Two spins

Spin state $|s\rangle$ for two spins is a unit vector in $\mathbb{C}^4$.

$$|s\rangle = \begin{pmatrix} c_{++} \\ c_{+-} \\ c_{-+} \\ c_{--} \end{pmatrix}, \quad |c_{++}|^2 + |c_{+-}|^2 + |c_{-+}|^2 + |c_{--}|^2 = 1$$

Spin measurement probabilities are the transition probabilities from $|s\rangle$ to an eigenstate.

For spin measurements in the z direction we have

$$\begin{aligned}
\Pr\left(S_{1z} = +\tfrac{\hbar}{2} \text{ and } S_{2z} = +\tfrac{\hbar}{2}\right) &= |\langle z_{++}|s\rangle|^2 = |c_{++}|^2 \\
\Pr\left(S_{1z} = +\tfrac{\hbar}{2} \text{ and } S_{2z} = -\tfrac{\hbar}{2}\right) &= |\langle z_{+-}|s\rangle|^2 = |c_{+-}|^2 \\
\Pr\left(S_{1z} = -\tfrac{\hbar}{2} \text{ and } S_{2z} = +\tfrac{\hbar}{2}\right) &= |\langle z_{-+}|s\rangle|^2 = |c_{-+}|^2 \\
\Pr\left(S_{1z} = -\tfrac{\hbar}{2} \text{ and } S_{2z} = -\tfrac{\hbar}{2}\right) &= |\langle z_{--}|s\rangle|^2 = |c_{--}|^2
\end{aligned}$$

where the eigenstates are

$$z_{++} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad z_{+-} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad z_{-+} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad z_{--} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Spin operators for the first spin ($\otimes$ is kronecker product).

$$S_{1x} = \frac{\hbar}{2}\sigma_x \otimes I = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$S_{1y} = \frac{\hbar}{2}\sigma_y \otimes I = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$S_{1z} = \frac{\hbar}{2}\sigma_z \otimes I = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Spin operators for the second spin.

$$S_{2x} = \frac{\hbar}{2} I \otimes \sigma_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$S_{2y} = \frac{\hbar}{2} I \otimes \sigma_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$S_{2z} = \frac{\hbar}{2} I \otimes \sigma_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Expectation values for the first spin.

$$\begin{aligned}
\langle S_{1x} \rangle &= \langle s | S_{1x} | s \rangle = \tfrac{\hbar}{2} \left(c_{++} c_{-+}^* + c_{++}^* c_{-+} + c_{+-} c_{--}^* + c_{+-}^* c_{--} \right) \\
\langle S_{1y} \rangle &= \langle s | S_{1y} | s \rangle = \tfrac{i\hbar}{2} \left(c_{++} c_{-+}^* - c_{++}^* c_{-+} + c_{+-} c_{--}^* - c_{+-}^* c_{--} \right) \\
\langle S_{1z} \rangle &= \langle s | S_{1z} | s \rangle = \tfrac{\hbar}{2} \left(|c_{++}|^2 + |c_{+-}|^2 - |c_{-+}|^2 - |c_{--}|^2 \right)
\end{aligned}$$

Expectation values for the second spin.

$$\begin{aligned}
\langle S_{2x} \rangle &= \langle s | S_{2x} | s \rangle = \tfrac{\hbar}{2} \left(c_{++} c_{+-}^* + c_{++}^* c_{+-} + c_{-+} c_{--}^* + c_{-+}^* c_{--} \right) \\
\langle S_{2y} \rangle &= \langle s | S_{2y} | s \rangle = \tfrac{i\hbar}{2} \left(c_{++} c_{+-}^* - c_{++}^* c_{+-} + c_{-+} c_{--}^* - c_{-+}^* c_{--} \right) \\
\langle S_{2z} \rangle &= \langle s | S_{2z} | s \rangle = \tfrac{\hbar}{2} \left(|c_{++}|^2 - |c_{+-}|^2 + |c_{-+}|^2 - |c_{--}|^2 \right)
\end{aligned}$$

