

15.1.2. Given that

$$\hat{a}^\dagger \hat{a} |\psi_n\rangle = n |\psi_n\rangle$$

and

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$$

show that

$$\hat{a}^\dagger \hat{a} (\hat{a}^\dagger |\psi_n\rangle) = (n+1) (\hat{a}^\dagger |\psi_n\rangle)$$

Noting that linear operators are associative we have

$$\hat{a}^\dagger \hat{a} (\hat{a}^\dagger |\psi_n\rangle) = \hat{a}^\dagger (\hat{a}\hat{a}^\dagger) |\psi_n\rangle$$

Then by the commutator relation given above we have

$$\hat{a}^\dagger \hat{a} (\hat{a}^\dagger |\psi_n\rangle) = \hat{a}^\dagger (\hat{a}^\dagger \hat{a} + 1) |\psi_n\rangle$$

By the number operator given above we have

$$\hat{a}^\dagger \hat{a} (\hat{a}^\dagger |\psi_n\rangle) = \hat{a}^\dagger (n+1) |\psi_n\rangle$$

Numbers commute with operators hence

$$\hat{a}^\dagger \hat{a} (\hat{a}^\dagger |\psi_n\rangle) = (n+1) \hat{a}^\dagger |\psi_n\rangle$$