Harmonic oscillator propagator

Let H be the harmonic oscillator Hamiltonian.

$$H = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}$$

Let $K(b,a)$ be the propagator for the harmonic oscillator.

$$K(b,a) = \langle x_b| \exp\left(-\frac{iHt}{\hbar}\right)|x_a\rangle \tag{1}$$

This is van Kortryk's identity.

$$\exp\left(-\frac{iHt}{\hbar}\right) = \exp\left(-\frac{im\omega}{2\hbar}\hat{x}^2\tan\frac{\omega t}{2}\right)\exp\left(-\frac{i}{2m\omega\hbar}\hat{p}^2\sin(\omega t)\right)\exp\left(-\frac{im\omega}{2\hbar}\hat{x}^2\tan\frac{\omega t}{2}\right) \tag{2}$$

Substitute (2) into (1) to obtain

$$K(b,a) = \langle x_b|\exp\left(-\frac{im\omega}{2\hbar}\hat{x}^2\tan\frac{\omega t}{2}\right)\exp\left(-\frac{i}{2m\omega\hbar}\hat{p}^2\sin(\omega t)\right)\exp\left(-\frac{im\omega}{2\hbar}\hat{x}^2\tan\frac{\omega t}{2}\right)|x_a\rangle$$

Replace operator $\hat{x}$ with its eigenvalues.

$$K(b,a) = \exp\left(-\frac{im\omega}{2\hbar}x_b^2\tan\frac{\omega t}{2}\right)\langle x_b|\exp\left(-\frac{i}{2m\omega\hbar}\hat{p}^2\sin(\omega t)\right)|x_a\rangle\exp\left(-\frac{im\omega}{2\hbar}x_a^2\tan\frac{\omega t}{2}\right) \tag{3}$$

Let

$$K_0(b,a) = \langle x_b|\exp\left(-\frac{i}{2m\omega\hbar}\hat{p}^2\sin(\omega t)\right)|x_a\rangle$$

Substitute T for $\sin(\omega t)/\omega$.

$$K_0(b,a) = \langle x_b|\exp\left(-\frac{iT}{2m\hbar}\hat{p}^2\right)|x_a\rangle$$

K_0 is now a free particle propagator hence

$$K_0(b,a) = \left(\frac{m}{2\pi i\hbar T}\right)^{1/2}\exp\left(\frac{im}{2\hbar T}(x_b-x_a)^2\right)$$

Substitute $\sin(\omega t)/\omega$ for T.

$$K_0(b,a) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega t)}\right)^{1/2}\exp\left(\frac{im\omega}{2\hbar\sin(\omega t)}(x_b-x_a)^2\right) \tag{4}$$

Substitute (4) into (3) and combine exponentials.

$$K(b,a) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega t)}\right)^{1/2}\exp\left(-\frac{im\omega}{2\hbar}\left(x_b^2+x_a^2\right)\tan\frac{\omega t}{2} + \frac{im\omega}{2\hbar\sin(\omega t)}(x_b-x_a)^2\right)$$

By the identity

$$\cot\theta = -\tan\frac{\theta}{2} + \frac{1}{\sin\theta}$$

we have

$$K(b,a) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega t)}\right)^{1/2} \exp\left(\frac{im\omega}{2\hbar}\left(x_b^2 + x_a^2\right)\cot(\omega t) - \frac{im\omega}{\hbar\sin(\omega t)}x_b x_a\right)$$