Bohr radius

Let $E(r)$ be total electron energy for a hydrogen atom (kinetic energy plus potential energy).

$$E(r) = \frac{\hbar^2}{2m_e r^2} - \frac{e^2}{4\pi\varepsilon_0 r}$$

Energy is minimized where

$$\frac{d}{dr}E(r) = 0$$

Noting that

$$\frac{d}{dr}E(r) = -\frac{\hbar^2}{m_e r^3} + \frac{e^2}{4\pi\varepsilon_0 r^2}$$

find r such that

$$\frac{d}{dr}E(r) = -\frac{\hbar^2}{m_e r^3} + \frac{e^2}{4\pi\varepsilon_0 r^2} = 0$$

Multiply both sides by r^3.

$$-\frac{\hbar^2}{m_e} + \frac{e^2 r}{4\pi\varepsilon_0} = 0$$

Rewrite as

$$\frac{e^2 r}{4\pi\varepsilon_0} = \frac{\hbar^2}{m_e}$$

Hence energy is minimized for

$$r = \frac{4\pi\varepsilon_0\hbar^2}{e^2 m_e} \equiv a_0$$