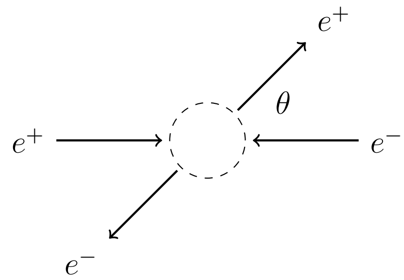
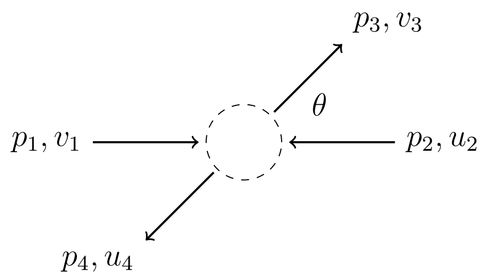
Bhabha scattering is the result of interactions between positrons and electrons. The following diagram represents a collider experiment with collinear electron and positron beams.



Here is the same diagram with momentum and spinor labels.



In a typical collider experiment the momentum vectors are

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \qquad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} \qquad p_3 = \begin{pmatrix} E \\ p\sin\theta\cos\phi \\ p\sin\theta\sin\phi \\ p\cos\theta \end{pmatrix} \qquad p_4 = \begin{pmatrix} E \\ -p\sin\theta\cos\phi \\ -p\sin\theta\sin\phi \\ -p\cos\theta \end{pmatrix}$$

where $E = \sqrt{p^2 + m^2}$. The spinors are

$$v_{11} = \begin{pmatrix} p \\ 0 \\ E+m \\ 0 \end{pmatrix} \quad u_{21} = \begin{pmatrix} E+m \\ 0 \\ -p \\ 0 \end{pmatrix} \quad v_{31} = \begin{pmatrix} p_3^z \\ p_3^x + ip_3^y \\ E+m \\ 0 \end{pmatrix} \quad u_{41} = \begin{pmatrix} E+m \\ 0 \\ p_4^z \\ p_4^x + ip_4^y \end{pmatrix}$$

$$v_{12} = \begin{pmatrix} 0 \\ -p \\ 0 \\ E+m \end{pmatrix} \quad u_{22} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ p \end{pmatrix} \quad v_{32} = \begin{pmatrix} p_3^x - ip_3^y \\ -p_3^z \\ 0 \\ E+m \end{pmatrix} \quad u_{42} = \begin{pmatrix} 0 \\ E+m \\ p_4^x - ip_4^y \\ -p_4^z \end{pmatrix}$$

The spinors shown above are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^4$ will be used.

This is the probability density for Bhabha scattering.

$$|\mathcal{M}(s_1, s_2, s_3, s_4)|^2 = \frac{e^4}{N} \left| -\frac{1}{t}(\bar{v}_1\gamma^\mu v_3)(\bar{u}_4\gamma_\mu u_2) + \frac{1}{s}(\bar{v}_1\gamma^\nu u_2)(\bar{u}_4\gamma_\nu v_3) \right|^2$$

Symbol s_j selects the spin (up or down) of spinor j. Symbol e is electron charge. Symbols s and t are Mandelstam variables $s = (p_1 + p_2)^2$ and $t = (p_1 - p_3)^2$.

Let

$$a_1 = (\bar{v}_1 \gamma^\mu v_3)(\bar{u}_4 \gamma_\mu u_2) \qquad a_2 = (\bar{v}_1 \gamma^\nu u_2)(\bar{u}_4 \gamma_\nu v_3)$$

Then

$$\begin{aligned}
|\mathcal{M}(s_1, s_2, s_3, s_4)|^2 &= \frac{e^4}{N} \left| -\frac{a_1}{t} + \frac{a_2}{s} \right|^2 \\
&= \frac{e^4}{N} \left(-\frac{a_1}{t} + \frac{a_2}{s} \right) \left(-\frac{a_1}{t} + \frac{a_2}{s} \right)^* \\
&= \frac{e^4}{N} \left(\frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{st} - \frac{a_1^* a_2}{st} + \frac{a_2 a_2^*}{s^2} \right)
\end{aligned}$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is computed by summing $|\mathcal{M}|^2$ over all spin states and dividing by the number of inbound states. There are four inbound states.

