8-3. Show that Q_α^c, Q_α^s are normal coordinates corresponding to standing wave normal modes $\cos(2\pi\alpha j/N)$ and $\sin(2\pi\alpha j/N)$, in the sense that (for N odd)

$$q_j(t) = \sqrt{\frac{2}{N}}\left(\frac{1}{2}Q_0^c(t) + \sum_{\alpha=1}^{(N-1)/2}\left(Q_\alpha^c(t)\cos\frac{2\pi\alpha j}{N} + Q_\alpha^s(t)\sin\frac{2\pi\alpha j}{N}\right)\right)$$

(8.82)

Consider the following equations.

$$Q_\alpha(t) = \frac{1}{\sqrt{N}}\sum_{k=1}^{N}q_k(t)\left(\cos\frac{2\pi\alpha k}{N} - i\sin\frac{2\pi\alpha k}{N}\right)$$

(8.77)

$$Q_\alpha^c = \frac{1}{\sqrt{2}}(Q_\alpha + Q_\alpha^*)$$

(8.79)

$$Q_\alpha^s = \frac{i}{\sqrt{2}}(Q_\alpha - Q_\alpha^*)$$

(8.80)

Rewrite (8.82) as

$$q_j = \frac{1}{\sqrt{2N}}Q_0^c + \frac{1}{N}\sum_{\alpha=1}^{(N-1)/2}\sum_{k=1}^{N}q_k(T_1 + T_2 + T_3 - T_4)$$

(1)

where

$$T_1 = \cos\frac{2\pi\alpha k}{N}\cos\frac{2\pi\alpha j}{N} - i\sin\frac{2\pi\alpha k}{N}\cos\frac{2\pi\alpha j}{N}$$

$$T_2 = \cos\frac{2\pi\alpha k}{N}\cos\frac{2\pi\alpha j}{N} + i\sin\frac{2\pi\alpha k}{N}\cos\frac{2\pi\alpha j}{N}$$

$$T_3 = i\cos\frac{2\pi\alpha k}{N}\sin\frac{2\pi\alpha j}{N} + \sin\frac{2\pi\alpha k}{N}\sin\frac{2\pi\alpha j}{N}$$

$$T_4 = i\cos\frac{2\pi\alpha k}{N}\sin\frac{2\pi\alpha j}{N} - \sin\frac{2\pi\alpha k}{N}\sin\frac{2\pi\alpha j}{N}$$

It follows that

$$T_1 + T_2 + T_3 - T_4 = 2\cos\frac{2\pi\alpha k}{N}\cos\frac{2\pi\alpha j}{N} + 2\sin\frac{2\pi\alpha k}{N}\sin\frac{2\pi\alpha j}{N}$$

By trigonometric identities

$$T_1 + T_2 + T_3 - T_4 = 2\cos\left(\frac{2\pi\alpha}{N}(j-k)\right) \tag{2}$$

Substitute (2) into (1) to obtain

$$q_j = \frac{1}{\sqrt{2N}}Q_0^c + \frac{2}{N}\sum_{\alpha=1}^{(N-1)/2}\sum_{k=1}^{N}q_k\cos\left(\frac{2\pi\alpha}{N}(j-k)\right)$$

Note that

$$Q_0^c = \sqrt{\frac{2}{N}}\sum_{k=1}^{N}q_k$$

Hence

$$q_j = \frac{1}{N}\sum_{k=1}^{N}q_k + \frac{2}{N}\sum_{\alpha=1}^{(N-1)/2}\sum_{k=1}^{N}q_k\cos\left(\frac{2\pi\alpha}{N}(j-k)\right)$$

Rewrite as

$$q_j = \frac{1}{N}\sum_{k=1}^{N}q_k\left(1 + 2\sum_{\alpha=1}^{(N-1)/2}\cos\left(\frac{2\pi\alpha}{N}(j-k)\right)\right) \tag{3}$$

For the sum over α we have

$$\sum_{\alpha=1}^{(N-1)/2}\cos\left(\frac{2\pi\alpha}{N}(j-k)\right) = \begin{cases}(N-1)/2 & j=k \\ -1/2 & j\neq k\end{cases}$$

Hence for $j = k$

$$1 + 2\sum_{\alpha=1}^{(N-1)/2}\cos\left(\frac{2\pi\alpha}{N}(j-k)\right) = N$$

and for $j \neq k$

$$1 + 2\sum_{\alpha=1}^{(N-1)/2}\cos\left(\frac{2\pi\alpha}{N}(j-k)\right) = 0$$

Therefore (3) is shown to be true.