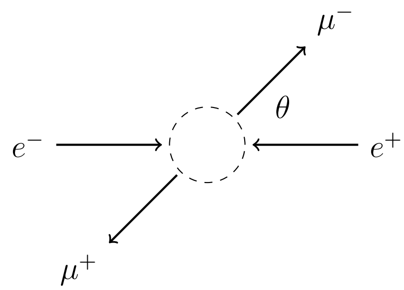
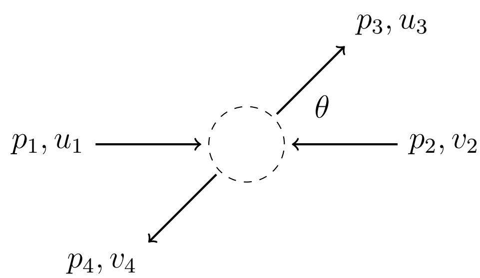
A high energy electron and positron collision can create two muons.



Here is the same diagram with momentum and spinor labels.



In a typical collider experiment the momentum vectors are

$$
\underset{\text{inbound electron}}{p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}} \quad
\underset{\text{inbound positron}}{p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix}} \quad
\underset{\text{outbound muon}}{p_3 = \begin{pmatrix} E \\ \rho\sin\theta\cos\phi \\ \rho\sin\theta\sin\phi \\ \rho\cos\theta \end{pmatrix}} \quad
\underset{\text{outbound anti-muon}}{p_4 = \begin{pmatrix} E \\ -\rho\sin\theta\cos\phi \\ -\rho\sin\theta\sin\phi \\ -\rho\cos\theta \end{pmatrix}}
$$

where E is beam energy, $p = \sqrt{E^2 - m^2}$, $\rho = \sqrt{E^2 - M^2}$, m is electron mass $0.51\,\text{MeV}$, and M is muon mass $106\,\text{MeV}$. The spinors are

$$
\underset{\substack{\text{inbound electron}\\\text{spin up}}}{u_{11} = \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix}} \quad
\underset{\substack{\text{inbound positron}\\\text{spin up}}}{v_{21} = \begin{pmatrix} -p \\ 0 \\ E+m \\ 0 \end{pmatrix}} \quad
\underset{\substack{\text{outbound muon}\\\text{spin up}}}{u_{31} = \begin{pmatrix} E+M \\ 0 \\ p_3^z \\ p_3^x + ip_3^y \end{pmatrix}} \quad
\underset{\substack{\text{outbound anti-muon}\\\text{spin up}}}{v_{41} \begin{pmatrix} p_4^z \\ p_4^x + ip_4^y \\ E+M \\ 0 \end{pmatrix}}
$$

$$
\underset{\substack{\text{inbound electron}\\\text{spin down}}}{u_{12} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix}} \quad
\underset{\substack{\text{inbound positron}\\\text{spin down}}}{v_{22} = \begin{pmatrix} 0 \\ p \\ 0 \\ E+m \end{pmatrix}} \quad
\underset{\substack{\text{outbound muon}\\\text{spin down}}}{u_{32} = \begin{pmatrix} 0 \\ E+M \\ p_3^x - ip_3^y \\ -p_3^z \end{pmatrix}} \quad
\underset{\substack{\text{outbound anti-muon}\\\text{spin down}}}{v_{42} = \begin{pmatrix} p_4^x - ip_4^y \\ -p_4^z \\ 0 \\ E+M \end{pmatrix}}
$$

The last digit in a spinor subscript is 1 for spin up and 2 for spin down. Note that the spinors are not normalized. A combined spinor normalization constant $N = (E+m)^2(E+M)^2$ will be used instead.

This is the probability density for spin state $abcd$. Symbol e is electron charge and $s = (p_1 + p_2)^2 = 4E^2$.

$$\left|\mathcal{M}_{abcd}\right|^2 = \frac{e^4}{Ns^2}\left|(\bar{u}_{3c}\gamma_\mu v_{4d})(\bar{v}_{2b}\gamma^\mu u_{1a})\right|^2$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is computed by summing $|\mathcal{M}_{abcd}|^2$ over all spin states and dividing by the number of inbound states. There are four inbound states.

