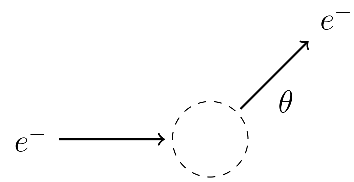
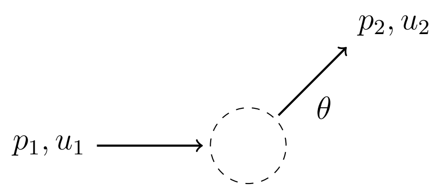
Rutherford Scattering

Consider an electron scattered by an atomic nucleus.[1]



Here is the same diagram with momentum and spinor labels.



For a typical Rutherford scattering experiment, the momentum vectors are

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \qquad p_2 = \begin{pmatrix} E \\ p \sin\theta \cos\phi \\ p \sin\theta \sin\phi \\ p \cos\theta \end{pmatrix}$$

Symbol p is electron momentum, m is electron mass, and $E = \sqrt{p^2 + m^2}$. The spinors are

$$u_{11} = \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix} \qquad u_{12} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix} \qquad u_{21} = \begin{pmatrix} E+m \\ 0 \\ p_{2z} \\ p_{2x} + ip_{2y} \end{pmatrix} \qquad u_{22} = \begin{pmatrix} 0 \\ E+m \\ p_{2x} - ip_{2y} \\ -p_{2z} \end{pmatrix}$$

The second digit in a spinor subscript is the spin state. The spinors are not normalized. Instead, a combined spinor normalization constant $N = (E+m)^2$ is used.

The following formula computes a probability density $|\mathcal{M}_{jk}|^2$ for Rutherford scattering where j is the spin state of the inbound electron and k is the spin state of the outbound electron.

$$|\mathcal{M}_{jk}|^2 = \frac{Z^2 e^4}{q^4} \frac{1}{N} \left| \bar{u}_{2k} \gamma^0 u_{1j} \right|^2$$

[1] The original Rutherford scattering experiment in 1911 used alpha particles, not electrons. However, scattering of any charged particle by Coulomb interaction is now known as Rutherford scattering. The first Rutherford scattering experiment using electrons appears to have been done by F. L. Arnot, then a student of Rutherford, in 1929.

Symbol Z is the atomic number of the nucleus, e is electron charge, and q is momentum transfer such that $q^4 = (p_1 - p_2)^4 = 16p^4 \sin^4(\theta/2)$.

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is computed by summing $|\mathcal{M}_{jk}|^2$ over all four spin states and then dividing by the number of inbound states. There are two inbound states.

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2}|\mathcal{M}_{jk}|^2 \\
&= \frac{Z^2e^4}{2q^4}\frac{1}{N}\sum_{j=1}^{2}\sum_{k=1}^{2}\left|\bar{u}_{2k}\gamma^0 u_{1j}\right|^2 \\
&= \frac{Z^2e^4}{2q^4}\operatorname{Tr}\Big((\not{p}_1 + m)\gamma^0(\not{p}_2 + m)\gamma^0\Big) \\
&= \frac{2Z^2e^4}{q^4}\left(E^2 + m^2 + p^2\cos\theta\right)
\end{aligned}$$

Run "rutherford-scattering-1.txt" to verify the following formulas.

$$\frac{1}{N}\sum_{j=1}^{2}\sum_{k=1}^{2}\left|\bar{u}_{2k}\gamma^0 u_{1j}\right|^2 = \operatorname{Tr}\Big((\not{p}_1 + m)\gamma^0(\not{p}_2 + m)\gamma^0\Big) = 4(E^2 + m^2 + p^2\cos\theta)$$

$$q^4 = (p_1 - p_2)^4 = 16p^4\sin^4(\theta/2)$$

$$4\sin^4(\theta/2) = (\cos\theta - 1)^2$$

Low energy approximation

For low energy electrons such that $p \ll m$ we have

$$E^2 + m^2 + p^2\cos\theta \approx 2m^2$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{Z^2e^4m^2}{4p^4\sin^4(\theta/2)}$$

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{16\pi^2}$$

By substituting $e^2 = 4\pi\alpha$ and $4\sin^4(\theta/2) = (\cos\theta - 1)^2$ we have

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2m^2}{p^4(\cos\theta - 1)^2}$$

We can integrate $d\sigma$ to obtain a cumulative distribution function. Recall that $d\Omega = \sin\theta\,d\theta\,d\phi$ hence

$$d\sigma = \frac{Z^2\alpha^2m^2}{p^4(\cos\theta - 1)^2}\sin\theta\,d\theta\,d\phi$$

Let $I(\xi)$ be the following definite integral.

$$\begin{aligned}
I(\xi) &= \frac{p^4}{Z^2\alpha^2 m^2}\int_0^{2\pi}\int_a^{\xi} d\sigma \\
&= 2\pi\int_a^{\xi}\frac{1}{(\cos\theta-1)^2}\sin\theta\,d\theta \\
&= 2\pi\left(\frac{1}{\cos\theta-1}\right)\Bigg|_a^{\xi} \\
&= 2\pi\left(\frac{1}{\cos\xi-1}-\frac{1}{\cos a-1}\right), \qquad a\le\xi\le\pi
\end{aligned}$$

The minimum angle $a>0$ is required because $d\sigma$ is undefined for $\theta=0$.

Let C be the normalization constant $C=I(\pi)$. Then the cumulative distribution function $F(\theta)$ is

$$F(\theta)=C^{-1}I(\theta), \qquad a\le\theta\le\pi$$

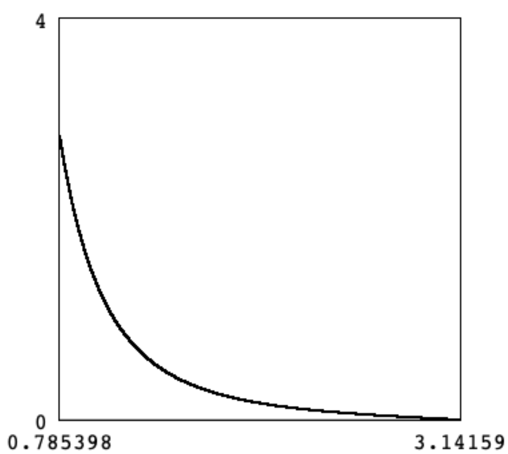
The probability of observing a scattering angle θ such that $\theta_1\le\theta\le\theta_2$ is

$$P(\theta_1\le\theta\le\theta_2)=F(\theta_2)-F(\theta_1)$$

The probability density $f(\theta)$ is the derivative of $F(\theta)$.

$$f(\theta)=\frac{dF(\theta)}{d\theta}=C^{-1}\frac{dI(\theta)}{d\theta}=C^{-1}\frac{2\pi\sin\theta}{(\cos\theta-1)^2}$$

Run “rutherford-scattering-3.txt” to draw a graph of $f(\theta)$ for $a=\pi/4=45°$.



The following table shows the corresponding probability distribution for three bins.

θ_1	θ_2	$P(\theta_1\le\theta\le\theta_2)$
0°	45°	–
45°	90°	0.83
90°	135°	0.14
135°	180°	0.03