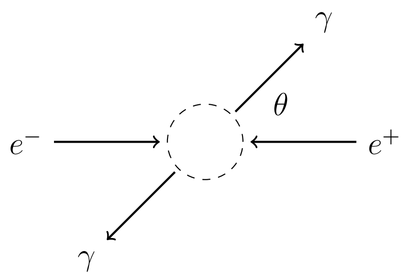
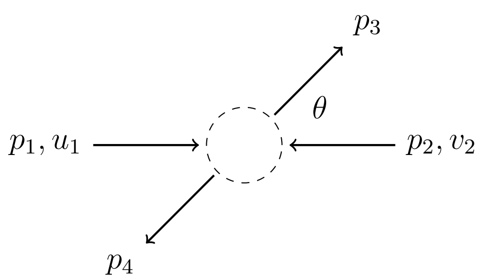
ELECTRON POSITRON ANNIHILATION

Electron positron annihilation creates two photons.



Here is the same diagram with momentum and spinor labels.



In a typical collider experiment the momentum vectors are

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \qquad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} \qquad p_3 = \begin{pmatrix} E \\ E\sin\theta\cos\phi \\ E\sin\theta\sin\phi \\ E\cos\theta \end{pmatrix} \qquad p_4 = \begin{pmatrix} E \\ -E\sin\theta\cos\phi \\ -E\sin\theta\sin\phi \\ -E\cos\theta \end{pmatrix}$$

where $E = \sqrt{p^2 + m^2}$.

The spinors are

$$u_{11} = \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix} \qquad u_{12} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix} \qquad v_{21} = \begin{pmatrix} -p \\ 0 \\ E+m \\ 0 \end{pmatrix} \qquad v_{22} = \begin{pmatrix} 0 \\ p \\ 0 \\ E+m \end{pmatrix}$$

The last digit in a spinor subscript is 1 for spin up and 2 for spin down. Note that the spinors are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^2$ will be used.

This is the probability density for annihilation. The formula is from Feynman diagrams.

$$|\mathcal{M}(s_1, s_2)|^2 = \frac{e^4}{N} \left| -\frac{\bar{v}_2 \gamma^\mu (\not{q}_1 + m) \gamma^\nu u_1}{t - m^2} - \frac{\bar{v}_2 \gamma^\nu (\not{q}_2 + m) \gamma^\mu u_1}{u - m^2} \right|^2$$

Symbol s_j selects the spin (up or down) of spinor j. Symbol e is electron charge. Symbol $q_1 = p_1 - p_3$ and $q_2 = p_1 - p_4$. Symbols t and u are Mandelstam variables $t = q_1^2 = (p_1 - p_3)^2$ and $u = q_2^2 = (p_1 - p_4)^2$.

Let

$$a_1 = \bar{v}_2\gamma^\mu(\not{q}_1 + m)\gamma^\nu u_1 \qquad a_2 = \bar{v}_2\gamma^\nu(\not{q}_2 + m)\gamma^\mu u_1$$

Then

$$\begin{aligned}
|\mathcal{M}(s_1,s_2)|^2 &= \frac{e^4}{N}\left|-\frac{a_1}{t-m^2} - \frac{a_2}{u-m^2}\right|^2 \\
&= \frac{e^4}{N}\left(-\frac{a_1}{t-m^2} - \frac{a_2}{u-m^2}\right)\left(-\frac{a_1}{t-m^2} - \frac{a_2}{u-m^2}\right)^* \\
&= \frac{e^4}{N}\left(\frac{a_1a_1^*}{(t-m^2)^2} + \frac{a_1a_2^*}{(t-m^2)(u-m^2)} + \frac{a_1^*a_2}{(t-m^2)(u-m^2)} + \frac{a_2a_2^*}{(u-m^2)^2}\right)
\end{aligned}$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is computed by summing $|\mathcal{M}|^2$ over all spin and polarization states and then dividing by the number of inbound states. There are four inbound states. The sum over polarization states is already accomplished by contraction of aa^* over μ and ν.

