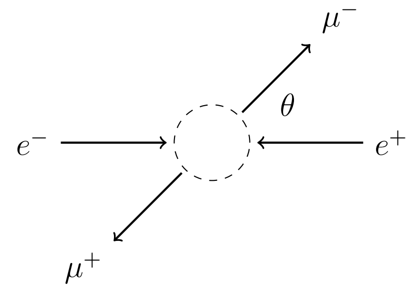
Muon pair production

Muon pair production is the interaction $e^- + e^+ \to \mu^- + \mu^+$.



Define the following momentum vectors and spinors. Symbol E is beam energy. Symbol p is electron momentum $p = \sqrt{E^2 - m^2}$ where m is electron mass 0.51 MeV. Symbol ρ is muon mementum $\rho = \sqrt{E^2 - M^2}$ where M is muon mass 106 MeV.

$$p_1 = \underset{\substack{\text{inbound}\\ \text{electron}}}{\begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}} \qquad p_2 = \underset{\substack{\text{inbound}\\ \text{positron}}}{\begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix}} \qquad p_3 = \underset{\substack{\text{outbound}\\ \text{muon}}}{\begin{pmatrix} E \\ \rho\sin\theta\cos\phi \\ \rho\sin\theta\sin\phi \\ \rho\cos\theta \end{pmatrix}} \qquad p_4 = \underset{\substack{\text{outbound}\\ \text{anti-muon}}}{\begin{pmatrix} E \\ -\rho\sin\theta\cos\phi \\ -\rho\sin\theta\sin\phi \\ -\rho\cos\theta \end{pmatrix}}$$

$$u_{11} = \underset{\substack{\text{inbound electron}\\ \text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix}} \qquad v_{21} = \underset{\substack{\text{inbound positron}\\ \text{spin up}}}{\begin{pmatrix} -p \\ 0 \\ E+m \\ 0 \end{pmatrix}} \qquad u_{31} = \underset{\substack{\text{outbound muon}\\ \text{spin up}}}{\begin{pmatrix} E+M \\ 0 \\ p_3^z \\ p_3^x + ip_3^y \end{pmatrix}} \qquad v_{41} = \underset{\substack{\text{outbound anti-muon}\\ \text{spin up}}}{\begin{pmatrix} p_4^z \\ p_4^x + ip_4^y \\ E+M \\ 0 \end{pmatrix}}$$

$$u_{12} = \underset{\substack{\text{inbound electron}\\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix}} \qquad v_{22} = \underset{\substack{\text{inbound positron}\\ \text{spin down}}}{\begin{pmatrix} 0 \\ p \\ 0 \\ E+m \end{pmatrix}} \qquad u_{32} = \underset{\substack{\text{outbound muon}\\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+M \\ p_3^x - ip_3^y \\ -p_3^z \end{pmatrix}} \qquad v_{42} = \underset{\substack{\text{outbound anti-muon}\\ \text{spin down}}}{\begin{pmatrix} p_4^x - ip_4^y \\ -p_4^z \\ 0 \\ E+M \end{pmatrix}}$$

The spinors are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^2(E+M)^2$ will be used.

This is the probability density for spin state *abcd*. The formula is derived from Feynman diagrams for muon pair production.

$$\left|\mathcal{M}_{abcd}\right|^2 = \frac{e^4}{Ns^2}\left|(\bar{u}_{3c}\gamma_\mu v_{4d})(\bar{v}_{2b}\gamma^\mu u_{1a})\right|^2$$

Symbol e is electron charge and

$$s = (p_1 + p_2)^2 = 4E^2$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is computed by summing $|\mathcal{M}_{abcd}|^2$ over all spin states and dividing by the number of inbound states. There are four inbound states.

$$
\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}\left|\mathcal{M}_{abcd}\right|^2 \\
&= \frac{e^4}{4Ns^2}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}\left|(\bar{u}_{3c}\gamma_\mu v_{4d})(\bar{v}_{2b}\gamma^\mu u_{1a})\right|^2
\end{aligned}
$$

The Casimir trick uses matrix arithmetic to compute sums.

$$
\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4s^2}\operatorname{Tr}\left((\not{p}_3 + M)\gamma^\mu(\not{p}_4 - M)\gamma^\nu\right)\operatorname{Tr}\left((\not{p}_2 - m)\gamma_\mu(\not{p}_1 + m)\gamma_\nu\right)
$$

The following formula is equivalent to the Casimir trick. (Recall that $a\cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$
\begin{aligned}
\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4s^2}\Big(&32(p_1\cdot p_3)(p_2\cdot p_4) + 32(p_1\cdot p_4)(p_2\cdot p_3) \\
&+ 32m^2(p_3\cdot p_4) + 32M^2(p_1\cdot p_2) + 64m^2M^2\Big)
\end{aligned}
$$

