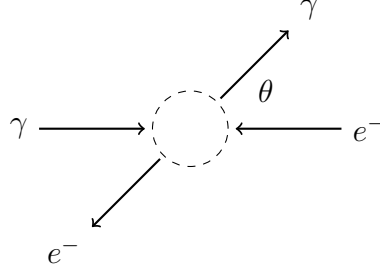


Compton scattering

Compton scattering is the interaction $e^- + \gamma \rightarrow e^- + \gamma$.



In the center-of-mass frame we have the following momentum vectors where ω is incident energy and E is total energy $E = \sqrt{\omega^2 + m^2}$. Polar angle θ is the observed scattering angle. Azimuth angle ϕ cancels out in scattering calculations.

$$p_1 = \begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix}, \quad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -\omega \end{pmatrix}, \quad p_3 = \begin{pmatrix} \omega \\ \omega \sin \theta \cos \phi \\ \omega \sin \theta \sin \phi \\ \omega \cos \theta \end{pmatrix}, \quad p_4 = \begin{pmatrix} E \\ -\omega \sin \theta \cos \phi \\ -\omega \sin \theta \sin \phi \\ -\omega \cos \theta \end{pmatrix}$$

inbound photon inbound electron outbound photon outbound electron

Spinors for the inbound electron.

$$u_{21} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ -\omega \\ 0 \end{pmatrix}, \quad u_{22} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ 0 \\ \omega \end{pmatrix}$$

inbound electron spin up inbound electron spin down

Spinors for the outbound electron.

$$u_{41} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix}, \quad u_{42} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}$$

outbound electron spin up outbound electron spin down

Let a be the spin state of the inbound electron and let b be the spin state of the outbound electron such that subscript $ba \in \{11, 12, 21, 22\}$.

The probability amplitude \mathcal{M}_{ba} for spin state ba is

$$\mathcal{M}_{ba} = \mathcal{M}_{1ba} + \mathcal{M}_{2ba}$$

where

$$\mathcal{M}_{1ba} = \frac{\bar{u}_{4b}(-ie\gamma^\mu)(\not{p}_1 + m)(-ie\gamma^\nu)u_{2a}}{(p_1 + p_2)^2 - m^2}, \quad \mathcal{M}_{2ba} = \frac{\bar{u}_{4b}(-ie\gamma^\nu)(\not{p}_2 + m)(-ie\gamma^\mu)u_{2a}}{(p_1 - p_4)^2 - m^2}$$

and

$$\not{q}_1 = (p_1 + p_2)^\alpha g_{\alpha\beta} \gamma^\beta, \quad \not{q}_2 = (p_4 - p_1)^\alpha g_{\alpha\beta} \gamma^\beta$$

The expected probability $\langle |\mathcal{M}|^2 \rangle$ is the average over all four spin states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{a=1}^2 \sum_{b=1}^2 |\mathcal{M}_{ba}|^2$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{a=1}^2 \sum_{b=1}^2 \left(\frac{\mathcal{M}_{1ba} \mathcal{M}_{1ba}^*}{(s - m^2)^2} + \frac{\mathcal{M}_{1ba} \mathcal{M}_{2ba}^* + \mathcal{M}_{2ba} \mathcal{M}_{1ba}^*}{(s - m^2)(u - m^2)} + \frac{\mathcal{M}_{2ba} \mathcal{M}_{2ba}^*}{(u - m^2)^2} \right)$$

where s and u are Mandelstam variables

$$s = (p_1 + p_2)^2, \quad u = (p_1 - p_4)^2$$

To understand how $\mathcal{M}_{1ba} \mathcal{M}_{1ba}^*$ is calculated, write \mathcal{M}_{1ba} in component form.

$$(\mathcal{M}_{1ba})^{\mu\nu} = \frac{(\bar{u}_{4b})_\alpha (-ie\gamma^{\mu\alpha}) (\not{q}_1 + m)^\beta_\rho (-ie\gamma^{\nu\rho}) (u_{2a})^\sigma}{s - m^2}$$

To sum over μ and ν of the product, $g_{\mu\nu}$ is needed to lower indices.

$$\mathcal{M}_{1ba} \mathcal{M}_{1ba}^* = (\mathcal{M}_{1ba})^{\mu\nu} (\mathcal{M}_{1ba}^*)_{\mu\nu} = (\mathcal{M}_{1ba})^{\mu\nu} g_{\mu\alpha} (\mathcal{M}_{1ba}^*)^{\alpha\beta} g_{\beta\nu}$$

Similarly for $\mathcal{M}_{2ba} \mathcal{M}_{2ba}^*$. For \mathcal{M}_{2ba} the index order is ν followed by μ hence

$$\mathcal{M}_{1ba} \mathcal{M}_{2ba}^* = (\mathcal{M}_{1ba})^{\mu\nu} (\mathcal{M}_{2ba}^*)_{\nu\mu} = (\mathcal{M}_{1ba})^{\mu\nu} g_{\nu\beta} (\mathcal{M}_{2ba}^*)^{\beta\alpha} g_{\alpha\mu}$$

