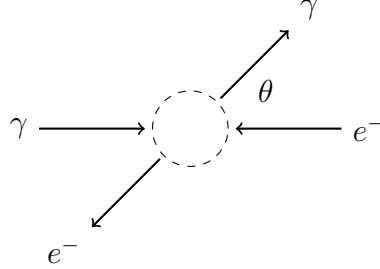


# Compton scattering

Compton scattering is the interaction  $e^- + \gamma \rightarrow e^- + \gamma$ .



In the center-of-mass frame we have the following momentum vectors where  $E = \sqrt{\omega^2 + m^2}$ .

$$\begin{aligned}
 p_1 &= \begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix} & p_2 &= \begin{pmatrix} E \\ 0 \\ 0 \\ -\omega \end{pmatrix} & p_3 &= \begin{pmatrix} \omega \\ \omega \sin \theta \cos \phi \\ \omega \sin \theta \sin \phi \\ \omega \cos \theta \end{pmatrix} & p_4 &= \begin{pmatrix} E \\ -\omega \sin \theta \cos \phi \\ -\omega \sin \theta \sin \phi \\ -\omega \cos \theta \end{pmatrix} \\
 &\text{inbound photon} & &\text{inbound electron} & &\text{outbound photon} & &\text{outbound electron}
 \end{aligned}$$

Spinors for the inbound electron.

$$\begin{aligned}
 u_{21} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ -\omega \\ 0 \end{pmatrix} & u_{22} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ 0 \\ \omega \end{pmatrix} \\
 &\text{inbound electron spin up} & &\text{inbound electron spin down}
 \end{aligned}$$

Spinors for the outbound electron.

$$\begin{aligned}
 u_{41} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix} & u_{42} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix} \\
 &\text{outbound electron spin up} & &\text{outbound electron spin down}
 \end{aligned}$$

The probability amplitude  $\mathcal{M}_{ab}$  for spin state  $ab$  is

$$\mathcal{M}_{ba} = \mathcal{M}_{1ab} + \mathcal{M}_{2ab}$$

where

$$\mathcal{M}_{1ab} = \frac{\bar{u}_{4b}(-ie\gamma^\mu)(\not{q}_1 + m)(-ie\gamma^\nu)u_{2a}}{s - m^2}, \quad \mathcal{M}_{2ab} = \frac{\bar{u}_{4b}(-ie\gamma^\nu)(\not{q}_2 + m)(-ie\gamma^\mu)u_{2a}}{u - m^2}$$

Symbols  $\not{q}_1$  and  $\not{q}_2$  are momentum transfer.

$$\begin{aligned}
 \not{q}_1 &= (p_1 + p_2)^\alpha g_{\alpha\beta} \gamma^\beta \\
 \not{q}_2 &= (p_4 - p_1)^\alpha g_{\alpha\beta} \gamma^\beta
 \end{aligned}$$

Symbols  $s$  and  $u$  are Mandelstam variables.

$$\begin{aligned}s &= (p_1 + p_2)^2 \\ u &= (p_1 - p_4)^2\end{aligned}$$

In component form (note that indices  $\mu$  and  $\nu$  are interchanged for  $\mathcal{M}_{2ab}$ )

$$\begin{aligned}(\mathcal{M}_{1ab})^{\mu\nu} &= \frac{(\bar{u}_{4b})_\alpha (-ie\gamma^{\mu\alpha}_\beta)(\not{q}_1 + m)^\beta_\rho (-ie\gamma^{\nu\rho}_\sigma)(u_{2a})^\sigma}{s - m^2} \\ (\mathcal{M}_{2ab})^{\nu\nu} &= \frac{(\bar{u}_{4b})_\alpha (-ie\gamma^{\nu\alpha}_\beta)(\not{q}_2 + m)^\beta_\rho (-ie\gamma^{\mu\rho}_\sigma)(u_{2a})^\sigma}{u - m^2}\end{aligned}$$

The expected probability density  $\langle |\mathcal{M}|^2 \rangle$  is the average of spin states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{a=1}^2 \sum_{b=1}^2 |\mathcal{M}_{ab}|^2$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{a=1}^2 \sum_{b=1}^2 (\mathcal{M}_{1ab}\mathcal{M}_{1ab}^* + \mathcal{M}_{1ab}\mathcal{M}_{2ab}^* + \mathcal{M}_{2ab}\mathcal{M}_{1ab}^* + \mathcal{M}_{2ab}\mathcal{M}_{2ab}^*)$$

Metric tensor  $g_{\mu\nu}$  is required to sum over indices  $\mu$  and  $\nu$ .

