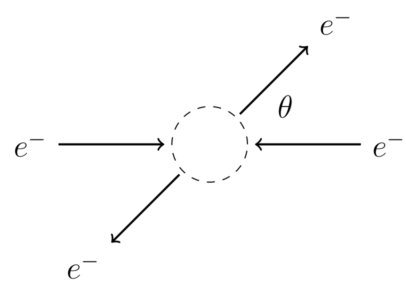
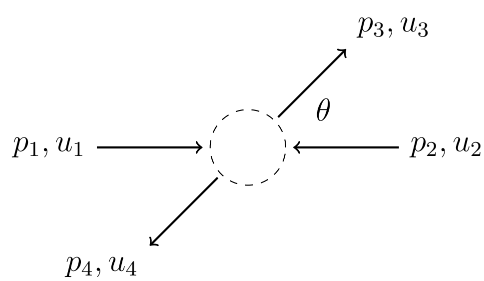
MOLLER SCATTERING

Moller scattering is the interaction of two electrons.



Here is the same diagram with momentum and spinor labels.



In center of mass coordinates the momentum vectors are

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \quad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} \quad p_3 = \begin{pmatrix} E \\ p\sin\theta\cos\phi \\ p\sin\theta\sin\phi \\ p\cos\theta \end{pmatrix} \quad p_4 = \begin{pmatrix} E \\ -p\sin\theta\cos\phi \\ -p\sin\theta\sin\phi \\ -p\cos\theta \end{pmatrix}$$

where $p = \sqrt{E^2 - m^2}$. The spinors are

$$u_{11} = \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix} \quad u_{21} = \begin{pmatrix} E+m \\ 0 \\ -p \\ 0 \end{pmatrix} \quad u_{31} = \begin{pmatrix} E+m \\ 0 \\ p_3^z \\ p_3^x + ip_3^y \end{pmatrix} \quad u_{41} = \begin{pmatrix} E+m \\ 0 \\ p_4^z \\ p_4^x + ip_4^y \end{pmatrix}$$

$$u_{12} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix} \quad u_{22} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ p \end{pmatrix} \quad u_{32} = \begin{pmatrix} 0 \\ E+m \\ p_3^x - ip_3^y \\ -p_3^z \end{pmatrix} \quad u_{42} = \begin{pmatrix} 0 \\ E+m \\ p_4^x - ip_4^y \\ -p_4^z \end{pmatrix}$$

The last digit in a spinor subscript is 1 for spin up and 2 for spin down. Note that the spinors are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^4$ will be used where needed.

This is the probability density for Moller scattering. The formula is from Feynman diagrams.

$$|\mathcal{M}(s_1,s_2,s_3,s_4)|^2 = \frac{e^4}{N}\left|\frac{1}{t}(\bar{u}_3\gamma^\mu u_1)(\bar{u}_4\gamma_\mu u_2) - \frac{1}{u}(\bar{u}_4\gamma^\nu u_1)(\bar{u}_3\gamma_\nu u_2)\right|^2$$

Symbol s_j selects the spin (up or down) of spinor j. Symbol e is electron charge. Symbols t and u are Mandelstam variables $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$.

Let

$$a_1 = (\bar{u}_3\gamma^\mu u_1)(\bar{u}_4\gamma_\mu u_2) \qquad a_2 = (\bar{u}_4\gamma^\nu u_1)(\bar{u}_3\gamma_\nu u_2)$$

Then

$$\begin{aligned}
|\mathcal{M}(s_1,s_2,s_3,s_4)|^2 &= \frac{e^4}{N}\left|\frac{a_1}{t} - \frac{a_2}{u}\right|^2 \\
&= \frac{e^4}{N}\left(\frac{a_1}{t} - \frac{a_2}{u}\right)\left(\frac{a_1}{t} - \frac{a_2}{u}\right)^* \\
&= \frac{e^4}{N}\left(\frac{a_1a_1^*}{t^2} - \frac{a_1a_2^*}{tu} - \frac{a_1^*a_2}{tu} + \frac{a_2a_2^*}{u^2}\right)
\end{aligned}$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is computed by summing $|\mathcal{M}|^2$ over all spin states and dividing by the number of inbound states. There are four inbound states.

