

Rutherford scattering 1

Find the scattering cross section for the Coulomb potential

$$V(r) = -\frac{Ze^2}{4\pi\epsilon_0 r}$$

Start with the Born approximation for scattering amplitude $f(\mathbf{p})$ where \mathbf{p} is momentum transfer.

$$f(\mathbf{p}) = \frac{m}{2\pi\hbar^2} \int \exp\left(\frac{i\mathbf{p} \cdot \mathbf{r}}{\hbar}\right) V(\mathbf{r}) d\mathbf{r}$$

Convert to polar coordinates where $p = |\mathbf{p}|$.

$$f(\mathbf{p}) = \frac{m}{2\pi\hbar^2} \int_0^\infty \int_0^\pi \int_0^{2\pi} \exp\left(\frac{ipr \cos \theta}{\hbar}\right) V(r, \theta, \phi) r^2 \sin \theta dr d\theta d\phi$$

Substitute the Coulomb potential

$$V(r, \theta, \phi) = -\frac{Ze^2}{4\pi\epsilon_0 r}$$

to obtain

$$f(\mathbf{p}) = -\frac{mZe^2}{8\pi^2\epsilon_0\hbar^2} \int_0^\infty \int_0^\pi \int_0^{2\pi} \exp\left(\frac{ipr \cos \theta}{\hbar}\right) r \sin \theta dr d\theta d\phi$$

Integrate over ϕ (multiply by 2π).

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\epsilon_0\hbar^2} \int_0^\infty \int_0^\pi \exp\left(\frac{ipr \cos \theta}{\hbar}\right) r \sin \theta dr d\theta$$

Transform the integral over θ into an integral over u where $u = \cos \theta$ and $du = -\sin \theta d\theta$. The minus sign in du is canceled by interchanging integration limits $\cos 0 = 1$ and $\cos \pi = -1$.

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\epsilon_0\hbar^2} \int_0^\infty \int_{-1}^1 \exp\left(\frac{ipru}{\hbar}\right) r dr du$$

Solve the integral over u .

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\epsilon_0\hbar^2} \int_0^\infty \left[\frac{\hbar}{ipr} \exp\left(\frac{ipru}{\hbar}\right) \right]_{u=-1}^{u=1} r dr$$

Cancel r and evaluate the limits.

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\epsilon_0\hbar^2} \int_0^\infty \frac{\hbar}{ip} \left[\exp\left(\frac{ipr}{\hbar}\right) - \exp\left(-\frac{ipr}{\hbar}\right) \right] dr$$

Solve the integral over r .

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\epsilon_0\hbar^2} \frac{\hbar}{ip} \left[\frac{\hbar}{ip} \exp\left(\frac{ipr}{\hbar}\right) + \frac{\hbar}{ip} \exp\left(-\frac{ipr}{\hbar}\right) \right]_{r=0}^{r=\infty}$$

The exponentials fail to converge at the upper limit. The workaround is to go back and multiply the integrand by $\exp(-\epsilon r) \approx 1$.

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\epsilon_0\hbar^2} \int_0^\infty \frac{\hbar}{ip} \left[\exp\left(\frac{ipr}{\hbar} - \epsilon r\right) - \exp\left(-\frac{ipr}{\hbar} - \epsilon r\right) \right] dr$$

Solve the modified integral.

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\epsilon_0\hbar^2} \frac{\hbar}{ip} \left[\frac{1}{ip/\hbar - \epsilon} \exp\left(\frac{ipr}{\hbar} - \epsilon r\right) + \frac{1}{ip/\hbar + \epsilon} \exp\left(-\frac{ipr}{\hbar} - \epsilon r\right) \right]_{r=0}^{r=\infty}$$

Evaluate the limits. Now the exponentials vanish at the upper limit.

$$f(\mathbf{p}) = -\frac{mZe^2}{4\pi\epsilon_0\hbar^2} \frac{\hbar}{ip} \left(-\frac{1}{ip/\hbar - \epsilon} - \frac{1}{ip/\hbar + \epsilon} \right) \quad (1)$$

Set $\epsilon = 0$ to obtain

$$f(\mathbf{p}) = -\frac{mZe^2}{2\pi\epsilon_0 p^2}$$

Substitute $e^2 = 4\pi\epsilon_0\alpha\hbar c$.

$$f(\mathbf{p}) = -\frac{2mZ\alpha\hbar c}{p^2}$$

Note that \mathbf{p} is momentum transfer such that

$$p^2 = |\mathbf{p}|^2 = 4mE(1 - \cos\theta)$$

Hence

$$f(\theta) = -\frac{Z\alpha\hbar c}{2E(1 - \cos\theta)}$$

Calculate the cross section.

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{Z^2\alpha^2(\hbar c)^2}{4E^2(1 - \cos\theta)^2} \quad (2)$$

Note that

$$(1 - \cos\theta)^2 = 4\sin^4(\theta/2)$$

Hence equation (2) is equivalent to

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2(\hbar c)^2}{16E^2\sin^4(\theta/2)} \quad (3)$$