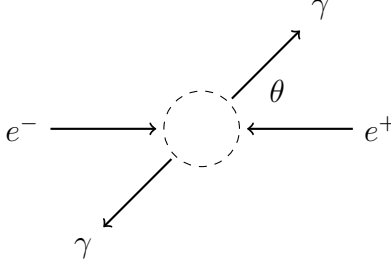
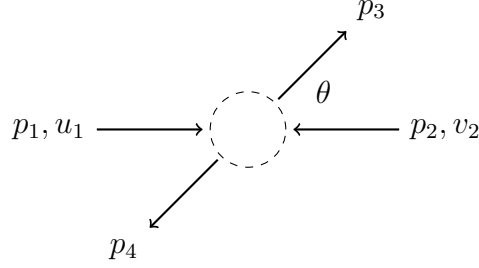


Electron positron annihilation creates two photons.



Here is the same diagram with momentum and spinor labels.



In a typical collider experiment the momentum vectors are

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \quad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} \quad p_3 = \begin{pmatrix} E \\ E \sin \theta \cos \phi \\ E \sin \theta \sin \phi \\ E \cos \theta \end{pmatrix} \quad p_4 = \begin{pmatrix} E \\ -E \sin \theta \cos \phi \\ -E \sin \theta \sin \phi \\ -E \cos \theta \end{pmatrix}$$

where  $E = \sqrt{p^2 + m^2}$ .

The spinors are

$$u_{11} = \begin{pmatrix} E + m \\ 0 \\ p \\ 0 \end{pmatrix} \quad u_{12} = \begin{pmatrix} 0 \\ E + m \\ 0 \\ -p \end{pmatrix} \quad v_{21} = \begin{pmatrix} -p \\ 0 \\ E + m \\ 0 \end{pmatrix} \quad v_{22} = \begin{pmatrix} 0 \\ p \\ 0 \\ E + m \end{pmatrix}$$

inbound electron, spin up      inbound electron, spin down      inbound positron, spin up      inbound positron, spin down

The spinors shown above are not individually normalized. Instead, a combined spinor normalization constant  $N = (E + m)^2$  will be used.

The following formula computes a probability density  $|\mathcal{M}_{ab}|^2$  for annihilation where  $a$  is the spin state of the inbound electron and  $b$  is the spin state of the inbound positron. The formula is from Feynman diagrams.

$$|\mathcal{M}_{ab}|^2 = \frac{e^4}{N} \left| -\frac{\bar{v}_{2b} \gamma^\mu (\not{q}_1 + m) \gamma^\nu u_{1a}}{t - m^2} - \frac{\bar{v}_{2b} \gamma^\nu (\not{q}_2 + m) \gamma^\mu u_{1a}}{u - m^2} \right|^2$$

Symbol  $e$  is electron charge. Symbol  $q_1 = p_1 - p_3$  and  $q_2 = p_1 - p_4$ . Symbols  $t$  and  $u$  are Mandelstam variables  $t = (p_1 - p_3)^2 = q_1^2$  and  $u = (p_1 - p_4)^2 = q_2^2$ .

Let

$$a_1 = \bar{v}_{2b}\gamma^\mu(\not{q}_1 + m)\gamma^\nu u_{1a} \quad a_2 = \bar{v}_{2b}\gamma^\nu(\not{q}_2 + m)\gamma^\mu u_{1a}$$

Then

$$\begin{aligned} |\mathcal{M}_{ab}|^2 &= \frac{e^4}{N} \left| -\frac{a_1}{t-m^2} - \frac{a_2}{u-m^2} \right|^2 \\ &= \frac{e^4}{N} \left( -\frac{a_1}{t-m^2} - \frac{a_2}{u-m^2} \right) \left( -\frac{a_1}{t-m^2} - \frac{a_2}{u-m^2} \right)^* \\ &= \frac{e^4}{N} \left( \frac{a_1 a_1^*}{(t-m^2)^2} + \frac{a_1 a_2^*}{(t-m^2)(u-m^2)} + \frac{a_1^* a_2}{(t-m^2)(u-m^2)} + \frac{a_2 a_2^*}{(u-m^2)^2} \right) \end{aligned}$$

The expected probability density  $\langle |\mathcal{M}|^2 \rangle$  is computed by summing  $|\mathcal{M}_{ab}|^2$  over all spin and polarization states and then dividing by the number of inbound states. There are four inbound states. The sum over polarization states is already accomplished by contraction of  $aa^*$  over  $\mu$  and  $\nu$ .

