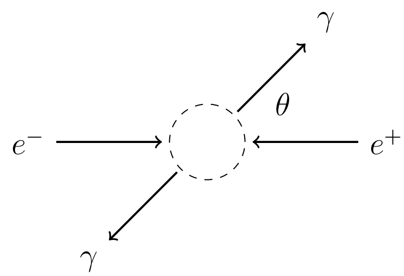
# Annihilation

Annihilation is the interaction $e^- + e^+ \to \gamma + \gamma$.




In the center-of-mass frame we have the following momentum vectors where $E = \sqrt{p^2 + m^2}$.

$$
p_1 = \underbrace{\begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}}_{\substack{\text{inbound}\\\text{electron}}} \qquad
p_2 = \underbrace{\begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix}}_{\substack{\text{inbound}\\\text{positron}}} \qquad
p_3 = \underbrace{\begin{pmatrix} E \\ E\sin\theta\cos\phi \\ E\sin\theta\sin\phi \\ E\cos\theta \end{pmatrix}}_{\substack{\text{outbound}\\\text{photon}}} \qquad
p_4 = \underbrace{\begin{pmatrix} E \\ -E\sin\theta\cos\phi \\ -E\sin\theta\sin\phi \\ -E\cos\theta \end{pmatrix}}_{\substack{\text{outbound}\\\text{photon}}}
$$

Spinors for the inbound electron.

$$
u_{11} = \frac{1}{\sqrt{E+m}} \underbrace{\begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix}}_{\substack{\text{inbound electron}\\\text{spin up}}} \qquad
u_{12} = \frac{1}{\sqrt{E+m}} \underbrace{\begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix}}_{\substack{\text{inbound electron}\\\text{spin down}}}
$$

Spinors for the inbound positron.

$$
v_{21} = \frac{1}{\sqrt{E+m}} \underbrace{\begin{pmatrix} -p \\ 0 \\ E+m \\ 0 \end{pmatrix}}_{\substack{\text{inbound positron}\\\text{spin up}}} \qquad
v_{22} = \frac{1}{\sqrt{E+m}} \underbrace{\begin{pmatrix} 0 \\ p \\ 0 \\ E+m \end{pmatrix}}_{\substack{\text{inbound positron}\\\text{spin down}}}
$$

Let $a$ be the spin state of the inbound electron and let $b$ be the spin state of the inbound positron such that subscript $ba \in \{11, 12, 21, 22\}$. The probability amplitude $\mathcal{M}_{ba}$ for spin state $ba$ is

$$
\mathcal{M}_{ba} = \mathcal{M}_{1ba} + \mathcal{M}_{2ba}
$$

where

$$
\mathcal{M}_{1ba} = \frac{\bar{v}_{2b}(-ie\gamma^\mu)(\not{q}_1 + m)(-ie\gamma^\nu)u_{1a}}{t - m^2}, \quad
\mathcal{M}_{2ba} = \frac{\bar{v}_{2b}(-ie\gamma^\nu)(\not{q}_2 + m)(-ie\gamma^\mu)u_{1a}}{u - m^2}
$$

Symbol $e$ is elementary charge and

$$\not{q}_1 = (p_1 - p_3)^\alpha g_{\alpha\beta}\gamma^\beta$$
$$\not{q}_2 = (p_1 - p_4)^\alpha g_{\alpha\beta}\gamma^\beta$$

$$t = (p_1 - p_3)^2$$
$$u = (p_1 - p_4)^2$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is the average probability density for all four spin states.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}|\mathcal{M}_{ba}|^2$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\left(\frac{\mathcal{M}_{1ba}\mathcal{M}_{1ba}^*}{(t-m^2)^2} + \frac{\mathcal{M}_{1ba}\mathcal{M}_{2ba}^* + \mathcal{M}_{2ba}\mathcal{M}_{1ba}^*}{(t-m^2)(u-m^2)} + \frac{\mathcal{M}_{2ba}\mathcal{M}_{2ba}^*}{(u-m^2)^2}\right)$$

