Eigenvalues of angular momentum

We will derive eigenvalues of L^2 and L_z from the following commutation relations.

$$[L_x, L_y] = i\hbar L_z$$
$$[L_y, L_z] = i\hbar L_x$$
$$[L_z, L_x] = i\hbar L_y$$
$$[L^2, L_z] = 0$$

Start by defining ladder operators L_+ and L_-.

$$L_+ = L_x + iL_y$$
$$L_- = L_x - iL_y$$

We have the following commutation relations for ladder operators.

$$[L_z, L_+] = [L_z, L_x] + i[L_z, L_y]$$
$$= i\hbar L_y + i(-i\hbar L_x)$$
$$= \hbar L_+$$

$$[L_z, L_-] = [L_z, L_x] - i[L_z, L_y]$$
$$= i\hbar L_y - i(-i\hbar L_x)$$
$$= -\hbar L_-$$

We also have

$$L_-L_+ = (L_x - iL_y)(L_x + iL_y)$$
$$= L_x^2 + L_y^2 + i[L_x, L_y]$$
$$= L^2 - L_z^2 - \hbar L_z$$

$$L_+L_- = (L_x + iL_y)(L_x - iL_y)$$
$$= L_x^2 + L_y^2 - i[L_x, L_y]$$
$$= L^2 - L_z^2 + \hbar L_z$$

Operators L^2 and L_z commute hence they share eigenfunctions ψ.

Let λ be an eigenvalue of L^2 and let μ be an eigenvalue of L_z such that

$$L^2\psi = \lambda\psi$$

and

$$L_z\psi = \mu\psi$$

We will now show that

$$\lambda \geq \mu^2$$

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By definition of L^2 we have
$$L^2\psi = \left(L_x^2 + L_y^2 + L_z^2\right)\psi$$

Substitute λ for L^2 and μ for L_z to obtain
$$\lambda\psi = \left(L_x^2 + L_y^2 + \mu^2\right)\psi$$

Rewrite as
$$\left(L_x^2 + L_y^2\right)\psi = (\lambda - \mu^2)\psi$$

The eigenvalues of squared Hermitian operators are nonnegative hence $\lambda - \mu^2 \geq 0$. Hence
$$\lambda \geq \mu^2$$

The property $\lambda \geq \mu^2$ means that μ has an upper limit.

Let μ_m be the maximum μ. Then for eigenfunction ψ_m we have
$$L_z\psi_m = \mu_m\psi_m$$

Apply L_+ to both sides.
$$L_+L_z\psi_m = \mu_m L_+\psi_m$$

Expand the left hand side.
$$(L_zL_+ - L_zL_+ + L_+L_z)\psi_m = \mu_m L_+\psi_m$$

Substitute $\hbar L_+$ for $[L_z, L_+]$.
$$L_zL_+\psi_m - \hbar L_+\psi_m = \mu_m L_+\psi_m$$

Hence
$$L_zL_+\psi_m = (\mu_m + \hbar)L_+\psi_m$$

Because μ_m is the maximum eigenvalue and $\mu_m + \hbar > \mu_m$ we must have
$$L_+\psi_m = 0$$

Consequently
$$L_-L_+\psi_m = 0$$

Recalling that
$$L_-L_+ = L^2 - L_z^2 - \hbar L_z$$

we have
$$(L^2 - L_z^2 - \hbar L_z)\psi_m = (\lambda - \mu_m^2 - \hbar\mu_m)\psi_m = 0$$

Hence
$$\lambda = \mu_m^2 + \hbar\mu_m \tag{1}$$

Let μ_k be the minimum μ. Then for eigenfunction ψ_k we have
$$L_z\psi_k = \mu_k\psi_k$$

Apply L_- to both sides.
$$L_- L_z \psi_k = \mu_k L_- \psi_k$$

Expand the left hand side.
$$(L_z L_- - L_z L_- + L_- L_z)\psi_k = \mu_m L_- \psi_k$$

Substitute $-\hbar L_-$ for $[L_z, L_-]$.
$$L_z L_- \psi_k + \hbar L_- \psi_k = \mu_k L_- \psi_k$$

Hence
$$L_z L_- \psi_k = (\mu_k - \hbar) L_- \psi_k$$

Because μ_k is the minimum eigenvalue and $\mu_k - \hbar < \mu_k$ we must have
$$L_- \psi_k = 0$$

Consequently
$$L_+ L_- \psi_k = 0$$

Recalling that
$$L_+ L_- = L^2 - L_z^2 + \hbar L_z$$

we have
$$(L^2 - L_z^2 + \hbar L_z)\psi_k = (\lambda - \mu_k^2 + \hbar \mu_k)\psi_k = 0$$

Hence
$$\lambda = \mu_k^2 - \hbar \mu_k \tag{2}$$

By equivalence of (1) and (2) we have
$$\mu_m^2 + \hbar \mu_m - \mu_k^2 + \hbar \mu_k = 0 \tag{3}$$

By ladder operators there is an integer n such that
$$\mu_m = \mu_k + n\hbar$$

Substitute $\mu_k + n\hbar$ for μ_m in (3) to obtain
$$\mu_k^2 + 2\mu_k n\hbar + n^2 \hbar^2 + \hbar \mu_k + n\hbar^2 - \mu_k^2 + \hbar \mu_k = 0$$

Cancel μ_k^2 and rewrite the remaining terms as
$$2\mu_k(n+1)\hbar + n(n+1)\hbar^2 = 0$$

Divide through by $(n+1)\hbar$ to obtain
$$2\mu_k + n\hbar = 0$$

Hence
$$\mu_k = -\frac{n\hbar}{2}$$

and
$$\mu_m = \mu_k + n\hbar = \frac{n\hbar}{2}$$

Define quantum number l as
$$l = \frac{n}{2} = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \dots$$

Then
$$\mu_m = l\hbar$$

By equation (1) we have

$$\lambda = \mu_m^2 + \mu_m \hbar = (l\hbar)^2 + l\hbar^2 = l(l+1)\hbar^2$$

Hence $l(l+1)\hbar^2$ are eigenvalues of L^2.

$$L^2\psi = \lambda\psi = l(l+1)\hbar^2\psi$$

For a given l, operator L_z has eigenvalues

$$\mu = \mu_k, \dots, \mu_m = -l\hbar, (-l+1)\hbar, \dots, (l-1)\hbar, l\hbar$$

Define quantum number m as

$$m = -l, -l+1, \dots, l-1, l$$

Then
$$\mu = m\hbar$$

Hence $m\hbar$ are eigenvalues of L_z.

$$L_z\psi = \mu\psi = m\hbar\psi$$