Spin operators from scratch

By definition of expectation value we have

$$\langle S_x \rangle = \tfrac{\hbar}{2} \Pr\left(\tfrac{\hbar}{2}\right) + \left(-\tfrac{\hbar}{2}\right) \Pr\left(-\tfrac{\hbar}{2}\right)$$

Rewrite probability as amplitude squared.

$$\langle S_x \rangle = \tfrac{\hbar}{2} \Big| \langle x_+ | s \rangle \Big|^2 - \tfrac{\hbar}{2} \Big| \langle x_- | s \rangle \Big|^2$$

Equivalently

$$\langle S_x \rangle = \tfrac{\hbar}{2} \langle s | x_+ \rangle \langle x_+ | s \rangle - \tfrac{\hbar}{2} \langle s | x_- \rangle \langle x_- | s \rangle$$

Factor out spin state $|s\rangle$.

$$\langle S_x \rangle = \langle s | \Big(\tfrac{\hbar}{2} |x_+\rangle\langle x_+| - \tfrac{\hbar}{2} |x_-\rangle\langle x_-| \Big) |s\rangle$$

Rewrite as

$$\langle S_x \rangle = \langle s | S_x | s \rangle$$

where

$$S_x = \tfrac{\hbar}{2} |x_+\rangle\langle x_+| - \tfrac{\hbar}{2} |x_-\rangle\langle x_-|$$

From basis states

$$x_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad x_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

we have outer products

$$|x_+\rangle\langle x_+| = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad |x_-\rangle\langle x_-| = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Hence

$$S_x = \frac{\hbar}{4} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{\hbar}{4} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Repeat for S_y and S_z.