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{1}{4} \sum_{a=1}^2 \sum_{b=1}^2 |\mathcal{M}_{ab}|^2 \\ &= \frac{e^4}{4N} \sum_{a=1}^2 \sum_{b=1}^2 \left( \frac{a_1 a_1^*}{(t-m^2)^2} + \frac{a_1 a_2^*}{(t-m^2)(u-m^2)} + \frac{a_1^* a_2}{(t-m^2)(u-m^2)} + \frac{a_2 a_2^*}{(u-m^2)^2} \right) \end{aligned}$$

Use the Casimir trick to replace sums over spins with matrix products.

$$\begin{aligned} f_{11} &= \frac{1}{N} \sum_{a=1}^2 \sum_{b=1}^2 a_1 a_1^* = \text{Tr} \left( (\not{p}_1 + m) \gamma^\mu (\not{q}_1 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\nu (\not{q}_1 + m) \gamma_\mu \right) \\ f_{12} &= \frac{1}{N} \sum_{a=1}^2 \sum_{b=1}^2 a_1 a_2^* = \text{Tr} \left( (\not{p}_1 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\mu (\not{q}_1 + m) \gamma_\nu \right) \\ f_{22} &= \frac{1}{N} \sum_{a=1}^2 \sum_{b=1}^2 a_2 a_2^* = \text{Tr} \left( (\not{p}_1 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\nu (\not{q}_2 + m) \gamma_\mu \right) \end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left( \frac{f_{11}}{(t-m^2)^2} + \frac{f_{12}}{(t-m^2)(u-m^2)} + \frac{f_{12}^*}{(t-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2} \right)$$

Run “annihilation-1.txt” to verify the Casimir trick for electron positron annihilation.

The following momentum formulas are equivalent to the Casimir trick. (Recall that  $a \cdot b = a^\mu g_{\mu\nu} b^\nu$ )

$$\begin{aligned} f_{11} &= 16(p_1 \cdot p_1)(p_1 \cdot p_2) - 32(p_1 \cdot p_1)(p_2 \cdot p_3) - 16(p_1 \cdot p_2)(p_3 \cdot p_3) + 32(p_1 \cdot p_3)(p_2 \cdot p_3) \\ &\quad - 48m^2(p_1 \cdot p_2) + 64m^2(p_1 \cdot p_3) + 64m^2(p_2 \cdot p_3) - 64m^2(p_3 \cdot p_3) - 64m^4 \\ f_{12} &= -32(p_1 \cdot p_1)(p_1 \cdot p_2) + 32(p_1 \cdot p_2)(p_1 \cdot p_3) + 32(p_1 \cdot p_2)(p_1 \cdot p_4) - 32(p_1 \cdot p_2)(p_3 \cdot p_4) \\ &\quad - 48m^2(p_1 \cdot p_1) + 48m^2(p_1 \cdot p_2) + 32m^2(p_1 \cdot p_3) + 32m^2(p_1 \cdot p_4) \\ &\quad - 16m^2(p_2 \cdot p_3) - 16m^2(p_2 \cdot p_4) - 16m^2(p_3 \cdot p_4) + 32m^4 \\ f_{22} &= 16(p_1 \cdot p_1)(p_1 \cdot p_2) - 32(p_1 \cdot p_1)(p_2 \cdot p_4) - 16(p_1 \cdot p_2)(p_4 \cdot p_4) + 32(p_1 \cdot p_4)(p_2 \cdot p_4) \\ &\quad - 48m^2(p_1 \cdot p_2) + 64m^2(p_1 \cdot p_4) + 64m^2(p_2 \cdot p_4) - 64m^2(p_4 \cdot p_4) - 64m^4 \end{aligned}$$

In Mandelstam variables  $s = (p_1 + p_2)^2$ ,  $t = (p_1 - p_3)^2$ ,  $u = (p_1 - p_4)^2$  the formulas are

$$\begin{aligned}f_{11} &= 8tu - 24tm^2 - 8um^2 - 8m^4 \\f_{12} &= 8sm^2 - 32m^4 \\f_{22} &= 8tu - 8tm^2 - 24um^2 - 8m^4\end{aligned}$$

