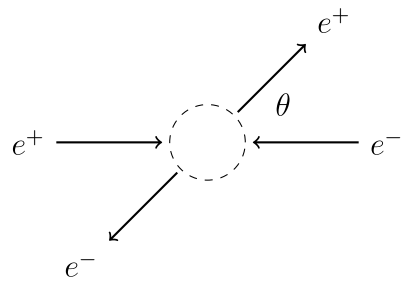
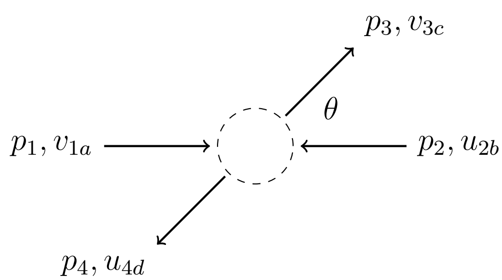
Bhabha scattering

The following diagram shows the geometry of a Bhabha scattering experiment.



Here is the same diagram with momentum and spinor labels.



In a typical collider experiment the momentum vectors are

$$p_1 = \underset{\text{inbound positron}}{\begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}} \qquad p_2 = \underset{\text{inbound electron}}{\begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix}} \qquad p_3 = \underset{\text{outbound positron}}{\begin{pmatrix} E \\ p\sin\theta\cos\phi \\ p\sin\theta\sin\phi \\ p\cos\theta \end{pmatrix}} \qquad p_4 = \underset{\text{outbound electron}}{\begin{pmatrix} E \\ -p\sin\theta\cos\phi \\ -p\sin\theta\sin\phi \\ -p\cos\theta \end{pmatrix}}$$

Symbol p is incident momentum. Symbol E is total energy $E = \sqrt{p^2 + m^2}$ where m is electron mass. Polar angle θ is the observed scattering angle. Azimuth angle ϕ cancels out in scattering calculations.

The spinors are

$$v_{11} = \underset{\substack{\text{inbound positron} \\ \text{spin up}}}{\begin{pmatrix} p \\ 0 \\ E+m \\ 0 \end{pmatrix}} \qquad u_{21} = \underset{\substack{\text{inbound electron} \\ \text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ -p \\ 0 \end{pmatrix}} \qquad v_{31} = \underset{\substack{\text{outbound positron} \\ \text{spin up}}}{\begin{pmatrix} p_3^z \\ p_3^x + ip_3^y \\ E+m \\ 0 \end{pmatrix}} \qquad u_{41} = \underset{\substack{\text{outbound electron} \\ \text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ p_4^z \\ p_4^x + ip_4^y \end{pmatrix}}$$

$$v_{12} = \underset{\substack{\text{inbound positron} \\ \text{spin down}}}{\begin{pmatrix} 0 \\ -p \\ 0 \\ E+m \end{pmatrix}} \qquad u_{22} = \underset{\substack{\text{inbound electron} \\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ p \end{pmatrix}} \qquad v_{32} = \underset{\substack{\text{outbound positron} \\ \text{spin down}}}{\begin{pmatrix} p_3^x - ip_3^y \\ -p_3^z \\ 0 \\ E+m \end{pmatrix}} \qquad u_{42} = \underset{\substack{\text{outbound electron} \\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ p_4^x - ip_4^y \\ -p_4^z \end{pmatrix}}$$

Spinor subscripts have 1 for spin up and 2 for spin down. The spinors are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^4$ will be used.

This is the probability density for spin state $abcd$. The formula is derived from Feynman diagrams for Bhabha scattering.

$$|\mathcal{M}_{abcd}|^2 = \frac{e^4}{N}\left|-\frac{1}{t}(\bar{v}_{1a}\gamma^\mu v_{3c})(\bar{u}_{4d}\gamma_\mu u_{2b}) + \frac{1}{s}(\bar{v}_{1a}\gamma^\nu u_{2b})(\bar{u}_{4d}\gamma_\nu v_{3c})\right|^2$$

Symbol e is electron charge. Symbols s and t are Mandelstam variables

$$\begin{aligned} s &= (p_1+p_2)^2 \\ t &= (p_1-p_3)^2 \end{aligned}$$

Let

$$a_1 = (\bar{v}_{1a}\gamma^\mu v_{3c})(\bar{u}_{4d}\gamma_\mu u_{2b}), \quad a_2 = (\bar{v}_{1a}\gamma^\nu u_{2b})(\bar{u}_{4d}\gamma_\nu v_{3c})$$

