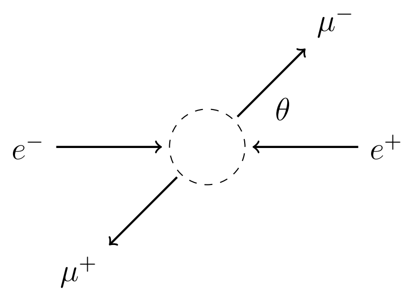
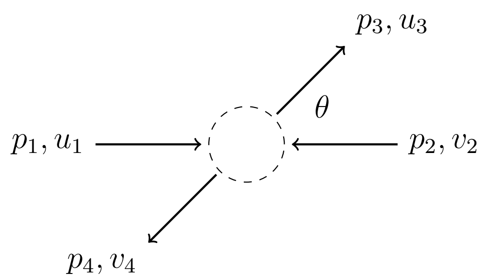
# Muon pair production

A high energy electron and positron collision can create two muons.




Here is the same diagram with momentum and spinor labels.




In a typical collider experiment the momentum vectors are

$$p_1 = \underset{\text{inbound electron}}{\begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}} \qquad p_2 = \underset{\text{inbound positron}}{\begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix}} \qquad p_3 = \underset{\text{outbound muon}}{\begin{pmatrix} E \\ \rho\sin\theta\cos\phi \\ \rho\sin\theta\sin\phi \\ \rho\cos\theta \end{pmatrix}} \qquad p_4 = \underset{\text{outbound anti-muon}}{\begin{pmatrix} E \\ -\rho\sin\theta\cos\phi \\ -\rho\sin\theta\sin\phi \\ -\rho\cos\theta \end{pmatrix}}$$

Symbol $E$ is beam energy, $p = \sqrt{E^2 - m^2}$, $\rho = \sqrt{E^2 - M^2}$, $m$ is electron mass $0.51\,\text{MeV}$, and $M$ is muon mass $106\,\text{MeV}$.

The spinors are

$$u_{11} = \underset{\substack{\text{inbound electron}\\ \text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix}} \qquad v_{21} = \underset{\substack{\text{inbound positron}\\ \text{spin up}}}{\begin{pmatrix} -p \\ 0 \\ E+m \\ 0 \end{pmatrix}} \qquad u_{31} = \underset{\substack{\text{outbound muon}\\ \text{spin up}}}{\begin{pmatrix} E+M \\ 0 \\ p_3^z \\ p_3^x + ip_3^y \end{pmatrix}} \qquad v_{41} = \underset{\substack{\text{outbound anti-muon}\\ \text{spin up}}}{\begin{pmatrix} p_4^z \\ p_4^x + ip_4^y \\ E+M \\ 0 \end{pmatrix}}$$

$$u_{12} = \underset{\substack{\text{inbound electron}\\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix}} \qquad v_{22} = \underset{\substack{\text{inbound positron}\\ \text{spin down}}}{\begin{pmatrix} 0 \\ p \\ 0 \\ E+m \end{pmatrix}} \qquad u_{32} = \underset{\substack{\text{outbound muon}\\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+M \\ p_3^x - ip_3^y \\ -p_3^z \end{pmatrix}} \qquad v_{42} = \underset{\substack{\text{outbound anti-muon}\\ \text{spin down}}}{\begin{pmatrix} p_4^x - ip_4^y \\ -p_4^z \\ 0 \\ E+M \end{pmatrix}}$$

Spinor subscripts have 1 for spin up and 2 for spin down. The spinors are not individually normalized. Instead, a combined spinor normalization constant $N=(E+m)^2(E+M)^2$ will be used.

