

# Probability current

Let  $\mathbf{J}$  be the probability current

$$\mathbf{J} = \frac{i\hbar}{2m} (\Psi \nabla \Psi^* - \Psi^* \nabla \Psi)$$

Show that

$$\nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t} |\Psi|^2$$

Recall that

$$\nabla \cdot \nabla = \nabla^2$$

By the product rule for divergence we have

$$\nabla \cdot (\Psi \nabla \Psi^*) = \nabla \Psi \cdot \nabla \Psi^* + \Psi \nabla^2 \Psi^*$$

Hence

$$\nabla \cdot \mathbf{J} = \frac{i\hbar}{2m} (\Psi \nabla^2 \Psi^* - \Psi^* \nabla^2 \Psi) \quad (1)$$

For the time derivative we have

$$\frac{\partial}{\partial t} |\Psi|^2 = \frac{\partial}{\partial t} (\Psi^* \Psi) = \Psi^* \frac{\partial}{\partial t} \Psi + \Psi \frac{\partial}{\partial t} \Psi^* \quad (2)$$

Recall the Schrödinger equation

$$\frac{\partial}{\partial t} \Psi = \frac{i\hbar}{2m} \nabla^2 \Psi - \frac{i}{\hbar} V \Psi$$

and

$$\frac{\partial}{\partial t} \Psi^* = \left( \frac{\partial}{\partial t} \Psi \right)^* = -\frac{i\hbar}{2m} \nabla^2 \Psi^* + \frac{i}{\hbar} V \Psi^*$$

Substitute the Schrödinger equation into (2) to obtain

$$\begin{aligned} \frac{\partial}{\partial t} |\Psi|^2 &= \Psi^* \left( \frac{i\hbar}{2m} \nabla^2 \Psi - \frac{i}{\hbar} V \Psi \right) + \Psi \left( -\frac{i\hbar}{2m} \nabla^2 \Psi^* + \frac{i}{\hbar} V \Psi^* \right) \\ &= \frac{i\hbar}{2m} (\Psi^* \nabla^2 \Psi - \Psi \nabla^2 \Psi^*) \end{aligned} \quad (3)$$

Hence by (1) and (3)

$$\nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t} |\Psi|^2 \quad (4)$$