

## Fermion spin

Fermion spin state  $|s\rangle$  is a normalized vector in  $\mathbb{C}^2$ .

$$|s\rangle = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}, \quad |c_+|^2 + |c_-|^2 = 1$$

Spin measurement probabilities are the transition probabilities from  $|s\rangle$  to an eigenstate.

For spin measurements in the  $z$  direction we have

$$\begin{aligned} \Pr(S_z = +\frac{\hbar}{2}) &= |\langle z_+ | s \rangle|^2 \\ \Pr(S_z = -\frac{\hbar}{2}) &= |\langle z_- | s \rangle|^2 \end{aligned}$$

Define the  $z$  eigenstates as

$$|z_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |z_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

By definition of expectation value we have

$$\langle S_z \rangle = \frac{\hbar}{2} \Pr(S_z = +\frac{\hbar}{2}) - \frac{\hbar}{2} \Pr(S_z = -\frac{\hbar}{2})$$

Rewrite as

$$\langle S_z \rangle = \frac{\hbar}{2} |\langle z_+ | s \rangle|^2 - \frac{\hbar}{2} |\langle z_- | s \rangle|^2$$

Rewrite again as

$$\langle S_z \rangle = \frac{\hbar}{2} \langle s | z_+ \rangle \langle z_+ | s \rangle - \frac{\hbar}{2} \langle s | z_- \rangle \langle z_- | s \rangle$$

Then by

$$\langle S_z \rangle = \langle s | S_z | s \rangle$$

we have

$$S_z = \frac{\hbar}{2} |z_+\rangle \langle z_+| - \frac{\hbar}{2} |z_-\rangle \langle z_-| = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

From the commutator

$$S_+ S_- - S_- S_+ = 2\hbar S_z$$

we have

$$S_+ S_- - S_- S_+ = \hbar^2 |z_+\rangle \langle z_+| - \hbar^2 |z_-\rangle \langle z_-|$$

Rewrite as

$$S_+ S_- - S_- S_+ = \hbar^2 |z_+\rangle \langle z_- | z_-\rangle \langle z_+| - \hbar^2 |z_-\rangle \langle z_+ | z_+\rangle \langle z_-|$$

Hence

$$\begin{aligned} S_+ &= \hbar |z_+\rangle \langle z_-| = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ S_- &= \hbar |z_-\rangle \langle z_+| = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Then by

$$\begin{aligned}S_+ &= S_x + iS_y \\S_- &= S_x - iS_y\end{aligned}$$

we obtain

$$\begin{aligned}S_x &= \frac{S_+ + S_-}{2} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\S_y &= \frac{S_+ - S_-}{2i} = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\end{aligned}$$

By solving for the eigenstates in

$$\begin{aligned}S_x|x_{\pm}\rangle &= \pm\frac{\hbar}{2}|x_{\pm}\rangle \\S_y|y_{\pm}\rangle &= \pm\frac{\hbar}{2}|y_{\pm}\rangle\end{aligned}$$

we obtain

$$\begin{aligned}|x_+\rangle &= \frac{|z_+\rangle + |z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\|x_-\rangle &= \frac{|z_+\rangle - |z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}|y_+\rangle &= \frac{|z_+\rangle + i|z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \\|y_-\rangle &= \frac{|z_+\rangle - i|z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}\end{aligned}$$

1. Verify that

$$\begin{aligned}S_x &= \frac{\hbar}{2}(|x_+\rangle\langle x_+| - |x_-\rangle\langle x_-|) \\S_y &= \frac{\hbar}{2}(|y_+\rangle\langle y_+| - |y_-\rangle\langle y_-|) \\S_z &= \frac{\hbar}{2}(|z_+\rangle\langle z_+| - |z_-\rangle\langle z_-|)\end{aligned}$$

```
xp = sqrt(1/2) (1,1)
xm = sqrt(1/2) (1,-1)
```

```
yp = sqrt(1/2) (1,i)
ym = sqrt(1/2) (1,-i)
```

```
zp = (1,0)
zm = (0,1)
```

```
Sx = hbar / 2 ((0,1),(1,0))
Sy = hbar / 2 ((0,-i),(i,0))
Sz = hbar / 2 ((1,0),(0,-1))
```

```
check(Sx == hbar / 2 (outer(xp,conj(xp)) - outer(xm,conj(xm))))
check(Sy == hbar / 2 (outer(yp,conj(yp)) - outer(ym,conj(ym))))
check(Sz == hbar / 2 (outer(zp,conj(zp)) - outer(zm,conj(zm))))
```

2. Let  $|s\rangle$  be the following spin state.

$$|s\rangle = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$$

Verify that  $|s\rangle$  is normalized and that

$$\langle \mathbf{S} \rangle = \langle s | \mathbf{S} | s \rangle = \frac{\hbar}{2} \begin{pmatrix} \frac{4}{9} \\ \frac{8}{9} \\ \frac{1}{9} \end{pmatrix}$$

Note: In component form we have

$$\langle s | \mathbf{S} | s \rangle = s_{\beta}^* S^{\alpha\beta}_{\gamma} s^{\gamma}$$

Eigenmath requires a transpose so that the  $\beta$  indices are adjacent.

$$\langle s | \mathbf{S} | s \rangle = s_{\beta}^* S^{\beta\alpha}_{\gamma} s^{\gamma}$$

```
s = (1/3 - 2/3 i, 2/3)
```

```
check(dot(conj(s),s) == 1)
```

```
Sx = hbar / 2 ((0,1),(1,0))
Sy = hbar / 2 ((0,-i),(i,0))
Sz = hbar / 2 ((1,0),(0,-1))
```

```
S = (Sx,Sy,Sz)
```

```
check(dot(conj(s),transpose(S),s) == hbar / 2 (4/9, 8/9, 1/9))
```

