

Feynman and Hibbs problem 3-9

Find the kernel for a particle in a constant external field f where the Lagrangian is

$$L = \frac{m}{2} \dot{x}(t)^2 + fx(t)$$

We have

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = m\ddot{x}(t)$$

and

$$\frac{\partial L}{\partial x} = f$$

By equation (2.7) which is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$$

we have

$$\ddot{x}(t) = 0$$

Hence velocity is constant. It follows that $x(t)$ is velocity times time.

$$x(t) = \left(\frac{x_b - x_a}{t_b - t_a} \right) (t - t_a) + x_a$$

By equation (2.1) which is

$$S = \int_{t_a}^{t_b} L dt$$

we have

$$S_{cl} = \int_{t_a}^{t_b} \frac{m}{2} \dot{x}(t)^2 dt + \int_{t_a}^{t_b} fx(t) dt$$

Equation (2.8) provides the following solution to the first integral.

$$\int_{t_a}^{t_b} \frac{m}{2} \dot{x}(t)^2 dt = \frac{m(x_b - x_a)^2}{2(t_b - t_a)} \quad (1)$$

Solve the second integral.

$$\int_{t_a}^{t_b} fx(t) dt = \frac{(x_b + x_a)(t_b - t_a)f}{2} \quad (2)$$

Action S_{cl} is the sum of (1) and (2).

$$S_{cl} = \frac{m(x_b - x_a)^2}{2(t_b - t_a)} + \frac{(x_b + x_a)(t_b - t_a)f}{2}$$

By equation (3.51) which is

$$K(b, a) = F(t_b - t_a) \exp\left(\frac{iS_{cl}}{\hbar}\right)$$

we have

$$K(b, a) = F(t_b - t_a) \exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)} + \frac{i(x_b + x_a)(t_b - t_a)f}{2\hbar}\right) \quad (3)$$

We now proceed to compute F . By equation (2.31) which is

$$K(b, a) = \int_{-\infty}^{\infty} K(b, c)K(c, a) dx_c$$

we have

$$\begin{aligned} K(b, a) &= \int_{-\infty}^{\infty} F(t_b - t_c) \exp\left(\frac{im(x_b - x_c)^2}{2\hbar(t_b - t_c)} + \frac{i(x_b + x_c)(t_b - t_c)f}{2\hbar}\right) \\ &\quad \times F(t_c - t_a) \exp\left(\frac{im(x_c - x_a)^2}{2\hbar(t_c - t_a)} + \frac{i(x_c + x_a)(t_c - t_a)f}{2\hbar}\right) dx_c \end{aligned} \quad (4)$$

Rewrite equation (4) as

$$K(b, a) = F(t_b - t_c)F(t_c - t_a) \int_{-\infty}^{\infty} \exp(iAx_c^2 + iBx_c + iC) \quad (5)$$

where

$$A = \frac{m}{2\hbar} \left(\frac{1}{t_b - t_c} + \frac{1}{t_c - t_a} \right) \quad (6)$$

$$B = \frac{(t_b - t_a)f}{2\hbar} - \frac{m}{\hbar} \left(\frac{x_b}{t_b - t_c} + \frac{x_a}{t_c - t_a} \right) \quad (7)$$

$$C = \frac{f}{2\hbar} (x_b(t_b - t_c) + x_a(t_c - t_a)) + \frac{m}{2\hbar} \left(\frac{x_b^2}{t_b - t_c} + \frac{x_a^2}{t_c - t_a} \right) \quad (8)$$

From the following integral solution formula

$$\int_{-\infty}^{\infty} \exp(iAx_c^2 + iBx_c + iC) dx_c = \left(-\frac{\pi}{iA}\right)^{1/2} \exp\left(-\frac{iB^2}{4A} + iC\right)$$

we have

$$\begin{aligned} \int_{-\infty}^{\infty} \exp(iAx_c^2 + iBx_c + iC) dx_c &= \left(-\frac{2\pi\hbar(t_b - t_c)(t_c - t_a)}{im(t_b - t_a)}\right)^{1/2} \\ &\times \exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)} + \frac{i(x_b + x_a)(t_b - t_a)f}{2\hbar} - \frac{i(t_b - t_a)(t_b - t_c)(t_c - t_a)f^2}{8\hbar m}\right) \end{aligned} \quad (9)$$

Substitute (9) into (5).

$$\begin{aligned} K(b, a) &= F(t_b - t_c)F(t_c - t_a) \left(-\frac{2\pi\hbar(t_b - t_c)(t_c - t_a)}{im(t_b - t_a)}\right)^{1/2} \\ &\times \exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)} + \frac{i(x_b + x_a)(t_b - t_a)f}{2\hbar} - \frac{i(t_b - t_a)(t_b - t_c)(t_c - t_a)f^2}{8\hbar m}\right) \end{aligned} \quad (10)$$

Equating (3) with (10) reduces to

$$\begin{aligned} F(t_b - t_a) &= F(t_b - t_c)F(t_c - t_a) \\ &\times \left(-\frac{2\pi\hbar(t_b - t_c)(t_c - t_a)}{im(t_b - t_a)}\right)^{1/2} \exp\left(-\frac{i(t_b - t_a)(t_b - t_c)(t_c - t_a)f^2}{8\hbar m}\right) \end{aligned} \quad (11)$$

From problem 3-7, let

$$F(t) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} g(t) \quad (12)$$

Substitute (12) into (11) to obtain

$$\begin{aligned} &\left(\frac{m}{2\pi i\hbar(t_b - t_a)}\right)^{1/2} g(t_b - t_a) \\ &= \left(\frac{m}{2\pi i\hbar(t_b - t_c)}\right)^{1/2} g(t_b - t_c) \left(\frac{m}{2\pi i\hbar(t_c - t_a)}\right)^{1/2} g(t_c - t_a) \\ &\times \left(-\frac{2\pi\hbar(t_b - t_c)(t_c - t_a)}{im(t_b - t_a)}\right)^{1/2} \exp\left(-\frac{i(t_b - t_a)(t_b - t_c)(t_c - t_a)f^2}{8\hbar m}\right) \end{aligned}$$

The coefficients cancel leaving

$$g(t_b - t_a) = g(t_b - t_c)g(t_c - t_a) \exp \left(-\frac{i(t_b - t_a)(t_b - t_c)(t_c - t_a)f^2}{8\hbar m} \right) \quad (13)$$

Note that

$$(t_b - t_a)^3 = (t_b - t_c)^3 + 3(t_b - t_a)(t_b - t_c)(t_c - t_a) + (t_c - t_a)^3 \quad (14)$$

Solve for $g(t)$ by completing the cube.

$$g(t) = \exp \left(-\frac{it^3 f^2}{24\hbar m} \right) \quad (15)$$

Substitute (15) into (12).

$$F(t) = \left(\frac{m}{2\pi i \hbar t} \right)^{1/2} \exp \left(-\frac{it^3 f^2}{24\hbar m} \right) \quad (16)$$

Substitute (16) into (3).

$$\begin{aligned} K(b, a) = & \left(\frac{m}{2\pi i \hbar (t_b - t_a)} \right)^{1/2} \\ & \times \exp \left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)} + \frac{i(x_b + x_a)(t_b - t_a)f}{2\hbar} - \frac{i(t_b - t_a)^3 f^2}{24\hbar m} \right) \end{aligned}$$