Let $|\Psi\rangle$ be a coherent state where $\bar{n}$ is the expected number of photons.

$$|\Psi\rangle = \sum_{n=0}^{\infty} \sqrt{\frac{\bar{n}^n \exp(-\bar{n})}{n!}} \exp\left(-i\left(n+\tfrac{1}{2}\right)\omega t\right)|n\rangle$$

Operator $\hat{a}$ is an eigenfunction of $|\Psi\rangle$.

$$\begin{gathered}\hat{a}|\Psi\rangle = \sqrt{\bar{n}}\exp(-i\omega t)|\Psi\rangle \\ \langle\Psi|\hat{a}^\dagger = (\hat{a}|\Psi\rangle)^\dagger = \sqrt{\bar{n}}\exp(i\omega t)\langle\Psi|\end{gathered}$$

Let $\hat{E}$ be the electric field operator

$$\hat{E} = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}(\hat{a} - \hat{a}^\dagger)$$

Note that $\hat{E}$ is Hermitian.

$$\hat{E} = \hat{E}^\dagger$$

The expected electric field is

$$\langle\hat{E}\rangle = \langle\Psi|\hat{E}|\Psi\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\langle\Psi|(\hat{a} - \hat{a}^\dagger)|\Psi\rangle$$

By distributive law

$$\langle\hat{E}\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\left(\langle\Psi|\hat{a}|\Psi\rangle - \langle\Psi|\hat{a}^\dagger|\Psi\rangle\right)$$

Substitute eigenvalues for operators.

$$\langle\hat{E}\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\left(\sqrt{\bar{n}}\exp(-i\omega t)\langle\Psi|\Psi\rangle - \sqrt{\bar{n}}\exp(i\omega t)\langle\Psi|\Psi\rangle\right)$$

By $\langle\Psi|\Psi\rangle = 1$ we have

$$\langle\hat{E}\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\left(\sqrt{\bar{n}}\exp(-i\omega t) - \sqrt{\bar{n}}\exp(i\omega t)\right)$$

Recalling that

$$2\sin(\omega t) = i\exp(-i\omega t) - i\exp(i\omega t)$$

we have

$$\langle \hat{E} \rangle = \sqrt{\frac{2\bar{n}\hbar\omega}{\epsilon_0}} \sin(\omega t)$$

Let $\hat{B}$ be the magnetic field operator

$$\hat{B} = \sqrt{\frac{\hbar\omega\mu_0}{2}} (\hat{a} + \hat{a}^\dagger)$$

By deduction similar to that for $\langle \hat{E} \rangle$ we obtain

$$\langle \hat{B} \rangle = \sqrt{2\bar{n}\hbar\omega\mu_0} \cos(\omega t)$$

The energy of an electromagnetic wave is

$$U = \frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{1}{2\mu_0} |\mathbf{B}|^2$$

For linear polarization and a suitable rotation matrix R we have

$$R\mathbf{E} = \begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix}, \quad R\mathbf{B} = \begin{pmatrix} 0 \\ B \\ 0 \end{pmatrix}$$

Hence in the rotated frame

$$U = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2$$

For a quantum field we have

$$U = \frac{\epsilon_0}{2} \langle \hat{E}^2 \rangle + \frac{1}{2\mu_0} \langle \hat{B}^2 \rangle$$

where

$$\langle \hat{E}^2 \rangle = \langle \Psi | \hat{E}\hat{E} | \Psi \rangle = -\frac{\hbar\omega}{2\epsilon_0} \langle \Psi | (\hat{a} - \hat{a}^\dagger)(\hat{a} - \hat{a}^\dagger) | \Psi \rangle$$

$$\langle \hat{B}^2 \rangle = \langle \Psi | \hat{B}\hat{B} | \Psi \rangle = \frac{\hbar\omega\mu_0}{2} \langle \Psi | (\hat{a} + \hat{a}^\dagger)(\hat{a} + \hat{a}^\dagger) | \Psi \rangle$$

For the coherent state

$$\begin{aligned}
\langle\Psi|\hat{a}\hat{a}|\Psi\rangle &= \left(\sqrt{\bar{n}}\exp(-i\omega t)\right)^2 && = \bar{n}\exp(-2i\omega t) \\
\langle\Psi|\hat{a}\hat{a}^\dagger|\Psi\rangle &= \langle\Psi|(\hat{a}^\dagger\hat{a}+1)|\Psi\rangle = \langle\Psi|a^\dagger\hat{a}|\Psi\rangle + \langle\Psi|\Psi\rangle && = \bar{n}+1 \\
\langle\Psi|\hat{a}^\dagger\hat{a}|\Psi\rangle &= \left(\sqrt{\bar{n}}\exp(i\omega t)\right)\left(\sqrt{\bar{n}}\exp(-i\omega t)\right) && = \bar{n} \\
\langle\Psi|\hat{a}^\dagger\hat{a}^\dagger|\Psi\rangle &= \left(\sqrt{\bar{n}}\exp(i\omega t)\right)^2 && = \bar{n}\exp(2i\omega t)
\end{aligned}$$

