9-5. The momentum in the field is given by

$$\frac{1}{4\pi c}\int \mathbf{E}\times\mathbf{B}\,d^3\mathbf{r}$$

In the absence of matter (so $\phi_{\mathbf{k}}=0$), show that this is

$$i\int \mathbf{k}\,(\mathbf{a}_{\mathbf{k}}^*\cdot\dot{\mathbf{a}}_{\mathbf{k}})\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

Note that

$$\mathbf{E}\times\mathbf{B}=(E_yB_z-E_zB_y)\mathbf{i}+(E_zB_x-E_xB_z)\mathbf{j}+(E_xB_y-E_yB_x)\mathbf{k} \tag{1}$$

where

$$\mathbf{i}=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad \mathbf{j}=\begin{pmatrix}0\\1\\0\end{pmatrix},\quad \mathbf{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

Consider equation (9.9).

$$\mathbf{E}=-\nabla\phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} \tag{9.9}$$

By hypothesis the scalar potential vanishes leaving

$$\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}=-\frac{1}{c}\frac{\partial A_x}{\partial t}\mathbf{i}-\frac{1}{c}\frac{\partial A_y}{\partial t}\mathbf{j}-\frac{1}{c}\frac{\partial A_z}{\partial t}\mathbf{k} \tag{2}$$

Consider equation (9.7).

$$\mathbf{B}=\nabla\times\mathbf{A} \tag{9.7}$$

It follows that

$$\mathbf{B}=\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\mathbf{k} \tag{3}$$

Substitute (2) and (3) into (1) to obtain

$$\mathbf{E}\times\mathbf{B}=F_x\mathbf{i}+F_y\mathbf{j}+F_z\mathbf{k}$$

where