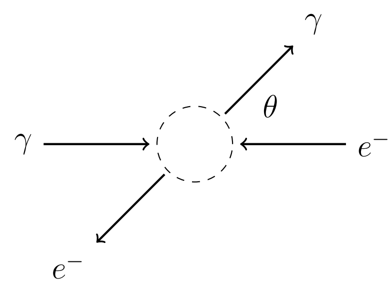
Compton scattering

Compton scattering is the interaction $e^- + \gamma \to e^- + \gamma$.



Define the following momentum vectors and spinors. Symbol ω is incident energy. Symbol E is total energy $E = \sqrt{\omega^2 + m^2}$ where m is electron mass. Polar angle θ is the observed scattering angle. Azimuth angle ϕ cancels out in scattering calculations.

$$p_1 = \underset{\text{inbound photon}}{\begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix}} \qquad p_2 = \underset{\text{inbound electron}}{\begin{pmatrix} E \\ 0 \\ 0 \\ -\omega \end{pmatrix}} \qquad p_3 = \underset{\text{outbound photon}}{\begin{pmatrix} \omega \\ \omega \sin\theta \cos\phi \\ \omega \sin\theta \sin\phi \\ \omega \cos\theta \end{pmatrix}} \qquad p_4 = \underset{\text{outbound electron}}{\begin{pmatrix} E \\ -\omega \sin\theta \cos\phi \\ -\omega \sin\theta \sin\phi \\ -\omega \cos\theta \end{pmatrix}}$$

$$u_{21} = \underset{\text{inbound electron spin up}}{\begin{pmatrix} E+m \\ 0 \\ -\omega \\ 0 \end{pmatrix}} \qquad u_{41} = \underset{\text{outbound electron spin up}}{\begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix}}$$

$$u_{22} = \underset{\text{inbound electron spin down}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ \omega \end{pmatrix}} \qquad u_{42} = \underset{\text{outbound electron spin down}}{\begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}}$$

The spinors are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^2$ will be used.

This is the probability density for spin state ab. The formula is derived from Feynman diagrams for Compton scattering.

$$|\mathcal{M}_{ab}|^2 = \frac{e^4}{N} \left| -\frac{\bar{u}_{4b}\gamma^\mu(\not{q}_1 + m)\gamma^\nu u_{2a}}{s - m^2} - \frac{\bar{u}_{4b}\gamma^\nu(\not{q}_2 + m)\gamma^\mu u_{2a}}{u - m^2} \right|^2$$

Symbol e is electron charge and

$$\not{q}_1 = (p_1 + p_2)^\mu g_{\mu\nu} \gamma^\nu$$
$$\not{q}_2 = (p_4 - p_1)^\mu g_{\mu\nu} \gamma^\nu$$

Symbols s and u are Mandelstam variables

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = (E + \omega)^2 \\
u &= (p_1 - p_4)^2 = (p_1 - p_4)^\mu g_{\mu\nu} (p_1 - p_4)^\nu
\end{aligned}
$$

Let

$$
a_1 = \bar{u}_{4b} \gamma^\mu (\not{q}_1 + m) \gamma^\nu u_{2a}, \quad a_2 = \bar{u}_{4b} \gamma^\nu (\not{q}_2 + m) \gamma^\mu u_{2a}
$$

Then

$$
\begin{aligned}
|\mathcal{M}_{ab}|^2 &= \frac{e^4}{N} \left| -\frac{a_1}{s - m^2} - \frac{a_2}{u - m^2} \right|^2 \\
&= \frac{e^4}{N} \left(-\frac{a_1}{s - m^2} - \frac{a_2}{u - m^2} \right) \left(-\frac{a_1}{s - m^2} - \frac{a_2}{u - m^2} \right)^* \\
&= \frac{e^4}{N} \left(\frac{a_1 a_1^*}{(s - m^2)^2} + \frac{a_1 a_2^*}{(s - m^2)(u - m^2)} + \frac{a_1^* a_2}{(s - m^2)(u - m^2)} + \frac{a_2 a_2^*}{(u - m^2)^2} \right)
\end{aligned}
$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is computed by summing $|\mathcal{M}_{ab}|^2$ over all spin and polarization states and then dividing by the number of inbound states. There are four inbound states. The sum over polarizations is already accomplished by contraction of aa^* over μ and ν.

