Green's function

In this section we will find the Green's function $G(\mathbf{x})$ such that

$$\left(\nabla^2 + k^2\right) G(\mathbf{x}) = \delta^3(\mathbf{x}) \tag{1}$$

Let $g(\mathbf{y})$ be the Fourier transform of $G(\mathbf{x})$ such that

$$G(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \exp(i\mathbf{x}\cdot\mathbf{y}) g(\mathbf{y})\, d\mathbf{y} \tag{2}$$

Substitute (2) into (1) to obtain

$$\left(\nabla^2 + k^2\right) \left[\frac{1}{(2\pi)^{\frac{3}{2}}} \int \exp(i\mathbf{x}\cdot\mathbf{y}) g(\mathbf{y})\, d\mathbf{y}\right] = \delta(\mathbf{x})$$

By linearity of differentiation the $(\nabla^2 + k^2)$ term can be moved inside the integral.

$$\frac{1}{(2\pi)^{\frac{3}{2}}} \int \left(\nabla^2 + k^2\right) \exp(i\mathbf{x}\cdot\mathbf{y}) g(\mathbf{y})\, d\mathbf{y} = \delta(\mathbf{x}) \tag{3}$$

Noting that

$$\nabla^2 \exp(i\mathbf{x}\cdot\mathbf{y}) = -y^2 \exp(i\mathbf{x}\cdot\mathbf{y})$$

and

$$\delta(\mathbf{x}) = \frac{1}{(2\pi)^3} \int \exp(i\mathbf{x}\cdot\mathbf{y})\, d\mathbf{y}$$

we have by substitution into (3)

$$\frac{1}{(2\pi)^{\frac{3}{2}}} \int \left(-y^2 + k^2\right) \exp(i\mathbf{x}\cdot\mathbf{y}) g(\mathbf{y})\, d\mathbf{y} = \frac{1}{(2\pi)^3} \int \exp(i\mathbf{x}\cdot\mathbf{y})\, d\mathbf{y}$$

Solve for $g(\mathbf{y})$.

$$g(\mathbf{y}) = \frac{1}{(2\pi)^{\frac{3}{2}} (k^2 - y^2)}$$

Substitute for $g(\mathbf{y})$ in (2) to obtain

$$G(\mathbf{x}) = \frac{1}{(2\pi)^3} \int \frac{\exp(i\mathbf{x}\cdot\mathbf{y})}{k^2 - y^2}\, d\mathbf{y}$$

Change to polar coordinates where $x = |\mathbf{x}|$, $y = |\mathbf{y}|$, and θ and ϕ are the angular distance from $\mathbf{x}$ to $\mathbf{y}$.

$$G(\mathbf{x}) = \frac{1}{(2\pi)^3} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{\exp(ixy\cos\theta)}{k^2 - y^2}\, y^2 \sin\theta\, dy\, d\theta\, d\phi$$

For the integrals over θ and ϕ we have

$$\int_0^\pi \int_0^{2\pi} \exp(ixy\cos\theta) \sin\theta\, d\theta\, d\phi = \frac{4\pi \sin(xy)}{xy}$$

Hence

$$G(\mathbf{x}) = \frac{1}{2\pi^2 x} \int_0^\infty \frac{y \sin(xy)}{k^2 - y^2}\, dy$$

Noting that $y \sin(xy)$ is an even function of y we can change the integral limits as follows.

$$G(\mathbf{x}) = \frac{1}{4\pi^2 x} \int_{-\infty}^\infty \frac{y \sin(xy)}{k^2 - y^2}\, dy$$

Negate the denominator.

$$G(\mathbf{x}) = \frac{1}{4\pi^2 x} \int_{-\infty}^\infty -\frac{y \sin(xy)}{y^2 - k^2}\, dy$$

Change the sine function to exponential form and factor the denominator.

$$G(\mathbf{x}) = \frac{i}{8\pi^2 x} \left[\int_{-\infty}^\infty \frac{y \exp(ixy)}{(y-k)(y+k)}\, dy - \int_{-\infty}^\infty \frac{y \exp(-ixy)}{(y-k)(y+k)}\, dy \right]$$

