

What is the mean lifetime of $2p$ hydrogen?

Let us begin by writing down the wave function ψ for hydrogen.

$$\psi_{nlm}(r, \theta, \phi) = R_{nl}(r)Y_{lm}(\theta, \phi)$$

where

$$\begin{aligned} R_{nl}(r) &= \frac{2}{n^2} \left(\frac{(n-l-1)!}{(n+l)!} \right)^{1/2} \left(\frac{2r}{na_0} \right)^l L_{n-l-1}^{2l+1} \left(\frac{2r}{na_0} \right) \exp \left(-\frac{r}{na_0} \right) a_0^{-3/2} \\ L_n^m(x) &= (n+m)! \sum_{k=0}^n \frac{(-x)^k}{(n-k)!(m+k)!k!} \\ Y_{lm}(\theta, \phi) &= (-1)^m \left(\frac{2l+1}{4\pi} \right)^{1/2} \left(\frac{(l-m)!}{(l+m)!} \right)^{1/2} P_l^m(\cos \theta) \exp(im\phi) \\ P_l^m(x) &= \frac{1}{2^l l!} (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l \\ a_0 &= \frac{4\pi\epsilon_0 \hbar^2}{e^2 \mu} \approx 0.529 \times 10^{-10} \text{ meter} \end{aligned}$$

The state $2p$ means that $n = 2$ and $l = 1$. That leaves three ways to choose m hence all of the following processes correspond to the transition $2p \rightarrow 1s$.

$$\left. \begin{array}{l} \psi_{2,1,1} \\ \psi_{2,1,0} \\ \psi_{2,1,-1} \end{array} \right\} \rightarrow \psi_{100} + \text{photon}$$

By Fermi's Golden Rule and using a dipole approximation we have for spontaneous emission rate A_{21}

$$A_{21} = \frac{e^2}{3\pi\epsilon_0 \hbar c^3} \omega_{21}^3 |r_{21}|^2 \quad (1)$$

where

$$\begin{aligned}
\omega_{21} &= \frac{E_2 - E_1}{\hbar}, \quad E_n = -\frac{e^2}{8\pi\epsilon_0 a_0 n^2} \\
|r_{21}|^2 &= |x_{21}|^2 + |y_{21}|^2 + |z_{21}|^2 \\
x_{21} &= \int_0^{2\pi} \int_0^\pi \int_0^\infty x f_{21} dV, \quad y_{21} = \int_0^{2\pi} \int_0^\pi \int_0^\infty y f_{21} dV, \quad z_{21} = \int_0^{2\pi} \int_0^\pi \int_0^\infty z f_{21} dV \\
x &= r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta \\
f_{21} &= \psi_{100}^* \psi_{210} = \frac{r \cos \theta}{4\sqrt{2}\pi a_0^4} \exp\left(-\frac{3r}{2a_0}\right) \\
dV &= r^2 \sin \theta dr d\theta d\phi
\end{aligned}$$

For the calculation of $|r_{21}|^2$ we obtain

$$x_{21} = 0, \quad y_{21} = 0, \quad z_{21} = \frac{2^7}{3^5} \sqrt{2} a_0$$

hence

$$|r_{21}|^2 = |z_{21}|^2 = \frac{2^{15}}{3^{10}} a_0^2 = \frac{32768}{59049} a_0^2$$

By equation (1) the transition rate is

$$A_{21} = 6.26 \times 10^8 \text{ second}^{-1}$$

Hence the mean lifetime is

$$\frac{1}{A_{21}} = 1.60 \times 10^{-9} \text{ second}$$