Lecture 2
"In this lecture, I introduce states describing any number of particles and define operators acting on these states. I argue that causality requires that the theory be written in terms of "observables," local Hermitian operators that commute at spacelike positions. This leads us to a theory of quantum fields."

2.1
Causality problems are resolved by quantum fields.
Non-relativistic momentum eigenstates.

$$\langle \mathbf{q}|\mathbf{p}\rangle = \delta^3(\mathbf{p}-\mathbf{q})$$

How do states transform under Lorentz transformations?
Let Λ be the Lorentz transformation $\Lambda^\mu{}_\nu$.
Let $U(\Lambda)$ be an operator that depends on Λ.
The operator $U(\Lambda)$ is required to be unitary, hence

$$U^\dagger(\Lambda)U(\Lambda) = 1$$

The measure d^3p in the following integral is not Lorentz invariant.

$$1 = \int d^3p\, |\mathbf{p}\rangle\langle\mathbf{p}|$$

Let there be a new set of states $|p\rangle$ based on 4-vectors p^μ such that

$$p^\mu = (E_\mathbf{p}, \mathbf{p})$$

where

$$E_\mathbf{p} = \sqrt{\mathbf{p}^2 + m^2}$$

Skipping over the derivation, the following integral is Lorentz invariant.

$$1 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\, |p\rangle\langle p|$$

We define (dp) to be the Lorentz invariant measure

$$(dp) = \frac{d^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}$$

We also have

$$|p\rangle = (2\pi)^{3/2}\sqrt{2E_{\mathbf{p}}}\,|\mathbf{p}\rangle$$

and

$$\langle q|p\rangle = (2\pi)^3\,2E_{\mathbf{p}}\,\delta^3(\mathbf{p}-\mathbf{q})$$

We define $\langle q|p\rangle$ to be the Lorentz invariant delta function. By Lorentz invariance we have

$$\langle q|p\rangle = \langle q'|p'\rangle$$

where

$$\begin{aligned}|p'\rangle &= U(\Lambda)|p\rangle\\ |q'\rangle &= U(\Lambda)|q\rangle\end{aligned}$$

2.2

Lorentz transformations can act on states or operators but not both.
Let P^μ be an operator and let p^μ be its 4-vector eigenvalue such that

$$P^\mu|p\rangle = p^\mu|p\rangle$$

Schrodinger picture: States transform, operators do not:

$$\begin{aligned}|\psi\rangle &\longmapsto U(\Lambda)|\psi\rangle\\ O &\longmapsto O\end{aligned}$$

Compute a matrix element:

$$\langle q|P^\mu|p\rangle \longmapsto \langle q'|P^\mu|p'\rangle$$

Heisenberg picture: Operators transform, states do not:

$$\begin{aligned}|\psi\rangle &\longmapsto |\psi\rangle\\ O &\longmapsto U^\dagger(\Lambda)\,O\,U(\Lambda)\end{aligned}$$

Compute a matrix element:

$$\langle q|P^\mu|p\rangle \longmapsto \langle q|U^\dagger(\Lambda)\,P^\mu\,U(\Lambda)|p\rangle = \langle q'|P^\mu|p'\rangle$$

Hence both pictures yield the same matrix element.

2.3
The operator $U(\Lambda)$ needs to have the following properties:
1. If $\Lambda_1 \cdot \Lambda_2 = \Lambda_3$ then $U(\Lambda_1)U(\Lambda_2) = U(\Lambda_3)$.
2. $U(\Lambda^{-1}) = U^{-1}(\Lambda)$
The operator $U(\Lambda)$ is a "unitary representation" of the Lorentz group.

2.4 [24:36]
How to construct states that describe more than one particle.
Let $|p_1, p_2\rangle$ be an eigenstate of a two-particle system.

$$P^\mu |p_1, p_2\rangle = (p_1 + p_2)^\mu |p_1, p_2\rangle$$

We want the particles to be indistinguishable hence $p_1^2 = p_2^2 = m^2$ and

$$\left.\begin{array}{l} |p_1, p_2\rangle \\ |p_2, p_1\rangle \end{array}\right\} \text{represent the same state}$$

There are only two possible choices (proved by Wigner)

$$|p_2, p_1\rangle = \begin{cases} +|p_1, p_2\rangle & \text{bosons} \\ -|p_1, p_2\rangle & \text{fermions} \end{cases}$$

The rest of the lecture will focus on bosons.
For bosons, the order of the p's in

$$|p_1, \ldots, p_n\rangle$$

doesn't make any difference.
The overlap of two-particle states is

$$\langle q_1, q_2 | p_1, p_2\rangle = \langle q_1|p_1\rangle\langle q_2|p_2\rangle + \langle q_1|p_2\rangle\langle q_2|p_1\rangle$$

