Feynman and Hibbs problem 4-2

For a particle of charge e in a magnetic field the Lagrangian is

$$L(\dot{\mathbf{x}}, \mathbf{x}) = \frac{m}{2}\dot{\mathbf{x}}^2 + \frac{e}{c}\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}, t) - e\phi(\mathbf{x}, t)$$

where $\dot{\mathbf{x}}$ is the velocity vector, c is the velocity of light, and $\mathbf{A}$ and ϕ are the vector and scalar potentials. Show that the corresponding Schrodinger equation is

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar}\left(\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right) \cdot \left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right)\psi + e\phi\psi\right)$$

From equation (4.3)

$$\psi(\mathbf{x}, t+\epsilon) = \frac{1}{A}\int_{\mathbb{R}^3} \exp\left(\frac{i\epsilon}{\hbar} L\left(\frac{\mathbf{x}-\mathbf{y}}{\epsilon}, \frac{\mathbf{x}+\mathbf{y}}{2}\right)\right)\psi(\mathbf{y}, t)\, dy_1\, dy_2\, dy_3 \qquad (1)$$

This is the Lagrangian with arguments from (1).

$$L\left(\frac{\mathbf{x}-\mathbf{y}}{\epsilon}, \frac{\mathbf{x}+\mathbf{y}}{2}\right) \\ = \frac{m}{2\epsilon^2}(\mathbf{x}-\mathbf{y})^2 + \frac{e}{c\epsilon}(\mathbf{x}-\mathbf{y}) \cdot \mathbf{A}\left(\frac{\mathbf{x}+\mathbf{y}}{2}, t\right) - e\phi\left(\frac{\mathbf{x}+\mathbf{y}}{2}, t\right)$$

Hence

$$\psi(\mathbf{x}, t+\epsilon) = \frac{1}{A}\int_{\mathbb{R}^3} \\ \exp\left(\frac{im}{2\hbar\epsilon}(\mathbf{x}-\mathbf{y})^2 + \frac{ie}{\hbar c}(\mathbf{x}-\mathbf{y}) \cdot \mathbf{A}\left(\frac{\mathbf{x}+\mathbf{y}}{2}, t\right) - \frac{ie\epsilon}{\hbar}\phi\left(\frac{\mathbf{x}+\mathbf{y}}{2}, t\right)\right) \\ \times \psi(\mathbf{y}, t)\, dy_1\, dy_2\, dy_3$$

Let

$$\mathbf{y} = \mathbf{x} + \boldsymbol{\eta}$$

Then

$$\mathbf{x}-\mathbf{y} = \boldsymbol{\eta}, \quad \frac{\mathbf{x}+\mathbf{y}}{2} = \mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, \quad dy_1\, dy_2\, dy_3 = d\eta_1\, d\eta_2\, d\eta_3$$

Hence

$$\psi(\mathbf{x},t+\epsilon)=\frac{1}{A}\int_{\mathbb{R}^3}\exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2+\frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)$$
$$\times\,\psi(\mathbf{x}+\boldsymbol{\eta},t)\,d\eta_1\,d\eta_2\,d\eta_3$$

Factor the exponential.

$$\psi(\mathbf{x},t+\epsilon)=\frac{1}{A}\int_{\mathbb{R}^3}$$
$$\exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\exp\left(\frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)\exp\left(-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)$$
$$\times\,\psi(\mathbf{x}+\boldsymbol{\eta},t)\,d\eta_1\,d\eta_2\,d\eta_3 \tag{2}$$

From the identity $\exp(i\theta)=\cos(\theta)+i\sin(\theta)$ we have

$$\exp\left(-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)$$
$$=\cos\left(-\frac{e\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)+i\sin\left(-\frac{e\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)$$

Then for small ϵ

$$\exp\left(-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)\approx 1-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)$$

The authors write that the $\boldsymbol{\eta}$ term can be dropped "because the error is of higher order than ϵ." Hence

$$\exp\left(-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)\approx 1-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x},t\right) \tag{3}$$

Substitute (3) into (2).

$$\psi(\mathbf{x},t+\epsilon)=\frac{1}{A}\left(1-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x},t\right)\right)\int_{\mathbb{R}^3}$$
$$\exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\exp\left(\frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)\psi(\mathbf{x}+\boldsymbol{\eta},t)\,d\eta_1\,d\eta_2\,d\eta_3$$

Let

$$T=-\frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}(\mathbf{x},t)$$

Then (justify this)

$$\exp\left(\frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)\approx\left(1+T+\tfrac{1}{2}T^2\right)$$

