9-5. The momentum in the field is given by

$$\frac{1}{4\pi c}\int \mathbf{E}\times\mathbf{B}\,d^3\mathbf{r}$$

In the absence of matter (so $\phi_{\mathbf{k}}=0$), show that this is

$$i\int \mathbf{k}\,(\mathbf{a}^*_{\mathbf{k}}\cdot\dot{\mathbf{a}}_{\mathbf{k}})\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

From equations (9.26) and (9.27) with $\phi=0$ we have

$$\mathbf{E}\,d^3\mathbf{r}=\sqrt{4\pi}\dot{\mathbf{a}}_{\mathbf{k}}\,\frac{d^3\mathbf{k}}{(2\pi)^3}\qquad \mathbf{B}\,d^3\mathbf{r}=\sqrt{4\pi}ic\mathbf{k}\times\mathbf{a}_{\mathbf{k}}\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

Hence

$$\begin{aligned}\mathbf{E}\times\mathbf{B}\,d^3\mathbf{r}&=\sqrt{4\pi}\dot{\mathbf{a}}_{\mathbf{k}}\times\left(\sqrt{4\pi}ic\mathbf{k}\times\mathbf{a}_{\mathbf{k}}\right)\frac{d^3\mathbf{k}}{(2\pi)^3}\\&=4\pi ic(\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{a}_{\mathbf{k}})\mathbf{k}\,\frac{d^3\mathbf{k}}{(2\pi)^3}-4\pi ic(\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{k})\mathbf{a}_{\mathbf{k}}\,\frac{d^3\mathbf{k}}{(2\pi)^3}\end{aligned}$$

By orthogonality of $\mathbf{E}$ and $\mathbf{k}$ (see problem 9-1), $\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{k}=0$ hence

$$\mathbf{E}\times\mathbf{B}\,d^3\mathbf{r}=4\pi ic(\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{a}_{\mathbf{k}})\mathbf{k}\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$