For the momentum vectors given above the result is

$$
\langle|\mathcal{M}|^2\rangle = e^4\left(1 + \cos^2\theta + \frac{m^2 + M^2}{E^2}\sin^2\theta + \frac{m^2M^2}{E^4}\cos^2\theta\right)
$$

For high energy experiments $E \gg M$ a useful approximation is

$$
\langle|\mathcal{M}|^2\rangle = e^4\left(1 + \cos^2\theta\right)
$$

Cross section

The differential cross section is

$$
\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{4(4\pi\varepsilon_0)^2 s}, \quad s = (p_1 + p_2)^2 = 4E^2
$$

For high energy experiments we have

$$
\langle|\mathcal{M}|^2\rangle = e^4\left(1 + \cos^2\theta\right)
$$

Substitute for $\langle|\mathcal{M}|^2\rangle$.

$$
\frac{d\sigma}{d\Omega} = \frac{e^4}{4(4\pi\varepsilon_0)^2 s}\left(1 + \cos^2\theta\right)
$$

Noting that

$$
e^2 = 4\pi\varepsilon_0\alpha\hbar c
$$

we can also write

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{4s}\left(1+\cos^2\theta\right)$$

We can integrate $d\sigma$ to obtain a cumulative distribution function. Let $I(\theta)$ be the following integral of $d\sigma$. (The $\sin\theta$ is from $d\Omega = \sin\theta\, d\theta\, d\phi$.)

$$I(\theta) = \int \left(1+\cos^2\theta\right)\sin\theta\, d\theta$$

The result is

$$I(\theta) = -\frac{\cos^3\theta}{3} - \cos\theta$$

The cumulative distribution function is

$$F(\theta) = \frac{I(\theta)-I(0)}{I(\pi)-I(0)} = -\frac{\cos^3\theta}{8} - \frac{3\cos\theta}{8} + \frac{1}{2}, \quad 0 \le \theta \le \pi$$

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

Let N be the number of scattering events from an experiment. Then the number of scattering events in the interval θ_1 to θ_2 is predicted to be

$$N\left(F(\theta_2) - F(\theta_1)\right)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{3}{8}\left(1+\cos^2\theta\right)\sin\theta$$

Note that if we had carried through the $\alpha^2(\hbar c)^2/4s$ in $I(\theta)$, it would have cancelled out in $F(\theta)$.

Data from SLAC PEP experiment

See www.hepdata.net/record/ins216031, Table 1, $s = (29.0\,\text{GeV})^2$.

x	y
-0.925	67.08
-0.85	58.67
-0.75	54.66
-0.65	51.72
-0.55	43.70
-0.45	41.12
-0.35	39.71
-0.25	35.34
-0.15	33.35
-0.05	34.69
0.05	34.05
0.15	34.48
0.25	34.66
0.35	35.23
0.45	35.60
0.55	40.13
0.65	42.56
0.75	46.37
0.85	49.28
0.925	55.70

Data x and y have the following relationship with the differential cross section formula.

$$x = \cos\theta, \quad y = s\frac{d\sigma}{d\cos\theta} = 2\pi s\frac{d\sigma}{d\Omega}$$

The cross section formula is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right) \times (\hbar c)^2$$

To compute predicted values $\hat{y}$, multiply by 10^{37} to convert square meters to nanobarns.

$$\hat{y} = 2\pi s\frac{d\sigma}{d\Omega} = \frac{\pi\alpha^2}{2}\left(1+x^2\right) \times (\hbar c)^2 \times 10^{37}$$

The following table shows predicted values $\hat{y}$.

x	y	$\hat{y}$
-0.925	67.08	60.44
-0.85	58.67	56.10
-0.75	54.66	50.89
-0.65	51.72	46.33
-0.55	43.70	42.42
-0.45	41.12	39.17
-0.35	39.71	36.56
-0.25	35.34	34.61
-0.15	33.35	33.30
-0.05	34.69	32.65
0.05	34.05	32.65
0.15	34.48	33.30
0.25	34.66	34.61
0.35	35.23	36.56
0.45	35.60	39.17
0.55	40.13	42.42
0.65	42.56	46.33
0.75	46.37	50.89
0.85	49.28	56.10
0.925	55.70	60.44

The coefficient of determination R^2 measures how well predicted values fit the data.

$$R^2 = 1 - \frac{\sum(y - \hat{y})^2}{\sum(y - \bar{y})^2} = 0.87$$

The result indicates that the model $d\sigma$ explains 87% of the variance in the data.

