Exercise 4.5. Take any unit 3-vector $\vec{n}$ and form the operator

$$\mathbf{H} = \frac{\hbar\omega}{2}\sigma \cdot \vec{n}$$

Find the energy eigenvalues and eigenvectors by solving the time-independent Schrodinger equation. Recall that Eq. 3.23 gives $\sigma \cdot \vec{n}$ in component form.

Here is equation (3.23).

$$\sigma_n = \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix} \tag{3.23}$$

Then by hypothesis

$$\mathbf{H} = \frac{\hbar\omega}{2}\begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}$$

Equation (4.28) is the time-independent Schrodinger equation.

$$\mathbf{H}|E_j\rangle = E_j|E_j\rangle \tag{4.28}$$

The eigenvalues E_j of $\mathbf{H}$ are solutions to $\det(\mathbf{H} - E_j\mathbf{I}) = 0$. Hence

$$\begin{aligned}
&\det(\mathbf{H} - E_j\mathbf{I}) \\
&= \begin{vmatrix} \frac{\hbar\omega}{2}n_z - E_j & \frac{\hbar\omega}{2}(n_x - in_y) \\ \frac{\hbar\omega}{2}(n_x + in_y) & -\frac{\hbar\omega}{2}n_z - E_j \end{vmatrix} \\
&= \left(\frac{\hbar\omega}{2}n_z - E_j\right)\left(-\frac{\hbar\omega}{2}n_z - E_j\right) - \frac{\hbar\omega}{2}(n_x - in_y)\frac{\hbar\omega}{2}(n_x + in_y) \\
&= E_j^2 - \left(\frac{\hbar\omega}{2}\right)^2\left(n_x^2 + n_y^2 + n_z^2\right) = 0
\end{aligned}$$

By hypothesis $\vec{n}$ is a unit vector hence

$$E_j^2 - \left(\frac{\hbar\omega}{2}\right)^2 = 0$$

Therefore the eigenvalues are

$$E_1 = \frac{\hbar\omega}{2}, \qquad E_2 = -\frac{\hbar\omega}{2}$$

To find the eigenvectors, let

$$|E_j\rangle = \begin{pmatrix} \cos\alpha \\ \sin\alpha\exp(i\phi) \end{pmatrix}$$

Then by (4.28) we have

$$\frac{\hbar\omega}{2}\begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha\exp(i\phi) \end{pmatrix} = E_j\begin{pmatrix} \cos\alpha \\ \sin\alpha\exp(i\phi) \end{pmatrix}$$

From the first row we have

$$\frac{\hbar\omega}{2}n_z\cos\alpha + \frac{\hbar\omega}{2}(n_x - in_y)\sin\alpha\exp(i\phi) = E_j\cos\alpha \tag{1}$$

From exercise 3.4 let

$$\begin{aligned} n_x &= \sin\theta\cos\phi \\ n_y &= \sin\theta\sin\phi \\ n_z &= \cos\theta \end{aligned} \tag{2}$$

Substitute (2) into (1) to obtain

$$\frac{\hbar\omega}{2}\cos\theta\cos\alpha + \frac{\hbar\omega}{2}\sin\theta(\cos\phi - i\sin\phi)\sin\alpha\exp(i\phi) = E_j\cos\alpha$$

Noting that $\cos\phi - i\sin\phi = \exp(-i\phi)$ we have

$$\frac{\hbar\omega}{2}\cos\theta\cos\alpha + \frac{\hbar\omega}{2}\sin\theta\sin\alpha = E_j\cos\alpha$$

Then by angle difference identity we have

$$\frac{\hbar\omega}{2}\cos(\theta - \alpha) = E_j\cos\alpha \tag{3}$$

Substitute $E_j = E_1 = \hbar\omega/2$ into (3) to obtain

$$\cos(\theta - \alpha) = \cos\alpha$$

It follows that

$$\alpha = \frac{\theta}{2}$$

Hence

$$|E_1\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\exp(i\phi) \end{pmatrix}$$

Substitute $E_j = E_2 = -\hbar\omega/2$ into (3) to obtain

$$\cos(\theta - \alpha) = -\cos\alpha$$

Rewrite as

$$\sin(\theta - \alpha + \pi/2) = \sin(\alpha - \pi/2)$$

It follows that

$$\alpha = \frac{\theta}{2} + \frac{\pi}{2}$$

Hence

$$|E_2\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2} + \frac{\pi}{2}\right) \\ \sin\left(\frac{\theta}{2} + \frac{\pi}{2}\right)\exp(i\phi) \end{pmatrix} = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}\exp(i\phi) \end{pmatrix}$$