$$\begin{aligned}\mathcal{M}_{1ab}\mathcal{M}_{1ab}^* &= (\mathcal{M}_{1ab})^{\mu\nu}(\mathcal{M}_{1ab}^*)_{\mu\nu} = (\mathcal{M}_{1ab})^{\mu\nu} [g_{\mu\alpha}(\mathcal{M}_{1ab}^*)^{\alpha\beta}g_{\beta\nu}] \\ \mathcal{M}_{1ab}\mathcal{M}_{2ab}^* &= (\mathcal{M}_{1ab})^{\mu\nu}(\mathcal{M}_{2ab}^*)_{\nu\mu} = (\mathcal{M}_{1ab})^{\mu\nu} [g_{\nu\alpha}(\mathcal{M}_{2ab}^*)^{\alpha\beta}g_{\alpha\mu}] \\ \mathcal{M}_{2ab}\mathcal{M}_{2ab}^* &= (\mathcal{M}_{2ab})^{\nu\mu}(\mathcal{M}_{2ab}^*)_{\nu\mu} = (\mathcal{M}_{2ab})^{\nu\mu} [g_{\nu\alpha}(\mathcal{M}_{2ab}^*)^{\alpha\beta}g_{\beta\mu}]\end{aligned}$$

The Casimir trick uses matrix arithmetic to sum over spin states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left( \frac{f_{11}}{(s - m^2)^2} + \frac{2f_{12}}{(s - m^2)(u - m^2)} + \frac{f_{22}}{(u - m^2)^2} \right) \quad (1)$$

where

$$\begin{aligned}f_{11} &= \text{Tr} \left( (\not{p}_2 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu \right) \\ f_{12} &= \text{Tr} \left( (\not{p}_2 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu \right) \\ f_{22} &= \text{Tr} \left( (\not{p}_2 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\nu(\not{q}_2 + m)\gamma_\mu \right)\end{aligned}$$

The following formulas are equivalent to the Casimir trick. (Recall that  $a \cdot b = a^\mu g_{\mu\nu} b^\nu$ )

$$\begin{aligned}f_{11} &= 32(p_1 \cdot p_2)(p_1 \cdot p_4) + 64m^2(p_1 \cdot p_2) - 32m^2(p_1 \cdot p_3) - 32m^2(p_1 \cdot p_4) + 32m^4 \\ f_{12} &= 16m^2(p_1 \cdot p_2) - 16m^2(p_1 \cdot p_4) + 32m^4 \\ f_{22} &= 32(p_1 \cdot p_2)(p_1 \cdot p_4) + 32m^2(p_1 \cdot p_2) - 32m^2(p_1 \cdot p_3) - 64m^2(p_1 \cdot p_4) + 32m^4\end{aligned}$$

In Mandelstam variables

$$\begin{aligned} f_{11} &= -8su + 24sm^2 + 8um^2 + 8m^4 \\ f_{12} &= 8sm^2 + 8um^2 + 16m^4 \\ f_{22} &= -8su + 8sm^2 + 24um^2 + 8m^4 \end{aligned} \tag{2}$$

Compton scattering experiments are typically done in the lab frame where the electron is at rest. Define Lorentz boost  $\Lambda$  for transforming momentum vectors to the lab frame.

$$\Lambda = \begin{pmatrix} E/m & 0 & 0 & \omega/m \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega/m & 0 & 0 & E/m \end{pmatrix}$$

The electron is at rest in the lab frame.

$$\Lambda p_2 = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Mandelstam variables are invariant under a boost.

$$\begin{aligned} s &= (p_1 + p_2)^2 = (\Lambda p_1 + \Lambda p_2)^2 \\ t &= (p_1 - p_3)^2 = (\Lambda p_1 - \Lambda p_3)^2 \\ u &= (p_1 - p_4)^2 = (\Lambda p_1 - \Lambda p_4)^2 \end{aligned}$$

In the lab frame, let  $\omega_L$  be the angular frequency of the incident photon and let  $\omega'_L$  be the angular frequency of the scattered photon.

$$\begin{aligned} \omega_L &= \Lambda p_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2}{m} + \frac{\omega E}{m} \\ \omega'_L &= \Lambda p_3 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2 \cos \theta}{m} + \frac{\omega E}{m} \end{aligned}$$

It can be shown that

$$\begin{aligned} s &= m^2 + 2m\omega_L \\ t &= 2m(\omega'_L - \omega_L) \\ u &= m^2 - 2m\omega'_L \end{aligned} \tag{3}$$

Then by (1), (2), and (3) we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left( \frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \left( \frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1 \right)^2 - 1 \right)$$

Lab scattering angle  $\theta_L$  is given by the Compton equation

$$\cos \theta_L = \frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1$$

Hence

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= 2e^4 \left( \frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \cos^2 \theta_L - 1 \right) \\ &= 2e^4 \left( \frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} - \sin^2 \theta_L \right) \end{aligned}$$

### Cross section

Now that we have derived  $\langle |\mathcal{M}|^2 \rangle$  we can investigate the angular distribution of scattered photons. For simplicity let us drop the  $L$  subscript from lab variables. From now on the symbols  $\omega$ ,  $\omega'$ , and  $\theta$  will be lab frame variables.