When  $E \gg m$  a useful approximation is to set  $m = 0$  and obtain

$$\begin{aligned}f_{11} &= 8tu \\f_{12} &= 0 \\f_{22} &= 8tu\end{aligned}$$

For  $m = 0$  the Mandelstam variables are

$$\begin{aligned}s &= 4E^2 \\t &= -2E^2(1 - \cos \theta) = -4E^2 \sin^2(\theta/2) \\u &= -2E^2(1 + \cos \theta) = -4E^2 \cos^2(\theta/2)\end{aligned}$$

The corresponding expected probability density is

$$\begin{aligned}\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left( \frac{8tu}{t^2} + \frac{8tu}{u^2} \right) \\&= 2e^4 \left( \frac{u}{t} + \frac{t}{u} \right) \\&= 2e^4 \left( \frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right)\end{aligned}$$

Substituting  $e^4 = 16\pi^2\alpha^2$  we have

$$\langle |\mathcal{M}|^2 \rangle = 32\pi^2\alpha^2 \left( \frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right)$$

The resulting differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{64\pi^2 s} = \frac{\alpha^2}{8E^2} \left( \frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right)$$

Run “annihilation-2.txt” to verify.

We can integrate  $d\sigma$  to obtain a cumulative distribution function. Recall that

$$d\Omega = \sin \theta \, d\theta \, d\phi$$

Hence

$$d\sigma = \frac{\alpha^2}{8E^2} \left( \frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right) \sin \theta \, d\theta \, d\phi$$

Let  $I(\theta)$  be the following integral of  $d\sigma$ .

$$\begin{aligned} I(\theta) &= \left( \frac{8E^2}{\alpha^2} \right) \frac{1}{2\pi} \int_0^{2\pi} \int d\sigma \\ &= \int \left( \frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right) \sin \theta d\theta, \quad a \leq \theta \leq \pi - a \end{aligned}$$

Angular support is limited to an arbitrary  $a > 0$  because  $I(0)$  and  $I(\pi)$  are undefined. Assume that  $I(\theta) - I(a)$  is computable given  $\theta$  by either symbolic or numerical integration.

Let  $C$  be the normalization constant

$$C = I(\pi - a) - I(a)$$

Then the cumulative distribution function  $F(\theta)$  is

$$F(\theta) = \frac{I(\theta) - I(a)}{C}, \quad a \leq \theta \leq \pi - a$$

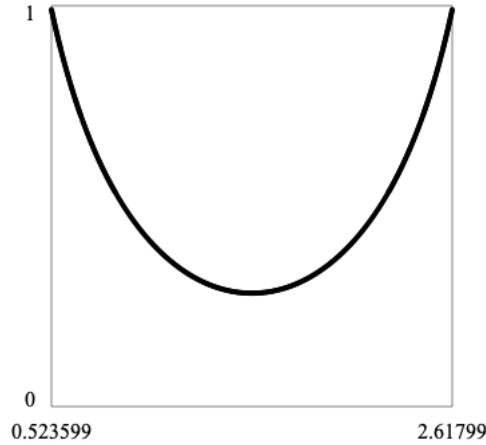
The probability of observing scattering events in the interval  $\theta_1$  to  $\theta_2$  can now be computed.

$$P(\theta_1 \leq \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

Probability density function  $f(\theta)$  is the derivative of  $F(\theta)$ .

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{C} \left( \frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right) \sin \theta, \quad a \leq \theta \leq \pi - a$$

Run “annihilation-4.txt” to plot  $f(\theta)$  for  $a = \pi/6 = 30^\circ$ .



Probability distribution for  $30^\circ$  bins ( $a = 30^\circ$ ).

$\theta_1$	$\theta_2$	$P(\theta_1 \leq \theta \leq \theta_2)$
$0^\circ$	$30^\circ$	—
$30^\circ$	$60^\circ$	0.33
$60^\circ$	$90^\circ$	0.17
$90^\circ$	$120^\circ$	0.17
$120^\circ$	$150^\circ$	0.33
$150^\circ$	$180^\circ$	—

## DESY PETRA data

The following table shows DESY PETRA electron positron annihilation data.<sup>1</sup>

$x$	$y$
0.0502	0.09983
0.1505	0.10791
0.2509	0.12026
0.3512	0.13002
0.4516	0.17681
0.5521	0.1957
0.6526	0.279
0.7312	0.33204

