Fermi's golden rule

Find the transition rate $\Gamma_{1\to 2}$ by deriving Fermi's golden rule.

Let the perturbing Hamiltonian be

$$H_1(x,t) = 2V(x)\cos(\omega t + \phi)$$

Let $\Psi(x,t)$ be the following linear combination of the two eigenstates.

$$\Psi(x,t) = c_1(t)\psi_1(x)\exp(-i\omega_1 t) + c_2(t)\psi_2(x)\exp(-i\omega_2 t)$$

We need to solve for $c_2(t)$ to find the transition rate.

From the time-dependent Schrodinger equation we have

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = H_0(x)\Psi(x,t) + H_1(x,t)\Psi(x,t)$$

Expand the left-hand side.

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\Psi(x,t) = i\hbar\frac{\partial c_1(t)}{\partial t}\psi_1(x)\exp(-i\omega_1 t) &+ \boxed{\hbar\omega_1 c_1(t)\psi_1(x)\exp(-i\omega_1 t)} \\ + i\hbar\frac{\partial c_2(t)}{\partial t}\psi_2(x)\exp(-i\omega_2 t) &+ \boxed{\hbar\omega_2 c_2(t)\psi_2(x)\exp(-i\omega_2 t)} \end{aligned}$$

Noting that $\hbar\omega_1 = E_1$ and $\hbar\omega_2 = E_2$, the boxed terms cancel with $H_0(x)\Psi(x,t)$ leaving

$$i\hbar\frac{\partial c_1(t)}{\partial t}\psi_1(x)\exp(-i\omega_1 t) + i\hbar\frac{\partial c_2(t)}{\partial t}\psi_2(x)\exp(-i\omega_2 t) = H_1(x,t)\Psi(x,t)$$

Substitute for $H_1(x)\Psi(x,t)$.

$$\begin{aligned} &i\hbar\frac{\partial c_1(t)}{\partial t}\psi_1(x)\exp(-i\omega_1 t) + i\hbar\frac{\partial c_2(t)}{\partial t}\psi_2(x)\exp(-i\omega_2 t) \\ &\qquad = 2\cos(\omega t + \phi)\Big[c_1(t)V(x)\psi_1(x)\exp(-i\omega_1 t) + c_2(t)V(x)\psi_2(x)\exp(-i\omega_2 t)\Big] \end{aligned}$$

Take the inner product of $\psi_2^*(x)$ with both sides to obtain

$$i\hbar\frac{\partial c_2(t)}{\partial t}\exp(-i\omega_2 t) = 2\cos(\omega t + \phi)\Big[c_1(t)M_{21}\exp(-i\omega_1 t) + c_2(t)M_{22}\exp(-i\omega_2 t)\Big]$$

Symbols M_{21} and M_{22} are the following matrix elements.

$$M_{21} = \int \psi_2^*(x)V(x)\psi_1(x)\,dx$$

$$M_{22} = \int \psi_2^*(x)V(x)\psi_2(x)\,dx$$

Multiply both sides by $\exp(i\omega_2 t)$.

$$i\hbar\frac{\partial c_2(t)}{\partial t} = 2\cos(\omega t+\phi)\Big[c_1(t)M_{21}\exp\big(i(\omega_2-\omega_1)t\big)+c_2(t)M_{22}\Big]$$

Let the initial state be $\Psi(x,0)=\psi_1(x)$ hence the initial conditions are

$$c_1(0)=1, \quad c_2(0)=0$$

For time t near the origin use the approximations $c_1(t)=1$ and $c_2(t)=0$ to obtain

$$i\hbar\frac{\partial c_2(t)}{\partial t} = 2\cos(\omega t+\phi)M_{21}\exp\big(i(\omega_2-\omega_1)t\big)$$

Solve for $c_2(t)$ by integrating.

$$c_2(t)=\frac{2M_{21}}{i\hbar}\int_0^t \cos(\omega t'+\phi)\exp\big(i(\omega_2-\omega_1)t'\big)\,dt'$$