$$
\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{1}{4} \sum_{a=1}^{2} \sum_{b=1}^{2} |\mathcal{M}_{ab}|^2 \\
&= \frac{e^4}{4N} \sum_{a=1}^{2} \sum_{b=1}^{2} \left(\frac{a_1 a_1^*}{(s - m^2)^2} + \frac{a_1 a_2^*}{(s - m^2)(u - m^2)} + \frac{a_1^* a_2}{(s - m^2)(u - m^2)} + \frac{a_2 a_2^*}{(u - m^2)^2} \right)
\end{aligned}
$$

The Casimir trick uses matrix arithmetic to compute sums.

$$
\begin{aligned}
f_{11} &= \frac{1}{N} \sum_{a=1}^{2} \sum_{b=1}^{2} a_1 a_1^* = \mathrm{Tr} \left((\not{p}_2 + m) \gamma^\mu (\not{q}_1 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\nu (\not{q}_1 + m) \gamma_\mu \right) \\
f_{12} &= \frac{1}{N} \sum_{a=1}^{2} \sum_{b=1}^{2} a_1 a_2^* = \mathrm{Tr} \left((\not{p}_2 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\mu (\not{q}_1 + m) \gamma_\nu \right) \\
f_{22} &= \frac{1}{N} \sum_{a=1}^{2} \sum_{b=1}^{2} a_2 a_2^* = \mathrm{Tr} \left((\not{p}_2 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\nu (\not{q}_2 + m) \gamma_\mu \right)
\end{aligned}
$$

Hence

$$
\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left(\frac{f_{11}}{(s - m^2)^2} + \frac{f_{12}}{(s - m^2)(u - m^2)} + \frac{f_{12}^*}{(s - m^2)(u - m^2)} + \frac{f_{22}}{(u - m^2)^2} \right) \tag{1}
$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$
\begin{aligned}
f_{11} &= 32(p_1 \cdot p_2)(p_1 \cdot p_4) + 64m^2(p_1 \cdot p_2) - 32m^2(p_1 \cdot p_3) - 32m^2(p_1 \cdot p_4) + 32m^4 \\
f_{12} &= 16m^2(p_1 \cdot p_2) - 16m^2(p_1 \cdot p_4) + 32m^4 \\
f_{22} &= 32(p_1 \cdot p_2)(p_1 \cdot p_4) + 32m^2(p_1 \cdot p_2) - 32m^2(p_1 \cdot p_3) - 64m^2(p_1 \cdot p_4) + 32m^4
\end{aligned}
$$

For Mandelstam variables

$$\begin{aligned} s &= (p_1 + p_2)^2 \\ t &= (p_1 - p_3)^2 \\ u &= (p_1 - p_4)^2 \end{aligned}$$

the formulas are

$$\begin{aligned} f_{11} &= -8su + 24sm^2 + 8um^2 + 8m^4 \\ f_{12} &= 8sm^2 + 8um^2 + 16m^4 \\ f_{22} &= -8su + 8sm^2 + 24um^2 + 8m^4 \end{aligned} \tag{2}$$

Compton scattering experiments are typically done in the lab frame where the electron is at rest. Lorentz boost Λ transforms momentum vectors to the lab frame.

$$\Lambda = \begin{pmatrix} E/m & 0 & 0 & \omega/m \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega/m & 0 & 0 & E/m \end{pmatrix}, \quad \Lambda p_2 = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Mandelstam variables are invariant under a boost.

$$\begin{aligned} s &= (p_1 + p_2)^2 = (\Lambda p_1 + \Lambda p_2)^2 \\ t &= (p_1 - p_3)^2 = (\Lambda p_1 - \Lambda p_3)^2 \\ u &= (p_1 - p_4)^2 = (\Lambda p_1 - \Lambda p_4)^2 \end{aligned}$$

In the lab frame, let ω_L be the angular frequency of the incident photon and let ω'_L be the angular frequency of the scattered photon.

$$\begin{aligned} \omega_L &= \Lambda p_1 \cdot (1,0,0,0) = \frac{\omega^2}{m} + \frac{\omega E}{m} \\ \omega'_L &= \Lambda p_3 \cdot (1,0,0,0) = \frac{\omega^2 \cos\theta}{m} + \frac{\omega E}{m} \end{aligned}$$

It follows that

$$\begin{aligned} s &= (p_1 + p_2)^2 = m^2 + 2m\omega_L \\ t &= (p_1 - p_3)^2 = 2m(\omega'_L - \omega_L) \\ u &= (p_1 - p_4)^2 = m^2 - 2m\omega'_L \end{aligned}$$

Then from equations (1) and (2)

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \left(\frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1 \right)^2 - 1 \right)$$

Lab scattering angle θ_L is given by the Compton formula.

$$\cos\theta_L = \frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1$$

Hence

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \cos^2\theta_L - 1 \right) \\ &= 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} - \sin^2\theta_L \right) \end{aligned}$$

Cross section

Now that we have derived $\langle|\mathcal{M}|^2\rangle$ we can investigate the angular distribution of scattered photons. For simplicity let us drop the L subscript from lab variables. From now on the symbols ω, ω', and θ will be lab frame variables.