Data  $x$  and  $y$  have the following relationship with the differential cross section formula.

$$x = \cos \theta \quad y = \frac{d\sigma}{d\Omega}$$

To compute predicted values  $\hat{y}$  from the cross section formula, use  $\sqrt{s} = 2E = 14.0 \text{ GeV}$ . Multiply by  $(\hbar c)^2$  to convert to SI and multiply by  $10^{37}$  to convert square meters to nanobarns.

$$\hat{y} = \frac{\alpha^2}{2(14.0)^2} \left( \frac{1+x}{1-x} + \frac{1-x}{1+x} \right) \times (\hbar c)^2 \times 10^{37}$$

The following table shows predicted values  $\hat{y}$  based on the above formula.

$x$	$y$	$\hat{y}$
0.0502	0.09983	0.106325
0.1505	0.10791	0.110694
0.2509	0.12026	0.120005
0.3512	0.13002	0.135559
0.4516	0.17681	0.159996
0.5521	0.1957	0.198562
0.6526	0.279	0.262745
0.7312	0.33204	0.348884

The coefficient of determination  $R^2$  measures how well predicted values fit the real data.

$$R^2 = 1 - \frac{\sum (y - \hat{y})^2}{\sum (y - \bar{y})^2} = 0.98$$

The result indicates that the model  $d\sigma$  explains 98% of the variance in the data.

Run “annihilation-3.txt” to verify.

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<sup>1</sup>[www.hepdata.net/record/ins191231](http://www.hepdata.net/record/ins191231) (Table 2, 14.0 GeV)

## Notes

Here are some notes on how the scripts work.

To convert  $a_1$  and  $a_2$  to Eigenmath code, it is instructive to write  $a_1$  and  $a_2$  in full component form.

$$a_1^{\mu\nu} = \bar{v}_{2\alpha} \gamma^{\mu\alpha}{}_{\beta} (\not{q}_1 + m)^{\beta}{}_{\rho} \gamma^{\nu\rho}{}_{\sigma} u_1^{\sigma} \quad a_2^{\nu\mu} = \bar{v}_{2\alpha} \gamma^{\nu\alpha}{}_{\beta} (\not{q}_2 + m)^{\beta}{}_{\rho} \gamma^{\mu\rho}{}_{\sigma} u_1^{\sigma}$$

Transpose the  $\gamma$  tensors to form inner products over  $\alpha$  and  $\rho$ .

$$a_1^{\mu\nu} = \bar{v}_{2\alpha} \gamma^{\alpha\mu}{}_{\beta} (\not{q}_1 + m)^{\beta}{}_{\rho} \gamma^{\rho\nu}{}_{\sigma} u_1^{\sigma} \quad a_2^{\nu\mu} = \bar{v}_{2\alpha} \gamma^{\alpha\nu}{}_{\beta} (\not{q}_2 + m)^{\beta}{}_{\rho} \gamma^{\rho\mu}{}_{\sigma} u_1^{\sigma}$$

Convert transposed  $\gamma$  to Eigenmath code.

$$\gamma^{\alpha\mu}{}_{\beta} \rightarrow \text{gammaT} = \text{transpose}(\text{gamma})$$

Then to compute  $a_1$  we have

$$a_1 = \bar{v}_{2\alpha} \gamma^{\alpha\mu}{}_{\beta} (\not{q}_1 + m)^{\beta}{}_{\rho} \gamma^{\rho\nu}{}_{\sigma} u_1^{\sigma} \rightarrow \text{a1} = \text{dot}(\text{v2bar}[\text{s2}], \text{gammaT}, \text{qslash1} + \text{m I}, \text{gammaT}, \text{u1}[\text{s1}])$$

where  $s_1$  and  $s_2$  are spin indices. Similarly for  $a_2$  we have

$$a_2 = \bar{v}_{2\alpha} \gamma^{\alpha\mu}{}_{\beta} (\not{q}_2 + m)^{\beta}{}_{\rho} \gamma^{\rho\nu}{}_{\sigma} u_1^{\sigma} \rightarrow \text{a2} = \text{dot}(\text{v2bar}[\text{s2}], \text{gammaT}, \text{qslash2} + \text{m I}, \text{gammaT}, \text{u1}[\text{s1}])$$

