1 Anticommutation

Consider the following eigenstates of a hypothetical quantum system.[1]

$$
\begin{aligned}
|00\rangle &= (1\ 0\ 0\ 0) && \text{no fermions}\\
|10\rangle &= (0\ 1\ 0\ 0) && \text{one fermion in state 1}\\
|01\rangle &= (0\ 0\ 1\ 0) && \text{one fermion in state 2}\\
|11\rangle &= (0\ 0\ 0\ 1) && \text{two fermions, one in state 1, one in state 2}
\end{aligned}
$$

Creation and annihilation operators are formed from outer products of state vectors. Sign changes make the operators antisymmetric.

$$
\begin{aligned}
\hat{b}_1^\dagger &= |10\rangle\langle 00| - |11\rangle\langle 01| && \text{Create one fermion in state 1}\\
\hat{b}_1 &= |00\rangle\langle 10| - |01\rangle\langle 11| && \text{Annihilate one fermion in state 1}\\
\hat{b}_2^\dagger &= |01\rangle\langle 00| + |11\rangle\langle 10| && \text{Create one fermion in state 2}\\
\hat{b}_2 &= |00\rangle\langle 01| + |10\rangle\langle 11| && \text{Annihilate one fermion in state 2}
\end{aligned}
$$

The operators in matrix form.

$$
\hat{b}_1^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad
\hat{b}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad
\hat{b}_2^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad
\hat{b}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

Verify anticommutation relations of the operators.

$$\hat{b}_j\hat{b}_k + \hat{b}_k\hat{b}_j = 0$$

$$\hat{b}_j^\dagger\hat{b}_k^\dagger + \hat{b}_k^\dagger\hat{b}_j^\dagger = 0$$

$$\hat{b}_j\hat{b}_k^\dagger + \hat{b}_k^\dagger\hat{b}_j = \delta_{jk}$$

[1]Adapted from problem 16.1.1 of "Quantum Mechanics for Scientists and Engineers."
`https://ee.stanford.edu/~dabm/QMbook.html`

2 Wavefunction operator

Consider the following eigenstates of a hypothetical quantum system.[2]

$$
\begin{aligned}
|00\rangle &= (1\ 0\ 0\ 0) && \text{no fermions} \\
|10\rangle &= (0\ 1\ 0\ 0) && \text{one fermion in state } \phi_1 \\
|01\rangle &= (0\ 0\ 1\ 0) && \text{one fermion in state } \phi_2 \\
|11\rangle &= (0\ 0\ 0\ 1) && \text{two fermions, one in state } \phi_1\text{, one in state } \phi_2
\end{aligned}
$$

Let fermion states ϕ_n be modeled by a one dimensional box of length L.

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

Creation and annihilation operators are formed from outer products of state vectors. Sign changes make the operators antisymmetric.

$$
\begin{aligned}
\hat{b}_1^\dagger &= |10\rangle\langle 00| - |11\rangle\langle 01| && \text{Create one fermion in state } \phi_1 \\
\hat{b}_1 &= |00\rangle\langle 10| - |01\rangle\langle 11| && \text{Annihilate one fermion in state } \phi_1 \\
\hat{b}_2^\dagger &= |01\rangle\langle 00| + |11\rangle\langle 10| && \text{Create one fermion in state } \phi_2 \\
\hat{b}_2 &= |00\rangle\langle 01| + |10\rangle\langle 11| && \text{Annihilate one fermion in state } \phi_2
\end{aligned}
$$

Given the wavefunction operator

$$\hat{\psi} = \frac{1}{\sqrt{2}} \sum_{n,m} \phi_n(x)\phi_m(y)\hat{b}_n\hat{b}_m$$

show that

$$\hat{\psi}|11\rangle = \frac{1}{\sqrt{2}}\big(\phi_1(x)\phi_2(y) - \phi_1(y)\phi_2(x)\big)|00\rangle$$

[2] Adapted from problem 16.2.1 of "Quantum Mechanics for Scientists and Engineers." `https://ee.stanford.edu/~dabm/QMbook.html`

3 Position operator

Consider the following eigenstates of a hypothetical quantum system.

$\lvert 00\rangle = (1\ 0\ 0\ 0)$	no fermions
$\lvert 10\rangle = (0\ 1\ 0\ 0)$	one fermion in state ϕ_1
$\lvert 01\rangle = (0\ 0\ 1\ 0)$	one fermion in state ϕ_2
$\lvert 11\rangle = (0\ 0\ 0\ 1)$	two fermions, one in state ϕ_1, one in state ϕ_2

Let fermion states ϕ_n be modeled by a one dimensional box of length L.

$$\phi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

Creation and annihilation operators are formed from outer products of state vectors. Sign changes make the operators antisymmetric.

$\hat{b}_1^\dagger = \lvert 10\rangle\langle 00\rvert - \lvert 11\rangle\langle 01\rvert$	Create one fermion in state ϕ_1
$\hat{b}_1 = \lvert 00\rangle\langle 10\rvert - \lvert 01\rangle\langle 11\rvert$	Annihilate one fermion in state ϕ_1
$\hat{b}_2^\dagger = \lvert 01\rangle\langle 00\rvert + \lvert 11\rangle\langle 10\rvert$	Create one fermion in state ϕ_2
$\hat{b}_2 = \lvert 00\rangle\langle 01\rvert + \lvert 10\rangle\langle 11\rvert$	Annihilate one fermion in state ϕ_2

Let $\hat{r}$ be the position operator

$$\hat{r} = \sum_{n,m} r_{nm}\hat{b}_n^\dagger\hat{b}_m$$

where

$$r_{nm} = \int_0^L \phi_n^*(x)x\phi_m(x)\,dx$$

Note that for a one dimensional box

$$r_{nn} = \langle x\rangle = \tfrac{1}{2}L$$

Verify that

$$\begin{aligned}
\langle 10\rvert\hat{r}\lvert 10\rangle &= r_{11}\\
\langle 10\rvert\hat{r}\lvert 01\rangle &= r_{12}\\
\langle 01\rvert\hat{r}\lvert 10\rangle &= r_{21}\\
\langle 01\rvert\hat{r}\lvert 01\rangle &= r_{22}
\end{aligned}$$