This is the probability density for spin state $abcd$. Symbol $e$ is electron charge and $s=(p_1+p_2)^2=4E^2$. The formula is derived from Feynman diagrams for muon pair production.

$$\left|\mathcal{M}_{abcd}\right|^2=\frac{e^4}{Ns^2}\left|(\bar{u}_{3c}\gamma_\mu v_{4d})(\bar{v}_{2b}\gamma^\mu u_{1a})\right|^2$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is computed by summing $|\mathcal{M}_{abcd}|^2$ over all spin states and dividing by the number of inbound states. There are four inbound states.

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle&=\frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}\left|\mathcal{M}_{abcd}\right|^2\\&=\frac{e^4}{4Ns^2}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}\left|(\bar{u}_{3c}\gamma_\mu v_{4d})(\bar{v}_{2b}\gamma^\mu u_{1a})\right|^2\end{aligned}$$

The Casimir trick uses matrix arithmetic to compute sums.

$$\langle|\mathcal{M}|^2\rangle=\frac{e^4}{4s^2}\operatorname{Tr}\Big((\not{p}_3+M)\gamma^\mu(\not{p}_4-M)\gamma^\nu\Big)\operatorname{Tr}\Big((\not{p}_2-m)\gamma_\mu(\not{p}_1+m)\gamma_\nu\Big)$$

The following formula is equivalent to the Casimir trick. (Recall that $a\cdot b=a^\mu g_{\mu\nu}b^\nu$)

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle=\frac{e^4}{4s^2}\Big(&32(p_1\cdot p_3)(p_2\cdot p_4)+32(p_1\cdot p_4)(p_2\cdot p_3)\\&+32m^2(p_3\cdot p_4)+32M^2(p_1\cdot p_2)+64m^2M^2\Big)\end{aligned}$$

For the momentum vectors given above the result is

$$\langle|\mathcal{M}|^2\rangle=e^4\left(1+\cos^2\theta+\frac{m^2+M^2}{E^2}\sin^2\theta+\frac{m^2M^2}{E^4}\cos^2\theta\right)$$

## Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega}=\frac{\langle|\mathcal{M}|^2\rangle}{4(4\pi\varepsilon_0)^2s}$$

For high energy experiments $E\gg M$ a useful approximation is

$$\langle|\mathcal{M}|^2\rangle=e^4\left(1+\cos^2\theta\right)\quad\text{and}\quad s=4E^2$$

Hence

$$\frac{d\sigma}{d\Omega}=\frac{e^4}{4(4\pi\varepsilon_0)^2s}\left(1+\cos^2\theta\right)$$

Noting that

$$e^2 = 4\pi\varepsilon_0\alpha\hbar c$$

we can also write

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{4s}\left(1+\cos^2\theta\right)$$

We can integrate $d\sigma$ to obtain a cumulative distribution function. Recall that

$$d\Omega = \sin\theta\,d\theta\,d\phi$$

Hence

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{4s}\left(1+\cos^2\theta\right)\sin\theta\,d\theta\,d\phi$$

Let $I(\theta)$ be the following integral of $d\sigma$.

$$I(\theta) = \int \left(1+\cos^2\theta\right)\sin\theta\,d\theta$$

The result is

$$I(\theta) = -\frac{\cos^3\theta}{3} - \cos\theta$$

The cumulative distribution function is

$$F(\theta) = \frac{I(\theta)-I(0)}{I(\pi)-I(0)} = -\frac{\cos^3\theta}{8} - \frac{3\cos\theta}{8} + \frac{1}{2}, \quad 0 \le \theta \le \pi$$

The probability of observing scattering events in the interval $\theta_1$ to $\theta_2$ is

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

## Data from SLAC PEP experiment

See www.hepdata.net/record/ins216031, Table 1, $s = (29.0\,\mathrm{GeV})^2$.

