8-4. Show that the ground-state wave function for the Lagrangian of equation (8.78) can be written

$$\Phi_0 = A\exp\left(-\frac{1}{2\hbar}\sum_{\alpha=1}^{N-1}\omega_\alpha Q_\alpha^* Q_\alpha\right) \tag{8.83}$$

(where A is a constant) by starting with the wave function in terms of the real variables Q_α^c and Q_α^s.

$$L = \frac{1}{2}\sum_{\alpha=0}^{N-1}\left(\dot{Q}_\alpha^*\dot{Q}_\alpha - \omega_\alpha^2 Q_\alpha^* Q_\alpha\right) \tag{8.78}$$

Consider the following equation from p. 216.

$$Q_\alpha = \frac{1}{\sqrt{2}}(Q_\alpha^c - iQ_\alpha^s)$$

It follows that

$$Q_\alpha^* Q_\alpha = \frac{1}{2}(Q_\alpha^c)^2 + \frac{1}{2}(Q_\alpha^s)^2 \tag{1}$$

Substitute (1) into (8.78).

$$L = \frac{1}{4}\sum_{\alpha=0}^{N-1}\left((\dot{Q}_\alpha^c)^2 + (\dot{Q}_\alpha^s)^2 - \omega_\alpha^2(Q_\alpha^c)^2 - \omega_\alpha^2(Q_\alpha^s)^2\right) \tag{2}$$

From equation (2.7),

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}} = \frac{\partial L}{\partial Q} \tag{3}$$

Use (2) and (3) to obtains the following equations of motion.

$$\ddot{Q}_\alpha^c = -\omega_\alpha^2 Q_\alpha^c \qquad \ddot{Q}_\alpha^s = -\omega_\alpha^2 Q_\alpha^s \tag{4}$$

From (8.7), let ϕ_0 be the ground state wave function associated with (4).

$$\phi_0(x_\alpha) = \exp\left(-\frac{\omega_\alpha x_\alpha^2}{2\hbar}\right)$$

Then by equation (8.62),

$$\Phi_0 = \prod_{\alpha=0}^{N-1}\phi_0(Q_\alpha^c)\phi_0(Q_\alpha^s) = \exp\left(-\frac{1}{2\hbar}\sum_{\alpha=0}^{N-1}\omega_\alpha(Q_\alpha^c + Q_\alpha^s)\right)$$