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{1}{4} \sum_{s_1=1}^{2} \sum_{s_2=1}^{2} \sum_{s_3=1}^{2} \sum_{s_4=1}^{2} |\mathcal{M}(s_1, s_2, s_3, s_4)|^2 \\
&= \frac{e^4}{4} \sum_{s_1=1}^{2} \sum_{s_2=1}^{2} \sum_{s_3=1}^{2} \sum_{s_4=1}^{2} \frac{1}{N} \left(\frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{st} - \frac{a_1^* a_2}{st} + \frac{a_2 a_2^*}{s^2} \right)
\end{aligned}$$

Use the Casimir trick to replace sums over spins with matrix products.

$$\begin{aligned}
f_{11} &= \frac{1}{N} \sum_{\text{spins}} a_1 a_1^* = \operatorname{Tr}\left((\not{p}_1 - m)\gamma^\mu (\not{p}_3 - m)\gamma^\nu \right) \operatorname{Tr}\left((\not{p}_4 + m)\gamma_\mu (\not{p}_2 + m)\gamma_\nu \right) \\
f_{12} &= \frac{1}{N} \sum_{\text{spins}} a_1 a_2^* = \operatorname{Tr}\left((\not{p}_1 - m)\gamma^\mu (\not{p}_2 + m)\gamma^\nu (\not{p}_4 + m)\gamma_\mu (\not{p}_3 - m)\gamma_\nu \right) \\
f_{22} &= \frac{1}{N} \sum_{\text{spins}} a_2 a_2^* = \operatorname{Tr}\left((\not{p}_1 - m)\gamma^\mu (\not{p}_2 + m)\gamma^\nu \right) \operatorname{Tr}\left((\not{p}_4 + m)\gamma_\mu (\not{p}_3 - m)\gamma_\nu \right)
\end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left(\frac{f_{11}}{t^2} - \frac{f_{12}}{st} - \frac{f_{12}^*}{st} + \frac{f_{22}}{s^2} \right)$$

Run "bhabha-scattering-1.txt" to verify the Casimir trick.

The following momentum formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$\begin{aligned}
f_{11} &= 32(p_1 \cdot p_2)(p_3 \cdot p_4) + 32(p_1 \cdot p_4)(p_2 \cdot p_3) - 32m^2(p_1 \cdot p_3) - 32m^2(p_2 \cdot p_4) + 64m^4 \\
f_{12} &= -32(p_1 \cdot p_4)(p_2 \cdot p_3) - 16m^2(p_1 \cdot p_2) + 16m^2(p_1 \cdot p_3) - 16m^2(p_1 \cdot p_4) \\
&\qquad - 16m^2(p_2 \cdot p_3) + 16m^2(p_2 \cdot p_4) - 16m^2(p_3 \cdot p_4) - 32m^4 \\
f_{22} &= 32(p_1 \cdot p_3)(p_2 \cdot p_4) + 32(p_1 \cdot p_4)(p_2 \cdot p_3) + 32m^2(p_1 \cdot p_2) + 32m^2(p_3 \cdot p_4) + 64m^4
\end{aligned}$$

In Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$ the formulas are

$$\begin{aligned}
f_{11} &= 8s^2 + 8u^2 - 64sm^2 - 64um^2 + 192m^4 \\
f_{12} &= -8u^2 + 64um^2 - 96m^4 \\
f_{22} &= 8t^2 + 8u^2 - 64tm^2 - 64um^2 + 192m^4
\end{aligned}$$

High energy approximation

When $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$
\begin{aligned}
f_{11} &= 8s^2 + 8u^2 \\
f_{12} &= -8u^2 \\
f_{22} &= 8t^2 + 8u^2
\end{aligned}
$$

For $m = 0$ the Mandelstam variables are

$$
\begin{aligned}
s &= 4E^2 \\
t &= -2E^2(1 - \cos\theta) = -4E^2 \sin^2(\theta/2) \\
u &= -2E^2(1 + \cos\theta) = -4E^2 \cos^2(\theta/2)
\end{aligned}
$$

