(13.4) The Lagrangian for electromagnetism in vacuo is $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$. Show that this can be rewritten as

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right) \tag{13.42}$$

and hence show that using the transverse projection operator, it may be expressed as

$$\mathcal{L} = \frac{1}{2}A^\mu P^T_{\mu\nu}\partial^2 A^\nu \tag{13.43}$$

This shows that $\mathcal{L}$ only includes the transverse components of the field, squaring with the idea of electromagnetic waves only representing vibrations transverse to the direction of propagation.

Consider equation (5.28).

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5.28}$$

It follows that

$$F^{\mu\nu}F_{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

Expand the right-hand side.

$$F^{\mu\nu}F_{\mu\nu} = \partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu - \partial^\nu A^\mu \partial_\mu A_\nu + \partial^\nu A^\mu \partial_\nu A_\mu$$

Make sums explicit.

$$F^{\mu\nu}F_{\mu\nu} = \underbrace{\sum_{\mu,\nu} \partial^\mu A^\nu \partial_\mu A_\nu}_{\Sigma_1} - \underbrace{\sum_{\mu,\nu} \partial^\mu A^\nu \partial_\nu A_\mu}_{\Sigma_2} - \underbrace{\sum_{\mu,\nu} \partial^\nu A^\mu \partial_\mu A_\nu}_{\Sigma_3} + \underbrace{\sum_{\mu,\nu} \partial^\nu A^\mu \partial_\nu A_\mu}_{\Sigma_4}$$

Noting that $\Sigma_1 = \Sigma_4$ and $\Sigma_2 = \Sigma_3$ we have

$$F^{\mu\nu}F_{\mu\nu} = 2\sum_{\mu,\nu} \partial^\mu A^\nu \partial_\mu A_\nu - 2\sum_{\mu,\nu} \partial^\mu A^\nu \partial_\nu A_\mu$$

Hence

$$F^{\mu\nu}F_{\mu\nu} = 2\left(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu\right)$$

and

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{2}\left(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu\right)$$