

Atomic transitions 5

From the previous section the transition rate is

$$R_{a \rightarrow b} = \frac{\pi e^2}{\varepsilon_0 \hbar^2} |\langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle|^2 \rho(\omega_0)$$

Noting that

$$\boldsymbol{\epsilon} \cdot \mathbf{r} = \epsilon_x x + \epsilon_y y + \epsilon_z z$$

we have

$$|\langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle|^2 = |\epsilon_x \langle \psi_b | x | \psi_a \rangle + \epsilon_y \langle \psi_b | y | \psi_a \rangle + \epsilon_z \langle \psi_b | z | \psi_a \rangle|^2$$

We will now average over $\boldsymbol{\epsilon}$ to eliminate it. Let

$$\begin{aligned} e_x &= \sin \theta \cos \phi \\ e_y &= \sin \theta \sin \phi \\ e_z &= \cos \theta \end{aligned}$$

Integrate over θ and ϕ to average.

$$\begin{aligned} \overline{|\langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle|^2} &= \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} |\epsilon_x \langle \psi_b | x | \psi_a \rangle + \epsilon_y \langle \psi_b | y | \psi_a \rangle + \epsilon_z \langle \psi_b | z | \psi_a \rangle|^2 \sin \theta d\theta d\phi \\ &= \frac{1}{3} \left(|\langle \psi_b | x | \psi_a \rangle|^2 + |\langle \psi_b | y | \psi_a \rangle|^2 + |\langle \psi_b | z | \psi_a \rangle|^2 \right) \\ &= \frac{1}{3} |\langle \psi_b | \mathbf{r} | \psi_a \rangle|^2 \end{aligned} \tag{1}$$

Hence

$$R_{a \rightarrow b} = \frac{\pi e^2}{3\varepsilon_0 \hbar^2} |\langle \psi_b | \mathbf{r} | \psi_a \rangle|^2 \rho(\omega_0)$$