9-9. For a complicated system moving nonrelativistically

$$(j_{1,\mathbf{k}})_{NM} = \sum_i \left(q_i \mathbf{e}_1 \cdot \dot{\mathbf{x}}_i \exp(-i\mathbf{k}\cdot\mathbf{x}_i)\right)_{NM}$$

where $\mathbf{e}_1$ is a unit vector in the direction of the polarization of the light and q_i and $\mathbf{x}_i$ are the charge and position of the ith particle. Assume the wavelength of the light is very large compared with the size of the atom, i.e., that the absolute square of the wave function describing the position of the ith electron falls to zero over a distance small compared with $1/k$. Show that we can then approximate $\exp(-i\mathbf{k}\cdot\mathbf{x}_i)$ by unity and write the matrix element as

$$(j_{1,\mathbf{k}})_{NM} = i\omega \mathbf{e}_1 \cdot \boldsymbol{\mu}_{NM} \tag{9.57}$$

where

$$\boldsymbol{\mu}_{NM} = \sum_i (q_i \mathbf{x}_i)_{NM} \tag{9.58}$$

The function $\boldsymbol{\mu}_{NM}$ is called the *matrix element of the electric dipole moment* of the atom, and the approximation used to derive equation (9.57) is called the *dipole approximation*. Show that the probability to emit light in any direction per unit time is

$$\frac{dP}{dt} = \frac{4\omega^3}{3\hbar c^3} \left|\boldsymbol{\mu}_{NM}\right|^2 \tag{9.59}$$

(Integrate equation (9.54) over all directions, remembering that $\mathbf{e}_1$ is perpendicular to $\mathbf{k}$ and that there are two possible directions of polarization.)

$$\frac{dP}{dt} = \frac{\omega}{2\pi\hbar c^3} \left|j_{1,\mathbf{q}}\right|^2_{NM} \, d\Omega \tag{9.54}$$

Adapted from problem 7-12,

$$\dot{\mathbf{x}}_i = -\frac{i}{\hbar}(\mathbf{x}_i \hat{H} - \hat{H}\mathbf{x}_i) = i\omega \mathbf{x}_i$$

The squared magnitude of $i\omega$ in (9.57) is ω^2. It follows that

$$\int \frac{\omega}{2\pi\hbar c^3}\left(\left|j_{1,\mathbf{k}}\right|^2_{NM} + \left|j_{2,\mathbf{k}}\right|^2_{NM}\right) d\Omega = 2\left|\boldsymbol{\mu}_{NM}\right|^2 \int_0^{2\pi}\int_0^{\pi} \frac{\omega^3}{2\pi\hbar c^3} \sin\theta \, d\theta \, d\phi$$

From the following integrals

$$\int_0^\pi \sin\theta \, d\theta = 2 \qquad \int_0^{2\pi} d\phi = 2\pi$$

the combined multiplier is 4π hence

$$\frac{dP}{dt} = \frac{4\omega^3}{\hbar c^3} \left| \boldsymbol{\mu}_{NM} \right|^2$$