

4-12. Carry out the integral in equation (4.64) by completing the square. Show that the correct free particle kernel (i.e., the three dimensional version of equation 3.3) results.

$$K_0(\mathbf{x}_b, t_b, \mathbf{x}_a, t_a) = \int \exp\left(\frac{i}{\hbar} \mathbf{p} \cdot (\mathbf{x}_b - \mathbf{x}_a)\right) \exp\left(-\frac{i}{\hbar} \frac{p^2}{2m} (t_b - t_a)\right) \frac{d^3 \mathbf{p}}{(2\pi\hbar)^3} \quad (4.64)$$

Let

$$T = t_b - t_a, \quad (X_1, X_2, X_3) = \mathbf{x}_b - \mathbf{x}_a$$

In component notation

$$\mathbf{p} \cdot (\mathbf{x}_b - \mathbf{x}_a) = p_1 X_1 + p_2 X_2 + p_3 X_3, \quad p^2 = \mathbf{p} \cdot \mathbf{p} = p_1^2 + p_2^2 + p_3^2$$

Substitute into (4.64) to obtain

$$K_0(\mathbf{x}_b, t_b, \mathbf{x}_a, t_a) = \frac{1}{(2\pi\hbar)^3} \times \int_{\mathbb{R}^3} \exp\left[\frac{i}{\hbar} \left(p_1 X_1 + p_2 X_2 + p_3 X_3 - \frac{p_1^2 + p_2^2 + p_3^2}{2m} T\right)\right] dp_1 dp_2 dp_3$$

Rewrite as

$$K_0(\mathbf{x}_b, t_b, \mathbf{x}_a, t_a) = \frac{1}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} \exp\left(\frac{i}{\hbar} p_1 X_1 - \frac{i}{\hbar} \frac{p_1^2}{2m} T\right) \times \exp\left(\frac{i}{\hbar} p_2 X_2 - \frac{i}{\hbar} \frac{p_2^2}{2m} T\right) \exp\left(\frac{i}{\hbar} p_3 X_3 - \frac{i}{\hbar} \frac{p_3^2}{2m} T\right) dp_1 dp_2 dp_3$$

Let

$$A = \frac{i}{2m\hbar} T \quad B = \frac{i}{\hbar} X_k$$

Then

$$\begin{aligned} \int_{-\infty}^{\infty} \exp(-Ap_k^2 + Bp_k) dp_k &= \left(\frac{\pi}{A}\right)^{1/2} \exp\left(\frac{B^2}{4A}\right) \\ &= \left(\frac{2\pi m\hbar}{iT}\right)^{1/2} \exp\left(\frac{imX_k^2}{2\hbar T}\right) \end{aligned} \quad (1)$$

Hence

$$K_0(\mathbf{x}_b, t_b, \mathbf{x}_a, t_a) = \frac{1}{(2\pi\hbar)^3} \prod_{k=1}^3 \left(\frac{2\pi m \hbar}{iT} \right)^{1/2} \exp \left(\frac{imX_k^2}{2\hbar T} \right)$$

Combine coefficients and convert to vector notation.

$$K_0(\mathbf{x}_b, t_b, \mathbf{x}_a, t_a) = \left(\frac{m}{2\pi i \hbar (t_b - t_a)} \right)^{3/2} \exp \left(\frac{im(\mathbf{x}_b - \mathbf{x}_a)^2}{2\hbar(t_b - t_a)} \right) \quad (2)$$