Hydrogen selection rules

Verify the following selection rules by computing spontaneous emission rates.

$$|\Delta l| = 1, \quad |\Delta m| \leq 1$$

The spontaneous emission rate is

$$A_{i\to f} = \frac{4\alpha\omega_{i\to f}^3 |R_{i\to f}|^2}{3c^2}$$

where

$$\omega_{i\to f} = \frac{E_f - E_i}{\hbar}, \quad E_n = -\frac{\alpha\hbar c}{2n^2 a_0}$$

$$|R_{i\to f}|^2 = |x_{i\to f}|^2 + |y_{i\to f}|^2 + |z_{i\to f}|^2$$

$$x_{i\to f} = \int_0^\infty \int_0^\pi \int_0^{2\pi} x\psi_f^*\psi_i \, dV, \quad y_{i\to f} = \int_0^\infty \int_0^\pi \int_0^{2\pi} y\psi_f^*\psi_i \, dV, \quad z_{i\to f} = \int_0^\infty \int_0^\pi \int_0^{2\pi} z\psi_f^*\psi_i \, dV$$

and

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

The wave function for hydrogen is

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi)$$

where

$$R_{nl}(r) = \frac{2}{n^2}\sqrt{\frac{(n-l-1)!}{(n+l)!}} \left(\frac{2r}{na_0}\right)^l L_{n-l-1}^{2l+1}\left(\frac{2r}{na_0}\right) \exp\left(-\frac{r}{na_0}\right) a_0^{-3/2}$$

$$L_n^m(x) = (n+m)! \sum_{k=0}^n \frac{(-x)^k}{(n-k)!(m+k)!k!}$$

$$Y_{lm}(\theta,\phi) = (-1)^m \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta)\exp(im\phi)$$

$$P_l^m(\cos\theta) = \begin{cases} \left(\dfrac{\sin\theta}{2}\right)^m \displaystyle\sum_{k=0}^{l-m} (-1)^k \frac{(l+m+k)!}{(l-m-k)!(m+k)!k!} \left(\frac{1-\cos\theta}{2}\right)^k, & m \geq 0 \\ (-1)^m \dfrac{(l+m)!}{(l-m)!} P_l^{|m|}(\cos\theta), & m < 0 \end{cases}$$