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4(4\pi\varepsilon_0)^2 s}\left(\frac{\omega'}{\omega}\right)^2 \langle|\mathcal{M}|^2\rangle$$

where

$$s = m^2 + 2m\omega = (mc^2)^2 + 2(mc^2)(\hbar\omega)$$

For the lab frame we have

$$\langle|\mathcal{M}|^2\rangle = 2e^4\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right)$$

Substitute for $\langle|\mathcal{M}|^2\rangle$.

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\varepsilon_0)^2 s}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right)$$

Noting that

$$e^2 = 4\pi\varepsilon_0\alpha\hbar c$$

we can also write

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right)$$

The scattered photon frequency ω' is computed from the Compton equation.

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1-\cos\theta)}$$

We can integrate $d\sigma$ to obtain a cumulative distribution function. Let $I(\theta)$ be the following integral of $d\sigma$. (The $\sin\theta$ is due to $d\Omega = \sin\theta\, d\theta\, d\phi$.)

$$I(\theta) = \int\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right)\sin\theta\, d\theta$$

The result is

$$\begin{aligned} I(\theta) = -\frac{\cos\theta}{R^2} &+ \log\big(1 + R(1-\cos\theta)\big)\left(\frac{1}{R} - \frac{2}{R^2} - \frac{2}{R^3}\right) \\ &- \frac{1}{2R\big(1+R(1-\cos\theta)\big)^2} + \frac{1}{1+R(1-\cos\theta)}\left(-\frac{2}{R^2} - \frac{1}{R^3}\right) \end{aligned}$$

where

$$R = \frac{\hbar\omega}{mc^2}$$

The cumulative distribution function is

$$F(\theta) = \frac{I(\theta) - I(0)}{I(\pi) - I(0)}, \quad 0 \le \theta \le \pi$$

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

Let N be the number of scattering events from an experiment. Then the number of scattering events in the interval θ_1 to θ_2 is predicted to be

$$N\left(F(\theta_2) - F(\theta_1)\right)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi) - I(0)} \left(\frac{\omega'}{\omega}\right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right) \sin\theta$$

Note that if we had carried through the $\alpha^2(\hbar c)^2/2s$ in $I(\theta)$, it would have cancelled out in $F(\theta)$.

Thomson scattering

For $\hbar\omega \ll mc^2$ we have

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos\theta)} \approx \omega$$

Hence we can use the approximations

$$\omega = \omega' \quad \text{and} \quad s = (mc^2)^2$$

to obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2\hbar^2}{2m^2c^2}\left(1 + \cos^2\theta\right)$$

which is the formula for Thomson scattering.

High energy approximation

For $\omega \gg m$ a useful approximation is to set $m = 0$ and obtain

$$\begin{aligned} f_{11} &= -8su \\ f_{12} &= 0 \\ f_{22} &= -8su \end{aligned}$$

Hence

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{e^4}{4}\left(\frac{-8su}{s^2}+\frac{-8su}{u^2}\right)\\ &= 2e^4\left(-\frac{u}{s}-\frac{s}{u}\right)\end{aligned}$$

Also for $m=0$ the Mandelstam variables s and u are

$$\begin{aligned}s &= 4\omega^2\\ u &= -2\omega^2(\cos\theta+1)\end{aligned}$$

Hence

$$\langle|\mathcal{M}|^2\rangle = 2e^4\left(\frac{\cos\theta+1}{2}+\frac{2}{\cos\theta+1}\right)$$

Notes

Here are a few notes regarding the Eigenmath scripts.