Then

$$\begin{aligned} |\mathcal{M}_{abcd}|^2 &= \frac{e^4}{N}\left|-\frac{a_1}{t}+\frac{a_2}{s}\right|^2 \\ &= \frac{e^4}{N}\left(-\frac{a_1}{t}+\frac{a_2}{s}\right)\left(-\frac{a_1}{t}+\frac{a_2}{s}\right)^* \\ &= \frac{e^4}{N}\left(\frac{a_1a_1^*}{t^2} - \frac{a_1a_2^*}{st} - \frac{a_1^*a_2}{st} + \frac{a_2a_2^*}{s^2}\right) \end{aligned}$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is computed by summing $|\mathcal{M}_{abcd}|^2$ over all spin states and then dividing by the number of inbound states. There are four inbound states.

$$\begin{aligned} \langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}|\mathcal{M}_{abcd}|^2 \\ &= \frac{e^4}{4N}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}\left(\frac{a_1a_1^*}{t^2} - \frac{a_1a_2^*}{st} - \frac{a_1^*a_2}{st} + \frac{a_2a_2^*}{s^2}\right) \end{aligned}$$

The Casimir trick uses matrix arithmetic to compute sums.

$$\begin{aligned} f_{11} &= \frac{1}{N}\sum_{abcd} a_1a_1^* = \mathrm{Tr}\left((\not{p}_1 - m)\gamma^\mu(\not{p}_3 - m)\gamma^\nu\right)\mathrm{Tr}\left((\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\right) \\ f_{12} &= \frac{1}{N}\sum_{abcd} a_1a_2^* = \mathrm{Tr}\left((\not{p}_1 - m)\gamma^\mu(\not{p}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{p}_3 - m)\gamma_\nu\right) \\ f_{22} &= \frac{1}{N}\sum_{abcd} a_2a_2^* = \mathrm{Tr}\left((\not{p}_1 - m)\gamma^\mu(\not{p}_2 + m)\gamma^\nu\right)\mathrm{Tr}\left((\not{p}_4 + m)\gamma_\mu(\not{p}_3 - m)\gamma_\nu\right) \end{aligned}$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left(\frac{f_{11}}{t^2} - \frac{f_{12}}{st} - \frac{f_{12}^*}{st} + \frac{f_{22}}{s^2}\right)$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$
\begin{aligned}
f_{11} &= 32(p_1 \cdot p_2)^2 + 32(p_1 \cdot p_4)^2 - 64m^2(p_1 \cdot p_3) + 64m^4 \\
f_{12} &= -32(p_1 \cdot p_4)^2 - 32m^2(p_1 \cdot p_2) + 32m^2(p_1 \cdot p_3) - 32m^2(p_1 \cdot p_4) - 32m^4 \\
f_{22} &= 32(p_1 \cdot p_3)^2 + 32(p_1 \cdot p_4)^2 + 64m^2(p_1 \cdot p_2) + 64m^4
\end{aligned}
$$

Using Mandelstam variables

$$
\begin{aligned}
s &= (p_1 + p_2)^2 \\
t &= (p_1 - p_3)^2 \\
u &= (p_1 - p_4)^2
\end{aligned}
$$

the formulas are

$$
\begin{aligned}
f_{11} &= 8s^2 + 8u^2 - 64sm^2 - 64um^2 + 192m^4 \\
f_{12} &= -8u^2 + 64um^2 - 96m^4 \\
f_{22} &= 8t^2 + 8u^2 - 64tm^2 - 64um^2 + 192m^4
\end{aligned}
$$

High energy approximation

For high energy experiments $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$
\begin{aligned}
f_{11} &= 8s^2 + 8u^2 \\
f_{12} &= -8u^2 \\
f_{22} &= 8t^2 + 8u^2
\end{aligned}
$$

Hence

$$
\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left(\frac{f_{11}}{t^2} - \frac{f_{12}}{st} - \frac{f_{12}^*}{st} + \frac{f_{22}}{s^2} \right) \\
&= \frac{e^4}{4} \left(\frac{8s^2 + 8u^2}{t^2} - \frac{-8u^2}{st} - \frac{-8u^2}{st} + \frac{8t^2 + 8u^2}{s^2} \right) \\
&= 2e^4 \left(\frac{s^2 + u^2}{t^2} + \frac{2u^2}{st} + \frac{t^2 + u^2}{s^2} \right)
\end{aligned}
$$

Combine terms so $\langle |\mathcal{M}|^2 \rangle$ has a common denominator.

$$
\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{s^2\left(s^2 + u^2\right) + 2stu^2 + t^2\left(t^2 + u^2\right)}{s^2 t^2} \right)
$$

For $m = 0$ the Mandelstam variables are

$$
\begin{aligned}
s &= 4E^2 \\
t &= 2E^2(\cos\theta - 1) \\
u &= -2E^2(\cos\theta + 1)
\end{aligned}
$$