Hence

$$\begin{aligned}\psi(\mathbf{x},t+\epsilon)=\frac{1}{A}&\left(1-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x},t\right)\right)\\&\times\int_{\mathbb{R}^3}\exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\left(1+T+\tfrac{1}{2}T^2\right)\psi(\mathbf{x}+\boldsymbol{\eta},t)\,d\eta_1\,d\eta_2\,d\eta_3\end{aligned}\tag{4}$$

Next we will use the following Taylor series approximations.

$$\begin{aligned}\psi(\mathbf{x},t+\epsilon)&\approx\psi(\mathbf{x},t)+\epsilon\frac{\partial\psi}{\partial t}\\\psi(\mathbf{x}+\boldsymbol{\eta},t)&\approx\psi(\mathbf{x},t)+\boldsymbol{\eta}\cdot\nabla\psi+\tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi)\end{aligned}\tag{5}$$

Note: In component notation

$$\boldsymbol{\eta}\cdot\nabla\psi=\eta_1\frac{\partial\psi}{\partial x_1}+\eta_2\frac{\partial\psi}{\partial x_2}+\eta_2\frac{\partial\psi}{\partial x_2}$$

and

$$\begin{aligned}\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi)=\eta_1^2\frac{\partial^2\psi}{\partial x_1^2}+\eta_2^2\frac{\partial^2\psi}{\partial x_2^2}+\eta_3^2\frac{\partial^2\psi}{\partial x_3^2}&\\+2\eta_1\eta_2\frac{\partial^2\psi}{\partial x_1\partial x_2}+2\eta_1\eta_3\frac{\partial^2\psi}{\partial x_1\partial x_3}+2\eta_2\eta_3\frac{\partial^2\psi}{\partial x_2\partial x_3}&\end{aligned}$$

Substitute the approximations (5) into (4).

$$\begin{aligned}\psi(\mathbf{x},t)+\epsilon\frac{\partial\psi}{\partial t}=\frac{1}{A}&\left(1-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x},t\right)\right)\int_{\mathbb{R}^3}\exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\\\times\left(1+T+\tfrac{1}{2}T^2\right)&\left(\psi(\mathbf{x},t)+\boldsymbol{\eta}\cdot\nabla\psi+\tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi)\right)\,d\eta_1\,d\eta_2\,d\eta_3\end{aligned}\tag{6}$$

To solve the above integral, we will use the following formulas provided by the authors.

$$I_k=\int_{-\infty}^{\infty}\exp\left(\frac{im\eta_k^2}{2\hbar\epsilon}\right)\,d\eta_k=\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{1/2}\qquad\text{FH (4.7)}$$

$$J_k=\int_{-\infty}^{\infty}\eta_k\exp\left(\frac{im\eta_k^2}{2\hbar\epsilon}\right)\,d\eta_k=0\qquad\text{FH (4.9)}$$

$$K_k=\int_{-\infty}^{\infty}\eta_k^2\exp\left(\frac{im\eta_k^2}{2\hbar\epsilon}\right)\,d\eta_k=\frac{i\hbar\epsilon}{m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{1/2}\qquad\text{FH (4.10)}$$

Hence

$$\begin{aligned}
&\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\psi(\mathbf{x},t)\,d\eta_1\,d\eta_2\,d\eta_3 \\
&= I_1 I_2 I_3 \psi(\mathbf{x},t) \\
&= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\psi(\mathbf{x},t)
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\boldsymbol{\eta}\cdot\nabla\psi(\mathbf{x},t)\,d\eta_1\,d\eta_2\,d\eta_3 \\
&= J_1 I_2 I_3 \frac{\partial\psi}{\partial x_1} + I_1 J_2 I_3 \frac{\partial\psi}{\partial x_2} + I_1 I_2 J_3 \frac{\partial\psi}{\partial x_3} \\
&= 0
\end{aligned} \tag{8}$$

$$\begin{aligned}
&\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi)\,d\eta_1\,d\eta_2\,d\eta_3 \\
&= \tfrac{1}{2}K_1 I_2 I_3 \frac{\partial^2\psi}{\partial x_1^2} + \tfrac{1}{2}I_1 K_2 I_3 \frac{\partial^2\psi}{\partial x_2^2} + \tfrac{1}{2}I_1 I_2 K_3 \frac{\partial^2\psi}{\partial x_3^2} \\
&+ J_1 J_2 I_1 \frac{\partial^2\psi}{\partial x_1 x_2} + J_1 I_2 J_3 \frac{\partial^2\psi}{\partial x_1 x_3} + I_1 J_2 J_3 \frac{\partial^2\psi}{\partial x_2 x_3} \\
&= \frac{i\hbar\epsilon}{2m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\nabla^2\psi
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)T\psi(\mathbf{x},t)\,d\eta_1\,d\eta_2\,d\eta_3 \\
&= J_1 I_2 I_3 \frac{-ie}{\hbar c}A_1\psi + I_1 J_2 I_3 \frac{-ie}{\hbar c}A_2\psi + I_1 I_2 J_3 \frac{-ie}{\hbar c}A_3\psi \\
&= 0
\end{aligned} \tag{10}$$

