Atomic transitions 2

Start with the perturbing Hamiltonian where E_0 is the amplitude of the electric field.

$$H_1(\mathbf{r},t) = -\frac{eE_0\boldsymbol{\epsilon}\cdot\mathbf{p}}{m\omega}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)$$

In exponential form

$$H_1(\mathbf{r},t) = -\frac{eE_0\boldsymbol{\epsilon}\cdot\mathbf{p}}{m\omega}\left(\tfrac{1}{2}\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)+\tfrac{1}{2}\exp(-i\mathbf{k}\cdot\mathbf{r}+i\omega t)\right)$$

Given the initial condition $c_b(0)=0$ the first-order approximation for $c_b(t)$ is

$$c_b(t) = -\frac{i}{\hbar}\int_0^t \langle\psi_b|H_1(\mathbf{r},t')|\psi_a\rangle\exp(i\omega_0 t')\,dt',\quad \omega_0=\frac{E_b-E_a}{\hbar}$$

Factor the integrand.

$$\begin{aligned}c_b(t) = \frac{ieE_0}{2m\hbar\omega}\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\int_0^t\exp(-i\omega t')\exp(i\omega_0 t')\,dt'\\ +\frac{ieE_0}{2m\hbar\omega}\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(-i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\int_0^t\exp(i\omega t')\exp(i\omega_0 t')\,dt'\end{aligned}$$

Solve the integrals to obtain

$$\begin{aligned}c_b(t) = \frac{eE_0}{2m\hbar\omega}\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\frac{\exp\big(i(\omega_0-\omega)t\big)-1}{\omega_0-\omega}\\ +\frac{eE_0}{2m\hbar\omega}\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(-i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\frac{\exp\big(i(\omega_0+\omega)t\big)-1}{\omega_0+\omega}\end{aligned}\tag{1}$$

As an approximation, discard the second term since the first term dominates for $\omega\approx\omega_0$.

$$c_b(t) = \frac{eE_0}{2m\hbar\omega}\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\frac{\exp\big(i(\omega_0-\omega)t\big)-1}{\omega_0-\omega}$$

Rewrite $c_b(t)$ in the form of a sine function.

$$c_b(t) = \frac{ieE_0}{m\hbar\omega}\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\frac{\sin\big(\frac{1}{2}(\omega_0-\omega)t\big)}{\omega_0-\omega}\exp\big(\tfrac{i}{2}(\omega_0-\omega)t\big)\tag{2}$$

Verify dimensions.

$$H_1(\mathbf{r},t)\propto\frac{\overset{e}{\mathrm{C}}\;\;\overset{E_0}{\mathrm{N\,C^{-1}}}\;\;\overset{\boldsymbol{\epsilon}\cdot\mathbf{p}}{\mathrm{kg\,m\,s^{-1}}}}{\underset{\mathrm{kg}}{m}\;\;\underset{\mathrm{s^{-1}}}{\omega}}=\mathrm{N\,m}=\mathrm{J}$$

$$c_b(t)\propto\frac{\overset{e}{\mathrm{C}}\;\;\overset{E_0}{\mathrm{N\,C^{-1}}}}{\underset{\mathrm{kg}}{m}\;\;\underset{\mathrm{J\,s}}{\hbar}\;\;\underset{\mathrm{s^{-1}}}{\omega}}\times\frac{\overset{\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle}{\mathrm{kg\,m\,s^{-1}}}}{\underset{\mathrm{s^{-1}}}{\omega_0-\omega}}=\frac{\mathrm{N\,m}}{\mathrm{J}}=1$$

Wave functions ψ_a and ψ_b have dimension meter$^{-1/2}$ hence they cancel with $dx\propto$ meter in the integral leaving units of momentum due to $\mathbf{p}$.