

Hydrogen transition 1

Start with the perturbing Hamiltonian (E_0 is peak electric field in newtons per coulomb)

$$H_1(\mathbf{r}, t) = -\frac{eE_0\boldsymbol{\epsilon} \cdot \mathbf{p}}{m\omega} \cos(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

In exponential form

$$H_1(\mathbf{r}, t) = -\frac{eE_0\boldsymbol{\epsilon} \cdot \mathbf{p}}{m\omega} \left(\frac{1}{2} \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t) + \frac{1}{2} \exp(-i\mathbf{k} \cdot \mathbf{r} + i\omega t) \right)$$

Given the initial condition $c_b(0) = 0$ the first-order approximation for $c_b(t)$ is

$$c_b(t) = -\frac{i}{\hbar} \int_0^t \langle \psi_b | H_1(\mathbf{r}, t') | \psi_a \rangle \exp(i\omega_0 t') dt', \quad \omega_0 = \frac{E_b - E_a}{\hbar}$$

Factor the integrand.

$$\begin{aligned} c_b(t) = & \frac{ieE_0}{2\hbar m\omega} \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle \int_0^t \exp(-i\omega t') \exp(i\omega_0 t') dt' \\ & + \frac{ieE_0}{2\hbar m\omega} \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(-i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle \int_0^t \exp(i\omega t') \exp(i\omega_0 t') dt' \end{aligned}$$

Solve the integrals to obtain

$$\begin{aligned} c_b(t) = & \frac{eE_0}{2\hbar m\omega} \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle \frac{\exp(i(\omega_0 - \omega)t) - 1}{\omega_0 - \omega} \\ & + \frac{eE_0}{2\hbar m\omega} \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(-i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle \frac{\exp(i(\omega_0 + \omega)t) - 1}{\omega_0 + \omega} \quad (1) \end{aligned}$$

As an approximation, discard the second term since the first term dominates for $\omega \approx \omega_0$.

$$c_b(t) = \frac{eE_0}{2\hbar m\omega} \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle \frac{\exp(i(\omega_0 - \omega)t) - 1}{\omega_0 - \omega}$$

Rewrite $c_b(t)$ in the form of a sine function.

$$c_b(t) = \frac{ieE_0}{\hbar m\omega} \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle \frac{\sin\left(\frac{1}{2}(\omega_0 - \omega)t\right)}{\omega_0 - \omega} \exp\left(\frac{i}{2}(\omega_0 - \omega)t\right) \quad (2)$$

Verify dimensions.

$$\begin{aligned} \frac{eE_0\boldsymbol{\epsilon} \cdot \mathbf{p}}{m\omega} &= \frac{\frac{e}{\text{coulomb}} \frac{E_0}{\text{newton coulomb}^{-1}} \frac{\boldsymbol{\epsilon} \cdot \mathbf{p}}{\text{momentum}}}{\frac{m}{\text{kilogram}} \frac{\omega}{\text{second}^{-1}}} = \text{joule} \\ c_b(t) &= \frac{\frac{e}{\text{coulomb}} \frac{E_0}{\text{newton coulomb}^{-1}} \frac{\langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle}{\text{momentum}}}{\frac{\hbar}{\text{joule second}} \frac{m}{\text{kilogram}} \frac{\omega}{\text{second}^{-1}}} = 1 \end{aligned}$$