Matrix mechanics 2

Show that for hydrogen[1]

$$L_1 = \begin{array}{cc} & \begin{array}{cccc} \psi_{1,0,0} & \psi_{2,1,-1} & \psi_{2,1,0} & \psi_{2,1,1} \end{array} \\ \begin{array}{c} \psi_{1,0,0} \\ \psi_{2,1,-1} \\ \psi_{2,1,0} \\ \psi_{2,1,1} \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{pmatrix} \end{array}$$

$$L_2 = \begin{array}{cc} & \begin{array}{cccc} \psi_{1,0,0} & \psi_{2,1,-1} & \psi_{2,1,0} & \psi_{2,1,1} \end{array} \\ \begin{array}{c} \psi_{1,0,0} \\ \psi_{2,1,-1} \\ \psi_{2,1,0} \\ \psi_{2,1,1} \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i\hbar}{\sqrt{2}} & 0 \\ 0 & -\frac{i\hbar}{\sqrt{2}} & 0 & \frac{i\hbar}{\sqrt{2}} \\ 0 & 0 & -\frac{i\hbar}{\sqrt{2}} & 0 \end{pmatrix} \end{array}$$

$$L_3 = \begin{array}{cc} & \begin{array}{cccc} \psi_{1,0,0} & \psi_{2,1,-1} & \psi_{2,1,0} & \psi_{2,1,1} \end{array} \\ \begin{array}{c} \psi_{1,0,0} \\ \psi_{2,1,-1} \\ \psi_{2,1,0} \\ \psi_{2,1,1} \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\hbar & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hbar \end{pmatrix} \end{array}$$

and

$$L^2 = \begin{array}{cc} & \begin{array}{cccc} \psi_{1,0,0} & \psi_{2,1,-1} & \psi_{2,1,0} & \psi_{2,1,1} \end{array} \\ \begin{array}{c} \psi_{1,0,0} \\ \psi_{2,1,-1} \\ \psi_{2,1,0} \\ \psi_{2,1,1} \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\hbar^2 & 0 & 0 \\ 0 & 0 & 2\hbar^2 & 0 \\ 0 & 0 & 0 & 2\hbar^2 \end{pmatrix} \end{array}$$

Verify

$$L^2 = {L_1}^2 + {L_2}^2 + {L_3}^2$$

and

$$\begin{aligned} L_2 L_3 - L_3 L_2 &= i\hbar L_1 \\ L_3 L_1 - L_1 L_3 &= i\hbar L_2 \\ L_1 L_2 - L_2 L_1 &= i\hbar L_3 \end{aligned}$$

[1]See p. 73 of *Quantum Mechanics in Matrix Form* by Günter Ludyk.

In spherical coordinates the angular momentum operators are

$$\hat{L}_1 = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$$
$$\hat{L}_2 = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$
$$\hat{L}_3 = -i\hbar\frac{\partial}{\partial\phi}$$

and

$$\hat{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$