

Small rotation

Let J be rotation operators.

$$J_1 = \frac{1}{\hbar}L_1, \quad J_2 = \frac{1}{\hbar}L_2, \quad J_3 = \frac{1}{\hbar}L_3$$

Let U be the unitary transformation

$$U = 1 - i\epsilon J_3 - \frac{1}{2}\epsilon^2 J_3^2$$

1. Show that to order ϵ^2

$$\begin{aligned} U^{-1}X_1U &= \left(1 - \frac{1}{2}\epsilon^2\right) X_1 - \epsilon X_2 \\ U^{-1}X_2U &= \left(1 - \frac{1}{2}\epsilon^2\right) X_2 + \epsilon X_1 \\ U^{-1}X_3U &= X_3 \end{aligned}$$

2. Show that to order ϵ^2

$$\begin{aligned} U^{-1}P_1U &= \left(1 - \frac{1}{2}\epsilon^2\right) P_1 - \epsilon P_2 \\ U^{-1}P_2U &= \left(1 - \frac{1}{2}\epsilon^2\right) P_2 + \epsilon P_1 \\ U^{-1}P_3U &= P_3 \end{aligned}$$

3. Show that to order ϵ^2

$$\begin{aligned} U^{-1}L_1U &= \left(1 - \frac{1}{2}\epsilon^2\right) L_1 - \epsilon L_2 \\ U^{-1}L_2U &= \left(1 - \frac{1}{2}\epsilon^2\right) L_2 + \epsilon L_1 \\ U^{-1}L_3U &= L_3 \end{aligned}$$

4. Show that to order ϵ^2

$$U^{-1}HU = H$$

where

$$H = \frac{1}{2m} (P_1^2 + P_2^2 + P_3^2)$$