Bell's theorem

Consider two observers A and B. Each observer has an apparatus to measure a spin. Each apparatus can be set in one of two orientations, 0 or 1. For independent expectation values we have

$$\langle A_j\rangle\langle B_k\rangle = \langle A_j B_k\rangle, \quad j,k \in \{0,1\}$$

Now consider all minimum and maximum expectation values along with a clever formula.

$\langle A_0\rangle$	$\langle A_1\rangle$	$\langle B_0\rangle$	$\langle B_1\rangle$	$\langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle$
1	1	1	1	2
1	1	1	−1	2
1	1	−1	1	−2
1	1	−1	−1	−2
1	−1	1	1	2
1	−1	1	−1	−2
1	−1	−1	1	2
1	−1	−1	−1	−2
−1	1	1	1	−2
−1	1	1	−1	2
−1	1	−1	1	−2
−1	1	−1	−1	2
−1	−1	1	1	−2
−1	−1	1	−1	−2
−1	−1	−1	1	2
−1	−1	−1	−1	2

Since the table is for all minimum and maximum values we have by inspection the range

$$-2 \le \langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle \le 2 \tag{1}$$

Now suppose a third apparatus generates two spins in the following singlet state.

$$|s\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

One spin is sent to A and the other is sent to B.

Let

$$A_0 = \sigma_z, \quad A_1 = \sigma_x, \quad B_0 = -\frac{\sigma_x + \sigma_z}{\sqrt{2}}, \quad B_1 = \frac{\sigma_x - \sigma_z}{\sqrt{2}}$$

Then for the singlet state we have

$$\langle A_0B_0\rangle = \frac{1}{\sqrt{2}}, \quad \langle A_0B_1\rangle = \frac{1}{\sqrt{2}}, \quad \langle A_1B_0\rangle = \frac{1}{\sqrt{2}}, \quad \langle A_1B_1\rangle = -\frac{1}{\sqrt{2}}$$

Hence

$$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle = 2\sqrt{2} \tag{2}$$

The result in (2) conflicts with (1) because for the singlet state

$$\langle A_j \rangle \langle B_k \rangle \neq \langle A_j B_k \rangle \tag{3}$$

Hence no theory that rejects (3) can explain quantum entanglement.

Exercises

1. Verify equation (2).