Let A_{nm} be the transition rate for spontaneous emission where the transition is from energy level E_n to a lower level E_m. Heisenberg, in his 1925 paper regarded as the origin of modern quantum mechanics, discovered the following formula for A_{nm}.

$$A_{nm} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3}\,\omega_{nm}^3\,|r_{nm}|^2$$

The transition frequency ω_{nm} is given by Bohr's frequency condition.

$$\omega_{nm} = \frac{1}{\hbar}(E_n - E_m)$$

The transition probability (multiplied by a physical constant) is

$$|r_{nm}|^2 = |x_{nm}|^2 + |y_{nm}|^2 + |z_{nm}|^2$$

For wave functions ψ in spherical coordinates we have the following transition amplitudes.

$$\begin{aligned}
x_{nm} &= \int \psi_m^*\,(r\sin\theta\cos\phi)\,\psi_n\,dV \\
y_{nm} &= \int \psi_m^*\,(r\sin\theta\sin\phi)\,\psi_n\,dV \\
z_{nm} &= \int \psi_m^*\,(r\cos\theta)\,\psi_n\,dV
\end{aligned}$$

Let us compute A_{21} for hydrogen. The energy levels for hydrogen are

$$E_n = -\frac{e^2}{8\pi\varepsilon_0 a_0 n^2}$$

where a_0 is the Bohr radius

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{e^2 m_e} = 5.29\times10^{-11}\,\text{meter}$$

For the transition frequency we have

$$\omega_{21} = \frac{1}{\hbar}(E_2 - E_1) = 1.55\times10^{16}\,\text{second}^{-1}$$

To compute the transition probability $|r_{21}|^2$ we need to consider all four eigenstates for $n = 2$.

n	ℓ	m_ℓ
2	1	1
2	1	-1
2	1	0
2	0	0

The following table shows the probability for every possible transition of ψ_2 to ψ_1.

$$
\begin{array}{lcccc}
 & \psi_{2,1,1} \to \psi_{1,0,0} & \psi_{2,1,-1} \to \psi_{1,0,0} & \psi_{2,1,0} \to \psi_{1,0,0} & \psi_{2,0,0} \to \psi_{1,0,0} \\
x_{21} = & -\frac{128}{243}\,a_0 & \frac{128}{243}\,a_0 & 0 & 0 \\
y_{21} = & -\frac{128}{243}i\,a_0 & -\frac{128}{243}i\,a_0 & 0 & 0 \\
z_{21} = & 0 & 0 & \frac{128}{243}\sqrt{2}\,a_0 & 0 \\
|r_{21}|^2 = & \frac{32768}{59049}\,a_0^2 & \frac{32768}{59049}\,a_0^2 & \frac{32768}{59049}\,a_0^2 & 0
\end{array}
$$

The transition $\psi_{2,0,0} \to \psi_{1,0,0}$ has zero probability. For the remaining transitions, the probability $|r_{21}|^2$ is independent of m_ℓ.

Now that we have $|r_{21}|^2$ we can compute a numerical value for A_{21}.

$$
A_{21} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3} \times \omega_{21}^3 \times \tfrac{32768}{59049}\,a_0^2 = 6.27 \times 10^8\,\text{second}^{-1}
$$

Here is A_{21} as a product of fundamental constants.

$$
A_{21} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3} \times \underbrace{\left(\frac{3e^4 m_e}{128\pi^2\varepsilon_0^2\hbar^3}\right)^3}_{\omega_{21}^3} \times \underbrace{\frac{32768}{59049}\left(\frac{4\pi\varepsilon_0\hbar^2}{e^2 m_e}\right)^2}_{|r_{21}|^2} = \frac{e^{10} m_e}{26244\,\pi^5\varepsilon_0^5\hbar^6 c^3}
$$

The parameters $n = 2$ and $m = 1$ contribute the following numerical factor to A_{21}.

$$
\underbrace{\left(-\frac{1}{2^2} + \frac{1}{1^2}\right)^3}_{\text{from } (E_2 - E_1)^3} \times \underbrace{\frac{32768}{59049}}_{\text{from } |r_{21}|^2} = \frac{512}{2187} = \frac{2^9}{3^7}
$$

Multiplying out numerical factors yields the numerical factor shown above for A_{21}.

$$
\frac{1}{3} \times \underbrace{\left(\frac{1}{32}\right)^3}_{\text{from } (E_n - E_m)^3} \times \underbrace{4^2}_{\text{from } a_0^2} \times \frac{512}{2187} = \frac{1}{26244} = \frac{1}{2^2 3^8}
$$

Let us analyze the units involved in computing A_{nm}. For the coefficient of A_{nm} we have

$$
\frac{e^2}{3\pi\varepsilon_0\hbar c^3} \propto \frac{\underbrace{\text{ampere}^2\,\text{second}^2}_{e^2}}{\underbrace{\left(\frac{\text{ampere}^2\,\text{second}^4}{\text{kilogram meter}^3}\right)}_{\varepsilon_0}\underbrace{\left(\frac{\text{kilogram meter}^2}{\text{second}}\right)}_{\hbar}\underbrace{\left(\frac{\text{meter}^3}{\text{second}^3}\right)}_{c^3}} = \frac{\text{second}^2}{\text{meter}^2}
$$

For the transition frequency we have

$$
\omega_{21} = \frac{3e^4 m_e}{128\pi^2\varepsilon_0^2\hbar^3} \propto \frac{\underbrace{\left(\text{ampere}^4\,\text{second}^4\right)}_{e^4}\underbrace{\text{kilogram}}_{m_e}}{\underbrace{\left(\frac{\text{ampere}^4\,\text{second}^8}{\text{kilogram}^2\,\text{meter}^6}\right)}_{\varepsilon_0^2}\underbrace{\left(\frac{\text{kilogram}^3\,\text{meter}^6}{\text{second}^3}\right)}_{\hbar^3}} = \text{second}^{-1}
$$

For the Bohr radius we have

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{e^2 m_e} \propto \frac{\underbrace{\left(\frac{\text{ampere}^2\,\text{second}^4}{\text{kilogram meter}^3}\right)}_{\varepsilon_0}\underbrace{\left(\frac{\text{kilogram}^2\,\text{meter}^4}{\text{second}^2}\right)}_{\hbar^2}}{\underbrace{\left(\text{ampere}^2\,\text{second}^2\right)}_{e^2}\underbrace{\text{kilogram}}_{m_e}} = \text{meter}$$

Hence

$$A_{nm} \propto \frac{\text{second}^2}{\text{meter}^2} \times \underbrace{\text{second}^{-3}}_{\omega_{nm}^3} \times \underbrace{\text{meter}^2}_{a_0^2} = \text{second}^{-1}$$

The coefficients B_{12} (absorption) and B_{21} (induced emission) can be computed from A_{21}.

$$B_{21} = \frac{c^2}{2h\nu^3}\,A_{21} = \frac{4.25 \times 10^{58}}{\nu^3}$$

$$B_{12} = \frac{g_2}{g_1}\,B_{21} = \frac{6}{2}\,B_{21} = \frac{1.28 \times 10^{59}}{\nu^3}$$

Symbol g_n is the multiplicity associated with energy level n.

$$g = (2s+1)(2\ell+1)$$