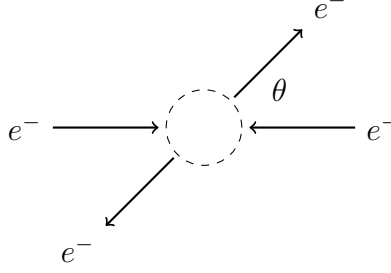
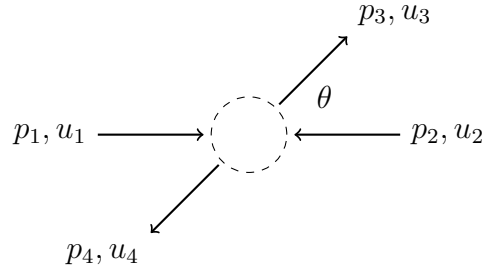


The following diagram shows the geometry of a typical collider experiment that generates electron scattering data.



Here is the same diagram with momentum labels  $p$  and Dirac spinor labels  $u$ .



In center of mass coordinates the momentum vectors are

$$\begin{aligned}
 p_1 &= \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} & p_2 &= \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} & p_3 &= \begin{pmatrix} E \\ p \sin \theta \cos \phi \\ p \sin \theta \sin \phi \\ p \cos \theta \end{pmatrix} & p_4 &= \begin{pmatrix} E \\ -p \sin \theta \cos \phi \\ -p \sin \theta \sin \phi \\ -p \cos \theta \end{pmatrix} \\
 \text{particle 1} & & \text{particle 2} & & \text{particle 3} & & \text{particle 4}
 \end{aligned}$$

Symbol  $p$  is incident momentum,  $E$  is total energy  $E = \sqrt{p^2 + m^2}$ , and  $m$  is electron mass. Polar angle  $\theta$  is the observed scattering angle. Azimuth angle  $\phi$  cancels out in scattering calculations.

The spinors are

$$\begin{aligned}
 u_{11} &= \begin{pmatrix} E + m \\ 0 \\ p \\ 0 \end{pmatrix} & u_{12} &= \begin{pmatrix} 0 \\ E + m \\ 0 \\ -p \end{pmatrix} & u_{21} &= \begin{pmatrix} E + m \\ 0 \\ -p \\ 0 \end{pmatrix} & u_{22} &= \begin{pmatrix} 0 \\ E + m \\ 0 \\ p \end{pmatrix} \\
 \text{particle 1} & & \text{particle 1} & & \text{particle 2} & & \text{particle 2} \\
 \text{spin up} & & \text{spin down} & & \text{spin up} & & \text{spin down} \\
 \\
 u_{31} &= \begin{pmatrix} E + m \\ 0 \\ p_{3z} \\ p_{3x} + ip_{3y} \end{pmatrix} & u_{32} &= \begin{pmatrix} 0 \\ E + m \\ p_{3x} - ip_{3y} \\ -p_{3z} \end{pmatrix} & u_{41} &= \begin{pmatrix} E + m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix} & u_{42} &= \begin{pmatrix} 0 \\ E + m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix} \\
 \text{particle 3} & & \text{particle 3} & & \text{particle 4} & & \text{particle 4} \\
 \text{spin up} & & \text{spin down} & & \text{spin up} & & \text{spin down}
 \end{aligned}$$

The spinors shown above are not individually normalized. Instead, a combined spinor normalization constant  $N = (E + m)^4$  will be used.

The following formula computes a probability density  $|\mathcal{M}_{abcd}|^2$  for electron scattering where the subscripts  $abcd$  are the spin states of the electrons.

$$|\mathcal{M}_{abcd}|^2 = \frac{e^4}{N} \left| \frac{1}{t} (\bar{u}_{3c} \gamma^\mu u_{1a}) (\bar{u}_{4d} \gamma_\mu u_{2b}) - \frac{1}{u} (\bar{u}_{4d} \gamma^\nu u_{1a}) (\bar{u}_{3c} \gamma_\nu u_{2b}) \right|^2$$

from Feynman diagram with  
photon exchange  
no electron interchange

from Feynman diagram with  
photon exchange  
electron interchange

Symbol  $e$  is electron charge. Symbols  $t$  and  $u$  are Mandelstam variables

$$t = (p_1 - p_3)^2$$

$$u = (p_1 - p_4)^2$$

Let

$$a_1 = (\bar{u}_{3c} \gamma^\mu u_{1a}) (\bar{u}_{4d} \gamma_\mu u_{2b}) \quad a_2 = (\bar{u}_{4d} \gamma^\nu u_{1a}) (\bar{u}_{3c} \gamma_\nu u_{2b})$$

