

Rutherford scattering 2

Find the cross section for Rutherford scattering with a screened Coulomb potential

$$V(r) = -\frac{Ze^2}{r} \exp\left(-\frac{r}{a}\right)$$

Start with

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2\epsilon_0^2} \left(\frac{m}{2\pi\hbar^2}\right)^2 |Q|^2, \quad Q = \int \exp\left(\frac{i\mathbf{p} \cdot \mathbf{r}}{\hbar}\right) V(\mathbf{r}) d^3\mathbf{r}$$

Convert Q to polar coordinates.

$$Q = \int_0^{2\pi} \int_0^\pi \int_0^\infty \exp\left(\frac{ipr \cos \theta}{\hbar}\right) V(r, \theta, \phi) r^2 \sin \theta dr d\theta d\phi$$

Substitute the screened Coulomb potential for $V(r, \theta, \phi)$ and note r^2 becomes r .

$$Q = -Ze^2 \int_0^{2\pi} \int_0^\pi \int_0^\infty \exp\left(\frac{ipr \cos \theta}{\hbar}\right) \exp\left(-\frac{r}{a}\right) r \sin \theta dr d\theta d\phi$$

Integrate over ϕ .

$$Q = -2\pi Ze^2 \int_0^\pi \int_0^\infty \exp\left(\frac{ipr \cos \theta}{\hbar}\right) \exp\left(-\frac{r}{a}\right) r \sin \theta dr d\theta$$

Transform the integral over θ to an integral over y where $y = \cos \theta$, $dy = -\sin \theta d\theta$.

$$Q = -2\pi Ze^2 \int_{-1}^1 \int_0^\infty \exp\left(\frac{ipry}{\hbar}\right) \exp\left(-\frac{r}{a}\right) r dr dy$$

Solve the integral over y (note r in the integrand cancels).

$$Q = -2\pi Ze^2 \int_0^\infty \frac{\hbar}{ip} \left[\exp\left(\frac{ipr}{\hbar}\right) - \exp\left(-\frac{ipr}{\hbar}\right) \right] \exp\left(-\frac{r}{a}\right) dr$$

Solve the integral over r .

$$Q = -2\pi Ze^2 \frac{\hbar}{ip} \left[\frac{1}{ip/\hbar - 1/a} \exp\left(\frac{ipr}{\hbar} - \frac{r}{a}\right) + \frac{1}{ip/\hbar + 1/a} \exp\left(-\frac{ipr}{\hbar} - \frac{r}{a}\right) \right]_0^\infty$$

Evaluate the limits.

$$Q = -2\pi Ze^2 \frac{\hbar}{ip} \left[-\frac{1}{ip/\hbar - 1/a} - \frac{1}{ip/\hbar + 1/a} \right] = -\frac{4\pi Ze^2}{(p/\hbar)^2 + (1/a)^2} \quad (1)$$

The cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2\epsilon_0^2} \left(\frac{m}{2\pi\hbar^2}\right)^2 |Q|^2 = \frac{1}{64\pi^2\epsilon_0^2} \frac{4m^2 Z^2 e^4}{[p^2 + (\hbar/a)^2]^2}$$

Substitute $(4\pi\epsilon_0\alpha\hbar c)^2$ for e^4 .

$$\frac{d\sigma}{d\Omega} = \frac{m^2 Z^2 \alpha^2 (\hbar c)^2}{[p^2 + (\hbar/a)^2]^2}$$

Symbol p is momentum transfer $|\mathbf{p}_i| - |\mathbf{p}_f|$ such that

$$p^2 = 2mE(1 - \cos \theta)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{m^2 Z^2 \alpha^2 (\hbar c)^2}{[2mE(1 - \cos \theta) + (\hbar/a)^2]^2}$$

Cancel m^2 in the numerator.

$$\frac{d\sigma}{d\Omega} = \frac{Z^2 \alpha^2 (\hbar c)^2}{[2E(1 - \cos \theta) + \frac{1}{m}(\hbar/a)^2]^2} \quad (2)$$

Let $a \rightarrow \infty$ to obtain the ordinary Rutherford cross section.