Electroweak model

The following differential cross section formula from electroweak theory results in a better fit to the data.[1]

$$\frac{d\sigma}{d\Omega} = F(s)\big(1 + \cos^2\theta\big) + G(s)\cos\theta$$

where

$$F(s) = \frac{\alpha^2}{4s}\left(1 + \frac{g_V^2}{\sqrt{2}\pi}\left(\frac{m_Z^2}{s - m_Z^2}\right)\left(\frac{sG}{\alpha}\right) + \frac{(g_A^2 + g_V^2)^2}{8\pi^2}\left(\frac{m_Z^2}{s - m_Z^2}\right)^2\left(\frac{sG}{\alpha}\right)^2\right)$$

$$G(s) = \frac{\alpha^2}{4s}\left(\frac{\sqrt{2}g_A^2}{\pi}\left(\frac{m_Z^2}{s - m_Z^2}\right)\left(\frac{sG}{\alpha}\right) + \frac{g_A^2 g_V^2}{\pi^2}\left(\frac{m_Z^2}{s - m_Z^2}\right)^2\left(\frac{sG}{\alpha}\right)^2\right)$$

[1] F. Mandl and G. Shaw, *Quantum Field Theory Revised Edition,* 316.

and

$$
\begin{aligned}
g_A &= -0.5 \\
g_V &= -0.0348 \\
m_Z &= 91.17\,\text{GeV} \\
G &= 1.166 \times 10^{-5}\,\text{GeV}^{-2}
\end{aligned}
$$

The corresponding formula for $\hat{y}$ is

$$\hat{y} = 2\pi \left[F(s)(1+x^2) + G(s)x\right] \times (\hbar c)^2 \times 10^{37}$$

where $\sqrt{s} = 29\,\text{GeV}$ is the center of mass collision energy. Here are the predicted values $\hat{y}$ based on the above formula.

x	y	$\hat{y}$
−0.925	67.08	65.59
−0.85	58.67	60.84
−0.75	54.66	55.07
−0.65	51.72	49.96
−0.55	43.70	45.49
−0.45	41.12	41.69
−0.35	39.71	38.53
−0.25	35.34	36.02
−0.15	33.35	34.17
−0.05	34.69	32.97
0.05	34.05	32.42
0.15	34.48	32.53
0.25	34.66	33.28
0.35	35.23	34.69
0.45	35.60	36.75
0.55	40.13	39.47
0.65	42.56	42.83
0.75	46.37	46.85
0.85	49.28	51.52
0.925	55.70	55.45

The coefficient of determination R^2 is

$$R^2 = 1 - \frac{\sum(y-\hat{y})^2}{\sum(y-\bar{y})^2} = 0.98$$

The result indicates that electroweak theory explains 98% of the variance in the data.

Notes

Here are a few notes about how the demo script works.

In component notation, traces are sums over a repeated index, in this case α.

$$\begin{aligned}\mathrm{Tr}\left((\not{p}_3 + M)\gamma^\mu(\not{p}_4 - M)\gamma^\nu\right) &= (\not{p}_3 + M)^\alpha{}_\beta\gamma^{\mu\beta}{}_\rho(\not{p}_4 - M)^\rho{}_\sigma\gamma^{\nu\sigma}{}_\alpha \\ \mathrm{Tr}\left((\not{p}_2 - m)\gamma_\mu(\not{p}_1 + m)\gamma_\nu\right) &= (\not{p}_2 - m)^\alpha{}_\beta\gamma_\mu{}^\beta{}_\rho(\not{p}_1 + m)^\rho{}_\sigma\gamma_\nu{}^\sigma{}_\alpha\end{aligned}$$

To convert the above formulas to Eigenmath code, the γ tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply γ^μ by the metric tensor to lower the index.

$\gamma^{\beta\mu}{}_\rho \quad\rightarrow$ `gammaT = transpose(gamma)`

$\gamma^\beta{}_{\mu\rho} \quad\rightarrow$ `gammaL = transpose(dot(gmunu,gamma))`

Define the following 4×4 matrices.

$(\not{p}_1 + m) \quad\rightarrow$ `X1 = pslash1 + m I`

$(\not{p}_2 - m) \quad\rightarrow$ `X2 = pslash2 - m I`

$(\not{p}_3 + M) \quad\rightarrow$ `X3 = pslash3 + M I`

$(\not{p}_4 - M) \quad\rightarrow$ `X4 = pslash4 - M I`

Then

$(\not{p}_3 + M)^\alpha{}_\beta\gamma^{\mu\beta}{}_\rho(\not{p}_4 - M)^\rho{}_\sigma\gamma^{\nu\sigma}{}_\alpha \quad\rightarrow$ `T1 = contract(dot(X3,gammaT,X4,gammaT),1,4)`

$(\not{p}_2 - m)^\alpha{}_\beta\gamma_\mu{}^\beta{}_\rho(\not{p}_1 + m)^\rho{}_\sigma\gamma_\nu{}^\sigma{}_\alpha \quad\rightarrow$ `T2 = contract(dot(X2,gammaL,X1,gammaL),1,4)`

Next, multiply matrices and sum over repeated indices. The dot function sums over ν then the contract function sums over μ. The transpose makes the ν indices adjacent as required by the dot function.

$\mathrm{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu)\,\mathrm{Tr}(\cdots\gamma_\mu\cdots\gamma_\nu) \quad\rightarrow$ `contract(dot(T1,transpose(T2)))`