Exercise 6.9. Prove that the four vectors $|sing\rangle$, $|T_1\rangle$, $|T_2\rangle$, and $|T_3\rangle$ are eigenvectors of $\vec{\sigma} \cdot \vec{\tau}$. What are their eigenvalues?

Recall that

$$\vec{\sigma} \cdot \vec{\tau} = \sigma_x \tau_x + \sigma_y \tau_y + \sigma_z \tau_z$$

Let A and B be the sets

$$A = \{|uu\rangle, |ud\rangle, |du\rangle, |dd\rangle\}$$
$$B = \{|sing\rangle, |T_1\rangle, |T_2\rangle, |T_3\rangle\}$$

By Table 1, the vectors in A are eigenvectors of σ and τ operators. By table closure, the vectors in A are also eigenvectors of compositions of σ and τ. Since the vectors in B are linear combinations of the vectors in A, then by linearity the vectors in B must also be eigenvectors. Finally, by linearity of the $\vec{\sigma} \cdot \vec{\tau}$ operator, the vectors in A and B must be eigenvectors of $\vec{\sigma} \cdot \vec{\tau}$.

By Table 1 we obtain the following eigenvalues.

	$\lvert sing\rangle$	$\lvert T_1\rangle$	$\lvert T_2\rangle$	$\lvert T_3\rangle$
$\sigma_x \tau_x$	-1	1	1	-1
$\sigma_y \tau_y$	-1	1	-1	1
$\sigma_z \tau_z$	-1	-1	1	1
$\vec{\sigma} \cdot \vec{\tau}$	-3	1	1	1