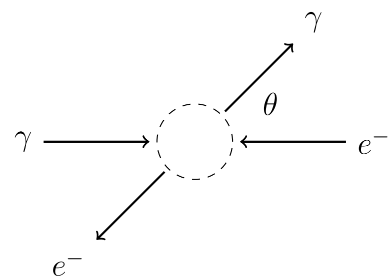
Compton scattering

Compton scattering is the interaction $e^- + \gamma \to e^- + \gamma$.



In the center-of-mass frame we have the following momentum vectors where $E = \sqrt{\omega^2 + m^2}$.

$$
p_1 = \underset{\substack{\text{inbound}\\ \text{photon}}}{\begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix}} \qquad
p_2 = \underset{\substack{\text{inbound}\\ \text{electron}}}{\begin{pmatrix} E \\ 0 \\ 0 \\ -\omega \end{pmatrix}} \qquad
p_3 = \underset{\substack{\text{outbound}\\ \text{photon}}}{\begin{pmatrix} \omega \\ \omega \sin\theta\cos\phi \\ \omega\sin\theta\sin\phi \\ \omega\cos\theta \end{pmatrix}} \qquad
p_4 = \underset{\substack{\text{outbound}\\ \text{electron}}}{\begin{pmatrix} E \\ -\omega \sin\theta\cos\phi \\ -\omega\sin\theta\sin\phi \\ -\omega\cos\theta \end{pmatrix}}
$$

Spinors for the inbound electron.

$$
u_{21} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound electron}\\ \text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ -\omega \\ 0 \end{pmatrix}} \qquad
u_{22} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound electron}\\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ \omega \end{pmatrix}}
$$

Spinors for the outbound electron.

$$
u_{41} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{outbound electron}\\ \text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix}} \qquad
u_{42} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{outbound electron}\\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}}
$$

Let a be the spin state of the inbound electron and let b be the spin state of the outbound electron such that subscript $ba \in \{11, 12, 21, 22\}$. The probability amplitude $\mathcal{M}_{ba}$ for spin state ba is

$$
\mathcal{M}_{ba} = \mathcal{M}_{1ba} + \mathcal{M}_{2ba}
$$

where

$$
\mathcal{M}_{1ba} = \frac{\bar{u}_{4b}(-ie\gamma^\mu)(\not{q}_1 + m)(-ie\gamma^\nu)u_{2a}}{s - m^2}, \quad
\mathcal{M}_{2ba} = \frac{\bar{u}_{4b}(-ie\gamma^\nu)(\not{q}_2 + m)(-ie\gamma^\mu)u_{2a}}{u - m^2}
$$

Symbol e is elementary charge and

$$\not{q}_1 = (p_1 + p_2)^\alpha g_{\alpha\beta}\gamma^\beta$$
$$\not{q}_2 = (p_4 - p_1)^\alpha g_{\alpha\beta}\gamma^\beta$$

$$s = (p_1 + p_2)^2$$
$$u = (p_1 - p_4)^2$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is the average probability density for all four spin states.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}|\mathcal{M}_{ba}|^2$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\left(\mathcal{M}_{1ba}\mathcal{M}^*_{1ba} + \mathcal{M}_{1ba}\mathcal{M}^*_{2ba} + \mathcal{M}_{2ba}\mathcal{M}^*_{1ba} + \mathcal{M}_{2ba}\mathcal{M}^*_{2ba}\right)$$

To understand how $\mathcal{M}_{1ba}\mathcal{M}^*_{1ba}$ is calculated, write $\mathcal{M}_{1ba}$ in component form.

$$(\mathcal{M}_{1ba})^{\mu\nu} = \frac{(\bar{u}_{4b})_\alpha(-ie\gamma^{\mu\alpha}{}_\beta)(\not{q}_1 + m)^\beta{}_\rho(-ie\gamma^{\nu\rho}{}_\sigma)(u_{2a})^\sigma}{s - m^2}$$

Metric tensor $g_{\mu\nu}$ is required to sum over indices μ and ν.

$$\mathcal{M}_{1ba}\mathcal{M}^*_{1ba} = (\mathcal{M}_{1ba})^{\mu\nu}(\mathcal{M}^*_{1ba})_{\mu\nu} = (\mathcal{M}_{1ba})^{\mu\nu}g_{\mu\alpha}(\mathcal{M}^*_{1ba})^{\alpha\beta}g_{\beta\nu}$$

Similarly for $\mathcal{M}_{2ba}\mathcal{M}^*_{2ba}$. For $\mathcal{M}_{2ba}$ the index order is ν followed by μ hence

$$\mathcal{M}_{1ba}\mathcal{M}^*_{2ba} = (\mathcal{M}_{1ba})^{\mu\nu}(\mathcal{M}^*_{2ba})_{\nu\mu} = (\mathcal{M}_{1ba})^{\mu\nu}g_{\nu\beta}(\mathcal{M}^*_{2ba})^{\beta\alpha}g_{\alpha\mu}$$

The Casimir trick uses matrix arithmetic to sum over spin states.

