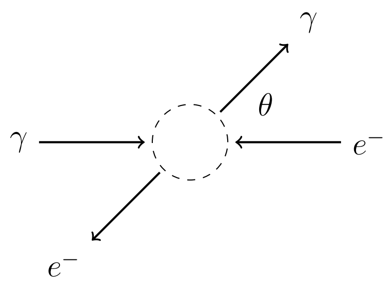
Compton scattering

Compton scattering is the interaction $e^- + \gamma \to e^- + \gamma$.



In the center-of-mass frame we have the following momentum vectors where ω is incident energy and E is total energy $E = \sqrt{\omega^2 + m^2}$. Polar angle θ is the observed scattering angle. Azimuth angle ϕ cancels out in scattering calculations.

$$p_1 = \underset{\substack{\text{inbound}\\\text{photon}}}{\begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix}}, \quad p_2 = \underset{\substack{\text{inbound}\\\text{electron}}}{\begin{pmatrix} E \\ 0 \\ 0 \\ -\omega \end{pmatrix}}, \quad p_3 = \underset{\substack{\text{outbound}\\\text{photon}}}{\begin{pmatrix} \omega \\ \omega\sin\theta\cos\phi \\ \omega\sin\theta\sin\phi \\ \omega\cos\theta \end{pmatrix}}, \quad p_4 = \underset{\substack{\text{outbound}\\\text{electron}}}{\begin{pmatrix} E \\ -\omega\sin\theta\cos\phi \\ -\omega\sin\theta\sin\phi \\ -\omega\cos\theta \end{pmatrix}}$$

Spinors for the inbound electron.

$$u_{21} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound electron}\\\text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ -\omega \\ 0 \end{pmatrix}}, \quad u_{22} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound electron}\\\text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ \omega \end{pmatrix}}$$

Spinors for the outbound electron.

$$u_{41} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{outbound electron}\\\text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix}}, \quad u_{42} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{outbound electron}\\\text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}}$$

Let a be the spin state of the inbound electron and let b be the spin state of the outbound electron such that subscript $ba \in \{11, 12, 21, 22\}$.

The probability amplitude $\mathcal{M}_{ba}$ for spin state ba is

$$\mathcal{M}_{ba} = \mathcal{M}_{1ba} + \mathcal{M}_{2ba}$$

where

$$\mathcal{M}_{1ba} = \frac{\bar{u}_{4b}(-ie\gamma^\mu)(\not{q}_1 + m)(-ie\gamma^\nu)u_{2a}}{(p_1+p_2)^2 - m^2}, \quad \mathcal{M}_{2ba} = \frac{\bar{u}_{4b}(-ie\gamma^\nu)(\not{q}_2 + m)(-ie\gamma^\mu)u_{2a}}{(p_1-p_4)^2 - m^2}$$

and

$$\not{q}_1 = (p_1 + p_2)^\alpha g_{\alpha\beta}\gamma^\beta, \quad \not{q}_2 = (p_4 - p_1)^\alpha g_{\alpha\beta}\gamma^\beta$$

The expected probability $\langle|\mathcal{M}|^2\rangle$ is the average over all four spin states.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\mathcal{M}_{ba}\mathcal{M}_{ba}^*$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\left(\frac{\mathcal{M}_{1ba}\mathcal{M}_{1ba}^*}{(s-m^2)^2} + \frac{\mathcal{M}_{1ba}\mathcal{M}_{2ba}^* + \mathcal{M}_{2ba}\mathcal{M}_{1ba}^*}{(s-m^2)(u-m^2)} + \frac{\mathcal{M}_{2ba}\mathcal{M}_{2ba}^*}{(u-m^2)^2}\right)$$

where s and u are the Mandelstam variables

$$s = (p_1 + p_2)^2, \quad u = (p_1 - p_4)^2$$

To understand how $\mathcal{M}_{1ba}\mathcal{M}_{1ba}^*$ is calculated, write $\mathcal{M}_{1ba}$ in component form.

$$(\mathcal{M}_{1ba})^{\mu\nu} = \frac{(\bar{u}_{4b})_\alpha(-ie\gamma^{\mu\alpha}{}_\beta)(\not{q}_1 + m)^\beta{}_\rho(-ie\gamma^{\nu\rho}{}_\sigma)(u_{2a})^\sigma}{s-m^2}$$

