

This is the Dirac equation.

$$i\hbar \left( \frac{1}{c} \gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z} \right) \psi = mc\psi$$

The following set of gamma matrices is known as the “Dirac representation.”

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Let  $\phi$  be the field

$$\phi(x, y, z, t) = p_x x + p_y y + p_z z - Et$$

where

$$E = \sqrt{p_x^2 c^2 + p_y^2 c^2 + p_z^2 c^2 + m^2 c^4}$$

The four positive wave solutions to the Dirac equation are

$$\begin{aligned} \psi_1 &= \begin{pmatrix} E/c + mc \\ 0 \\ p_z \\ p_x + ip_y \end{pmatrix} \exp\left(\frac{i\phi}{\hbar}\right) & \psi_2 &= \begin{pmatrix} 0 \\ E/c + mc \\ p_x - ip_y \\ -p_z \end{pmatrix} \exp\left(\frac{i\phi}{\hbar}\right) \\ \psi_3 &= \begin{pmatrix} p_z \\ p_x + ip_y \\ E/c - mc \\ 0 \end{pmatrix} \exp\left(\frac{i\phi}{\hbar}\right) & \psi_4 &= \begin{pmatrix} p_x - ip_y \\ -p_z \\ 0 \\ E/c - mc \end{pmatrix} \exp\left(\frac{i\phi}{\hbar}\right) \end{aligned}$$

The four negative wave solutions are

$$\begin{aligned} \psi_5 &= \begin{pmatrix} E/c - mc \\ 0 \\ p_z \\ p_x + ip_y \end{pmatrix} \exp\left(-\frac{i\phi}{\hbar}\right) & \psi_6 &= \begin{pmatrix} 0 \\ E/c - mc \\ p_x - ip_y \\ -p_z \end{pmatrix} \exp\left(-\frac{i\phi}{\hbar}\right) \\ \psi_7 &= \begin{pmatrix} p_z \\ p_x + ip_y \\ E/c + mc \\ 0 \end{pmatrix} \exp\left(-\frac{i\phi}{\hbar}\right) & \psi_8 &= \begin{pmatrix} p_x - ip_y \\ -p_z \\ 0 \\ E/c + mc \end{pmatrix} \exp\left(-\frac{i\phi}{\hbar}\right) \end{aligned}$$

The negative wave solutions flip the sign of the  $mc^2$  term.

The following solutions are used in quantum electrodynamics.

- $\psi_1$  fermion, spin up
- $\psi_2$  fermion, spin down
- $\psi_7$  anti-fermion, spin up
- $\psi_8$  anti-fermion, spin down

Here is a check of physical units. The momenta  $p_x$ ,  $p_y$ , and  $p_z$  have units of

$$\frac{\text{kilogram meter}}{\text{second}}$$

Hence

$$p_x x \propto \frac{\text{kilogram meter}^2}{\text{second}}$$

For the time-dependent term

$$Et \propto \frac{\text{kilogram meter}^2}{\text{second}^2} \times \text{second} = \frac{\text{kilogram meter}^2}{\text{second}}$$

We have for the reduced Planck constant

$$\hbar \propto \frac{\text{kilogram meter}^2}{\text{second}}$$

Hence  $\phi/\hbar$  is dimensionless as required by the exponential function.

$$\frac{p_x x - Et}{\hbar} \propto \frac{\text{kilogram meter}^2}{\text{second}} \times \frac{\text{second}}{\text{kilogram meter}^2} = 1$$

The derivatives introduce inverse units.

$$\frac{\partial \psi}{\partial t} \propto \frac{1}{\text{second}} \quad \frac{\partial \psi}{\partial x} \propto \frac{1}{\text{meter}}$$

Hence

$$\frac{\hbar}{c} \frac{\partial \psi}{\partial t} \propto \frac{\text{kilogram meter}}{\text{second}}$$

and

$$\hbar \frac{\partial \psi}{\partial x} \propto \frac{\text{kilogram meter}}{\text{second}}$$

The resulting units match the right-hand side of the Dirac equation.

$$mc \propto \frac{\text{kilogram meter}}{\text{second}}$$