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{s_1=1}^{2}\sum_{s_2=1}^{2}|\mathcal{M}(s_1,s_2)|^2 \\
&= \frac{e^4}{4}\sum_{s_1=1}^{2}\sum_{s_2=1}^{2}\frac{1}{N}\left(\frac{a_1a_1^*}{(t-m^2)^2} + \frac{a_1a_2^*}{(t-m^2)(u-m^2)} + \frac{a_1^*a_2}{(t-m^2)(u-m^2)} + \frac{a_2a_2^*}{(u-m^2)^2}\right)
\end{aligned}$$

Use the Casimir trick to replace sums over spins with matrix products.

$$\begin{aligned}
f_{11} &= \frac{1}{N}\sum_{\text{spins}} a_1a_1^* = \operatorname{Tr}\Big((\not{p}_1 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\Big) \\
f_{12} &= \frac{1}{N}\sum_{\text{spins}} a_1a_2^* = \operatorname{Tr}\Big((\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\Big) \\
f_{22} &= \frac{1}{N}\sum_{\text{spins}} a_2a_2^* = \operatorname{Tr}\Big((\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_2 + m)\gamma_\mu\Big)
\end{aligned}$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left(\frac{f_{11}}{(t-m^2)^2} + \frac{f_{12}}{(t-m^2)(u-m^2)} + \frac{f_{12}^*}{(t-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2}\right)$$

Run "annihilation-1.txt" to verify the Casimir trick for electron positron annihilation.

The following momentum formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$\begin{aligned}
f_{11} &= 16(p_1\cdot p_1)(p_1\cdot p_2) - 32(p_1\cdot p_1)(p_2\cdot p_3) - 16(p_1\cdot p_2)(p_3\cdot p_3) + 32(p_1\cdot p_3)(p_2\cdot p_3) \\
&\quad - 48m^2(p_1\cdot p_2) + 64m^2(p_1\cdot p_3) + 64m^2(p_2\cdot p_3) - 64m^2(p_3\cdot p_3) - 64m^4 \\
f_{12} &= -32(p_1\cdot p_1)(p_1\cdot p_2) + 32(p_1\cdot p_2)(p_1\cdot p_3) + 32(p_1\cdot p_2)(p_1\cdot p_4) - 32(p_1\cdot p_2)(p_3\cdot p_4) \\
&\quad - 48m^2(p_1\cdot p_1) + 48m^2(p_1\cdot p_2) + 32m^2(p_1\cdot p_3) + 32m^2(p_1\cdot p_4) \\
&\quad - 16m^2(p_2\cdot p_3) - 16m^2(p_2\cdot p_4) - 16m^2(p_3\cdot p_4) + 32m^4 \\
f_{22} &= 16(p_1\cdot p_1)(p_1\cdot p_2) - 32(p_1\cdot p_1)(p_2\cdot p_4) - 16(p_1\cdot p_2)(p_4\cdot p_4) + 32(p_1\cdot p_4)(p_2\cdot p_4) \\
&\quad - 48m^2(p_1\cdot p_2) + 64m^2(p_1\cdot p_4) + 64m^2(p_2\cdot p_4) - 64m^2(p_4\cdot p_4) - 64m^4
\end{aligned}$$

In Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$ the formulas are

$$
\begin{aligned}
f_{11} &= 8tu - 24tm^2 - 8um^2 - 8m^4 \\
f_{12} &= 8sm^2 - 32m^4 \\
f_{22} &= 8tu - 8tm^2 - 24um^2 - 8m^4
\end{aligned}
$$

When $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$
\begin{aligned}
f_{11} &= 8tu \\
f_{12} &= 0 \\
f_{22} &= 8tu
\end{aligned}
$$

For $m = 0$ the Mandelstam variables are

$$
\begin{aligned}
s &= 4E^2 \\
t &= -2E^2(1 - \cos\theta) = -4E^2 \sin^2(\theta/2) \\
u &= -2E^2(1 + \cos\theta) = -4E^2 \cos^2(\theta/2)
\end{aligned}
$$

The corresponding expected probability density is

$$
\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4}\left(\frac{8tu}{t^2} + \frac{8tu}{u^2}\right) \\
&= 2e^4\left(\frac{u}{t} + \frac{t}{u}\right) \\
&= 2e^4\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)
\end{aligned}
$$

Recall that $e^2 = 4\pi\alpha$ hence

$$
\langle |\mathcal{M}|^2 \rangle = 32\pi^2\alpha^2\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)
$$

The resulting differential cross section is

$$
\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{64\pi^2 s} = \frac{\alpha^2}{8E^2}\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)
$$

Run “annihilation-2.txt” to verify.

We can integrate $d\sigma$ to obtain a cumulative distribution function.

Let

$$
I(\xi) = 2\pi \int_\alpha^\xi \frac{d\sigma}{d\Omega} \sin\theta \, d\theta, \quad \alpha \le \xi \le \pi - \alpha
$$

for some $\alpha > 0$. The support region is restricted because $d\sigma$ is undefined at $\theta = 0$ and $\theta = \pi$.

