Let ϕ be the field

$$\phi(x,y,z,t) = p_x x + p_y y + p_z z - Et$$

where

$$E = \sqrt{p_x^2 c^2 + p_y^2 c^2 + p_z^2 c^2 + m^2 c^4}$$

The following solutions to the Dirac equation are used to model fermions.

$$\underbrace{\psi_1 = \begin{pmatrix} E + mc^2 \\ 0 \\ p_z c \\ p_x c + i p_y c \end{pmatrix} \exp\left(\frac{i\phi}{\hbar}\right)}_{\text{fermion spin up}} \qquad \underbrace{\psi_2 = \begin{pmatrix} 0 \\ E + mc^2 \\ p_x c - i p_y c \\ -p_z c \end{pmatrix} \exp\left(\frac{i\phi}{\hbar}\right)}_{\text{fermion spin down}}$$

$$\underbrace{\psi_7 = \begin{pmatrix} p_z c \\ p_x c + i p_y c \\ E + mc^2 \\ 0 \end{pmatrix} \exp\left(-\frac{i\phi}{\hbar}\right)}_{\text{anti-fermion spin up}} \qquad \underbrace{\psi_8 = \begin{pmatrix} p_x c - i p_y c \\ -p_z c \\ 0 \\ E + mc^2 \end{pmatrix} \exp\left(-\frac{i\phi}{\hbar}\right)}_{\text{anti-fermion spin down}}$$

A spinor is the vector part of each solution. The following eight spinors are used for scattering calculations. The u spinors are fermions from ψ_1 and ψ_2. The v spinors are anti-fermions from ψ_7 and ψ_8. The last digit of the u or v subscript is 1 for spin up and 2 for spin down.

$$u_{11} = \begin{pmatrix} E_1 + m_1 c^2 \\ 0 \\ p_{1z} c \\ p_{1x} c + i p_{1y} c \end{pmatrix} \quad v_{21} = \begin{pmatrix} p_{2z} c \\ p_{2x} c + i p_{2y} c \\ E_2 + m_2 c^2 \\ 0 \end{pmatrix} \quad u_{31} = \begin{pmatrix} E_3 + m_3 c^2 \\ 0 \\ p_{3z} c \\ p_{3x} c + i p_{3y} c \end{pmatrix} \quad v_{41} = \begin{pmatrix} p_{4z} c \\ p_{4x} c + i p_{4y} c \\ E_4 + m_4 c^2 \\ 0 \end{pmatrix}$$

$$u_{12} = \begin{pmatrix} 0 \\ E_1 + m_1 c^2 \\ p_{1x} c - i p_{1y} c \\ -p_{1z} c \end{pmatrix} \quad v_{22} = \begin{pmatrix} p_{2x} c - i p_{2y} c \\ -p_{2z} c \\ 0 \\ E_2 + m_2 c^2 \end{pmatrix} \quad u_{32} = \begin{pmatrix} 0 \\ E_3 + m_3 c^2 \\ p_{3x} c - i p_{3y} c \\ -p_{3z} c \end{pmatrix} \quad v_{42} = \begin{pmatrix} p_{4x} c - i p_{4y} c \\ -p_{4z} c \\ 0 \\ E_4 + m_4 c^2 \end{pmatrix}$$

These are the associated momentum vectors.

$$p_1 = \begin{pmatrix} E_1 \\ p_{1x} \\ p_{1y} \\ p_{1z} \end{pmatrix} \quad p_2 = \begin{pmatrix} E_2 \\ p_{2x} \\ p_{2y} \\ p_{2z} \end{pmatrix} \quad p_3 = \begin{pmatrix} E_3 \\ p_{3x} \\ p_{3y} \\ p_{3z} \end{pmatrix} \quad p_4 = \begin{pmatrix} E_4 \\ p_{4x} \\ p_{4y} \\ p_{4z} \end{pmatrix}$$

Spinors are solutions to the following momentum-space Dirac equation with $\not{p} = p \cdot (c^{-1}\gamma^0, \gamma^1, \gamma^2, \gamma^3)$.

$$(\not{p} - mc)u = 0 \qquad (\not{p} + mc)v = 0$$

Up and down spinors have the following "completeness property."

$$u_{11}\bar{u}_{11} + u_{12}\bar{u}_{12} = (E_1 + m_1 c^2)(\not{p}_1 + m_1 c) \qquad v_{21}\bar{v}_{21} + v_{22}\bar{v}_{22} = (E_2 + m_2 c^2)(\not{p}_2 - m_2 c)$$

The adjoint of a spinor is $\bar{u} = c^{-1} u^\dagger \gamma^0$. The adjoint is a row vector hence $u\bar{u}$ is an outer product.