# Spin direction vector

Expectation of spin operators is a projection of spin state $|s\rangle$ onto Euclidean space.

$$\langle x\rangle = \langle s|\sigma_x|s\rangle, \quad \langle y\rangle = \langle s|\sigma_y|s\rangle, \quad \langle z\rangle = \langle s|\sigma_z|s\rangle$$

Hence the spin direction vector is

$$\mathbf{u} = \begin{pmatrix} \langle x\rangle \\ \langle y\rangle \\ \langle z\rangle \end{pmatrix} = \langle s|\boldsymbol{\sigma}|s\rangle, \quad \boldsymbol{\sigma} = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}$$

Let $\theta$ and $\phi$ be polar and azimuth angles such that

$$\mathbf{u} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

Then

$$|s\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\exp(i\phi) \end{pmatrix} = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

where

$$c_+ = \sqrt{\frac{\langle z\rangle + 1}{2}}, \quad c_- = \sqrt{\frac{1 - \langle z\rangle}{2}}\frac{\langle x\rangle + i\langle y\rangle}{\sqrt{\langle x\rangle^2 + \langle y\rangle^2}}$$

Example. Let

$$|s\rangle = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$$

Then

$$\mathbf{u} = \begin{pmatrix} \langle x\rangle \\ \langle y\rangle \\ \langle z\rangle \end{pmatrix} = \begin{pmatrix} \frac{4}{9} \\ \frac{8}{9} \\ \frac{1}{9} \end{pmatrix}$$

and

$$c_+ = \sqrt{\frac{\langle z\rangle + 1}{2}} = \frac{\sqrt{5}}{3}, \quad c_- = \sqrt{\frac{1 - \langle z\rangle}{2}}\frac{\langle x\rangle + i\langle y\rangle}{\sqrt{\langle x\rangle^2 + \langle y\rangle^2}} = \frac{2 + 4i}{3\sqrt{5}}$$

The results $c_+$ and $c_-$ differ from the original $|s\rangle$ but they do represent the same state.

The spin eigenstates are

$$\begin{array}{lll} |x_+\rangle = \frac{1}{\sqrt{2}}(1,1) & |y_+\rangle = \frac{1}{\sqrt{2}}(1,i) & |z_+\rangle = (1,0) \\ |x_-\rangle = \frac{1}{\sqrt{2}}(1,-1) & |y_-\rangle = \frac{1}{\sqrt{2}}(1,-i) & |z_-\rangle = (0,1) \end{array}$$

Hence for

$$|\chi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\exp(i\phi) \end{pmatrix}$$

we have

$$\begin{aligned}
\Pr\left(S_x = +\tfrac{\hbar}{2}\right) &= |\langle x_+|\chi|\rangle|^2 = \tfrac{1}{2} + \tfrac{1}{2}\sin\theta\cos\phi \\
\Pr\left(S_x = -\tfrac{\hbar}{2}\right) &= |\langle x_-|\chi|\rangle|^2 = \tfrac{1}{2} - \tfrac{1}{2}\sin\theta\cos\phi \\[1em]
\Pr\left(S_y = +\tfrac{\hbar}{2}\right) &= |\langle y_+|\chi|\rangle|^2 = \tfrac{1}{2} + \tfrac{1}{2}\sin\theta\sin\phi \\
\Pr\left(S_y = -\tfrac{\hbar}{2}\right) &= |\langle y_-|\chi|\rangle|^2 = \tfrac{1}{2} - \tfrac{1}{2}\sin\theta\sin\phi \\[1em]
\Pr\left(S_z = +\tfrac{\hbar}{2}\right) &= |\langle z_+|\chi|\rangle|^2 = \cos^2\tfrac{\theta}{2} \\
\Pr\left(S_z = -\tfrac{\hbar}{2}\right) &= |\langle z_-|\chi|\rangle|^2 = \sin^2\tfrac{\theta}{2}
\end{aligned}$$