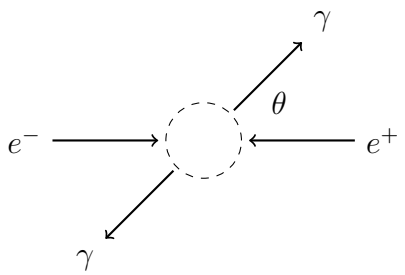
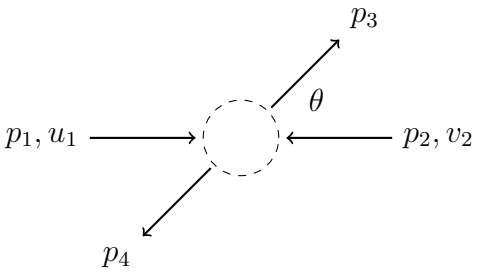
ELECTRON POSITRON ANNIHILATION

Electron positron annihilation creates two photons.



Here is the same diagram with momentum and spinor labels.



In a typical collider experiment the momentum vectors are

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \qquad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} \qquad p_3 = \begin{pmatrix} E \\ E\sin\theta\cos\phi \\ E\sin\theta\sin\phi \\ E\cos\theta \end{pmatrix} \qquad p_4 = \begin{pmatrix} E \\ -E\sin\theta\cos\phi \\ -E\sin\theta\sin\phi \\ -E\cos\theta \end{pmatrix}$$

where $E = \sqrt{p^2 + m^2}$.

The spinors are

$$u_{11} = \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix} \qquad u_{12} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix} \qquad v_{21} = \begin{pmatrix} -p \\ 0 \\ E+m \\ 0 \end{pmatrix} \qquad v_{22} = \begin{pmatrix} 0 \\ p \\ 0 \\ E+m \end{pmatrix}$$

inbound electron, spin up | inbound electron, spin down | inbound positron, spin up | inbound positron, spin down

The spinors shown above are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^2$ will be used.

The following formula computes a probability density $|\mathcal{M}_{ab}|^2$ for annihilation where a is the spin state of the inbound electron and b is the spin state of the inbound positron. The formula is from Feynman diagrams.

$$|\mathcal{M}_{ab}|^2 = \frac{e^4}{N}\left|-\frac{\bar{v}_{2b}\gamma^\mu(\not{q}_1 + m)\gamma^\nu u_{1a}}{t - m^2} - \frac{\bar{v}_{2b}\gamma^\nu(\not{q}_2 + m)\gamma^\mu u_{1a}}{u - m^2}\right|^2$$

Symbol e is electron charge. Symbol $q_1 = p_1 - p_3$ and $q_2 = p_1 - p_4$. Symbols t and u are Mandelstam variables $t = (p_1 - p_3)^2 = q_1^2$ and $u = (p_1 - p_4)^2 = q_2^2$.

Let

$$a_1 = \bar{v}_{2b}\gamma^\mu(\not{q}_1 + m)\gamma^\nu u_{1a} \qquad a_2 = \bar{v}_{2b}\gamma^\nu(\not{q}_2 + m)\gamma^\mu u_{1a}$$

Then

$$\begin{aligned}
|\mathcal{M}_{ab}|^2 &= \frac{e^4}{N}\left|-\frac{a_1}{t-m^2} - \frac{a_2}{u-m^2}\right|^2 \\
&= \frac{e^4}{N}\left(-\frac{a_1}{t-m^2} - \frac{a_2}{u-m^2}\right)\left(-\frac{a_1}{t-m^2} - \frac{a_2}{u-m^2}\right)^* \\
&= \frac{e^4}{N}\left(\frac{a_1 a_1^*}{(t-m^2)^2} + \frac{a_1 a_2^*}{(t-m^2)(u-m^2)} + \frac{a_1^* a_2}{(t-m^2)(u-m^2)} + \frac{a_2 a_2^*}{(u-m^2)^2}\right)
\end{aligned}$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is computed by summing $|\mathcal{M}_{ab}|^2$ over all spin and polarization states and then dividing by the number of inbound states. There are four inbound states. The sum over polarization states is already accomplished by contraction of aa^* over μ and ν.

