Rutherford scattering 1

Use the following formula to compute the cross section for Rutherford scattering.

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2\varepsilon_0^2}\left(\frac{mQ}{4\pi\hbar^2}\right)^2, \quad Q = \int \exp\left(\frac{i\mathbf{p}\cdot\mathbf{r}}{\hbar}\right) V(\mathbf{r})\, d\mathbf{r}^3$$

Convert Q to polar coordinates.

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) V(r,\theta,\phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

For Rutherford scattering $V(\mathbf{r})$ is the Coulomb potential.

$$V(\mathbf{r}) = V(r) = -\frac{Ze^2}{r}$$

Substitute the Coulomb potential for $V(r,\theta,\phi)$ and note r^2 becomes r.

$$Q = -Ze^2\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) r\sin\theta\, dr\, d\theta\, d\phi$$

Integrate over ϕ.

$$Q = -2\pi Ze^2\int_0^{\pi}\int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) r\sin\theta\, dr\, d\theta$$

Change the complex exponential to rectangular form.

$$Q = -2\pi Ze^2\int_0^{\pi}\int_0^{\infty} \left[\cos\left(\frac{pr\cos\theta}{\hbar}\right) + i\sin\left(\frac{pr\cos\theta}{\hbar}\right)\right] V(r)\, r\sin\theta\, dr\, d\theta$$

By the integrals

$$\int_0^{\pi} \cos\big(a\cos(\theta)\big)\sin\theta\, d\theta = \frac{2\sin a}{a}, \quad \int_0^{\pi} \sin\big(a\cos(\theta)\big)\sin\theta\, d\theta = 0$$

we obtain (note r in the integrand is canceled)

$$Q = -\frac{4\pi\hbar Ze^2}{p}\int_0^{\infty} \sin\left(\frac{pr}{\hbar}\right) dr$$

To solve the integral, multiply the integrand by $\exp(-\epsilon r)$.

$$Q = -\frac{4\pi\hbar Ze^2}{p}\int_0^{\infty} \sin\left(\frac{pr}{\hbar}\right)\exp(-\epsilon r)\, dr$$

Convert the integrand to exponential form.

$$Q = -\frac{4\pi\hbar Ze^2}{p}\int_0^{\infty} \frac{i}{2}\left[\exp\left(-\frac{ipr}{\hbar} - \epsilon r\right) - \exp\left(\frac{ipr}{\hbar} - \epsilon r\right)\right] dr$$

Solve the integral.

$$Q = -\frac{4\pi\hbar Z e^2}{p}\frac{i}{2}\left(\frac{1}{-ip/\hbar - \epsilon} - \frac{1}{ip/\hbar - \epsilon}\right) \tag{1}$$

Set $\epsilon = 0$.

$$Q = -\frac{4\pi\hbar Z e^2}{p}\left(-\frac{\hbar}{p}\right)$$

Hence

$$Q = \frac{4\pi\hbar^2 Z e^2}{p^2}$$

Compute the differential cross section.

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2\varepsilon_0^2}\left(\frac{mQ}{4\pi\hbar^2}\right)^2 = \frac{1}{16\pi^2\varepsilon_0^2}\frac{m^2 Z^2 e^4}{p^4} \tag{2}$$

Substitute $16\pi^2\varepsilon_0^2\alpha^2(\hbar c)^2$ for e^4.

$$\frac{d\sigma}{d\Omega} = \frac{m^2 Z^2 \alpha^2 (\hbar c)^2}{p^4}$$

Symbol p is momentum transfer $|\mathbf{p}_i| - |\mathbf{p}_f|$ such that

$$p^2 = 2mE(\cos\theta - 1)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2(\hbar c)^2}{4E^2(\cos\theta - 1)^2} \tag{3}$$

Noting that

$$4\sin^4\frac{\theta}{2} = (\cos(\theta) - 1)^2$$

we also have

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2(\hbar c)^2}{16E^2\sin^4(\theta/2)}$$

Experimental data

The following data is from Geiger and Marsden's 1913 paper (y is the number of scattering events).

θ	y
150	22.2
135	27.4
120	33.0
105	47.3
75	136
60	320
45	989
37.5	1760
30	5260
22.5	20300
15	105400

Let

$$x_i = \frac{1}{\sin^4(\theta_i/2)}$$

The scattering probability for angle θ_i is

$$\Pr(\theta_i) = \frac{x_i}{\sum x}$$

Predicted values $\hat{y}_i$ are the product $\Pr(\theta_i)$ times the total number of scattering events.

$$\sum_i x_i = 4529, \quad \sum_i y_i = 134295$$

$$\hat{y}_i = \underbrace{\frac{1}{4529}}_{\substack{\text{normalization}\\ \text{constant}}} \times \underbrace{x_i}_{\substack{\text{relative}\\ \text{probability}}} \times \underbrace{134295}_{\text{total events}}$$

The following table shows the predicted values $\hat{y}$.

θ	y	$\hat{y}$
150	22.2	34.1
135	27.4	40.7
120	33.0	52.7
105	47.3	74.9
75	136	216
60	320	474
45	989	1383
37.5	1760	2778
30	5260	6608
22.5	20300	20471
15	105400	102162

The coefficient of determination R^2 measures how well predicted values fit the data.

$$R^2 = 1 - \frac{\sum_i (y_i - \hat{y}_i)^2}{\sum_i (y_i - \bar{y})^2} = 0.999$$

The result indicates that x explains 99.9% of the variance in the data.