

Field momentum

For empty space show that

$$\frac{1}{4\pi c} \int \mathbf{E} \times \mathbf{B} d^3\mathbf{r} = i \int \mathbf{k} (\mathbf{a}_{\mathbf{k}}^* \cdot \dot{\mathbf{a}}_{\mathbf{k}}) \frac{d^3\mathbf{k}}{(2\pi)^3}$$

This is the electric field vector.

$$\mathbf{E}(\mathbf{r}, t) = \int \left(-i\mathbf{k}\phi_{\mathbf{k}} - \sqrt{4\pi} \dot{\mathbf{a}}_{\mathbf{k}} \right) \exp(i\mathbf{k} \cdot \mathbf{r}) \frac{d^3\mathbf{k}}{(2\pi)^3}$$

Set $\phi_{\mathbf{k}} = 0$ for empty space.

$$\mathbf{E}(\mathbf{r}, t) = -\sqrt{4\pi} \int \dot{\mathbf{a}}_{\mathbf{k}} \exp(i\mathbf{k} \cdot \mathbf{r}) \frac{d^3\mathbf{k}}{(2\pi)^3}$$

This is the magnetic field vector.

$$\mathbf{B}(\mathbf{r}, t) = \sqrt{4\pi} ic \int (\mathbf{k} \times \mathbf{a}_{\mathbf{k}}) \exp(i\mathbf{k} \cdot \mathbf{r}) \frac{d^3\mathbf{k}}{(2\pi)^3}$$

Take the complex conjugate of \mathbf{B} .

$$\mathbf{B}(\mathbf{r}, t) = -\sqrt{4\pi} ic \int (\mathbf{k} \times \mathbf{a}_{\mathbf{k}}^*) \exp(-i\mathbf{k} \cdot \mathbf{r}) \frac{d^3\mathbf{k}}{(2\pi)^3}$$

Hence

$$\mathbf{E} \times \mathbf{B} = 4\pi ic \iint \dot{\mathbf{a}}_{\mathbf{k}} \times (\mathbf{k}' \times \mathbf{a}_{\mathbf{k}'}^*) \exp(i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}) \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\mathbf{k}'}{(2\pi)^3}$$

Integrate over \mathbf{r} to change the exponential to a delta function.

$$\int \mathbf{E} \times \mathbf{B} d^3\mathbf{r} = 4\pi ic \iint \dot{\mathbf{a}}_{\mathbf{k}} \times (\mathbf{k}' \times \mathbf{a}_{\mathbf{k}'}^*) (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\mathbf{k}'}{(2\pi)^3}$$

The delta function vanishes except for $\mathbf{k} = \mathbf{k}'$.

$$\int \mathbf{E} \times \mathbf{B} d^3\mathbf{r} = 4\pi ic \int \dot{\mathbf{a}}_{\mathbf{k}} \times (\mathbf{k} \times \mathbf{a}_{\mathbf{k}}) \frac{d^3\mathbf{k}}{(2\pi)^3}$$

Expand the triple product.

$$\int \mathbf{E} \times \mathbf{B} d\mathbf{r} = 4\pi ic \int ((\dot{\mathbf{a}}_{\mathbf{k}} \cdot \mathbf{a}_{\mathbf{k}}^*) \mathbf{k} - (\dot{\mathbf{a}}_{\mathbf{k}} \cdot \mathbf{k}) \mathbf{a}_{\mathbf{k}}^*) \frac{d^3\mathbf{k}}{(2\pi)^3}$$

The inner product $\dot{\mathbf{a}}_{\mathbf{k}} \cdot \mathbf{k}$ vanishes by orthogonality hence

$$\int \mathbf{E} \times \mathbf{B} d\mathbf{r} = 4\pi ic \int (\dot{\mathbf{a}}_{\mathbf{k}} \cdot \mathbf{a}_{\mathbf{k}}^*) \mathbf{k} \frac{d^3\mathbf{k}}{(2\pi)^3}$$