Harmonic oscillator propagator 2

Consider the harmonic oscillator eigenstate

$$\psi_n(x,t) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right) \exp\left(-\frac{m\omega x^2}{2\hbar} - i\left(n+\tfrac{1}{2}\right)\omega t\right)$$

and the harmonic oscillator propagator

$$K(x_b,t_b,x_a,t_a) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega T)}\right)^{\frac{1}{2}} \exp\left[\frac{im\omega}{2\hbar\sin(\omega T)}\left(x_a^2\cos(\omega T) - 2x_a x_b + x_b^2\cos(\omega T)\right)\right]$$

where $T = t_b - t_a$.

We should have

$$\psi_n(x_b,T) = \int_{-\infty}^{\infty} K(x_b,T,x_a,0)\psi_n(x_a,0)\,dx_a$$

Try for $n=1$.

$$\psi_1(x_a,0) = \sqrt{2}\left(\frac{m^2\omega^3}{\pi\hbar^3}\right)^{\frac{1}{4}} x_a \exp\left(-\frac{m\omega x_a^2}{2\hbar}\right)$$

Hence

$$\psi_1(x_b,T) = \sqrt{2}\left(\frac{m^2\omega^3}{\pi\hbar^3}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} K(x_b,T,x_a,0)\,x_a \exp\left(-\frac{m\omega x_a^2}{2\hbar}\right)$$

Substitute for K.

$$\begin{aligned}\psi_1(x_b,T) = \sqrt{2}\left(\frac{m^2\omega^3}{\pi\hbar^3}\right)^{\frac{1}{4}} &\left(\frac{m\omega}{2\pi i\hbar\sin(\omega T)}\right)^{\frac{1}{2}} \\ &\times \int_{-\infty}^{\infty} \exp\left(\frac{im\omega x_a^2}{2\hbar\sin(\omega T)} - \frac{im\omega x_a}{\hbar\sin(\omega T)} + \frac{im\omega x_b^2\cos(\omega T)}{2\hbar\sin(\omega T)}\right) dx_a\end{aligned}$$

Let

$$a = -\frac{im\omega}{2\hbar\sin(\omega T)},\quad b = -\frac{im\omega}{\hbar\sin(\omega T)},\quad c = \frac{im\omega x_b^2\cos(\omega T)}{2\hbar\sin(\omega T)}$$

so that

$$\psi_1(x_b,T) = \sqrt{2}\left(\frac{m^2\omega^3}{\pi\hbar^3}\right)^{\frac{1}{4}} \left(\frac{m\omega}{2\pi i\hbar\sin(\omega T)}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp(-ax_a^2 + bx_a + c)\,dx_a$$

Solve the integral.

$$\begin{aligned}\psi_1(x_b,T) &= \sqrt{2}\left(\frac{m^2\omega^3}{\pi\hbar^3}\right)^{\frac{1}{4}} \left(\frac{m\omega}{2\pi i\hbar\sin(\omega T)}\right)^{\frac{1}{2}} \frac{\sqrt{\pi}}{2}\frac{b}{a^{3/2}} \exp\left(\frac{b^2}{4a} + c\right) \\ &= \sqrt{2}\left(\frac{m^3\omega^3}{\pi\hbar^3}\right)^{\frac{1}{4}} x_b \exp\left(-\frac{m\omega x_b^2}{2\hbar} - \tfrac{3}{2}i\omega T\right) \\ &= \psi_1(x_b,T)\end{aligned}$$