

## Rutherford scattering 2

Find the cross section for Rutherford scattering with a screened Coulomb potential

$$V(r) = -\frac{Ze^2}{r} \exp\left(-\frac{r}{a}\right)$$

Start with

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2\epsilon_0^2} \left( \frac{mQ}{2\pi\hbar^2} \right)^2, \quad Q = \int \exp\left(\frac{i\mathbf{p} \cdot \mathbf{r}}{\hbar}\right) V(\mathbf{r}) d^3\mathbf{r}$$

Convert  $Q$  to polar coordinates.

$$Q = \int_0^{2\pi} \int_0^\pi \int_0^\infty \exp\left(\frac{ipr \cos \theta}{\hbar}\right) V(r, \theta, \phi) r^2 \sin \theta dr d\theta d\phi$$

Substitute the screened Coulomb potential for  $V(r, \theta, \phi)$  and note  $r^2$  becomes  $r$ .

$$Q = -Ze^2 \int_0^{2\pi} \int_0^\pi \int_0^\infty \exp\left(\frac{ipr \cos \theta}{\hbar}\right) \exp\left(-\frac{r}{a}\right) r \sin \theta dr d\theta d\phi$$

Integrate over  $\phi$ .

$$Q = -2\pi Ze^2 \int_0^\pi \int_0^\infty \exp\left(\frac{ipr \cos \theta}{\hbar}\right) \exp\left(-\frac{r}{a}\right) r \sin \theta dr d\theta$$

Transform the integral over  $\theta$  to an integral over  $y$  where  $y = \cos \theta$ ,  $dy = -\sin \theta d\theta$ .

$$Q = -2\pi Ze^2 \int_{-1}^1 \int_0^\infty \exp\left(\frac{ipry}{\hbar}\right) \exp\left(-\frac{r}{a}\right) r dr dy$$

Solve the integral over  $y$  (note  $r$  in the integrand cancels).

$$Q = -2\pi Ze^2 \int_0^\infty \frac{\hbar}{ip} \left[ \exp\left(\frac{ipr}{\hbar}\right) - \exp\left(-\frac{ipr}{\hbar}\right) \right] \exp\left(-\frac{r}{a}\right) dr$$

Solve the integral over  $r$ .

$$Q = -2\pi Ze^2 \frac{\hbar}{ip} \left[ \frac{1}{ip/\hbar - 1/a} \exp\left(\frac{ipr}{\hbar} - \frac{r}{a}\right) + \frac{1}{ip/\hbar + 1/a} \exp\left(-\frac{ipr}{\hbar} - \frac{r}{a}\right) \right]_0^\infty$$

Evaluate the limits.

$$Q = -2\pi Ze^2 \frac{\hbar}{ip} \left[ -\frac{1}{ip/\hbar - 1/a} - \frac{1}{ip/\hbar + 1/a} \right] = -\frac{4\pi Ze^2}{(p/\hbar)^2 + (1/a)^2} \quad (1)$$

The cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2\epsilon_0^2} \left( \frac{mQ}{2\pi\hbar^2} \right)^2 = \frac{1}{64\pi^2\epsilon_0^2} \frac{4m^2 Z^2 e^4}{[p^2 + (\hbar/a)^2]^2}$$

Substitute  $(4\pi\epsilon_0\alpha\hbar c)^2$  for  $e^4$ .

$$\frac{d\sigma}{d\Omega} = \frac{m^2 Z^2 \alpha^2 (\hbar c)^2}{[p^2 + (\hbar/a)^2]^2}$$

Symbol  $p$  is momentum transfer  $|\mathbf{p}_i| - |\mathbf{p}_f|$  such that

$$p^2 = 2mE(1 - \cos \theta)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{m^2 Z^2 \alpha^2 (\hbar c)^2}{[2mE(1 - \cos \theta) + (\hbar/a)^2]^2}$$

Cancel  $m^2$  in the numerator.

$$\frac{d\sigma}{d\Omega} = \frac{Z^2 \alpha^2 (\hbar c)^2}{[2E(1 - \cos \theta) + \frac{1}{m}(\hbar/a)^2]^2} \quad (2)$$

Let  $a \rightarrow \infty$  to obtain the ordinary Rutherford cross section.