Addition of angular momentum 2

In spherical coordinates

$$\begin{aligned} L_x &= i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) \\ L_y &= i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) \\ L_z &= -i\hbar\frac{\partial}{\partial\phi} \end{aligned}$$

and

$$L^2 = L_x^2 + L_y^2 + L_z^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

Recall that

$$L_z Y_{lm} = m\hbar Y_{lm}$$

and

$$S_z\chi_\pm = \pm\tfrac{1}{2}\hbar\chi_\pm$$

Hence for $\Psi = Y_{lm}\chi_\pm$ we have

$$\begin{aligned} J_z\Psi &= L_z\Psi + S_z\Psi \\ &= m\hbar\Psi \pm \tfrac{1}{2}\hbar\Psi \\ &= \left(m \pm \tfrac{1}{2}\right)\hbar\Psi \end{aligned}$$