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left(\frac{f_{11}}{(s-m^2)^2} + \frac{2f_{12}}{(s-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2}\right) \tag{1}$$

where

$$f_{11} = \mathrm{Tr}\Big((\not{p}_2 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\Big)$$
$$f_{12} = \mathrm{Tr}\Big((\not{p}_2 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\Big)$$
$$f_{22} = \mathrm{Tr}\Big((\not{p}_2 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\nu(\not{q}_2 + m)\gamma_\mu\Big)$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$f_{11} = 32(p_1\cdot p_2)(p_1\cdot p_4) + 64m^2(p_1\cdot p_2) - 32m^2(p_1\cdot p_3) - 32m^2(p_1\cdot p_4) + 32m^4$$
$$f_{12} = 16m^2(p_1\cdot p_2) - 16m^2(p_1\cdot p_4) + 32m^4$$
$$f_{22} = 32(p_1\cdot p_2)(p_1\cdot p_4) + 32m^2(p_1\cdot p_2) - 32m^2(p_1\cdot p_3) - 64m^2(p_1\cdot p_4) + 32m^4$$

In Mandelstam variables

$$
\begin{aligned}
f_{11} &= -8su + 24sm^2 + 8um^2 + 8m^4 \\
f_{12} &= 8sm^2 + 8um^2 + 16m^4 \\
f_{22} &= -8su + 8sm^2 + 24um^2 + 8m^4
\end{aligned} \tag{2}
$$

Compton scattering experiments are typically done in the lab frame where the electron is at rest. Define Lorentz boost Λ for transforming momentum vectors to the lab frame.

$$
\Lambda = \begin{pmatrix} E/m & 0 & 0 & \omega/m \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega/m & 0 & 0 & E/m \end{pmatrix}
$$

The electron is at rest in the lab frame.

$$
\Lambda p_2 = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

Mandelstam variables are invariant under a boost.

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = (\Lambda p_1 + \Lambda p_2)^2 \\
t &= (p_1 - p_3)^2 = (\Lambda p_1 - \Lambda p_3)^2 \\
u &= (p_1 - p_4)^2 = (\Lambda p_1 - \Lambda p_4)^2
\end{aligned}
$$

In the lab frame, let ω_L be the angular frequency of the incident photon and let ω_L' be the angular frequency of the scattered photon.

$$
\omega_L = \Lambda p_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2}{m} + \frac{\omega E}{m}
$$

$$
\omega_L' = \Lambda p_3 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2 \cos\theta}{m} + \frac{\omega E}{m}
$$

It can be shown that

$$
\begin{aligned}
s &= m^2 + 2m\omega_L \\
t &= 2m(\omega_L' - \omega_L) \\
u &= m^2 - 2m\omega_L'
\end{aligned} \tag{3}
$$

Then by (1), (2), and (3) we have

$$
\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{\omega_L}{\omega_L'} + \frac{\omega_L'}{\omega_L} + \left(\frac{m}{\omega_L} - \frac{m}{\omega_L'} + 1 \right)^2 - 1 \right)
$$

Lab scattering angle θ_L is given by the Compton equation

$$\cos\theta_L = \frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1$$

Hence

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= 2e^4\left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \cos^2\theta_L - 1\right)\\ &= 2e^4\left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} - \sin^2\theta_L\right)\end{aligned}$$

Cross section

Now that we have derived $\langle|\mathcal{M}|^2\rangle$ we can investigate the angular distribution of scattered photons. For simplicity let us drop the L subscript from lab variables. From now on the symbols ω, ω', and θ will be lab frame variables.

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4(4\pi\varepsilon_0)^2 s}\left(\frac{\omega'}{\omega}\right)^2\langle|\mathcal{M}|^2\rangle$$

where

$$s = m^2 + 2m\omega = (mc^2)^2 + 2(mc^2)(\hbar\omega)$$

and ω' is given by the Compton equation

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1-\cos\theta)}$$

For the lab frame we have

$$\langle|\mathcal{M}|^2\rangle = 2e^4\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right)$$

Hence in the lab frame

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\varepsilon_0)^2 s}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right)$$

Noting that

$$e^2 = 4\pi\varepsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right)$$

Noting that

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

we also have

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{2s}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta\right)\sin\theta\,d\theta\,d\phi$$

Let $S(\theta_1,\theta_2)$ be the following integral of $d\sigma$.

