(3.2) Show that for the simple harmonic oscillator:

(a) $\left[\hat{a}, (\hat{a}^\dagger)^n\right] = n(\hat{a}^\dagger)^{n-1}$,
(b) $\langle 0|\hat{a}^n(\hat{a}^\dagger)^m|0\rangle = n!\delta_{nm}$,
(c) $\langle m|\hat{a}^\dagger|n\rangle = \sqrt{n+1}\delta_{m,n+1}$,
(d) $\langle m|\hat{a}|n\rangle = \sqrt{n}\delta_{m,n-1}$.

(a) We have for $r = m + n - 1$

$$\begin{aligned}\left[\hat{a}, (\hat{a}^\dagger)^n\right]|m\rangle &= \hat{a}(\hat{a}^\dagger)^n|m\rangle - (\hat{a}^\dagger)^n\hat{a}|m\rangle \\ &= \sqrt{m+n}\left(\prod_{k=1}^{n}\sqrt{m+k}\right)|r\rangle - \left(\prod_{k=1}^{n}\sqrt{m-1+k}\right)\sqrt{m}|r\rangle\end{aligned}$$

It follows that

$$\left[\hat{a}, (\hat{a}^\dagger)^n\right]|m\rangle = (m+n)\left(\prod_{k=1}^{n-1}\sqrt{m+k}\right)|r\rangle - m\left(\prod_{k=2}^{n}\sqrt{m-1+k}\right)|r\rangle$$

Noting that

$$\prod_{k=1}^{n-1}\sqrt{m+k} = \prod_{k=2}^{n}\sqrt{m-1+k}$$

we have

$$\left[\hat{a}, (\hat{a}^\dagger)^n\right]|m\rangle = n\left(\prod_{k=1}^{n-1}\sqrt{m+k}\right)|m+n-1\rangle$$

Hence

$$\left[\hat{a}, (\hat{a}^\dagger)^n\right]|m\rangle = n(\hat{a}^\dagger)^{n-1}|m\rangle$$

(b) We have

$$\langle 0|\hat{a}^n(\hat{a}^\dagger)^m|0\rangle = \left(\prod_{j=1}^{n}\sqrt{j}\right)\left(\prod_{k=1}^{m}\sqrt{k}\right)\langle n|m\rangle$$

Hence

$$\langle 0|\hat{a}^n(\hat{a}^\dagger)^m|0\rangle = \begin{cases} n!, & n = m \\ 0, & n \neq m \end{cases}$$

(c) We have

$$\langle m|\hat{a}^\dagger|n\rangle = \sqrt{n+1}\langle m|n+1\rangle$$

Hence

$$\langle m|\hat{a}^\dagger|n\rangle = \begin{cases} \sqrt{n+1}, & m = n+1 \\ 0, & m \neq n+1 \end{cases}$$

(d) We have

$$\langle m|\hat{a}|n\rangle = \sqrt{n}\langle m|n-1\rangle$$

Hence

$$\langle m|\hat{a}|n\rangle = \begin{cases} \sqrt{n}, & m = n-1 \\ 0, & m \neq n-1 \end{cases}$$