Spin flip

From exercise 5.8 of "Quantum Mechanics" by Richard Fitzpatrick.

Consider an electron at rest in the following magnetic field $\mathbf{B}$.

$$\mathbf{B} = B_0 \cos(\omega t) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Find the minimum B_0 such that $\langle S_x \rangle$ ranges from $-\frac{\hbar}{2}$ to $+\frac{\hbar}{2}$.

The Hamiltonian is

$$H = \frac{e}{m} \mathbf{B} \cdot \mathbf{S}$$

Hence by hypothesis

$$H = \frac{e}{m} B_0 \cos(\omega t) S_z$$

Let $|s\rangle$ be the following spin state.

$$|s\rangle = \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$$

By the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |s\rangle = H|s\rangle$$

In component form

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} c_1(t) &= +\frac{e\hbar}{2m} B_0 \cos(\omega t) c_1(t) \\ i\hbar \frac{\partial}{\partial t} c_2(t) &= -\frac{e\hbar}{2m} B_0 \cos(\omega t) c_2(t) \end{aligned}$$

Solve for $c_1(t)$ and $c_2(t)$ and normalize so that $\langle s|s\rangle = |c_1(t)|^2 + |c_2(t)|^2 = 1$.

$$\begin{aligned} c_1(t) &= \frac{1}{\sqrt{2}} \exp\left[-\frac{ie}{2m\omega} B_0 \sin(\omega t)\right] \\ c_2(t) &= \frac{1}{\sqrt{2}} \exp\left[+\frac{ie}{2m\omega} B_0 \sin(\omega t)\right] \end{aligned} \tag{1}$$

Having obtained $|s\rangle$ we can now solve for $\langle S_x \rangle$.

$$\langle S_x \rangle = \langle s|S_x|s\rangle = \frac{\hbar}{2} \cos\left[\frac{e}{m\omega} B_0 \sin(\omega t)\right] \tag{2}$$

At time $t = 0$

$$\langle S_x \rangle = \frac{\hbar}{2}$$

To obtain $\langle S_x \rangle = -\frac{\hbar}{2}$ we must have

$$\frac{e}{m\omega} B_0 \sin(\omega t) = \pi$$

Taking $\sin(\omega t) = 1$ we have

$$B_0 = \frac{\pi m \omega}{e}$$

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