Atomic transitions 3

From the previous section

$$c_b(t) = \frac{ieE_0}{m\hbar\omega}\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\frac{\sin\big(\frac{1}{2}(\omega_0-\omega)t\big)}{\omega_0-\omega}\exp\big(\tfrac{i}{2}(\omega_0-\omega)t\big)$$

The transition probability is

$$\Pr_{a\to b}(t) = |c_b(t)|^2 = \frac{e^2E_0^2}{m^2\hbar^2\omega^2}\big|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\big|^2\frac{\sin^2\big(\frac{1}{2}(\omega_0-\omega)t\big)}{(\omega_0-\omega)^2}$$

We are now going to pivot from E_0 to a full radiation field. Let u be energy density such that

$$E_0^2 = \frac{2u}{\varepsilon_0}$$

By substitution

$$\Pr_{a\to b}(t) = \frac{2u}{\varepsilon_0}\frac{e^2}{m^2\hbar^2\omega^2}\big|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\big|^2\frac{\sin^2\big(\frac{1}{2}(\omega_0-\omega)t\big)}{(\omega_0-\omega)^2}$$

For a full radiation field

$$u = \int_{-\infty}^{\infty}\rho(\omega)\,d\omega$$

where $\rho(\omega)$ is energy per volume per hertz. By substitution

$$\Pr_{a\to b}(t) = \frac{2}{\varepsilon_0}\frac{e^2}{m^2\hbar^2}\big|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\big|^2\int_{-\infty}^{\infty}\frac{\rho(\omega)}{\omega^2}\frac{\sin^2\big(\frac{1}{2}(\omega_0-\omega)t\big)}{(\omega_0-\omega)^2}\,d\omega$$

Because the integrand is sharply peaked at $\omega=\omega_0$, we are going to make the following move. Substitute ω_0 for ω in $\rho(\omega)/\omega^2$. That makes the term a constant so it can be moved outside the integral. We now have

$$\Pr_{a\to b}(t) = \frac{2}{\varepsilon_0}\frac{e^2}{m^2\hbar^2}\big|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\big|^2\frac{\rho(\omega_0)}{\omega_0^2}\int_{-\infty}^{\infty}\frac{\sin^2\big(\frac{1}{2}(\omega_0-\omega)t\big)}{(\omega_0-\omega)^2}\,d\omega$$

Use the following change of variable to solve the integral.

$$y = \frac{1}{2}(\omega-\omega_0)t,\quad dy = \frac{t}{2}\,d\omega,\quad \omega-\omega_0 = \frac{2y}{t}$$

Noting that the integral of a sinc function is π we have

$$\int_{-\infty}^{\infty}\frac{\sin^2\big(\frac{1}{2}(\omega_0-\omega)t\big)}{(\omega_0-\omega)^2}\,d\omega = \frac{t}{2}\int_{-\infty}^{\infty}\frac{\sin^2(-y)}{y^2}\,dy = \frac{\pi}{2}t$$

Hence

$$\Pr_{a\to b}(t) = \frac{\pi e^2\rho(\omega_0)}{\varepsilon_0 m^2\hbar^2\omega_0^2}\big|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\big|^2 t$$

The transition rate is

$$R_{a\to b} = \frac{d}{dt}\Pr_{a\to b}(t) = \frac{\pi e^2 \rho(\omega_0)}{\varepsilon_0 m^2 \hbar^2 \omega_0^2}\left|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\right|^2$$

Verify dimensions.

$$R_{a\to b} \propto \frac{\underset{\mathrm{C}^2}{e^2}\quad \underset{\mathrm{J\,m^{-3}\,s}}{\rho(\omega_0)}}{\underset{\mathrm{C^2\,J^{-1}\,m^{-1}}}{\varepsilon_0}\quad \underset{\mathrm{kg}^2}{m^2}\quad \underset{\mathrm{J^2\,s^2}}{\hbar^2}\quad \underset{\mathrm{s}^{-2}}{\omega_0^2}} \times \underset{\mathrm{kg^2\,m^2\,s^{-2}}}{\left|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\right|^2} = \mathrm{s}^{-1}$$