

# Fermi's golden rule

Find the transition rate  $\Gamma_{1 \rightarrow 2}$  by deriving Fermi's golden rule.

Let the perturbing Hamiltonian be

$$H_1(x, t) = 2V(x) \cos(\omega t + \phi)$$

Let  $\Psi(x, t)$  be the following linear combination of the two eigenstates.

$$\Psi(x, t) = c_1(t)\psi_1(x) \exp(-i\omega_1 t) + c_2(t)\psi_2(x) \exp(-i\omega_2 t)$$

We need to solve for  $c_2(t)$  to find the transition rate.

From the time-dependent Schrodinger equation we have

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = H_0(x)\Psi(x, t) + H_1(x, t)\Psi(x, t)$$

The left-hand side works out to be

$$\begin{aligned} LHS = \left( i\hbar \frac{\partial c_1(t)}{\partial t} + \hbar\omega_1 c_1(t) \right) \psi_1(x) \exp(-i\omega_1 t) \\ + \left( i\hbar \frac{\partial c_2(t)}{\partial t} + \hbar\omega_2 c_2(t) \right) \psi_2(x) \exp(-i\omega_2 t) \end{aligned}$$

Using the time-independent Schrodinger equation the right-hand side is

$$RHS = c_1(t) \underbrace{E_1}_{\text{cancels}} \psi_1(x) \exp(-i\omega_1 t) + c_2(t) \underbrace{E_2}_{\text{cancels}} \psi_2(x) \exp(-i\omega_2 t) + H_1(x, t)\Psi(x, t)$$

The indicated terms cancel by the substitutions  $E_1 = \hbar\omega_1$  and  $E_2 = \hbar\omega_2$  leaving

$$LHS = i\hbar \frac{\partial c_1(t)}{\partial t} \psi_1(x) \exp(-i\omega_1 t) + i\hbar \frac{\partial c_2(t)}{\partial t} \psi_2(x) \exp(-i\omega_2 t)$$

and

$$RHS = H_1(x, t)\Psi(x, t)$$

Hence by  $LHS = RHS$  we have

$$i\hbar \frac{\partial c_1(t)}{\partial t} \psi_1(x) \exp(-i\omega_1 t) + i\hbar \frac{\partial c_2(t)}{\partial t} \psi_2(x) \exp(-i\omega_2 t) = H_1(x, t)\Psi(x, t)$$

Take the inner product of  $\psi_2^*(x)$  with the above equation to obtain

$$i\hbar \frac{\partial c_2(t)}{\partial t} \exp(-i\omega_2 t) = 2 \cos(\omega t + \phi) (c_1(t) M_{21} \exp(-i\omega_1 t) + c_2(t) M_{22} \exp(-i\omega_2 t))$$

where  $M_{21}$  and  $M_{22}$  are the matrix elements

$$\begin{aligned} M_{21} &= \int \psi_2^*(x) V(x) \psi_1(x) dx \\ M_{22} &= \int \psi_2^*(x) V(x) \psi_2(x) dx \end{aligned}$$

Cancel exponentials.

$$i\hbar \frac{\partial c_2(t)}{\partial t} = 2 \cos(\omega t + \phi) \left( c_1(t) M_{21} \exp(i(\omega_2 - \omega_1)t) + 2c_2(t) M_{22} \right)$$

Let the initial state be  $\Psi(x, 0) = \psi_1(x)$  hence the initial conditions are

$$c_1(0) = 1, \quad c_2(0) = 0$$

Then for time  $t$  near the origin we can use the approximation

$$i\hbar \frac{\partial c_2(t)}{\partial t} = 2 \cos(\omega t + \phi) M_{21} \exp(i(\omega_2 - \omega_1)t)$$

Solve for  $c_2(t)$  by integrating.

$$c_2(t) = \frac{2M_{21}}{i\hbar} \int_0^t \cos(\omega t' + \phi) \exp(i(\omega_2 - \omega_1)t') dt'$$

