In his 1917 paper, "On the Quantum Theory of Radiation," Einstein uses the following argument to derive Planck's law. The argument requires induced emission, a process which was a theoretical discovery by Einstein. Prior to Einstein, no one was aware that induced emission existed.

Consider a gas at temperature T. Let N be the number of molecules in the gas and let N_n be the number of molecules with internal energy ε_n. By the Maxwell-Boltzmann distribution we have

$$\frac{N_n}{N} = p_n \exp\left(-\frac{\varepsilon_n}{kT}\right) \tag{1}$$

where k is Boltzmann's constant. The coefficient p_n is a statistical weighting factor that does not depend on T.

Let us now consider the processes by which a molecule transitions between energy levels. The processes are absorption, induced emission, and spontaneous emission. Let ε_m be an energy level such that $\varepsilon_m > \varepsilon_n$. Let B_{nm}, B_{mn}, and A_{mn} be coefficients of transition rates such that

$$\underset{\substack{\text{absorption}\\ \varepsilon_n\to\varepsilon_m}}{\frac{dN_n}{dt} = B_{nm}N_n\rho(\nu,T)} \qquad \underset{\substack{\text{induced emission}\\ \varepsilon_m\to\varepsilon_n}}{\frac{dN_m}{dt} = B_{mn}N_m\rho(\nu,T)} \qquad \underset{\substack{\text{spontaneous emission}\\ \varepsilon_m\to\varepsilon_n}}{\frac{dN_m}{dt} = A_{mn}N_m}$$

Absorption and induced emission are proportional to $\rho(\nu, T)$ which is the radiant energy density of the gas as a function of radiant frequency ν and temperature T. The A and B coefficients are presumed to not depend on temperature T.

At equilibrium, transition rates between ε_m and ε_n are equal.

$$\underset{\substack{\text{absorption}\\ \varepsilon_n\to\varepsilon_m}}{B_{nm}N_n\rho(\nu,T)} = \underset{\substack{\text{induced emission}\\ \varepsilon_m\to\varepsilon_n}}{B_{mn}N_m\rho(\nu,T)} + \underset{\substack{\text{spontaneous emission}\\ \varepsilon_m\to\varepsilon_n}}{A_{mn}N_m}$$

Divide through by N.

$$\underset{\substack{\text{absorption}\\ \varepsilon_n\to\varepsilon_m}}{B_{nm}\frac{N_n}{N}\rho(\nu,T)} = \underset{\substack{\text{induced emission}\\ \varepsilon_m\to\varepsilon_n}}{B_{mn}\frac{N_m}{N}\rho(\nu,T)} + \underset{\substack{\text{spontaneous emission}\\ \varepsilon_m\to\varepsilon_n}}{A_{mn}\frac{N_m}{N}}$$

Then by the Maxwell-Boltzmann distribution (1) we have

$$\underset{\substack{\text{absorption}\\ \varepsilon_n\to\varepsilon_m}}{B_{nm}p_n\rho(\nu,T)\exp\left(-\frac{\varepsilon_n}{kT}\right)} = \underset{\substack{\text{induced emission}\\ \varepsilon_m\to\varepsilon_n}}{B_{mn}p_m\rho(\nu,T)\exp\left(-\frac{\varepsilon_m}{kT}\right)} + \underset{\substack{\text{spontaneous emission}\\ \varepsilon_m\to\varepsilon_n}}{A_{mn}p_m\exp\left(-\frac{\varepsilon_m}{kT}\right)} \tag{2}$$

Multiply both sides by $\exp(\varepsilon_m/kT)$.

$$\underset{\substack{\text{absorption}\\ \varepsilon_n\to\varepsilon_m}}{B_{nm}p_n\rho(\nu,T)\exp\left(\frac{\varepsilon_m-\varepsilon_n}{kT}\right)} = \underset{\substack{\text{induced emission}\\ \varepsilon_m\to\varepsilon_n}}{B_{mn}p_m\rho(\nu,T)} + \underset{\substack{\text{spontaneous emission}\\ \varepsilon_m\to\varepsilon_n}}{A_{mn}p_m}$$

Note that for increasing T we have

$$\lim_{T\to\infty}\exp\left(\frac{\varepsilon_m-\varepsilon_n}{kT}\right) = 1$$

It follows that for $T \to \infty$ the equilibrium formula is

$$B_{nm} p_n \rho(\nu, T) = B_{mn} p_m \rho(\nu, T) + A_{mn} p_m$$

Divide through by $\rho(\nu, T)$.