3. Let  $|s\rangle$  be the following spin state.

$$|s\rangle = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$$

Verify the following measurement probabilities for  $|s\rangle$ .

$$\begin{aligned} \Pr(S_x = +\frac{\hbar}{2}) &= |\langle x_+ | s \rangle|^2 = \frac{13}{18} \\ \Pr(S_x = -\frac{\hbar}{2}) &= |\langle x_- | s \rangle|^2 = \frac{5}{18} \end{aligned}$$

$$\begin{aligned} \Pr(S_y = +\frac{\hbar}{2}) &= |\langle y_+ | s \rangle|^2 = \frac{17}{18} \\ \Pr(S_y = -\frac{\hbar}{2}) &= |\langle y_- | s \rangle|^2 = \frac{1}{18} \end{aligned}$$

$$\begin{aligned} \Pr(S_z = +\frac{\hbar}{2}) &= |\langle z_+ | s \rangle|^2 = \frac{5}{9} \\ \Pr(S_z = -\frac{\hbar}{2}) &= |\langle z_- | s \rangle|^2 = \frac{4}{9} \end{aligned}$$

```
s = (1/3 - 2/3 i, 2/3)
```

```
xp = sqrt(1/2) (1,1)
xm = sqrt(1/2) (1,-1)
```

```
yp = sqrt(1/2) (1,i)
ym = sqrt(1/2) (1,-i)
```

```
zp = (1,0)
zm = (0,1)
```

```
Pr(a,b) = dot(conj(a),b) dot(conj(b),a)
```

```
check(Pr(xp,s) == 13/18)
check(Pr(xm,s) == 5/18)
```

```
check(Pr(yp,s) == 17/18)
check(Pr(ym,s) == 1/18)
```

```
check(Pr(zp,s) == 5/9)
check(Pr(zm,s) == 4/9)
```

4. Let  $|s\rangle$  be the following spin state.

$$|s\rangle = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$$

Verify that the following spin state  $|\chi\rangle$  is indistinguishable from  $|s\rangle$ .

$$|\chi\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \exp(i\phi) \end{pmatrix}$$

where

$$\cos(\theta/2) = \sqrt{\frac{\langle z \rangle + 1}{2}} = \frac{\sqrt{5}}{3}$$

and

$$\sin(\theta/2) \exp(i\phi) = \sqrt{\frac{1 - \langle z \rangle}{2}} \frac{\langle x \rangle + i\langle y \rangle}{\sqrt{\langle x \rangle^2 + \langle y \rangle^2}} = \frac{2 + 4i}{3\sqrt{5}}$$

with

$$\begin{aligned} \langle x \rangle &= \frac{2}{\hbar} \langle S_x \rangle \\ \langle y \rangle &= \frac{2}{\hbar} \langle S_y \rangle \\ \langle z \rangle &= \frac{2}{\hbar} \langle S_z \rangle \end{aligned}$$

```
s = (1/3 - 2/3 i, 2/3)
```

```
Sx = hbar / 2 ((0,1),(1,0))
Sy = hbar / 2 ((0,-i),(i,0))
Sz = hbar / 2 ((1,0),(0,-1))
```

```
S = (Sx,Sy,Sz)
```

```
x = 2 / hbar dot(conj(s),Sx,s)
y = 2 / hbar dot(conj(s),Sy,s)
z = 2 / hbar dot(conj(s),Sz,s)
```

```
cp = sqrt((z + 1) / 2)
cm = sqrt((1 - z) / 2) (x + i y) / sqrt(x^2 + y^2)
```

```
check(cp == sqrt(5) / 3)
check(cm == (2 + 4 i) / (3 sqrt(5)))
```

```
chi = (cp,cm)
```

```
check(dot(conj(s),transpose(S),s) == dot(conj(chi),transpose(S),chi))
```

5. Verify the following spin commutation relations using  $\mathbf{S}\psi = (\mathbf{r} \times \mathbf{p})\psi$ .

$$[S_x, S_y] = i\hbar S_z$$

$$[S_y, S_z] = i\hbar S_x$$

$$[S_z, S_x] = i\hbar S_y$$

$$[S^2, S_x] = 0$$

$$[S^2, S_y] = 0$$

$$[S^2, S_z] = 0$$

$$[S_+, S_-] = 2\hbar S_z$$

where

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

and

$$S_+ = S_x + iS_y$$

$$S_- = S_x - iS_y$$

```

Sx(psi) = -i hbar (y d(psi,z) - z d(psi,y))
Sy(psi) = -i hbar (z d(psi,x) - x d(psi,z))
Sz(psi) = -i hbar (x d(psi,y) - y d(psi,x))

psi = Psi()

check(Sx(Sy(psi)) - Sy(Sx(psi)) == i hbar Sz(psi))
check(Sy(Sz(psi)) - Sz(Sy(psi)) == i hbar Sx(psi))
check(Sz(Sx(psi)) - Sx(Sz(psi)) == i hbar Sy(psi))

S2(psi) = Sx(Sx(psi)) + Sy(Sy(psi)) + Sz(Sz(psi))

check(S2(Sx(psi)) - Sx(S2(psi)) == 0)
check(S2(Sy(psi)) - Sy(S2(psi)) == 0)
check(S2(Sz(psi)) - Sz(S2(psi)) == 0)

Sp(psi) = Sx(psi) + i Sy(psi)
Sm(psi) = Sx(psi) - i Sy(psi)

check(Sp(Sm(psi)) - Sm(Sp(psi)) == 2 hbar Sz(psi))

```