

Atomic transitions 7

From the previous section the spontaneous emission rate is

$$A_{b \rightarrow a} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3} \omega_{ba}^3 |\langle \psi_a | \mathbf{r} | \psi_b \rangle|^2$$

For hydrogen $2p \rightarrow 1s$ we have

$$A_{21} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3} \omega_{21}^3 |\langle \psi_{100} | \mathbf{r} | \psi_{210} \rangle|^2$$

Noting that

$$e^2 = 4\pi\varepsilon_0\hbar c\alpha$$

we can also write

$$A_{21} = \frac{4\alpha}{3c^2} \omega_{21}^3 |\langle \psi_{100} | \mathbf{r} | \psi_{210} \rangle|^2 \quad (1)$$

Verify dimensions:

$$A_{21} \propto (\text{m/s})^{-2} \times \text{s}^{-3} \times \text{m}^2 = \text{s}^{-1} \text{ (or hertz)}$$

For angular frequency ω_{21} we have

$$\omega_{21} = \frac{E_2 - E_1}{\hbar}$$

and for hydrogen

$$E_n = -\frac{\alpha\hbar c}{2n^2 a_0}$$

Hence

$$\omega_{21} = \frac{3\alpha c}{8a_0}$$

For the transition “probability” we have

$$|\langle \psi_{100} | \mathbf{r} | \psi_{210} \rangle|^2 = |x_{21}|^2 + |y_{21}|^2 + |z_{21}|^2$$

where

$$\begin{aligned} x_{21} &= \int_0^\infty \int_0^\pi \int_0^{2\pi} \psi_{100}^* x \psi_{210} dV \\ y_{21} &= \int_0^\infty \int_0^\pi \int_0^{2\pi} \psi_{100}^* y \psi_{210} dV \\ z_{21} &= \int_0^\infty \int_0^\pi \int_0^{2\pi} \psi_{100}^* z \psi_{210} dV \end{aligned}$$

and

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta, \quad dV = r^2 \sin \theta dr d\theta d\phi$$

The integrals work out to be

$$x_{21} = 0, \quad y_{21} = 0, \quad z_{21} = \frac{2^7}{3^5} \sqrt{2} a_0$$

hence

$$|\langle \psi_{100} | \mathbf{r} | \psi_{210} \rangle|^2 = |z_{21}|^2 = \frac{2^{15}}{3^{10}} a_0^2$$

By equation (1) the spontaneous emission rate is

$$A_{21} = \frac{2^8}{3^8} \frac{\alpha^4 c}{a_0} = 6.26 \times 10^8 \text{ s}^{-1}$$

Noting that

$$a_0 = \frac{\hbar}{\alpha \mu c}$$

we can also write

$$A_{21} = \frac{2^8}{3^8} \frac{\alpha^5 \mu c^2}{\hbar} = 6.26 \times 10^8 \text{ s}^{-1}$$

where μ is reduced electron mass

$$\mu = \frac{m_e m_p}{m_e + m_p}$$