

The following amplitudes are the matrix elements for position  $X$  and momentum  $P$ .

$$X_{kj} = \int_{-\infty}^{\infty} \psi_k^* x \psi_j dx$$

$$P_{kj} = \int_{-\infty}^{\infty} \psi_k^* \left( -i\hbar \frac{d}{dx} \right) \psi_j dx$$

For  $4 \times 4$  matrices we have

$$X = \left( \frac{\hbar}{2m\omega} \right)^{1/2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$P = i \left( \frac{\hbar m \omega}{2} \right)^{1/2} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$H = \frac{P^2}{2m} + \frac{1}{2} m \omega^2 X^2 = \begin{pmatrix} \frac{1}{2} \hbar \omega & 0 & 0 & 0 \\ 0 & \frac{3}{2} \hbar \omega & 0 & 0 \\ 0 & 0 & \frac{5}{2} \hbar \omega & 0 \\ 0 & 0 & 0 & \frac{7}{2} \hbar \omega \end{pmatrix}$$

Note:  $H_{33}$  cannot be computed using  $4 \times 4$  matrices. The value  $\frac{7}{2} \hbar \omega$  is the corrected eigenvalue.

Consider the following eigenfunction.

$$\Psi = \sum_k c_k \psi_k$$

Let us compute the expected value of  $x$  for a system in state  $\Psi$ .

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^* x \Psi dx$$

Expand the integrand.

$$\begin{aligned} \langle x \rangle &= \int_{-\infty}^{\infty} \left( \sum_k c_k^* \psi_k^* \right) x \left( \sum_j c_j \psi_j \right) dx \\ &= \sum_k \sum_j c_k^* c_j \int_{-\infty}^{\infty} \psi_k^* x \psi_j dx \\ &= \sum_k \sum_j c_k^* c_j X_{kj} \end{aligned}$$

Hence

$$\langle x \rangle = \begin{pmatrix} c_0^* & c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} X_{00} & X_{01} & X_{02} & X_{03} \\ X_{10} & X_{11} & X_{12} & X_{13} \\ X_{20} & X_{21} & X_{22} & X_{23} \\ X_{30} & X_{31} & X_{32} & X_{33} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$