Then

$$\begin{aligned} |\mathcal{M}_{abcd}|^2 &= \frac{e^4}{N} \left| \frac{a_1}{t} - \frac{a_2}{u} \right|^2 \\ &= \frac{e^4}{N} \left( \frac{a_1}{t} - \frac{a_2}{u} \right) \left( \frac{a_1}{t} - \frac{a_2}{u} \right)^* \\ &= \frac{e^4}{N} \left( \frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{tu} - \frac{a_1^* a_2}{tu} + \frac{a_2 a_2^*}{u^2} \right) \end{aligned}$$

The expected probability density  $\langle |\mathcal{M}|^2 \rangle$  is computed by summing  $|\mathcal{M}_{abcd}|^2$  over all spin states and dividing by the number of inbound states. There are four inbound states.

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{1}{4} \sum_{a=1}^2 \sum_{b=1}^2 \sum_{c=1}^2 \sum_{d=1}^2 |\mathcal{M}_{abcd}|^2 \\ &= \frac{e^4}{4N} \sum_{a=1}^2 \sum_{b=1}^2 \sum_{c=1}^2 \sum_{d=1}^2 \left( \frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{tu} - \frac{a_1^* a_2}{tu} + \frac{a_2 a_2^*}{u^2} \right) \end{aligned}$$

Use the Casimir trick to replace sums over spins with matrix products.

$$\begin{aligned} f_{11} &= \frac{1}{N} \sum_{abcd} a_1 a_1^* = \text{Tr} \left( (\not{p}_3 + m) \gamma^\mu (\not{p}_1 + m) \gamma^\nu \right) \text{Tr} \left( (\not{p}_4 + m) \gamma_\mu (\not{p}_2 + m) \gamma_\nu \right) \\ f_{12} &= \frac{1}{N} \sum_{abcd} a_1 a_2^* = \text{Tr} \left( (\not{p}_3 + m) \gamma^\mu (\not{p}_1 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\mu (\not{p}_2 + m) \gamma_\nu \right) \\ f_{22} &= \frac{1}{N} \sum_{abcd} a_2 a_2^* = \text{Tr} \left( (\not{p}_4 + m) \gamma^\mu (\not{p}_1 + m) \gamma^\nu \right) \text{Tr} \left( (\not{p}_3 + m) \gamma_\mu (\not{p}_2 + m) \gamma_\nu \right) \end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left( \frac{f_{11}}{t^2} - \frac{f_{12}}{tu} - \frac{f_{12}^*}{tu} + \frac{f_{22}}{u^2} \right)$$

The following formulas are equivalent to the Casimir trick. (Recall that  $a \cdot b = a^\mu g_{\mu\nu} b^\nu$ )

$$\begin{aligned} f_{11} &= 32(p_1 \cdot p_2)^2 + 32(p_1 \cdot p_4)^2 - 64m^2(p_1 \cdot p_3) + 64m^4 \\ f_{12} &= -32(p_1 \cdot p_2)^2 + 32m^2(p_1 \cdot p_2) + 32m^2(p_1 \cdot p_3) + 32m^2(p_1 \cdot p_4) - 32m^4 \\ f_{22} &= 32(p_1 \cdot p_2)^2 + 32(p_1 \cdot p_3)^2 - 64m^2(p_1 \cdot p_4) + 64m^4 \end{aligned}$$

In Mandelstam variables

$$\begin{aligned} s &= (p_1 + p_2)^2 \\ t &= (p_1 - p_3)^2 \\ u &= (p_1 - p_4)^2 \end{aligned}$$

the formulas are

$$\begin{aligned} f_{11} &= 8s^2 + 8u^2 - 64sm^2 - 64um^2 + 192m^4 \\ f_{12} &= -8s^2 + 64sm^2 - 96m^4 \\ f_{22} &= 8s^2 + 8t^2 - 64sm^2 - 64tm^2 + 192m^4 \end{aligned}$$

## High energy approximation

When  $E \gg m$  a useful approximation is to set  $m = 0$  and obtain

$$\begin{aligned} f_{11} &= 8s^2 + 8u^2 \\ f_{12} &= -8s^2 \\ f_{22} &= 8s^2 + 8t^2 \end{aligned}$$

Hence

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left( \frac{f_{11}}{t^2} - \frac{f_{12}}{tu} - \frac{f_{12}^*}{tu} + \frac{f_{22}}{u^2} \right) \\ &= \frac{e^4}{4} \left( \frac{8s^2 + 8u^2}{t^2} - \frac{-8s^2}{tu} - \frac{-8s^2}{tu} + \frac{8s^2 + 8t^2}{u^2} \right) \\ &= 2e^4 \left( \frac{s^2 + u^2}{t^2} + \frac{2s^2}{tu} + \frac{s^2 + t^2}{u^2} \right) \end{aligned}$$

Combine terms so  $\langle |\mathcal{M}|^2 \rangle$  has a common denominator.

