Dirac spinors

Let ϕ be the field

$$\phi = p_x x + p_y y + p_z z - Et$$

where

$$E = \sqrt{p_x^2 c^2 + p_y^2 c^2 + p_z^2 c^2 + m^2 c^4}$$

Fermion fields are the following solutions to the Dirac equation.

$$\psi_1 = \underbrace{\begin{pmatrix} E/c + mc \\ 0 \\ p_z \\ p_x + ip_y \end{pmatrix}}_{\text{fermion spin up}} \exp\left(\frac{i\phi}{\hbar}\right) \qquad \psi_2 = \underbrace{\begin{pmatrix} 0 \\ E/c + mc \\ p_x - ip_y \\ -p_z \end{pmatrix}}_{\text{fermion spin down}} \exp\left(\frac{i\phi}{\hbar}\right)$$

$$\psi_7 = \underbrace{\begin{pmatrix} p_z \\ p_x + ip_y \\ E/c + mc \\ 0 \end{pmatrix}}_{\text{anti-fermion spin up}} \exp\left(-\frac{i\phi}{\hbar}\right) \qquad \psi_8 = \underbrace{\begin{pmatrix} p_x - ip_y \\ -p_z \\ 0 \\ E/c + mc \end{pmatrix}}_{\text{anti-fermion spin down}} \exp\left(-\frac{i\phi}{\hbar}\right)$$

A spinor is the vector part of ψ.

$$u_1 = \underbrace{\begin{pmatrix} E/c + mc \\ 0 \\ p_z \\ p_x + ip_y \end{pmatrix}}_{\text{fermion spin up}} \qquad u_2 = \underbrace{\begin{pmatrix} 0 \\ E/c + mc \\ p_x - ip_y \\ -p_z \end{pmatrix}}_{\text{fermion spin down}}$$

$$v_1 = \underbrace{\begin{pmatrix} p_z \\ p_x + ip_y \\ E/c + mc \\ 0 \end{pmatrix}}_{\text{anti-fermion spin up}} \qquad v_2 = \underbrace{\begin{pmatrix} p_x - ip_y \\ -p_z \\ 0 \\ E/c + mc \end{pmatrix}}_{\text{anti-fermion spin down}}$$

This is the spacetime momentum vector p.

$$p = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

Spinors are solutions to the momentum-space Dirac equations

$$\not{p}u = mcu, \quad \not{p}v = -mcv$$

where

$$\not{p} = p^\mu g_{\mu\nu} \gamma^\nu$$

Up and down spinors have the following completeness property.

$$
\begin{aligned}
u_1\bar{u}_1 + u_2\bar{u}_2 &= (E/c + mc)(\not{p} + mc) \\
v_1\bar{v}_1 + v_2\bar{v}_2 &= (E/c + mc)(\not{p} - mc)
\end{aligned}
$$

Spinor adjoints are

$$
\bar{u} = u^\dagger\gamma^0, \quad \bar{v} = v^\dagger\gamma^0
$$

hence $u\bar{u}$ and $v\bar{v}$ are outer products that form 4×4 matrices.