Hence

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= 2e^4\left(\frac{32E^8\cos^4\theta + 192E^8\cos^2\theta + 288E^8}{64E^8(\cos\theta - 1)^2}\right)\\ &= e^4\left(\frac{\cos^4\theta + 6\cos^2\theta + 9}{(\cos\theta - 1)^2}\right)\\ &= e^4\left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2\end{aligned}$$

Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{4(4\pi\varepsilon_0)^2 s}$$

For the high energy approximation $m = 0$ we have

$$\langle|\mathcal{M}|^2\rangle = e^4\left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2 \quad \text{and} \quad s = 4E^2$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{4(4\pi\varepsilon_0)^2 s}\left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2$$

Noting that

$$e^2 = 4\pi\varepsilon_0\alpha\hbar c$$

we can also write

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{4s}\left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2$$

We can integrate $d\sigma$ to obtain a cumulative distribution function. Recall that

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

Hence

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{4s}\left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2 \sin\theta\, d\theta\, d\phi$$

Let $I(\theta)$ be the following integral of $d\sigma$.

$$I(\theta) = \int\left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2 \sin\theta\, d\theta$$

The result is

$$I(\theta) = \frac{16}{\cos\theta - 1} - \frac{\cos^3\theta}{3} - \cos^2\theta - 9\cos\theta - 16\log(1 - \cos\theta)$$

The cumulative distribution function is

$$F(\theta) = \frac{I(\theta) - I(a)}{I(\pi) - I(a)}, \quad a \le \theta \le \pi$$

Angular support is limited to an arbitrary $a > 0$ because $I(0)$ is undefined.

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

Data from SLAC SPEAR experiment

The following Bhabha scattering data is adapted from SLAC-PUB-1501.

k	$x_k,\ x_{k+1}$	y
1	$0.6, 0.5$	4432
2	$0.5, 0.4$	2841
3	$0.4, 0.3$	2045
4	$0.3, 0.2$	1420
5	$0.2, 0.1$	1136
6	$0.1, 0.0$	852
7	$0.0, -0.1$	656
8	$-0.1, -0.2$	625
9	$-0.2, -0.3$	511
10	$-0.3, -0.4$	455
11	$-0.4, -0.5$	402
12	$-0.5, -0.6$	398

Column k is the bin number, column y is the number of scattering events, and

$$x_k = \cos\theta_k$$

The cumulative distribution function for this experiment is

$$F(\theta) = \frac{I(\theta) - I(\theta_1)}{I(\theta_{13}) - I(\theta_1)}$$

where

$$\theta_{13} = \arccos(-0.6), \quad \theta_1 = \arccos(0.6)$$

The scattering probability P_k is

$$P_k = F\left(\arccos(x_{k+1})\right) - F\left(\arccos(x_k)\right)$$

Multiply P_k by total scattering events to obtain predicted number of events $\hat{y}_k$.

$$\sum y_k = 15773, \quad \hat{y}_k = 15773\, P_k$$

Bin	$x_k,\ x_{k+1}$	y	$\hat{y}$
1	$0.6, 0.5$	4432	4598
2	$0.5, 0.4$	2841	2880
3	$0.4, 0.3$	2045	1955
4	$0.3, 0.2$	1420	1410
5	$0.2, 0.1$	1136	1068
6	$0.1, 0.0$	852	843
7	$0.0, -0.1$	656	689
8	$-0.1, -0.2$	625	582
9	$-0.2, -0.3$	511	505
10	$-0.3, -0.4$	455	450
11	$-0.4, -0.5$	402	411
12	$-0.5, -0.6$	398	382

The coefficient of determination R^2 measures how well predicted values fit the data.

$$R^2 = 1 - \frac{\sum(y - \hat{y})^2}{\sum(y - \bar{y})^2} = 0.997$$

The result indicates that $F(\theta)$ explains 99.7% of the variance in the data.