To understand how $\mathcal{M}_{1ba}\mathcal{M}_{1ba}^*$ is calculated, write $\mathcal{M}_{1ba}$ in component form.

$$(\mathcal{M}_{1ba})^{\mu\nu} = \frac{(\bar{v}_{2b})_\alpha(-ie\gamma^{\mu\alpha}{}_\beta)(\not{q}_1 + m)^\beta{}_\rho(-ie\gamma^{\nu\rho}{}_\sigma)(u_{1a})^\sigma}{t - m^2}$$

Metric tensor $g_{\mu\nu}$ is required to sum over indices $\mu$ and $\nu$.

$$\mathcal{M}_{1ba}\mathcal{M}_{1ba}^* = (\mathcal{M}_{1ba})^{\mu\nu}(\mathcal{M}_{1ba}^*)_{\mu\nu} = (\mathcal{M}_{1ba})^{\mu\nu}g_{\mu\alpha}(\mathcal{M}_{1ba}^*)^{\alpha\beta}g_{\beta\nu}$$

Similarly for $\mathcal{M}_{2ba}\mathcal{M}_{2ba}^*$. For $\mathcal{M}_{2ba}$ the index order is $\nu$ followed by $\mu$ hence

$$\mathcal{M}_{1ba}\mathcal{M}_{2ba}^* = (\mathcal{M}_{1ba})^{\mu\nu}(\mathcal{M}_{2ba}^*)_{\nu\mu} = (\mathcal{M}_{1ba})^{\mu\nu}g_{\nu\beta}(\mathcal{M}_{2ba}^*)^{\beta\alpha}g_{\alpha\mu}$$

The Casimir trick uses matrix arithmetic to sum over spin states.

$$f_{11} = \sum_{a=1}^{2}\sum_{b=1}^{2}\mathcal{M}_{1ba}\mathcal{M}_{1ba}^* = e^4\,\mathrm{Tr}\left((\not{p}_1 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\right)$$
$$f_{12} = \sum_{a=1}^{2}\sum_{b=1}^{2}\mathcal{M}_{1ba}\mathcal{M}_{2ba}^* = e^4\,\mathrm{Tr}\left((\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\right)$$
$$f_{22} = \sum_{a=1}^{2}\sum_{b=1}^{2}\mathcal{M}_{2ba}\mathcal{M}_{2ba}^* = e^4\,\mathrm{Tr}\left((\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_2 + m)\gamma_\mu\right)$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\left(\frac{f_{11}}{(t-m^2)^2} + \frac{2f_{12}}{(t-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2}\right)$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$f_{11} = e^4\left(32(p_1 \cdot p_3)(p_1 \cdot p_4) - 32m^2(p_1 \cdot p_2) + 64m^2(p_1 \cdot p_3) + 32m^2(p_1 \cdot p_4) - 64m^4\right)$$
$$f_{12} = e^4\left(16m^2(p_1 \cdot p_3) + 16m^2(p_1 \cdot p_4) - 32m^4\right)$$
$$f_{22} = e^4\left(32(p_1 \cdot p_3)(p_1 \cdot p_4) - 32m^2(p_1 \cdot p_2) + 32m^2(p_1 \cdot p_3) + 64m^2(p_1 \cdot p_4) - 64m^4\right)$$

In Mandelstam variables

$$
\begin{aligned}
f_{11} &= e^4\left(8tu - 24tm^2 - 8um^2 - 8m^4\right)\\
f_{12} &= e^4\left(8sm^2 - 32m^4\right)\\
f_{22} &= e^4\left(8tu - 8tm^2 - 24um^2 - 8m^4\right)
\end{aligned}
$$

For high energy experiments such that $E \gg m$ let $m = 0$ and obtain

$$
\begin{aligned}
f_{11} &= e^4\, 8tu\\
f_{12} &= 0\\
f_{22} &= e^4\, 8tu
\end{aligned}
$$