The cumulative distribution function is

$$
F(\theta) = \frac{I(\theta)}{I(\pi)}, \quad \alpha \le \theta \le \pi - \alpha
$$

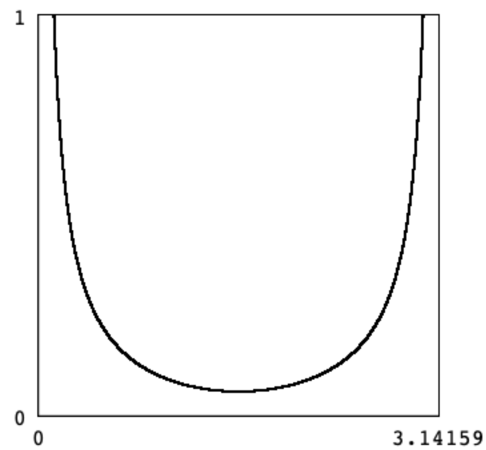
Hence

$$P(\theta_1 \leq \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{\sin(\theta)}{I(\pi)} \left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta} \right), \quad \alpha \leq \theta \leq \pi - \alpha$$

Run "annihilation-4.txt" to plot $f(\theta)$ for $\alpha = \pi/180$.



Here is a probability distribution for 20° bins with $\alpha = 20°$.

θ_1	θ_2	$P(\theta_1 \leq \theta \leq \theta_2)$
0°	20°	–
20°	40°	0.25
40°	60°	0.13
60°	80°	0.08
80°	100°	0.07
100°	120°	0.08
120°	140°	0.13
140°	160°	0.25
160°	180°	–

The following table shows DESY-PETRA electron positron annihilation data.[1]

x	y
0.0502	0.09983
0.1505	0.10791
0.2509	0.12026
0.3512	0.13002
0.4516	0.17681
0.5521	0.1957
0.6526	0.279
0.7312	0.33204

Data x and y have the following relationship with the differential cross section formula.

$$x = \cos\theta \qquad y = \frac{d\sigma}{d\Omega}$$

To compute predicted values $\hat{y}$ from the cross section formula, use $\sqrt{s} = 2E = 14.0\,\text{GeV}$. Multiply by $(\hbar c)^2$ to convert to SI and multiply by 10^{37} to convert square meters to nanobarns.

$$\hat{y} = \frac{\alpha^2}{2(14.0)^2}\left(\frac{1+x}{1-x} + \frac{1-x}{1+x}\right) \times (\hbar c)^2 \times 10^{37}$$

The following table shows predicted values $\hat{y}$ based on the above formula.

x	y	$\hat{y}$
0.0502	0.09983	0.106325
0.1505	0.10791	0.110694
0.2509	0.12026	0.120005
0.3512	0.13002	0.135559
0.4516	0.17681	0.159996
0.5521	0.1957	0.198562
0.6526	0.279	0.262745
0.7312	0.33204	0.348884

The coefficient of determination R^2 measures how well predicted values fit the real data.

$$R^2 = 1 - \frac{\sum(y-\hat{y})^2}{\sum(y-\bar{y})^2} = 0.98$$

The result indicates that the model $d\sigma$ explains 98% of the variance in the data.

Run "annihilation-3.txt" to verify.

[1] `www.hepdata.net/record/ins191231` (Table 2, 14.0 GeV)

Here are some notes on how the scripts work.

To convert a_1 and a_2 to Eigenmath code, it is instructive to write a_1 and a_2 in full component form.

$$a_1^{\mu\nu} = \bar{v}_{2\alpha}\gamma^{\mu\alpha}{}_\beta(\not{q}_1 + m)^\beta{}_\rho\gamma^{\nu\rho}{}_\sigma u_1^\sigma \qquad a_2^{\nu\mu} = \bar{v}_{2\alpha}\gamma^{\nu\alpha}{}_\beta(\not{q}_2 + m)^\beta{}_\rho\gamma^{\mu\rho}{}_\sigma u_1^\sigma$$

Transpose the γ tensors to form inner products over α and ρ.

$$a_1^{\mu\nu} = \bar{v}_{2\alpha}\gamma^{\alpha\mu}{}_\beta(\not{q}_1 + m)^\beta{}_\rho\gamma^{\rho\nu}{}_\sigma u_1^\sigma \qquad a_2^{\nu\mu} = \bar{v}_{2\alpha}\gamma^{\alpha\nu}{}_\beta(\not{q}_2 + m)^\beta{}_\rho\gamma^{\rho\mu}{}_\sigma u_1^\sigma$$