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}\left|\mathcal{M}_{abcd}\right|^2 \\ &= \frac{e^4}{4Ns^2}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}\left|(\bar{u}_{3c}\gamma_\mu v_{4d})(\bar{v}_{2b}\gamma^\mu u_{1a})\right|^2\end{aligned}$$

Another way to compute $\langle|\mathcal{M}|^2\rangle$ is to use the following Casimir trick.

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4s^2}\operatorname{Tr}\Big((\not{p}_3 + M)\gamma^\mu(\not{p}_4 - M)\gamma^\nu\Big)\operatorname{Tr}\Big((\not{p}_2 - m)\gamma_\mu(\not{p}_1 + m)\gamma_\nu\Big)$$

Here is a third way to compute $\langle|\mathcal{M}|^2\rangle$.

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4s^2}\Big(&32(p_1\cdot p_3)(p_2\cdot p_4) + 32(p_1\cdot p_4)(p_2\cdot p_3) \\ &+ 32m^2(p_3\cdot p_4) + 32M^2(p_1\cdot p_2) + 64m^2M^2\Big)\end{aligned}$$

For the momentum vectors given above the result is

$$\langle|\mathcal{M}|^2\rangle = e^4\left(1 + \cos^2\theta + \frac{m^2 + M^2}{E^2}\sin^2\theta + \frac{m^2M^2}{E^4}\cos^2\theta\right)$$

The Stanford Linear Collider had a collision energy of $2E = 91$ GeV. For beam energies such as SLC where $E \gg M$ the above equation can be approximated as

$$\langle|\mathcal{M}|^2\rangle = e^4\big(1 + \cos^2\theta\big)$$

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{64\pi^2 s} = \frac{e^4}{256\pi^2E^2}\big(1 + \cos^2\theta\big)$$

Recall that $e^2 = 4\pi\alpha$ hence

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}\big(1 + \cos^2\theta\big)$$

We can integrate $d\sigma$ to obtain a cumulative distribution function. Recall that

$$d\Omega = \sin\theta\,d\theta\,d\phi$$

Hence

$$d\sigma = \frac{\alpha^2}{16E^2}\big(1 + \cos^2\theta\big)\sin\theta\,d\theta\,d\phi$$

Let $I(\theta)$ be the integral of $d\sigma$.

$$I(\theta) = \frac{\alpha^2}{16E^2} \int_0^{2\pi} \int \left(1 + \cos^2\theta\right) \sin\theta \, d\theta \, d\phi$$

The result is

$$I(\theta) = \frac{\pi\alpha^2}{8E^2} \left(-\frac{1}{3}\cos^3\theta - \cos\theta + \frac{4}{3}\right)$$

The cumulative distribution function is

$$F(\theta) = \frac{I(\theta) - I(0)}{I(\pi) - I(0)} = -\frac{1}{8}\cos^3\theta - \frac{3}{8}\cos\theta + \frac{1}{2}, \quad 0 \le \theta \le \pi$$

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

Data from SLAC PEP experiment

See www.hepdata.net/record/ins216031, Table 1, 29.0 GeV.

x	y
-0.925	67.08
-0.85	58.67
-0.75	54.66
-0.65	51.72
-0.55	43.70
-0.45	41.12
-0.35	39.71
-0.25	35.34
-0.15	33.35
-0.05	34.69
0.05	34.05
0.15	34.48
0.25	34.66
0.35	35.23
0.45	35.60
0.55	40.13
0.65	42.56
0.75	46.37
0.85	49.28
0.925	55.70

Data x and y have the following relationship with cross section parameters.

$$x = \cos\theta, \quad y = (2E)^2 \frac{d\sigma}{d\cos\theta}$$

The differential cross section for muon production is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}(1 + \cos^2\theta)$$

