Fun trick 1

Show that

$$[p^2, \mathbf{r}] = -2i\hbar\mathbf{p}$$

where

$$\mathbf{r} = \otimes \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{p} = -i\hbar\nabla, \quad p^2 = \mathbf{p}\cdot\mathbf{p} = -\hbar^2\nabla^2$$

Operator $\mathbf{r}$ forms the outer product of its operand with the vector (x, y, z).

Expanding the commutator we have

$$\begin{aligned}
[p^2, \mathbf{r}] &= p^2\mathbf{r} - \mathbf{r}p^2 \\
&= \mathbf{p}\cdot\mathbf{p}\mathbf{r} - \mathbf{r}\mathbf{p}\cdot\mathbf{p} \\
&= \mathrm{Tr}[\mathbf{ppr} - \mathbf{rpp}] \\
&= \mathrm{Tr}[\mathbf{ppr} - \mathbf{prp} + \mathbf{prp} - \mathbf{rpp}] && \text{trick!} \\
&= \mathrm{Tr}[\mathbf{p}(\mathbf{pr} - \mathbf{rp}) + (\mathbf{pr} - \mathbf{rp})\mathbf{p}] \\
&= (-i\hbar)\mathbf{p} + (-i\hbar)\mathbf{p} \\
&= -2i\hbar\mathbf{p}
\end{aligned}$$

The trick is introducing null term $\mathbf{prp} - \mathbf{prp}$ so that the operators can be factored. Trace operator Tr contracts on the first and second indices.

Verify the following formulas.

$$[p^2, \mathbf{r}] = -2i\hbar\mathbf{p} \tag{1}$$

$$[p^2, \mathbf{r}] = \mathrm{Tr}[\mathbf{ppr} - \mathbf{rpp}] \tag{2}$$

$$[p^2, \mathbf{r}] = \mathrm{Tr}[\mathbf{ppr} - \mathbf{prp} + \mathbf{prp} - \mathbf{rpp}] \tag{3}$$

$$\mathbf{pr} - \mathbf{rp} = -i\hbar\mathbf{I} \tag{4}$$

$$\mathbf{p}\cdot\mathbf{p} = \mathrm{Tr}[\mathbf{pp}] \tag{5}$$

From the Hamiltonian

$$H(\mathbf{r}, t) = \frac{p^2}{2m} + V(\mathbf{r}, t)$$

we have for p^2

$$p^2 = 2m(H - V)$$

Then by substitution (V cancels because it commutes with $\mathbf{r}$)

$$[H, \mathbf{r}] = -\frac{i\hbar}{m}\mathbf{p}$$

Hence we have for momentum $\mathbf{p}$

$$\mathbf{p} = \frac{i}{\hbar}m[H, \mathbf{r}]$$