

# Wien's displacement law

For absolute temperature  $T$ , Wien's displacement law gives the peak EM radiation wavelength as

$$\lambda_{\text{peak}} = \frac{b}{T}$$

where  $b$  is Wien's displacement constant

$$b = 2.898 \times 10^{-3} \text{ kelvin meter}$$

Wien estimated the value of  $b$  empirically. However,  $b$  has an exact value that can be derived from Planck's law

$$u = \frac{2hc^2}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{kT\lambda}\right) - 1}$$

Take the derivative of  $u$  with respect to  $\lambda$ .

$$\frac{du}{d\lambda} = \frac{2h^2c^3 \exp\left(\frac{hc}{kT\lambda}\right)}{kT\lambda^7 \left(\exp\left(\frac{hc}{kT\lambda}\right) - 1\right)^2} - \frac{10hc^2}{\lambda^6 \left(\exp\left(\frac{hc}{kT\lambda}\right) - 1\right)}$$

Peak intensity is at  $du/d\lambda = 0$ .

$$\frac{2h^2c^3 \exp\left(\frac{hc}{kT\lambda}\right)}{kT\lambda^7 \left(\exp\left(\frac{hc}{kT\lambda}\right) - 1\right)^2} - \frac{10hc^2}{\lambda^6 \left(\exp\left(\frac{hc}{kT\lambda}\right) - 1\right)} = 0$$

Multiply both sides by

$$\frac{\lambda^6}{2hc^2 \left(\exp\left(\frac{hc}{kT\lambda}\right) - 1\right)}$$

to obtain

$$\frac{hc \exp\left(\frac{hc}{kT\lambda}\right)}{kT\lambda \left(\exp\left(\frac{hc}{kT\lambda}\right) - 1\right)} - 5 = 0$$

Define

$$x = \frac{hc}{kT\lambda}$$

and write

$$\frac{xe^x}{e^x - 1} - 5 = 0$$

Multiply both sides by  $e^x - 1$  to obtain

$$xe^x - 5e^x + 5 = 0 \tag{1}$$

The solution to (1) is

$$x = W_0(-5e^{-5}) + 5 \approx 4.96511$$

where  $W_0$  is the Lambert  $W$  function

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1} z^n}{n!}$$

Hence

$$b = \lambda T = \frac{hc}{kx} = 0.00289777 \text{ kelvin meter}$$