

Feynman and Hibbs problem 3-12

If the wave function for a harmonic oscillator (at $t = 0$) is

$$\psi(x, 0) = \exp\left(-\frac{m\omega}{2\hbar}(x - a)^2\right)$$

then, using equation (3.42) and the results of problem 3-8, show that

$$\psi(x, T) = \exp\left(-\frac{i\omega T}{2} - \frac{m\omega}{2\hbar}(x^2 - 2ax \exp(-i\omega T) + a^2 \cos(\omega T) \exp(-i\omega T))\right)$$

and find the probability density $|\psi|^2$.

Adapted from equation (3.42)

$$\psi(x, T) = \int_{-\infty}^{\infty} K(x, T; x_c, 0) \psi(x_c, 0) dx_c$$

Adapted from problem 3-8

$$K = \left(\frac{m\omega}{2\pi i \hbar \sin(\omega T)}\right)^{1/2} \exp\left(\frac{im\omega}{2\hbar \sin(\omega T)}((x^2 + x_c^2) \cos(\omega T) - 2xx_c)\right)$$

Hence

$$\begin{aligned} \psi(x, T) = \left(\frac{m\omega}{2\pi i \hbar \sin(\omega T)}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left(\frac{im\omega}{2\hbar \sin(\omega T)}((x^2 + x_c^2) \cos(\omega T) - 2xx_c)\right) \\ \times \exp\left(-\frac{m\omega}{2\hbar}(x_c - a)^2\right) dx_c \end{aligned}$$

Rewrite as

$$\psi(x, T) = \left(\frac{m\omega}{2\pi i \hbar \sin(\omega T)}\right)^{1/2} \int_{-\infty}^{\infty} \exp(Ax_c^2 + Bx_c + C) dx_c$$

where

$$A = \frac{m\omega}{2\hbar} \left(\frac{i \cos(\omega T)}{\sin(\omega T)} - 1\right) \quad (1)$$

$$B = \frac{m\omega}{\hbar} \left(a - \frac{ix}{\sin(\omega T)}\right) \quad (2)$$

$$C = \frac{m\omega}{2\hbar} \left(\frac{ix^2 \cos(\omega T)}{\sin(\omega T)} - a^2\right) \quad (3)$$

Solve the integral.

$$\int_{-\infty}^{\infty} \exp(Ax_c^2 + Bx_c + C) = \left(-\frac{\pi}{A}\right)^{1/2} \exp\left(-\frac{B^2}{4A} + C\right)$$

$$-\frac{\pi}{A} = -\frac{2\pi\hbar \sin(\omega T)}{im\omega \cos(\omega T) - m\omega \sin(\omega T)} \quad (4)$$

$$-\frac{B^2}{4A} + C = -\frac{m\omega}{2\hbar} (x^2 - 2ax \exp(-i\omega T) + a^2 \cos(\omega T) \exp(-i\omega T)) \quad (5)$$

It can be shown that

$$\frac{m\omega}{2\pi i\hbar \sin(\omega T)} \times \left(-\frac{\pi}{A}\right) = \exp(-i\omega T) \quad (6)$$

from K

Hence

$$\psi(x, T) = \exp\left(-\frac{i\omega T}{2}\right)$$

$$\times \exp\left(-\frac{m\omega}{2\hbar} (x^2 - 2ax \exp(-i\omega T) + a^2 \cos(\omega T) \exp(-i\omega T))\right)$$