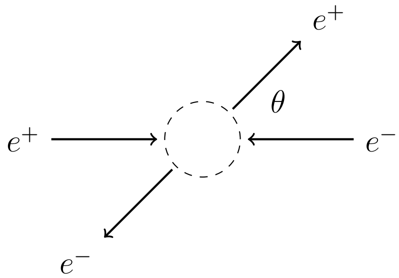
Bhabha scattering

Bhabha scattering is the interaction $e^- + e^+ \to e^- + e^+$.



In the center-of-mass frame we have the following momentum vectors where $E = \sqrt{p^2 + m^2}$.

$$p_1 = \underset{\text{inbound } e^+}{\begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}} \qquad p_2 = \underset{\text{inbound } e^-}{\begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix}} \qquad p_3 = \underset{\text{outbound } e^+}{\begin{pmatrix} E \\ p\sin\theta\cos\phi \\ p\sin\theta\sin\phi \\ p\cos\theta \end{pmatrix}} \qquad p_4 = \underset{\text{outbound } e^-}{\begin{pmatrix} E \\ -p\sin\theta\cos\phi \\ -p\sin\theta\sin\phi \\ -p\cos\theta \end{pmatrix}}$$

Spinors for the inbound positron.

$$v_{11} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound } e^+ \\ \text{spin up}}}{\begin{pmatrix} p \\ 0 \\ E+m \\ 0 \end{pmatrix}} \qquad v_{12} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound } e^+ \\ \text{spin down}}}{\begin{pmatrix} 0 \\ -p \\ 0 \\ E+m \end{pmatrix}}$$

Spinors for the inbound electron.

$$u_{21} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound } e^- \\ \text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ -p \\ 0 \end{pmatrix}} \qquad u_{22} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound } e^- \\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ p \end{pmatrix}}$$

Spinors for the outbound positron.

$$v_{31} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{outbound } e^+ \\ \text{spin up}}}{\begin{pmatrix} p_{3z} \\ p_{3x} + ip_{3y} \\ E+m \\ 0 \end{pmatrix}} \qquad v_{32} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{outbound } e^+ \\ \text{spin down}}}{\begin{pmatrix} p_{3x} - ip_{3y} \\ -p_{3z} \\ 0 \\ E+m \end{pmatrix}}$$

Spinors for the outbound electron.

$$u_{41} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{outbound } e^- \\ \text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix}} \qquad u_{42} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{outbound } e^- \\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}}$$

The probability amplitude $\mathcal{M}_{abcd}$ for spin state $abcd$ is

$$\mathcal{M}_{abcd} = \mathcal{M}_{1abcd} + \mathcal{M}_{2abcd}$$

where

$$\mathcal{M}_{1abcd} = \frac{e^2}{s}(\bar{v}_{1a}\gamma^\mu u_{2b})(\bar{u}_{4d}\gamma_\mu v_{3c}), \quad \mathcal{M}_{2abcd} = -\frac{e^2}{t}(\bar{v}_{1a}\gamma^\nu v_{3c})(\bar{u}_{4d}\gamma_\nu u_{2b})$$

Symbol e is elementary charge and

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is the average of spin states.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}|\mathcal{M}_{abcd}|^2$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{abcd}(\mathcal{M}_{1abcd}\mathcal{M}^*_{1abcd} + \mathcal{M}_{1abcd}\mathcal{M}^*_{2abcd} + \mathcal{M}_{2abcd}\mathcal{M}^*_{1abcd} + \mathcal{M}_{2abcd}\mathcal{M}^*_{2abcd})$$

The Casimir trick uses matrix arithmetic to sum over spin states.

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left(\frac{f_{11}}{s^2} - \frac{2f_{12}}{st} + \frac{f_{22}}{t^2}\right)$$

where

$$f_{11} = \mathrm{Tr}\left((\not{p}_1 - m)\gamma^\mu(\not{p}_2 + m)\gamma^\nu\right)\mathrm{Tr}\left((\not{p}_4 + m)\gamma_\mu(\not{p}_3 - m)\gamma_\nu\right)$$
$$f_{12} = \mathrm{Tr}\left((\not{p}_1 - m)\gamma^\mu(\not{p}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{p}_3 - m)\gamma_\nu\right)$$
$$f_{22} = \mathrm{Tr}\left((\not{p}_1 - m)\gamma^\mu(\not{p}_3 - m)\gamma^\nu\right)\mathrm{Tr}\left((\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\right)$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$f_{11} = 32(p_1 \cdot p_3)^2 + 32(p_1 \cdot p_4)^2 + 64m^2(p_1 \cdot p_2) + 64m^4$$
$$f_{12} = -32(p_1 \cdot p_4)^2 - 32m^2(p_1 \cdot p_2) + 32m^2(p_1 \cdot p_3) - 32m^2(p_1 \cdot p_4) - 32m^4$$
$$f_{22} = 32(p_1 \cdot p_2)^2 + 32(p_1 \cdot p_4)^2 - 64m^2(p_1 \cdot p_3) + 64m^4$$

For $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$f_{11} = 64E^4(\cos^2\theta + 1)$$
$$f_{12} = -32E^4(\cos\theta + 1)^2$$
$$f_{22} = 32E^4(\cos\theta + 1)^2 + 128E^4$$

For $m = 0$ the Mandelstam variables are

$$s = 4E^2$$
$$t = 2E^2(\cos\theta - 1)$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left(\frac{f_{11}}{s^2} - \frac{2f_{12}}{st} + \frac{f_{22}}{t^2}\right) = e^4\left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2$$

Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{4(4\pi\varepsilon_0)^2 s}$$

where

$$s = (p_1 + p_2)^2 = 4E^2$$

For high energy experiments we have

$$\langle|\mathcal{M}|^2\rangle = e^4 \left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2$$

Hence for high energy experiments

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{4(4\pi\varepsilon_0)^2 s} \left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2$$

Noting that

$$e^2 = 4\pi\varepsilon_0 \alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{4s} \left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2$$

Noting that

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

we also have

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{4s} \left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2 \sin\theta\, d\theta\, d\phi$$

Let $S(\theta_1, \theta_2)$ be the following integral of $d\sigma$.

$$S(\theta_1, \theta_2) = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{\pi\alpha^2(\hbar c)^2}{2s} [I(\theta_2) - I(\theta_1)]$$

where

$$I(\theta) = \frac{16}{\cos\theta - 1} - \frac{\cos^3\theta}{3} - \cos^2\theta - 9\cos\theta - 16\log(1 - \cos\theta)$$

The cumulative distribution function is

$$F(\theta) = \frac{S(a,\theta)}{S(a,\pi)} = \frac{I(\theta) - I(a)}{I(\pi) - I(a)}, \quad a \le \theta \le \pi$$

Angular support is reduced by an arbitrary angle $a > 0$ because $I(0)$ is undefined.

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 < \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi) - I(a)} \left(\frac{\cos^2\theta + 3}{\cos\theta - 1} \right)^2 \sin\theta$$

Data from SLAC SPEAR experiment

The following Bhabha scattering data is from SLAC-PUB-1501.

k	x_k	x_{k+1}	y
1	0.6	0.5	4432
2	0.5	0.4	2841
3	0.4	0.3	2045
4	0.3	0.2	1420
5	0.2	0.1	1136
6	0.1	0.0	852
7	0.0	−0.1	656
8	−0.1	−0.2	625
9	−0.2	−0.3	511
10	−0.3	−0.4	455
11	−0.4	−0.5	402
12	−0.5	−0.6	398

Column k is the bin number, column y is the number of scattering events, and

$$x_k = \cos\theta_k$$

The cumulative distribution function for this experiment is

$$F(\theta) = \frac{I(\theta) - I(\theta_1)}{I(\theta_{13}) - I(\theta_1)}$$

where

$$\theta_{13} = \arccos(-0.6), \quad \theta_1 = \arccos(0.6)$$

The scattering probability P_k is

$$P_k = F\left(\arccos(x_{k+1})\right) - F\left(\arccos(x_k)\right)$$

Multiply P_k by total scattering events to obtain predicted number of events $\hat{y}_k$.

$$\sum y_k = 15773, \quad \hat{y}_k = 15773\, P_k$$

The following table shows the predicted scattering events $\hat{y}$.

k	x_k	x_{k+1}	y	$\hat{y}$
1	0.6	0.5	4432	4598
2	0.5	0.4	2841	2880
3	0.4	0.3	2045	1955
4	0.3	0.2	1420	1410
5	0.2	0.1	1136	1068
6	0.1	0.0	852	843
7	0.0	−0.1	656	689
8	−0.1	−0.2	625	582
9	−0.2	−0.3	511	505
10	−0.3	−0.4	455	450
11	−0.4	−0.5	402	411
12	−0.5	−0.6	398	382

The coefficient of determination R^2 measures how well predicted values fit the data.

$$R^2 = 1 - \frac{\sum(y - \hat{y})^2}{\sum(y - \bar{y})^2} = 0.997$$

The result indicates that $F(\theta)$ explains 99.7% of the variance in the data.

Data from DESY PETRA experiment

See www.hepdata.net/record/ins191231, Table 3, 14.0 GeV.

x	y
−0.7300	0.10115
−0.6495	0.12235
−0.5495	0.11258
−0.4494	0.09968
−0.3493	0.14749
−0.2491	0.14017
−0.1490	0.18190
−0.0488	0.22964
0.0514	0.25312
0.1516	0.30998
0.2520	0.40898
0.3524	0.62695
0.4529	0.91803
0.5537	1.51743
0.6548	2.56714
0.7323	4.30279

Data x and y have the following relationship with the cross section formula.

$$x = \cos\theta, \quad y = \frac{d\sigma}{d\Omega} \text{ in units of nanobarns}$$

The cross section formula is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2 \times (\hbar c)^2$$

To compute predicted values $\hat{y}$, multiply by 10^{37} to convert square meters to nanobarns.

$$\hat{y} = \frac{\alpha^2}{4s}\left(\frac{x^2 + 3}{x - 1}\right)^2 \times (\hbar c)^2 \times 10^{37}$$

The following table shows predicted values $\hat{y}$ for $s = (14.0\,\text{GeV})^2$.

x	y	$\hat{y}$
−0.7300	0.10115	0.110296
−0.6495	0.12235	0.113816
−0.5495	0.11258	0.120101
−0.4494	0.09968	0.129075
−0.3493	0.14749	0.141592
−0.2491	0.14017	0.158934
−0.1490	0.18190	0.182976
−0.0488	0.22964	0.216737
0.0514	0.25312	0.264989
0.1516	0.30998	0.335782
0.2520	0.40898	0.443630
0.3524	0.62695	0.615528
0.4529	0.91803	0.907700
0.5537	1.51743	1.451750
0.6548	2.56714	2.609280
0.7323	4.30279	4.615090

The coefficient of determination R^2 measures how well predicted values fit the data.

$$R^2 = 1 - \frac{\sum(y - \hat{y})^2}{\sum(y - \bar{y})^2} = 0.995$$

The result indicates that the model $d\sigma$ explains 99.5% of the variance in the data.