

Harmonic oscillator propagator

Consider the harmonic oscillator eigenstate

$$\psi_n(x, t) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi \hbar} \right)^{\frac{1}{4}} H_n \left(\sqrt{\frac{m\omega}{\hbar}} x \right) \exp \left(-\frac{m\omega x^2}{2\hbar} - i \left(n + \frac{1}{2} \right) \omega t \right)$$

and the harmonic oscillator propagator

$$K(x_b, t_b, x_a, t_a) = \left(\frac{m\omega}{2\pi i \hbar \sin(\omega T)} \right)^{\frac{1}{2}} \exp \left[\frac{im\omega}{2\hbar \sin(\omega T)} (x_a^2 \cos(\omega T) - 2x_a x_b + x_b^2 \cos(\omega T)) \right]$$

where $T = t_b - t_a$.

We should have

$$\psi_n(x_b, t_b) = \int_{-\infty}^{\infty} K(x_b, t_b, x_a, t_a) \psi_n(x_a, t_a) dx_a$$

Try for $n = 1$.

$$\psi_1(x_a, t_a) = \sqrt{2} \left(\frac{m^2 \omega^3}{\pi \hbar^3} \right)^{\frac{1}{4}} x_a \exp \left(-\frac{m\omega x_a^2}{2\hbar} - \frac{3i\omega t_a}{2} \right)$$

By the identity

$$\int_{-\infty}^{\infty} y \exp(-ay^2 + by) dy = \frac{\sqrt{\pi}}{2} \frac{b}{a^{3/2}} \exp \left(\frac{b^2}{4a} \right)$$

the path integral is

$$\begin{aligned} I = \int_{-\infty}^{\infty} K(x_b, t_b, x_a, t_a) \psi_1(x_a, t_a) dx_a &= \frac{1}{2} \left(-\frac{m^9 \omega^9}{\pi \hbar^9} \right)^{\frac{1}{4}} \left(\frac{m\omega}{2\hbar} \sin(\omega T) - \frac{im\omega}{2\hbar} \cos(\omega T) \right)^{-\frac{3}{2}} \\ &\times x_b \exp \left(-\frac{m\omega x_b^2}{2\hbar (\sin(\omega T)^2 - i \sin(\omega T) \cos(\omega T))} + \frac{im\omega x_b^2 \cos(\omega T)}{2\hbar \sin(\omega T)} - \frac{3i\omega t_a}{2} \right) \end{aligned}$$

Substitute $t_b - T$ for t_a and substitute $-\pi/(2\omega)$ for T . The result is

$$I = \psi_1(x_b, t_b)$$

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