Bell's theorem

Consider two observers A and B. Each observer has an apparatus for measuring spin. Each apparatus can be set in one of two orientations labeled 0 and 1. For independent random variables A and B we have

$$\langle A\rangle\langle B\rangle = \langle AB\rangle$$

Now consider all minimum and maximum expectation values along with a special formula.

$\langle A_0\rangle$	$\langle A_1\rangle$	$\langle B_0\rangle$	$\langle B_1\rangle$	$\langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle$
1	1	1	1	2
1	1	1	−1	2
1	1	−1	1	−2
1	1	−1	−1	−2
1	−1	1	1	2
1	−1	1	−1	−2
1	−1	−1	1	2
1	−1	−1	−1	−2
−1	1	1	1	−2
−1	1	1	−1	2
−1	1	−1	1	−2
−1	1	−1	−1	2
−1	−1	1	1	−2
−1	−1	1	−1	−2
−1	−1	−1	1	2
−1	−1	−1	−1	2

Since the table is for all minimum and maximum values we have by inspection the range

$$-2 \le \langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle \le 2 \tag{1}$$

Now suppose a third apparatus generates two spins in the following singlet state.

$$|s\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

One spin is sent to A and the other is sent to B.

Let

$$A_0 = \sigma_z, \quad A_1 = \sigma_x, \quad B_0 = -\frac{\sigma_x + \sigma_z}{\sqrt{2}}, \quad B_1 = \frac{\sigma_x - \sigma_z}{\sqrt{2}}$$

Then for the singlet state we have

$$\langle A_0B_0\rangle = \frac{1}{\sqrt{2}}, \quad \langle A_0B_1\rangle = \frac{1}{\sqrt{2}}, \quad \langle A_1B_0\rangle = \frac{1}{\sqrt{2}}, \quad \langle A_1B_1\rangle = -\frac{1}{\sqrt{2}}$$

Hence

$$\langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle = 2\sqrt{2} \tag{2}$$

The result in (2) conflicts with (1) because for the singlet state the random variables are not independent. Any theory that asserts independent random variables is falsified by (2).

Exercises

1. Verify equation (2).

2. Verify that for the singlet state

$$\langle A_0 \rangle = 0, \quad \langle A_1 \rangle = 0, \quad \langle B_0 \rangle = 0, \quad \langle B_1 \rangle = 0$$

Hence $\langle A \rangle \langle B \rangle \neq \langle AB \rangle$ for the singlet state.