9-7. Show, for the vacuum state, the expectation value of $\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}}$ is $(\hbar/2kc)\delta_{\mathbf{k},\mathbf{q}}$ and that of $\bar{a}_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}}$ is $(\hbar/2kc)\delta_{-\mathbf{k},\mathbf{q}}$. Develop a formula for the expectation of $(\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{k}})^r$ for integral r and explain thereby how the expectation of such quantities as $(\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{k}})^r(\bar{a}^*_{1,\mathbf{q}}\bar{a}_{1,\mathbf{q}})^s$ can be got for $\mathbf{q} \neq \mathbf{k}$. Show that the expectation of $(\bar{a}_{1,\mathbf{k}})^2$ or $(\bar{a}^*_{1,\mathbf{k}})^2$ vanishes. Show that the expectation of $(\bar{a}_{1,\mathbf{k}})^2$ or $(\bar{a}^*_{1,\mathbf{k}})^2$ vanishes. Show that the expectation of the product of any odd number of $\bar{a}$'s is zero and that you can compute the expectation value of any product of $\bar{a}$'s or $\bar{a}^*$'s for the vacuum state.

We will use the following table of integrals.

$$\int_{-\infty}^{\infty} \exp(-ax^2 + b)\,dx = \sqrt{\frac{\pi}{a}}\exp(b) \tag{1}$$

$$\int_{-\infty}^{\infty} x\exp(-ax^2 + b)\,dx = 0 \tag{2}$$

$$\int_{-\infty}^{\infty} x^2\exp(-ax^2 + b)\,dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}\exp(b) \tag{3}$$

For simplicity of notation, let

$$A = \bar{a}^c_{1,\mathbf{k}} \qquad B = \bar{a}^s_{1,\mathbf{k}} \qquad C = \bar{a}^c_{1,\mathbf{q}} \qquad D = \bar{a}^s_{1,\mathbf{q}}$$

These formulas convert $\bar{a}$ to sine and cosine modes.

$$\bar{a}_{1,\mathbf{k}} = \frac{1}{\sqrt{2}}(A - iB) \qquad \bar{a}_{1,\mathbf{q}} = \frac{1}{\sqrt{2}}(C - iD) \tag{4}$$

Adapted from equation (8.84)

$$\langle\Phi_0|f|\Phi_0\rangle = \int\limits_{-\infty}^{\infty}\cdots\int\limits_{-\infty}^{\infty} f|\Phi_0|^2\,dA\,dB\,dC\,dD$$

The following $|\Phi_0|^2$ is adapted from equation (9.43). Symbol q is a mode (physical unit meter^{-1}), not an electric charge. Note that we *could* include other modes in addition to k and q. However, integrals over unused modes are cancelled by the normalization constant.

$$|\Phi_0|^2 = \Phi_0^*\Phi_0 = \exp\left(-\frac{kc}{\hbar}A^2 - \frac{kc}{\hbar}B^2 - \frac{qc}{\hbar}C^2 - \frac{qc}{\hbar}D^2\right)$$

Compute the normalization constant K.

$$K = \langle \Phi_0 | 1 | \Phi_0 \rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\Phi_0|^2 \, dA \, dB \, dC \, dD$$

By integral (1) for each factor in the measure,

$$K = \left(\frac{\pi \hbar}{kc} \right)^{1/2} \left(\frac{\pi \hbar}{kc} \right)^{1/2} \left(\frac{\pi \hbar}{qc} \right)^{1/2} \left(\frac{\pi \hbar}{qc} \right)^{1/2}$$

Compute the expectation of $\bar{a}^*_{1,\mathbf{k}} \bar{a}_{1,\mathbf{k}}$. From (4) we have

$$\bar{a}^*_{1,\mathbf{k}} \bar{a}_{1,\mathbf{k}} = \frac{A^2 + B^2}{2}$$

Hence

$$\langle \Phi_0 | \bar{a}^*_{1,\mathbf{k}} \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle = \frac{1}{K} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\frac{A^2 + B^2}{2} \right) |\Phi_0|^2 \, dA \, dB \, dC \, dD$$

Rewrite as

$$\begin{aligned} \langle \Phi_0 | \bar{a}^*_{1,\mathbf{k}} \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle = \frac{1}{2K} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} A^2 |\Phi_0|^2 \, dA \, dB \, dC \, dD \\ + \frac{1}{2K} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} B^2 |\Phi_0|^2 \, dA \, dB \, dC \, dD \end{aligned}$$

By integrals (1) and (3) we have

$$\langle \Phi_0 | \bar{a}^*_{1,\mathbf{k}} \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle = \frac{1}{K} \frac{\hbar}{2kc} \left(\frac{\pi \hbar}{kc} \right)^{1/2} \left(\frac{\pi \hbar}{kc} \right)^{1/2} \left(\frac{\pi \hbar}{qc} \right)^{1/2} \left(\frac{\pi \hbar}{qc} \right)^{1/2}$$

