

# 1 Anticommutation

Consider the following eigenstates of a hypothetical quantum system.<sup>1</sup>

$$\begin{aligned} |00\rangle &= (1 \ 0 \ 0 \ 0) && \text{no fermions} \\ |10\rangle &= (0 \ 1 \ 0 \ 0) && \text{one fermion in state 1} \\ |01\rangle &= (0 \ 0 \ 1 \ 0) && \text{one fermion in state 2} \\ |11\rangle &= (0 \ 0 \ 0 \ 1) && \text{two fermions, one in state 1, one in state 2} \end{aligned}$$

Creation and annihilation operators are formed from outer products of state vectors. Sign changes make the operators antisymmetric.

$$\begin{aligned} \hat{b}_1^\dagger &= |10\rangle\langle 00| - |11\rangle\langle 01| && \text{Create one fermion in state 1} \\ \hat{b}_1 &= |00\rangle\langle 10| - |01\rangle\langle 11| && \text{Annihilate one fermion in state 1} \\ \hat{b}_2^\dagger &= |01\rangle\langle 00| + |11\rangle\langle 10| && \text{Create one fermion in state 2} \\ \hat{b}_2 &= |00\rangle\langle 01| + |10\rangle\langle 11| && \text{Annihilate one fermion in state 2} \end{aligned}$$

The operators in matrix form.

$$\hat{b}_1^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \hat{b}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \hat{b}_2^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \hat{b}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Verify anticommutation relations of the operators.

$$\hat{b}_j \hat{b}_k + \hat{b}_k \hat{b}_j = 0$$

$$\hat{b}_j^\dagger \hat{b}_k^\dagger + \hat{b}_k^\dagger \hat{b}_j^\dagger = 0$$

$$\hat{b}_j \hat{b}_k^\dagger + \hat{b}_k^\dagger \hat{b}_j = \delta_{jk}$$

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<sup>1</sup>Adapted from problem 16.1.1 of “Quantum Mechanics for Scientists and Engineers.”  
<https://ee.stanford.edu/~dabm/QMbook.html>

## 2 Wavefunction operator

Consider the following eigenstates of a hypothetical quantum system.<sup>2</sup>

$ 00\rangle = (1\ 0\ 0\ 0)$	no fermions
$ 10\rangle = (0\ 1\ 0\ 0)$	one fermion in state $\phi_1$
$ 01\rangle = (0\ 0\ 1\ 0)$	one fermion in state $\phi_2$
$ 11\rangle = (0\ 0\ 0\ 1)$	two fermions, one in state $\phi_1$ , one in state $\phi_2$

Let fermion states  $\phi_n$  be modeled by a one dimensional box of length  $L$ .

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

Creation and annihilation operators are formed from outer products of state vectors. Sign changes make the operators antisymmetric.

$\hat{b}_1^\dagger =  10\rangle\langle 00  -  11\rangle\langle 01 $	Create one fermion in state $\phi_1$
$\hat{b}_1 =  00\rangle\langle 10  -  01\rangle\langle 11 $	Annihilate one fermion in state $\phi_1$
$\hat{b}_2^\dagger =  01\rangle\langle 00  +  11\rangle\langle 10 $	Create one fermion in state $\phi_2$
$\hat{b}_2 =  00\rangle\langle 01  +  10\rangle\langle 11 $	Annihilate one fermion in state $\phi_2$

Given the wavefunction operator

$$\hat{\psi} = \frac{1}{\sqrt{2}} \sum_{n,m} \phi_n(x) \phi_m(y) \hat{b}_n \hat{b}_m$$

show that

$$\hat{\psi}|11\rangle = \frac{1}{\sqrt{2}}(\phi_1(x)\phi_2(y) - \phi_1(y)\phi_2(x))|00\rangle$$

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<sup>2</sup>Adapted from problem 16.2.1 of “Quantum Mechanics for Scientists and Engineers.”  
<https://ee.stanford.edu/~dabm/QMbook.html>

### 3 Position operator

Consider the following eigenstates of a hypothetical quantum system.

