Muon decay

From the Particle Data Group website[1]

μ MEAN LIFE τ

Measurements with an error $> 0.001 \times 10^{-6}$ s have been omitted.

VALUE (10^{-6} s)	*DOCUMENT ID*	*TECN*	*CHG*	*COMMENT*
2.1969811±0.0000022 OUR AVERAGE				
2.1969803±0.0000021±0.0000007 [1]	TISHCHENKO 13	CNTR	+	Surface μ^+ at PSI
2.197083 ±0.000032 ±0.000015	BARCZYK 08	CNTR	+	Muons from π^+ decay at rest
2.197013 ±0.000021 ±0.000011	CHITWOOD 07	CNTR	+	Surface μ^+ at PSI
2.197078 ±0.000073	BARDIN 84	CNTR	+	
2.197025 ±0.000155	BARDIN 84	CNTR	−	
2.19695 ±0.00006	GIOVANETTI 84	CNTR	+	
2.19711 ±0.00008	BALANDIN 74	CNTR	+	
2.1973 ±0.0003	DUCLOS 73	CNTR	+	
• • • We do not use the following data for averages, fits, limits, etc. • • •				
2.1969803±0.0000022	WEBBER 11	CNTR	+	Surface μ^+ at PSI

[1] TISHCHENKO 13 uses 1.6×10^{12} μ^+ events and supersedes WEBBER 11.

From "V minus A" theory we have the following formula for muon lifetime τ.

$$\tau = \frac{96\pi^2 h}{G_F^2 \left(m_\mu c^2\right)^5}$$

Symbol G_F is Fermi coupling constant, m_μ is muon mass.

From NIST we have

$$\begin{aligned}
G_F &= 1.1663787 \times 10^{-5}\ \text{GeV}^{-2} \\
m_\mu &= 1.883531627 \times 10^{-28}\ \text{kilogram} \\
h &= 6.62607015 \times 10^{-34}\ \text{joule second (exact)} \\
c &= 299792458\ \text{meter second}^{-1}\ \text{(exact)} \\
1\,\text{eV} &= 1.602176634 \times 10^{-19}\ \text{joule (exact)}
\end{aligned}$$

Hence

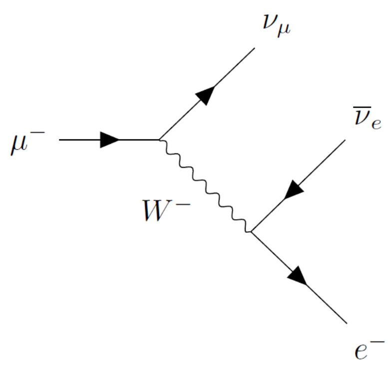
$$\tau = \frac{96\pi^2 h}{G_F^2 \left(m_\mu c^2\right)^5} = 2.18735 \times 10^{-6}\,\text{second}$$

The result is a bit smaller than the observed value from Particle Data Group.

$$\frac{\tau}{\text{observed value}} = \frac{2.18735 \times 10^{-6}\,\text{second}}{2.19698 \times 10^{-6}\ \text{second}} = 0.9956$$

As the following diagram shows, a muon decays into a muon neutrino, an electron antineutrino, and an electron.

[1] https://pdg.lbl.gov/2020/listings/rpp2020-list-muon.pdf



Particle	Momentum	Mass	Spin up	Spin down
Muon μ^-	p_1	m_1	u_{11}	u_{12}
Muon neutrino ν_μ	p_2	m_2	u_{21}	u_{22}
Electron anti-neutrino $\bar{\nu}_e$	p_3	m_3	v_{31}	v_{32}
Electron e^-	p_4	m_4	u_{41}	u_{42}

For $E_n = \sqrt{|\mathbf{p}_n|^2 + {m_n}^2}$ we have

$$p_1 = \underset{\mu^-}{\begin{pmatrix} E_1 \\ p_{1x} \\ p_{1y} \\ p_{1z} \end{pmatrix}} \qquad p_2 = \underset{\nu_\mu}{\begin{pmatrix} E_2 \\ p_{2x} \\ p_{2y} \\ p_{2z} \end{pmatrix}} \qquad p_3 = \underset{\bar{\nu}_e}{\begin{pmatrix} E_3 \\ p_{3x} \\ p_{3y} \\ p_{3z} \end{pmatrix}} \qquad p_4 = \underset{e^-}{\begin{pmatrix} E_4 \\ p_{4x} \\ p_{4y} \\ p_{4z} \end{pmatrix}}$$

Spinors for the muon.

