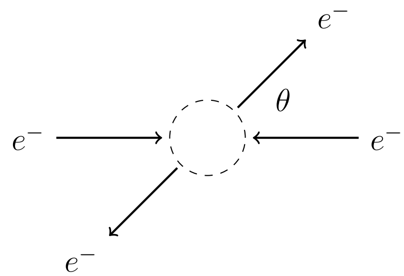
Moller scattering

Moller scattering is the process $e^- + e^- \to e^- + e^-$.



The following momentum vectors are for the center-of-mass frame with $E = \sqrt{p^2 + m^2}$.

$$
p_1 = \underset{e^- \longrightarrow}{\begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}} \qquad
p_2 = \underset{\longleftarrow e^-}{\begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix}} \qquad
p_3 = \underset{e^- \nearrow}{\begin{pmatrix} E \\ p\sin\theta\cos\phi \\ p\sin\theta\sin\phi \\ p\cos\theta \end{pmatrix}} \qquad
p_4 = \underset{\swarrow e^-}{\begin{pmatrix} E \\ -p\sin\theta\cos\phi \\ -p\sin\theta\sin\phi \\ -p\cos\theta \end{pmatrix}}
$$

Spinors for p_1.

$$
u_{11} = \frac{1}{\sqrt{E+m}} \underset{\text{spin up}}{\begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix}} \qquad
u_{12} = \frac{1}{\sqrt{E+m}} \underset{\text{spin down}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix}}
$$

Spinors for p_2.

$$
u_{21} = \frac{1}{\sqrt{E+m}} \underset{\text{spin up}}{\begin{pmatrix} E+m \\ 0 \\ -p \\ 0 \end{pmatrix}} \qquad
u_{22} = \frac{1}{\sqrt{E+m}} \underset{\text{spin down}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ p \end{pmatrix}}
$$

Spinors for p_3.

$$
u_{31} = \frac{1}{\sqrt{E+m}} \underset{\text{spin up}}{\begin{pmatrix} E+m \\ 0 \\ p_{3z} \\ p_{3x} + ip_{3y} \end{pmatrix}} \qquad
u_{32} = \frac{1}{\sqrt{E+m}} \underset{\text{spin down}}{\begin{pmatrix} 0 \\ E+m \\ p_{3x} - ip_{3y} \\ -p_{3z} \end{pmatrix}}
$$

Spinors for p_4.

$$
u_{41} = \frac{1}{\sqrt{E+m}} \underset{\text{spin up}}{\begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix}} \qquad
u_{42} = \frac{1}{\sqrt{E+m}} \underset{\text{spin down}}{\begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}}
$$

The scattering amplitude $\mathcal{M}_{abcd}$ for spin state $abcd$ is

$$\mathcal{M}_{abcd} = \mathcal{M}_{1abcd} + \mathcal{M}_{2abcd}$$

where

$$\mathcal{M}_{1abcd} = \frac{e^2}{t} \underbrace{(\bar{u}_{3c}\gamma^\mu u_{1a})(\bar{u}_{4d}\gamma_\mu u_{2b})}_{\text{no electron interchange}}, \quad \mathcal{M}_{2abcd} = -\frac{e^2}{u} \underbrace{(\bar{u}_{4d}\gamma^\nu u_{1a})(\bar{u}_{3c}\gamma_\nu u_{2b})}_{\text{electron interchange}}$$

Symbols t and u are Mandelstam variables.

$$t = (p_1 - p_3)^2$$
$$u = (p_1 - p_4)^2$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is the sum of squared amplitudes divided by the number of inbound states.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4} \sum_{abcd} |\mathcal{M}_{abcd}|^2$$

Expand the summand and label the terms. By hermiticity $\boxed{2} = \boxed{3}$.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4} \sum_{abcd} \Big(\underbrace{\mathcal{M}_{1abcd}\mathcal{M}^*_{1abcd}}_{\boxed{1}} + \underbrace{\mathcal{M}_{1abcd}\mathcal{M}^*_{2abcd}}_{\boxed{2}} + \underbrace{\mathcal{M}_{2abcd}\mathcal{M}^*_{1abcd}}_{\boxed{3}} + \underbrace{\mathcal{M}_{2abcd}\mathcal{M}^*_{2abcd}}_{\boxed{4}} \Big)$$

The following Casimir trick uses matrix arithmetic to sum over spin states.

$$\sum_{abcd} \boxed{1} = \frac{e^4}{t^2} \operatorname{Tr}\Big[(\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu\Big] \operatorname{Tr}\Big[(\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\Big]$$
$$\sum_{abcd} \boxed{2} = -\frac{e^4}{tu} \operatorname{Tr}\Big[(\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\Big]$$
$$\sum_{abcd} \boxed{4} = \frac{e^4}{u^2} \operatorname{Tr}\Big[(\not{p}_4 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu\Big] \operatorname{Tr}\Big[(\not{p}_3 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\Big]$$