Hence

$$
\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{e^4}{4}\left(\frac{8tu}{t^2} + \frac{8tu}{u^2}\right)\\
&= 2e^4\left(\frac{u}{t} + \frac{t}{u}\right)
\end{aligned}
$$

For $m = 0$ the Mandelstam variables are

$$
\begin{aligned}
t &= -2E^2(1 - \cos\theta)\\
u &= -2E^2(1 + \cos\theta)
\end{aligned}
$$

Hence

$$
\langle|\mathcal{M}|^2\rangle = 2e^4\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)
$$

**Cross section**

The differential cross section is

$$
\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{4(4\pi\varepsilon_0)^2 s}
$$

where

$$
s = (p_1 + p_2)^2 = 4E^2
$$

For high energy experiments we have

$$
\langle|\mathcal{M}|^2\rangle = 2e^4\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)
$$

Hence for high energy experiments

$$
\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\varepsilon_0)^2 s}\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)
$$

Noting that

$$
e^2 = 4\pi\varepsilon_0\alpha\hbar c
$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s}\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)$$

Noting that

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

we also have

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{2s}\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)\sin\theta\, d\theta\, d\phi$$

Let $S(\theta_1, \theta_2)$ be the following integral of $d\sigma$.

$$S(\theta_1, \theta_2) = \int_0^{2\pi}\int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{\pi\alpha^2(\hbar c)^2}{s}\left[I(\theta_2) - I(\theta_1)\right]$$

where

$$I(\theta) = 2\cos\theta + 2\log(1-\cos\theta) - 2\log(1+\cos\theta)$$

The cumulative distribution function is

$$F(\theta) = \frac{S(a,\theta)}{S(a,\pi-a)} = \frac{I(\theta)-I(a)}{I(\pi-a)-I(a)}, \quad a \le \theta \le \pi - a$$

Angular support is reduced by an arbitrary angle $a > 0$ because $I(0)$ and $I(\pi)$ are undefined.

The probability of observing scattering events in the interval $\theta_1$ to $\theta_2$ is

$$P(\theta_1 < \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi-a)-I(a)}\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)\sin\theta$$

### Data from DESY PETRA experiment

See www.hepdata.net/record/ins191231, Table 2, 14.0 GeV.

| $x$ | $y$ |
|---|---|
| 0.0502 | 0.09983 |
| 0.1505 | 0.10791 |
| 0.2509 | 0.12026 |
| 0.3512 | 0.13002 |
| 0.4516 | 0.17681 |
| 0.5521 | 0.19570 |
| 0.6526 | 0.27900 |
| 0.7312 | 0.33204 |

Data $x$ and $y$ have the following relationship with the differential cross section formula.

$$x = \cos\theta, \quad y = \frac{d\sigma}{d\Omega}$$

The cross section formula is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right) \times (\hbar c)^2$$

To compute predicted values $\hat{y}$, multiply by $10^{37}$ to convert square meters to nanobarns.

$$\hat{y} = \frac{\alpha^2}{2s}\left(\frac{1+x}{1-x} + \frac{1-x}{1+x}\right) \times (\hbar c)^2 \times 10^{37}$$

The following table shows predicted values $\hat{y}$ for $s = (14.0\,\mathrm{GeV})^2$.

| $x$ | $y$ | $\hat{y}$ |
|---|---|---|
| 0.0502 | 0.09983 | 0.106325 |
| 0.1505 | 0.10791 | 0.110694 |
| 0.2509 | 0.12026 | 0.120005 |
| 0.3512 | 0.13002 | 0.135559 |
| 0.4516 | 0.17681 | 0.159996 |
| 0.5521 | 0.19570 | 0.198562 |
| 0.6526 | 0.27900 | 0.262745 |
| 0.7312 | 0.33204 | 0.348884 |

The coefficient of determination $R^2$ measures how well predicted values fit the data.

$$R^2 = 1 - \frac{\sum(y-\hat{y})^2}{\sum(y-\bar{y})^2} = 0.98$$

The result indicates that the model $d\sigma$ explains 98% of the variance in the data.