3-12. If the wave function for a harmonic oscillator is (at $t = 0$)

$$\psi(x, 0) = \exp\left(-\frac{m\omega}{2\hbar}(x - a)^2\right) \tag{3.67}$$

then using equation (3.42) and the results of problem 3-8, show that

$$\psi(x, T) =$$
$$\exp\left(-\frac{i\omega T}{2} - \frac{m\omega}{2\hbar}\left(x^2 - 2ax\exp(-i\omega T) + a^2\cos(\omega T)\exp(-i\omega T))\right)\right)$$
$$\tag{3.68}$$

and find the probability distribution $|\psi|^2$.

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From problem 3-8

$$K(x, T, x_c, 0) =$$
$$\left(\frac{m\omega}{2\pi i\hbar\sin(\omega T)}\right)^{1/2}\exp\left(\frac{im\omega}{2\hbar\sin(\omega T)}\left((x^2 + x_c^2)\cos(\omega T) - 2xx_c\right)\right) \quad (2)$$

From equation (3.42)

$$\psi(x, T) = \int_{-\infty}^{\infty} K(x, T, x_c, 0)\psi(x_c, 0)\, dx_c \tag{3}$$

Substitute (3.68) and (2) into (3) to obtain

$$\psi(x, T) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega T)}\right)^{1/2}$$
$$\times \int_{-\infty}^{\infty} \exp\left(\frac{im\omega}{2\hbar\sin(\omega T)}\left((x^2 + x_c^2)\cos(\omega T) - 2xx_c\right)\right)$$
$$\times \exp\left(-\frac{m\omega}{2\hbar}(x_c - a)^2\right)\, dx_c$$

Rewrite as

$$\psi(x, T) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega T)}\right)^{1/2}\int_{-\infty}^{\infty}\exp(Ax_c^2 + Bx_c + C)\, dx_c \tag{4}$$

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where

$$A = \frac{m\omega}{2\hbar} \left(\frac{i\cos(\omega T)}{\sin(\omega T)} - 1 \right) = \frac{im\omega \exp(i\omega T)}{2\hbar \sin(\omega T)}$$

$$B = \frac{m\omega}{\hbar} \left(a - \frac{ix}{\sin(\omega T)} \right)$$

$$C = \frac{m\omega}{2\hbar} \left(\frac{ix^2 \cos(\omega T)}{\sin(\omega T)} - a^2 \right)$$

Solve the integral.

$$\int_{-\infty}^{\infty} \exp(Ax_c^2 + Bx_c + C) = \left(-\frac{\pi}{A} \right)^{1/2} \exp\left(-\frac{B^2}{4A} + C \right) \qquad (5)$$

where

$$-\frac{\pi}{A} = \frac{2\pi i \hbar \sin(\omega T)}{m\omega \exp(i\omega T)} \qquad (6)$$

and

$$-\frac{B^2}{4A} + C = -\frac{m\omega}{2\hbar} \left(x^2 - 2ax \exp(-i\omega T) + a^2 \cos(\omega T) \exp(-i\omega T) \right) \qquad (7)$$

Note that

$$\underbrace{\frac{m\omega}{2\pi i \hbar \sin(\omega T)}}_{\text{from equation (4)}} \times \underbrace{\frac{2\pi i \hbar \sin(\omega T)}{m\omega \exp(i\omega T)}}_{\text{from equation (6)}} = \exp(-i\omega T) \qquad (8)$$

Substitute the solved integral (5) into (3) to obtain

$$\psi(x, T) = \exp\left(-\frac{i\omega T}{2} \right)$$

$$\times \exp\left(-\frac{m\omega}{2\hbar} \left(x^2 - 2ax \exp(-i\omega T) + a^2 \cos(\omega T) \exp(-i\omega T) \right) \right)$$