

(37.1) (a) Verify eqn 37.4.

(b) Show that the rotation matrix

$$D(\theta^3) = e^{-\frac{i}{2}\sigma^3\theta^3} = \begin{pmatrix} e^{-\frac{i\theta^3}{2}} & 0 \\ 0 & e^{\frac{i\theta^3}{2}} \end{pmatrix} \quad (37.22)$$

(a) This is equation (37.4).

$$D(\theta^1) = \exp\left(-\frac{i}{2}\sigma^1\theta^1\right) = I \cos \frac{\theta^1}{2} - i\sigma^1 \sin \frac{\theta^1}{2} \quad (37.4)$$

To verify (37.4), start with equation (37.2).

$$D(\boldsymbol{\theta}) = \exp\left(-\frac{i}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\theta}\right) \quad (37.2)$$

Convert  $\boldsymbol{\sigma} \cdot \boldsymbol{\theta}$  to component notation.

$$D(\boldsymbol{\theta}) = \exp\left(-\frac{i}{2}(\sigma^1\theta^1 + \sigma^2\theta^2 + \sigma^3\theta^3)\right)$$

Then for  $\theta^2 = \theta^3 = 0$  we have

$$D(\theta^1) = \exp\left(-\frac{i}{2}\sigma^1\theta^1\right)$$

Convert to rectangular coordinates.

$$D(\theta^1) = \cos\left(\frac{1}{2}\sigma^1\theta^1\right) - i \sin\left(\frac{1}{2}\sigma^1\theta^1\right) \quad (1)$$

Noting that

$$(\sigma^1)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

we have

$$(\sigma^1)^{2n} = I^n = I$$

and

$$(\sigma^1)^{2n+1} = I^{2n}\sigma^1 = \sigma^1$$

Hence by considering the Taylor expansion of sine and cosine in (1) we have

$$D(\theta^1) = I \cos \frac{\theta^1}{2} - i\sigma^1 \sin \frac{\theta^1}{2}$$

(b) Note that

$$(\sigma^3)^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Hence by the same argument as in part (a) we have

$$D(\theta^3) = I \cos \frac{\theta^3}{2} - i\sigma^3 \sin \frac{\theta^3}{2}$$

Rewrite in matrix form.

$$D(\theta^3) = \begin{pmatrix} \cos \frac{\theta^3}{2} - i \sin \frac{\theta^3}{2} & 0 \\ 0 & \cos \frac{\theta^3}{2} + i \sin \frac{\theta^3}{2} \end{pmatrix} = \begin{pmatrix} e^{-\frac{i\theta^3}{2}} & 0 \\ 0 & e^{\frac{i\theta^3}{2}} \end{pmatrix}$$