Expected euclidean spin vectors.

$$\langle \mathbf{S}_1 \rangle = \langle s | \mathbf{S}_1 | s \rangle = \begin{pmatrix} \langle S_{1x} \rangle \\ \langle S_{1y} \rangle \\ \langle S_{1z} \rangle \end{pmatrix}, \quad \langle \mathbf{S}_2 \rangle = \langle s | \mathbf{S}_2 | s \rangle = \begin{pmatrix} \langle S_{2x} \rangle \\ \langle S_{2y} \rangle \\ \langle S_{2z} \rangle \end{pmatrix}$$

To convert $\langle \mathbf{S}_1 \rangle$ and $\langle \mathbf{S}_2 \rangle$ to a spin state $|s\rangle$, let θ_1 and θ_2 be polar angles and let ϕ_1 and ϕ_2 be azimuth angles such that

$$\langle \mathbf{S}_1 \rangle = \frac{\hbar}{2} \begin{pmatrix} \sin\theta_1 \cos\phi_1 \\ \sin\theta_1 \sin\phi_1 \\ \cos\theta_1 \end{pmatrix}, \quad \langle \mathbf{S}_2 \rangle = \frac{\hbar}{2} \begin{pmatrix} \sin\theta_2 \cos\phi_2 \\ \sin\theta_2 \sin\phi_2 \\ \cos\theta_2 \end{pmatrix}$$

Then

$$|s_1\rangle = \begin{pmatrix} \cos(\theta_1/2) \\ \sin(\theta_1/2) \exp(i\phi_1) \end{pmatrix}, \quad |s_2\rangle = \begin{pmatrix} \cos(\theta_2/2) \\ \sin(\theta_2/2) \exp(i\phi_2) \end{pmatrix}$$

and

$$|s\rangle = |s_1\rangle \otimes |s_2\rangle = \begin{pmatrix} \cos(\theta_1/2) \cos(\theta_2/2) \\ \cos(\theta_1/2) \sin(\theta_2/2) \exp(i\phi_2) \\ \sin(\theta_1/2) \cos(\theta_2/2) \exp(i\phi_1) \\ \sin(\theta_1/2) \sin(\theta_2/2) \exp(i\phi_1) \exp(i\phi_2) \end{pmatrix}$$

Spin angular momentum magnitude squared operator.

$$(\mathbf{S}_1 + \mathbf{S}_2)^2 = (S_{1x} + S_{2x})^2 + (S_{1y} + S_{2y})^2 + (S_{1z} + S_{2z})^2 = \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Expectation value for spin angular momentum magnitude squared.

$$\langle s|(\mathbf{S}_1 + \mathbf{S}_2)^2|s\rangle = \hbar^2 \left(2|c_{++}|^2 + |c_{+-} + c_{-+}|^2 + 2|c_{--}|^2\right)$$

Exercises

1. Verify spin operators for two spins.

2. Verify expectation values for two spins.

3. Let

$$|s_1\rangle = \begin{pmatrix} \cos(\theta_1/2) \\ \sin(\theta_1/2)\exp(i\phi_1) \end{pmatrix}, \quad |s_2\rangle = \begin{pmatrix} \cos(\theta_2/2) \\ \sin(\theta_2/2)\exp(i\phi_2) \end{pmatrix}$$

and

$$|s\rangle = |s_1\rangle \otimes |s_2\rangle$$

Verify that

$$\langle s|\mathbf{S}_1|s\rangle = \frac{\hbar}{2} \begin{pmatrix} \sin\theta_1 \cos\phi_1 \\ \sin\theta_1 \sin\phi_1 \\ \cos\theta_1 \end{pmatrix}, \quad \langle s|\mathbf{S}_2|s\rangle = \frac{\hbar}{2} \begin{pmatrix} \sin\theta_2 \cos\phi_2 \\ \sin\theta_2 \sin\phi_2 \\ \cos\theta_2 \end{pmatrix}$$