Harmonic oscillator propagator

Consider the harmonic oscillator eigenstate

$$\psi_n(x,t) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right) \exp\left(-\frac{m\omega x^2}{2\hbar} - i\left(n+\tfrac{1}{2}\right)\omega t\right)$$

and the harmonic oscillator propagator

$$K(x_b,t_b,x_a,t_a) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega T)}\right)^{\frac{1}{2}} \exp\left[\frac{im\omega}{2\hbar\sin(\omega T)}\left(x_a^2\cos(\omega T) - 2x_a x_b + x_b^2\cos(\omega T)\right)\right]$$

where $T = t_b - t_a$.

We should have

$$\psi_n(x_b,T) = \int_{-\infty}^{\infty} K(x_b,T,x_a,0)\psi_n(x_a,0)\,dx_a$$

Try for $n=1$.

$$\psi_1(x_a,0) = \sqrt{2}\left(\frac{m^2\omega^3}{\pi\hbar^3}\right)^{\frac{1}{4}} x_a \exp\left(-\frac{m\omega x_a^2}{2\hbar}\right)$$

By the identity

$$\int_{-\infty}^{\infty} y\exp\left(-ay^2+by\right)\,dy = \frac{\sqrt{\pi}}{2}\frac{b}{a^{3/2}}\exp\left(\frac{b^2}{4a}\right)$$

the path integral is

$$I = \int_{-\infty}^{\infty} K(x_b,T,x_a,0)\psi_1(x_a,0)\,dx_a = \sqrt{2}\left(\frac{m^3\omega^3}{\pi\hbar^3}\right)^{\frac{1}{4}} x_b\exp\left(-\frac{m\omega x_b^2}{2\hbar} - \tfrac{3}{2}i\omega T\right)$$

Hence

$$I = \psi_1(x_b,T)$$

Click here to verify.

Ref. Feynman and Hibbs problem 3-12.