In general the overlap is the sum over all the possible ways the p's could be equal to the q's.

$$\langle q_1, \ldots, q_n | p_1, \ldots, p_n\rangle = \underbrace{\langle q_1|p_1\rangle \cdots \langle q_n|p_n\rangle + \text{permutations}}_{n!\ \text{terms}}$$

2.5 [30:48]

Define a new space of all number of particles.

$$\Big\{|0\rangle, |p\rangle, |p_1, p_2\rangle, \cdots\Big\} = \text{Fock Space}$$

The no-particle state $|0\rangle$ is the "vacuum" state or "ground" state.

$$\langle 0|0\rangle = 1$$

States with different number of particles are orthogonal, their overlap is zero.

$$\langle 0|p\rangle = 0$$

Next, define operators on this space of states.
Creation operator:

$$|p_1, \ldots, p_n\rangle = \alpha^\dagger(p_1)\cdots\alpha^\dagger(p_n)|0\rangle$$

and

$$\alpha^\dagger(k)|p_1, \ldots, p_n\rangle = |k, p_1, \ldots, p_n\rangle$$

Let's work out the following:

$$\begin{aligned}\langle q_1, \ldots, q_m|\alpha(k)|p_1, \ldots, p_n\rangle &= \langle p_1, \ldots, p_n|\alpha^\dagger(k)|q_1, \ldots, q_m\rangle^* \\ &= \langle p_1, \ldots, p_n|k, q_1, \ldots, q_m\rangle^*\end{aligned}$$

which is nonzero only if $n = m + 1$.
What is the formula for α?

$$\alpha(k)|p_1, \ldots, p_n\rangle = \sum_{i=1}^{n}\langle k|p_i\rangle|p_1, \ldots, \widehat{p_i}, \ldots, p_n\rangle$$

where the hat means omit p_i and

$$\langle k|p_i\rangle = \begin{cases} 1 & k = p_i \\ 0 & k \neq p_i \end{cases}$$

Annihilation of the no-particle state yields scalar zero.

$$\alpha(k)|0\rangle = 0$$

Now compute commutators.

Order doesn't matter for bosons hence

$$[\alpha^\dagger(p), \alpha^\dagger(q)] = 0$$

and

$$[\alpha(p), \alpha(q)] = 0$$

However

$$[\alpha(p), \alpha^\dagger(q)] \neq 0$$

Let's work this out.

$$\begin{aligned}\alpha(p)\alpha^\dagger(q)|p_1, \ldots, p_n\rangle &= \alpha(p)|q, p_1, \ldots, p_n\rangle \\ &= \langle p|q\rangle|p_1, \ldots, p_n\rangle + \sum_{i=1}^{n}\langle p|p_i\rangle|q, p_1, \ldots, \widehat{p_i}, \ldots, p_n\rangle\end{aligned}$$

Now repeat in the other order.

$$\begin{aligned}\alpha^\dagger(q)\alpha(p)|p_1, \ldots, p_n\rangle &= \alpha^\dagger(q)\sum_{i=1}^{n}\langle p|p_i\rangle|p_1, \ldots, \widehat{p_i}, \ldots, p_n\rangle \\ &= \sum_{i=1}^{n}\langle p|p_i\rangle|q, p_1, \ldots, \widehat{p_i}, \ldots, p_n\rangle\end{aligned}$$

We get almost exactly the same thing but not quite. Put it all together to get

$$[\alpha(p), \alpha^\dagger(q)] = \langle q|p\rangle$$

where $\langle q|p\rangle$ is the Lorentz invariant delta function.

2.6 [45:54]
Recall the 4-momentum operator P^μ with eigenvalues p^μ.

$$P^\mu|p\rangle = p^\mu|p\rangle$$

and

$$P^\mu|p_1, \ldots, p_n\rangle = (p_1 + \cdots + p_n)^\mu|p_1, \ldots, p_n\rangle$$

Let's see what P^μ is in terms of creation and annihilation operators.

$$P^\mu = \int (dp)\, p^\mu\, \alpha^\dagger(p)\alpha(p)$$

where (dp) is the Lorentz invariant measure.
Check for one-particle state:

$$\begin{aligned}P^\mu|p\rangle &= \int (dk)\, k^\mu\, \alpha^\dagger(k)\alpha(k)|p\rangle \\ &= \int (dk)\, k^\mu\, \alpha^\dagger(k)\langle k|p\rangle|0\rangle \\ &= p^\mu \alpha^\dagger(p)|0\rangle \\ &= p^\mu|p\rangle\end{aligned}$$

Note that the delta function $\langle k|p\rangle$ reduces the integral to just $k = p$.
Left as an exercise to check for n-particle state.
And here it is.

