

From equations (5.45) and (5.43)

$$\mathcal{A}f(x) = \int_{-\infty}^{\infty} G_A(x, x') f(x') dx' \quad (1)$$

Hence

$$\int_{-\infty}^{\infty} g^*(x) \mathcal{A}f(x) dx = \int_{-\infty}^{\infty} g^*(x) \left(\int_{-\infty}^{\infty} G_A(x, x') f(x') dx' \right) dx$$

Combine factors.

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_A(x, x') g^*(x) f(x') dx' dx$$

By equation (5.44) substitute $G_A^*(x', x)$ for $G_A(x, x')$.

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_A^*(x', x) g^*(x) f(x') dx' dx$$

By complex conjugation of a product

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (G_A(x', x) g(x))^* f(x') dx' dx$$

Factor out $f(x') dx'$.

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} (G_A(x', x) g(x))^* dx \right) f(x') dx'$$

Then by equation (1)

$$= \int_{-\infty}^{\infty} (\mathcal{A}g(x'))^* f(x') dx'$$