

Electromagnetic tensor

This is the standard model for an EM field.

$$\mathbf{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}, \quad A^\mu = \begin{pmatrix} \phi \\ A_x \\ A_y \\ A_z \end{pmatrix}, \quad A_\mu = g_{\mu\nu} A^\nu = \begin{pmatrix} \phi \\ -A_x \\ -A_y \\ -A_z \end{pmatrix}$$

$$g_{\mu\nu} = ((1,0,0,0), (0,-1,0,0), (0,0,-1,0), (0,0,0,-1))$$

$$\begin{aligned} \mathbf{A} &= (A_x(), A_y(), A_z()) \\ \mathbf{A}^\mu &= (\phi(), A_x(), A_y(), A_z()) \\ \mathbf{A}_\mu &= \text{dot}(g_{\mu\nu}, \mathbf{A}^\nu) \end{aligned}$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\begin{aligned} \mathbf{B} &= \text{curl}(\mathbf{A}) \\ \mathbf{E} &= -\text{d}(\phi(), (x,y,z)) - \text{d}(\mathbf{A}, t) \end{aligned}$$

$$\begin{aligned} B_x &= B[1] \\ B_y &= B[2] \\ B_z &= B[3] \end{aligned}$$

$$\begin{aligned} E_x &= E[1] \\ E_y &= E[2] \\ E_z &= E[3] \end{aligned}$$

This is the electromagnetic tensor.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = A_{\nu,\mu} - A_{\mu,\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$\begin{aligned} X &= (t, x, y, z) \\ F_{dd} &= \text{d}(\mathbf{A}_d, X) \\ F_{dd} &= \text{transpose}(F_{dd}) - F_{dd} \end{aligned}$$

$$\mathbf{T} = ((0, E_x, E_y, E_z), (-E_x, 0, -B_z, B_y), (-E_y, B_z, 0, -B_x), (-E_z, -B_y, B_x, 0))$$

$$\text{check}(F_{dd} == \mathbf{T})$$

Check the following relations.

$$F_{\mu\nu} F^{\mu\nu} = 2(\mathbf{B}^2 - \mathbf{E}^2), \quad \det(F_{\mu\nu}) = \det(F^{\mu\nu}) = (\mathbf{B} \cdot \mathbf{E})^2$$

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Fuu = dot(gmunu,Fdd,gmunu)
T = contract(dot(transpose(Fdd),Fuu))
check(T == 2 dot(B,B) - 2 dot(E,E))
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check(det(Fdd) == dot(B,E)^2)
check(det(Fuu) == dot(B,E)^2)
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This is the vector current.

$$J^\nu = \partial_\mu F^{\mu\nu} = F^{\mu\nu}{}_{,\mu}$$

Gradient increases rank by one. The new index is the rightmost index, hence the contraction is over the first and third indices.

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Ju = contract(d(Fuu,X),1,3)
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Check the following relations.

$$\partial_\mu J^\mu = J^\mu{}_{,\mu} = 0, \quad \mathbf{J} = \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t}$$

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check(contract(d(Ju,X)) == 0)
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Jx = Ju[2]
Jy = Ju[3]
Jz = Ju[4]
J = (Jx,Jy,Jz)
check(J == curl(B) - d(E,t))
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