

Quantum angular momentum

For $l = 1$ and $l = 2$ find the eigenfunctions of L_x where

$$L_x = i\hbar \left(\sin \phi \frac{\partial}{\partial \theta} + \cos \phi \cot \theta \frac{\partial}{\partial \phi} \right)$$

In other words, find the eigenfunctions $\psi_{l,m}$ such that $L_x \psi_{l,m} = m\hbar \psi_{l,m}$.

Using ladder operators we have

$$L_x Y_{l,m} = \frac{1}{2}(L_+ + L_-)Y_{l,m} = \frac{\hbar}{2} (c_{l,m}^+ Y_{l,m+1} + c_{l,m}^- Y_{l,m-1})$$

where

$$c_{l,m}^+ = \sqrt{(l-m)(l+m+1)} = \sqrt{l(l+1) - m(m+1)}$$

$$c_{l,m}^- = \sqrt{(l+m)(l-m+1)} = \sqrt{l(l+1) - m(m-1)}$$

Hence for $l = 1$ we have

$$\begin{pmatrix} L_x Y_{1,1} \\ L_x Y_{1,0} \\ L_x Y_{1,-1} \end{pmatrix} = \hbar M_x \begin{pmatrix} Y_{1,1} \\ Y_{1,0} \\ Y_{1,-1} \end{pmatrix}$$

where

$$M_x = \frac{1}{2} \begin{pmatrix} 0 & c_{1,0}^+ & 0 \\ c_{1,1}^- & 0 & c_{1,-1}^+ \\ 0 & c_{1,0}^- & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix}$$

These are the eigenvectors x_m of M_x where m is the eigenvalue.

$$x_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}^T, \quad M_x x_1 = x_1$$

$$x_0 = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}^T, \quad M_x x_0 = 0$$

$$x_{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}^T, \quad M_x x_{-1} = -x_{-1}$$

The eigenfunctions $\psi_{l,m}$ are

$$\psi_{1,m} = x_m \cdot \begin{pmatrix} Y_{1,1} \\ Y_{1,0} \\ Y_{1,-1} \end{pmatrix}$$

Hence

$$\psi_{1,1} = \frac{1}{2}Y_{1,1} + \frac{1}{\sqrt{2}}Y_{1,0} + \frac{1}{2}Y_{1,-1}, \quad L_x \psi_{1,1} = \hbar \psi_{1,1}$$

$$\psi_{1,0} = -\frac{1}{\sqrt{2}}Y_{1,1} + \frac{1}{\sqrt{2}}Y_{1,-1}, \quad L_x \psi_{1,0} = 0$$

$$\psi_{1,-1} = -\frac{1}{2}Y_{1,1} + \frac{1}{\sqrt{2}}Y_{1,0} - \frac{1}{2}Y_{1,-1}, \quad L_x \psi_{1,-1} = -\hbar \psi_{1,-1}$$

For $l = 2$

$$M_x = \frac{1}{2} \begin{pmatrix} 0 & c_{2,1}^+ & 0 & 0 & 0 \\ c_{2,2}^- & 0 & c_{2,0}^+ & 0 & 0 \\ 0 & c_{2,1}^- & 0 & c_{2,-1}^+ & 0 \\ 0 & 0 & c_{2,0}^- & 0 & c_{2,-2}^+ \\ 0 & 0 & 0 & c_{2,-1}^- & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{3/2} & 0 & 0 \\ 0 & \sqrt{3/2} & 0 & \sqrt{3/2} & 0 \\ 0 & 0 & \sqrt{3/2} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

These are the eigenvectors x_m of M_x where m is the eigenvalue.

$$\begin{aligned} x_2 &= \left(\frac{1}{4} \quad \frac{1}{2} \quad \frac{\sqrt{3}}{2\sqrt{2}} \quad \frac{1}{2} \quad \frac{1}{4} \right)^T, & M_x x_2 &= 2x_2 \\ x_1 &= \left(-\frac{1}{2} \quad -\frac{1}{2} \quad 0 \quad \frac{1}{2} \quad \frac{1}{2} \right)^T, & M_x x_1 &= x_1 \\ x_0 &= \left(\frac{\sqrt{3}}{2\sqrt{2}} \quad 0 \quad -\frac{1}{2} \quad 0 \quad \frac{\sqrt{3}}{2\sqrt{2}} \right)^T, & M_x x_0 &= 0 \\ x_{-1} &= \left(-\frac{1}{2} \quad \frac{1}{2} \quad 0 \quad -\frac{1}{2} \quad \frac{1}{2} \right)^T, & M_x x_{-1} &= -x_{-1} \\ x_{-2} &= \left(-\frac{1}{4} \quad \frac{1}{2} \quad -\frac{\sqrt{3}}{2\sqrt{2}} \quad \frac{1}{2} \quad -\frac{1}{4} \right)^T, & M_x x_{-2} &= -2x_{-2} \end{aligned}$$

The eigenfunctions $\psi_{l,m}$ are

$$\psi_{2,m} = x_m \cdot \begin{pmatrix} Y_{2,2} \\ Y_{2,1} \\ Y_{2,0} \\ Y_{2,-1} \\ Y_{2,-2} \end{pmatrix}$$

Hence

$$\begin{aligned} \psi_{2,2} &= \frac{1}{4}Y_{2,2} + \frac{1}{2}Y_{2,1} + \frac{\sqrt{3}}{2\sqrt{2}}Y_{2,0} + \frac{1}{2}Y_{2,-1} + \frac{1}{4}Y_{2,-2}, & L_x \psi_{2,2} &= 2\hbar\psi_{2,2} \\ \psi_{2,1} &= -\frac{1}{2}Y_{2,2} - \frac{1}{2}Y_{2,1} + \frac{1}{2}Y_{2,-1} + \frac{1}{2}Y_{2,-2}, & L_x \psi_{2,1} &= \hbar\psi_{2,1} \\ \psi_{2,0} &= \frac{\sqrt{3}}{2\sqrt{2}}Y_{2,2} - \frac{1}{2}Y_{2,0} + \frac{\sqrt{3}}{2\sqrt{2}}Y_{2,-2}, & L_x \psi_{2,0} &= 0 \\ \psi_{2,-1} &= -\frac{1}{2}Y_{2,2} + \frac{1}{2}Y_{2,1} - \frac{1}{2}Y_{2,-1} + \frac{1}{2}Y_{2,-2}, & L_x \psi_{2,-1} &= -\hbar\psi_{2,-1} \\ \psi_{2,-2} &= -\frac{1}{4}Y_{2,2} + \frac{1}{2}Y_{2,1} - \frac{\sqrt{3}}{2\sqrt{2}}Y_{2,0} + \frac{1}{2}Y_{2,-1} - \frac{1}{4}Y_{2,-2}, & L_x \psi_{2,-2} &= -2\hbar\psi_{2,-2} \end{aligned}$$