Feynman and Hibbs problem 4-1

Show that for a single particle moving in three dimensions in a potential energy $V(\mathbf{x},t)$ the Schrodinger equation is

$$\frac{\partial\psi(\mathbf{x},t)}{\partial t} = -\frac{i}{\hbar}\left(-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x},t) + V(\mathbf{x},t)\psi(\mathbf{x},t)\right)$$

This is the Lagrangian.

$$L(\dot{\mathbf{x}},\mathbf{x}) = \frac{m}{2}\dot{\mathbf{x}}^2 - V(\mathbf{x},t) \tag{1}$$

Extend equation (4.3) from one dimension to three dimensions.

$$\psi(\mathbf{x},t+\epsilon) = \frac{1}{A}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \exp\left(\frac{i}{\hbar}\epsilon L\left(\frac{\mathbf{x}-\mathbf{y}}{\epsilon},\frac{\mathbf{x}+\mathbf{y}}{2}\right)\right)\psi(\mathbf{y},t)\,dy_1\,dy_2\,dy_3$$

From (1) we have

$$L\left(\frac{\mathbf{x}-\mathbf{y}}{\epsilon},\frac{\mathbf{x}+\mathbf{y}}{2}\right) = \frac{m}{2\epsilon^2}(\mathbf{x}-\mathbf{y})^2 - V\left(\frac{\mathbf{x}+\mathbf{y}}{2},t\right)$$

Hence

$$\psi(\mathbf{x},t+\epsilon) = \frac{1}{A}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \exp\left(\frac{i}{\hbar}\left(\frac{m}{2\epsilon}(\mathbf{x}-\mathbf{y})^2 - \epsilon V\left(\frac{\mathbf{x}+\mathbf{y}}{2},t\right)\right)\right)$$
$$\times\,\psi(\mathbf{y},t)\,dy_1\,dy_2\,dy_3$$

Let

$$\mathbf{y} = \mathbf{x} + \boldsymbol{\eta}$$

Then

$$(\mathbf{x}-\mathbf{y})^2 = \boldsymbol{\eta}^2$$

and

$$\frac{\mathbf{x}+\mathbf{y}}{2} = \mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}$$

Hence

$$\psi(\mathbf{x}, t+\epsilon) = \frac{1}{A} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(\frac{i}{\hbar}\left(\frac{m}{2\epsilon}\boldsymbol{\eta}^2 - \epsilon V\left(\mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, t\right)\right)\right) \times \psi(\mathbf{x}+\boldsymbol{\eta}, t)\, d\eta_1\, d\eta_2\, d\eta_3$$

Factor the exponential.

$$\psi(\mathbf{x}, t+\epsilon) = \frac{1}{A} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \exp\left(-\frac{i\epsilon}{\hbar} V\left(\mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, t\right)\right) \times \psi(\mathbf{x}+\boldsymbol{\eta}, t)\, d\eta_1\, d\eta_2\, d\eta_3 \quad (2)$$

Now we are going to use an approximation for the second exponential. From the identity $\exp(i\theta) = \cos(\theta) + i\sin(\theta)$ we have

$$\exp\left(-\frac{i\epsilon}{\hbar} V\left(\mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, t\right)\right) = \cos\left(-\frac{\epsilon}{\hbar} V\left(\mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, t\right)\right) + i\sin\left(-\frac{\epsilon}{\hbar} V\left(\mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, t\right)\right)$$

For very small ϵ we have the approximation

$$\exp\left(-\frac{i\epsilon}{\hbar} V\left(\mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, t\right)\right) \approx 1 - \frac{i\epsilon}{\hbar} V\left(\mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, t\right)$$

The authors write that the $\frac{1}{2}\boldsymbol{\eta}$ term can be dropped "because the error is of higher order than ϵ." Hence

$$\exp\left(-\frac{i\epsilon}{\hbar} V\left(\mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, t\right)\right) \approx 1 - \frac{i\epsilon}{\hbar} V\left(\mathbf{x}, t\right) \quad (3)$$