To sum over μ and ν of the product, $g_{\mu\nu}$ is needed to lower indices.

$$\mathcal{M}_{1ba}\mathcal{M}_{1ba}^* = (\mathcal{M}_{1ba})^{\mu\nu}(\mathcal{M}_{1ba}^*)_{\mu\nu} = (\mathcal{M}_{1ba})^{\mu\nu}g_{\mu\alpha}(\mathcal{M}_{1ba}^*)^{\alpha\beta}g_{\beta\nu}$$

Similarly for $\mathcal{M}_{2ba}\mathcal{M}_{2ba}^*$. For $\mathcal{M}_{2ba}$ the index order is ν followed by μ hence

$$\mathcal{M}_{1ba}\mathcal{M}_{2ba}^* = (\mathcal{M}_{1ba})^{\mu\nu}(\mathcal{M}_{2ba}^*)_{\nu\mu} = (\mathcal{M}_{1ba})^{\mu\nu}g_{\nu\beta}(\mathcal{M}_{2ba}^*)^{\beta\alpha}g_{\alpha\mu}$$

The Casimir trick uses matrix arithmetic to sum over spin states.

$$f_{11} = \sum_{a=1}^{2}\sum_{b=1}^{2}\mathcal{M}_{1ba}\mathcal{M}_{1ba}^* = e^4\,\mathrm{Tr}\left((\not{p}_2 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\right)$$

$$f_{12} = \sum_{a=1}^{2}\sum_{b=1}^{2}\mathcal{M}_{1ba}\mathcal{M}_{2ba}^* = e^4\,\mathrm{Tr}\left((\not{p}_2 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\right)$$

$$f_{22} = \sum_{a=1}^{2}\sum_{b=1}^{2}\mathcal{M}_{2ba}\mathcal{M}_{2ba}^* = e^4\,\mathrm{Tr}\left((\not{p}_2 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\nu(\not{q}_2 + m)\gamma_\mu\right)$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\left(\frac{f_{11}}{(s-m^2)^2} + \frac{f_{12} + f_{12}^*}{(s-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2}\right) \tag{1}$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$f_{11} = e^4\left(32(p_1\cdot p_2)(p_1\cdot p_4) + 64m^2(p_1\cdot p_2) - 32m^2(p_1\cdot p_3) - 32m^2(p_1\cdot p_4) + 32m^4\right)$$
$$f_{12} = e^4\left(16m^2(p_1\cdot p_2) - 16m^2(p_1\cdot p_4) + 32m^4\right)$$
$$f_{22} = e^4\left(32(p_1\cdot p_2)(p_1\cdot p_4) + 32m^2(p_1\cdot p_2) - 32m^2(p_1\cdot p_3) - 64m^2(p_1\cdot p_4) + 32m^4\right)$$

In Mandelstam variables

$$\begin{aligned} f_{11} &= e^4 \left(-8su + 24sm^2 + 8um^2 + 8m^4\right) \\ f_{12} &= e^4 \left(8sm^2 + 8um^2 + 16m^4\right) \\ f_{22} &= e^4 \left(-8su + 8sm^2 + 24um^2 + 8m^4\right) \end{aligned} \tag{2}$$

Compton scattering experiments are typically done in the lab frame where the electron is at rest. Define Lorentz boost Λ for transforming momentum vectors to the lab frame.

$$\Lambda = \begin{pmatrix} E/m & 0 & 0 & \omega/m \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega/m & 0 & 0 & E/m \end{pmatrix}$$

The electron is at rest in the lab frame.

$$\Lambda p_2 = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Mandelstam variables are invariant under a boost.

$$\begin{aligned} s &= (p_1 + p_2)^2 = (\Lambda p_1 + \Lambda p_2)^2 \\ t &= (p_1 - p_3)^2 = (\Lambda p_1 - \Lambda p_3)^2 \\ u &= (p_1 - p_4)^2 = (\Lambda p_1 - \Lambda p_4)^2 \end{aligned}$$

In the lab frame, let ω_L be the angular frequency of the incident photon and let ω'_L be the angular frequency of the scattered photon.

$$\omega_L = \Lambda p_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2}{m} + \frac{\omega E}{m}$$

$$\omega'_L = \Lambda p_3 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2 \cos\theta}{m} + \frac{\omega E}{m}$$

