

Harmonic oscillator coherent state

A coherent state minimizes uncertainty. The ground state is a coherent state. To make the ground state more interesting, parameters r and θ are added to shift $\langle x \rangle$ and $\langle p \rangle$ from zero. Shifting $\langle x \rangle$ and $\langle p \rangle$ makes the coherent state time-dependent.

$$\psi_{n,r,\theta}(x,t) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} H_n \left(\sqrt{\frac{m\omega}{\hbar}} (x - \langle x \rangle) \right) \\ \times \exp \left[-\frac{m\omega}{2\hbar} (x - \langle x \rangle)^2 + \frac{i}{\hbar} \langle p \rangle \left(x - \frac{\langle x \rangle}{2} \right) - i \left(n + \frac{1}{2} \right) \omega t \right]$$

Parameters r and θ are polar coordinates in phase space such that

$$\langle x \rangle = \sqrt{\frac{2\hbar}{m\omega}} r \cos(\omega t + \theta), \quad \langle p \rangle = -\sqrt{2m\hbar\omega} r \sin(\omega t + \theta)$$

Note that $\psi_{0,0,\theta}(x,0)$ is equivalent to the ordinary ground state.

$$\psi_{0,0,\theta}(x,0) = \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} \exp \left(-\frac{m\omega x^2}{2\hbar} \right)$$

Exercises

1. Verify

$$i\hbar \frac{d}{dt} \psi = \hat{H} \psi$$

2. Verify

$$\int_{-\infty}^{\infty} \psi^* \psi dx = 1$$

3. Verify for the ground state

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{\hbar}{2m\omega}}$$

and

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{m\hbar\omega}{2}}$$

Hence $\Delta x \Delta p$ is the minimum allowed by the uncertainty principle.

$$\Delta x \Delta p = \frac{\hbar}{2}$$