7-6. Show, for a particle moving in three-dimensional space x, y, z,

$$\langle (x_{k+1}-x_k)^2 \rangle = \langle (y_{k+1}-y_k)^2 \rangle = \langle (z_{k+1}-z_k)^2 \rangle = -\frac{\hbar\epsilon}{im}\langle 1 \rangle \tag{7.50}$$

$$\begin{aligned}\langle (x_{k+1}-x_k)(y_{k+1}-y_k) \rangle &= \langle (x_{k+1}-x_k)(z_{k+1}-z_k) \rangle \\ &= \langle (y_{k+1}-y_k)(z_{k+1}-z_k) \rangle = 0\end{aligned} \tag{7.51}$$

The action for a particle in three-dimensional space is

$$S = \int_{t_a}^{t_b} \left(\frac{m}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - V(x,y,z) \right) dt$$

By extending equation (7.39) to three dimensions we have

$$\nabla_k S = \frac{\partial S}{\partial x_k}\mathbf{i} + \frac{\partial S}{\partial y_k}\mathbf{j} + \frac{\partial S}{\partial z_k}\mathbf{k}$$

where

$$\begin{aligned}\frac{\partial S}{\partial x_k} &= -m\left(\frac{x_{k+1}-x_k}{\epsilon} - \frac{x_k - x_{k-1}}{\epsilon}\right) - \epsilon \frac{\partial V}{\partial x}\bigg|_{x_k,y_k,z_k} \\ \frac{\partial S}{\partial y_k} &= -m\left(\frac{y_{k+1}-y_k}{\epsilon} - \frac{y_k - y_{k-1}}{\epsilon}\right) - \epsilon \frac{\partial V}{\partial y}\bigg|_{x_k,y_k,z_k} \\ \frac{\partial S}{\partial z_k} &= -m\left(\frac{z_{k+1}-z_k}{\epsilon} - \frac{z_k - z_{k-1}}{\epsilon}\right) - \epsilon \frac{\partial V}{\partial z}\bigg|_{x_k,y_k,z_k}\end{aligned}$$

Let

$$F = x_k + y_k + z_k$$

Then

$$\nabla_k F = \frac{\partial F}{\partial x_k}\mathbf{i} + \frac{\partial F}{\partial y_k}\mathbf{j} + \frac{\partial F}{\partial z_k}\mathbf{k} = \mathbf{i}+\mathbf{j}+\mathbf{k}$$

and

$$\langle \nabla_k F \rangle = \langle 1 \rangle \mathbf{i} + \langle 1 \rangle \mathbf{j} + \langle 1 \rangle \mathbf{k}$$

By equation (7.43)

$$\begin{aligned}\left\langle F\frac{\partial S}{\partial x_k}\right\rangle \mathbf{i} &= \left\langle x_k\frac{\partial S}{\partial x_k}\right\rangle \mathbf{i} \\ \left\langle F\frac{\partial S}{\partial y_k}\right\rangle \mathbf{j} &= \left\langle y_k\frac{\partial S}{\partial y_k}\right\rangle \mathbf{j} \\ \left\langle F\frac{\partial S}{\partial z_k}\right\rangle \mathbf{k} &= \left\langle z_k\frac{\partial S}{\partial z_k}\right\rangle \mathbf{k}\end{aligned}$$

By equation (7.33)

$$\langle 1\rangle\mathbf{i} = \left\langle x_k \frac{\partial S}{\partial x_k}\right\rangle \mathbf{i}$$
$$\langle 1\rangle\mathbf{j} = \left\langle y_k \frac{\partial S}{\partial y_k}\right\rangle \mathbf{j}$$
$$\langle 1\rangle\mathbf{k} = \left\langle z_k \frac{\partial S}{\partial z_k}\right\rangle \mathbf{k}$$

Then by the same arguments that led to equation (7.49) we have

$$\left\langle (x_{k+1} - x_k)^2 \right\rangle \mathbf{i} = -\frac{\hbar\epsilon}{im}\langle 1\rangle\mathbf{i}$$
$$\left\langle (y_{k+1} - y_k)^2 \right\rangle \mathbf{j} = -\frac{\hbar\epsilon}{im}\langle 1\rangle\mathbf{j}$$
$$\left\langle (z_{k+1} - z_k)^2 \right\rangle \mathbf{k} = -\frac{\hbar\epsilon}{im}\langle 1\rangle\mathbf{k}$$

Hence (7.50) is shown to be true.

Let

$$F = x_k y_k z_k$$

Then

$$\langle \nabla_k F\rangle = \langle y_k z_k\rangle\mathbf{i} + \langle x_k z_k\rangle\mathbf{j} + \langle x_k y_k\rangle\mathbf{k}$$

By equation (7.43)

$$\left\langle y_k z_k \frac{\partial S}{\partial x_k}\right\rangle \mathbf{i} = 0$$
$$\left\langle x_k z_k \frac{\partial S}{\partial y_k}\right\rangle \mathbf{j} = 0$$
$$\left\langle x_k y_k \frac{\partial S}{\partial z_k}\right\rangle \mathbf{k} = 0$$

Then by equation (7.33)

$$\langle \nabla_k F\rangle = \langle F \nabla_k S\rangle \tag{7.33}$$

we have

$$\langle y_k z_k\rangle = 0$$
$$\langle x_k z_k\rangle = 0$$
$$\langle x_k y_k\rangle = 0$$

The above argument can be repeated for all combinations of subscripts k and $k+1$ hence (7.51) is shown to be true.