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left( \frac{u^2(s^2 + u^2) + 2s^2tu + t^2(s^2 + t^2)}{t^2u^2} \right)$$

For  $m = 0$  the Mandelstam variables are

$$\begin{aligned} s &= 4E^2 \\ t &= 2E^2(\cos \theta - 1) \\ u &= -2E^2(\cos \theta + 1) \end{aligned}$$

Hence

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= 2e^4 \left( \frac{32E^8 \cos^4 \theta + 192E^8 \cos^2 \theta + 288E^8}{16E^8 (\cos \theta - 1)^2 (\cos \theta + 1)^2} \right) \\
&= 4e^4 \frac{(\cos^2 \theta + 3)^2}{(\cos \theta - 1)^2 (\cos \theta + 1)^2} \\
&= 4e^4 \left( \frac{\cos^2 \theta + 3}{\cos^2 \theta - 1} \right)^2
\end{aligned}$$

The following equivalent formula can also be used.

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= 2e^4 \left( \frac{s^2 + u^2}{t^2} + \frac{2s^2}{tu} + \frac{s^2 + t^2}{u^2} \right) \\
&= 2e^4 \left( \underbrace{\frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)}}_{\substack{\text{from Feynman diagram with} \\ \text{photon exchange} \\ \text{no electron interchange}}} + \underbrace{\frac{2}{\sin^2(\theta/2) \cos^2(\theta/2)}}_{\text{interaction term}} + \underbrace{\frac{1 + \sin^4(\theta/2)}{\cos^4(\theta/2)}}_{\substack{\text{from Feynman diagram with} \\ \text{photon exchange} \\ \text{electron interchange}}} \right)
\end{aligned}$$

For example, see A. Zee, p. 134.

## Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{64\pi^2 s} = \frac{e^4}{16\pi^2 s} \left( \frac{\cos^2 \theta + 3}{\cos^2 \theta - 1} \right)^2, \quad s \gg m$$

Substituting  $e^4 = 16\pi^2 \alpha^2$  yields

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s} \left( \frac{\cos^2 \theta + 3}{\cos^2 \theta - 1} \right)^2$$

We can integrate  $d\sigma$  to obtain a cumulative distribution function. Recall that

$$d\Omega = \sin \theta d\theta d\phi$$

Hence

$$d\sigma = \frac{\alpha^2}{s} \left( \frac{\cos^2 \theta + 3}{\cos^2 \theta - 1} \right)^2 \sin \theta d\theta d\phi$$

Let  $I(\theta)$  be the following integral of  $d\sigma$ .

$$\begin{aligned}
I(\theta) &= \frac{s}{2\pi\alpha^2} \int_0^{2\pi} \int d\sigma \\
&= \int \left( \frac{\cos^2 \theta + 3}{\cos^2 \theta - 1} \right)^2 \sin \theta d\theta, \quad a \leq \theta \leq \pi - a
\end{aligned}$$

Angular support is limited to an arbitrary  $a > 0$  because  $I(0)$  and  $I(\pi)$  are undefined. Assume that  $I(\theta) - I(a)$  is computable given  $\theta$  by either symbolic or numerical integration.

Let  $C$  be the normalization constant

$$C = I(\pi - a) - I(a)$$

Then the cumulative distribution function  $F(\theta)$  is

$$F(\theta) = \frac{I(\theta) - I(a)}{C}, \quad a \leq \theta \leq \pi - a$$

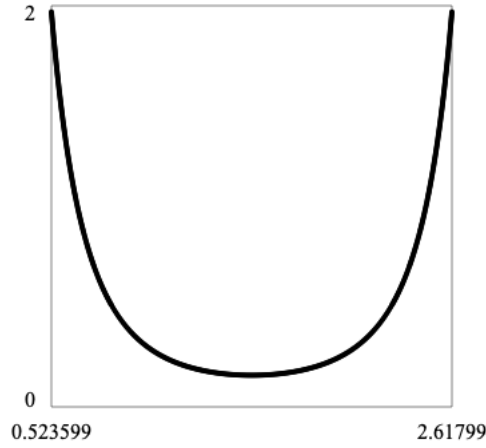
The probability of observing scattering events in the interval  $\theta_1$  to  $\theta_2$  can now be computed.

$$P(\theta_1 \leq \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

Probability density function  $f(\theta)$  is the derivative of  $F(\theta)$ .