The Casimir trick uses matrix arithmetic to sum over spin states.

$$\begin{aligned} f_{11} &= \sum_{a=1}^2 \sum_{b=1}^2 \mathcal{M}_{1ba} \mathcal{M}_{1ba}^* = e^4 \text{Tr} \left((\not{p}_2 + m) \gamma^\mu (\not{q}_1 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\nu (\not{q}_1 + m) \gamma_\mu \right) \\ f_{12} &= \sum_{a=1}^2 \sum_{b=1}^2 \mathcal{M}_{1ba} \mathcal{M}_{2ba}^* = e^4 \text{Tr} \left((\not{p}_2 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\mu (\not{q}_1 + m) \gamma_\nu \right) \\ f_{22} &= \sum_{a=1}^2 \sum_{b=1}^2 \mathcal{M}_{2ba} \mathcal{M}_{2ba}^* = e^4 \text{Tr} \left((\not{p}_2 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\nu (\not{q}_2 + m) \gamma_\mu \right) \end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \left(\frac{f_{11}}{(s - m^2)^2} + \frac{2f_{12}}{(s - m^2)(u - m^2)} + \frac{f_{22}}{(u - m^2)^2} \right) \quad (1)$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$\begin{aligned} f_{11} &= e^4 (32(p_1 \cdot p_2)(p_1 \cdot p_4) + 64m^2(p_1 \cdot p_2) - 32m^2(p_1 \cdot p_3) - 32m^2(p_1 \cdot p_4) + 32m^4) \\ f_{12} &= e^4 (16m^2(p_1 \cdot p_2) - 16m^2(p_1 \cdot p_4) + 32m^4) \\ f_{22} &= e^4 (32(p_1 \cdot p_2)(p_1 \cdot p_4) + 32m^2(p_1 \cdot p_2) - 32m^2(p_1 \cdot p_3) - 64m^2(p_1 \cdot p_4) + 32m^4) \end{aligned}$$

In Mandelstam variables

$$\begin{aligned} f_{11} &= e^4 (-8su + 24sm^2 + 8um^2 + 8m^4) \\ f_{12} &= e^4 (8sm^2 + 8um^2 + 16m^4) \\ f_{22} &= e^4 (-8su + 8sm^2 + 24um^2 + 8m^4) \end{aligned} \tag{2}$$

Compton scattering experiments are typically done in the lab frame where the electron is at rest. Define Lorentz boost Λ for transforming momentum vectors to the lab frame.

$$\Lambda = \begin{pmatrix} E/m & 0 & 0 & \omega/m \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega/m & 0 & 0 & E/m \end{pmatrix}$$

The electron is at rest in the lab frame.

$$\Lambda p_2 = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Mandelstam variables are invariant under a boost.

$$\begin{aligned} s &= (p_1 + p_2)^2 = (\Lambda p_1 + \Lambda p_2)^2 \\ t &= (p_1 - p_3)^2 = (\Lambda p_1 - \Lambda p_3)^2 \\ u &= (p_1 - p_4)^2 = (\Lambda p_1 - \Lambda p_4)^2 \end{aligned}$$

In the lab frame, let ω_L be the angular frequency of the incident photon and let ω'_L be the angular frequency of the scattered photon.

$$\begin{aligned} \omega_L &= \Lambda p_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2}{m} + \frac{\omega E}{m} \\ \omega'_L &= \Lambda p_3 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2 \cos \theta}{m} + \frac{\omega E}{m} \end{aligned}$$

It can be shown that

$$\begin{aligned} s &= m^2 + 2m\omega_L \\ t &= 2m(\omega'_L - \omega_L) \\ u &= m^2 - 2m\omega'_L \end{aligned} \tag{3}$$

Then by (1), (2), and (3) we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \left(\frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1 \right)^2 - 1 \right)$$

Lab scattering angle θ_L is given by the Compton equation

$$\cos \theta_L = \frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1$$

Hence

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \cos^2 \theta_L - 1 \right) \\ &= 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} - \sin^2 \theta_L \right) \end{aligned}$$

Cross section

Now that we have derived $\langle |\mathcal{M}|^2 \rangle$ we can investigate the angular distribution of scattered photons. For simplicity let us drop the L subscript from lab variables. From now on the symbols ω , ω' , and θ will be lab frame variables.