Let us compute predicted values $\hat{y}$ from the cross section formula. Start by finding the relationship between $d\Omega$ and $d\cos\theta$. Since $1 + \cos^2\theta$ has no dependence on ϕ we have

$$\int_\Omega (1 + \cos^2\theta)\, d\Omega = \int_0^{2\pi} \int_0^{\pi} (1 + \cos^2\theta) \sin\theta \, d\theta \, d\phi = 2\pi \int_0^{\pi} (1 + \cos^2\theta) \sin\theta \, d\theta$$

Hence

$$d\Omega = 2\pi \sin\theta \, d\theta = -2\pi \, d\cos\theta$$

We want positive cross sections so drop the minus sign and set

$$\frac{d\sigma}{d\cos\theta} = 2\pi \frac{d\sigma}{d\Omega}$$

We can now write

$$\begin{aligned} y &= (2E)^2 \frac{d\sigma}{d\cos\theta} \\ &= (2E)^2 (2\pi) \frac{d\sigma}{d\Omega} \\ &= (2E)^2 (2\pi) \frac{\alpha^2}{16E^2}(1 + \cos^2\theta) \\ &= \frac{\pi\alpha^2}{2}(1 + \cos^2\theta) \end{aligned}$$

Multiply by $(\hbar c)^2$ to convert to SI and multiply by 10^{37} to convert square meters to nanobarns.

$$y = \frac{\pi\alpha^2}{2}(1 + \cos^2\theta) \times (\hbar c)^2 \times 10^{37}$$

Replace $\cos\theta$ with explanatory variable x to obtain $\hat{y}$.

$$\hat{y} = \frac{\pi\alpha^2}{2}(1 + x^2) \times (\hbar c)^2 \times 10^{37}$$

Here are the predicted values $\hat{y}$ based on the above formula.

x	y	$\hat{y}$
-0.925	67.08	60.44
-0.85	58.67	56.10
-0.75	54.66	50.89
-0.65	51.72	46.33
-0.55	43.70	42.42
-0.45	41.12	39.17
-0.35	39.71	36.56
-0.25	35.34	34.61
-0.15	33.35	33.30
-0.05	34.69	32.65
0.05	34.05	32.65
0.15	34.48	33.30
0.25	34.66	34.61
0.35	35.23	36.56
0.45	35.60	39.17
0.55	40.13	42.42
0.65	42.56	46.33
0.75	46.37	50.89
0.85	49.28	56.10
0.925	55.70	60.44

The coefficient of determination R^2 measures how well predicted values fit the real data.

$$R^2 = 1 - \frac{\sum(y-\hat{y})^2}{\sum(y-\bar{y})^2} = 0.87$$

The result indicates that the model $d\sigma$ explains 87% of the variance in the data.

Electroweak model

The following differential cross section formula from electroweak theory results in a better fit to the data.[1]

$$\frac{d\sigma}{d\Omega} = F(s)(1+\cos^2\theta) + G(s)\cos\theta$$

where

$$F(s) = \frac{\alpha^2}{4s}\left(1 + \frac{g_V^2}{\sqrt{2}\pi}\left(\frac{m_Z^2}{s-m_Z^2}\right)\left(\frac{sG}{\alpha}\right) + \frac{(g_A^2+g_V^2)^2}{8\pi^2}\left(\frac{m_Z^2}{s-m_Z^2}\right)^2\left(\frac{sG}{\alpha}\right)^2\right)$$

$$G(s) = \frac{\alpha^2}{4s}\left(\frac{\sqrt{2}g_A^2}{\pi}\left(\frac{m_Z^2}{s-m_Z^2}\right)\left(\frac{sG}{\alpha}\right) + \frac{g_A^2 g_V^2}{\pi^2}\left(\frac{m_Z^2}{s-m_Z^2}\right)^2\left(\frac{sG}{\alpha}\right)^2\right)$$

[1] F. Mandl and G. Shaw, *Quantum Field Theory Revised Edition,* 316.

and

$$
\begin{aligned}
g_A &= -0.5 \\
g_V &= -0.0348 \\
m_Z &= 91.17\,\text{GeV} \\
G &= 1.166 \times 10^{-5}\,\text{GeV}^{-2}
\end{aligned}
$$