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{C} \left( \frac{\cos^2 \theta + 3}{\cos^2 \theta - 1} \right)^2 \sin \theta$$

This is a graph of  $f(\theta)$  for  $a = \pi/6 = 30^\circ$ .



Probability distribution for  $30^\circ$  bins ( $a = 30^\circ$ ).

$\theta_1$	$\theta_2$	$P(\theta_1 \leq \theta \leq \theta_2)$
$0^\circ$	$30^\circ$	—
$30^\circ$	$60^\circ$	0.40
$60^\circ$	$90^\circ$	0.10
$90^\circ$	$120^\circ$	0.10
$120^\circ$	$150^\circ$	0.40
$150^\circ$	$180^\circ$	—

## Notes

In component notation, the trace operators of the Casimir trick become sums over a repeated index, in this case  $\alpha$ .

$$\begin{aligned}
f_{11} &= \left( (\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \right) \left( (\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \right) \\
f_{12} &= (\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_2 + m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha \\
f_{22} &= \left( (\not{p}_4 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \right) \left( (\not{p}_3 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \right)
\end{aligned}$$

To convert the above formulas to Eigenmath code, the  $\gamma$  tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply  $\gamma^\mu$  by the metric tensor to lower the index.

$$\begin{aligned}\gamma^{\beta\mu}{}_\rho &\rightarrow \text{gammaT} = \text{transpose}(\text{gamma}) \\ \gamma^\beta{}_{\mu\rho} &\rightarrow \text{gammaL} = \text{transpose}(\text{dot}(\text{gmunu}, \text{gamma}))\end{aligned}$$

Define the following  $4 \times 4$  matrices.

$$\begin{aligned}(\not{p}_1 + m) &\rightarrow X1 = \text{pslash1} + m \text{ I} \\ (\not{p}_2 + m) &\rightarrow X2 = \text{pslash2} + m \text{ I} \\ (\not{p}_3 + m) &\rightarrow X3 = \text{pslash3} + m \text{ I} \\ (\not{p}_4 + m) &\rightarrow X4 = \text{pslash4} + m \text{ I}\end{aligned}$$

Then for  $f_{11}$  we have the following Eigenmath code. The contract function sums over  $\alpha$ .

$$\begin{aligned}(\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha &\rightarrow T1 = \text{contract}(\text{dot}(X3, \text{gammaT}, X1, \text{gammaT}), 1, 4) \\ (\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha &\rightarrow T2 = \text{contract}(\text{dot}(X4, \text{gammaL}, X2, \text{gammaL}), 1, 4)\end{aligned}$$

Next, multiply then sum over repeated indices. The dot function sums over  $\nu$  then the contract function sums over  $\mu$ . The transpose makes the  $\nu$  indices adjacent as required by the dot function.

$$f_{11} = \text{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu) \text{Tr}(\cdots \gamma_\mu \cdots \gamma_\nu) \rightarrow \text{contract}(\text{dot}(T1, \text{transpose}(T2)))$$

Follow suit for  $f_{22}$ .

$$\begin{aligned}(\not{p}_4 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha &\rightarrow T1 = \text{contract}(\text{dot}(X4, \text{gammaT}, X1, \text{gammaT}), 1, 4) \\ (\not{p}_3 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha &\rightarrow T2 = \text{contract}(\text{dot}(X3, \text{gammaL}, X2, \text{gammaL}), 1, 4)\end{aligned}$$

Then

$$f_{22} = \text{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu) \text{Tr}(\cdots \gamma_\mu \cdots \gamma_\nu) \rightarrow \text{contract}(\text{dot}(T1, \text{transpose}(T2)))$$

The calculation of  $f_{12}$  begins with

$$\begin{aligned}(\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_2 + m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha \\ \rightarrow T = \text{contract}(\text{dot}(X3, \text{gammaT}, X1, \text{gammaT}, X4, \text{gammaL}, X2, \text{gammaL}), 1, 6)\end{aligned}$$

Then sum over repeated indices  $\mu$  and  $\nu$ .

$$f_{12} = \text{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu \cdots \gamma_\mu \cdots \gamma_\nu) \rightarrow \text{contract}(\text{contract}(T, 1, 3))$$