

Let $|\Psi\rangle$ be a coherent state where \bar{n} is the expected number of photons.

$$|\Psi\rangle = \sum_{n=0}^{\infty} \sqrt{\frac{\bar{n}^n \exp(-\bar{n})}{n!}} \exp\left(-i\left(n + \frac{1}{2}\right)\omega t\right) |n\rangle$$

It can be shown that¹

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}} \exp(-i\omega t)|\Psi\rangle$$

It follows that

$$\langle\Psi|\hat{a}^\dagger = (\hat{a}|\Psi\rangle)^\dagger = \sqrt{\bar{n}} \exp(i\omega t)\langle\Psi|$$

Let \hat{E} be the electric field operator

$$\hat{E} = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}(\hat{a} - \hat{a}^\dagger)$$

The expected electric field for the coherent state is

$$\langle\hat{E}\rangle = \langle\Psi|\hat{E}|\Psi\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\langle\Psi|(\hat{a} - \hat{a}^\dagger)|\Psi\rangle$$

Hence

$$\langle\hat{E}\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}} (\sqrt{\bar{n}} \exp(-i\omega t)\langle\Psi|\Psi\rangle - \sqrt{\bar{n}} \exp(i\omega t)\langle\Psi|\Psi\rangle)$$

By $\langle\Psi|\Psi\rangle = 1$ we have

$$\langle\hat{E}\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}} (\sqrt{\bar{n}} \exp(-i\omega t) - \sqrt{\bar{n}} \exp(i\omega t))$$

Recalling that

$$2\sin(\omega t) = i\exp(-i\omega t) - i\exp(i\omega t)$$

we have

$$\langle\hat{E}\rangle = \sqrt{\frac{2\hbar\omega\bar{n}}{\epsilon_0}} \sin(\omega t)$$

Hence the peak amplitude is proportional to $\sqrt{\bar{n}}$.

¹See *Quantum Mechanics for Scientists and Engineers* problem 15.6.1.

The total energy of the electromagnetic field per unit volume is

$$U = \frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{1}{2\mu_0} |\mathbf{B}|^2$$

For linearly polarized light we can by suitable rotation obtain

$$\mathbf{E} = \begin{pmatrix} E_x \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ B_y \\ 0 \end{pmatrix}$$

Hence in the rotated frame

$$U = \frac{\epsilon_0}{2} E_x^2 + \frac{1}{2\mu_0} B_y^2$$

For the quantum field we have

$$U = \frac{\epsilon_0}{2} \langle \hat{E}_x^2 \rangle + \frac{1}{2\mu_0} \langle \hat{B}_y^2 \rangle$$

where

$$\hat{E}_x = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}(\hat{a} - \hat{a}^\dagger), \quad \hat{B}_y = \sqrt{\frac{\hbar\omega\mu_0}{2}}(\hat{a} + \hat{a}^\dagger)$$

For the coherent state we have

$$\begin{aligned} \langle \Psi | \hat{a} \hat{a} | \Psi \rangle &= (\sqrt{\bar{n}} \exp(-i\omega t))^2 &&= \bar{n} \exp(-2i\omega t) \\ \langle \Psi | \hat{a} \hat{a}^\dagger | \Psi \rangle &= \langle \Psi | (\hat{a}^\dagger \hat{a} + 1) | \Psi \rangle &&= \bar{n} + 1 \\ \langle \Psi | \hat{a}^\dagger \hat{a} | \Psi \rangle &= (\sqrt{\bar{n}} \exp(i\omega t)) (\sqrt{\bar{n}} \exp(-i\omega t)) &&= \bar{n} \\ \langle \Psi | \hat{a}^\dagger \hat{a}^\dagger | \Psi \rangle &= (\sqrt{\bar{n}} \exp(i\omega t))^2 &&= \bar{n} \exp(2i\omega t) \end{aligned}$$

Hence

$$\begin{aligned} \langle \hat{E}_x^2 \rangle &= \langle \Psi | \hat{E}_x \hat{E}_x | \Psi \rangle = -\frac{\hbar\omega}{2\epsilon_0} \langle \Psi | (\hat{a} - \hat{a}^\dagger)(\hat{a} - \hat{a}^\dagger) | \Psi \rangle \\ &= -\frac{\hbar\omega}{2\epsilon_0} (\bar{n} \exp(-2i\omega t) + \bar{n} \exp(2i\omega t) - 2\bar{n} - 1) \end{aligned}$$

and

$$\begin{aligned} \langle \hat{B}_y^2 \rangle &= \langle \Psi | \hat{B}_y \hat{B}_y | \Psi \rangle = \frac{\hbar\omega\mu_0}{2} \langle \Psi | (\hat{a} + \hat{a}^\dagger)(\hat{a} + \hat{a}^\dagger) | \Psi \rangle \\ &= \frac{\hbar\omega\mu_0}{2} (\bar{n} \exp(-2i\omega t) + \bar{n} \exp(2i\omega t) + 2\bar{n} + 1) \end{aligned}$$

Noting that

$$\begin{aligned} 4 \sin(\omega t)^2 &= -\exp(-2i\omega t) - \exp(2i\omega t) + 2 \\ 4 \cos(\omega t)^2 &= \exp(-2i\omega t) + \exp(2i\omega t) + 2 \end{aligned}$$

we have

$$\begin{aligned} \langle \hat{E}_x^2 \rangle &= \frac{2\bar{n}\hbar\omega}{\epsilon_0} \sin(\omega t)^2 + \frac{\hbar\omega}{2\epsilon_0} \\ \langle \hat{B}_y^2 \rangle &= 2\bar{n}\hbar\omega\mu_0 \cos(\omega t)^2 + \frac{\hbar\omega\mu_0}{2} \end{aligned}$$

Rewrite as

$$\begin{aligned} \frac{\epsilon_0}{2} \langle \hat{E}_x^2 \rangle &= \bar{n}\hbar\omega \sin(\omega t)^2 + \frac{\hbar\omega}{4} \\ \frac{1}{2\mu_0} \langle \hat{B}_y^2 \rangle &= \bar{n}\hbar\omega \cos(\omega t)^2 + \frac{\hbar\omega}{4} \end{aligned}$$

Hence the total energy per unit volume is

$$U = \frac{\epsilon_0}{2} \langle \hat{E}_x^2 \rangle + \frac{1}{2\mu_0} \langle \hat{B}_y^2 \rangle = \left(\bar{n} + \frac{1}{2} \right) \hbar\omega$$