

Problem 15.1.1. Prove the relation

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$$

for the harmonic oscillator raising and lowering operators, starting from their definitions

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}} \left( -\frac{d}{d\xi} + \xi \right) \quad \text{and} \quad \hat{a} = \frac{1}{\sqrt{2}} \left( \frac{d}{d\xi} + \xi \right)$$

For  $\hat{a}\hat{a}^\dagger$  we have

$$\hat{a}\hat{a}^\dagger\psi = \frac{1}{2} \left( \frac{d}{d\xi} + \xi \right) \left( -\frac{d}{d\xi} + \xi \right) \psi$$

Expand the right-hand side.

$$\begin{aligned} \hat{a}\hat{a}^\dagger\psi &= \frac{1}{2} \left( -\frac{d^2}{d\xi^2}\psi + \frac{d}{d\xi}(\xi\psi) - \xi\frac{d}{d\xi}\psi + \xi^2\psi \right) \\ &= \frac{1}{2} \left( -\frac{d^2}{d\xi^2}\psi + \psi + \xi^2\psi \right) \end{aligned} \tag{1}$$

For  $\hat{a}^\dagger\hat{a}$  we have

$$\hat{a}^\dagger\hat{a}\psi = \frac{1}{2} \left( -\frac{d}{d\xi} + \xi \right) \left( \frac{d}{d\xi} + \xi \right) \psi$$

Expand the right-hand side.

$$\begin{aligned} \hat{a}^\dagger\hat{a}\psi &= \frac{1}{2} \left( -\frac{d^2}{d\xi^2}\psi - \frac{d}{d\xi}(\xi\psi) + \xi\frac{d}{d\xi}\psi + \xi^2\psi \right) \\ &= \frac{1}{2} \left( -\frac{d^2}{d\xi^2}\psi - \psi + \xi^2\psi \right) \end{aligned} \tag{2}$$

Subtract (2) from (1).

$$\hat{a}\hat{a}^\dagger\psi - \hat{a}^\dagger\hat{a}\psi = \frac{1}{2}\psi - \left( -\frac{1}{2}\psi \right) = \psi$$

Hence

$$[\hat{a}, \hat{a}^\dagger] = 1$$