Bell's theorem

The following theorem of independent random variables is the key to understanding Bell's theorem. If two random variables A and B are independent (uncorrelated) then

$$\langle A \rangle \langle B \rangle = \langle AB \rangle$$

Consider two machines A and B that measure spin. Each machine can be set in one of two orientations labeled 0 and 1. Assuming the measurements are uncorrelated we have the following table of expectation values and a clever formula.

$\langle A_0 \rangle$	$\langle A_1 \rangle$	$\langle B_0 \rangle$	$\langle B_1 \rangle$	$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle$
1	1	1	1	2
1	1	1	-1	2
1	1	-1	1	-2
1	1	-1	-1	-2
1	-1	1	1	2
1	-1	1	-1	-2
1	-1	-1	1	2
1	-1	-1	-1	-2
-1	1	1	1	-2
-1	1	1	-1	2
-1	1	-1	1	-2
-1	1	-1	-1	2
-1	-1	1	1	-2
-1	-1	1	-1	-2
-1	-1	-1	1	2
-1	-1	-1	-1	2

Since the table is for all minimum and maximum values we have by inspection

$$-2 \leq \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2 \qquad (1)$$

Now suppose a third machine generates two spins in the following entangled state.

$$|s\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

One spin is sent to A and the other is sent to B.

Let

$$A_0 = \sigma_z, \quad A_1 = \sigma_x, \quad B_0 = -\frac{\sigma_x + \sigma_z}{\sqrt{2}}, \quad B_1 = \frac{\sigma_x - \sigma_z}{\sqrt{2}}$$

Then for the entangled state $|s\rangle$ we have

$$\langle A_0 B_0 \rangle = \frac{1}{\sqrt{2}}, \quad \langle A_0 B_1 \rangle = \frac{1}{\sqrt{2}}, \quad \langle A_1 B_0 \rangle = \frac{1}{\sqrt{2}}, \quad \langle A_1 B_1 \rangle = -\frac{1}{\sqrt{2}}$$

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Hence
$$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle = 2\sqrt{2} \tag{2}$$

The result in (2) conflicts with (1) because for an entangled state the random variables are not independent. Any theory that asserts A and B are independent is constrained by (1) and falsified by (2). Hence Bell's theorem: No local theory can explain quantum mechanics. (The term "local" means A and B are independent.)

Exercises

1. Verify equation (2).

2. Verify that for the singlet state $|s\rangle$ given above we have

$$\langle A_0 \rangle = 0, \quad \langle A_1 \rangle = 0, \quad \langle B_0 \rangle = 0, \quad \langle B_1 \rangle = 0.$$

Hence $\langle A \rangle \langle B \rangle \neq \langle AB \rangle$ for the singlet state.

3. There are three additional entangled states.

$$|s_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad |s_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad |s_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Verify that A and B are correlated for all entangled states.