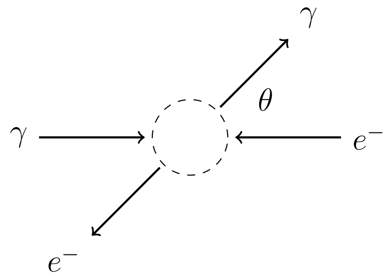
Compton scattering

Compton scattering is the interaction $e^- + \gamma \rightarrow e^- + \gamma$.



In the center-of-mass frame we have the following momentum vectors where $E = \sqrt{\omega^2 + m^2}$.

$$p_1 = \underset{\substack{\text{inbound}\\\text{photon}}}{\begin{pmatrix}\omega\\0\\0\\\omega\end{pmatrix}} \qquad p_2 = \underset{\substack{\text{inbound}\\\text{electron}}}{\begin{pmatrix}E\\0\\0\\-\omega\end{pmatrix}} \qquad p_3 = \underset{\substack{\text{outbound}\\\text{photon}}}{\begin{pmatrix}\omega\\\omega\sin\theta\cos\phi\\\omega\sin\theta\sin\phi\\\omega\cos\theta\end{pmatrix}} \qquad p_4 = \underset{\substack{\text{outbound}\\\text{electron}}}{\begin{pmatrix}E\\-\omega\sin\theta\cos\phi\\-\omega\sin\theta\sin\phi\\-\omega\cos\theta\end{pmatrix}}$$

Spinors for the inbound electron.

$$u_{21} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound electron}\\\text{spin up}}}{\begin{pmatrix}E+m\\0\\-\omega\\0\end{pmatrix}} \qquad u_{22} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound electron}\\\text{spin down}}}{\begin{pmatrix}0\\E+m\\0\\\omega\end{pmatrix}}$$

Spinors for the outbound electron.

$$u_{41} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{outbound electron}\\\text{spin up}}}{\begin{pmatrix}E+m\\0\\p_{4z}\\p_{4x}+ip_{4y}\end{pmatrix}} \qquad u_{42} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{outbound electron}\\\text{spin down}}}{\begin{pmatrix}0\\E+m\\p_{4x}-ip_{4y}\\-p_{4z}\end{pmatrix}}$$

The scattering amplitude ${\mathcal{M}_{ab}}^{\mu\nu}$ for spin ab and polarization $\mu\nu$ is

$${\mathcal{M}_{ba}}^{\mu\nu} = {\mathcal{M}_{1ab}}^{\mu\nu} + {\mathcal{M}_{2ab}}^{\nu\mu}$$

where

$${\mathcal{M}_{1ab}}^{\mu\nu} = \frac{\bar{u}_{4b}(-ie\gamma^\mu)(\not{q}_1 + m)(-ie\gamma^\nu)u_{2a}}{s - m^2}$$

$${\mathcal{M}_{2ab}}^{\nu\mu} = \frac{\bar{u}_{4b}(-ie\gamma^\nu)(\not{q}_2 + m)(-ie\gamma^\mu)u_{2a}}{u - m^2}$$

Matrices $\not{q}_1$ and $\not{q}_2$ represent momentum transfer.

$$\not{q}_1 = (p_1 + p_2)^\alpha g_{\alpha\beta}\gamma^\beta$$
$$\not{q}_2 = (p_4 - p_1)^\alpha g_{\alpha\beta}\gamma^\beta$$

Scalars s and u are Mandelstam variables.

$$s = (p_1 + p_2)^2$$
$$u = (p_1 - p_4)^2$$

In component form (note that indices μ and ν are interchanged for $\mathcal{M}_{2ab}$)

$$\mathcal{M}_{1ab}{}^{\mu\nu} = \frac{(\bar{u}_{4b})_\alpha(-ie\gamma^{\mu\alpha}{}_\beta)(\not{q}_1 + m)^\beta{}_\rho(-ie\gamma^{\nu\rho}{}_\sigma)(u_{2a})^\sigma}{s - m^2}$$
$$\mathcal{M}_{2ab}{}^{\nu\nu} = \frac{(\bar{u}_{4b})_\alpha(-ie\gamma^{\nu\alpha}{}_\beta)(\not{q}_2 + m)^\beta{}_\rho(-ie\gamma^{\mu\rho}{}_\sigma)(u_{2a})^\sigma}{u - m^2}$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is the average over spin and polarization states.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{a,b}\sum_{\mu,\nu}\left|\mathcal{M}_{ab}{}^{\mu\nu}\right|^2$$

Summing over μ and ν requires $g_{\mu\nu}$ to lower indices.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{a,b}\mathcal{M}_{ab}{}^{\mu\nu}\left(g_{\mu\alpha}\mathcal{M}_{ab}{}^{\alpha\beta}g_{\beta\nu}\right)^*$$

Substitute $\mathcal{M}_{1ab} + \mathcal{M}_{2ab}$ for $\mathcal{M}_{ab}$. (Note that $P_{12} = P_{21}^*$ hence by the property that probabilities are real we have $P_{12} = P_{21}$.)

