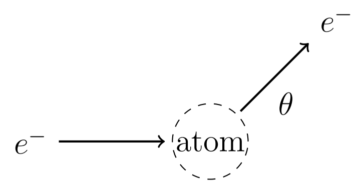
Rutherford scattering 3

Rutherford scattering is the interaction $e^- + \text{atom} \to e^- + \text{atom}$.



Momentum vectors for Rutherford scattering where $E = \sqrt{p^2 + m^2}$.

$$p_1 = \underset{\substack{\text{inbound}\\ \text{electron}}}{\begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}}, \quad p_2 = \underset{\substack{\text{outbound}\\ \text{electron}}}{\begin{pmatrix} E \\ p\sin\theta\cos\phi \\ p\sin\theta\sin\phi \\ p\cos\theta \end{pmatrix}}$$

Spinors for the inbound electron.

$$u_{11} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound electron}\\ \text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix}}, \quad u_{12} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound electron}\\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix}}$$

Spinors for the outbound electron.

$$u_{21} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{outbound electron}\\ \text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ p_{2z} \\ p_{2x} + ip_{2y} \end{pmatrix}}, \quad u_{22} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{outbound electron}\\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ p_{2x} - ip_{2y} \\ -p_{2z} \end{pmatrix}}$$

The probability density $|\mathcal{M}_{ab}|^2$ for spin state ab.

$$|\mathcal{M}_{ab}|^2 = \frac{Z^2 e^4}{q^4} \left|\bar{u}_{2b}\gamma^0 u_{1a}\right|^2$$

q is momentum transfer such that

$$q^4 = (p_1 - p_2)^4 = \left[(p_1 - p_2)^\mu g_{\mu\nu}(p_1 - p_2)^\nu\right]^2 = 4p^4(\cos\theta - 1)^2$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is the average of spin states.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{2}\sum_{a=1}^{2}\sum_{b=1}^{2}|\mathcal{M}_{ab}|^2$$

The Casimir trick uses matrix arithmetic to sum over spin states.

$$\langle|\mathcal{M}|^2\rangle = \frac{Z^2e^4}{2q^4}\,\mathrm{Tr}\left[(\not{p}_1+m)\gamma^0(\not{p}_2+m)\gamma^0\right]$$

The result is

$$\langle|\mathcal{M}|^2\rangle = \frac{2Z^2e^4}{q^4}\left(E^2+m^2+p^2\cos\theta\right)$$

For low energy experiments such that $p \ll m$ we can use the approximation

$$E^2+m^2+p^2\cos\theta \approx 2m^2$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{4Z^2e^4m^2}{q^4}$$

Substituting $e^4 = 16\pi^2\alpha^2$ and $q^4 = 4p^4(\cos\theta-1)^2$ we have

$$\langle|\mathcal{M}|^2\rangle = \frac{16\pi^2Z^2\alpha^2m^2}{p^4(\cos\theta-1)^2}$$

Cross section

The differential cross section for Rutherford scattering is

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{16\pi^2}$$

For low energy experiments we have

$$\langle|\mathcal{M}|^2\rangle = \frac{16\pi^2Z^2\alpha^2m^2}{p^4(\cos\theta-1)^2}$$

Hence for low energy experiments

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2m^2}{p^4(\cos\theta-1)^2}$$

Noting that

$$d\Omega = \sin\theta\,d\theta\,d\phi$$

we also have

$$d\sigma = \frac{Z^2\alpha^2m^2}{p^4(\cos\theta-1)^2}\sin\theta\,d\theta\,d\phi$$

Let $S(\theta_1,\theta_2)$ be the following surface integral of $d\sigma$.

$$S(\theta_1,\theta_2) = \int_0^{2\pi}\int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{2\pi Z^2 \alpha^2 m^2}{p^4} \big(I(\theta_2) - I(\theta_1) \big)$$

where

$$I(\theta) = \frac{1}{\cos\theta - 1}$$

The cumulative distribution function is

$$F(\theta) = \frac{S(a, \theta)}{S(a, \pi)} = \frac{I(\theta) - I(a)}{I(\pi) - I(a)} = \frac{2(\cos a - \cos\theta)}{(1 + \cos a)(1 - \cos\theta)}, \quad a \le \theta \le \pi$$

Angular support is reduced by an arbitrary angle $a > 0$ because $I(0)$ is undefined.

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi) - I(a)} \frac{1}{(\cos\theta - 1)^2} \sin\theta$$

Notes

1. The original Rutherford scattering experiment in 1911 used alpha particles, not electrons. However, scattering of any charged particle by Coulomb interaction is now known as Rutherford scattering. The first Rutherford scattering experiment using electrons appears to have been done by F. L. Arnot, then a student of Rutherford, in 1929.

2. Lancaster and Blundell page 356 has

$$\frac{d\sigma}{d\Omega} = \frac{Z^2 \alpha^2}{4 m^2 \mathbf{v}^4 \sin^4(\theta/2)}$$

Noting that

$$\frac{1}{m^2 \mathbf{v}^4} = \frac{m^2}{m^4 \mathbf{v}^4} = \frac{m^2}{p^4}$$

and

$$4 \sin^4(\theta/2) = (\cos\theta - 1)^2$$

we have

$$\frac{Z^2 \alpha^2}{4 m^2 \mathbf{v}^4 \sin^4(\theta/2)} = \frac{Z^2 \alpha^2 m^2}{p^4 (\cos\theta - 1)^2}$$