

Addition of angular momentum 2

In spherical coordinates

$$\begin{aligned}L_x &= i\hbar \left(\sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right) \\L_y &= i\hbar \left(-\cos \phi \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right) \\L_z &= -i\hbar \frac{\partial}{\partial \phi}\end{aligned}$$

and

$$L^2 = -\hbar^2 \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right)$$

Recall that

$$L_z Y_{lm} = m\hbar Y_{lm}$$

Hence for $\Psi = Y_{lm}\chi_{\pm}$ we have

$$J_z \Psi = L_z \Psi + S_z \Psi = m\hbar \Psi \pm \frac{1}{2}\hbar \Psi$$