$$S(\theta_1,\theta_2)=\int_0^{2\pi}\int_{\theta_1}^{\theta_2}d\sigma$$

The solution is

$$S(\theta_1,\theta_2)=\frac{\pi\alpha^2(\hbar c)^2}{s}[I(\theta_2)-I(\theta_1)]$$

where

$$\begin{aligned}I(\theta)=-\frac{\cos\theta}{R^2}+\log\big(1+R(1-\cos\theta)\big)\left(\frac{1}{R}-\frac{2}{R^2}-\frac{2}{R^3}\right)\\
-\frac{1}{2R\big(1+R(1-\cos\theta)\big)^2}+\frac{1}{1+R(1-\cos\theta)}\left(-\frac{2}{R^2}-\frac{1}{R^3}\right)\end{aligned}$$

and

$$R=\frac{\hbar\omega}{mc^2}$$

The cumulative distribution function is

$$F(\theta)=\frac{S(0,\theta)}{S(0,\pi)}=\frac{I(\theta)-I(0)}{I(\pi)-I(0)},\quad 0\le\theta\le\pi$$

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1<\theta\le\theta_2)=F(\theta_2)-F(\theta_1)$$

The probability density function is

$$f(\theta)=\frac{dF(\theta)}{d\theta}=\frac{1}{I(\pi)-I(0)}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta\right)\sin\theta$$

Thomson scattering

For $\hbar\omega\ll mc^2$ we have

$$\omega'=\frac{\omega}{1+\frac{\hbar\omega}{mc^2}(1-\cos\theta)}\approx\omega$$

Hence we can use the approximations

$$\omega=\omega'\quad\text{and}\quad s=(mc^2)^2$$

to obtain

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2\hbar^2}{2m^2c^2}\left(1+\cos^2\theta\right)$$

which is the formula for Thomson scattering.

High energy approximation

For $\omega \gg m$ a useful approximation is to set $m = 0$ and obtain

$$\begin{aligned} f_{11} &= -8su \\ f_{12} &= 0 \\ f_{22} &= -8su \end{aligned}$$

Hence

$$\begin{aligned} \langle|\mathcal{M}|^2\rangle &= \frac{e^4}{4}\left(\frac{-8su}{s^2} + \frac{-8su}{u^2}\right) \\ &= 2e^4\left(-\frac{u}{s} - \frac{s}{u}\right) \end{aligned}$$

The Mandelstam variables for $m = 0$ are

$$\begin{aligned} s &= 4\omega^2 \\ u &= -2\omega^2(\cos\theta + 1) \end{aligned}$$

Hence

$$\langle|\mathcal{M}|^2\rangle = 2e^4\left(\frac{\cos\theta + 1}{2} + \frac{2}{\cos\theta + 1}\right)$$

Data from a CERN LEP experiment

See "Compton Scattering of Quasi-Real Virtual Photons at LEP," arxiv.org/abs/hep-ex/0504012.

x	y
-0.74	13380
-0.60	7720
-0.47	6360
-0.34	4600
-0.20	4310
-0.07	3700
0.06	3640
0.20	3340
0.33	3500
0.46	3010
0.60	3310
0.73	3330

The data are for the center of mass frame and have the following relationship with the differential cross section formula.

$$x = \cos\theta, \quad y = \frac{d\sigma}{d\cos\theta} = 2\pi\frac{d\sigma}{d\Omega}$$

For the high energy approximation we have

$$\langle|\mathcal{M}|^2\rangle = 2e^4\left(\frac{\cos\theta+1}{2}+\frac{2}{\cos\theta+1}\right)$$

The corresponding cross section formula is

$$\frac{d\sigma}{d\Omega}=\frac{\langle|\mathcal{M}|^2\rangle}{64\pi^2 s}=\frac{e^4}{32\pi^2 s}\left(\frac{\cos\theta+1}{2}+\frac{2}{\cos\theta+1}\right),\qquad s\gg m$$

Substituting $e^4=16\pi^2\alpha^2$ yields

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2}{2s}\left(\frac{\cos\theta+1}{2}+\frac{2}{\cos\theta+1}\right)$$

Multiply by 2π to obtain

$$\frac{d\sigma}{d\cos\theta}=\frac{\pi\alpha^2}{s}\left(\frac{\cos\theta+1}{2}+\frac{2}{\cos\theta+1}\right)$$

To compute predicted values $\hat{y}$ from the above formula, multiply by $(hc)^2$ to convert to SI and multiply by 10^{40} to convert square meters to picobarns.

$$\hat{y}=\frac{\pi\alpha^2}{s}\left(\frac{x+1}{2}+\frac{2}{x+1}\right)\times(hc)^2\times 10^{40}$$

The following table shows $\hat{y}$ for $s=(40\,\mathrm{GeV})^2$.

x	y	$\hat{y}$
−0.74	13380	12573
−0.60	7720	8358
−0.47	6360	6491
−0.34	4600	5401
−0.20	4310	4661
−0.07	3700	4204
0.06	3640	3884
0.20	3340	3643
0.33	3500	3486
0.46	3010	3375
0.60	3310	3295
0.73	3330	3248

The coefficient of determination R^2 measures how well predicted values fit the data.

$$R^2=1-\frac{\sum(y-\hat{y})^2}{\sum(y-\bar{y})^2}=0.97$$

The result indicates that the model $d\sigma$ explains 97% of the variance in the data.