$ 00\rangle = (1\ 0\ 0\ 0)$	no fermions
$ 10\rangle = (0\ 1\ 0\ 0)$	one fermion in state $\phi_1$
$ 01\rangle = (0\ 0\ 1\ 0)$	one fermion in state $\phi_2$
$ 11\rangle = (0\ 0\ 0\ 1)$	two fermions, one in state $\phi_1$ , one in state $\phi_2$

Let fermion states  $\phi_n$  be modeled by a one dimensional box of length  $L$ .

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

Creation and annihilation operators are formed from outer products of state vectors. Sign changes make the operators antisymmetric.

$\hat{b}_1^\dagger =  10\rangle\langle 00  -  11\rangle\langle 01 $	Create one fermion in state $\phi_1$
$\hat{b}_1 =  00\rangle\langle 10  -  01\rangle\langle 11 $	Annihilate one fermion in state $\phi_1$
$\hat{b}_2^\dagger =  01\rangle\langle 00  +  11\rangle\langle 10 $	Create one fermion in state $\phi_2$
$\hat{b}_2 =  00\rangle\langle 01  +  10\rangle\langle 11 $	Annihilate one fermion in state $\phi_2$

Let  $\hat{r}$  be the position operator

$$\hat{r} = \sum_{n,m} r_{nm} \hat{b}_n^\dagger \hat{b}_m$$

where

$$r_{nm} = \int_0^L \phi_n^*(x) x \phi_m(x) dx$$

Note that for a one dimensional box

$$r_{nn} = \langle x \rangle = \frac{1}{2}L$$

Verify that

$$\begin{aligned} \langle 10|\hat{r}|10\rangle &= r_{11} \\ \langle 10|\hat{r}|01\rangle &= r_{12} \\ \langle 01|\hat{r}|10\rangle &= r_{21} \\ \langle 01|\hat{r}|01\rangle &= r_{22} \end{aligned}$$

## 4 Superposition of eigenstates

Consider a system with the following eigenstates.

$$\begin{aligned} |0\rangle &= (1\ 0\ 0\ 0) && \text{no electrons} \\ |1\rangle &= (0\ 1\ 0\ 0) && \text{one electron in state } \phi_1 \\ |2\rangle &= (0\ 0\ 1\ 0) && \text{one electron in state } \phi_2 \\ |3\rangle &= (0\ 0\ 0\ 1) && \text{two electrons, one in state } \phi_1, \text{ one in state } \phi_2 \end{aligned}$$

Then for the wavefunction basis

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

and for  $L = 10^{-9}$  meters we have

$$\hat{E} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.38\text{ eV} & 0 & 0 \\ 0 & 0 & 1.50\text{ eV} & 0 \\ 0 & 0 & 0 & 6.55\text{ eV} \end{pmatrix}$$

Let  $|\xi\rangle$  be the state vector

$$|\xi\rangle = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

The expected energy is

$$\langle\xi|\hat{E}|\xi\rangle = \frac{0\text{ eV}}{4} + \frac{0.38\text{ eV}}{4} + \frac{1.50\text{ eV}}{4} + \frac{6.55\text{ eV}}{4} = 2.11\text{ eV}$$

For the system we are considering, the result of a single measurement is either 0 eV, 0.38 eV, 1.50 eV, or 6.55 eV. The value 2.11 eV is the expected average across multiple measurements. Recall that a measurement causes the system to exit state  $|\xi\rangle$  and enter an eigenstate  $|0\rangle$ ,  $|1\rangle$ ,  $|2\rangle$ , or  $|3\rangle$  corresponding to the measured eigenvalue. The system must be put back in state  $|\xi\rangle$  before the next measurement.

To use a slot machine analogy, state  $|\xi\rangle$  is like the wheels spinning. Observing the system makes the wheels stop. The stopped wheels are in an eigenstate  $|0\rangle$ ,  $|1\rangle$ ,  $|2\rangle$ , or  $|3\rangle$ . Once they are stopped the wheels don't change, they remain in the same eigenstate. You have to pull the lever to get the wheels spinning again.