm^2 + 24um^2 + 8m^4
\end{aligned} \tag{3}$$

Scattering experiments are typically done in the lab frame. Define Lorentz boost Λ for transforming momentum vectors to the lab frame.

$$\Lambda = \begin{pmatrix} E/m & 0 & 0 & \omega/m \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega/m & 0 & 0 & E/m \end{pmatrix}$$

The electron is at rest in the lab frame.

$$\Lambda p_2 = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Mandelstam variables are invariant under a boost.

$$\begin{aligned}
s &= (p_1 + p_2)^2 = (\Lambda p_1 + \Lambda p_2)^2 \\
t &= (p_1 - p_3)^2 = (\Lambda p_1 - \Lambda p_3)^2 \\
u &= (p_1 - p_4)^2 = (\Lambda p_1 - \Lambda p_4)^2
\end{aligned} \tag{4}$$

In the lab frame, let ω_L be the angular frequency of the incident photon and let ω_L' be the angular frequency of the scattered photon.

$$\omega_L = \Lambda p_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2}{m} + \frac{\omega E}{m}$$

$$\omega_L' = \Lambda p_3 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2 \cos\theta}{m} + \frac{\omega E}{m}$$

It can be shown that

$$s = m^2 + 2m\omega_L$$
$$t = 2m(\omega_L' - \omega_L) \tag{5}$$
$$u = m^2 - 2m\omega_L'$$

Then by (1), (3), and (5) we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left[\frac{\omega_L}{\omega_L'} + \frac{\omega_L'}{\omega_L} + \left(\frac{m}{\omega_L} - \frac{m}{\omega_L'} + 1 \right)^2 - 1 \right] \tag{6}$$

Lab scattering angle θ_L is given by the Compton equation

$$\cos\theta_L = \frac{m}{\omega_L} - \frac{m}{\omega_L'} + 1$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{\omega_L}{\omega_L'} + \frac{\omega_L'}{\omega_L} + \cos^2\theta_L - 1 \right)$$
$$= 2e^4 \left(\frac{\omega_L}{\omega_L'} + \frac{\omega_L'}{\omega_L} - \sin^2\theta_L \right)$$

Now that we have derived $\langle |\mathcal{M}|^2 \rangle$ we can investigate the angular distribution of scattered photons. For simplicity let us drop the L subscript from lab variables. From now on the symbols ω, ω', and θ will be lab frame variables.

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4(4\pi\varepsilon_0)^2 s} \left(\frac{\omega'}{\omega} \right)^2 \langle |\mathcal{M}|^2 \rangle$$

where

$$s = m^2 + 2m\omega = (mc^2)^2 + 2(mc^2)(\hbar\omega)$$

and ω' is given by the Compton equation

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos\theta)}$$

For the lab frame we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta \right)$$

Hence in the lab frame

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\varepsilon_0)^2 s} \left(\frac{\omega'}{\omega} \right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta \right)$$

Substituting

$$e^2 = 4\pi\varepsilon_0 \alpha \hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s} \left(\frac{\omega'}{\omega} \right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta \right)$$

which is the Klein-Nishina formula.