It follows that

$$
s^2 t^2 = 256 E^8 \sin^4(\theta/2)
$$

The corresponding expected probability density is

$$
\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left(\frac{f_{11}}{t^2} - \frac{f_{12}}{st} - \frac{f_{12}^*}{st} + \frac{f_{22}}{s^2} \right) \\
&= \frac{e^4}{4s^2t^2} \left(s^2 f_{11} - st f_{12} - st f_{12}^* + t^2 f_{22} \right) \\
&= \frac{e^4}{4s^2t^2} \left(s^2 \left(8s^2 + 8u^2\right) + 16stu^2 + t^2 \left(8t^2 + 8u^2\right) \right) \\
&= \frac{e^4}{1024 E^8 \sin^4(\theta/2)} \left(256 E^8 \cos^4\theta + 1536 E^8 \cos^2\theta + 2304 E^8 \right) \\
&= \frac{e^4}{4 \sin^4(\theta/2)} \left(\cos^4\theta + 6\cos^2\theta + 9 \right) \\
&= \frac{e^4}{4} \frac{\left(\cos^2\theta + 3\right)^2}{\sin^4(\theta/2)}
\end{aligned}
$$

Run “bhabha-scattering-2.txt” to verify.

Cross section

This is the differential cross section for Bhabha scattering.

$$
\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{64\pi^2 s} = \frac{e^4}{1024\pi^2 E^2} \frac{\left(\cos^2\theta + 3\right)^2}{\sin^4(\theta/2)}
$$

Substituting $e^4 = 16\pi^2\alpha^2$ yields

$$
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{64E^2} \frac{\left(\cos^2\theta + 3\right)^2}{\sin^4(\theta/2)}
$$

We can integrate $d\sigma$ to obtain a cumulative distribution function. Recall that

$$
d\Omega = \sin\theta \, d\theta \, d\phi
$$

Hence

$$d\sigma = \frac{\alpha^2}{64E^2}\frac{\left(\cos^2\theta + 3\right)^2}{\sin^4(\theta/2)}\sin\theta\,d\theta\,d\phi$$

Let $I(\theta)$ be the following integral.

$$\begin{aligned} I(\theta) &= \left(\frac{64E^2}{\alpha^2}\right)\frac{1}{2\pi}\int_0^{2\pi}\int d\sigma \\ &= \int \frac{\left(\cos^2\theta + 3\right)^2}{\sin^4(\theta/2)}\sin\theta\,d\theta, \quad a \le \xi \le \pi \end{aligned}$$

Angular support is limited to an arbitrary $a > 0$ because $I(0)$ is undefined.

Let C be the normalization constant $C = I(\pi) - I(a)$. Then the cumulative distribution function $F(\theta)$ is

$$F(\theta) = \frac{I(\theta) - I(a)}{C}, \quad a \le \theta \le \pi$$

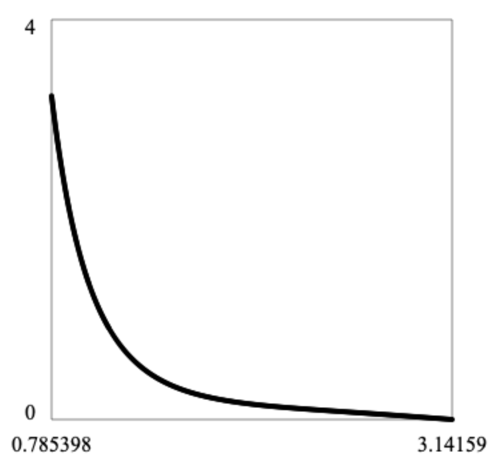
The probability of observing scattering events in the interval θ_1 to θ_2 can now be computed.

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

Probability density function $f(\theta)$ is the derivative of $F(\theta)$.

$$f(\theta) = \frac{dF(\theta)}{d\theta} = C^{-1}\frac{dI(\theta)}{d\theta} = C^{-1}\frac{\left(\cos^2\theta + 3\right)^2}{\sin^4(\theta/2)}\sin\theta$$

Run “bhabha-scattering-3.txt” to draw $f(\theta)$ for $a = \pi/4 = 45°$.