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) T\boldsymbol{\eta}\cdot\nabla\psi \, d\eta_1 \, d\eta_2 \, d\eta_3 \\
&= \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \\
&\times \frac{-ie}{\hbar c}(\eta_1 A_1 + \eta_2 A_2 + \eta_3 A_3)\left(\eta_1\frac{\partial\psi}{\partial x_1} + \eta_2\frac{\partial\psi}{\partial x_2} + \eta_3\frac{\partial\psi}{\partial x_3}\right) d\eta_1 \, d\eta_2 \, d\eta_3 \\
&= K_1 I_2 I_3 \frac{-ieA_1}{\hbar c}\frac{\partial\psi}{\partial x_1} + J_1 J_2 I_3 \frac{-ieA_1}{\hbar c}\frac{\partial\psi}{\partial x_2} + J_1 I_2 J_3 \frac{-ieA_1}{\hbar c}\frac{\partial\psi}{\partial x_3} \\
&+ J_1 J_2 I_3 \frac{-ieA_2}{\hbar c}\frac{\partial\psi}{\partial x_1} + I_1 K_2 I_3 \frac{-ieA_2}{\hbar c}\frac{\partial\psi}{\partial x_2} + I_1 J_2 J_3 \frac{-ieA_2}{\hbar c}\frac{\partial\psi}{\partial x_3} \\
&+ J_1 I_2 J_3 \frac{-ieA_3}{\hbar c}\frac{\partial\psi}{\partial x_1} + I_1 J_2 J_3 \frac{-ieA_3}{\hbar c}\frac{\partial\psi}{\partial x_2} + I_1 I_2 K_3 \frac{-ieA_3}{\hbar c}\frac{\partial\psi}{\partial x_3} \\
&= \frac{i\hbar\epsilon}{m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\frac{-ie}{\hbar c}\left(A_1\frac{\partial\psi}{\partial x_1} + A_2\frac{\partial\psi}{\partial x_2} + A_3\frac{\partial\psi}{\partial x_3}\right) \\
&= \frac{i\hbar\epsilon}{m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\frac{-ie}{\hbar c}\mathbf{A}\nabla\psi
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) T\tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi) \, d\eta_1 \, d\eta_2 \, d\eta_3 \\
&= \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\frac{-ie}{\hbar c}(\eta_1 A_1 + \eta_2 A_2 + \eta_3 A_3)\times \\
&\frac{1}{2}\left(\eta_1^2\frac{\partial^2\psi}{\partial x_1^2} + \eta_2^2\frac{\partial^2\psi}{\partial x_2^2} + \eta_3^2\frac{\partial^2\psi}{\partial x_3^2} + 2\eta_1\eta_2\frac{\partial^2\psi}{\partial x_1 x_2} + 2\eta_1\eta_3\frac{\partial^2\psi}{\partial x_1 x_3} + 2\eta_2\eta_3\frac{\partial^2\psi}{\partial x_2 x_3}\right) \\
&= 0
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \tfrac{1}{2}T^2\psi(\mathbf{x},t)\,d\eta_1\,d\eta_2\,d\eta_3 \\
&= \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\left(-\frac{e^2}{2\hbar^2c^2}\right)(\eta_1A_1+\eta_2A_2+\eta_3A_3)^2\psi(\mathbf{x},t)\,d\eta_1\,d\eta_2\,d\eta_3 \\
&= \frac{i\hbar\epsilon}{m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\left(-\frac{e^2}{2\hbar^2c^2}\right)(A_1^2+A_2^2+A_3^2)\psi(\mathbf{x},t) \\
&= \frac{i\hbar\epsilon}{m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\left(-\frac{e^2}{2\hbar^2c^2}\right)\mathbf{A}^2\psi
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \tfrac{1}{2}T^2\boldsymbol{\eta}\cdot\nabla\psi\,d\eta_1\,d\eta_2\,d\eta_3 \\
&= \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\left(-\frac{e^2}{2\hbar^2c^2}\right) \\
&\times (\eta_1A_1+\eta_2A_2+\eta_3A_3)^2\left(\eta_1\frac{\partial\psi}{\partial x_1}+\eta_2\frac{\partial\psi}{\partial x_2}+\eta_3\frac{\partial\psi}{\partial x_3}\right)\,d\eta_1\,d\eta_2\,d\eta_3 \\
&= 0
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \tfrac{1}{2}T^2\tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi)\,d\eta_1\,d\eta_2\,d\eta_3 \\
&= \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\left(-\frac{e^2}{2\hbar^2c^2}\right)(\eta_1A_1+\eta_2A_2+\eta_3A_3)^2 \times \\
&\frac{1}{2}\left(\eta_1^2\frac{\partial^2\psi}{\partial x_1^2}+\eta_2^2\frac{\partial^2\psi}{\partial x_2^2}+\eta_3^2\frac{\partial^2\psi}{\partial x_3^2}+2\eta_1\eta_2\frac{\partial^2\psi}{\partial x_1x_2}+2\eta_1\eta_3\frac{\partial^2\psi}{\partial x_1x_3}+2\eta_2\eta_3\frac{\partial^2\psi}{\partial x_2x_3}\right) \\
&= \frac{e^2\epsilon^2}{2m^2c^2}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \\
&\times\left(\frac{1}{2}\mathbf{A}^2\nabla^2\psi+\mathbf{A}\nabla\nabla\psi\mathbf{A}-\frac{3}{2}\left(A_1^2\frac{\partial^2}{\partial x_1^2}+A_2^2\frac{\partial^2}{\partial x_2^2}+A_3^2\frac{\partial^2}{\partial x_3^2}\right)\right)
\end{aligned} \tag{15}
$$