The expectation $\bar{n}+1$ for $\hat{a}\hat{a}^\dagger$ is from the commutator

$$\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$$

Using the expectation values derived above we now have

$$\begin{aligned}
\langle\hat{E}^2\rangle &= -\frac{\hbar\omega}{2\epsilon_0}\left(\bar{n}\exp(-2i\omega t) + \bar{n}\exp(2i\omega t) - 2\bar{n} - 1\right) \\
\langle\hat{B}^2\rangle &= \frac{\hbar\omega\mu_0}{2}\left(\bar{n}\exp(-2i\omega t) + \bar{n}\exp(2i\omega t) + 2\bar{n} + 1\right)
\end{aligned}$$

Noting that

$$\begin{aligned}
-4\sin(\omega t)^2 &= \exp(-2i\omega t) + \exp(2i\omega t) - 2 \\
4\cos(\omega t)^2 &= \exp(-2i\omega t) + \exp(2i\omega t) + 2
\end{aligned}$$

we have

$$\begin{aligned}
\langle\hat{E}^2\rangle &= -\frac{\hbar\omega}{2\epsilon_0}\left(-4\bar{n}\sin(\omega t)^2 - 1\right) \\
\langle\hat{B}^2\rangle &= \frac{\hbar\omega\mu_0}{2}\left(4\bar{n}\cos(\omega t)^2 + 1\right)
\end{aligned}$$

Rewrite as

$$\begin{aligned}
\frac{\epsilon_0}{2}\langle\hat{E}^2\rangle &= \hbar\omega\left(\bar{n}\sin(\omega t)^2 + \tfrac{1}{4}\right) \\
\frac{1}{2\mu_0}\langle\hat{B}^2\rangle &= \hbar\omega\left(\bar{n}\cos(\omega t)^2 + \tfrac{1}{4}\right)
\end{aligned}$$

Hence the total energy per unit volume is

$$U = \frac{\epsilon_0}{2}\langle\hat{E}^2\rangle + \frac{1}{2\mu_0}\langle\hat{B}^2\rangle = \hbar\omega\left(\bar{n} + \tfrac{1}{2}\right)$$

Check units.

$$\hbar\omega = h\nu \propto \text{joule second} \times \frac{1}{\text{second}} = \text{joule}$$

We will now show that

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}}\exp(-i\omega t)|\Psi\rangle$$

Apply operator $\hat{a}$ to coherent state $|\Psi\rangle$ to obtain

$$\hat{a}|\Psi\rangle = \sum_{n=0}^{\infty}\sqrt{\frac{\bar{n}^n\exp(-\bar{n})}{n!}}\exp\left(-i\left(n+\tfrac{1}{2}\right)\omega t\right)\sqrt{n}|n-1\rangle$$

The $n = 0$ term vanishes hence the sum can start from $n = 1$.

$$\hat{a}|\Psi\rangle = \sum_{n=1}^{\infty}\sqrt{\frac{\bar{n}^n\exp(-\bar{n})}{n!}}\exp\left(-i\left(n+\tfrac{1}{2}\right)\omega t\right)\sqrt{n}|n-1\rangle$$

The $\sqrt{n}$ cancels with n factorial.

$$\hat{a}|\Psi\rangle = \sum_{n=1}^{\infty}\sqrt{\frac{\bar{n}^n\exp(-\bar{n})}{(n-1)!}}\exp\left(-i\left(n+\tfrac{1}{2}\right)\omega t\right)|n-1\rangle$$

Factor out $\sqrt{\bar{n}}\exp(-i\omega t)$.

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}}\exp(-i\omega t)\sum_{n=1}^{\infty}\sqrt{\frac{\bar{n}^{n-1}\exp(-\bar{n})}{(n-1)!}}\exp\left(-i\left(n-\tfrac{1}{2}\right)\omega t\right)|n-1\rangle$$

Substitute $n + 1$ for index n.

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}}\exp(-i\omega t)\sum_{n=0}^{\infty}\sqrt{\frac{\bar{n}^n\exp(-\bar{n})}{n!}}\exp\left(-i\left(n+\tfrac{1}{2}\right)\omega t\right)|n\rangle$$

Hence

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}}\exp(-i\omega t)|\Psi\rangle$$

Energy states

For state $|n\rangle$ we have

$$\begin{aligned}\langle n|\hat{a}\hat{a}|n\rangle &= 0 \\ \langle n|\hat{a}\hat{a}^{\dagger}|n\rangle &= n+1 \\ \langle n|\hat{a}^{\dagger}\hat{a}|n\rangle &= n \\ \langle n|\hat{a}^{\dagger}\hat{a}^{\dagger}|n\rangle &= 0\end{aligned}$$

Hence

$$\begin{aligned}\langle \hat{E}^2\rangle &= -\frac{\hbar\omega}{2\epsilon_0}(-2n-1) \\ \langle \hat{B}^2\rangle &= \frac{\hbar\omega\mu_0}{2}(2n+1)\end{aligned}$$

Rewrite as

$$\begin{aligned}\frac{\epsilon_0}{2}\langle \hat{E}^2\rangle &= \hbar\omega\left(\tfrac{1}{2}n+\tfrac{1}{4}\right) \\ \frac{1}{2\mu_0}\langle \hat{B}^2\rangle &= \hbar\omega\left(\tfrac{1}{2}n+\tfrac{1}{4}\right)\end{aligned}$$

Hence the total energy per unit volume is

$$U = \frac{\epsilon_0}{2}\langle \hat{E}^2\rangle + \frac{1}{2\mu_0}\langle \hat{B}^2\rangle = \hbar\omega\left(n+\tfrac{1}{2}\right)$$