$$B_{nm} p_n = B_{mn} p_m + \frac{A_{mn} p_m}{\rho(\nu, T)}$$

Energy density $\rho(\nu, T)$ increases with temperature T hence $A_{mn} p_m / \rho(\nu, T)$ vanishes for $T \to \infty$ leaving

$$B_{nm} p_n = B_{mn} p_m \tag{3}$$

Einstein reasoned that equation (2) is true in general based on the assumption that the factors involved do not depend on T. By substitution in the absorption term we can now eliminate $B_{nm} p_n$ and obtain

$$\underset{\substack{\text{absorption} \\ \varepsilon_n \to \varepsilon_m}}{B_{mn} p_m \rho(\nu, T) \exp\left(\frac{\varepsilon_m - \varepsilon_n}{kT}\right)} = \underset{\substack{\text{induced emission} \\ \varepsilon_m \to \varepsilon_n}}{B_{mn} p_m \rho(\nu, T)} + \underset{\substack{\text{spontaneous emission} \\ \varepsilon_m \to \varepsilon_n}}{A_{mn} p_m}$$

Divide both sides by $B_{mn} p_m$.

$$\underset{\substack{\text{absorption} \\ \varepsilon_n \to \varepsilon_m}}{\rho(\nu, T) \exp\left(\frac{\varepsilon_m - \varepsilon_n}{kT}\right)} = \underset{\substack{\text{induced emission} \\ \varepsilon_m \to \varepsilon_n}}{\rho(\nu, T)} + \underset{\substack{\text{spontaneous emission} \\ \varepsilon_m \to \varepsilon_n}}{\frac{A_{mn}}{B_{mn}}}$$

Rearrange terms.

$$\underset{\substack{\text{absorption} \\ \varepsilon_n \to \varepsilon_m}}{\rho(\nu, T) \exp\left(\frac{\varepsilon_m - \varepsilon_n}{kT}\right)} - \underset{\substack{\text{induced emission} \\ \varepsilon_m \to \varepsilon_n}}{\rho(\nu, T)} = \underset{\substack{\text{spontaneous emission} \\ \varepsilon_m \to \varepsilon_n}}{\frac{A_{mn}}{B_{mn}}}$$

Factor out $\rho(\nu, T)$.

$$\rho(\nu, T) \left(\exp\left(\frac{\varepsilon_m - \varepsilon_n}{kT}\right) - 1 \right) = \frac{A_{mn}}{B_{mn}}$$

Solve for $\rho(\nu, T)$.

$$\rho(\nu, T) = \frac{A_{mn}}{B_{mn}} \frac{1}{\exp\left(\frac{\varepsilon_m - \varepsilon_n}{kT}\right) - 1}$$

We now consider the limit of $\rho(\nu, T)$ as $\varepsilon_m - \varepsilon_n \to \infty$.

$$\lim_{\varepsilon_m - \varepsilon_n \to \infty} \rho(\nu, T) = \frac{A_{mn}}{B_{mn}} \exp\left(-\frac{\varepsilon_m - \varepsilon_n}{kT}\right)$$

Then by equivalence with Wien's law (which is accurate for large ν)

$$\rho_{\text{wien}}(\nu, T) = \frac{2h\nu^3}{c^2} \exp\left(-\frac{h\nu}{kT}\right)$$

we have

$$\frac{A_{mn}}{B_{mn}} = \frac{2h\nu^3}{c^2} \tag{4}$$

and

$$\varepsilon_m - \varepsilon_n = h\nu$$

Then by substitution we obtain Planck's law.

$$\begin{aligned}\rho(\nu, T) &= \frac{A_{mn}}{B_{mn}} \frac{1}{\exp\left(\frac{\varepsilon_m - \varepsilon_n}{kT}\right) - 1} \\ &= \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1}\end{aligned}$$

Let us now consider the values of the A and B coefficients. The coefficient for spontaneous emission can be computed from quantum mechanics. For example, for hydrogen we have

$$A_{21} = \frac{e^{10} m_e}{26244\,\pi^5 \varepsilon_0^5 \hbar^6 c^3} = 6.27 \times 10^8\,\text{second}^{-1}$$

The coefficient for induced emission can be obtained from equation (4).

$$B_{mn} = \frac{c^2}{2h\nu^3} A_{mn}$$

The coefficient for absorption can be computed from equation (3).

$$B_{nm} = \frac{p_m}{p_n} B_{mn}$$

The ratio p_m/p_n is equal to g_m/g_n where g_m is the multiplicity associated with energy level m.

$$g = (2s + 1)(2\ell + 1)$$