

# Rotating wave approximation

Let  $\Psi(\mathbf{r}, t)$  be the following wave function for a two state system.

$$\Psi(\mathbf{r}, t) = \psi_a(\mathbf{r})c_a(t) \exp(-\frac{i}{\hbar}E_a t) + \psi_b(\mathbf{r})c_b(t) \exp(-\frac{i}{\hbar}E_b t)$$

Let  $\hat{H}(\mathbf{r}, t)$  be the Hamiltonian

$$\hat{H}(\mathbf{r}, t) = \hat{H}_0(\mathbf{r}) + \hat{H}_1(\mathbf{r}, t)$$

where

$$\hat{H}_0\psi_a = E_a\psi_a, \quad \hat{H}_0\psi_b = E_b\psi_b, \quad \hat{H}_0\Psi = (E_a + E_b)\Psi$$

It was shown that if  $\Psi$  is a solution to the Schrödinger equation then

$$\frac{d}{dt}c_a(t) = -\frac{i}{\hbar}\langle\psi_a|\hat{H}_1|\psi_b\rangle \exp(-i\omega_0 t)c_b(t), \quad \frac{d}{dt}c_b(t) = -\frac{i}{\hbar}\langle\psi_b|\hat{H}_1|\psi_a\rangle \exp(i\omega_0 t)c_a(t) \quad (1)$$

where

$$\omega_0 = \frac{E_b - E_a}{\hbar}$$

Let  $\hat{H}_1(\mathbf{r}, t)$  be the perturbation

$$\hat{H}_1(\mathbf{r}, t) = \hat{V}(\mathbf{r}) \cos(\omega t)$$

Then

$$\langle\psi_a|\hat{H}_1|\psi_b\rangle = \langle\psi_a|\hat{V}|\psi_b\rangle \left[ \frac{1}{2} \exp(i\omega t) + \frac{1}{2} \exp(-i\omega t) \right]$$

The rotating wave approximation discards the second term and asserts

$$\langle\psi_a|\hat{H}_1|\psi_b\rangle = \frac{1}{2}\langle\psi_a|\hat{V}|\psi_b\rangle \exp(i\omega t) \quad (2)$$

Substitute equation (2) into (1) to obtain

$$\frac{d}{dt}c_a(t) = -\frac{i}{2\hbar}\langle\psi_a|\hat{V}|\psi_b\rangle \exp(i(\omega - \omega_0)t)c_b(t) \quad (3)$$

and

$$\frac{d}{dt}c_b(t) = -\frac{i}{2\hbar}\langle\psi_b|\hat{V}|\psi_a\rangle \exp(i(\omega_0 - \omega)t)c_a(t) \quad (4)$$

Use Laplace transforms to solve for  $c_b(t)$  with initial conditions  $c_a(0) = 1$  and  $c_b(0) = 0$ .

$$c_b(t) = -\frac{i}{2\hbar\omega_r}\langle\psi_b|\hat{V}|\psi_a\rangle \sin(\omega_r t) \exp\left(\frac{i}{2}(\omega_0 - \omega)t\right)$$

Symbol  $\omega_r$  is the Rabi flopping frequency

$$\omega_r = \frac{1}{2}\sqrt{(\omega_0 - \omega)^2 + |\langle\psi_a|\hat{V}|\psi_b\rangle|^2/\hbar^2}$$

Use the latter part of equation (1) and the solution for  $c_b(t)$  to solve for  $c_a(t)$ .

$$c_a(t) = \left[ \cos(\omega_r t) + i \left( \frac{\omega_0 - \omega}{2\omega_r} \right) \sin(\omega_r t) \right] \exp \left( -\frac{i}{2}(\omega_0 - \omega)t \right)$$

Rewrite  $\omega_r$  as

$$\omega_r = \frac{1}{2\hbar} \sqrt{\hbar^2(\omega_0 - \omega)^2 + |\langle \psi_a | \hat{V} | \psi_b \rangle|^2}$$

and note that for

$$\hbar^2(\omega_0 - \omega)^2 \gg |\langle \psi_a | \hat{V} | \psi_b \rangle|^2$$

we have

$$\omega_r \approx \frac{1}{2}|\omega_0 - \omega|$$

Using this approximation, the transition probability  $P_{a \rightarrow b}(t) = |c_b(t)|^2$  becomes identical to the result from first order perturbation theory.