Path integral step by step

① Let H be the following Hamiltonian where $V \equiv V(x,t)$.

$$H = \frac{\hat{p}^2}{2m} + V$$

② Let K_j be the amplitude to go from x_j to x_{j+1} in time δt where $\delta t = T/N$.

$$K_j = \langle x_{j+1} | \exp\left(-\frac{i}{\hbar}H\delta t\right) | x_j \rangle$$

(3) By the identity

$$\int |p\rangle\langle p|\,dp = 1$$

we can write

$$K_{j} = \int \langle x_{j+1} | \exp \left(-\frac{i}{\hbar} H \delta t \right) | p \rangle \langle p | x_{j} \rangle dp$$

(4) Replace operator \hat{p} with its eigenvalue p.

$$K_{j} = \int \exp\left(-\frac{i}{\hbar} \left(\frac{p^{2}}{2m} + V\right) \delta t\right) \langle x_{j+1} | p \rangle \langle p | x_{j} \rangle dp$$

(5) By the identity

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right)$$

we can write

$$K_{j} = \frac{1}{2\pi\hbar} \int \left(-\frac{i}{\hbar} \left(\frac{p^{2}}{2m} + V \right) \delta t \right) \exp\left(\frac{i}{\hbar} p x_{j+1} \right) \exp\left(-\frac{i}{\hbar} p x_{j} \right) dp$$

(6) Combine exponentials.

$$K_{j} = \frac{1}{2\pi\hbar} \int \exp\left(-\frac{i}{\hbar} \left(\frac{p^{2}}{2m} + V\right) \delta t + \frac{i}{\hbar} p(x_{j+1} - x_{j})\right) dp$$

(7) The V term does not depend on p so factor it out of the integrand.

$$K_{j} = \frac{1}{2\pi\hbar} \exp\left(-\frac{i}{\hbar}V\delta t\right) \int \exp\left(-\frac{i}{\hbar}\frac{p^{2}}{2m}\delta t + \frac{i}{\hbar}p(x_{j+1} - x_{j})\right) dp$$

(8) To solve the integral, use the identity

$$\int_{-\infty}^{\infty} \exp(-ay^2 + by) \, dy = \left(\frac{\pi}{a}\right)^{1/2} \exp\left(\frac{b^2}{4a}\right)$$

with

$$a = \frac{i}{\hbar} \frac{\delta t}{2m}, \quad b = \frac{i}{\hbar} (x_{j+1} - x_j)$$

to obtain

$$K_{j} = \frac{1}{2\pi\hbar} \exp\left(-\frac{i}{\hbar}V\delta t\right) \left(\frac{2\pi m\hbar}{i\delta t}\right)^{1/2} \exp\left(\frac{i}{\hbar}\frac{m}{2}\frac{(x_{j+1} - x_{j})^{2}}{\delta t}\right)$$

(9) Combine coefficients and exponentials.

$$K_{j} = \left(\frac{m}{2\pi i\hbar \delta t}\right)^{1/2} \exp\left(\frac{i}{\hbar} \frac{m}{2} \frac{(x_{j+1} - x_{j})^{2}}{\delta t} - \frac{i}{\hbar} V \delta t\right)$$

(10) Factor out i/\hbar and δt .

$$K_{j} = \left(\frac{m}{2\pi i\hbar \delta t}\right)^{1/2} \exp\left[\frac{i}{\hbar} \left(\frac{m}{2} \left(\frac{x_{j+1} - x_{j}}{\delta t}\right)^{2} - V\right) \delta t\right]$$

① Let K(b,a) be the amplitude to go from x_a at time zero to x_b at time T where $x_0 \equiv x_a$ and $x_N \equiv x_b$.

$$K(b,a) = \int dx_{N-1} \cdots \int dx_2 \int dx_1 K_{N-1} \cdots K_2 K_1 K_0$$

 \bigcirc Substitute for the K_i .

$$K(b,a) = \int dx_{N-1} \cdots \int dx_2 \int dx_1 \left(\frac{m}{2\pi i\hbar \delta t}\right)^{N/2} \exp\left[\frac{i}{\hbar} \sum_{j=0}^{N-1} \left(\frac{m}{2} \left(\frac{x_{j+1} - x_j}{\delta t}\right)^2 - V\right) \delta t\right]$$

(13) In the limit we have

$$\lim_{N \to \infty} \sum_{j=0}^{N-1} \left(\frac{m}{2} \left(\frac{x_{j+1} - x_j}{\delta t} \right)^2 - V \right) \delta t = \int_0^T \left(\frac{m}{2} \left(\frac{d}{dt} x(t) \right)^2 - V \right) dt$$

Hence

$$K(b,a) = \int dx_{N-1} \cdots \int dx_2 \int dx_1 \left(\frac{m}{2\pi i\hbar \delta t}\right)^{N/2} \exp\left[\frac{i}{\hbar} \int_0^T \left(\frac{m}{2} \left(\frac{d}{dt}x(t)\right)^2 - V\right) dt\right]$$

(14) Define

$$\int Dx(t) = \lim_{N \to \infty} \int dx_{N-1} \cdots \int dx_2 \int dx_1 \left(\frac{m}{2\pi i\hbar \delta t}\right)^{N/2}$$

and write

$$K(b,a) = \int Dx(t) \, \exp \left[\frac{i}{\hbar} \int_0^T \left(\frac{m}{2} \left(\frac{d}{dt} x(t) \right)^2 - V \right) \, dt \right]$$