

3-7. Using equations (3.52) and (2.31), express $F(t+s)$ in terms of $F(t)$ and $F(s)$, where $t = t_b - t_c$ and $s = t_c - t_a$.

From equation (3.52) and $t + s = t_b - t_a$

$$K(b, a) = F(t + s) \exp \left(\frac{im(x_b - x_a)^2}{2\hbar(t + s)} \right) \quad (1)$$

From equation (2.31)

$$K(b, a) = \int_{-\infty}^{\infty} K(b, c) K(c, a) dx_c$$

Hence

$$K(b, a) = F(t)F(s) \int_{-\infty}^{\infty} \exp \left(\frac{im(x_b - x_c)^2}{2\hbar t} \right) \exp \left(\frac{im(x_c - x_a)^2}{2\hbar s} \right) dx_c$$

Solve the integral.

$$K(b, a) = F(t)F(s) \left(\frac{2\pi i \hbar t s}{m(t + s)} \right)^{1/2} \exp \left(\frac{im(x_b - x_a)^2}{2\hbar(t + s)} \right) \quad (2)$$

Equating (1) with (2) causes the exponentials to cancel leaving

$$F(t + s) = F(t)F(s) \left(\frac{2\pi i \hbar t s}{m(t + s)} \right)^{1/2} \quad (3)$$

Show that if

$$F(t) = \left(\frac{m}{2\pi i \hbar t} \right)^{1/2} f(t)$$

then the new function $f(t)$ must satisfy

$$f(t + s) = f(t)f(s)$$

By substitution

$$F(t + s) = \left(\frac{m}{2\pi i \hbar (t + s)} \right)^{1/2} f(t + s)$$

and

$$F(t)F(s) = \left(\frac{m}{2\pi i \hbar t} \right)^{1/2} \left(\frac{m}{2\pi i \hbar s} \right)^{1/2} f(t)f(s)$$

Then by (3) we have

$$\begin{aligned} \left(\frac{m}{2\pi i \hbar (t+s)} \right)^{1/2} f(t+s) = \\ \left(\frac{m}{2\pi i \hbar t} \right)^{1/2} \left(\frac{m}{2\pi i \hbar s} \right)^{1/2} f(t) f(s) \left(\frac{2\pi i \hbar t s}{m(t+s)} \right)^{1/2} \end{aligned}$$

The coefficients cancel leaving

$$f(t+s) = f(t)f(s) \tag{4}$$