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}|\mathcal{M}_{ab}|^2 \\
&= \frac{e^4}{4N}\sum_{a=1}^{2}\sum_{b=1}^{2}\left(\frac{a_1 a_1^*}{(t-m^2)^2} + \frac{a_1 a_2^*}{(t-m^2)(u-m^2)} + \frac{a_1^* a_2}{(t-m^2)(u-m^2)} + \frac{a_2 a_2^*}{(u-m^2)^2}\right)
\end{aligned}$$

Use the Casimir trick to replace sums over spins with matrix products.

$$f_{11} = \frac{1}{N}\sum_{a=1}^{2}\sum_{b=1}^{2} a_1 a_1^* = \mathrm{Tr}\left((\not{p}_1 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\right)$$

$$f_{12} = \frac{1}{N}\sum_{a=1}^{2}\sum_{b=1}^{2} a_1 a_2^* = \mathrm{Tr}\left((\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\right)$$

$$f_{22} = \frac{1}{N}\sum_{a=1}^{2}\sum_{b=1}^{2} a_2 a_2^* = \mathrm{Tr}\left((\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_2 + m)\gamma_\mu\right)$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left(\frac{f_{11}}{(t-m^2)^2} + \frac{f_{12}}{(t-m^2)(u-m^2)} + \frac{f_{12}^*}{(t-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2}\right)$$

Run "annihilation-1.txt" to verify the Casimir trick for electron positron annihilation.

The following momentum formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$\begin{aligned}
f_{11} = {} & 16(p_1\cdot p_1)(p_1\cdot p_2) - 32(p_1\cdot p_1)(p_2\cdot p_3) - 16(p_1\cdot p_2)(p_3\cdot p_3) + 32(p_1\cdot p_3)(p_2\cdot p_3) \\
& - 48m^2(p_1\cdot p_2) + 64m^2(p_1\cdot p_3) + 64m^2(p_2\cdot p_3) - 64m^2(p_3\cdot p_3) - 64m^4 \\
f_{12} = {} & -32(p_1\cdot p_1)(p_1\cdot p_2) + 32(p_1\cdot p_2)(p_1\cdot p_3) + 32(p_1\cdot p_2)(p_1\cdot p_4) - 32(p_1\cdot p_2)(p_3\cdot p_4) \\
& - 48m^2(p_1\cdot p_1) + 48m^2(p_1\cdot p_2) + 32m^2(p_1\cdot p_3) + 32m^2(p_1\cdot p_4) \\
& - 16m^2(p_2\cdot p_3) - 16m^2(p_2\cdot p_4) - 16m^2(p_3\cdot p_4) + 32m^4 \\
f_{22} = {} & 16(p_1\cdot p_1)(p_1\cdot p_2) - 32(p_1\cdot p_1)(p_2\cdot p_4) - 16(p_1\cdot p_2)(p_4\cdot p_4) + 32(p_1\cdot p_4)(p_2\cdot p_4) \\
& - 48m^2(p_1\cdot p_2) + 64m^2(p_1\cdot p_4) + 64m^2(p_2\cdot p_4) - 64m^2(p_4\cdot p_4) - 64m^4
\end{aligned}$$

In Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$ the formulas are

$$\begin{aligned} f_{11} &= 8tu - 24tm^2 - 8um^2 - 8m^4 \\ f_{12} &= 8sm^2 - 32m^4 \\ f_{22} &= 8tu - 8tm^2 - 24um^2 - 8m^4 \end{aligned}$$

When $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$\begin{aligned} f_{11} &= 8tu \\ f_{12} &= 0 \\ f_{22} &= 8tu \end{aligned}$$

For $m = 0$ the Mandelstam variables are

$$\begin{aligned} s &= 4E^2 \\ t &= -2E^2(1 - \cos\theta) = -4E^2 \sin^2(\theta/2) \\ u &= -2E^2(1 + \cos\theta) = -4E^2 \cos^2(\theta/2) \end{aligned}$$

The corresponding expected probability density is

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left(\frac{8tu}{t^2} + \frac{8tu}{u^2} \right) \\ &= 2e^4 \left(\frac{u}{t} + \frac{t}{u} \right) \\ &= 2e^4 \left(\frac{1 + \cos\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{1 + \cos\theta} \right) \end{aligned}$$

Substituting $e^4 = 16\pi^2\alpha^2$ we have

$$\langle |\mathcal{M}|^2 \rangle = 32\pi^2\alpha^2 \left(\frac{1 + \cos\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{1 + \cos\theta} \right)$$

The resulting differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{64\pi^2 s} = \frac{\alpha^2}{8E^2} \left(\frac{1 + \cos\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{1 + \cos\theta} \right)$$

Run "annihilation-2.txt" to verify.

We can integrate $d\sigma$ to obtain a cumulative distribution function. Recall that

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

Hence

$$d\sigma = \frac{\alpha^2}{8E^2} \left(\frac{1 + \cos\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{1 + \cos\theta} \right) \sin\theta \, d\theta \, d\phi$$

Let $I(\theta)$ be the following integral of $d\sigma$.

$$\begin{aligned} I(\theta) &= \left(\frac{8E^2}{\alpha^2}\right)\frac{1}{2\pi}\int_0^{2\pi}\int d\sigma \\ &= \int\left(\frac{1+\cos\theta}{1-\cos\theta}+\frac{1-\cos\theta}{1+\cos\theta}\right)\sin\theta\, d\theta, \quad a \le \theta \le \pi - a \end{aligned}$$

Angular support is limited to an arbitrary $a > 0$ because $I(0)$ and $I(\pi)$ are undefined.

