

(1.2) Consider the functionals $H[f] = \int G(x, y)f(y) dy$, $I[f] = \int_{-1}^1 f(x) dx$ and $J[f] = \int \left(\frac{\partial f}{\partial y}\right)^2 dy$ of the function f . Find the functional derivatives $\frac{\delta H[f]}{\delta f(z)}$, $\frac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)}$ and $\frac{\delta J[f]}{\delta f(x)}$.

For the first functional derivative we have

$$\frac{\delta H[f]}{\delta f(z)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\int G(x, y)(f(y) + \epsilon \delta(y - z)) dy - \int G(x, y)f(y) dy \right)$$

The integrals of $G(x, y)f(y)$ cancel and ϵ cancels.

$$\frac{\delta H[f]}{\delta f(z)} = \int G(x, y)\delta(y - z) dy = G(x, z)$$

For the second functional derivative we have

$$\frac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)} = \frac{\delta}{\delta f(x_0)} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\int_{-1}^1 (f(x) + \epsilon \delta(x - x_1))^3 dx - \int_{-1}^1 f(x)^3 dx \right)$$

Terms involving ϵ^2 and ϵ^3 vanish in the limit hence

$$\begin{aligned} \frac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)} &= \\ \frac{\delta}{\delta f(x_0)} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} &\left(\int_{-1}^1 f(x)^3 dx + \epsilon \int_{-1}^1 3f(x)^2 \delta(x - x_1) dx - \int_{-1}^1 f(x)^3 dx \right) \end{aligned}$$

The integrals of $f(x)^3$ cancel and ϵ cancels.

$$\begin{aligned} \frac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)} &= \frac{\delta}{\delta f(x_0)} \int_{-1}^1 3f(x)^2 \delta(x - x_1) dx \\ &= \frac{\delta}{\delta f(x_0)} \begin{cases} 3f(x_1)^2 & -1 \leq x_1 \leq 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (1)$$

Now do the second derivative.

$$\frac{\delta}{\delta f(x_0)} (3f(x_1)^2) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(3(f(x_1) + \epsilon \delta(x_1 - x_0))^2 - 3f(x_1)^2 \right)$$

The term involving ϵ^2 vanishes in the limit hence

$$\begin{aligned} \frac{\delta}{\delta f(x_0)} (3f(x_1)^2) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (3f(x_1)^2 + 6f(x_1)\epsilon\delta(x_1 - x_0) - 3f(x_1)^2) \\ &= \begin{cases} 6f(x_1) & x_0 = x_1 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (2)$$

Substitute (2) into (1) to obtain

$$\frac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)} = \begin{cases} 6f(x_1) & x_0 = x_1 \text{ and } -1 \leq x_1 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

For the third functional derivative we have

$$\frac{\delta J[f]}{\delta f(x)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\int \left(\frac{\partial f}{\partial y} + \epsilon \frac{\partial \delta(y-x)}{\partial y} \right)^2 dy - \int \left(\frac{\partial f}{\partial y} \right)^2 dy \right)$$

The term involving ϵ^2 vanishes in the limit hence

$$\frac{\delta J[f]}{\delta f(x)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\int \left(\frac{\partial f}{\partial y} \right)^2 dy + 2\epsilon \int \frac{\partial f}{\partial y} \frac{\partial \delta(y-x)}{\partial y} dy - \int \left(\frac{\partial f}{\partial y} \right)^2 dy \right)$$

After cancellations

$$\frac{\delta J[f]}{\delta f(x)} = 2 \int \frac{\partial f}{\partial y} \frac{\partial \delta(y-x)}{\partial y} dy$$

Let

$$u = \frac{\partial f}{\partial y} \quad v = \delta(y-x)$$

Then

$$du = \frac{\partial^2 f}{\partial y^2} dy \quad dv = \frac{\partial \delta(y-x)}{\partial y} dy$$

and

$$\frac{\delta J[f]}{\delta f(x)} = 2 \int u dv$$

Integrate by parts.

$$\begin{aligned} \frac{\delta J[f]}{\delta f(x)} &= 2uv - 2 \int v du \\ &= 2 \frac{\partial f}{\partial y} \delta(y-x) - 2 \int \delta(y-x) \frac{\partial^2 f}{\partial y^2} dy \\ &= -2 \frac{\partial^2 f}{\partial x^2} \quad \text{see equation (1.19)} \end{aligned}$$