Exercise 6.10. A system of two spins has the Hamiltonian

$$\mathbf{H} = \frac{\omega}{2}\vec{\sigma} \cdot \vec{\tau}$$

What are the possible energies of the system, and what are the eigenvectors of the Hamiltonian?

Suppose the system starts in the state $|uu\rangle$. What is the state at any later time? Answer the same question for initial states of $|ud\rangle$, $|du\rangle$, and $|dd\rangle$.

Consider equation (4.28).

$$\mathbf{H}|E_j\rangle = E_j|E_j\rangle \tag{4.28}$$

By (4.28) the possible energies are the eigenvalues of $\mathbf{H}$. From Exercise 6.9 the eigenvalues of $\vec{\sigma} \cdot \vec{\tau}$ are -3 and 1. Hence the possible energies are $-3\omega/2$ and $\omega/2$. Also from Exercise 6.9 the eigenvectors are $|sing\rangle$, $|T_1\rangle$, $|T_2\rangle$, and $|T_3\rangle$.

Without using Exercise 6.9 we have

$$\mathbf{H} = \frac{\omega}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The characteristic polynomial of $\mathbf{H}$ is

$$|\mathbf{H} - \lambda\mathbf{I}| = \lambda^4 - \tfrac{3}{2}\lambda^2\omega^2 + \lambda\omega^3 - \tfrac{3}{16}\omega^4 = 0$$

The roots of the characteristic polynomial are $\lambda = -3\omega/2$ and $\lambda = \omega/2$. These λ are the eigenvalues of $\mathbf{H}$ and hence the possible energies.

Consider equation (4.30).

$$\alpha_j(t) = \alpha_j(0)\exp\left(-\frac{iE_jt}{\hbar}\right) \tag{4.30}$$

Note that $|uu\rangle$ and $|dd\rangle$ are eigenvectors of $\mathbf{H}$.

$$\mathbf{H}|uu\rangle = \frac{\omega}{2}|uu\rangle$$

$$\mathbf{H}|dd\rangle = \frac{\omega}{2}|dd\rangle$$

By (4.30) with $E_{uu} = E_{dd} = \omega/2$ we have

$$|\Psi_{uu}(t)\rangle = \exp\left(-\frac{i\omega t}{2\hbar}\right)|uu\rangle$$
$$|\Psi_{dd}(t)\rangle = \exp\left(-\frac{i\omega t}{2\hbar}\right)|dd\rangle$$

For the initial state $|ud\rangle$ we have

$$|ud\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}|T_1\rangle + \frac{1}{\sqrt{2}}|sing\rangle$$

By equation (4.30) with $E_{T_1} = \omega/2$ and $E_{sing} = -3\omega/2$ we have

$$|\Psi_{ud}(t)\rangle = \frac{1}{\sqrt{2}}\exp\left(-\frac{i\omega t}{2\hbar}\right)|T_1\rangle + \frac{1}{\sqrt{2}}\exp\left(\frac{3i\omega t}{2\hbar}\right)|sing\rangle$$
$$= \frac{1}{2}\begin{pmatrix} 0 \\ \exp(-i\omega t/2\hbar) + \exp(3i\omega t/2\hbar) \\ \exp(-i\omega t/2\hbar) - \exp(3i\omega t/2\hbar) \\ 0 \end{pmatrix}$$

For the initial state $|du\rangle$ we have

$$|du\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}|T_1\rangle - \frac{1}{\sqrt{2}}|sing\rangle$$

By equation (4.30) with $E_{T_1} = \omega/2$ and $E_{sing} = -3\omega/2$ we have

$$|\Psi_{du}(t)\rangle = \frac{1}{\sqrt{2}}\exp\left(-\frac{i\omega t}{2\hbar}\right)|T_1\rangle - \frac{1}{\sqrt{2}}\exp\left(\frac{3i\omega t}{2\hbar}\right)|sing\rangle$$
$$= \frac{1}{2}\begin{pmatrix} 0 \\ \exp(-i\omega t/2\hbar) - \exp(3i\omega t/2\hbar) \\ \exp(-i\omega t/2\hbar) + \exp(3i\omega t/2\hbar) \\ 0 \end{pmatrix}$$