

This is the Dirac equation with $c = 1$ and $\hbar = 1$.

$$i \left(\gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z} \right) \psi = m\psi$$

There are lots of ways to choose the gamma matrices. The following gamma matrices are the “Dirac representation.”

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The wave function ψ has angular frequency ω equal to the energy of the particle.

$$\omega = \sqrt{k_x^2 + k_y^2 + k_z^2 + m^2}$$

There are four positive frequency solutions that are linearly independent.

$$\begin{aligned} \psi_1 &= \begin{pmatrix} \omega + m \\ 0 \\ k_z \\ k_x + ik_y \end{pmatrix} \exp(ik_x x + ik_y y + ik_z z - i\omega t) & \psi_2 &= \begin{pmatrix} 0 \\ \omega + m \\ k_x - ik_y \\ -k_z \end{pmatrix} \exp(ik_x x + ik_y y + ik_z z - i\omega t) \\ \psi_3 &= \begin{pmatrix} k_z \\ k_x + ik_y \\ \omega - m \\ 0 \end{pmatrix} \exp(ik_x x + ik_y y + ik_z z - i\omega t) & \psi_4 &= \begin{pmatrix} k_x - ik_y \\ -k_z \\ 0 \\ \omega - m \end{pmatrix} \exp(ik_x x + ik_y y + ik_z z - i\omega t) \end{aligned}$$

There are four negative frequency solutions that are linearly independent. The negative frequency solutions flip the sign of m .

$$\begin{aligned} \psi_5 &= \begin{pmatrix} \omega - m \\ 0 \\ k_z \\ k_x + ik_y \end{pmatrix} \exp(-ik_x x - ik_y y - ik_z z + i\omega t) & \psi_6 &= \begin{pmatrix} 0 \\ \omega - m \\ k_x - ik_y \\ -k_z \end{pmatrix} \exp(-ik_x x - ik_y y - ik_z z + i\omega t) \\ \psi_7 &= \begin{pmatrix} k_z \\ k_x + ik_y \\ \omega + m \\ 0 \end{pmatrix} \exp(-ik_x x - ik_y y - ik_z z + i\omega t) & \psi_8 &= \begin{pmatrix} k_x - ik_y \\ -k_z \\ 0 \\ \omega + m \end{pmatrix} \exp(-ik_x x - ik_y y - ik_z z + i\omega t) \end{aligned}$$

The following solutions are used by quantum electrodynamics.

$$\begin{aligned} \psi_1 & \text{ Fermion, spin up} \\ \psi_2 & \text{ Fermion, spin down} \\ \\ \psi_7 & \text{ Anti-fermion, spin up} \\ \psi_8 & \text{ Anti-fermion, spin down} \end{aligned}$$