

# Rutherford scattering 1

Find the scattering cross section for Coulomb potential  $V(r)$ .

$$V(r) = -\frac{Ze^2}{4\pi\epsilon_0 r}$$

Start with the Born approximation.

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |Q|^2, \quad Q = \int \exp\left(\frac{i\mathbf{p} \cdot \mathbf{r}}{\hbar}\right) V(\mathbf{r}) d\mathbf{r}^3$$

Convert  $Q$  to polar coordinates.

$$Q = \int_0^{2\pi} \int_0^\pi \int_0^\infty \exp\left(\frac{ipr \cos \theta}{\hbar}\right) V(r, \theta, \phi) r^2 \sin \theta dr d\theta d\phi$$

Substitute the Coulomb potential for  $V(r, \theta, \phi)$  and note  $r^2$  becomes  $r$ .

$$Q = -\frac{Ze^2}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^\pi \int_0^\infty \exp\left(\frac{ipr \cos \theta}{\hbar}\right) r \sin \theta dr d\theta d\phi$$

Integrate over  $\phi$  (multiplies  $Q$  by  $2\pi$ ).

$$Q = -\frac{Ze^2}{2\epsilon_0} \int_0^\pi \int_0^\infty \exp\left(\frac{ipr \cos \theta}{\hbar}\right) r \sin \theta dr d\theta$$

Transform the integral over  $\theta$  to an integral over  $y$  where  $y = \cos \theta$  and  $dy = -\sin \theta d\theta$ . The minus sign in  $dy$  is canceled by interchanging integration limits  $\cos 0 = 1$  and  $\cos \pi = -1$ .

$$Q = -\frac{Ze^2}{2\epsilon_0} \int_{-1}^1 \int_0^\infty \exp\left(\frac{ipry}{\hbar}\right) r dr dy$$

Solve the integral over  $y$  (note  $r$  in the integrand cancels).

$$Q = -\frac{Ze^2}{2\epsilon_0} \int_0^\infty \frac{\hbar}{ip} \left[ \exp\left(\frac{ipr}{\hbar}\right) - \exp\left(-\frac{ipr}{\hbar}\right) \right] dr$$

Solve the integral over  $r$ .

$$Q = -\frac{Ze^2}{2\epsilon_0} \frac{\hbar}{ip} \left[ \frac{\hbar}{ip} \exp\left(\frac{ipr}{\hbar}\right) + \frac{\hbar}{ip} \exp\left(-\frac{ipr}{\hbar}\right) \right]_0^\infty$$

The first exponential is a problem so go back and multiply the integrand by  $\exp(-\epsilon r)$ .

$$Q = -\frac{Ze^2}{2\epsilon_0} \int_0^\infty \frac{\hbar}{ip} \left[ \exp\left(\frac{ipr}{\hbar} - \epsilon r\right) - \exp\left(-\frac{ipr}{\hbar} - \epsilon r\right) \right] dr$$

Solve the integral.

$$Q = -\frac{Ze^2}{2\epsilon_0} \frac{\hbar}{ip} \left[ \frac{1}{ip/\hbar - \epsilon} \exp\left(\frac{ipr}{\hbar} - \epsilon r\right) + \frac{1}{ip/\hbar + \epsilon} \exp\left(-\frac{ipr}{\hbar} - \epsilon r\right) \right]_0^\infty$$

Evaluate the limits.

$$Q = -\frac{Ze^2}{2\varepsilon_0} \frac{\hbar}{ip} \left( -\frac{1}{ip/\hbar - \epsilon} - \frac{1}{ip/\hbar + \epsilon} \right) = -\frac{Ze^2}{2\varepsilon_0} \frac{2}{(p/\hbar)^2 + \epsilon^2} \quad (1)$$

Set  $\epsilon = 0$  to obtain

$$Q = -\frac{Ze^2\hbar^2}{\varepsilon_0 p^2}$$

Calculate the cross section.

$$\frac{d\sigma}{d\Omega} = \left( \frac{m}{2\pi\hbar^2} \right)^2 |Q|^2 = \frac{m^2 Z^2 e^4}{4\pi^2 \varepsilon_0^2 p^4} \quad (2)$$

Substitute  $16\pi^2 \varepsilon_0^2 \alpha^2 \hbar^2 c^2$  for  $e^4$ .

$$\frac{d\sigma}{d\Omega} = \frac{4m^2 Z^2 \alpha^2 \hbar^2 c^2}{p^4}$$

Symbol  $p$  is momentum transfer  $|\mathbf{p}_i| - |\mathbf{p}_f|$  such that

$$p^2 = 4mE(1 - \cos \theta)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{Z^2 \alpha^2 \hbar^2 c^2}{4E^2 (1 - \cos \theta)^2} \quad (3)$$

Noting that

$$4 \sin^4 \frac{\theta}{2} = (1 - \cos \theta)^2$$

we have the alternative form of (3)

$$\frac{d\sigma}{d\Omega} = \frac{Z^2 \alpha^2 \hbar^2 c^2}{16E^2 \sin^4(\theta/2)}$$