Problem 15.6.1. Show that the coherent state in Eq. (15.94) is an eigenstate of the annihilation operator $\hat{a}_\lambda$, with eigenvalue $\sqrt{\bar{n}}\exp(-i\omega_\lambda t)$.

This is equation (15.94).

$$|\Psi_{\lambda\bar{n}}\rangle = \sum_{n_\lambda=0}^{\infty} c_{\lambda\bar{n}n} \exp\left[-i\left(n_\lambda + \tfrac{1}{2}\right)\omega_\lambda t\right]|n_\lambda\rangle \tag{15.94}$$

We will also need equation (15.95).

$$c_{\lambda\bar{n}n} = \sqrt{\frac{\bar{n}^{n_\lambda}\exp(-\bar{n})}{n_\lambda!}} \tag{15.95}$$

We want to show that

$$\hat{a}_\lambda|\Psi_{\lambda\bar{n}}\rangle = \sqrt{\bar{n}}\exp(-i\omega_\lambda t)|\Psi_{\lambda\bar{n}}\rangle$$

Apply operator $\hat{a}_\lambda$ to state $|\Psi_{\lambda\bar{n}}\rangle$ to obtain

$$\hat{a}_\lambda|\Psi_{\lambda\bar{n}}\rangle = \sum_{n_\lambda=0}^{\infty} c_{\lambda\bar{n}n} \exp\left[-i\left(n_\lambda + \tfrac{1}{2}\right)\omega_\lambda t\right]\sqrt{n_\lambda}|n_\lambda - 1\rangle$$

The $n_\lambda = 0$ term vanishes hence the sum can start from $n_\lambda = 1$.

$$\hat{a}_\lambda|\Psi_{\lambda\bar{n}}\rangle = \sum_{n_\lambda=1}^{\infty} c_{\lambda\bar{n}n} \exp\left[-i\left(n_\lambda + \tfrac{1}{2}\right)\omega_\lambda t\right]\sqrt{n_\lambda}|n_\lambda - 1\rangle$$

The $\sqrt{n_\lambda}$ cancels with the denominator in $c_{\lambda\bar{n}n}$.

$$\hat{a}_\lambda|\Psi_{\lambda\bar{n}}\rangle = \sum_{n_\lambda=1}^{\infty} \sqrt{\frac{\bar{n}^{n_\lambda}\exp(-\bar{n})}{(n_\lambda-1)!}} \exp\left[-i\left(n_\lambda + \tfrac{1}{2}\right)\omega_\lambda t\right]|n_\lambda - 1\rangle$$

On the right-hand side, factor out $\sqrt{\bar{n}}\exp(-i\omega_\lambda t)$.

$$\begin{aligned}&\hat{a}_\lambda|\Psi_{\lambda\bar{n}}\rangle = \\ &\qquad\sqrt{\bar{n}}\exp(-i\omega_\lambda t)\sum_{n_\lambda=1}^{\infty} \sqrt{\frac{\bar{n}^{n_\lambda-1}\exp(-\bar{n})}{(n_\lambda-1)!}} \exp\left[-i\left(n_\lambda - \tfrac{1}{2}\right)\omega_\lambda t\right]|n_\lambda - 1\rangle\end{aligned}$$

Substitute $n_\lambda + 1$ for index n_λ.

$$\hat{a}_\lambda|\Psi_{\lambda\bar{n}}\rangle = \sqrt{\bar{n}}\exp(-i\omega_\lambda t)\sum_{n_\lambda=0}^{\infty}\sqrt{\frac{\bar{n}^{n_\lambda}\exp(-\bar{n})}{n_\lambda!}}\exp\left[-i\left(n_\lambda+\tfrac{1}{2}\right)\omega_\lambda t\right]|n_\lambda\rangle$$

Hence

$$\hat{a}_\lambda|\Psi_{\lambda\bar{n}}\rangle = \sqrt{\bar{n}}\exp(-i\omega_\lambda t)|\Psi_{\lambda\bar{n}}\rangle$$