

Path integral step by step

- ① Let H be the following Hamiltonian where $V \equiv V(x, t)$.

$$H = \frac{\hat{p}^2}{2m} + V$$

- ② Let K_j be the amplitude to go from x_j to x_{j+1} in time δt where $\delta t = T/N$.

$$K_j = \langle x_{j+1} | \exp \left(-\frac{i}{\hbar} H \delta t \right) | x_j \rangle$$

- ③ By the identity

$$\int |p\rangle \langle p| dp = 1$$

we can write

$$K_j = \int \langle x_{j+1} | \exp \left(-\frac{i}{\hbar} H \delta t \right) | p \rangle \langle p | x_j \rangle dp$$

- ④ Replace operator \hat{p} with its eigenvalue p .

$$K_j = \int \exp \left(-\frac{i}{\hbar} \left(\frac{p^2}{2m} + V \right) \delta t \right) \langle x_{j+1} | p \rangle \langle p | x_j \rangle dp$$

- ⑤ By the identity

$$\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp \left(\frac{ipx}{\hbar} \right)$$

we can write

$$K_j = \frac{1}{2\pi\hbar} \int \left(-\frac{i}{\hbar} \left(\frac{p^2}{2m} + V \right) \delta t \right) \exp \left(\frac{i}{\hbar} px_{j+1} \right) \exp \left(-\frac{i}{\hbar} px_j \right) dp$$

- ⑥ Combine exponentials.

$$K_j = \frac{1}{2\pi\hbar} \int \exp \left(-\frac{i}{\hbar} \left(\frac{p^2}{2m} + V \right) \delta t + \frac{i}{\hbar} p(x_{j+1} - x_j) \right) dp$$

- ⑦ The V term does not depend on p so factor it out of the integrand.

$$K_j = \frac{1}{2\pi\hbar} \exp \left(-\frac{i}{\hbar} V \delta t \right) \int \exp \left(-\frac{i}{\hbar} \frac{p^2}{2m} \delta t + \frac{i}{\hbar} p(x_{j+1} - x_j) \right) dp$$

- ⑧ To solve the integral, use the identity

$$\int_{-\infty}^{\infty} \exp(-ay^2 + by) dy = \left(\frac{\pi}{a} \right)^{1/2} \exp \left(\frac{b^2}{4a} \right)$$

with

$$a = \frac{i}{\hbar} \frac{\delta t}{2m}, \quad b = \frac{i}{\hbar} (x_{j+1} - x_j)$$

to obtain

$$K_j = \frac{1}{2\pi\hbar} \exp \left(-\frac{i}{\hbar} V \delta t \right) \left(\frac{2\pi m \hbar}{i \delta t} \right)^{1/2} \exp \left(\frac{i}{\hbar} \frac{m}{2} \frac{(x_{j+1} - x_j)^2}{\delta t} \right)$$

- ⑨ Combine coefficients and exponentials.

$$K_j = \left(\frac{m}{2\pi i \hbar \delta t} \right)^{1/2} \exp \left(\frac{i}{\hbar} \frac{m}{2} \frac{(x_{j+1} - x_j)^2}{\delta t} - \frac{i}{\hbar} V \delta t \right)$$

⑩ Factor out i/\hbar and δt .

$$K_j = \left(\frac{m}{2\pi i \hbar \delta t} \right)^{1/2} \exp \left[\frac{i}{\hbar} \left(\frac{m}{2} \left(\frac{x_{j+1} - x_j}{\delta t} \right)^2 - V \right) \delta t \right]$$

⑪ Let $K(b, a)$ be the amplitude to go from x_a at time zero to x_b at time T where $x_0 \equiv x_a$ and $x_N \equiv x_b$.

$$K(b, a) = \int dx_{N-1} \cdots \int dx_2 \int dx_1 K_{N-1} \cdots K_2 K_1 K_0$$

⑫ Substitute for the K_j .

$$K(b, a) = \int dx_{N-1} \cdots \int dx_2 \int dx_1 \left(\frac{m}{2\pi i \hbar \delta t} \right)^{N/2} \exp \left[\frac{i}{\hbar} \sum_{j=0}^{N-1} \left(\frac{m}{2} \left(\frac{x_{j+1} - x_j}{\delta t} \right)^2 - V \right) \delta t \right]$$

⑬ In the limit we have

$$\lim_{N \rightarrow \infty} \sum_{j=0}^{N-1} \left(\frac{m}{2} \left(\frac{x_{j+1} - x_j}{\delta t} \right)^2 - V \right) \delta t = \int_0^T \left(\frac{m}{2} \left(\frac{d}{dt} x(t) \right)^2 - V \right) dt$$

Hence

$$K(b, a) = \int dx_{N-1} \cdots \int dx_2 \int dx_1 \left(\frac{m}{2\pi i \hbar \delta t} \right)^{N/2} \exp \left[\frac{i}{\hbar} \int_0^T \left(\frac{m}{2} \left(\frac{d}{dt} x(t) \right)^2 - V \right) dt \right]$$

⑭ Define

$$\int Dx(t) = \lim_{N \rightarrow \infty} \int dx_{N-1} \cdots \int dx_2 \int dx_1 \left(\frac{m}{2\pi i \hbar \delta t} \right)^{N/2}$$

and write

$$K(b, a) = \int Dx(t) \exp \left[\frac{i}{\hbar} \int_0^T \left(\frac{m}{2} \left(\frac{d}{dt} x(t) \right)^2 - V \right) dt \right]$$