Let

$$f_{11} = \operatorname{Tr}\Big[(\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu\Big] \operatorname{Tr}\Big[(\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\Big]$$
$$f_{12} = -\operatorname{Tr}\Big[(\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\Big]$$
$$f_{22} = \operatorname{Tr}\Big[(\not{p}_4 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu\Big] \operatorname{Tr}\Big[(\not{p}_3 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\Big]$$

so that

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4} \left(\frac{f_{11}}{t^2} + \frac{2f_{12}}{tu} + \frac{f_{22}}{u^2} \right)$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$f_{11} = 32(p_1 \cdot p_2)^2 + 32(p_1 \cdot p_4)^2 - 64(p_1 \cdot p_2)m^2 + 64(p_1 \cdot p_4)m^2$$
$$f_{12} = 32(p_1 \cdot p_2)^2 - 64(p_1 \cdot p_2)m^2$$
$$f_{22} = 32(p_1 \cdot p_2)^2 + 32(p_1 \cdot p_3)^2 - 64(p_1 \cdot p_2)m^2 + 64(p_1 \cdot p_3)m^2$$

In Mandelstam variables

$$\begin{aligned}
f_{11} &= 8s^2 + 8u^2 - 64sm^2 - 64um^2 + 192m^4 \\
f_{12} &= 8s^2 - 64sm^2 + 96m^4 \\
f_{22} &= 8s^2 + 8t^2 - 64sm^2 - 64tm^2 + 192m^4
\end{aligned}$$

For $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$\begin{aligned}
f_{11} &= 8s^2 + 8u^2 \\
f_{12} &= 8s^2 \\
f_{22} &= 8s^2 + 8t^2
\end{aligned}$$

For $m = 0$ the Mandelstam variables are

$$\begin{aligned}
s &= 4E^2 \\
t &= -2E^2(1 - \cos\theta) \\
u &= -2E^2(1 + \cos\theta)
\end{aligned}$$

Hence

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{e^4}{4}\left(\frac{f_{11}}{t^2} + \frac{2f_{12}}{tu} + \frac{f_{22}}{u^2}\right) \\
&= 2e^4\left(\frac{s^2 + u^2}{t^2} + \frac{2s^2}{tu} + \frac{s^2 + t^2}{u^2}\right) \\
&= 2e^4\left(\underbrace{\frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)}}_{\text{no electron interchange}} + \underbrace{\frac{2}{\sin^2(\theta/2)\cos^2(\theta/2)}}_{\text{interaction term}} + \underbrace{\frac{1 + \sin^4(\theta/2)}{\cos^4(\theta/2)}}_{\text{electron interchange}}\right)
\end{aligned}$$

The expected probability density can be written more compactly as

$$\langle|\mathcal{M}|^2\rangle = 4e^4\frac{(\cos^2\theta + 3)^2}{\sin^4\theta}$$

Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{4(4\pi\varepsilon_0)^2 s}$$

where

$$s = (p_1 + p_2)^2 = 4E^2$$

For high energy experiments we have

$$\langle|\mathcal{M}|^2\rangle = 4e^4\frac{(\cos^2\theta + 3)^2}{\sin^4\theta}$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{(4\pi\varepsilon_0)^2 s}\frac{(\cos^2\theta + 3)^2}{\sin^4\theta}$$

Noting that

$$e^2 = 4\pi\varepsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{s}\frac{(\cos^2\theta + 3)^2}{\sin^4\theta}$$

Noting that

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

we also have

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{s}\frac{(\cos^2\theta + 3)^2}{\sin^4\theta}\sin\theta\, d\theta\, d\phi$$

Let $S(\theta_1, \theta_2)$ be the following integral of $d\sigma$.

$$S(\theta_1, \theta_2) = \int_0^{2\pi}\int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{2\pi\alpha^2(\hbar c)^2}{s}[I(\theta_2) - I(\theta_1)]$$

where

$$I(\theta) = -\frac{8\cos\theta}{\sin^2\theta} - \cos\theta$$

The cumulative distribution function is

$$F(\theta) = \frac{S(a, \theta)}{S(a, \pi - a)} = \frac{I(\theta) - I(a)}{I(\pi - a) - I(a)}, \quad a \le \theta \le \pi - a$$

Angular support is reduced by an arbitrary angle $a > 0$ because $I(0)$ and $I(\pi)$ are undefined.

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 < \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi - a) - I(a)}\frac{(\cos^2\theta + 3)^2}{\sin^4\theta}\sin\theta$$

Note

A. Zee page 134 has the cross section

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4\pi}\right)^2 \frac{1}{8E^2} f(\theta)$$

where $f(\theta)$ is the probability density function

$$f(\theta) = \frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{2}{\sin^2(\theta/2)\cos^2(\theta/2)} + \frac{1 + \sin^4(\theta/2)}{\cos^4(\theta/2)}$$

The probability density function is equivalent to

$$f(\theta) = \frac{2(\cos^2\theta + 3)^2}{\sin^4\theta}$$

Hence for natural units $\varepsilon_0 = \hbar = c = 1$ and $e^2 = 4\pi\alpha$ the above cross section is equivalent to

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{4E^2}\frac{(\cos^2\theta + 3)^2}{\sin^4\theta}$$