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4(4\pi\epsilon_0)^2 s} \left( \frac{\omega'}{\omega} \right)^2 \langle |\mathcal{M}|^2 \rangle$$

where

$$s = m^2 + 2m\omega = (mc^2)^2 + 2(mc^2)(\hbar\omega)$$

and  $\omega'$  is given by the Compton equation

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos \theta)}$$

For the lab frame we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

Hence in the lab frame

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\epsilon_0)^2 s} \left( \frac{\omega'}{\omega} \right)^2 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

Noting that

$$e^2 = 4\pi\epsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s} \left( \frac{\omega'}{\omega} \right)^2 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

Noting that

$$d\Omega = \sin \theta d\theta d\phi$$

we also have

$$d\sigma = \frac{\alpha^2 (\hbar c)^2}{2s} \left( \frac{\omega'}{\omega} \right)^2 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right) \sin \theta d\theta d\phi$$

Let  $S(\theta_1, \theta_2)$  be the following integral of  $d\sigma$ .

$$S(\theta_1, \theta_2) = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{\pi \alpha^2 (\hbar c)^2}{s} [I(\theta_2) - I(\theta_1)]$$

where

$$I(\theta) = -\frac{\cos \theta}{R^2} + \log(1 + R(1 - \cos \theta)) \left( \frac{1}{R} - \frac{2}{R^2} - \frac{2}{R^3} \right) \\ - \frac{1}{2R(1 + R(1 - \cos \theta))^2} + \frac{1}{1 + R(1 - \cos \theta)} \left( -\frac{2}{R^2} - \frac{1}{R^3} \right)$$

and

$$R = \frac{\hbar \omega}{mc^2}$$

The cumulative distribution function is

$$F(\theta) = \frac{S(0, \theta)}{S(0, \pi)} = \frac{I(\theta) - I(0)}{I(\pi) - I(0)}, \quad 0 \leq \theta \leq \pi$$

The probability of observing scattering events in the interval  $\theta_1$  to  $\theta_2$  is

$$P(\theta_1 < \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi) - I(0)} \left( \frac{\omega'}{\omega} \right)^2 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right) \sin \theta$$

### Thomson scattering

For  $\hbar \omega \ll mc^2$  we have

$$\omega' = \frac{\omega}{1 + \frac{\hbar \omega}{mc^2} (1 - \cos \theta)} \approx \omega$$

Hence we can use the approximations

$$\omega = \omega' \quad \text{and} \quad s = (mc^2)^2$$

to obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \hbar^2}{2m^2 c^2} (1 + \cos^2 \theta)$$

which is the formula for Thomson scattering.

### High energy approximation

For  $\omega \gg m$  a useful approximation is to set  $m = 0$  and obtain

$$\begin{aligned} f_{11} &= -8su \\ f_{12} &= 0 \\ f_{22} &= -8su \end{aligned}$$

Hence

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left( \frac{-8su}{s^2} + \frac{-8su}{u^2} \right) \\ &= 2e^4 \left( -\frac{u}{s} - \frac{s}{u} \right) \end{aligned}$$

The Mandelstam variables for  $m = 0$  are

$$\begin{aligned} s &= 4\omega^2 \\ u &= -2\omega^2(\cos \theta + 1) \end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left( \frac{\cos \theta + 1}{2} + \frac{2}{\cos \theta + 1} \right)$$

In the center of mass frame

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{4s(4\pi\epsilon_o)^2} = \frac{e^4}{2s(4\pi\epsilon_o)^2} \left( \frac{\cos \theta + 1}{2} + \frac{2}{\cos \theta + 1} \right)$$

Substitute  $e^4 = (4\pi\epsilon_0\alpha\hbar c)^2$  to obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s} \left( \frac{\cos \theta + 1}{2} + \frac{2}{\cos \theta + 1} \right) \times (\hbar c)^2$$

It follows that

$$\frac{d\sigma}{d\cos \theta} = 2\pi \frac{d\sigma}{d\Omega} = \frac{\pi\alpha^2}{s} \left( \frac{\cos \theta + 1}{2} + \frac{2}{\cos \theta + 1} \right) \times (\hbar c)^2$$

Cf. equation (1) of [arxiv.org/pdf/hep-ex/0504012](https://arxiv.org/pdf/hep-ex/0504012).