By Cauchy's integral formula we have

$$\int_{-\infty}^\infty \frac{y \exp(ixy)}{y+k} \frac{1}{y-k}\, dy = i\pi \exp(ikx)$$

and

$$\int_{-\infty}^\infty \frac{y \exp(-ixy)}{y-k} \frac{1}{y+k}\, dy = -i\pi \exp(ikx)$$

Hence

$$\boxed{G(\mathbf{x}) = -\frac{\exp(ikx)}{4\pi x}} \tag{4}$$

where

$$x = |\mathbf{x}|$$

Verify that (4) satisfies (1).

We will need the following formula from Griffiths and Schroeter problem 10.8.

$$\nabla^2(1/r) = -4\pi\delta^3(\mathbf{r}) \tag{5}$$

Recall that $\nabla^2 = \nabla \cdot \nabla$ and for scalar functions f and g we have

$$\nabla \cdot (f \nabla g) = \nabla f \cdot \nabla g + f \nabla^2 g$$

Substituting $\mathbf{r}$ for $\mathbf{x}$ in (4) we have for the Laplacian of $G(\mathbf{r})$

$$\begin{aligned}
\nabla^2 G(\mathbf{r}) &= \nabla \cdot \nabla \left(-\frac{e^{ikr}}{4\pi r} \right) \\
&= -\frac{1}{4\pi} \nabla \cdot \left(\frac{1}{r} \nabla e^{ikr} + e^{ikr} \nabla \frac{1}{r} \right) \\
&= -\frac{1}{4\pi} \left(\nabla \frac{1}{r} \cdot \nabla e^{ikr} + \frac{1}{r} \nabla^2 e^{ikr} + \nabla e^{ikr} \cdot \nabla \frac{1}{r} + e^{ikr} \nabla^2 \frac{1}{r} \right)
\end{aligned} \tag{6}$$

In spherical coordinates the gradient of f is

$$\nabla f = \left(\frac{\partial f}{\partial r}\right)\mathbf{e}_r + \left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right)\mathbf{e}_\theta + \left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\right)\mathbf{e}_\phi$$

hence

$$\nabla\frac{1}{r}\cdot\nabla e^{ikr} = \left(-\frac{1}{r^2}\mathbf{e}_r + 0\mathbf{e}_\theta + 0\mathbf{e}_\phi\right)\cdot\left(ike^{ikr}\mathbf{e}_r + 0\mathbf{e}_\theta + 0\mathbf{e}_\phi\right) = -\frac{ike^{ikr}}{r^2} \tag{7}$$

In spherical coordinates the Laplacian of f is

$$\nabla^2 f = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

hence

$$\begin{aligned}
\frac{1}{r}\nabla^2 e^{ikr} &= \frac{1}{r^2}\frac{\partial^2}{\partial r^2}\left(re^{ikr}\right)\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(e^{ikr} + ikre^{ikr}\right)\\
&= \frac{1}{r^2}\left(2ike^{ikr} - k^2re^{ikr}\right)\\
&= \frac{2ike^{ikr}}{r^2} - \frac{k^2e^{ikr}}{r}
\end{aligned} \tag{8}$$

Substitute (5), (7), and (8) into (6) to obtain

$$\nabla^2 G(\mathbf{r}) = \frac{k^2e^{ikr}}{4\pi r} + \delta^3(\mathbf{r})e^{ikr}$$

Then by equation (4)

$$\nabla^2 G(\mathbf{r}) = -k^2 G(\mathbf{r}) + \delta^3(\mathbf{r})e^{ikr}$$

Noting that $e^{ikr} = 1$ at $r = 0$, the e^{ikr} term can be discarded leaving

$$\nabla^2 G(\mathbf{r}) = -k^2 G(\mathbf{r}) + \delta^3(\mathbf{r}) \tag{9}$$

Hence

$$\left(\nabla^2 + k^2\right)G(\mathbf{r}) = \delta^3(\mathbf{r})$$

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