The corresponding formula for $\hat{y}$ is

$$\hat{y} = 2\pi \left[F(s)(1+x^2) + G(s)x\right] \times (\hbar c)^2 \times 10^{37}$$

where $\sqrt{s} = 29\,\text{GeV}$ is the center of mass collision energy. Here are the predicted values $\hat{y}$ based on the above formula.

x	y	$\hat{y}$
−0.925	67.08	65.59
−0.85	58.67	60.84
−0.75	54.66	55.07
−0.65	51.72	49.96
−0.55	43.70	45.49
−0.45	41.12	41.69
−0.35	39.71	38.53
−0.25	35.34	36.02
−0.15	33.35	34.17
−0.05	34.69	32.97
0.05	34.05	32.42
0.15	34.48	32.53
0.25	34.66	33.28
0.35	35.23	34.69
0.45	35.60	36.75
0.55	40.13	39.47
0.65	42.56	42.83
0.75	46.37	46.85
0.85	49.28	51.52
0.925	55.70	55.45

The coefficient of determination R^2 is

$$R^2 = 1 - \frac{\sum(y-\hat{y})^2}{\sum(y-\bar{y})^2} = 0.98$$

The result indicates that electroweak theory explains 98% of the variance in the data.

Notes

Here are a few notes about how the scripts work.

In component notation the traces become sums over the repeated index α.

$$\operatorname{Tr}\Big((\not{p}_3+M)\gamma^\mu(\not{p}_4-M)\gamma^\nu\Big) = (\not{p}_3+M)^\alpha{}_\beta\gamma^{\mu\beta}{}_\rho(\not{p}_4-M)^\rho{}_\sigma\gamma^{\nu\sigma}{}_\alpha$$
$$\operatorname{Tr}\Big((\not{p}_2-m)\gamma_\mu(\not{p}_1+m)\gamma_\nu\Big) = (\not{p}_2-m)^\alpha{}_\beta\gamma_\mu{}^\beta{}_\rho(\not{p}_1+m)^\rho{}_\sigma\gamma_\nu{}^\sigma{}_\alpha$$

To convert the above formulas to Eigenmath code, the γ tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply γ^μ by the metric tensor to lower the index.

$\gamma^{\beta\mu}{}_\rho \quad\rightarrow\quad$ `gammaT = transpose(gamma)`

$\gamma^\beta{}_{\mu\rho} \quad\rightarrow\quad$ `gammaL = transpose(dot(gmunu,gamma))`

Define the following 4×4 matrices.

$(\not{p}_1+m) \quad\rightarrow\quad$ `X1 = pslash1 + m I`

$(\not{p}_2-m) \quad\rightarrow\quad$ `X2 = pslash2 - m I`

$(\not{p}_3+M) \quad\rightarrow\quad$ `X3 = pslash3 + M I`

$(\not{p}_4-M) \quad\rightarrow\quad$ `X4 = pslash4 - M I`

Then

$(\not{p}_3+M)^\alpha{}_\beta\gamma^{\mu\beta}{}_\rho(\not{p}_4-M)^\rho{}_\sigma\gamma^{\nu\sigma}{}_\alpha \quad\rightarrow\quad$ `T1 = contract(dot(X3,gammaT,X4,gammaT),1,4)`

$(\not{p}_2-m)^\alpha{}_\beta\gamma_\mu{}^\beta{}_\rho(\not{p}_1+m)^\rho{}_\sigma\gamma_\nu{}^\sigma{}_\alpha \quad\rightarrow\quad$ `T2 = contract(dot(X2,gammaL,X1,gammaL),1,4)`

Next, multiply matrices and sum over repeated indices. The dot function sums over ν then the contract function sums over μ. The transpose makes the ν indices adjacent as required by the dot function.

$\operatorname{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu)\operatorname{Tr}(\cdots\gamma_\mu\cdots\gamma_\nu) \quad\rightarrow\quad$ `contract(dot(T1,transpose(T2)))`