The following table shows the corresponding probability distribution for three bins.

θ_1	θ_2	$P(\theta_1 \le \theta \le \theta_2)$
0°	45°	–
45°	90°	0.83
90°	135°	0.13
135°	180°	0.04

SLAC data

The following Bhabha scattering data is adapted from SLAC-PUB-1501.

	Bin	x_k, x_{k+1}	y
(Smallest θ)	1	$0.6, 0.5$	4432
	2	$0.5, 0.4$	2841
	3	$0.4, 0.3$	2045
	4	$0.3, 0.2$	1420
	5	$0.2, 0.1$	1136
	6	$0.1, 0.0$	852
	7	$0.0, -0.1$	656
	8	$-0.1, -0.2$	625
	9	$-0.2, -0.3$	511
	10	$-0.3, -0.4$	455
	11	$-0.4, -0.5$	402
(Largest θ)	12	$-0.5, -0.6$	398

Data column y is the observed number of scattering events per bin.

To compute predicted counts $\hat{y}$, integrate the probability density function over each bin and multiply by the total number of observed counts.

$$P_k = C^{-1} \int_{\arccos(x_k)}^{\arccos(x_{k+1})} \frac{(\cos^2\theta + 3)^2}{\sin^4(\theta/2)} \sin\theta$$

$$\hat{y}_k = P_k \sum y$$

Bin	x_k, x_{k+1}	y	$\hat{y}$
1	$0.6, 0.5$	4432	4598
2	$0.5, 0.4$	2841	2880
3	$0.4, 0.3$	2045	1955
4	$0.3, 0.2$	1420	1410
5	$0.2, 0.1$	1136	1068
6	$0.1, 0.0$	852	843
7	$0.0, -0.1$	656	689
8	$-0.1, -0.2$	625	582
9	$-0.2, -0.3$	511	505
10	$-0.3, -0.4$	455	450
11	$-0.4, -0.5$	402	411
12	$-0.5, -0.6$	398	382

The coefficient of determination R^2 measures how well predicted values fit the real data. Let y be observed counts per bin and let $\hat{y}$ be predicted counts per bin. Then

$$R^2 = 1 - \frac{\sum(y-\hat{y})^2}{\sum(y-\bar{y})^2} = 0.997$$

The result indicates that the model $\langle|\mathcal{M}|^2\rangle$ explains 99.7% of the variance in the data.

Run "bhabha-scattering-4.txt" to verify.

DESY data

The following table shows DESY-PETRA Bhabha scattering data obtained from HEP Data.[1]

x	y
-0.73	0.10115
-0.6495	0.12235
-0.5495	0.11258
-0.4494	0.09968
-0.3493	0.14749
-0.2491	0.14017
-0.149	0.1819
-0.0488	0.22964
0.0514	0.25312
0.1516	0.30998
0.252	0.40898
0.3524	0.62695
0.4529	0.91803
0.5537	1.51743
0.6548	2.56714
0.7323	4.30279

Data x and y have the following relationship with the cross section model.

$$x = \cos\theta \qquad y = \frac{d\sigma}{d\Omega}$$

The differential cross section for Bhabha scattering is

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{64\pi^2 s} = \frac{\alpha^2}{2s}\left(\frac{s^2+u^2}{t^2} + \frac{2u^2}{st} + \frac{t^2+u^2}{s^2}\right)$$

The predicted cross section $\hat{y}$ is computed from data x and beam energy E as

$$\hat{y} = \frac{\alpha^2}{2s}\left(\frac{s^2+u^2}{t^2} + \frac{2u^2}{st} + \frac{t^2+u^2}{s^2}\right) \times (\hbar c)^2 \times 10^{37}$$

where

$$\begin{aligned} s &= 4E^2 \\ t &= -2E^2(1-x) \\ u &= -2E^2(1+x) \end{aligned}$$

Factor $(\hbar c)^2$ converts the result to SI and factor 10^{37} converts square meters to nanobarns.

The following table shows $\hat{y}$ for $E = 7.0\,\text{GeV}$.

