Line integral

There are two different kinds of line integrals, one for scalar fields and one for vector fields. The following table shows how both are based on the calculation of arc length.

	Abstract form	Computable form
Arc length	$\int_C ds$	$\int_a^b \lvert g'(t)\rvert\,dt$
Line integral, scalar field	$\int_C f\,ds$	$\int_a^b f(g(t))\,\lvert g'(t)\rvert\,dt$
Line integral, vector field	$\int_C (F\cdot u)\,ds$	$\int_a^b F(g(t))\cdot g'(t)\,dt$

For the vector field form, the symbol u is the unit tangent vector

$$u = \frac{g'(t)}{|g'(t)|}$$

The length of the tangent vector cancels with ds as follows.

$$\int_C (F\cdot u)\,ds = \int_a^b \left(F(g(t))\cdot\frac{g'(t)}{|g'(t)|}\right)\left(|g'(t)|\,dt\right) = \int_a^b F(g(t))\cdot g'(t)\,dt$$

Evaluate

$$\int_C x\,ds \quad \text{and} \quad \int_C x\,dx$$

where C is a straight line from $(0,0)$ to $(1,1)$.

What a difference the measure makes. The first integral is over a scalar field and the second is over a vector field. This can be understood when we recall that

$$ds = |g'(t)|\,dt$$

Hence for $\int_C x\,ds$ we have

```
x = t
y = t
g = (x,y)
defint(x abs(d(g,t)),t,0,1)
```

$$\frac{1}{2^{1/2}}$$

For $\int_C x\,dx$ we have

```
x = t
y = t
g = (x,y)
F = (x,0)
defint(dot(F,d(g,t)),t,0,1)
```

$\frac{1}{2}$

The following line integral problems are from *Advanced Calculus, Fifth Edition* by Wilfred Kaplan.

Evaluate $\int y^2\,dx$ along the straight line from $(0,0)$ to $(2,2)$.

```
x = 2t
y = 2t
g = (x,y)
F = (y^2,0)
defint(dot(F,d(g,t)),t,0,1)
```

$\frac{8}{3}$

Evaluate $\int z\,dx + x\,dy + y\,dz$ along the path $x = 2t + 1$, $y = t^2$, $z = 1 + t^3$, $0 \le t \le 1$.

```
x = 2t+1
y = t^2
z = 1+t^3
g = (x,y,z)
F = (z,x,y)
defint(dot(F,d(g,t)),t,0,1)
```

$\frac{163}{30}$