

Atomic transitions 3

From the previous section

$$c_b(t) = \frac{ieE_0}{m\hbar\omega} \langle \psi_b | \exp(i\mathbf{k} \cdot \mathbf{r}) \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \frac{\sin\left(\frac{1}{2}(\omega_0 - \omega)t\right)}{\omega_0 - \omega} \exp\left(\frac{i}{2}(\omega_0 - \omega)t\right)$$

The transition probability is

$$\text{Pr}_{a \rightarrow b}(t) = |c_b(t)|^2 = \frac{e^2 E_0^2}{m^2 \hbar^2 \omega^2} \left| \langle \psi_b | \exp(i\mathbf{k} \cdot \mathbf{r}) \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \right|^2 \frac{\sin^2\left(\frac{1}{2}(\omega_0 - \omega)t\right)}{(\omega_0 - \omega)^2}$$

We are now going to pivot from E_0 to a full radiation field. Let u be energy density such that

$$E_0^2 = \frac{2u}{\varepsilon_0}$$

By substitution

$$\text{Pr}_{a \rightarrow b}(t) = \frac{2u}{\varepsilon_0} \frac{e^2}{m^2 \hbar^2 \omega^2} \left| \langle \psi_b | \exp(i\mathbf{k} \cdot \mathbf{r}) \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \right|^2 \frac{\sin^2\left(\frac{1}{2}(\omega_0 - \omega)t\right)}{(\omega_0 - \omega)^2}$$

For a full radiation field

$$u = \int_{-\infty}^{\infty} \rho(\omega) d\omega$$

where $\rho(\omega)$ is energy per volume per hertz. By substitution

$$\text{Pr}_{a \rightarrow b}(t) = \frac{2}{\varepsilon_0} \frac{e^2}{m^2 \hbar^2} \left| \langle \psi_b | \exp(i\mathbf{k} \cdot \mathbf{r}) \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \right|^2 \int_{-\infty}^{\infty} \frac{\rho(\omega)}{\omega^2} \frac{\sin^2\left(\frac{1}{2}(\omega_0 - \omega)t\right)}{(\omega_0 - \omega)^2} d\omega$$

Because the integrand is sharply peaked at $\omega = \omega_0$, we are going to make the following move. Substitute ω_0 for ω in $\rho(\omega)/\omega^2$. That makes the term a constant so it can be moved outside the integral. We now have

$$\text{Pr}_{a \rightarrow b}(t) = \frac{2}{\varepsilon_0} \frac{e^2}{m^2 \hbar^2} \left| \langle \psi_b | \exp(i\mathbf{k} \cdot \mathbf{r}) \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \right|^2 \frac{\rho(\omega_0)}{\omega_0^2} \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{1}{2}(\omega_0 - \omega)t\right)}{(\omega_0 - \omega)^2} d\omega$$

Apply change of variable

$$y = \frac{1}{2}(\omega - \omega_0)t, \quad dy = \frac{t}{2} d\omega, \quad \omega - \omega_0 = \frac{2y}{t}$$

to obtain

$$\int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{1}{2}(\omega_0 - \omega)t\right)}{(\omega_0 - \omega)^2} d\omega = \frac{t}{2} \int_{-\infty}^{\infty} \frac{\sin^2(-y)}{y^2} dy = \frac{\pi}{2} t$$

Hence

$$\text{Pr}_{a \rightarrow b}(t) = \frac{\pi e^2}{\varepsilon_0 m^2 \hbar^2} \left| \langle \psi_b | \exp(i\mathbf{k} \cdot \mathbf{r}) \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \right|^2 \frac{\rho(\omega_0)}{\omega_0^2} t$$

The transition rate is

$$R_{a \rightarrow b} = \frac{d}{dt} \text{Pr}_{a \rightarrow b}(t) = \frac{\pi e^2}{\varepsilon_0 m^2 \hbar^2} \left| \langle \psi_b | \exp(i \mathbf{k} \cdot \mathbf{r}) \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \right|^2 \frac{\rho(\omega_0)}{\omega_0^2}$$

Verify dimensions.

$$R_{a \rightarrow b} \propto \frac{\frac{e^2}{\text{C}^2}}{\frac{\varepsilon_0}{\text{C}^2 \text{ J}^{-1} \text{ m}^{-1}} \frac{m^2}{\text{kg}^2} \frac{\hbar^2}{\text{J}^2 \text{ s}^2}} \times \left| \langle \psi_b | \exp(i \mathbf{k} \cdot \mathbf{r}) \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \right|^2 \times \frac{\frac{\rho(\omega_0)}{\text{J m}^{-3} \text{ s}}}{\frac{\omega_0^2}{\text{s}^{-2}}} = \text{s}^{-1}$$