The solution is

$$c_2(t) = -\frac{M_{21}}{\hbar}\left(\frac{\exp\big(i(\omega_2-\omega_1-\omega)t\big)-1}{\omega_2-\omega_1-\omega}\right)\exp(-i\phi) \\ -\frac{M_{21}}{\hbar}\left(\frac{\exp\big(i(\omega_2-\omega_1+\omega)t\big)-1}{\omega_2-\omega_1+\omega}\right)\exp(+i\phi) \quad (1)$$

For ω such that $\omega\approx\omega_2-\omega_1$ the first term dominates so discard the second term and write

$$c_2(t) = -\frac{M_{21}}{\hbar}\left(\frac{\exp\big(i(\omega_2-\omega_1-\omega)t\big)-1}{\omega_2-\omega_1-\omega}\right)\exp(-i\phi)$$

Rewrite using a sinc function.

$$c_2(t) = -\frac{it}{\hbar}M_{21}\exp\left(i\,\frac{\omega_2-\omega_1-\omega}{2}\,t-i\phi\right)\operatorname{sinc}\left(\frac{\omega_2-\omega_1-\omega}{2}\,t\right) \quad (2)$$

Hence the transition probability is

$$P(1\to 2)=|c_2(t)|^2=\frac{t^2}{\hbar^2}|M_{21}|^2\operatorname{sinc}^2\left(\frac{\omega_2-\omega_1-\omega}{2}\,t\right) \quad (3)$$

Integrate $P(1\to 2)$ to obtain the total transition probability $P_{tot}(1\to 2)$.

$$P_{tot}(1\to 2)=\frac{t^2}{\hbar^2}|M_{21}|^2\int_{E-\epsilon}^{E+\epsilon}\operatorname{sinc}^2\left(\frac{E'/\hbar-\omega}{2}\,t\right)g(E')\,dE'$$

where

$$E=\hbar(\omega_2-\omega_1)$$

and $g(E')$ is the density of photon states for energy E'.

Use the approximation $g(E') \approx g(\hbar\omega)$ to obtain

$$P_{tot}(1 \to 2) = \frac{t^2}{\hbar^2}|M_{21}|^2 g(\hbar\omega) \int_{E-\epsilon}^{E+\epsilon} \operatorname{sinc}^2\left(\frac{E'/\hbar - \omega}{2}\,t\right) dE'$$

Let

$$y = \frac{E'/\hbar - \omega}{2}\,t$$

It follows that

$$E' = \frac{2\hbar y}{t} + \hbar\omega$$

and

$$dE' = \frac{2\hbar}{t}\,dy$$

The integration limits transform as

$$E \pm \epsilon \to \frac{(E \pm \epsilon)/\hbar - \omega}{2}\,t = \frac{Et}{2\hbar} - \frac{\omega t}{2} \pm \frac{\epsilon t}{2\hbar} \approx \pm\frac{\epsilon t}{2\hbar}$$

Hence

$$P_{tot}(1 \to 2) = \frac{2t}{\hbar}|M_{21}|^2 g(\hbar\omega) \int_{-\epsilon t/2\hbar}^{\epsilon t/2\hbar} \operatorname{sinc}^2 y\,dy$$

Use the approximation

$$\int_{-\epsilon t/2\hbar}^{\epsilon t/2\hbar} \operatorname{sinc}^2 y\,dy \approx \int_{-\infty}^{\infty} \operatorname{sinc}^2 y\,dy = \pi$$

to obtain

$$P_{tot}(1 \to 2) = \frac{2\pi t}{\hbar}|M_{21}|^2 g(\hbar\omega)$$

The transition rate is the derivative of $P_{tot}(1 \to 2)$.

$$\Gamma_{1\to2} = \frac{d}{dt}P_{tot}(1 \to 2) = \frac{2\pi}{\hbar}|M_{21}|^2 g(\hbar\omega)$$