In component notation the product  $a_1 a_1^*$  is

$$a_1 a_1^* = a_1^{\mu\nu} a_1^{*\mu\nu}$$

To sum over  $\mu$  and  $\nu$  it is necessary to lower indices with the metric tensor. Also, transpose  $a_1^*$  to form an inner product with  $\nu$ .

$$a_1 a_1^* = a_1^{\mu\nu} a_{1\nu\mu}^*$$

Convert to Eigenmath code. The dot function sums over  $\nu$  and the contract function sums over  $\mu$ .

$$a_1 a_1^* \rightarrow \text{a11} = \text{contract}(\text{dot}(\text{a1}, \text{gmunu}, \text{transpose}(\text{conj}(\text{a1}))), \text{gmunu})$$

Similarly for  $a_2 a_2^*$  we have

$$a_2 a_2^* \rightarrow \text{a22} = \text{contract}(\text{dot}(\text{a2}, \text{gmunu}, \text{transpose}(\text{conj}(\text{a2}))), \text{gmunu})$$

The product  $a_1 a_2^*$  does not require a transpose because  $a_2 = a_2^{\nu\mu}$ .

$$a_1^{\mu\nu} a_{2\nu\mu}^* \rightarrow \text{a12} = \text{contract}(\text{dot}(\text{a1}, \text{gmunu}, \text{conj}(\text{a2})), \text{gmunu})$$

In component notation, a trace operator becomes a sum over an index, in this case  $\alpha$ .

$$\begin{aligned} f_{11} &= \text{Tr} \left( (\not{p}_1 + m) \gamma^\mu (\not{q}_1 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\nu (\not{q}_1 + m) \gamma_\mu \right) \\ &= (\not{p}_1 + m)^\alpha_\beta \gamma^{\mu\beta}_\rho (\not{q}_1 + m)^\rho_\sigma \gamma^{\nu\sigma}_\tau (\not{p}_2 - m)^\tau_\delta \gamma^\delta_{\nu\eta} (\not{q}_1 + m)^\eta_\xi \gamma^\xi_{\mu\alpha} \end{aligned}$$

As before, transpose  $\gamma$  tensors to form inner products.

$$f_{11} = (\not{p}_1 + m)^\alpha_\beta \gamma^{\beta\mu}_\rho (\not{q}_1 + m)^\rho_\sigma \gamma^{\sigma\nu}_\tau (\not{p}_2 - m)^\tau_\delta \gamma^\delta_{\nu\eta} (\not{q}_1 + m)^\eta_\xi \gamma^\xi_{\mu\alpha}$$

This is the code for transposing  $\gamma$ .

$$\begin{aligned} \gamma^{\beta\mu}_\beta &\rightarrow \text{gammaT} = \text{transpose}(\text{gamma}) \\ \gamma^\delta_{\nu\eta} &\rightarrow \text{gammaL} = \text{transpose}(\text{dot}(\text{gmunu}, \text{gamma})) \end{aligned}$$

To convert  $f_{11}$  to Eigenmath code, use an intermediate variable  $T$  for the inner product.

$$T^{\alpha\mu\nu}_{\nu\mu\alpha} \rightarrow \text{T} = \text{dot}(\text{P1}, \text{gammaT}, \text{Q1}, \text{gammaT}, \text{P2}, \text{gammaL}, \text{Q1}, \text{gammaL})$$

Now sum over the indices of  $T$ . The innermost contract sums over  $\nu$  then the next contract sums over  $\mu$ . Finally the outermost contract sums over  $\alpha$ .

$$f_{11} \rightarrow \text{f11} = \text{contract}(\text{contract}(\text{contract}(\text{T}, 3, 4), 2, 3))$$

Follow suit for  $f_{22}$ . For  $f_{12}$  the order of the rightmost  $\mu$  and  $\nu$  is reversed.

$$f_{12} = \text{Tr} \left( (\not{p}_1 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\mu (\not{q}_1 + m) \gamma_\nu \right)$$

The resulting inner product is  $T^{\alpha\mu\nu}_{\mu\nu\alpha}$  so the contraction is different.

$$f_{12} \rightarrow \text{f12} = \text{contract}(\text{contract}(\text{contract}(\text{T}, 3, 5), 2, 3))$$

The innermost contract sums over  $\nu$  followed by sum over  $\mu$  then sum over  $\alpha$ .