Start by writing out a_1 and a_2 in full component form.

$$a_1^{\mu\nu} = \bar{u}_{4\alpha}\gamma^{\mu\alpha}{}_\beta(\not{q}_1+m)^\beta{}_\rho\gamma^{\nu\rho}{}_\sigma u_2^\sigma, \quad a_2^{\nu\mu} = \bar{u}_{4\alpha}\gamma^{\nu\alpha}{}_\beta(\not{q}_2+m)^\beta{}_\rho\gamma^{\mu\rho}{}_\sigma u_2^\sigma$$

Transpose γ tensors to form inner products over α and ρ.

$$a_1^{\mu\nu} = \bar{u}_{4\alpha}\gamma^{\alpha\mu}{}_\beta(\not{q}_1+m)^\beta{}_\rho\gamma^{\rho\nu}{}_\sigma u_2^\sigma, \quad a_2^{\nu\mu} = \bar{u}_{4\alpha}\gamma^{\alpha\nu}{}_\beta(\not{q}_2+m)^\beta{}_\rho\gamma^{\rho\mu}{}_\sigma u_2^\sigma$$

Convert transposed γ to Eigenmath code.

$\gamma^{\alpha\mu}{}_\beta \quad \rightarrow$ `gammaT = transpose(gamma)`

Then to compute a_1 we have

$a_1 = \bar{u}_{4\alpha}\gamma^{\alpha\mu}{}_\beta(\not{q}_1+m)^\beta{}_\rho\gamma^{\rho\nu}{}_\sigma u_2^\sigma$

$\rightarrow$ `a1 = dot(u4bar[s4],gammaT,qslash1 + m I,gammaT,u2[s2])`

where s_2 and s_4 are spin indices. Similarly for a_2 we have

$a_2 = \bar{u}_{4\alpha}\gamma^{\alpha\nu}{}_\beta(\not{q}_2+m)^\beta{}_\rho\gamma^{\rho\mu}{}_\sigma u_2^\sigma$

$\rightarrow$ `a2 = dot(u4bar[s4],gammaT,qslash2 + m I,gammaT,u2[s2])`

In component notation the product $a_1a_1^*$ is

$$a_1a_1^* = a_1^{\mu\nu}a_1^{*\mu\nu}$$

To sum over μ and ν it is necessary to lower indices with the metric tensor. Also, transpose a_1^* to form an inner product with ν.

$$a_1a_1^* = a_1^{\mu\nu}a^*_{1\nu\mu}$$

Convert to Eigenmath code. The dot function sums over ν and the contract function sums over μ.

$$a_1 a_1^* \quad \rightarrow \quad \texttt{a11 = contract(dot(a1,gmunu,transpose(conj(a1)),gmunu))}$$

Similarly for $a_2 a_2^*$ we have

$$a_2 a_2^* \quad \rightarrow \quad \texttt{a22 = contract(dot(a2,gmunu,transpose(conj(a2)),gmunu))}$$

The product $a_1 a_2^*$ does not require a transpose because $a_1 a_2^* = a_1^{\mu\nu} a_2^{*\nu\mu}$.

$$a_1 a_2^* \quad \rightarrow \quad \texttt{a12 = contract(dot(a1,gmunu,conj(a2),gmunu))}$$

In component notation, a trace operator becomes a sum over an index, in this case α.

$$\begin{aligned} f_{11} &= \operatorname{Tr}\Big((\not{p}_2 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\Big) \\ &= (\not{p}_2 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{q}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\nu{}^\delta{}_\eta (\not{q}_1 + m)^\eta{}_\xi \gamma_\mu{}^\xi{}_\alpha \end{aligned}$$

As before, transpose γ tensors to form inner products.

$$f_{11} = (\not{p}_2 + m)^\alpha{}_\beta \gamma^{\beta\mu}{}_\rho (\not{q}_1 + m)^\rho{}_\sigma \gamma^{\sigma\nu}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma^\delta{}_{\nu\eta} (\not{q}_1 + m)^\eta{}_\xi \gamma^\xi{}_{\mu\alpha}$$

To convert to Eigenmath code, use an intermediate variable for the inner product.

$$T^{\alpha\mu\nu}{}_{\nu\mu\alpha} \quad \rightarrow \quad \texttt{T = dot(P2,gammaT,Q1,gammaT,P4,gammaL,Q1,gammaL)}$$

Now sum over the indices of T. The innermost contract sums over ν then the next contract sums over μ. Finally the outermost contract sums over α.

$$f_{11} \quad \rightarrow \quad \texttt{f11 = contract(contract(contract(T,3,4),2,3))}$$

Follow suit for f_{22}. For f_{12} the order of the rightmost μ and ν is reversed.

$$f_{12} = \operatorname{Tr}\Big((\not{p}_2 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\Big)$$

The resulting inner product is $T^{\alpha\mu\nu}{}_{\mu\nu\alpha}$ so the contraction is different.

$$f_{12} \quad \rightarrow \quad \texttt{f12 = contract(contract(contract(T,3,5),2,3))}$$

The innermost contract sums over ν followed by sum over μ then sum over α.