| $x$ | $y$ |
|---|---|
| $-0.925$ | 67.08 |
| $-0.85$ | 58.67 |
| $-0.75$ | 54.66 |
| $-0.65$ | 51.72 |
| $-0.55$ | 43.70 |
| $-0.45$ | 41.12 |
| $-0.35$ | 39.71 |
| $-0.25$ | 35.34 |
| $-0.15$ | 33.35 |
| $-0.05$ | 34.69 |
| 0.05 | 34.05 |
| 0.15 | 34.48 |
| 0.25 | 34.66 |
| 0.35 | 35.23 |
| 0.45 | 35.60 |
| 0.55 | 40.13 |
| 0.65 | 42.56 |
| 0.75 | 46.37 |
| 0.85 | 49.28 |
| 0.925 | 55.70 |

Data $x$ and $y$ have the following relationship with the differential cross section formula.

$$x = \cos\theta, \quad y = s\frac{d\sigma}{d\cos\theta} = 2\pi s\frac{d\sigma}{d\Omega}$$

The cross section formula is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right) \times (\hbar c)^2$$

To compute predicted values $\hat{y}$, multiply by $10^{37}$ to convert square meters to nanobarns.

$$\hat{y} = 2\pi s\frac{d\sigma}{d\Omega} = \frac{\pi\alpha^2}{2}\left(1+x^2\right) \times (\hbar c)^2 \times 10^{37}$$

The following table shows predicted values $\hat{y}$.

| $x$ | $y$ | $\hat{y}$ |
|---|---|---|
| $-0.925$ | 67.08 | 60.44 |
| $-0.85$ | 58.67 | 56.10 |
| $-0.75$ | 54.66 | 50.89 |
| $-0.65$ | 51.72 | 46.33 |
| $-0.55$ | 43.70 | 42.42 |
| $-0.45$ | 41.12 | 39.17 |
| $-0.35$ | 39.71 | 36.56 |
| $-0.25$ | 35.34 | 34.61 |
| $-0.15$ | 33.35 | 33.30 |
| $-0.05$ | 34.69 | 32.65 |
| 0.05 | 34.05 | 32.65 |
| 0.15 | 34.48 | 33.30 |
| 0.25 | 34.66 | 34.61 |
| 0.35 | 35.23 | 36.56 |
| 0.45 | 35.60 | 39.17 |
| 0.55 | 40.13 | 42.42 |
| 0.65 | 42.56 | 46.33 |
| 0.75 | 46.37 | 50.89 |
| 0.85 | 49.28 | 56.10 |
| 0.925 | 55.70 | 60.44 |

The coefficient of determination $R^2$ measures how well predicted values fit the data.

$$R^2 = 1 - \frac{\sum(y-\hat{y})^2}{\sum(y-\bar{y})^2} = 0.87$$

The result indicates that the model $d\sigma$ explains 87% of the variance in the data.

## Electroweak model

The following differential cross section formula from electroweak theory results in a better fit to the data.[1]

$$\frac{d\sigma}{d\Omega} = F(s)\big(1+\cos^2\theta\big) + G(s)\cos\theta$$

where

$$F(s) = \frac{\alpha^2}{4s}\left(1 + \frac{g_V^2}{\sqrt{2}\pi}\left(\frac{m_Z^2}{s-m_Z^2}\right)\left(\frac{sG}{\alpha}\right) + \frac{(g_A^2+g_V^2)^2}{8\pi^2}\left(\frac{m_Z^2}{s-m_Z^2}\right)^2\left(\frac{sG}{\alpha}\right)^2\right)$$

$$G(s) = \frac{\alpha^2}{4s}\left(\frac{\sqrt{2}g_A^2}{\pi}\left(\frac{m_Z^2}{s-m_Z^2}\right)\left(\frac{sG}{\alpha}\right) + \frac{g_A^2 g_V^2}{\pi^2}\left(\frac{m_Z^2}{s-m_Z^2}\right)^2\left(\frac{sG}{\alpha}\right)^2\right)$$

[1] F. Mandl and G. Shaw, *Quantum Field Theory Revised Edition,* 316.

and

$$
\begin{aligned}
g_A &= -0.5 \\
g_V &= -0.0348 \\
m_Z &= 91.17\,\mathrm{GeV} \\
G &= 1.166 \times 10^{-5}\,\mathrm{GeV}^{-2}
\end{aligned}
$$

