

6-6. Suppose the potential is that of a central force. Thus $V(\mathbf{r}) = V(r)$. Show that $v(\check{\mathbf{p}})$ can be written as

$$v(\check{\mathbf{p}}) = v(\check{p}) = \frac{4\pi\hbar}{\check{p}} \int_0^\infty \sin\left(\frac{\check{p}r}{\hbar}\right) V(r)r dr \quad (6.45)$$

Suppose $v(r)$ is the Coulomb potential $-Ze^2/r$. In this case the integral for $v(\check{p})$ is oscillatory at the upper limit. But convergence of the integral can be artificially forced by introducing the factor $e^{-\epsilon r}$ and then taking the limit of the result as $\epsilon \rightarrow 0$. Following through this calculation, show that the cross section corresponds to the Rutherford cross section

$$\frac{d\sigma_{\text{Ruth}}}{d\Omega} = \frac{4m^2 Z^2 e^4}{\check{p}^4} \quad (6.46)$$

Consider equation (6.39).

$$v(\check{\mathbf{p}}) = \int \exp\left(\frac{i\check{\mathbf{p}} \cdot \mathbf{r}}{\hbar}\right) V(\mathbf{r}) d^3\mathbf{r} \quad (6.39)$$

Convert (6.39) to polar coordinates.

$$v(\check{\mathbf{p}}) = \int_0^{2\pi} \int_0^\pi \int_0^\infty \exp\left(\frac{i\check{p}r \cos\theta}{\hbar}\right) V(r)r^2 \sin\theta dr d\theta d\phi$$

Integrate over ϕ .

$$v(\check{\mathbf{p}}) = 2\pi \int_0^\pi \int_0^\infty \exp\left(\frac{i\check{p}r \cos\theta}{\hbar}\right) V(r)r^2 \sin\theta dr d\theta$$

Convert the exponential to rectangular form.

$$v(\check{\mathbf{p}}) = 2\pi \int_0^\pi \int_0^\infty \left(\cos\left(\frac{\check{p}r \cos\theta}{\hbar}\right) + i \sin\left(\frac{\check{p}r \cos\theta}{\hbar}\right) \right) V(r)r^2 \sin\theta dr d\theta$$

Integrate over θ .

$$v(\check{\mathbf{p}}) = \frac{4\pi\hbar}{\check{p}} \int_0^\infty \sin\left(\frac{\check{p}r}{\hbar}\right) V(r)r dr$$

The above result is due to the integrals

$$\begin{aligned}\int_0^\pi \cos(A \cos \theta) \sin \theta \, d\theta &= \frac{2 \sin A}{A} \\ \int_0^\pi \sin(A \cos \theta) \sin \theta \, d\theta &= 0\end{aligned}$$

Given the Coulomb potential and integration trick we have

$$v(\check{p}) = -\frac{4\pi\hbar Z e^2}{\check{p}} \int_0^\infty \sin\left(\frac{\check{p}r}{\hbar}\right) \exp(-\epsilon r) \, dr$$

Solve the integral.

$$v(\check{p}) = -\frac{4\pi\hbar Z e^2}{\check{p}} \frac{\check{p}/\hbar}{(\check{p}/\hbar)^2 + \epsilon^2}$$

Let $\epsilon \rightarrow 0$.

$$v(\check{p}) = -\frac{4\pi\hbar^2 Z e^2}{\check{p}^2}$$

By equation (6.44)

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |v(\check{p})|^2 = \frac{4m^2 Z^2 e^4}{\check{p}^4}$$