3-12. Let the wave function for a harmonic oscillator at time $t = 0$ be

$$\psi(x,0) = \exp\left(-\frac{m\omega}{2\hbar}(x-a)^2\right) \tag{1}$$

Using equation (3.42) and the results of problem 3-8 show that at time $t = T$ the wave function is

$$\psi(x,T) = \\ \exp\left(-\frac{i\omega T}{2} - \frac{m\omega}{2\hbar}\left(x^2 - 2ax\exp(-i\omega T) + a^2\cos(\omega T)\exp(-i\omega T)\right)\right)$$

From problem 3-8

$$K(x,T,x_c,0) = \\ \left(\frac{m\omega}{2\pi i\hbar\sin(\omega T)}\right)^{1/2}\exp\left(\frac{im\omega}{2\hbar\sin(\omega T)}\left((x^2+x_c^2)\cos(\omega T) - 2xx_c\right)\right) \tag{2}$$

From equation (3.42)

$$\psi(x,T) = \int_{-\infty}^{\infty} K(x,T,x_c,0)\psi(x_c,0)\,dx_c \tag{3}$$

Substitute (1) and (2) into (3) to obtain

$$\begin{aligned}\psi(x,T) &= \left(\frac{m\omega}{2\pi i\hbar\sin(\omega T)}\right)^{1/2} \\ &\times \int_{-\infty}^{\infty}\exp\left(\frac{im\omega}{2\hbar\sin(\omega T)}\left((x^2+x_c^2)\cos(\omega T) - 2xx_c\right)\right) \\ &\times \exp\left(-\frac{m\omega}{2\hbar}(x_c-a)^2\right)\,dx_c\end{aligned}$$

Rewrite as

$$\psi(x,T) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega T)}\right)^{1/2}\int_{-\infty}^{\infty}\exp(Ax_c^2 + Bx_c + C)\,dx_c \tag{4}$$

where

$$\begin{aligned}A &= \frac{m\omega}{2\hbar}\left(\frac{i\cos(\omega T)}{\sin(\omega T)} - 1\right) = \frac{im\omega\exp(i\omega T)}{2\hbar\sin(\omega T)} \\ B &= \frac{m\omega}{\hbar}\left(a - \frac{ix}{\sin(\omega T)}\right) \\ C &= \frac{m\omega}{2\hbar}\left(\frac{ix^2\cos(\omega T)}{\sin(\omega T)} - a^2\right)\end{aligned}$$

Solve the integral.

$$\int_{-\infty}^{\infty} \exp(Ax_c^2 + Bx_c + C) = \left(-\frac{\pi}{A}\right)^{1/2} \exp\left(-\frac{B^2}{4A} + C\right) \tag{5}$$

where

$$-\frac{\pi}{A} = \frac{2\pi i\hbar \sin(\omega T)}{m\omega \exp(i\omega T)} \tag{6}$$

and

$$-\frac{B^2}{4A} + C = -\frac{m\omega}{2\hbar}\left(x^2 - 2ax\exp(-i\omega T) + a^2\cos(\omega T)\exp(-i\omega T)\right) \tag{7}$$

Note that

$$\underbrace{\frac{m\omega}{2\pi i\hbar \sin(\omega T)}}_{\text{from equation (4)}} \times \underbrace{\frac{2\pi i\hbar \sin(\omega T)}{m\omega \exp(i\omega T)}}_{\text{from equation (6)}} = \exp(-i\omega T) \tag{8}$$

Substitute the solved integral (5) into (3) to obtain

$$\begin{aligned}\psi(x,T) = {} & \exp\left(-\frac{i\omega T}{2}\right) \\ & \times \exp\left(-\frac{m\omega}{2\hbar}\left(x^2 - 2ax\exp(-i\omega T) + a^2\cos(\omega T)\exp(-i\omega T)\right)\right)\end{aligned}$$