Rutherford scattering 1

Use the following formula to compute the cross section for Rutherford scattering.

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2\varepsilon_0^2}\left(\frac{mQ}{4\pi\hbar^2}\right)^2, \quad Q = \int \exp\left(\frac{i\mathbf{p}\cdot\mathbf{r}}{\hbar}\right) V(\mathbf{r})\, d\mathbf{r}^3$$

Convert Q to polar coordinates.

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) V(r,\theta,\phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

For Rutherford scattering $V(\mathbf{r})$ is the Coulomb potential.

$$V(\mathbf{r}) = V(r) = -\frac{Ze^2}{r}$$

Substitute the Coulomb potential for $V(r,\theta,\phi)$ and note r^2 becomes r.

$$Q = -Ze^2\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) r\sin\theta\, dr\, d\theta\, d\phi$$

Integrate over ϕ.

$$Q = -2\pi Ze^2\int_0^{\pi}\int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) r\sin\theta\, dr\, d\theta$$

Change the complex exponential to rectangular form.

$$Q = -2\pi Ze^2\int_0^{\pi}\int_0^{\infty} \left[\cos\left(\frac{pr\cos\theta}{\hbar}\right) + i\sin\left(\frac{pr\cos\theta}{\hbar}\right)\right] V(r)\, r\sin\theta\, dr\, d\theta$$

By the integrals

$$\int_0^{\pi} \cos\big(a\cos(\theta)\big)\sin\theta\, d\theta = \frac{2\sin a}{a}, \quad \int_0^{\pi} \sin\big(a\cos(\theta)\big)\sin\theta\, d\theta = 0$$

we obtain (note r in the integrand is canceled)

$$Q = -\frac{4\pi\hbar Ze^2}{p}\int_0^{\infty} \sin\left(\frac{pr}{\hbar}\right) dr$$

To solve the integral, multiply the integrand by $\exp(-\epsilon r)$.

$$Q = -\frac{4\pi\hbar Ze^2}{p}\int_0^{\infty} \sin\left(\frac{pr}{\hbar}\right)\exp(-\epsilon r)\, dr$$

Convert the integrand to exponential form.

$$Q = -\frac{4\pi\hbar Ze^2}{p}\int_0^{\infty} \frac{i}{2}\left[\exp\left(-\frac{ipr}{\hbar} - \epsilon r\right) - \exp\left(\frac{ipr}{\hbar} - \epsilon r\right)\right] dr$$

Solve the integral.

$$Q = -\frac{4\pi\hbar Z e^2}{p}\frac{i}{2}\left(\frac{1}{-ip/\hbar - \epsilon} - \frac{1}{ip/\hbar - \epsilon}\right) \tag{1}$$

Set $\epsilon = 0$.

$$Q = -\frac{4\pi\hbar Z e^2}{p}\left(-\frac{\hbar}{p}\right)$$

Hence

$$Q = \frac{4\pi\hbar^2 Z e^2}{p^2}$$

Compute the differential cross section.

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2\varepsilon_0^2}\left(\frac{mQ}{4\pi\hbar^2}\right)^2 = \frac{1}{16\pi^2\varepsilon_0^2}\frac{m^2 Z^2 e^4}{p^4} \tag{2}$$

Substitute $16\pi^2\varepsilon_0^2\alpha^2(\hbar c)^2$ for e^4.

$$\frac{d\sigma}{d\Omega} = \frac{m^2 Z^2 \alpha^2 (\hbar c)^2}{p^4}$$

Symbol p is momentum transfer $|\mathbf{p}_i| - |\mathbf{p}_f|$ such that

$$p^2 = 2mE(\cos\theta - 1)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2(\hbar c)^2}{4E^2(\cos\theta - 1)^2} \tag{3}$$

Noting that

$$4\sin^4\frac{\theta}{2} = (\cos(\theta) - 1)^2$$

we also have

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2(\hbar c)^2}{16E^2\sin^4(\theta/2)}$$

Experimental data

The following data is from Geiger and Marsden's 1913 paper (y is the number of scattering events).

$\frac{1}{\sin^4(\theta/2)}$	y
1.15	22.2
1.38	27.4
1.79	33.0
2.53	47.3
7.25	136
16.0	320
46.6	989
93.7	1760
223	5260
690	20300
3445	105400

Predicted values are bin probability times total number of events.

$$\sum_i \frac{1}{\sin^4(\theta_i/2)} = 4528, \quad \sum_i y_i = 134295$$

$$\hat{y}_i = \underbrace{\frac{1}{4528\sin^4(\theta_i/2)}}_{\text{bin probability}} \times \underbrace{134295}_{\text{total events}}$$

The following table shows the predicted values.

$\frac{1}{\sin^4(\theta/2)}$	y	$\hat{y}$
1.15	22.2	34.1
1.38	27.4	40.9
1.79	33.0	53.1
2.53	47.3	75.0
7.25	136	215
16.0	320	474
46.6	989	1382
93.7	1760	2779
223	5260	6613
690	20300	20463
3445	105400	102165

The coefficient of determination R^2 measures how well predicted values fit the data.

$$R^2 = 1 - \frac{\sum_i (y_i - \hat{y}_i)^2}{\sum_i (y_i - \bar{y})^2} = 0.999$$

The result indicates that $d\sigma$ explains 99.9% of the variance in the data.