Substitute the solved integrals into (6) to obtain

$$
\psi(\mathbf{x},t)+\epsilon\frac{\partial\psi}{\partial t}=\frac{1}{A}\left(1-\frac{ie\epsilon}{\hbar}\phi(\mathbf{x},t)\right)I
$$

where

$$
\begin{aligned}
I = {} & \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\psi(\mathbf{x},t) && \text{from (7)} \\
& + \frac{i\hbar\epsilon}{2m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\nabla^2\psi && \text{from (9)} \\
& + \frac{i\hbar\epsilon}{m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\frac{-ie}{\hbar c}\mathbf{A}\nabla\psi && \text{from (11)} \\
& + \frac{i\hbar\epsilon}{m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\left(-\frac{e^2}{2\hbar^2c^2}\right)\mathbf{A}^2\psi && \text{from (13)} \\
& + 0 && \text{from (15)}
\end{aligned}
$$

The result from (15) is discarded because it is proportional to ϵ^2.

Simplify I as follows.

$$
I = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\left(\psi(\mathbf{x},t) + \frac{i\hbar\epsilon}{2m}\nabla^2\psi + \frac{e\epsilon}{mc}\mathbf{A}\nabla\psi - \frac{ie^2\epsilon}{2m\hbar c^2}\mathbf{A}^2\psi\right) \tag{16}
$$

In the limit as $\epsilon \to 0$ we have

$$
\psi(\mathbf{x},t) = \frac{1}{A}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\psi(\mathbf{x},t)
$$

hence

$$
A = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}
$$

Cancel A with the coefficient in I to obtain

$$
\begin{aligned}
\psi(\mathbf{x},t) + \epsilon\frac{\partial\psi}{\partial t} = {} & \left(1 - \frac{ie\epsilon}{\hbar}\phi(\mathbf{x},t)\right) \\
& \times\left(\psi(\mathbf{x},t) + \frac{i\hbar\epsilon}{2m}\nabla^2\psi + \frac{e\epsilon}{mc}\mathbf{A}\nabla\psi - \frac{ie^2\epsilon}{2m\hbar c^2}\mathbf{A}^2\psi\right)
\end{aligned}
$$

Expand the product and discard terms of order greater than ϵ.

$$
\begin{aligned}
& \psi(\mathbf{x},t) + \epsilon\frac{\partial\psi}{\partial t} \\
& \quad = \psi(\mathbf{x},t) + \frac{i\hbar\epsilon}{2m}\nabla^2\psi + \frac{e\epsilon}{mc}\mathbf{A}\nabla\psi - \frac{ie^2\epsilon}{2m\hbar c^2}\mathbf{A}^2\psi - \frac{ie\epsilon}{\hbar}\phi(\mathbf{x},t)\psi
\end{aligned}
$$

Cancel leading terms $\psi(\mathbf{x},t)$ and divide through by ϵ.

$$\frac{\partial\psi}{\partial t} = \frac{i\hbar}{2m}\nabla^2\psi + \frac{e}{mc}\mathbf{A}\nabla\psi - \frac{ie^2}{2m\hbar c^2}\mathbf{A}^2\psi - \frac{ie}{\hbar}\phi(\mathbf{x},t)\psi \tag{17}$$