Feynman and Hibbs problem 3-9
Find the kernel for a particle in a constant external field f where the Lagrangian is

$$L = \frac{m}{2}\dot{x}^2 + fx$$

We have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

and

$$\frac{\partial L}{\partial x} = 0$$

By equation (2.7)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

we have

$$\ddot{x} = 0$$

Hence velocity is constant. It follows that $x(t)$ is velocity times time.

$$x(t) = \left(\frac{x_b - x_a}{t_b - t_a}\right)(t - t_a) + x_a$$

Compute the action S_{cl}. From equation (2.1)

$$S_{cl} = \int_{t_a}^{t_b} \frac{m}{2}\dot{x}(t)^2\,dt + \int_{t_a}^{t_b} fx(t)\,dt$$

Use equation (2.8) to solve the first integral.

$$\int_{t_a}^{t_b} \frac{m}{2}\dot{x}(t)^2\,dt = \frac{m(x_b - x_a)^2}{2(t_b - t_a)} \tag{1}$$

Solve the second integral in S_{cl}.

$$\int_{t_a}^{t_b} fx(t)\,dt = \frac{(x_b + x_a)(t_b - t_a)f}{2} \tag{2}$$

Action S_{cl} is the sum of (1) and (2).

$$S_{cl} = \frac{m(x_b - x_a)^2}{2(t_b - t_a)} + \frac{(x_b + x_a)(t_b - t_a)f}{2}$$

By equation (3.51)

$$K(b,a) = F(t_b - t_a)\exp\left(\frac{iS_{cl}}{\hbar}\right)$$

we have

$$K(b,a) = F(t_b - t_a)\exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)} + \frac{i(x_b + x_a)(t_b - t_a)f}{2\hbar}\right) \tag{3}$$

We now proceed to compute F. By equation (2.31)

$$K(b,a) = \int_{-\infty}^{\infty} K(b,c)K(c,a)\,dx_c$$

we have

$$\begin{aligned} K(b,a) = \int_{-\infty}^{\infty} & F(t_b - t_c)\exp\left(\frac{im(x_b - x_c)^2}{2\hbar(t_b - t_c)} + \frac{i(x_b + x_c)(t_b - t_c)f}{2\hbar}\right) \\ & \times F(t_c - t_a)\exp\left(\frac{im(x_c - x_a)^2}{2\hbar(t_c - t_a)} + \frac{i(x_c + x_a)(t_c - t_a)f}{2\hbar}\right)\,dx_c \end{aligned} \tag{4}$$

Rewrite as

$$K(b,a) = F(t_b - t_c)F(t_c - t_a)\int_{-\infty}^{\infty} \exp(iAx_c^2 + iBx_c + iC) \tag{5}$$

where

$$A = \frac{m(t_b - t_a)}{2\hbar(t_b - t_c)(t_c - t_a)} \tag{6}$$

$$B = \frac{(t_b - t_a)f}{2\hbar} - \frac{m}{\hbar}\left(\frac{x_b}{t_b - t_c} + \frac{x_a}{t_c - t_a}\right) \tag{7}$$

$$C = \frac{f}{2\hbar}\big(x_b(t_b - t_c) + x_a(t_c - t_a)\big) + \frac{m}{2\hbar}\left(\frac{x_b^2}{t_b - t_c} + \frac{x_a^2}{t_c - t_a}\right) \tag{8}$$

Solve the integral in (5). From the formula

$$\int_{-\infty}^{\infty} \exp(iAx_c^2 + iBx_c + iC)\,dx_c = \left(-\frac{\pi}{iA}\right)^{1/2}\exp\left(-\frac{iB^2}{4A} + iC\right)$$

we have

$$\int_{-\infty}^{\infty} \exp(iAx_c^2 + iBx_c + iC)\,dx_c = \left(-\frac{2\pi\hbar(t_b-t_c)(t_c-t_a)}{im(t_b-t_a)}\right)^{1/2}$$
$$\times \exp\left(\frac{im(x_b-x_a)^2}{2\hbar(t_b-t_a)} + \frac{i(x_b+x_a)(t_b-t_a)f}{2\hbar} - \frac{i(t_b-t_a)(t_b-t_c)(t_c-t_a)f^2}{8\hbar m}\right)$$

Hence

$$K(b,a) = F(t_b-t_c)F(t_c-t_a)\left(-\frac{2\pi\hbar(t_b-t_c)(t_c-t_a)}{im(t_b-t_a)}\right)^{1/2}$$
$$\times \exp\left(\frac{im(x_b-x_a)^2}{2\hbar(t_b-t_a)} + \frac{i(x_b+x_a)(t_b-t_a)f}{2\hbar} - \frac{i(t_b-t_a)(t_b-t_c)(t_c-t_a)f^2}{8\hbar m}\right) \quad (9)$$

Equating (3) with (9) reduces to

$$F(t_b-t_a) = F(t_b-t_c)F(t_c-t_a)$$
$$\times \left(-\frac{2\pi\hbar(t_b-t_c)(t_c-t_a)}{im(t_b-t_a)}\right)^{1/2} \exp\left(-\frac{i(t_b-t_a)(t_b-t_c)(t_c-t_a)f^2}{8\hbar m}\right) \quad (10)$$

From problem 3-7, let

$$F(t) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \exp(at) \quad (11)$$

Then substituting (11) in (10) yields

$$\left(\frac{m}{2\pi i\hbar(t_b-t_a)}\right)^{1/2} \exp(a(t_b-t_a))$$
$$= \left(\frac{m}{2\pi i\hbar(t_b-t_c)}\right)^{1/2} \exp(a(t_b-t_c)) \left(\frac{m}{2\pi i\hbar(t_c-t_a)}\right)^{1/2} \exp(a(t_c-t_a))$$
$$\times \left(-\frac{2\pi\hbar(t_b-t_c)(t_c-t_a)}{im(t_b-t_a)}\right)^{1/2} \exp\left(-\frac{i(t_b-t_a)(t_b-t_c)(t_c-t_a)f^2}{8\hbar m}\right)$$

The coefficients cancel leaving

$$\exp(a(t_b-t_a)) = \exp(a(t_b-t_c))\exp(a(t_c-t_a))$$
$$\times \exp\left(-\frac{i(t_b-t_a)(t_b-t_c)(t_c-t_a)f^2}{8\hbar m}\right)$$