Electromagnetic tensor

This is the standard model for an EM field.

$$\mathbf{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}, \quad A^\mu = \begin{pmatrix} \phi \\ A_x \\ A_y \\ A_z \end{pmatrix}, \quad A_\mu = g_{\mu\nu} A^\nu = \begin{pmatrix} \phi \\ -A_x \\ -A_y \\ -A_z \end{pmatrix}$$

```
gmunu = ((1,0,0,0),(0,-1,0,0),(0,0,-1,0),(0,0,0,-1))

A = (Ax(),Ay(),Az())
Au = (phi(),Ax(),Ay(),Az())
Ad = dot(gmunu,Au)
```

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

```
B = curl(A)
E = -d(phi(),(x,y,z)) - d(A,t)

Bx = B[1]
By = B[2]
Bz = B[3]

Ex = E[1]
Ey = E[2]
Ez = E[3]
```

This is the electromagnetic tensor.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = A_{\nu,\mu} - A_{\mu,\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

```
X = (t,x,y,z)
Add = d(Ad,X)
Fdd = transpose(Add) - Add

T = ((0, Ex, Ey, Ez),
     (-Ex, 0, -Bz, By),
     (-Ey, Bz, 0, -Bx),
     (-Ez, -By, Bx, 0))

check(Fdd == T)
```

Check the following relations.

$$F_{\mu\nu} F^{\mu\nu} = 2\big(\mathbf{B}^2 - \mathbf{E}^2\big), \quad \det(F_{\mu\nu}) = \det(F^{\mu\nu}) = (\mathbf{B} \cdot \mathbf{E})^2$$

```
Fuu = dot(gmunu,Fdd,gmunu)
T = contract(dot(transpose(Fdd),Fuu))
check(T == 2 dot(B,B) - 2 dot(E,E))

check(det(Fdd) == dot(B,E)^2)
check(det(Fuu) == dot(B,E)^2)
```

This is the vector current.

$$J^\nu = \partial_\mu F^{\mu\nu} = F^{\mu\nu}{}_{,\mu}$$

Gradient increases rank by one. The new index is the rightmost index, hence the contraction is over the first and third indices.

```
Ju = contract(d(Fuu,X),1,3)
```

Check the following relations.

$$\partial_\mu J^\mu = J^\mu{}_{,\mu} = 0, \quad \mathbf{J} = \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t}$$

```
check(contract(d(Ju,X)) == 0)

Jx = Ju[2]
Jy = Ju[3]
Jz = Ju[4]
J = (Jx,Jy,Jz)
check(J == curl(B) - d(E,t))
```