Atomic transitions 3

From the previous section

$$c_b(t) = \frac{ieE_0}{m\hbar\omega}\langle\psi_b|\exp(i\mathbf{k}\cdot\mathbf{r})\boldsymbol{\epsilon}\cdot\mathbf{p}|\psi_a\rangle\frac{\sin\left(\frac{1}{2}(\omega_0-\omega)t\right)}{\omega_0-\omega}\exp\left(\frac{i}{2}(\omega_0-\omega)t\right)$$

The transition probability is

$$\Pr_{a\to b}(t) = |c_b(t)|^2 = \frac{e^2E_0^2}{m^2\hbar^2\omega^2}\left|\langle\psi_b|\exp(i\mathbf{k}\cdot\mathbf{r})\boldsymbol{\epsilon}\cdot\mathbf{p}|\psi_a\rangle\right|^2\frac{\sin^2\left(\frac{1}{2}(\omega_0-\omega)t\right)}{(\omega_0-\omega)^2}$$

We are now going to pivot from E_0 to a full radiation field. Let u be energy density such that

$$E_0^2 = \frac{2u}{\varepsilon_0}$$

By substitution

$$\Pr_{a\to b}(t) = \frac{2u}{\varepsilon_0}\frac{e^2}{m^2\hbar^2\omega^2}\left|\langle\psi_b|\exp(i\mathbf{k}\cdot\mathbf{r})\boldsymbol{\epsilon}\cdot\mathbf{p}|\psi_a\rangle\right|^2\frac{\sin^2\left(\frac{1}{2}(\omega_0-\omega)t\right)}{(\omega_0-\omega)^2}$$

For a full radiation field

$$u = \int_{-\infty}^{\infty}\rho(\omega)\,d\omega$$

where $\rho(\omega)$ is energy per volume per hertz. By substitution

$$\Pr_{a\to b}(t) = \frac{2}{\varepsilon_0}\frac{e^2}{m^2\hbar^2}\left|\langle\psi_b|\exp(i\mathbf{k}\cdot\mathbf{r})\boldsymbol{\epsilon}\cdot\mathbf{p}|\psi_a\rangle\right|^2\int_{-\infty}^{\infty}\frac{\rho(\omega)}{\omega^2}\frac{\sin^2\left(\frac{1}{2}(\omega_0-\omega)t\right)}{(\omega_0-\omega)^2}\,d\omega$$

Because the integrand is sharply peaked at $\omega=\omega_0$, we are going to make the following move. Substitute ω_0 for ω in $\rho(\omega)/\omega^2$. That makes the term a constant so it can be moved outside the integral. We now have

$$\Pr_{a\to b}(t) = \frac{2}{\varepsilon_0}\frac{e^2}{m^2\hbar^2}\left|\langle\psi_b|\exp(i\mathbf{k}\cdot\mathbf{r})\boldsymbol{\epsilon}\cdot\mathbf{p}|\psi_a\rangle\right|^2\frac{\rho(\omega_0)}{\omega_0^2}\int_{-\infty}^{\infty}\frac{\sin^2\left(\frac{1}{2}(\omega_0-\omega)t\right)}{(\omega_0-\omega)^2}\,d\omega$$

Apply change of variable

$$y = \frac{1}{2}(\omega-\omega_0)t, \quad dy = \frac{t}{2}\,d\omega, \quad \omega-\omega_0 = \frac{2y}{t}$$

to obtain

$$\int_{-\infty}^{\infty}\frac{\sin^2\left(\frac{1}{2}(\omega_0-\omega)t\right)}{(\omega_0-\omega)^2}\,d\omega = \frac{t}{2}\int_{-\infty}^{\infty}\frac{\sin^2(-y)}{y^2}\,dy = \frac{\pi}{2}t$$

Hence

$$\Pr_{a\to b}(t) = \frac{\pi e^2}{\varepsilon_0 m^2\hbar^2}\left|\langle\psi_b|\exp(i\mathbf{k}\cdot\mathbf{r})\boldsymbol{\epsilon}\cdot\mathbf{p}|\psi_a\rangle\right|^2\frac{\rho(\omega_0)}{\omega_0^2}t$$

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The transition rate is

$$R_{a \to b} = \frac{d}{dt} \Pr_{a \to b}(t) = \frac{\pi e^2}{\varepsilon_0 m^2 \hbar^2} \left| \langle \psi_b | \exp(i\mathbf{k} \cdot \mathbf{r}) \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \right|^2 \frac{\rho(\omega_0)}{\omega_0^2}$$

Verify dimensions.

$$R_{a \to b} \propto \frac{\dfrac{e^2}{\mathrm{C}^2}}{\dfrac{\varepsilon_0}{\mathrm{C}^2\,\mathrm{J}^{-1}\,\mathrm{m}^{-1}} \dfrac{m^2}{\mathrm{kg}^2} \dfrac{\hbar^2}{\mathrm{J}^2\,\mathrm{s}^2}} \times \frac{\left| \langle \psi_b | \exp(i\mathbf{k} \cdot \mathbf{r}) \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \right|^2}{\mathrm{kg}^2\,\mathrm{m}^2\,\mathrm{s}^{-2}} \times \frac{\dfrac{\rho(\omega_0)}{\mathrm{J}\,\mathrm{m}^{-3}\,\mathrm{s}}}{\dfrac{\omega_0^2}{\mathrm{s}^{-2}}} = \mathrm{s}^{-1}$$