

Exercise 4.6. Carry out the Schrodinger Ket recipe for a single spin. The Hamiltonian is  $\mathbf{H} = \frac{\omega\hbar}{2}\sigma_z$  and the final observable is  $\sigma_x$ . The initial state is given as  $|u\rangle$  (the state in which  $\sigma_z = +1$ ).

After time  $t$ , an experiment is done to measure  $\sigma_y$ . What are the possible outcomes and what are the probabilities for those outcomes?

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Note: Use  $|r\rangle$  for the initial state instead of  $|u\rangle$ . Otherwise, the result is time-independent.

Step 1 of the Schrodinger Ket recipe is obtain  $\mathbf{H}$ , which we already have by hypothesis.

$$\mathbf{H} = \frac{\hbar\omega}{2}\sigma_z = \frac{\hbar\omega}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Step 2. Prepare an initial state  $|\Psi(0)\rangle$ . By hypothesis the initial state is  $|r\rangle$ .

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle \quad (2.5)$$

Hence

$$|\Psi(0)\rangle = |r\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Step 3. Find the eigenvalues and eigenvectors of  $\mathbf{H}$  by solving the time-independent Schrodinger equation,

$$\mathbf{H}|E_j\rangle = E_j|E_j\rangle$$

The eigenvalues are obtained by solving the characteristic equation  $\det(\mathbf{H} - E_j\mathbf{I}) = 0$ .

$$\begin{aligned} \det(\mathbf{H} - E_j\mathbf{I}) &= \begin{vmatrix} \frac{\hbar\omega}{2} - E_j & 0 \\ 0 & -\frac{\hbar\omega}{2} - E_j \end{vmatrix} \\ &= \left(\frac{\hbar\omega}{2} - E_j\right) \left(-\frac{\hbar\omega}{2} - E_j\right) \\ &= E_j^2 - \left(\frac{\hbar\omega}{2}\right)^2 = 0 \end{aligned}$$

Hence the eigenvalues are

$$E_1 = \frac{\hbar\omega}{2}, \quad E_2 = -\frac{\hbar\omega}{2}$$

The eigenvectors of  $\mathbf{H}$  are  $|u\rangle$  and  $|d\rangle$ .

$$|E_1\rangle = |u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |E_2\rangle = |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Step 4. Use the initial state vector  $|\Psi(0)\rangle$ , along with the eigenvectors  $|E_j\rangle$  from step 3, to calculate the initial coefficients  $\alpha_j(0)$ :

$$\alpha_j(0) = \langle E_j | \Psi(0) \rangle$$

We have

$$\alpha_1(0) = \langle E_1 | r \rangle = \frac{1}{\sqrt{2}}$$

$$\alpha_2(0) = \langle E_2 | r \rangle = \frac{1}{\sqrt{2}}$$

Step 5. Rewrite  $|\Psi(0)\rangle$  in terms of the eigenvectors  $|E_j\rangle$  and the initial coefficients  $\alpha_j(0)$ :

$$|\Psi(0)\rangle = \sum_j \alpha_j(0) |E_j\rangle$$

We have

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Step 6. In the above equation, replace each  $\alpha_j(0)$  with  $\alpha_j(t)$  to capture its time-independence. As a result,  $|\Psi(0)\rangle$  becomes  $|\Psi(t)\rangle$ :

$$|\Psi(t)\rangle = \sum_j \alpha_j(t) |E_j\rangle$$

We have

$$|\Psi(t)\rangle = \alpha_1(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Step 7. Using Eq. 4.30, replace each  $\alpha_j(t)$  with  $\alpha_j(0) \exp(-iE_j t/\hbar)$ :

$$|\Psi(t)\rangle = \sum_j \alpha_j(0) \exp\left(-\frac{i}{\hbar} E_j t\right) |E_j\rangle \quad (4.34)$$

We have

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}} \exp\left(-\frac{i}{\hbar} E_1 t\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \exp\left(-\frac{i}{\hbar} E_2 t\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}} \exp\left(-\frac{i\omega t}{2}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \exp\left(\frac{i\omega t}{2}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This concludes the Schrodinger Ket recipe.

Recall that

$$|i\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}, \quad |o\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$$

The probability of measuring  $\sigma_y = 1$  is

$$P(1 \mid \sigma_y) = \langle i | \Psi(t) \rangle \langle \Psi(t) | i \rangle = \frac{1 + \sin \omega t}{2}$$

The probability of measuring  $\sigma_y = -1$  is

$$P(-1 \mid \sigma_y) = \langle o | \Psi(t) \rangle \langle \Psi(t) | o \rangle = \frac{1 - \sin \omega t}{2}$$