It can be shown that

$$\begin{aligned} s &= m^2 + 2m\omega_L \\ t &= 2m(\omega'_L - \omega_L) \\ u &= m^2 - 2m\omega'_L \end{aligned} \tag{3}$$

Then by (1), (2), and (3) we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \left(\frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1 \right)^2 - 1 \right)$$

Lab scattering angle θ_L is given by the Compton equation

$$\cos\theta_L = \frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1$$

Hence

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= 2e^4\left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \cos^2\theta_L - 1\right) \\ &= 2e^4\left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} - \sin^2\theta_L\right)\end{aligned}$$

Cross section

Now that we have derived $\langle|\mathcal{M}|^2\rangle$ we can investigate the angular distribution of scattered photons. For simplicity let us drop the L subscript from lab variables. From now on the symbols ω, ω', and θ will be lab frame variables.

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4(4\pi\varepsilon_0)^2 s}\left(\frac{\omega'}{\omega}\right)^2 \langle|\mathcal{M}|^2\rangle$$

where

$$s = m^2 + 2m\omega = (mc^2)^2 + 2(mc^2)(\hbar\omega)$$

and ω' is given by the Compton equation

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1-\cos\theta)}$$

For the lab frame we have

$$\langle|\mathcal{M}|^2\rangle = 2e^4\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right)$$

Hence in the lab frame

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\varepsilon_0)^2 s}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right)$$

Noting that

$$e^2 = 4\pi\varepsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right)$$

Noting that

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

we also have

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{2s}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta\right)\sin\theta\,d\theta\,d\phi$$

Let $S(\theta_1,\theta_2)$ be the following surface integral of $d\sigma$.

$$S(\theta_1,\theta_2) = \int_0^{2\pi}\int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1,\theta_2) = \frac{2\pi\alpha^2(\hbar c)^2}{2s}\big(I(\theta_2)-I(\theta_1)\big)$$

where

$$I(\theta) = -\frac{\cos\theta}{R^2} + \log\big(1+R(1-\cos\theta)\big)\left(\frac{1}{R}-\frac{2}{R^2}-\frac{2}{R^3}\right) - \frac{1}{2R\big(1+R(1-\cos\theta)\big)^2} + \frac{1}{1+R(1-\cos\theta)}\left(-\frac{2}{R^2}-\frac{1}{R^3}\right)$$

and

$$R = \frac{\hbar\omega}{mc^2}$$

The cumulative distribution function is

$$F(\theta) = \frac{S(0,\theta)}{S(0,\pi)} = \frac{I(\theta)-I(0)}{I(\pi)-I(0)}, \quad 0 \le \theta \le \pi$$

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

Let N be the total number of scattering events from an experiment. Then the number of scattering events in the interval θ_1 to θ_2 is predicted to be

$$NP(\theta_1 \le \theta \le \theta_2)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi)-I(0)}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta\right)\sin\theta$$

Thomson scattering

For $\hbar\omega \ll mc^2$ we have

$$\omega' = \frac{\omega}{1+\frac{\hbar\omega}{mc^2}(1-\cos\theta)} \approx \omega$$

Hence we can use the approximations

$$\omega = \omega' \quad \text{and} \quad s = (mc^2)^2$$

to obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2\hbar^2}{2m^2c^2}\left(1+\cos^2\theta\right)$$

which is the formula for Thomson scattering.

High energy approximation

For $\omega \gg m$ a useful approximation is to set $m = 0$ and obtain

$$\begin{aligned} f_{11} &= e^4(-8su) \\ f_{12} &= 0 \\ f_{22} &= e^4(-8su) \end{aligned}$$

Hence

$$\begin{aligned} \langle|\mathcal{M}|^2\rangle &= \frac{e^4}{4}\left(\frac{-8su}{s^2} + \frac{-8su}{u^2}\right) \\ &= 2e^4\left(-\frac{u}{s} - \frac{s}{u}\right) \end{aligned}$$

Also for $m = 0$ the Mandelstam variables s and u are

$$\begin{aligned} s &= 4\omega^2 \\ u &= -2\omega^2(\cos\theta + 1) \end{aligned}$$

Hence

$$\langle|\mathcal{M}|^2\rangle = 2e^4\left(\frac{\cos\theta + 1}{2} + \frac{2}{\cos\theta + 1}\right)$$