Angular momentum 2

The cross product in $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ demands rectangular coordinates. Hence for a wavefunction ψ in spherical coordinates, vectors $\mathbf{r}$ and $\mathbf{p}$ must be transformed in $\mathbf{L}\psi$.

Vector $\mathbf{r}$ transforms as

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}$$

To transform $\mathbf{p}$ we have by the chain rule

$$\begin{aligned}
\frac{\partial}{\partial x} &= \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\
\frac{\partial}{\partial y} &= \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\
\frac{\partial}{\partial z} &= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}
\end{aligned}$$

Hence

$$\begin{aligned}
p_x &= -i\hbar\frac{\partial}{\partial x} = -i\hbar\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right) \\
p_y &= -i\hbar\frac{\partial}{\partial y} = -i\hbar\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right) \\
p_z &= -i\hbar\frac{\partial}{\partial z} = -i\hbar\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)
\end{aligned}$$

Using the transformed coordinates

$$\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}$$

we have in spherical coordinates

$$\begin{aligned}
L_x &= yp_z - zp_y = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\theta\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right) \\
L_y &= zp_x - xp_z = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \frac{\cos\theta\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right) \\
L_z &= xp_y - yp_x = -i\hbar\frac{\partial}{\partial\phi}
\end{aligned}$$

and

$$L^2 = {L_x}^2 + {L_y}^2 + {L_z}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

Eigenmath script

Note

Griffith's book has a different approach.

Noting that in spherical coordinates

$$\mathbf{r} = r\mathbf{e}_r$$

and

$$\nabla = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

we have for angular momentum $\mathbf{L}$

$$\mathbf{L} = -i\hbar(\mathbf{r}\times\nabla) = -i\hbar\left[r(\mathbf{e}_r\times\mathbf{e}_r)\frac{\partial}{\partial r} + (\mathbf{e}_r\times\mathbf{e}_\theta)\frac{\partial}{\partial\theta} + (\mathbf{e}_r\times\mathbf{e}_\phi)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]$$

In rectangular coordinates

$$\mathbf{e}_r = \begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix},\quad \mathbf{e}_\theta = \begin{pmatrix}\cos\theta\cos\phi\\ \cos\theta\sin\phi\\ -\sin\theta\end{pmatrix},\quad \mathbf{e}_\phi = \begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix}$$

Hence for the cross products

$$\begin{aligned}\mathbf{e}_r\times\mathbf{e}_r &= 0\\ \mathbf{e}_r\times\mathbf{e}_\theta &= \mathbf{e}_\phi\\ \mathbf{e}_r\times\mathbf{e}_\phi &= -\mathbf{e}_\theta\end{aligned}$$

Angular momentum $\mathbf{L}$ now reduces to

$$\mathbf{L} = -i\hbar\left(\mathbf{e}_\phi\frac{\partial}{\partial\theta} - \mathbf{e}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)$$

Substitute for the basis vectors

$$\begin{aligned}\mathbf{e}_\theta &= \mathbf{e}_x\cos\theta\cos\phi + \mathbf{e}_y\cos\theta\sin\phi - \mathbf{e}_z\sin\theta\\ \mathbf{e}_\phi &= -\mathbf{e}_x\sin\phi + \mathbf{e}_y\cos\phi\end{aligned}$$

to obtain

$$\mathbf{L} = -i\hbar\left[(-\mathbf{e}_x\sin\phi + \mathbf{e}_y\cos\phi)\frac{\partial}{\partial\theta} - (\mathbf{e}_x\cos\theta\cos\phi + \mathbf{e}_y\cos\theta\sin\phi - \mathbf{e}_z\sin\theta)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]$$

In component form

$$\begin{aligned}L_x &= \mathbf{L}\cdot\mathbf{e}_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\theta\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\\ L_y &= \mathbf{L}\cdot\mathbf{e}_y = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \frac{\cos\theta\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\\ L_z &= \mathbf{L}\cdot\mathbf{e}_z = -i\hbar\frac{\partial}{\partial\phi}\end{aligned}$$