

Addition of angular momentum 2

In spherical coordinates

$$\begin{aligned} L_x &= i\hbar \left(\sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right) \\ L_y &= i\hbar \left(-\cos \phi \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right) \\ L_z &= -i\hbar \frac{\partial}{\partial \phi} \end{aligned}$$

and

$$L^2 = L_x^2 + L_y^2 + L_z^2 = -\hbar^2 \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right)$$

Recall that

$$L_z Y_{lm} = m\hbar Y_{lm}$$

and

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence for $\Psi = Y_{lm}\chi_+$ we have

$$\begin{aligned} J_z \Psi &= L_z \Psi + S_z \Psi \\ &= m\hbar \Psi + \frac{1}{2}\hbar \Psi \\ &= \left(m + \frac{1}{2}\right) \hbar \Psi \end{aligned}$$

and for $\Psi = Y_{lm}\chi_-$

$$\begin{aligned} J_z \Psi &= L_z \Psi + S_z \Psi \\ &= m\hbar \Psi - \frac{1}{2}\hbar \Psi \\ &= \left(m - \frac{1}{2}\right) \hbar \Psi \end{aligned}$$