The corresponding formula for $\hat{y}$ is

$$
\hat{y} = 2\pi \left[F(s)(1+x^2) + G(s)x\right] \times (\hbar c)^2 \times 10^{37}
$$

where $\sqrt{s} = 29\,\mathrm{GeV}$ is the center of mass collision energy. Here are the predicted values $\hat{y}$ based on the above formula.

| $x$ | $y$ | $\hat{y}$ |
|---|---|---|
| −0.925 | 67.08 | 65.59 |
| −0.85 | 58.67 | 60.84 |
| −0.75 | 54.66 | 55.07 |
| −0.65 | 51.72 | 49.96 |
| −0.55 | 43.70 | 45.49 |
| −0.45 | 41.12 | 41.69 |
| −0.35 | 39.71 | 38.53 |
| −0.25 | 35.34 | 36.02 |
| −0.15 | 33.35 | 34.17 |
| −0.05 | 34.69 | 32.97 |
| 0.05 | 34.05 | 32.42 |
| 0.15 | 34.48 | 32.53 |
| 0.25 | 34.66 | 33.28 |
| 0.35 | 35.23 | 34.69 |
| 0.45 | 35.60 | 36.75 |
| 0.55 | 40.13 | 39.47 |
| 0.65 | 42.56 | 42.83 |
| 0.75 | 46.37 | 46.85 |
| 0.85 | 49.28 | 51.52 |
| 0.925 | 55.70 | 55.45 |

The coefficient of determination $R^2$ is

$$
R^2 = 1 - \frac{\sum (y-\hat{y})^2}{\sum (y-\bar{y})^2} = 0.98
$$

The result indicates that electroweak theory explains 98% of the variance in the data.

## Notes

Here are a few notes about how the demo script works.

In component notation, traces are sums over a repeated index, in this case $\alpha$.

$$\begin{aligned}
\operatorname{Tr}\left((\not{p}_3 + M)\gamma^\mu(\not{p}_4 - M)\gamma^\nu\right) &= (\not{p}_3 + M)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_4 - M)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \\
\operatorname{Tr}\left((\not{p}_2 - m)\gamma_\mu(\not{p}_1 + m)\gamma_\nu\right) &= (\not{p}_2 - m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha
\end{aligned}$$

To convert the above formulas to Eigenmath code, the $\gamma$ tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply $\gamma^\mu$ by the metric tensor to lower the index.

$\gamma^{\beta\mu}{}_\rho \quad \rightarrow$ `gammaT = transpose(gamma)`

$\gamma^\beta{}_{\mu\rho} \quad \rightarrow$ `gammaL = transpose(dot(gmunu,gamma))`

Define the following $4 \times 4$ matrices.

$(\not{p}_1 + m) \quad \rightarrow$ `X1 = pslash1 + m I`

$(\not{p}_2 - m) \quad \rightarrow$ `X2 = pslash2 - m I`

$(\not{p}_3 + M) \quad \rightarrow$ `X3 = pslash3 + M I`

$(\not{p}_4 - M) \quad \rightarrow$ `X4 = pslash4 - M I`

Then

$(\not{p}_3 + M)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_4 - M)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \quad \rightarrow$ `T1 = contract(dot(X3,gammaT,X4,gammaT),1,4)`

$(\not{p}_2 - m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \quad \rightarrow$ `T2 = contract(dot(X2,gammaL,X1,gammaL),1,4)`

Next, multiply matrices and sum over repeated indices. The dot function sums over $\nu$ then the contract function sums over $\mu$. The transpose makes the $\nu$ indices adjacent as required by the dot function.

$\operatorname{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu)\operatorname{Tr}(\cdots\gamma_\mu\cdots\gamma_\nu) \quad \rightarrow$ `contract(dot(T1,transpose(T2)))`