

(8.1) Show that the form of the time evolution operator  $\hat{U}(t_2, t_1) = \exp[-i\hat{H}(t_2 - t_1)]$  (as given in eqn 8.7) exhibits properties 1–5 in Section 8.1.

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- (1)  $\hat{U}(t_1, t_1) = 1$
- (2)  $\hat{U}(t_3, t_2)\hat{U}(t_2, t_1) = \hat{U}(t_3, t_1)$
- (3)  $i\frac{d}{dt_2}\hat{U}(t_2, t_1) = \hat{H}\hat{U}(t_2, t_1)$
- (4)  $\hat{U}(t_1, t_2) = \hat{U}^{-1}(t_2, t_1)$
- (5)  $\hat{U}^\dagger(t_2, t_1)\hat{U}(t_2, t_1) = 1$

(1)

$$\begin{aligned}\hat{U}(t_1, t_1) &= \exp[-i\hat{H}(t_1 - t_1)] \\ &= \exp(0) \\ &= 1\end{aligned}$$

(2)

$$\begin{aligned}\hat{U}(t_3, t_2)\hat{U}(t_2, t_1) &= \exp[-i\hat{H}(t_3 - t_2)] \exp[-i\hat{H}(t_2 - t_1)] \\ &= \exp[-i\hat{H}(t_3 - t_1)] \\ &= \hat{U}(t_3, t_1)\end{aligned}$$

(3)

$$\begin{aligned}i\frac{d}{dt_2}\hat{U}(t_2, t_1) &= i\frac{d}{dt_2} \exp[-i\hat{H}(t_2 - t_1)] \\ &= i(-i\hat{H}) \exp[-i\hat{H}(t_2 - t_1)] \\ &= \hat{H}\hat{U}(t_2, t_1)\end{aligned}$$

(4)

$$\begin{aligned}\hat{U}(t_1, t_2) &= \exp[-i\hat{H}(t_1 - t_2)] \\ &= \exp[i\hat{H}(t_2 - t_1)] \\ &= \exp[-i\hat{H}(t_2 - t_1)]^{-1} \\ &= \hat{U}^{-1}(t_2, t_1)\end{aligned}$$

(5)

$$\begin{aligned}\hat{U}^\dagger(t_2, t_1)\hat{U}(t_2, t_1) &= \exp[-i\hat{H}(t_2 - t_1)]^{-1} \exp[-i\hat{H}(t_2 - t_1)] \\ &= \exp[i\hat{H}(t_2 - t_1)] \exp[-i\hat{H}(t_2 - t_1)] \\ &= \exp(0) \\ &= 1\end{aligned}$$