Substituting (3) into (2) yields

$$\psi(\mathbf{x}, t+\epsilon) = \frac{1}{A} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \left(1 - \frac{i\epsilon}{\hbar} V\left(\mathbf{x}, t\right)\right) \times \psi(\mathbf{x}+\boldsymbol{\eta}, t)\, d\eta_1\, d\eta_2\, d\eta_3 \quad (4)$$

Next we will use the following Taylor series approximations.

$$\begin{aligned} \psi(\mathbf{x}, t+\epsilon) &\approx \psi(\mathbf{x}, t) + \epsilon\frac{\partial\psi}{\partial t} \\ \psi(\mathbf{x}+\boldsymbol{\eta}, t) &\approx \psi(\mathbf{x}, t) + \boldsymbol{\eta}\cdot\nabla\psi + \tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi) \end{aligned} \tag{5}$$

Note: In component notation

$$\boldsymbol{\eta}\cdot\nabla\psi = \eta_1\frac{\partial\psi}{\partial x_1} + \eta_2\frac{\partial\psi}{\partial x_2} + \eta_2\frac{\partial\psi}{\partial x_2}$$

and

$$\begin{aligned} \boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi) = \eta_1^2\frac{\partial^2\psi}{\partial x_1^2} + \eta_2^2\frac{\partial^2\psi}{\partial x_2^2} + \eta_3^2\frac{\partial^2\psi}{\partial x_3^2} \\ + 2\eta_1\eta_2\frac{\partial^2\psi}{\partial x_1\partial x_2} + 2\eta_1\eta_3\frac{\partial^2\psi}{\partial x_1\partial x_3} + 2\eta_2\eta_3\frac{\partial^2\psi}{\partial x_2\partial x_3} \end{aligned}$$

Substitute the approximations (5) into (4).

$$\begin{aligned} \psi(\mathbf{x}, t) + \epsilon\frac{\partial\psi}{\partial t} = \frac{1}{A}\int_V \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\left(1 - \frac{i\epsilon}{\hbar}V\left(\mathbf{x}, t\right)\right) \\ \times\left(\psi(\mathbf{x}, t) + \boldsymbol{\eta}\cdot\nabla\psi + \tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi)\right)\, d\eta_1\, d\eta_2\, d\eta_3 \end{aligned}$$

Expand the integral.

$$\begin{aligned} \psi(\mathbf{x}, t) + \epsilon\frac{\partial\psi}{\partial t} = \frac{\psi(\mathbf{x}, t)}{A}\int_V \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\, d\eta_1\, d\eta_2\, d\eta_3 \\ + \frac{1}{A}\int_V \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\boldsymbol{\eta}\cdot\nabla\psi\, d\eta_1\, d\eta_2\, d\eta_3 \qquad (6) \\ + \frac{1}{2A}\int_V \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi)\, d\eta_1\, d\eta_2\, d\eta_3 \\ - \frac{i\epsilon}{A\hbar}V\left(\mathbf{x}, t\right)\psi(\mathbf{x}, t)\int_V \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\, d\eta_1\, d\eta_2\, d\eta_3 \\ - \frac{i\epsilon}{A\hbar}V\left(\mathbf{x}, t\right)\int_V \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\boldsymbol{\eta}\cdot\nabla\psi\, d\eta_1\, d\eta_2\, d\eta_3 \qquad (8) \\ - \frac{i\epsilon}{2A\hbar}V\left(\mathbf{x}, t\right)\int_V \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi)\, d\eta_1\, d\eta_2\, d\eta_3 \end{aligned}$$

From the identity

$$\int_{-\infty}^{\infty} \exp(ax^2)x\,dx = 0$$

the integrals (6) and (8) are zero.

$$\begin{aligned}
&\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \boldsymbol{\eta}\cdot\nabla\psi\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\quad = \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \eta_1 \frac{\partial\psi}{\partial x_1}\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad + \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \eta_2 \frac{\partial\psi}{\partial x_2}\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad + \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right) \eta_3 \frac{\partial\psi}{\partial x_3}\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\quad = 0
\end{aligned}$$