Rutherford scattering 1

Find the scattering cross section for the Coulomb potential

$$V(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r}$$

Start with the Born approximation for scattering amplitude $f(\mathbf{p})$ where $\mathbf{p}$ is momentum transfer.

$$f(\mathbf{p}) = \frac{m}{2\pi\hbar^2} \int \exp\left(\frac{i\mathbf{p}\cdot\mathbf{r}}{\hbar}\right) V(\mathbf{r})\, d\mathbf{r}$$

Convert to polar coordinates where $p = |\mathbf{p}|$.

$$f(\mathbf{p}) = \frac{m}{2\pi\hbar^2} \int_0^\infty \int_0^\pi \int_0^{2\pi} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) V(r,\theta,\phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Substitute the Coulomb potential

$$V(r,\theta,\phi) = -\frac{Ze^2}{4\pi\epsilon_0 r}$$

to obtain

$$f(\mathbf{p}) = -\frac{mZe^2}{8\pi^2\varepsilon_0\hbar^2} \int_0^\infty \int_0^\pi \int_0^{2\pi} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) r \sin\theta\, dr\, d\theta\, d\phi$$

Integrate over ϕ (multiply by 2π).

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\varepsilon_0\hbar^2} \int_0^\infty \int_0^\pi \exp\left(\frac{ipr\cos\theta}{\hbar}\right) r \sin\theta\, dr\, d\theta$$

Transform the integral over θ into an integral over y where $y = \cos\theta$ and $dy = -\sin\theta\, d\theta$. The minus sign in dy is canceled by interchanging integration limits $\cos 0 = 1$ and $\cos\pi = -1$.

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\varepsilon_0\hbar^2} \int_0^\infty \int_{-1}^1 \exp\left(\frac{ipry}{\hbar}\right) r\, dr\, dy$$

Solve the integral over y and note that r in the integrand cancels.

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\varepsilon_0\hbar^2} \int_0^\infty \frac{\hbar}{ip} \left[\exp\left(\frac{ipr}{\hbar}\right) - \exp\left(-\frac{ipr}{\hbar}\right)\right] dr$$

Solve the integral over r.

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\varepsilon_0\hbar^2} \frac{\hbar}{ip} \left[\frac{\hbar}{ip}\exp\left(\frac{ipr}{\hbar}\right) + \frac{\hbar}{ip}\exp\left(-\frac{ipr}{\hbar}\right)\right]_{r=0}^{r=\infty}$$

The exponentials fail to converge at the upper limit. The workaround is to go back and multiply the integrand by $\exp(-\epsilon r) \approx 1$.

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\varepsilon_0\hbar^2} \int_0^\infty \frac{\hbar}{ip} \left[\exp\left(\frac{ipr}{\hbar} - \epsilon r\right) - \exp\left(-\frac{ipr}{\hbar} - \epsilon r\right)\right] dr$$

Solve the modified integral.

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\varepsilon_0\hbar^2}\frac{\hbar}{ip}\left[\frac{1}{ip/\hbar-\epsilon}\exp\left(\frac{ipr}{\hbar}-\epsilon r\right)+\frac{1}{ip/\hbar+\epsilon}\exp\left(-\frac{ipr}{\hbar}-\epsilon r\right)\right]_{r=0}^{r=\infty}$$

Evaluate the limits. Now the exponentials vanish at the upper limit.

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\varepsilon_0\hbar^2}\frac{\hbar}{ip}\left(-\frac{1}{ip/\hbar-\epsilon}-\frac{1}{ip/\hbar+\epsilon}\right) \tag{1}$$

Set $\epsilon = 0$ to obtain

$$f(\mathbf{p}) = -\frac{mZe^2}{2\pi\varepsilon_0 p^2}$$

Substitute $e^2 = 4\pi\varepsilon_0\alpha\hbar c$.

$$f(\mathbf{p}) = -\frac{2mZ\alpha\hbar c}{p^2}$$

Note that $\mathbf{p}$ is momentum transfer such that

$$p^2 = |\mathbf{p}|^2 = 4mE(1-\cos\theta)$$

Hence

$$f(\theta) = -\frac{Z\alpha\hbar c}{2E(1-\cos\theta)}$$

Calculate the cross section.

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{Z^2\alpha^2(\hbar c)^2}{4E^2(1-\cos\theta)^2} \tag{2}$$

Note that

$$(1-\cos\theta)^2 = 4\sin^4(\theta/2)$$

Hence equation (2) is equivalent to

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2(\hbar c)^2}{16E^2\sin^4(\theta/2)} \tag{3}$$