Prove the following Gordon decomposition by direct calculation. Momentum vectors $p_1$ and $p_2$ have the same rest mass $m$. Each of the spins $s_1$ and $s_2$ can be either up or down.

$$\bar{u}(p_2, s_2)\gamma^\mu u(p_1, s_1) = \bar{u}(p_2, s_2)\left[\frac{(p_2 + p_1)^\mu}{2m} + i\sigma^{\mu\nu}\frac{(p_2 - p_1)_\nu}{2m}\right] u(p_1, s_1)$$

The following vectors and spinors are used. Spinors $u_{11}$ and $u_{21}$ are spin up, $u_{12}$ and $u_{22}$ are spin down.

$$p_1 = \begin{pmatrix} E_1 \\ p_{1x} \\ p_{1y} \\ p_{1z} \end{pmatrix} \quad u_{11} = \begin{pmatrix} E_1 + m \\ 0 \\ p_{1z} \\ p_{1x} + ip_{1y} \end{pmatrix} \quad u_{12} = \begin{pmatrix} 0 \\ E_1 + m \\ p_{1x} - ip_{1y} \\ -p_{1z} \end{pmatrix} \quad E_1 = \sqrt{p_{1x}^2 + p_{1y}^2 + p_{1z}^2 + m^2}$$

$$p_2 = \begin{pmatrix} E_2 \\ p_{2x} \\ p_{2y} \\ p_{2z} \end{pmatrix} \quad u_{21} = \begin{pmatrix} E_2 + m \\ 0 \\ p_{2z} \\ p_{2x} + ip_{2y} \end{pmatrix} \quad u_{22} = \begin{pmatrix} 0 \\ E_2 + m \\ p_{2x} - ip_{2y} \\ -p_{2z} \end{pmatrix} \quad E_2 = \sqrt{p_{2x}^2 + p_{2y}^2 + p_{2z}^2 + m^2}$$

Tensor $\sigma^{\mu\nu}$ is defined as

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right] = \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)$$

In component notation we have

$$\sigma^{\mu\nu\alpha}{}_\beta = \frac{i}{2}\left(\gamma^{\mu\alpha}{}_\rho\gamma^{\nu\rho}{}_\beta - \gamma^{\nu\alpha}{}_\rho\gamma^{\mu\rho}{}_\beta\right)$$

Let $T^{\mu\nu} = \gamma^\mu\gamma^\nu$. Transpose the first two indices of $\gamma^{\nu\rho}{}_\beta$ to form a dot product.

$$T^{\mu\nu\alpha}{}_\beta = \gamma^{\mu\alpha}{}_\rho\gamma^{\rho\nu}{}_\beta$$

Convert to code. The transpose on the second and third indices interchanges $\alpha$ and $\nu$.

$$T^{\mu\nu\alpha}{}_\beta = \texttt{transpose(dot(gamma,transpose(gamma)),2,3)}$$

Hence

$$\sigma^{\mu\nu} = \texttt{i/2 (T - transpose(T))}$$

where $\texttt{T} = T^{\mu\nu\alpha}{}_\beta$. Now convert $\sigma^{\mu\nu}(p_2 - p_1)_\nu$ to code.

$$\sigma^{\mu\nu}(p_2 - p_1)_\nu = \sigma^{\mu\alpha}{}_\beta{}^\nu g_{\nu\rho}(p_2 - p_1)^\rho = \texttt{dot(S,gmunu,p2 - p1)}$$

where $\texttt{S} = \sigma^{\mu\alpha}{}_\beta{}^\nu = \texttt{transpose(transpose(sigmamunu,2,3),3,4)}$.

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E1 = sqrt(p1x^2 + p1y^2 + p1z^2 + m^2)
E2 = sqrt(p2x^2 + p2y^2 + p2z^2 + m^2)
p1 = (E1, p1x, p1y, p1z)
p2 = (E2, p2x, p2y, p2z)
u11 = (E1 + m, 0, p1z, p1x + i p1y)
u12 = (0, E1 + m, p1x - i p1y, -p1z)
u21 = (E2 + m, 0, p2z, p2x + i p2y)
u22 = (0, E2 + m, p2x - i p2y, -p2z)
u1 = (u11,u12)
u2 = (u21,u22)
I = ((1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1))
gmunu = ((1,0,0,0),(0,-1,0,0),(0,0,-1,0),(0,0,0,-1))
gamma0 = ((1,0,0,0),(0,1,0,0),(0,0,-1,0),(0,0,0,-1))
gamma1 = ((0,0,0,1),(0,0,1,0),(0,-1,0,0),(-1,0,0,0))
gamma2 = ((0,0,0,-i),(0,0,i,0),(0,i,0,0),(-i,0,0,0))
gamma3 = ((0,0,1,0),(0,0,0,-1),(-1,0,0,0),(0,1,0,0))
gamma = (gamma0,gamma1,gamma2,gamma3)
u2bar = dot(conj(u2),gamma0) -- adjoint of u2
T = transpose(dot(gamma,transpose(gamma)),2,3)
sigmamunu = i/2 (T - transpose(T))
S = transpose(transpose(sigmamunu,2,3),3,4)
V = (outer(p2 + p1,I) + i dot(S,gmunu,p2 - p1)) / (2 m)
for(s1,1,2,for(s2,1,2,for(mu,1,4, -- for each spin state and gamma
  A = dot(u2bar[s2],gamma[mu],u1[s1]),
  B = dot(u2bar[s2],V[mu],u1[s1]),
  print(A==B) -- print 1 if A equals B
)))
```