

## Probability current

Let  $\mathbf{J}$  be the probability current

$$\mathbf{J} = \frac{i\hbar}{2m} (\Psi \nabla \Psi^* - \Psi^* \nabla \Psi) \quad (1)$$

Show that

$$\nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t} |\Psi|^2 \quad (2)$$

Recall that divergence of gradient equals Laplacian.

$$\nabla \cdot \nabla = \nabla^2$$

By the product rule for divergence

$$\begin{aligned} \nabla \cdot \Psi \nabla \Psi^* &= \nabla \Psi \cdot \nabla \Psi^* + \Psi \nabla^2 \Psi^* \\ \nabla \cdot \Psi^* \nabla \Psi &= \nabla \Psi^* \cdot \nabla \Psi + \Psi^* \nabla^2 \Psi \end{aligned}$$

Hence the divergence of  $\mathbf{J}$  in (1) is

$$\begin{aligned} \nabla \cdot \mathbf{J} &= \frac{i\hbar}{2m} (\nabla \cdot \Psi \nabla \Psi^* - \nabla \cdot \Psi^* \nabla \Psi) \\ &= \frac{i\hbar}{2m} (\Psi \nabla^2 \Psi^* - \Psi^* \nabla^2 \Psi) \end{aligned} \quad (3)$$

For the time derivative in (2) we have

$$\frac{\partial}{\partial t} |\Psi|^2 = \frac{\partial}{\partial t} (\Psi^* \Psi) = \Psi^* \frac{\partial}{\partial t} \Psi + \Psi \frac{\partial}{\partial t} \Psi^* \quad (4)$$

Recall the Schrödinger equation

$$\frac{\partial}{\partial t} \Psi = \frac{i\hbar}{2m} \nabla^2 \Psi - \frac{i}{\hbar} V \Psi$$

and

$$\frac{\partial}{\partial t} \Psi^* = \left( \frac{\partial}{\partial t} \Psi \right)^* = -\frac{i\hbar}{2m} \nabla^2 \Psi^* + \frac{i}{\hbar} V \Psi^*$$

Substitute the Schrödinger equation into (4) to obtain

$$\begin{aligned} \frac{\partial}{\partial t} |\Psi|^2 &= \Psi^* \left( \frac{i\hbar}{2m} \nabla^2 \Psi - \frac{i}{\hbar} V \Psi \right) + \Psi \left( -\frac{i\hbar}{2m} \nabla^2 \Psi^* + \frac{i}{\hbar} V \Psi^* \right) \\ &= \frac{i\hbar}{2m} (\Psi^* \nabla^2 \Psi - \Psi \nabla^2 \Psi^*) \end{aligned} \quad (5)$$

Substitute (5) into (3) to obtain

$$\nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t} |\Psi|^2 \quad (6)$$