(11.1) One of the criteria we had for a successful theory of a scalar field was that the commutator for space-like separations would be zero. Let's see if our scalar field has this feature. Show that

$$[\hat{\phi}(x), \hat{\phi}(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big(\exp\big(-ip\cdot(x-y)\big) - \exp(-ip\cdot(y-x)) \Big) \quad (11.51)$$

For space-like separation we are able to swap $(y - x)$ in the second term to $(x - y)$. This gives us zero, as required.

Consider equation (11.12).

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} \Big(\hat{a}_{\mathbf{p}} \exp(-ip \cdot x) + \hat{a}_{\mathbf{p}}^\dagger \exp(ip \cdot x) \Big) \quad (11.12)$$

It follows from (11.12) that

$$[\hat{\phi}(x), \hat{\phi}(y)] = \frac{1}{(2\pi)^3} \int \frac{d^3p}{(2E_p)^{\frac{1}{2}}} \int \frac{d^3q}{(2E_q)^{\frac{1}{2}}} (PQ - QP) \quad (1)$$

where

$$P = \hat{a}_{\mathbf{p}} \exp(-ip \cdot x) + \hat{a}_{\mathbf{p}}^\dagger \exp(ip \cdot x)$$
$$Q = \hat{a}_{\mathbf{q}} \exp(-iq \cdot y) + \hat{a}_{\mathbf{q}}^\dagger \exp(iq \cdot y)$$

Expanding the commutator in (1) we have

$$PQ - QP = [\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}] \exp(-ip \cdot x - iq \cdot y) + [\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}^\dagger] \exp(-ip \cdot x + iq \cdot y)$$
$$+ [\hat{a}_{\mathbf{p}}^\dagger, \hat{a}_{\mathbf{q}}] \exp(ip \cdot x - iq \cdot y) + [\hat{a}_{\mathbf{p}}^\dagger, \hat{a}_{\mathbf{q}}^\dagger] \exp(ip \cdot x + iq \cdot y)$$

Then from the commutation relations

$$[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}] = 0, \quad [\hat{a}_{\mathbf{p}}^\dagger, \hat{a}_{\mathbf{q}}^\dagger] = 0, \quad [\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}^\dagger] = \delta(\mathbf{p} - \mathbf{q})$$

we have

$$PQ - QP = \delta(\mathbf{p} - \mathbf{q}) \Big(\exp(-ip \cdot x + iq \cdot y) - \exp(ip \cdot x - iq \cdot y) \Big) \quad (2)$$

Substitute (2) into (1) to obtain

$$[\hat{\phi}(x), \hat{\phi}(y)] = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_p} \Big(\exp\big(-ip \cdot (x - y)\big) - \exp\big(-ip \cdot (y - x)\big) \Big)$$

1