Let C be the normalization constant

$$C = I(\pi - a) - I(a)$$

Then the cumulative distribution function $F(\theta)$ is

$$F(\theta) = \frac{I(\theta) - I(a)}{C}, \quad a \le \theta \le \pi - a$$

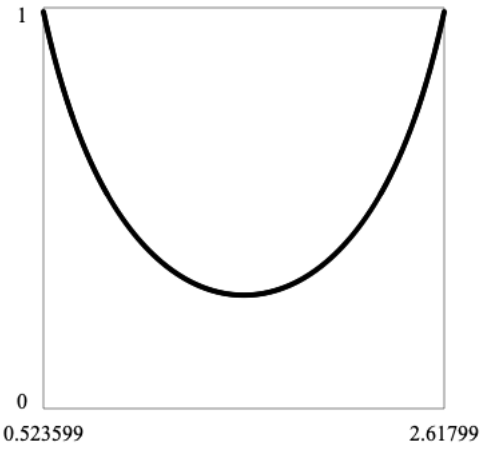
The probability of observing scattering events in the interval θ_1 to θ_2 can now be computed.

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

Probability density function $f(\theta)$ is the derivative of $F(\theta)$.

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{C}\left(\frac{1+\cos\theta}{1-\cos\theta}+\frac{1-\cos\theta}{1+\cos\theta}\right)\sin\theta, \quad a \le \theta \le \pi - a$$

Run "annihilation-4.txt" to plot $f(\theta)$ for $a = \pi/6 = 30°$.



This is the probability distribution for 30° bins and $a = 30°$.

θ_1	θ_2	$P(\theta_1 \le \theta \le \theta_2)$
0°	30°	–
30°	60°	0.33
60°	90°	0.17
90°	120°	0.17
120°	150°	0.33
150°	180°	–

The following table shows DESY-PETRA electron positron annihilation data.[1]

x	y
0.0502	0.09983
0.1505	0.10791
0.2509	0.12026
0.3512	0.13002
0.4516	0.17681
0.5521	0.1957
0.6526	0.279
0.7312	0.33204

Data x and y have the following relationship with the differential cross section formula.

$$x = \cos\theta \qquad y = \frac{d\sigma}{d\Omega}$$

To compute predicted values $\hat{y}$ from the cross section formula, use $\sqrt{s} = 2E = 14.0\,\text{GeV}$. Multiply by $(\hbar c)^2$ to convert to SI and multiply by 10^{37} to convert square meters to nanobarns.

$$\hat{y} = \frac{\alpha^2}{2(14.0)^2}\left(\frac{1+x}{1-x} + \frac{1-x}{1+x}\right) \times (\hbar c)^2 \times 10^{37}$$

The following table shows predicted values $\hat{y}$ based on the above formula.

x	y	$\hat{y}$
0.0502	0.09983	0.106325
0.1505	0.10791	0.110694
0.2509	0.12026	0.120005
0.3512	0.13002	0.135559
0.4516	0.17681	0.159996
0.5521	0.1957	0.198562
0.6526	0.279	0.262745
0.7312	0.33204	0.348884

The coefficient of determination R^2 measures how well predicted values fit the real data.

$$R^2 = 1 - \frac{\sum(y-\hat{y})^2}{\sum(y-\bar{y})^2} = 0.98$$

The result indicates that the model $d\sigma$ explains 98% of the variance in the data.

Run "annihilation-3.txt" to verify.

[1] `www.hepdata.net/record/ins191231` (Table 2, 14.0 GeV)

Here are some notes on how the scripts work.

To convert a_1 and a_2 to Eigenmath code, it is instructive to write a_1 and a_2 in full component form.

$$a_1^{\mu\nu} = \bar{v}_{2\alpha}\gamma^{\mu\alpha}{}_{\beta}(\not{q}_1 + m)^{\beta}{}_{\rho}\gamma^{\nu\rho}{}_{\sigma}u_1^{\sigma} \qquad a_2^{\nu\mu} = \bar{v}_{2\alpha}\gamma^{\nu\alpha}{}_{\beta}(\not{q}_2 + m)^{\beta}{}_{\rho}\gamma^{\mu\rho}{}_{\sigma}u_1^{\sigma}$$

Transpose the γ tensors to form inner products over α and ρ.

$$a_1^{\mu\nu} = \bar{v}_{2\alpha}\gamma^{\alpha\mu}{}_{\beta}(\not{q}_1 + m)^{\beta}{}_{\rho}\gamma^{\rho\nu}{}_{\sigma}u_1^{\sigma} \qquad a_2^{\nu\mu} = \bar{v}_{2\alpha}\gamma^{\alpha\nu}{}_{\beta}(\not{q}_2 + m)^{\beta}{}_{\rho}\gamma^{\rho\mu}{}_{\sigma}u_1^{\sigma}$$

