

Angular momentum 2

The cross product in $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ demands rectangular coordinates. Hence for a wavefunction ψ in spherical coordinates, vectors \mathbf{r} and \mathbf{p} must be transformed in $\mathbf{L}\psi$.

Vector \mathbf{r} transforms as

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \sin \theta \cos \phi \\ r \sin \theta \sin \phi \\ r \cos \theta \end{pmatrix}$$

To transform \mathbf{p} we have by the chain rule

$$\begin{aligned} \frac{\partial}{\partial x} &= \sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{\cos \theta \cos \phi}{r} \frac{\partial}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial y} &= \sin \theta \sin \phi \frac{\partial}{\partial r} + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} + \frac{\cos \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z} &= \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \end{aligned}$$

Hence

$$\begin{aligned} p_x &= -i\hbar \frac{\partial}{\partial x} = -i\hbar \left(\sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{\cos \theta \cos \phi}{r} \frac{\partial}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \\ p_y &= -i\hbar \frac{\partial}{\partial y} = -i\hbar \left(\sin \theta \sin \phi \frac{\partial}{\partial r} + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} + \frac{\cos \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \\ p_z &= -i\hbar \frac{\partial}{\partial z} = -i\hbar \left(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) \end{aligned}$$

Using the transformed coordinates

$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta \end{aligned}$$

we have in spherical coordinates

$$\begin{aligned} L_x &= yp_z - zp_y = i\hbar \left(\sin \phi \frac{\partial}{\partial \theta} + \frac{\cos \theta \cos \phi}{\sin \theta} \frac{\partial}{\partial \phi} \right) \\ L_y &= zp_x - xp_z = i\hbar \left(-\cos \phi \frac{\partial}{\partial \theta} + \frac{\cos \theta \sin \phi}{\sin \theta} \frac{\partial}{\partial \phi} \right) \\ L_z &= xp_y - yp_x = -i\hbar \frac{\partial}{\partial \phi} \end{aligned}$$

and

$$L^2 = L_x^2 + L_y^2 + L_z^2 = -\hbar^2 \left(\frac{\partial^2}{\partial \theta^2} + \frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right)$$

Eigenmath script

Note

The following construction is from section 4.3.2 of “Introduction to Quantum Mechanics” by Griffiths and Schroeter.

Noting that in spherical coordinates

$$\mathbf{r} = r\mathbf{e}_r$$

and

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}$$

we have for angular momentum \mathbf{L}

$$\mathbf{L} = -i\hbar(\mathbf{r} \times \nabla) = -i\hbar \left[r(\mathbf{e}_r \times \mathbf{e}_r) \frac{\partial}{\partial r} + (\mathbf{e}_r \times \mathbf{e}_\theta) \frac{\partial}{\partial \theta} + (\mathbf{e}_r \times \mathbf{e}_\phi) \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right]$$

In rectangular coordinates

$$\mathbf{e}_r = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix}, \quad \mathbf{e}_\theta = \begin{pmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ -\sin \theta \end{pmatrix}, \quad \mathbf{e}_\phi = \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix}$$

Hence for the cross products

$$\mathbf{e}_r \times \mathbf{e}_r = 0$$

$$\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_\phi$$

$$\mathbf{e}_r \times \mathbf{e}_\phi = -\mathbf{e}_\theta$$

Angular momentum \mathbf{L} now reduces to

$$\mathbf{L} = -i\hbar \left(\mathbf{e}_\phi \frac{\partial}{\partial \theta} - \mathbf{e}_\theta \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right)$$

Substitute for the basis vectors.

$$\mathbf{L} = -i\hbar \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix} \frac{\partial}{\partial \theta} + i\hbar \begin{pmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ -\sin \theta \end{pmatrix} \frac{1}{\sin \theta} \frac{\partial}{\partial \phi}$$

In component form

$$L_x = i\hbar \left(\sin \phi \frac{\partial}{\partial \theta} + \frac{\cos \theta \cos \phi}{\sin \theta} \frac{\partial}{\partial \phi} \right)$$

$$L_y = i\hbar \left(-\cos \phi \frac{\partial}{\partial \theta} + \frac{\cos \theta \sin \phi}{\sin \theta} \frac{\partial}{\partial \phi} \right)$$

$$L_z = -i\hbar \frac{\partial}{\partial \phi}$$