(11.1) One of the criteria we had for a successful theory of a scalar field was that the commutator for space-like separations would be zero. Let's see if our scalar field has this feature. Show that

$$[\hat{\phi}(x), \hat{\phi}(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(\exp\left(-ip \cdot (x-y) \right) - \exp(-ip \cdot (y-x)) \right) \quad (11.51)$$

For space-like separation we are able to swap $(y - x)$ in the second term to $(x - y)$. This gives us zero, as required.

Consider equation (11.12).

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_\mathbf{p})^{\frac{1}{2}}} \left(\hat{a}_\mathbf{p} \exp(-ip \cdot x) + \hat{a}_\mathbf{p}^\dagger \exp(ip \cdot x) \right) \quad (11.12)$$

It follows from (11.12) that

$$[\hat{\phi}(x), \hat{\phi}(y)] = \frac{1}{(2\pi)^3} \int \frac{d^3p}{(2E_p)^{\frac{1}{2}}} \int \frac{d^3q}{(2E_q)^{\frac{1}{2}}} (PQ - QP) \quad (1)$$

where

$$P = \hat{a}_\mathbf{p} \exp(-ip \cdot x) + \hat{a}_\mathbf{p}^\dagger \exp(ip \cdot x)$$
$$Q = \hat{a}_\mathbf{q} \exp(-iq \cdot y) + \hat{a}_\mathbf{q}^\dagger \exp(iq \cdot y)$$

Expanding the commutator in (1) we have

$$PQ - QP = [\hat{a}_\mathbf{p}, \hat{a}_\mathbf{q}] \exp(-ip \cdot x - iq \cdot y) + [\hat{a}_\mathbf{p}, \hat{a}_\mathbf{q}^\dagger] \exp(-ip \cdot x + iq \cdot y)$$
$$+ [\hat{a}_\mathbf{p}^\dagger, \hat{a}_\mathbf{q}] \exp(ip \cdot x - iq \cdot y) + [\hat{a}_\mathbf{p}^\dagger, \hat{a}_\mathbf{q}^\dagger] \exp(ip \cdot x + iq \cdot y)$$

Then from the commutation relations

$$[\hat{a}_\mathbf{p}, \hat{a}_\mathbf{q}] = 0, \quad [\hat{a}_\mathbf{p}^\dagger, \hat{a}_\mathbf{q}^\dagger] = 0, \quad [\hat{a}_\mathbf{p}, \hat{a}_\mathbf{q}^\dagger] = \delta(\mathbf{p} - \mathbf{q})$$

we have

$$PQ - QP = \delta(\mathbf{p} - \mathbf{q}) \left(\exp(-ip \cdot x + iq \cdot y) - \exp(ip \cdot x - iq \cdot y) \right) \quad (2)$$

Substitute (2) into (1) to obtain

$$[\hat{\phi}(x), \hat{\phi}(y)] = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_p} \left(\exp\left(-ip \cdot (x - y) \right) - \exp\left(-ip \cdot (y - x) \right) \right)$$

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