Convert transposed γ to Eigenmath code.

$$\gamma^{\alpha\mu}{}_\beta \quad \rightarrow \quad \texttt{gammaT = transpose(gamma)}$$

Then to compute a_1 we have

$$a_1 = \bar{v}_{2\alpha}\gamma^{\alpha\mu}{}_\beta(\not{q}_1 + m)^\beta{}_\rho\gamma^{\rho\nu}{}_\sigma u_1^\sigma$$
$$\rightarrow \quad \texttt{a1 = dot(v2bar[s2],gammaT,qslash1 + m I,gammaT,u1[s1])}$$

where s_1 and s_2 are spin indices. Similarly for a_2 we have

$$a_2 = \bar{v}_{2\alpha}\gamma^{\alpha\mu}{}_\beta(\not{q}_2 + m)^\beta{}_\rho\gamma^{\rho\nu}{}_\sigma u_1^\sigma$$
$$\rightarrow \quad \texttt{a2 = dot(v2bar[s2],gammaT,qslash2 + m I,gammaT,u1[s1])}$$

In component notation the product $a_1 a_1^*$ is

$$a_1 a_1^* = a_1^{\mu\nu} a_1^{*\mu\nu}$$

To sum over μ and ν it is necessary to lower indices with the metric tensor. Also, transpose a_1^* to form an inner product with ν.

$$a_1 a_1^* = a_1^{\mu\nu} a_{1\nu\mu}^*$$

Convert to Eigenmath code. The dot function sums over ν and the contract function sums over μ.

$$a_1 a_1^* \quad \rightarrow \quad \texttt{a11 = contract(dot(a1,gmunu,transpose(conj(a1)),gmunu))}$$

Similarly for $a_2 a_2^*$ we have

$$a_2 a_2^* \quad \rightarrow \quad \texttt{a22 = contract(dot(a2,gmunu,transpose(conj(a2)),gmunu))}$$

The product $a_1 a_2^*$ does not require a transpose because $a_2 = a_2^{\nu\mu}$.

$$a_1^{\mu\nu} a_{2\nu\mu}^* \quad \rightarrow \quad \texttt{a12 = contract(dot(a1,gmunu,conj(a2),gmunu))}$$

In component notation, a trace operator becomes a sum over an index, in this case α.

$$
\begin{aligned}
f_{11} &= \operatorname{Tr}\Big((\not{p}_1 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\Big) \\
&= (\not{p}_1 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{q}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_2 - m)^\tau{}_\delta \gamma_\nu{}^\delta{}_\eta (\not{q}_1 + m)^\eta{}_\xi \gamma_\mu{}^\xi{}_\alpha
\end{aligned}
$$

As before, transpose γ tensors to form inner products.

$$
f_{11} = (\not{p}_1 + m)^\alpha{}_\beta \gamma^{\beta\mu}{}_\rho (\not{q}_1 + m)^\rho{}_\sigma \gamma^{\sigma\nu}{}_\tau (\not{p}_2 - m)^\tau{}_\delta \gamma^\delta{}_{\nu\eta} (\not{q}_1 + m)^\eta{}_\xi \gamma^\xi{}_{\mu\alpha}
$$

This is the code for transposing γ.

$\gamma^{\beta\mu}{}_\beta \quad \rightarrow$
```
gammaT = transpose(gamma)
```
$\gamma^\delta{}_{\nu\eta} \quad \rightarrow$
```
gammaL = transpose(dot(gmunu,gamma))
```

To convert f_{11} to Eigenmath code, use an intermediate variable T for the inner product.

$T^{\alpha\mu\nu}{}_{\nu\mu\alpha} \quad \rightarrow$
```
T = dot(P1,gammaT,Q1,gammaT,P2,gammaL,Q1,gammaL)
```

Now sum over the indices of T. The innermost contract sums over ν then the next contract sums over μ. Finally the outermost contract sums over α.

$f_{11} \quad \rightarrow$
```
f11 = contract(contract(contract(T,3,4),2,3))
```

Follow suit for f_{22}. For f_{12} the order of the rightmost μ and ν is reversed.

$$
f_{12} = \operatorname{Tr}\Big((\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\Big)
$$

The resulting inner product is $T^{\alpha\mu\nu}{}_{\mu\nu\alpha}$ so the contraction is different.

$f_{12} \quad \rightarrow$
```
f12 = contract(contract(contract(T,3,5),2,3))
```

The innermost contract sums over ν followed by sum over μ then sum over α.