The radicals are cancelled by the normalization constant K.

$$\langle \Phi_0 | \bar{a}^*_{1,\mathbf{k}} \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle = \frac{\hbar}{2kc} \tag{5}$$

Compute the expectation of $\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}}$.

$$\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}} = \frac{AC + BD - iAD + iBC}{2}$$

Hence

$$\langle \Phi_0 | \bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}} | \Phi_0 \rangle = \frac{1}{K} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\frac{AC + BD - iAD + iBC}{2} \right) |\Phi_0|^2 \, dA \, dB \, dC \, dD$$

By integral (2) all terms are zero, hence

$$\langle \Phi_0 | \bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}} | \Phi_0 \rangle = 0 \tag{6}$$

Combine (5) and (6) to obtain

$$\langle \Phi_0 | \bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}} | \Phi_0 \rangle = \frac{\hbar}{2kc} \delta_{\mathbf{k},\mathbf{q}}$$

By equation (8.77)

$$\bar{a}^*_{1,\mathbf{k}} = \bar{a}_{1,-\mathbf{k}}$$

Hence

$$\langle \Phi_0 | \bar{a}_{1,-\mathbf{k}}\bar{a}_{1,\mathbf{q}} | \Phi_0 \rangle = \frac{\hbar}{2kc} \delta_{\mathbf{k},\mathbf{q}}$$

(9-7 cont'd) Develop a formula for the expectation of $(\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{k}})^r$ for integral r and explain thereby how the expectation of such quantities as $(\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{k}})^r(\bar{a}^*_{1,\mathbf{q}}\bar{a}_{1,\mathbf{q}})^s$ can be got for $\mathbf{q} \neq \mathbf{k}$.

By the binomial theorem

$$\left(\frac{A^2 + B^2}{2} \right)^r = \frac{1}{2^r} \sum_{j=0}^{r} \binom{r}{j} A^{2j} B^{2(r-j)} \tag{7}$$

To compute the expectation of (7) we need the following integral.

$$\begin{aligned} \int_{-\infty}^{\infty} x^{2n} \exp(-ax^2 + b) &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n a^n} \sqrt{\frac{\pi}{a}} \exp(b) \\ &= (2n-1)!! \left(\frac{1}{2a} \right)^n \sqrt{\frac{\pi}{a}} \exp(b) \end{aligned} \tag{8}$$

Given (8), define the following function F. (The $\sqrt{\pi/a}$ factor is left out because it gets cancelled by the normalization constant K.)

$$F(n) = (2n-1)!!\left(\frac{\hbar}{2kc}\right)^n$$

Note that

$$F(j)F(r-j) = (2j-1)!!\,(2r-2j-1)!!\left(\frac{\hbar}{2kc}\right)^j\left(\frac{\hbar}{2kc}\right)^{r-j}$$

It turns out that

$$\frac{1}{2^r}\sum_{j=0}^{r}\binom{r}{j}(2j-1)!!\,(2r-2j-1)!! = r!$$

Hence

$$\langle\Phi_0^*|(\bar{a}_{1,\mathbf{k}}^*\bar{a}_{1,\mathbf{k}})^r|\Phi_0\rangle = r!\left(\frac{\hbar}{2kc}\right)^r$$

Regarding the $\mathbf{q}\neq\mathbf{k}$ part of the problem, we have

$$\left(\frac{A^2+B^2}{2}\right)^r\left(\frac{C^2+D^2}{2}\right)^s = \left(\frac{1}{2^r}\sum_{j=0}^{r}\binom{r}{j}A^{2j}B^{2(r-j)}\right)\left(\frac{1}{2^s}\sum_{k=0}^{s}\binom{s}{k}C^{2k}D^{2(r-k)}\right)$$

Hence

$$\langle\Phi_0^*|(\bar{a}_{1,\mathbf{k}}^*\bar{a}_{1,\mathbf{k}})^r(\bar{a}_{1,\mathbf{q}}^*\bar{a}_{1,\mathbf{q}})^s|\Phi_0\rangle = r!\left(\frac{\hbar}{2kc}\right)^r s!\left(\frac{\hbar}{2qc}\right)^s$$

(9-7 cont'd) Show that the expectation of $(\bar{a}_{1,\mathbf{k}})^2$ or $(\bar{a}_{1,\mathbf{k}}^*)^2$ vanishes.

We have

$$(\bar{a}_{1,\mathbf{k}})^2 = \frac{A^2-B^2}{2} - iAB \qquad (\bar{a}_{1,\mathbf{k}}^*)^2 = \frac{A^2-B^2}{2} + iAB$$

The integrals of A^2 and $-B^2$ cancel each other. The integral of AB vanishes by integral (2).

(9-7 cont'd) Show that the expectation of the product of any odd number of $\bar{a}$'s is zero and that you can compute the expectation value of any product of $\bar{a}$'s or $\bar{a}^*$'s for the vacuum state.