Spin sign change

An electron is at rest in the following magnetic field.

$$\mathbf{B} = B_0 \cos(\omega t) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

What is the minimum B_0 that changes the sign of spin in the x direction?

These are the spin operators.

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This is the spin angular momentum operator.

$$\mathbf{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$$

This is the Hamiltonian.

$$H = \frac{ge}{2m}\mathbf{S} \cdot \mathbf{B} = \frac{ge}{2m} S_z B_0 \cos(\omega t)$$

Let $|s\rangle$ be the spin state

$$|s\rangle = \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$$

By the Schrodinger equation

$$i\hbar \frac{\partial}{\partial t}|s\rangle = H|s\rangle$$

we have

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} c_1(t) &= \frac{ge\hbar}{4m} B_0 \cos(\omega t) c_1(t) \\ i\hbar \frac{\partial}{\partial t} c_2(t) &= -\frac{ge\hbar}{4m} B_0 \cos(\omega t) c_2(t) \end{aligned}$$

Hence

$$\begin{aligned} c_1(t) &= C_1 \exp\left(-\frac{ige}{4m\omega} B_0 \sin(\omega t)\right) \\ c_2(t) &= C_2 \exp\left(\frac{ige}{4m\omega} B_0 \sin(\omega t)\right) \end{aligned} \tag{1}$$

where complex coefficients C_1 and C_2 have the general polar forms

$$C_1 = a_1 \exp(i\theta_1), \quad C_2 = a_2 \exp(i\theta_2), \quad |C_1|^2 + |C_2|^2 = a_1^2 + a_2^2 = 1$$

For the expectation value in the x direction we have

$$\langle S_x \rangle = \langle s|S_x|s\rangle = a_1 a_2 \hbar \cos\left(\frac{ge}{2m\omega} B_0 \sin(\omega t) - \theta_1 + \theta_2\right) \tag{2}$$

Note that the cosine function in (2) is always positive for

$$-\frac{\pi}{2} \leq \frac{ge}{2m\omega} B_0 \sin(\omega t) - \theta_1 + \theta_2 \leq \frac{\pi}{2}$$

For t such that $\sin(\omega t) = 1$ we have

$$\frac{ge}{2m\omega} B_0 - \theta_1 + \theta_2 \leq \frac{\pi}{2}$$

Hence $\langle S_x \rangle$ changes sign for some t when

$$B_0 > \frac{2m\omega}{ge} \left(\frac{\pi}{2} + \theta_1 - \theta_2 \right)$$

See exercise 10.6 of *Quantum Mechanics* (Lulu edition) by Richard Fitzpatrick.