Spherical harmonics

Verify

$$r^2\nabla^2 Y_{lm}(\theta,\phi) = -l(l+1)Y_{lm}(\theta,\phi)$$

where $Y_{lm}(\theta,\phi)$ are spherical harmonic functions

$$Y_{lm}(\theta,\phi) = (-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\exp(im\phi)$$

See arxiv.org/abs/1805.12125 for the following form of $P_l^m(\cos\theta)$.

$$P_l^m(\cos\theta) = \begin{cases} \left(\dfrac{\sin\theta}{2}\right)^m \displaystyle\sum_{k=0}^{l-m}(-1)^k\frac{(l+m+k)!}{(l-m-k)!(m+k)!k!}\left(\frac{1-\cos\theta}{2}\right)^k, & m \ge 0 \\ (-1)^m\dfrac{(l+m)!}{(l-m)!}P_l^{|m|}(\cos\theta), & m < 0 \end{cases}$$

Function $Y_{lm}(\theta,\phi)$ is independent of r hence

$$r^2\nabla^2 Y_{lm}(\theta,\phi) = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}Y_{lm}(\theta,\phi)\right] + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}Y_{lm}(\theta,\phi)$$