The solution is

$$c_2(t) = -\frac{M_{21}}{\hbar} \left( \frac{\exp(i(\omega_2 - \omega_1 - \omega)t) - 1}{\omega_2 - \omega_1 - \omega} \right) \exp(-i\phi) - \frac{M_{21}}{\hbar} \left( \frac{\exp(i(\omega_2 - \omega_1 + \omega)t) - 1}{\omega_2 - \omega_1 + \omega} \right) \exp(+i\phi) \quad (1)$$

For  $\omega$  such that  $\omega \approx \omega_2 - \omega_1$  the first term dominates so discard the second term and write

$$c_2(t) = -\frac{M_{21}}{\hbar} \left( \frac{\exp(i(\omega_2 - \omega_1 - \omega)t) - 1}{\omega_2 - \omega_1 - \omega} \right) \exp(-i\phi)$$

Rewrite using a sinc function.

$$c_2(t) = -\frac{it}{\hbar} M_{21} \exp \left( i \frac{\omega_2 - \omega_1 - \omega}{2} t - i\phi \right) \text{sinc} \left( \frac{\omega_2 - \omega_1 - \omega}{2} t \right) \quad (2)$$

Hence the transition probability is

$$P(1 \rightarrow 2) = |c_2(t)|^2 = \frac{t^2}{\hbar^2} |M_{21}|^2 \text{sinc}^2 \left( \frac{\omega_2 - \omega_1 - \omega}{2} t \right) \quad (3)$$

Integrate  $P(1 \rightarrow 2)$  to obtain the total transition probability  $P_{tot}(1 \rightarrow 2)$ .

$$P_{tot}(1 \rightarrow 2) = \frac{t^2}{\hbar^2} |M_{21}|^2 \int_{E-\epsilon}^{E+\epsilon} \text{sinc}^2 \left( \frac{E'/\hbar - \omega}{2} t \right) g(E') dE'$$

where

$$E = \hbar(\omega_2 - \omega_1)$$

and  $g(E')$  is the density of photon states for energy  $E'$ .

Use the approximation  $g(E') \approx g(\hbar\omega)$  to obtain

$$P_{tot}(1 \rightarrow 2) = \frac{t^2}{\hbar^2} |M_{21}|^2 g(\hbar\omega) \int_{E-\epsilon}^{E+\epsilon} \text{sinc}^2 \left( \frac{E'/\hbar - \omega}{2} t \right) dE'$$

Let

$$y = \frac{E'/\hbar - \omega}{2} t$$

It follows that

$$E' = \frac{2\hbar y}{t} + \hbar\omega$$

and

$$dE' = \frac{2\hbar}{t} dy$$

The integration limits transform as

$$E \pm \epsilon \rightarrow \frac{(E \pm \epsilon)/\hbar - \omega}{2} t = \frac{Et}{2\hbar} - \frac{\omega t}{2} \pm \frac{\epsilon t}{2\hbar} \approx \pm \frac{\epsilon t}{2\hbar}$$

Hence

$$P_{tot}(1 \rightarrow 2) = \frac{2t}{\hbar} |M_{21}|^2 g(\hbar\omega) \int_{-\epsilon t/2\hbar}^{\epsilon t/2\hbar} \text{sinc}^2 y dy$$

Use the approximation

$$\int_{-\epsilon t/2\hbar}^{\epsilon t/2\hbar} \text{sinc}^2 y dy \approx \int_{-\infty}^{\infty} \text{sinc}^2 y dy = \pi$$

to obtain

$$P_{tot}(1 \rightarrow 2) = \frac{2\pi t}{\hbar} |M_{21}|^2 g(\hbar\omega)$$

The transition rate is the derivative of  $P_{tot}(1 \rightarrow 2)$ .

$$\Gamma_{1 \rightarrow 2} = \frac{d}{dt} P_{tot}(1 \rightarrow 2) = \frac{2\pi}{\hbar} |M_{21}|^2 g(\hbar\omega)$$