Convert transposed γ to Eigenmath code.

$$\gamma^{\alpha\mu}{}_{\beta} \quad \rightarrow \quad \texttt{gammaT = transpose(gamma)}$$

Then to compute a_1 we have

$$a_1 = \bar{v}_{2\alpha}\gamma^{\alpha\mu}{}_{\beta}(\not{q}_1 + m)^{\beta}{}_{\rho}\gamma^{\rho\nu}{}_{\sigma}u_1^{\sigma}$$

$$\rightarrow \quad \texttt{a1 = dot(v2bar[s2],gammaT,qslash1 + m I,gammaT,u1[s1])}$$

where s_1 and s_2 are spin indices. Similarly for a_2 we have

$$a_2 = \bar{v}_{2\alpha}\gamma^{\alpha\mu}{}_{\beta}(\not{q}_2 + m)^{\beta}{}_{\rho}\gamma^{\rho\nu}{}_{\sigma}u_1^{\sigma}$$

$$\rightarrow \quad \texttt{a2 = dot(v2bar[s2],gammaT,qslash2 + m I,gammaT,u1[s1])}$$

In component notation the product $a_1 a_1^*$ is

$$a_1 a_1^* = a_1^{\mu\nu} a_1^{*\mu\nu}$$

To sum over μ and ν it is necessary to lower indices with the metric tensor. Also, transpose a_1^* to form an inner product with ν.

$$a_1 a_1^* = a_1^{\mu\nu} a_{1\nu\mu}^*$$

Convert to Eigenmath code. The dot function sums over ν and the contract function sums over μ.

$$a_1 a_1^* \quad \rightarrow \quad \texttt{a11 = contract(dot(a1,gmunu,transpose(conj(a1)),gmunu))}$$

Similarly for $a_2 a_2^*$ we have

$$a_2 a_2^* \quad \rightarrow \quad \texttt{a22 = contract(dot(a2,gmunu,transpose(conj(a2)),gmunu))}$$

The product $a_1 a_2^*$ does not require a transpose because $a_2 = a_2^{\nu\mu}$.

$$a_1^{\mu\nu} a_{2\nu\mu}^* \quad \rightarrow \quad \texttt{a12 = contract(dot(a1,gmunu,conj(a2),gmunu))}$$

In component notation, a trace operator becomes a sum over an index, in this case α.

$$\begin{aligned} f_{11} &= \mathrm{Tr}\left((\not{p}_1 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\right) \\ &= (\not{p}_1 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{q}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_2 - m)^\tau{}_\delta \gamma_\nu{}^\delta{}_\eta (\not{q}_1 + m)^\eta{}_\xi \gamma_\mu{}^\xi{}_\alpha \end{aligned}$$

As before, transpose γ tensors to form inner products.

$$f_{11} = (\not{p}_1 + m)^\alpha{}_\beta \gamma^{\beta\mu}{}_\rho (\not{q}_1 + m)^\rho{}_\sigma \gamma^{\sigma\nu}{}_\tau (\not{p}_2 - m)^\tau{}_\delta \gamma^\delta{}_{\nu\eta} (\not{q}_1 + m)^\eta{}_\xi \gamma^\xi{}_{\mu\alpha}$$

This is the code for transposing γ.

$$\begin{aligned} \gamma^{\beta\mu}{}_\beta \quad &\rightarrow \quad \texttt{gammaT = transpose(gamma)} \\ \gamma^\delta{}_{\nu\eta} \quad &\rightarrow \quad \texttt{gammaL = transpose(dot(gmunu,gamma))} \end{aligned}$$

To convert f_{11} to Eigenmath code, use an intermediate variable T for the inner product.

$$T^{\alpha\mu\nu}{}_{\nu\mu\alpha} \quad \rightarrow \quad \texttt{T = dot(P1,gammaT,Q1,gammaT,P2,gammaL,Q1,gammaL)}$$

Now sum over the indices of T. The innermost contract sums over ν then the next contract sums over μ. Finally the outermost contract sums over α.

$$f_{11} \quad \rightarrow \quad \texttt{f11 = contract(contract(contract(T,3,4),2,3))}$$

Follow suit for f_{22}. For f_{12} the order of the rightmost μ and ν is reversed.

$$f_{12} = \mathrm{Tr}\left((\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\right)$$

The resulting inner product is $T^{\alpha\mu\nu}{}_{\mu\nu\alpha}$ so the contraction is different.

$$f_{12} \quad \rightarrow \quad \texttt{f12 = contract(contract(contract(T,3,5),2,3))}$$

The innermost contract sums over ν followed by sum over μ then sum over α.