$$\begin{aligned}P^\mu|p_1,\dots,p_n\rangle &= \int (dk)\, k^\mu\, \alpha^\dagger(k)\alpha(k)|p_1,\dots,p_n\rangle \\ &= \int (dk)\, k^\mu\, \alpha^\dagger(k)\left(\sum_{i=1}^n \langle k|p_i\rangle|p_1,\dots,\widehat{p_i},\dots,p_n\rangle\right) \\ &= \sum_{i=1}^n \left(\int (dk)\, k^\mu\, \alpha^\dagger(k)\right)\langle k|p_i\rangle|p_1,\dots,\widehat{p_i},\dots,p_n\rangle \\ &= \sum_{i=1}^n (p_i)^\mu \alpha^\dagger(p_i)|p_1,\dots,\widehat{p_i},\dots,p_n\rangle \\ &= \sum_{i=1}^n (p_i)^\mu|p_1,\dots,p_n\rangle \\ &= (p_1+\cdots+p_n)^\mu|p_1,\dots,p_n\rangle\end{aligned}$$

2.7 [50:20]
Operators in position space.

$$\begin{aligned}\varphi^+(x) &= \int (dp)\, \alpha(p)e^{-ip\cdot x} \\ \varphi^-(x) &= \int (dp)\, \alpha^\dagger(p)e^{+ip\cdot x} = [\varphi^+(x)]^\dagger\end{aligned}$$

Note that these operators depend on time because p and x are 4-vectors.

Schrodinger picture (states transform)

$$
\begin{aligned}
|p\rangle &\longmapsto |\Lambda p\rangle \\
\alpha(p) &\longmapsto \alpha(p)
\end{aligned}
$$

Heisenberg picture (operators transform)

$$
\begin{aligned}
|p\rangle &\longmapsto |p\rangle \\
\alpha(p) &\longmapsto \alpha(\Lambda^{-1}p)
\end{aligned}
$$

Let's compute a matrix element in both Schrodinger and Heisenberg pictures.
In the rest frame

$$\langle 0|\alpha(p)|q\rangle = \langle p|q\rangle\langle 0|0\rangle = \langle p|q\rangle$$

because $\alpha(p)|q\rangle = |0\rangle$ for $p = q$.
Now compute the transformed matrix element in a moving frame:

$$
\begin{aligned}
\langle 0|\alpha(p)|q\rangle &\xmapsto{\text{SP}} \langle 0|\alpha(p)|\Lambda q\rangle = \langle p|\Lambda q\rangle \\
&\xmapsto{\text{HP}} \langle 0|\alpha(\Lambda^{-1}p)|q\rangle = \langle \Lambda^{-1}p|q\rangle
\end{aligned}
$$

Note that $\langle p|\Lambda q\rangle = \langle \Lambda^{-1}p|q\rangle$.
Let's stay with the Heisenberg picture.

$$\varphi^{+}(x) = \int (dp)\, \alpha(p) e^{-ip\cdot x}$$

How does this transform?

$$
\begin{aligned}
\xmapsto{\text{HP}} \int (dp)\, e^{-ip\cdot x}\alpha(\Lambda^{-1}p) &= \int (dp')\, e^{-ip'\cdot(\Lambda^{-1}x)}\, \alpha(p') \\
&= \varphi^{+}(\Lambda^{-1}x) \qquad \text{(see note)}
\end{aligned}
$$

where $p' = \Lambda^{-1}p$ and by Lorentz invariance measure $(dp) = (dp')$.
Note: The dummy integration variable p' was changed to p.
The transformation in the exponential has $p \cdot x = (\Lambda^{-1}p)\cdot(\Lambda^{-1}x)$ which will now be proved.
Recall that the dot product involves the spacetime metric η.

$$p \cdot x = p^T \eta x$$

Also recall that the metric is Lorentz invariant.

$$\begin{aligned}
\eta &= \Lambda^T \eta \Lambda \\
&= (\Lambda^T)^{-1} \Lambda^T \eta \Lambda \Lambda^{-1} \\
&= (\Lambda^T)^{-1} \eta \Lambda^{-1} \\
&= (\Lambda^{-1})^T \eta \Lambda^{-1} && \text{(see note)}
\end{aligned}$$

Hence

$$\begin{aligned}
(\Lambda^{-1} p) \cdot (\Lambda^{-1} x) &= (\Lambda^{-1} p)^T \eta \Lambda^{-1} x \\
&= p^T (\Lambda^{-1})^T \eta \Lambda^{-1} x \\
&= p^T \eta x \\
&= p \cdot x
\end{aligned}$$

Note: Recall that $(A^T)^{-1} = (A^{-1})^T$ for any non-singular matrix A.

Lecture 3
"In this lecture we reverse the process of the previous lecture. We apply the rules of quantum mechanics to scalar field theory, and show that this gives rise to a theory of particles."