(2.2) For the Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \lambda\hat{x}^4$, where λ is small, show by writing the Hamiltonian in terms of creation and annihilation operators and using perturbation theory, that the energy eigenvalues of all the levels are given by

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega + \frac{3\lambda}{4}\left(\frac{\hbar}{m\omega}\right)^2\left(2n^2 + 2n + 1\right) \tag{2.67}$$

For the above Hamiltonian, let $\hat{H} = \hat{H}_0 + \lambda\hat{V}$ with

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \qquad \hat{V} = \hat{x}^4$$

From perturbation theory we have

$$\begin{aligned}\left(\hat{H}_0 + \lambda\hat{V}\right)&\left(\psi_0 + \lambda\psi_1 + \lambda^2\psi_2 + \cdots\right) \\ &= \left(E_0 + \lambda E_1 + \lambda^2 E_2 + \cdots\right)\left(\psi_0 + \lambda\psi_1 + \lambda^2\psi_2 + \cdots\right)\end{aligned}$$

Because λ is small, discard powers of λ.

$$\hat{H}_0(\psi_0 + \lambda\psi_1) + \lambda\hat{V}\psi_0 = E_0(\psi_0 + \lambda\psi_1) + \lambda E_1\psi_0$$

Cancel $\hat{H}_0\psi_0$ with $E_0\psi_0$.

$$\lambda\hat{H}_0\psi_1 + \lambda\hat{V}\psi_0 = \lambda E_0\psi_1 + \lambda E_1\psi_0$$

Cancel λ.

$$\hat{H}_0\psi_1 + \hat{V}\psi_0 = E_0\psi_1 + E_1\psi_0$$

It follows that

$$\langle\psi_0|\hat{H}_0|\psi_1\rangle + \langle\psi_0|\hat{V}|\psi_0\rangle = E_0\langle\psi_0|\psi_1\rangle + E_1\langle\psi_0|\psi_0\rangle \tag{1}$$

Because $\hat{H}_0$ is Hermitian we have

$$\langle\psi_0|\hat{H}_0|\psi_1\rangle = \left(\langle\psi_1|\hat{H}_0|\psi_0\rangle\right)^\dagger = (E_0\langle\psi_1|\psi_0\rangle)^\dagger = E_0\langle\psi_0|\psi_1\rangle$$

Hence the first and third terms in equation (1) cancel leaving

$$\langle\psi_0|\hat{V}|\psi_0\rangle = E_1\langle\psi_0|\psi_0\rangle$$

It follows that

$$E_1 = \frac{\langle\psi_0|\hat{V}|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle}$$

Then for $|\psi_0\rangle = |n\rangle$ we have

$$E_1 = \frac{\langle n|\hat{V}|n\rangle}{\langle n|n\rangle} = \langle n|\hat{V}|n\rangle$$

Consider equation (2.12).

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right) \tag{2.12}$$

From $\hat{V} = \hat{x}^4$ and from equation (2.12) we have

$$E_1 = \langle n|\hat{V}|n\rangle = \langle n|\hat{x}^4|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^2 \langle n|\left(\hat{a} + \hat{a}^\dagger\right)^4|n\rangle$$

The following expectation values are from the expansion of $(\hat{a} + \hat{a}^\dagger)^4$. All other terms in the expansion vanish.

$$\begin{aligned}
\langle n|\hat{a}\hat{a}\hat{a}^\dagger\hat{a}^\dagger|n\rangle &= \sqrt{n+1}\sqrt{n+2}\sqrt{n+2}\sqrt{n+1} && = n^2 + 3n + 2 \\
\langle n|\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger|n\rangle &= \sqrt{n+1}\sqrt{n+1}\sqrt{n+1}\sqrt{n+1} && = n^2 + 2n + 1 \\
\langle n|\hat{a}\hat{a}^\dagger\hat{a}^\dagger\hat{a}|n\rangle &= \sqrt{n+1}\sqrt{n+1}\sqrt{n}\sqrt{n} && = n^2 + n \\
\langle n|\hat{a}^\dagger\hat{a}\hat{a}\hat{a}^\dagger|n\rangle &= \sqrt{n}\sqrt{n}\sqrt{n+1}\sqrt{n+1} && = n^2 + n \\
\langle n|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|n\rangle &= \sqrt{n}\sqrt{n}\sqrt{n}\sqrt{n} && = n^2 \\
\langle n|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|n\rangle &= \sqrt{n}\sqrt{n-1}\sqrt{n-1}\sqrt{n} && = n^2 - n
\end{aligned}$$

Hence

$$E_1 = \left(\frac{\hbar}{2m\omega}\right)^2 \left(6n^2 + 6n + 3\right)$$

Therefore

$$E_n = E_0 + \lambda E_1 = \hbar\omega\left(n + \frac{1}{2}\right) + \lambda\left(\frac{\hbar}{2m\omega}\right)^2 \left(6n^2 + 6n + 3\right)$$