Data from DESY PETRA experiment

See www.hepdata.net/record/ins191231, Table 3, 14.0 GeV.

x	y
-0.73	0.10115
-0.6495	0.12235
-0.5495	0.11258
-0.4494	0.09968
-0.3493	0.14749
-0.2491	0.14017
-0.149	0.1819
-0.0488	0.22964
0.0514	0.25312
0.1516	0.30998
0.252	0.40898
0.3524	0.62695
0.4529	0.91803
0.5537	1.51743
0.6548	2.56714
0.7323	4.30279

Data x and y have the following relationship with the differential cross section formula.

$$x = \cos\theta, \quad y = \frac{d\sigma}{d\Omega}$$

The cross section formula is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2 \times (\hbar c)^2$$

To compute predicted values $\hat{y}$, multiply by 10^{37} to convert square meters to nanobarns.

$$\hat{y} = \frac{\alpha^2}{4s}\left(\frac{x^2 + 3}{x - 1}\right)^2 \times (\hbar c)^2 \times 10^{37}$$

The following table shows predicted values $\hat{y}$ for $s = (14.0\,\mathrm{GeV})^2$.

x	y	$\hat{y}$
−0.73	0.10115	0.110296
−0.6495	0.12235	0.113816
−0.5495	0.11258	0.120101
−0.4494	0.09968	0.129075
−0.3493	0.14749	0.141592
−0.2491	0.14017	0.158934
−0.149	0.1819	0.182976
−0.0488	0.22964	0.216737
0.0514	0.25312	0.264989
0.1516	0.30998	0.335782
0.252	0.40898	0.44363
0.3524	0.62695	0.615528
0.4529	0.91803	0.9077
0.5537	1.51743	1.45175
0.6548	2.56714	2.60928
0.7323	4.30279	4.61509

The coefficient of determination R^2 measures how well predicted values fit the data.

$$R^2 = 1 - \frac{\sum(y - \hat{y})^2}{\sum(y - \bar{y})^2} = 0.995$$

The result indicates that the model $d\sigma$ explains 99.5% of the variance in the data.

Notes

Here are a few notes about how the Eigenmath scripts work. In component notation the trace operators of the Casimir trick become sums over the repeated index α.

$$f_{11} = \left((\not p_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not p_3 - m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha\right)\left((\not p_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not p_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha\right)$$

$$f_{12} = (\not p_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not p_2 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not p_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not p_3 - m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha$$

$$f_{22} = \left((\not p_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not p_2 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha\right)\left((\not p_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not p_3 - m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha\right)$$

To convert the above formulas to Eigenmath code, the γ tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply γ^μ by the metric tensor to lower the index.

$$\gamma^{\beta\mu}{}_\rho \quad \rightarrow \quad \texttt{gammaT = transpose(gamma)}$$
$$\gamma^{\beta}{}_{\mu\rho} \quad \rightarrow \quad \texttt{gammaL = transpose(dot(gmunu,gamma))}$$

Define the following 4×4 matrices.

$$(\not{p}_1 - m) \quad \rightarrow \quad \texttt{X1 = pslash1 - m I}$$
$$(\not{p}_2 + m) \quad \rightarrow \quad \texttt{X2 = pslash2 + m I}$$
$$(\not{p}_3 - m) \quad \rightarrow \quad \texttt{X3 = pslash3 - m I}$$
$$(\not{p}_4 + m) \quad \rightarrow \quad \texttt{X4 = pslash4 + m I}$$

Then for f_{11} we have the following Eigenmath code. The contract function sums over α.

$$(\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_3 - m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \quad \rightarrow \quad \texttt{T1 = contract(dot(X1,gammaT,X3,gammaT),1,4)}$$
$$(\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \quad \rightarrow \quad \texttt{T2 = contract(dot(X4,gammaL,X2,gammaL),1,4)}$$

Next, multiply then sum over repeated indices. The dot function sums over ν then the contract function sums over μ. The transpose makes the ν indices adjacent as required by the dot function.

$$f_{11} = \mathrm{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu)\,\mathrm{Tr}(\cdots \gamma_\mu \cdots \gamma_\nu) \quad \rightarrow \quad \texttt{f11 = contract(dot(T1,transpose(T2)))}$$

Follow suit for f_{22}.

$$(\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \quad \rightarrow \quad \texttt{T1 = contract(dot(X1,gammaT,X2,gammaT),1,4)}$$
$$(\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_3 - m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \quad \rightarrow \quad \texttt{T2 = contract(dot(X4,gammaL,X3,gammaL),1,4)}$$

Hence

$$f_{22} = \mathrm{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu)\,\mathrm{Tr}(\cdots \gamma_\mu \cdots \gamma_\nu) \quad \rightarrow \quad \texttt{f22 = contract(dot(T1,transpose(T2)))}$$

The calculation of f_{12} begins with

$$(\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_3 - m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha$$
$$\rightarrow \quad \texttt{T = contract(dot(X1,gammaT,X2,gammaT,X4,gammaL,X3,gammaL),1,6)}$$

Then sum over repeated indices μ and ν.

$$f_{12} = \mathrm{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu \cdots \gamma_\mu \cdots \gamma_\nu) \quad \rightarrow \quad \texttt{f12 = contract(contract(T,1,3))}$$