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4(4\pi\epsilon_0)^2 s} \left(\frac{\omega'}{\omega} \right)^2 \langle |\mathcal{M}|^2 \rangle$$

where

$$s = m^2 + 2m\omega = (mc^2)^2 + 2(mc^2)(\hbar\omega)$$

and ω' is given by the Compton equation

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos \theta)}$$

For the lab frame we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

Hence in the lab frame

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\epsilon_0)^2 s} \left(\frac{\omega'}{\omega} \right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

Noting that

$$e^2 = 4\pi\epsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s} \left(\frac{\omega'}{\omega} \right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

Noting that

$$d\Omega = \sin \theta d\theta d\phi$$

we also have

$$d\sigma = \frac{\alpha^2 (\hbar c)^2}{2s} \left(\frac{\omega'}{\omega} \right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right) \sin \theta d\theta d\phi$$

Let $S(\theta_1, \theta_2)$ be the following surface integral of $d\sigma$.

$$S(\theta_1, \theta_2) = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{2\pi\alpha^2(\hbar c)^2}{2s} (I(\theta_2) - I(\theta_1))$$

where

$$I(\theta) = -\frac{\cos \theta}{R^2} + \log(1 + R(1 - \cos \theta)) \left(\frac{1}{R} - \frac{2}{R^2} - \frac{2}{R^3} \right) - \frac{1}{2R(1 + R(1 - \cos \theta))^2} + \frac{1}{1 + R(1 - \cos \theta)} \left(-\frac{2}{R^2} - \frac{1}{R^3} \right)$$

and

$$R = \frac{\hbar\omega}{mc^2}$$

The cumulative distribution function is

$$F(\theta) = \frac{S(0, \theta)}{S(0, \pi)} = \frac{I(\theta) - I(0)}{I(\pi) - I(0)}, \quad 0 \leq \theta \leq \pi$$

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 \leq \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

Let N be the total number of scattering events from an experiment. Then the number of scattering events in the interval θ_1 to θ_2 is predicted to be

$$NP(\theta_1 \leq \theta \leq \theta_2)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi) - I(0)} \left(\frac{\omega'}{\omega} \right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right) \sin \theta$$

Thomson scattering

For $\hbar\omega \ll mc^2$ we have

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2} (1 - \cos \theta)} \approx \omega$$

Hence we can use the approximations

$$\omega = \omega' \quad \text{and} \quad s = (mc^2)^2$$

to obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \hbar^2}{2m^2 c^2} (1 + \cos^2 \theta)$$

which is the formula for Thomson scattering.

High energy approximation

For $\omega \gg m$ a useful approximation is to set $m = 0$ and obtain

$$\begin{aligned}f_{11} &= e^4(-8su) \\f_{12} &= 0 \\f_{22} &= e^4(-8su)\end{aligned}$$

Hence

$$\begin{aligned}\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left(\frac{-8su}{s^2} + \frac{-8su}{u^2} \right) \\&= 2e^4 \left(-\frac{u}{s} - \frac{s}{u} \right)\end{aligned}$$

Also for $m = 0$ the Mandelstam variables s and u are

$$\begin{aligned}s &= 4\omega^2 \\u &= -2\omega^2(\cos \theta + 1)\end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{\cos \theta + 1}{2} + \frac{2}{\cos \theta + 1} \right)$$

Data from a CERN LEP experiment

See “Compton Scattering of Quasi-Real Virtual Photons at LEP,” arxiv.org/abs/hep-ex/0504012.

x	y
-0.74	13380
-0.60	7720
-0.47	6360
-0.34	4600
-0.20	4310
-0.07	3700
0.06	3640
0.20	3340
0.33	3500
0.46	3010
0.60	3310
0.73	3330

The data are for the center of mass frame and have the following relationship with the differential cross section formula.

$$x = \cos \theta, \quad y = \frac{d\sigma}{d \cos \theta} = 2\pi \frac{d\sigma}{d\Omega}$$

For the high energy approximation we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{\cos \theta + 1}{2} + \frac{2}{\cos \theta + 1} \right)$$

The corresponding cross section formula is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{64\pi^2 s} = \frac{e^4}{32\pi^2 s} \left(\frac{\cos \theta + 1}{2} + \frac{2}{\cos \theta + 1} \right), \quad s \gg m$$

Substituting $e^4 = 16\pi^2 \alpha^2$ yields

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s} \left(\frac{\cos \theta + 1}{2} + \frac{2}{\cos \theta + 1} \right)$$

Multiply by 2π to obtain

$$\frac{d\sigma}{d\cos \theta} = \frac{\pi \alpha^2}{s} \left(\frac{\cos \theta + 1}{2} + \frac{2}{\cos \theta + 1} \right)$$

To compute predicted values \hat{y} from the above formula, multiply by $(hc)^2$ to convert to SI and multiply by 10^{40} to convert square meters to picobarns.

$$\hat{y} = \frac{\pi \alpha^2}{s} \left(\frac{x + 1}{2} + \frac{2}{x + 1} \right) \times (hc)^2 \times 10^{40}$$

The following table shows \hat{y} for $s = (40 \text{ GeV})^2$.

x	y	\hat{y}
-0.74	13380	12573
-0.60	7720	8358
-0.47	6360	6491
-0.34	4600	5401
-0.20	4310	4661
-0.07	3700	4204
0.06	3640	3884
0.20	3340	3643
0.33	3500	3486
0.46	3010	3375
0.60	3310	3295
0.73	3330	3248

The coefficient of determination R^2 measures how well predicted values fit the data.

$$R^2 = 1 - \frac{\sum (y - \hat{y})^2}{\sum (y - \bar{y})^2} = 0.97$$

The result indicates that the model $d\sigma$ explains 97% of the variance in the data.