Atomic transitions 4

From the previous section the transition rate is

$$R_{a \to b} = \frac{\pi e^2}{\varepsilon_0 m^2 \hbar^2} \left| \langle \psi_b | \exp(i\mathbf{k} \cdot \mathbf{r}) \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \right|^2 \frac{\rho(\omega_0)}{\omega_0^2}$$

Substitute the dipole approximation

$$\exp(i\mathbf{k} \cdot \mathbf{r}) \approx 1$$

to obtain

$$R_{a \to b} = \frac{\pi e^2}{\varepsilon_0 m^2 \hbar^2} \left| \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \right|^2 \frac{\rho(\omega_0)}{\omega_0^2}$$

By the identity

$$\mathbf{p} = \frac{im}{\hbar} [H_0, \mathbf{r}] \tag{1}$$

we have

$$\begin{aligned}
\langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle &= \frac{im}{\hbar} \langle \psi_b | \boldsymbol{\epsilon} \cdot [H_0, \mathbf{r}] | \psi_a \rangle \\
&= \frac{im}{\hbar} \langle \psi_b | \boldsymbol{\epsilon} \cdot H_0 \mathbf{r} | \psi_a \rangle - \frac{im}{\hbar} \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} H_0 | \psi_a \rangle \\
&= \frac{im}{\hbar} E_b \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle - \frac{im}{\hbar} E_a \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle \\
&= \frac{im}{\hbar} (E_b - E_a) \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle \\
&= im\omega_0 \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle
\end{aligned}$$

Hence

$$R_{a \to b} = \frac{\pi e^2}{\varepsilon_0 \hbar^2} \left| \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle \right|^2 \rho(\omega_0)$$

Verify dimensions.

$$R_{a \to b} \propto \frac{\dfrac{e^2}{\mathrm{C}^2}}{\dfrac{\epsilon_0}{\mathrm{C}^2\,\mathrm{J}^{-1}\,\mathrm{m}^{-1}} \quad \dfrac{\hbar^2}{\mathrm{J}^2\,\mathrm{s}^2}} \times \underset{\mathrm{m}^2}{\left| \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle \right|^2} \times \underset{\mathrm{J\,m^{-3}\,s}}{\rho(\omega_0)} = \mathrm{s}^{-1}$$

Eigenmath code

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