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{a,b}\Big[\underbrace{\mathcal{M}_{1ab}{}^{\mu\nu}\left(g_{\mu\alpha}\mathcal{M}_{1ab}{}^{\alpha\beta}g_{\beta\nu}\right)^*}_{P_{11ab}} + \underbrace{\mathcal{M}_{1ab}{}^{\mu\nu}\left(g_{\nu\alpha}\mathcal{M}_{2ab}{}^{\alpha\beta}g_{\beta\mu}\right)^*}_{P_{12ab}}$$
$$+ \underbrace{\mathcal{M}_{2ab}{}^{\nu\mu}\left(g_{\mu\alpha}\mathcal{M}_{1ab}{}^{\alpha\beta}g_{\beta\nu}\right)^*}_{P_{21ab}} + \underbrace{\mathcal{M}_{2ab}{}^{\nu\mu}\left(g_{\nu\alpha}\mathcal{M}_{2ab}{}^{\alpha\beta}g_{\beta\mu}\right)^*}_{P_{22ab}}\Big]$$

The Casimir trick uses matrix arithmetic to sum over spin and polarization states:

$$\sum_{a,b}P_{11ab} = \frac{e^4}{(s - m^2)^2}\operatorname{Tr}\left[(\not{p}_2 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\right]$$
$$\sum_{a,b}P_{12ab} = \frac{e^4}{(s - m^2)(u - m^2)}\operatorname{Tr}\left[(\not{p}_2 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\right]$$
$$\sum_{a,b}P_{22ab} = \frac{e^4}{(u - m^2)^2}\operatorname{Tr}\left[(\not{p}_2 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\nu(\not{q}_2 + m)\gamma_\mu\right]$$

Let

$$
\begin{aligned}
f_{11} &= \mathrm{Tr}\left[(\not{p}_2 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\right] \\
f_{12} &= \mathrm{Tr}\left[(\not{p}_2 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\right] \\
f_{22} &= \mathrm{Tr}\left[(\not{p}_2 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\nu(\not{q}_2 + m)\gamma_\mu\right]
\end{aligned}
$$

so that

$$
\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left[\frac{f_{11}}{(s-m^2)^2} + \frac{2f_{12}}{(s-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2}\right]
$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$
\begin{aligned}
f_{11} &= 32(p_1 \cdot p_2)(p_1 \cdot p_4) + 32(p_1 \cdot p_2)m^2 + 32m^4 \\
f_{12} &= 16(p_1 \cdot p_2)m^2 - 16(p_1 \cdot p_4)m^2 + 32m^4 \\
f_{22} &= 32(p_1 \cdot p_2)(p_1 \cdot p_4) - 32(p_1 \cdot p_4)m^2 + 32m^4
\end{aligned}
$$

In Mandelstam variables

$$
\begin{aligned}
f_{11} &= -8su + 24sm^2 + 8um^2 + 8m^4 \\
f_{12} &= 8sm^2 + 8um^2 + 16m^4 \\
f_{22} &= -8su + 8sm^2 + 24um^2 + 8m^4
\end{aligned}
\tag{2}
$$

Compton scattering experiments are typically done in the lab frame where the electron is at rest. Define Lorentz boost Λ for transforming momentum vectors to the lab frame.

$$
\Lambda = \begin{pmatrix} E/m & 0 & 0 & \omega/m \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega/m & 0 & 0 & E/m \end{pmatrix}
$$

The electron is at rest in the lab frame.

$$
\Lambda p_2 = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

Mandelstam variables are invariant under a boost.

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = (\Lambda p_1 + \Lambda p_2)^2 \\
t &= (p_1 - p_3)^2 = (\Lambda p_1 - \Lambda p_3)^2 \\
u &= (p_1 - p_4)^2 = (\Lambda p_1 - \Lambda p_4)^2
\end{aligned}
$$

In the lab frame, let ω_L be the angular frequency of the incident photon and let ω'_L be the angular frequency of the scattered photon.

$$\omega_L = \Lambda p_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2}{m} + \frac{\omega E}{m}$$

$$\omega'_L = \Lambda p_3 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2 \cos\theta}{m} + \frac{\omega E}{m}$$

It can be shown that

$$\begin{aligned} s &= m^2 + 2m\omega_L \\ t &= 2m(\omega'_L - \omega_L) \\ u &= m^2 - 2m\omega'_L \end{aligned} \tag{3}$$

Then by (1), (2), and (3) we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left[\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \left(\frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1 \right)^2 - 1 \right]$$

Lab scattering angle θ_L is given by the Compton equation

$$\cos\theta_L = \frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1$$

Hence

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \cos^2\theta_L - 1 \right) \\ &= 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} - \sin^2\theta_L \right) \end{aligned}$$

Cross section

Now that we have derived $\langle |\mathcal{M}|^2 \rangle$ we can investigate the angular distribution of scattered photons. For simplicity let us drop the L subscript from lab variables. From now on the symbols ω, ω', and θ will be lab frame variables.