$$u_{11} = \frac{1}{\sqrt{E_1 + m_1}} \underset{\mu^-\text{ spin up}}{\begin{pmatrix} E_1 + m_1 \\ 0 \\ p_{1z} \\ p_{1x} + ip_{1y} \end{pmatrix}} \qquad u_{12} = \frac{1}{\sqrt{E_1 + m_1}} \underset{\mu^-\text{ spin down}}{\begin{pmatrix} 0 \\ E_1 + m_1 \\ p_{1x} - ip_{1y} \\ -p_{1z} \end{pmatrix}}$$

Spinors for the muon neutrino.

$$u_{21} = \frac{1}{\sqrt{E_2 + m_2}} \underset{\nu_\mu\text{ spin up}}{\begin{pmatrix} E_2 + m_2 \\ 0 \\ p_{2z} \\ p_{2x} + ip_{2y} \end{pmatrix}} \qquad u_{22} = \frac{1}{\sqrt{E_2 + m_2}} \underset{\nu_\mu\text{ spin down}}{\begin{pmatrix} 0 \\ E_2 + m_2 \\ p_{2x} - ip_{2y} \\ -p_{2z} \end{pmatrix}}$$

Spinors for the electron anti-neutrino.

$$v_{31} = \frac{1}{\sqrt{E_3 + m_3}} \underset{\bar{\nu}_e\text{ spin up}}{\begin{pmatrix} p_{3z} \\ p_{3x} + ip_{3y} \\ E_3 + m_3 \\ 0 \end{pmatrix}} \qquad v_{32} = \frac{1}{\sqrt{E_3 + m_3}} \underset{\bar{\nu}_e\text{ spin down}}{\begin{pmatrix} p_{3x} - ip_{3y} \\ -p_{3z} \\ 0 \\ E_3 + m_3 \end{pmatrix}}$$

Spinors for the electron.

$$u_{41} = \frac{1}{\sqrt{E_4 + m_4}} \underset{e^- \text{ spin up}}{\begin{pmatrix} E_4 + m_4 \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix}} \qquad u_{42} = \frac{1}{\sqrt{E_4 + m_4}} \underset{e^- \text{ spin down}}{\begin{pmatrix} 0 \\ E_4 + m_4 \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}}$$

The decay amplitude $\mathcal{M}_{abcd}$ for spin state $abcd$ is

$$\mathcal{M}_{abcd} = \frac{G_F}{\sqrt{2}} \left(\bar{u}_{4d}\gamma^\mu(1-\gamma^5)v_{3c}\right) \left(\bar{u}_{2b}\gamma_\mu(1-\gamma^5)u_{1a}\right)$$

The expected probability $\langle|\mathcal{M}|^2\rangle$ is the average of spin states.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{2}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}|\mathcal{M}_{abcd}|^2$$

The Casimir trick uses matrix arithmetic to sum over spin states.

$$\langle|\mathcal{M}|^2\rangle = \frac{G_F^2}{4}\operatorname{Tr}\left(\not{p}_4\gamma^\mu(1-\gamma^5)\not{p}_3\gamma^\nu(1-\gamma^5)\right)\operatorname{Tr}\left(\not{p}_2\gamma_\mu(1-\gamma^5)\not{p}_1\gamma_\nu(1-\gamma^5)\right)$$

The result is a simple formula.

$$\langle|\mathcal{M}|^2\rangle = 64G_F^2(p_1\cdot p_3)(p_2\cdot p_4)$$

In the muon rest frame p_1 is fixed at $p_1 = (m_1, 0, 0, 0)$. The remaining momentum vectors are free to have any values that conserve energy and momentum. Muon decay rate Γ is the expectation value for all possible decay momenta. By Fermi's golden rule

$$\Gamma = \frac{1}{512\pi^5 m_\mu}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\langle|\mathcal{M}|^2\rangle\,\delta(p_1 - p_2 - p_3 - p_4)\,\frac{d^3p_2}{E_2}\,\frac{d^3p_3}{E_3}\,\frac{d^3p_4}{E_4}$$

Altogether there are nine integrals, three for each of p_2, p_3, and p_4. The delta function restricts the integration space to values that conserve energy and momentum.

It can be shown that

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

Muon lifetime τ is the inverse of decay rate.

$$\tau = \frac{1}{\Gamma} = \frac{192\pi^3}{G_F^2 m_\mu^5}$$

Change natural units to h and c.

$$\tau = \frac{96\pi^2 h}{G_F^2\left(m_\mu c^2\right)^5}$$