9-7. Show, for the vacuum state, the expectation value of $\bar{a}_{1,\mathbf{k}}^*\bar{a}_{1,\mathbf{q}}$ is $(\hbar/2kc)\delta_{\mathbf{k},\mathbf{q}}$ and that of $\bar{a}_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}}$ is $(\hbar/2kc)\delta_{-\mathbf{k},\mathbf{q}}$.

We will use the following integrals.

$$\int_{-\infty}^{\infty} \exp(-ax^2 + b)\, dx = \sqrt{\frac{\pi}{a}}\exp(b) \tag{1}$$

$$\int_{-\infty}^{\infty} x\exp(-ax^2 + b)\, dx = 0 \tag{2}$$

$$\int_{-\infty}^{\infty} x^2 \exp(-ax^2 + b)\, dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}\exp(b) \tag{3}$$

For simplicity of notation, let

$$A = \bar{a}_{1,\mathbf{k}}^c \qquad B = \bar{a}_{1,\mathbf{k}}^s \qquad C = \bar{a}_{1,\mathbf{q}}^c \qquad D = \bar{a}_{1,\mathbf{q}}^s$$

From problem 9-6

$$\bar{a}_{1,\mathbf{k}} = \frac{1}{\sqrt{2}}(A - iB)$$

$$\bar{a}_{1,\mathbf{q}} = \frac{1}{\sqrt{2}}(C - iD) \tag{4}$$

Adapted from equation (8.84)

$$\langle \Phi_0 | F | \Phi_0 \rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_0^* F \Phi_0 \, dA\, dB\, dC\, dD$$

Adapted from problem 9-6 with q as a mode number (not electric charge).

$$\Phi_0 = \exp\left(-\frac{kc}{4\hbar}A^2 - \frac{kc}{4\hbar}B^2 - \frac{qc}{4\hbar}C^2 - \frac{qc}{4\hbar}D^2\right)$$

It follows that

$$\Phi_0^*\Phi_0 = \exp\left(-\frac{kc}{2\hbar}A^2 - \frac{kc}{2\hbar}B^2 - \frac{qc}{2\hbar}C^2 - \frac{qc}{2\hbar}D^2\right)$$

Compute the normalization constant K.

$$K = \langle \Phi_0 | 1 | \Phi_0 \rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_0^*\Phi_0 \, dA\, dB\, dC\, dD$$

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Apply equation (1) for each factor in the measure (see problem 8-5).

$$K = \left(\frac{2\pi\hbar}{kc}\right)^{1/2} \left(\frac{2\pi\hbar}{kc}\right)^{1/2} \left(\frac{2\pi\hbar}{qc}\right)^{1/2} \left(\frac{2\pi\hbar}{qc}\right)^{1/2} = \frac{4\pi^2\hbar^2}{kqc^2}$$

Compute the expectation value for $\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}}$. From (4) we have

$$\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} = \frac{A^2 + B^2}{2}$$

Hence

$$\langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle = \frac{1}{K} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_0^* \Phi_0 \frac{A^2 + B^2}{2} \, dA \, dB \, dC \, dD$$

Rewrite as

$$\langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle = \frac{1}{2K} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_0^* \Phi_0 A^2 \, dA \, dB \, dC \, dD$$

$$+ \frac{1}{2K} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_0^* \Phi_0 B^2 \, dA \, dB \, dC \, dD$$

By equations (1) and (3) we have

$$\langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle = \frac{1}{K} \frac{\hbar}{kc} \left(\frac{2\pi\hbar}{kc}\right)^{1/2} \left(\frac{2\pi\hbar}{kc}\right)^{1/2} \left(\frac{2\pi\hbar}{qc}\right)^{1/2} \left(\frac{2\pi\hbar}{qc}\right)^{1/2}$$

Hence
$$\langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle = \frac{\hbar}{kc} \tag{5}$$

Compute the expectation value for $\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{q}}$. From (4) we have

$$\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{q}} = \frac{1}{2} AC - \frac{1}{2} BD - \frac{i}{2} AD - \frac{i}{2} BC$$