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4(4\pi\varepsilon_0)^2 s} \left(\frac{\omega'}{\omega} \right)^2 \langle |\mathcal{M}|^2 \rangle$$

where

$$s = m^2 + 2m\omega = (mc^2)^2 + 2(mc^2)(\hbar\omega)$$

and ω' is given by the Compton equation

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos\theta)}$$

For the lab frame we have

$$\langle|\mathcal{M}|^2\rangle = 2e^4 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right)$$

Hence in the lab frame

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\varepsilon_0)^2 s} \left(\frac{\omega'}{\omega}\right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right)$$

Noting that

$$e^2 = 4\pi\varepsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s} \left(\frac{\omega'}{\omega}\right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right)$$

Noting that

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

we also have

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{2s} \left(\frac{\omega'}{\omega}\right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right) \sin\theta\, d\theta\, d\phi$$

Let $S(\theta_1, \theta_2)$ be the following integral of $d\sigma$.

$$S(\theta_1, \theta_2) = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{\pi\alpha^2(\hbar c)^2}{s} [I(\theta_2) - I(\theta_1)]$$

where

$$I(\theta) = -\frac{\cos\theta}{R^2} + \log\big(1 + R(1 - \cos\theta)\big) \left(\frac{1}{R} - \frac{2}{R^2} - \frac{2}{R^3}\right) - \frac{1}{2R\big(1 + R(1 - \cos\theta)\big)^2} + \frac{1}{1 + R(1 - \cos\theta)} \left(-\frac{2}{R^2} - \frac{1}{R^3}\right)$$

and

$$R = \frac{\hbar\omega}{mc^2}$$

The cumulative distribution function is

$$F(\theta) = \frac{S(0, \theta)}{S(0, \pi)} = \frac{I(\theta) - I(0)}{I(\pi) - I(0)}, \quad 0 \le \theta \le \pi$$

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 < \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi) - I(0)} \left(\frac{\omega'}{\omega}\right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right) \sin\theta$$

Thomson scattering

For $\hbar\omega \ll mc^2$ we have

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos\theta)} \approx \omega$$

Hence we can use the approximations

$$\omega = \omega' \quad \text{and} \quad s = (mc^2)^2$$

to obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2\hbar^2}{2m^2c^2}\left(1 + \cos^2\theta\right)$$

which is the formula for Thomson scattering.

High energy approximation

For $\omega \gg m$ a useful approximation is to set $m = 0$ and obtain

$$\begin{aligned} f_{11} &= -8su \\ f_{12} &= 0 \\ f_{22} &= -8su \end{aligned}$$

Hence

$$\begin{aligned} \langle|\mathcal{M}|^2\rangle &= \frac{e^4}{4}\left(\frac{-8su}{s^2} + \frac{-8su}{u^2}\right) \\ &= 2e^4\left(-\frac{u}{s} - \frac{s}{u}\right) \end{aligned}$$

The Mandelstam variables for $m = 0$ are

$$\begin{aligned} s &= 4\omega^2 \\ u &= -2\omega^2(\cos\theta + 1) \end{aligned}$$

Hence

$$\langle|\mathcal{M}|^2\rangle = 2e^4\left(\frac{\cos\theta + 1}{2} + \frac{2}{\cos\theta + 1}\right)$$

In the center of mass frame

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{4s(4\pi\epsilon_o)^2} = \frac{e^4}{2s(4\pi\epsilon_o)^2}\left(\frac{\cos\theta + 1}{2} + \frac{2}{\cos\theta + 1}\right)$$

Substitute $e^4 = (4\pi\varepsilon_0 \alpha \hbar c)^2$ to obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s} \left(\frac{\cos\theta + 1}{2} + \frac{2}{\cos\theta + 1} \right) \times (\hbar c)^2$$

It follows that

$$\frac{d\sigma}{d\cos\theta} = 2\pi \frac{d\sigma}{d\Omega} = \frac{\pi\alpha^2}{s} \left(\frac{\cos\theta + 1}{2} + \frac{2}{\cos\theta + 1} \right) \times (\hbar c)^2$$

Cf. equation (1) of arxiv.org/pdf/hep-ex/0504012.