Probability current

Let $\mathbf{J}$ be the probability current

$$\mathbf{J} = \frac{i\hbar}{2m}\left(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi\right)$$

Show that

$$\nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t}|\Psi|^2$$

Recall that

$$\nabla \cdot \nabla = \nabla^2$$

By the product rule for divergence we have

$$\nabla \cdot (\Psi\nabla\Psi^*) = \nabla\Psi \cdot \nabla\Psi^* + \Psi\nabla^2\Psi^*$$

Hence

$$\nabla \cdot \mathbf{J} = \frac{i\hbar}{2m}\left(\Psi\nabla^2\Psi^* - \Psi^*\nabla^2\Psi\right) \tag{1}$$

For the time derivative we have

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{\partial}{\partial t}(\Psi^*\Psi) = \Psi^*\frac{\partial\Psi}{\partial t} + \Psi\frac{\partial\Psi^*}{\partial t} \tag{2}$$

Recall the Schrödinger equation

$$\frac{\partial\Psi}{\partial t} = \frac{i\hbar}{2m}\nabla^2\Psi - \frac{i}{\hbar}V\Psi \tag{3}$$

Substitute (3) into (2) to obtain

$$\begin{aligned}\frac{\partial}{\partial t}|\Psi|^2 &= \Psi^*\left(\frac{i\hbar}{2m}\nabla^2\Psi - \frac{i}{\hbar}V\Psi\right) + \Psi\left(-\frac{i\hbar}{2m}\nabla^2\Psi + \frac{i}{\hbar}V\Psi\right) \\ &= \frac{i\hbar}{2m}\left(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*\right) \qquad (4)\end{aligned}$$

Hence by (1) and (4)

$$\nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t}|\Psi|^2$$