Electromagnetic tensor

This is the standard model for an electromagnetic field.

$$\mathbf{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \qquad \mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$A^\mu = \begin{pmatrix} \phi \\ A_x \\ A_y \\ A_z \end{pmatrix} \qquad A_\mu = g_{\mu\nu} A^\nu = \begin{pmatrix} \phi \\ -A_x \\ -A_y \\ -A_z \end{pmatrix}$$

This is the electromagnetic tensor.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = A_{\nu,\mu} - A_{\mu,\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{1}$$

Verify that

$$F_{\mu\nu} F^{\mu\nu} = 2\big(\mathbf{B}^2 - \mathbf{E}^2\big) \tag{2}$$

Verify that

$$\det(F_{\mu\nu}) = \det(F^{\mu\nu}) = (\mathbf{B} \cdot \mathbf{E})^2 \tag{3}$$

This is the vector current.

$$J^\nu = \partial_\mu F^{\mu\nu} = F^{\mu\nu}{}_{,\mu}$$

Verify that

$$\partial_\mu J^\mu = J^\mu{}_{,\mu} = 0 \tag{4}$$

Verify that

$$\mathbf{J} = \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} \tag{5}$$

Eigenmath script