Cold emission

Consider the following potential energy function where Q is a positive constant.

$$V(x) = \begin{cases} 0, & x < 0 \\ V_0 - Qx, & x \geq 0 \end{cases}$$

Suppose a particle of mass m and energy $E < V_0$ is traveling from left to right along the x axis. The particle is in a potential energy barrier for

$$E \leq V_0 - Qx$$

Solving for x we have the particle in the barrier for

$$x \leq \frac{V_0 - E}{Q}$$

Hence the particle has a Schrodinger equation for each the following regions.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_1 = E\psi_1, \qquad\qquad x < 0$$
$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_2 + (V_0 - Qx)\psi_2 = E\psi_2, \qquad\qquad 0 \leq x \leq \frac{V_0 - E}{Q}$$
$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_3 = E\psi_3, \qquad\qquad x > \frac{V_0 - E}{Q}$$

Let ψ_1 and ψ_3 have the most general free-particle solutions.

$$\psi_1(x) = A\exp\left(i\sqrt{\frac{2mE}{\hbar^2}}x\right) + B\exp\left(-i\sqrt{\frac{2mE}{\hbar^2}}x\right)$$

$$\psi_3(x) = F\exp\left(i\sqrt{\frac{2mE}{\hbar^2}}x\right) + G\exp\left(-i\sqrt{\frac{2mE}{\hbar^2}}x\right)$$

Let $W = V_0 - E$ and use the WKB approximation to solve for ψ_2.

$$\psi_2(x) = C\exp\left(-\frac{1}{\hbar}\int\sqrt{2m(W-Qx)}\,dx\right) + D\exp\left(\frac{1}{\hbar}\int\sqrt{2m(W-Qx)}\,dx\right)$$
$$= C\exp\left(\frac{(2m(W-Qx))^{\frac{3}{2}}}{3Qm\hbar}\right) + D\exp\left(-\frac{(2m(W-Qx))^{\frac{3}{2}}}{3Qm\hbar}\right)$$

To simplify the formulas let

$$k = \frac{\sqrt{2mE}}{\hbar}, \quad \beta(x) = \frac{(2m(W-Qx))^{\frac{3}{2}}}{3Qm\hbar}$$

and write

$$\psi_1(x) = A \exp(ikx) + B \exp(-ikx)$$
$$\psi_2(x) = C \exp(\beta(x)) + D \exp(-\beta(x))$$
$$\psi_3(x) = F \exp(ikx) + G \exp(-ikx)$$

Exponentials of $-i$ represent particles moving from right to left. The B exponential represents a particle reflected from the boundary at $x = 0$. There is no reflection for $x > (V_0 - E)/Q$ hence $G = 0$.

Let us now solve for the coefficients using boundary conditions. Let $L = (V_0 - E)/Q$. Four boundary conditions are needed to ensure continuity at $x = 0$ and $x = L$.

$$\psi_1(0) = \psi_2(0)$$
$$\psi_1'(0) = \psi_2'(0)$$
$$\psi_2(L) = \psi_3(L)$$
$$\psi_2'(L) = \psi_3'(L)$$

From the boundary condition $\psi_2(L) = \psi_3(L)$ we have

$$C \exp(\beta(L)) + D \exp(-\beta(L)) = F \exp(ikL) \tag{1}$$

From the boundary condition $\psi_2'(L) = \psi_3'(L)$ we have

$$\beta'(L)C \exp(\beta(L)) - \beta'(L)D \exp(-\beta(L)) = ikF \exp(ikL) \tag{2}$$

Add $\beta'(L)$ times (1) to (2) to obtain

$$2\beta'(L)C \exp(\beta(L)) = (\beta'(L) + ik)F \exp(ikL)$$

Hence

$$C = \frac{(\beta'(L) + ik)F \exp(ikL - \beta(L))}{2\beta'(L)} \tag{3}$$