Spin state

The result of measuring spin is either $+\hbar/2$ or $-\hbar/2$. Let

$$\chi = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$$

For spin state χ and all three axes, find the probabilities of measuring $\pm\hbar/2$.

These are the eigenstates for spin.

$$\begin{aligned} |x_+\rangle &= \tfrac{1}{\sqrt{2}}(1,1) & |y_+\rangle &= \tfrac{1}{\sqrt{2}}(1,i) & |z_+\rangle &= (1,0) \\ |x_-\rangle &= \tfrac{1}{\sqrt{2}}(1,-1) & |y_-\rangle &= \tfrac{1}{\sqrt{2}}(1,-i) & |z_-\rangle &= (0,1) \end{aligned}$$

For the x direction we have

$$\Pr\left(+\tfrac{\hbar}{2}\right) = |\langle x_+|\chi\rangle|^2 = \tfrac{13}{18}, \quad \Pr\left(-\tfrac{\hbar}{2}\right) = |\langle x_-|\chi\rangle|^2 = \tfrac{5}{18}$$

For the y direction we have

$$\Pr\left(+\tfrac{\hbar}{2}\right) = |\langle y_+|\chi\rangle|^2 = \tfrac{17}{18}, \quad \Pr\left(-\tfrac{\hbar}{2}\right) = |\langle y_-|\chi\rangle|^2 = \tfrac{1}{18}$$

For the z direction we have

$$\Pr\left(+\tfrac{\hbar}{2}\right) = |\langle z_+|\chi\rangle|^2 = \tfrac{5}{9}, \quad \Pr\left(-\tfrac{\hbar}{2}\right) = |\langle z_-|\chi\rangle|^2 = \tfrac{4}{9}$$

Find $\langle S_x\rangle$, $\langle S_y\rangle$, and $\langle S_z\rangle$.

$$\begin{aligned} \langle x\rangle &= \langle\chi|\sigma_x|\chi\rangle = \tfrac{4}{9}, & \langle S_x\rangle &= \tfrac{\hbar}{2}\langle x\rangle = \tfrac{2}{9}\hbar \\ \langle y\rangle &= \langle\chi|\sigma_y|\chi\rangle = \tfrac{8}{9}, & \langle S_y\rangle &= \tfrac{\hbar}{2}\langle y\rangle = \tfrac{4}{9}\hbar \\ \langle z\rangle &= \langle\chi|\sigma_z|\chi\rangle = \tfrac{1}{9}, & \langle S_z\rangle &= \tfrac{\hbar}{2}\langle z\rangle = \tfrac{1}{18}\hbar \end{aligned}$$