Shielded Coulomb potential

Find the cross section for Rutherford scattering with

$$V(r) = -\frac{Ze^2}{r}\exp\left(-\frac{r}{a}\right)$$

Start with

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2\epsilon_0^2}\left(\frac{mQ}{2\pi\hbar^2}\right)^2, \quad Q = \int \exp\left(\frac{i\mathbf{p}\cdot\mathbf{r}}{\hbar}\right) V(\mathbf{r})\, d^3\mathbf{r}$$

Convert Q to polar coordinates.

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) V(r,\theta,\phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Substitute the shielded Coulomb potential for $V(r,\theta,\phi)$ and note r^2 becomes r.

$$Q = -Ze^2 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) \exp\left(-\frac{r}{a}\right) r\sin\theta\, dr\, d\theta\, d\phi$$

Integrate over ϕ.

$$Q = -2\pi Ze^2 \int_0^{\pi}\int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) \exp\left(-\frac{r}{a}\right) r\sin\theta\, dr\, d\theta$$

Change the complex exponential to rectangular form.

$$Q = -2\pi Ze^2 \int_0^{\pi}\int_0^{\infty} \left[\cos\left(\frac{pr\cos\theta}{\hbar}\right) + i\sin\left(\frac{pr\cos\theta}{\hbar}\right)\right] \exp\left(-\frac{r}{a}\right) r\sin\theta\, dr\, d\theta$$

By the definite integrals

$$\int_0^{\pi} \cos(a\cos\theta)\sin\theta\, d\theta = \frac{2\sin a}{a}, \qquad \int_0^{\pi} \sin(a\cos\theta)\sin\theta\, d\theta = 0$$

we have for the integral over θ (note r in the integrand is canceled)

$$Q = -\frac{4\pi Ze^2\hbar}{p}\int_0^{\infty} \sin\left(\frac{pr}{\hbar}\right)\exp\left(-\frac{r}{a}\right) dr$$

By the definite integral

$$\int_0^{\pi} \frac{\sin\theta}{\left(\sin^2(\theta/2)+a\right)^2}\, d\theta = \frac{2}{a^2+a}$$

we have for the integral over r

$$Q = -\frac{4\pi Ze^2\hbar}{p} \times \frac{p/\hbar}{(p/\hbar)^2 + (1/a)^2} = -\frac{4\pi Ze^2}{(p/\hbar)^2+(1/a)^2}$$