

8-3. Show that Q_α^c , Q_α^s are normal coordinates corresponding to standing wave normal modes $\cos(2\pi\alpha j/N)$ and $\sin(2\pi\alpha j/N)$, in the sense that (for N odd)

$$q_j(t) = \sqrt{\frac{2}{N}} \left(\frac{1}{2} Q_0^c(t) + \sum_{\alpha=1}^{(N-1)/2} \left(Q_\alpha^c(t) \cos \frac{2\pi\alpha j}{N} + Q_\alpha^s(t) \sin \frac{2\pi\alpha j}{N} \right) \right) \quad (8.82)$$

(Note: A typo in 8.82 is fixed.)

Consider the following equations.

$$Q_\alpha(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^N q_k(t) \left(\cos \frac{2\pi\alpha k}{N} - i \sin \frac{2\pi\alpha k}{N} \right) \quad (8.77)$$

$$Q_\alpha^c = \frac{1}{\sqrt{2}} (Q_\alpha + Q_\alpha^*) \quad (8.79)$$

$$Q_\alpha^s = \frac{i}{\sqrt{2}} (Q_\alpha - Q_\alpha^*) \quad (8.80)$$

Let

$$\begin{aligned} T_1 &= \cos \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N} - i \sin \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N} \\ T_2 &= \cos \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N} + i \sin \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N} \\ T_3 &= i \cos \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N} + \sin \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N} \\ T_4 &= i \cos \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N} - \sin \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N} \end{aligned}$$

Then

$$T = T_1 + T_2 + T_3 - T_4 = 2 \cos \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N} + 2 \sin \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N}$$

By trigonometric identities

$$T = 2 \cos \left(\frac{2\pi\alpha}{N} (j - k) \right)$$

It follows that

$$\begin{aligned}
\sum_{\alpha=1}^{(N-1)/2} \left(Q_{\alpha}^c \cos \frac{2\pi\alpha j}{N} + Q_{\alpha}^s \sin \frac{2\pi\alpha j}{N} \right) &= \sum_{\alpha=1}^{(N-1)/2} \frac{1}{\sqrt{N}} \sum_{k=1}^N q_k \frac{T}{\sqrt{2}} \\
&= \sqrt{\frac{2}{N}} \sum_{\alpha=1}^{(N-1)/2} \sum_{k=1}^N q_k \cos \left(\frac{2\pi\alpha}{N} (j - k) \right)
\end{aligned}$$