Exercise 6.9. Prove that the four vectors $|sing\rangle$, $|T_1\rangle$, $|T_2\rangle$, and $|T_3\rangle$ are eigenvectors of $\vec{\sigma} \cdot \vec{\tau}$. What are their eigenvalues?

Recall that

$$\vec{\sigma} \cdot \vec{\tau} = \sigma_x \tau_x + \sigma_y \tau_y + \sigma_z \tau_z$$

Let A and B be the sets

$$\begin{aligned} A &= \{|uu\rangle, |ud\rangle, |du\rangle, |dd\rangle\} \\ B &= \{|sing\rangle, |T_1\rangle, |T_2\rangle, |T_3\rangle\} \end{aligned}$$

By Table 1 on page 350, the vectors in A are eigenvectors of spin operators σ and τ. By closure of Table 1, the vectors in A are also eigenvectors of compositions of σ and τ.

The vectors in B are linear combinations of the vectors in A. Hence by linearity, the vectors in B are eigenvectors of σ and τ and compositions of σ and τ. Finally, by linearity of the $\vec{\sigma} \cdot \vec{\tau}$ operator, the vectors in A and B are eigenvectors of $\vec{\sigma} \cdot \vec{\tau}$.

By Table 1 we obtain the following eigenvalues.

	$\|sing\rangle$	$\|T_1\rangle$	$\|T_2\rangle$	$\|T_3\rangle$
$\sigma_x \tau_x$	-1	1	1	-1
$\sigma_y \tau_y$	-1	1	-1	1
$\sigma_z \tau_z$	-1	-1	1	1
$\vec{\sigma} \cdot \vec{\tau}$	-3	1	1	1