Exercise 3.3. Calculate the eigenvectors and eigenvalues of σ_n. *Hint:* Assume the eigenvector λ_1 has the form

$$\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

where α is an unknown parameter. Plug this vector into the eigenvalue equation and solve for α in terms of θ. Why did we use a single parameter α? Notice that our suggested column vector must have unit length.

Matrix σ_n is given on page 85.

$$\sigma_n = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Equation (3.5) is the eigenvalue equation where λ is an eigenvalue, $|\lambda\rangle$ is an eigenvector, and $\mathbf{M}$ is a matrix.

$$\mathbf{M}|\lambda\rangle = \lambda|\lambda\rangle \tag{3.5}$$

Following the above hint we have

$$\sigma_n|\lambda_1\rangle = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\alpha + \sin\theta\sin\alpha \\ \sin\theta\cos\alpha - \cos\theta\sin\alpha \end{pmatrix}$$

By sine and cosine angle difference formulas

$$\sigma_n|\lambda_1\rangle = \begin{pmatrix} \cos(\theta - \alpha) \\ \sin(\theta - \alpha) \end{pmatrix}$$

Then by equation (3.5) we have

$$\begin{pmatrix} \cos(\theta - \alpha) \\ \sin(\theta - \alpha) \end{pmatrix} = \lambda_1 \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

Taking $\lambda_1 = 1$ we have

$$\theta - \alpha = \alpha$$

Hence

$$\alpha = \frac{\theta}{2}$$