Harmonic oscillator coherent state

A coherent state minimizes uncertainty. The ground state is a coherent state. To make the ground state more interesting, parameters r and θ are added to shift $\langle x \rangle$ and $\langle p \rangle$ from zero. Time dependence is also added.

$$\psi_{n,r,\theta}(x,t) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\left(x - \langle x \rangle\right)\right)$$
$$\times \exp\left[-\frac{m\omega}{2\hbar}\left(x - \langle x \rangle\right)^2 + \frac{i}{\hbar}\langle p \rangle\left(x - \frac{\langle x \rangle}{2}\right) - i\left(n + \frac{1}{2}\right)\omega t\right]$$

Parameters r and θ are polar coordinates in phase space such that

$$\langle x \rangle = \sqrt{\frac{2\hbar}{m\omega}}\, r\cos(\omega t + \theta), \quad \langle p \rangle = -\sqrt{2m\hbar\omega}\, r\sin(\omega t + \theta)$$

Note that $\psi_{0,0,\theta}(x,0)$ is equivalent to the ordinary ground state.

$$\psi_{0,0,\theta}(x,0) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

Exercises

1. Verify for $n = 1$
$$i\hbar\frac{d}{dt}\psi(x,t) = \hat{H}\psi(x,t)$$

2. Verify for $n = 1$
$$\int_{-\infty}^{\infty} \psi^*(x,t)\psi(x,t)\,dx = 1$$

3. Verify for the ground state $\psi_0(x,t)$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{\hbar}{2m\omega}}$$

and

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{m\hbar\omega}{2}}$$

Hence $\Delta x \Delta p$ is the minimum allowed by the uncertainty principle.

$$\Delta x \Delta p = \frac{\hbar}{2}$$