Chapter 6

Start with equation (6.1).

$$K_V(b,a) = \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} \left(\tfrac{1}{2}m\dot{x}^2 - V(x(t),t)\right)\,dt\right)\mathcal{D}x(t) \tag{6.1}$$

Partition the integral.

$$K_V(b,a) = \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2\,dt - \frac{i}{\hbar}\int_{t_a}^{t_b} V(x(t),t)\,dt\right)\mathcal{D}x(t)$$

Factor the exponential.

$$K_V(b,a) = \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2\,dt\right)\exp\left(-\frac{i}{\hbar}\int_{t_a}^{t_b} V(x(t),t)\,dt\right)\mathcal{D}x(t)$$

Use t_c for the measure in the second integral.

$$K_V(b,a) = \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2\,dt\right)\exp\left(-\frac{i}{\hbar}\int_{t_a}^{t_b} V(x(t_c),t_c)\,dt_c\right)\mathcal{D}x(t)$$

Make the second exponential a power series.

$$K_V(b,a) = \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2\,dt\right)\times$$
$$\left(1 - \frac{i}{\hbar}\int_{t_a}^{t_b} V(x(t_c),t_c)\,dt_c + \frac{1}{2}\left(-\frac{i}{\hbar}\int_{t_a}^{t_b} V(x(t_c),t_c)\,dt_c\right)^2 + \cdots\right)\mathcal{D}x(t)$$

Expand the product.

$$\begin{aligned}
K_V(b,a) &= \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2\,dt\right)\mathcal{D}x(t) \\
&\quad - \frac{i}{\hbar}\int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2\,dt\right)\left(\int_{t_a}^{t_b} V(x(t_c),t_c)\,dt_c\right)\mathcal{D}x(t) \\
&\quad - \frac{1}{2\hbar^2}\int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2\,dt\right)\left(\int_{t_a}^{t_b} V(x(t_c),t_c)\,dt_c\right)^2\mathcal{D}x(t) + \cdots
\end{aligned}$$

Let

$$K_0(b,a) = \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2\,dt\right)\mathcal{D}x(t) \tag{6.5}$$

$$K^{(1)}(b,a) = -\frac{i}{\hbar}\int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2\,dt\right)\left(\int_{t_a}^{t_b} V(x(t_c),t_c)\,dt_c\right)\mathcal{D}x(t)$$

$$K^{(2)}(b,a) = -\frac{1}{2\hbar^2}\int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2\,dt\right)\left(\int_{t_a}^{t_b} V(x(t_c),t_c)\,dt_c\right)^2\mathcal{D}x(t)$$

Then equation (6.4) follows.

$$K_V(b,a) = K_0(b,a) + K^{(1)}(b,a) + K^{(2)}(b,a) + \cdots \tag{6.4}$$

Let us take a closer look at $K^{(1)}$. By the distributive law we can change the order of integration and obtain the following.

$$K^{(1)}(b,a) = -\frac{i}{\hbar}\int_{t_a}^{t_b}\int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2\,dt\right)V(x(t_c),t_c)\,\mathcal{D}x(t)\,dt_c$$

Let

$$I(t_c) = \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2\,dt\right)V(x(t_c),t_c)\,\mathcal{D}x(t)$$

so that

$$K^{(1)}(b,a) = -\frac{i}{\hbar}\int_{t_a}^{t_b} I(t_c)\,dt_c$$

We want to rewrite $I(t_c)$ as an integral over $x(t_c)$.

Let $x_c = x(t_c)$ and note that x_c can take on any value. In other words, for any $x_c \in (-\infty,\infty)$ there is a path from x_a to x_b that goes through x_c. Since $V(x_c,t_c)$ is a function of c only, the kernel for the path is a free particle from a to c and from c to b. Hence

$$I(t_c) = \int_{-\infty}^{\infty} K_0(x_b,t_b;x_c,t_c)V(x_c,t_c)K_0(x_c,t_c;x_a,t_a)\,dx_c$$

Or more compactly

$$I(t_c) = \int_{-\infty}^{\infty} K_0(b,c)V(c)K_0(c,a)\,dx_c$$

Hence

$$K^{(1)}(b,a) = -\frac{i}{\hbar}\int_{t_a}^{t_b}\int_{-\infty}^{\infty} K_0(b,c)V(c)K_0(c,a)\,dx_c\,dt_c$$