[1] `www.hepdata.net/record/ins191231` (Table 3, 14.0 GeV)

x	y	$\hat{y}$
-0.73	0.10115	0.110296
-0.6495	0.12235	0.113816
-0.5495	0.11258	0.120101
-0.4494	0.09968	0.129075
-0.3493	0.14749	0.141592
-0.2491	0.14017	0.158934
-0.149	0.1819	0.182976
-0.0488	0.22964	0.216737
0.0514	0.25312	0.264989
0.1516	0.30998	0.335782
0.252	0.40898	0.44363
0.3524	0.62695	0.615528
0.4529	0.91803	0.9077
0.5537	1.51743	1.45175
0.6548	2.56714	2.60928
0.7323	4.30279	4.61509

The coefficient of determination R^2 measures how well predicted values fit the real data.

$$R^2 = 1 - \frac{\sum(y-\hat{y})^2}{\sum(y-\bar{y})^2} = 0.995$$

The result indicates that the model $d\sigma$ explains 99.5% of the variance in the data.

Run “bhabha-scattering-5.txt” to verify.

Notes on Eigenmath scripts

Here are a few notes about how the Eigenmath scripts work. In component notation the trace operators of the Casimir trick become sums over the repeated index α.

$$\begin{aligned}
f_{11} &= \Big((\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_3 - m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha\Big)\Big((\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha\Big)\\
f_{12} &= (\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_3 - m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha\\
f_{22} &= \Big((\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha\Big)\Big((\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_3 - m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha\Big)
\end{aligned}$$

To convert the above formulas to Eigenmath code, the γ tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply γ^μ by the metric tensor to lower the index.

$\gamma^{\beta\mu}{}_\rho \quad \rightarrow$ `gammaT = transpose(gamma)`

$\gamma^\beta{}_{\mu\rho} \quad \rightarrow$ `gammaL = transpose(dot(gmunu,gamma))`

Define the following 4×4 matrices.

$(\not{p}_1 - m) \quad \rightarrow$ `X1 = pslash1 - m I`

$(\not{p}_2 + m) \quad \rightarrow$ `X2 = pslash2 + m I`

$(\not{p}_3 - m) \quad \rightarrow$ `X3 = pslash3 - m I`

$(\not{p}_4 + m) \quad \rightarrow$ `X4 = pslash4 + m I`

Then for f_{11} we have the following Eigenmath code. The contract function sums over α.

$$(\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_3 - m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \quad \rightarrow \quad \texttt{T1 = contract(dot(X1,gammaT,X3,gammaT),1,4)}$$
$$(\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \quad \rightarrow \quad \texttt{T2 = contract(dot(X4,gammaL,X2,gammaL),1,4)}$$

Next, multiply then sum over repeated indices. The dot function sums over ν then the contract function sums over μ. The transpose makes the ν indices adjacent as required by the dot function.

$$f_{11} = \operatorname{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu)\operatorname{Tr}(\cdots\gamma_\mu\cdots\gamma_\nu) \quad \rightarrow \quad \texttt{f11 = contract(dot(T1,transpose(T2)))}$$

Follow suit for f_{22}.

$$(\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \quad \rightarrow \quad \texttt{T1 = contract(dot(X1,gammaT,X2,gammaT),1,4)}$$
$$(\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_3 - m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \quad \rightarrow \quad \texttt{T2 = contract(dot(X4,gammaL,X3,gammaL),1,4)}$$

Hence

$$f_{22} = \operatorname{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu)\operatorname{Tr}(\cdots\gamma_\mu\cdots\gamma_\nu) \quad \rightarrow \quad \texttt{f22 = contract(dot(T1,transpose(T2)))}$$

The calculation of f_{12} begins with

$$(\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_3 - m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha$$
$$\rightarrow \quad \texttt{T = contract(dot(X1,gammaT,X2,gammaT,X4,gammaL,X3,gammaL),1,6)}$$

Then sum over repeated indices μ and ν.

$$f_{12} = \operatorname{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu\cdots\gamma_\mu\cdots\gamma_\nu) \quad \rightarrow \quad \texttt{f12 = contract(contract(T,1,3))}$$