4-12. Carry out the integral in equation (4.64) by completing the square. Show that the correct free particle kernel (i.e., the three dimensional version of equation 3.3) results.

$$K_0(\mathbf{x}_b, t_b, \mathbf{x}_a, t_a)$$
$$= \int \exp\left(\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{x}_b - \mathbf{x}_a)\right) \exp\left(-\frac{i}{\hbar}\frac{p^2}{2m}(t_b - t_a)\right) \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \quad (4.64)$$

Let
$$T = t_b - t_a$$
and
$$(X_1, X_2, X_3) = \mathbf{x}_b - \mathbf{x}_a$$

In component notation
$$\mathbf{p}\cdot(\mathbf{x}_b - \mathbf{x}_a) = p_1 X_1 + p_2 X_2 + p_3 X_3$$
and
$$p^2 = p_1^2 + p_2^2 + p_3^2$$

Rewrite equation (4.64) in component notation.

$$K_0(\mathbf{x}_b, t_b, \mathbf{x}_a, t_a) = \frac{1}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} \exp\left(\frac{i}{\hbar}p_1 X_1 - \frac{i}{\hbar}\frac{p_1^2}{2m}T\right)$$
$$\times \exp\left(\frac{i}{\hbar}p_2 X_2 - \frac{i}{\hbar}\frac{p_2^2}{2m}T\right) \exp\left(\frac{i}{\hbar}p_3 X_3 - \frac{i}{\hbar}\frac{p_3^2}{2m}T\right) dp_1\, dp_2\, dp_3$$

Let
$$A = -\frac{i}{2m\hbar}T$$
$$B = \frac{i}{\hbar}X_k$$

Then
$$\int_{-\infty}^{\infty} \exp(Ap_k^2 + Bp_k)\, dp_k = \left(-\frac{\pi}{A}\right)^{1/2} \exp\left(-\frac{B^2}{4A}\right)$$
$$= \left(-\frac{2\pi i m \hbar}{T}\right)^{1/2} \exp\left(\frac{imX_k^2}{2\hbar T}\right) \quad (1)$$

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Hence

$$K_0(\mathbf{x}_b, t_b, \mathbf{x}_a, t_a) = \frac{1}{(2\pi\hbar)^3} \prod_{k=1}^{3} \left(-\frac{2\pi im\hbar}{T}\right)^{1/2} \exp\left(\frac{imX_k^2}{2\hbar T}\right)$$

Combine coefficients and convert to vector notation.

$$K_0(\mathbf{x}_b, t_b, \mathbf{x}_a, t_a) = \left(-\frac{im}{2\pi\hbar(t_b - t_a)}\right)^{3/2} \exp\left(\frac{im(\mathbf{x}_b - \mathbf{x}_a)^2}{2\hbar(t_b - t_a)}\right) \qquad (2)$$