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{s_1=1}^{2}\sum_{s_2=1}^{2}\sum_{s_3=1}^{2}\sum_{s_4=1}^{2}|\mathcal{M}(s_1,s_2,s_3,s_4)|^2 \\
&= \frac{e^4}{4}\sum_{s_1=1}^{2}\sum_{s_2=1}^{2}\sum_{s_3=1}^{2}\sum_{s_4=1}^{2}\frac{1}{N}\left(\frac{a_1a_1^*}{t^2} - \frac{a_1a_2^*}{tu} - \frac{a_1^*a_2}{tu} + \frac{a_2a_2^*}{u^2}\right)
\end{aligned}$$

Use the Casimir trick to replace sums over spins with matrix products.

$$\begin{aligned}
f_{11} &= \frac{1}{N}\sum_{\text{spins}} a_1a_1^* = \operatorname{Tr}\Big((\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu\Big)\operatorname{Tr}\Big((\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\Big) \\
f_{12} &= \frac{1}{N}\sum_{\text{spins}} a_1a_2^* = \operatorname{Tr}\Big((\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\Big) \\
f_{22} &= \frac{1}{N}\sum_{\text{spins}} a_2a_2^* = \operatorname{Tr}\Big((\not{p}_4 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu\Big)\operatorname{Tr}\Big((\not{p}_3 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\Big)
\end{aligned}$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left(\frac{f_{11}}{t^2} - \frac{f_{12}}{tu} - \frac{f_{12}^*}{tu} + \frac{f_{22}}{u^2}\right)$$

Run “moller-scattering-1.txt” to verify the Casimir trick.

These formulas compute probability densities from dot products. Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$.

$$\begin{aligned}
f_{11} &= 32(p_1 \cdot p_2)(p_3 \cdot p_4) + 32(p_1 \cdot p_4)(p_2 \cdot p_3) - 32m^2(p_1 \cdot p_3) - 32m^2(p_2 \cdot p_4) + 64m^4 \\
f_{12} &= -32(p_1 \cdot p_2)(p_3 \cdot p_4) + 16m^2(p_1 \cdot p_2) + 16m^2(p_1 \cdot p_3) + 16m^2(p_1 \cdot p_4) \\
&\qquad + 16m^2(p_2 \cdot p_3) + 16m^2(p_2 \cdot p_4) + 16m^2(p_3 \cdot p_4) - 32m^4 \\
f_{22} &= 32(p_1 \cdot p_2)(p_3 \cdot p_4) + 32(p_1 \cdot p_3)(p_2 \cdot p_4) - 32m^2(p_1 \cdot p_4) - 32m^2(p_2 \cdot p_3) + 64m^4
\end{aligned}$$

In Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$ the formulas are

$$\begin{aligned}
f_{11} &= 8s^2 + 8u^2 - 64sm^2 - 64um^2 + 192m^4 \\
f_{12} &= -8s^2 + 64sm^2 - 96m^4 \\
f_{22} &= 8s^2 + 8t^2 - 64sm^2 - 64tm^2 + 192m^4
\end{aligned}$$

When $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$\begin{aligned}
f_{11} &= 8s^2 + 8u^2 \\
f_{12} &= -8s^2 \\
f_{22} &= 8s^2 + 8t^2
\end{aligned}$$

For $m = 0$ the Mandelstam variables are

$$\begin{aligned}
s &= 4E^2 \\
t &= -2E^2(1 - \cos\theta) = -4E^2 \sin^2(\theta/2) \\
u &= -2E^2(1 + \cos\theta) = -4E^2 \cos^2(\theta/2)
\end{aligned}$$

The corresponding expected probability density is

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left(\frac{8s^2 + 8u^2}{t^2} + \frac{16s^2}{tu} + \frac{8s^2 + 8t^2}{u^2} \right) \\
&= 2e^4 \left(\frac{s^2 + u^2}{t^2} + \frac{2s^2}{tu} + \frac{s^2 + t^2}{u^2} \right) \\
&= 2e^4 \left(\frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{8}{\sin^2\theta} + \frac{1 + \sin^4(\theta/2)}{\cos^4(\theta/2)} \right)
\end{aligned}$$

Run "moller-scattering-2.txt" to verify the formulas on this page.

In the center of mass frame the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{64\pi^2 s} = \frac{\alpha^2}{8E^2}\left(\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{8}{\sin^2\theta} + \frac{1+\sin^4(\theta/2)}{\cos^4(\theta/2)}\right)$$

We can integrate $d\sigma$ to obtain a normalized probability density function.

Let

$$I(\xi) = 2\pi\int_\alpha^\xi \frac{d\sigma}{d\Omega}\sin\theta\,d\theta, \qquad \alpha \le \xi \le \pi-\alpha$$

for some $\alpha > 0$. The support region is restricted because $d\sigma$ is undefined for $\theta = 0$ and $\theta = \pi$.

The cumulative distribution function is

$$F(\theta) = \frac{I(\theta)}{I(\pi-\alpha)}, \qquad \alpha \le \theta \le \pi-\alpha$$

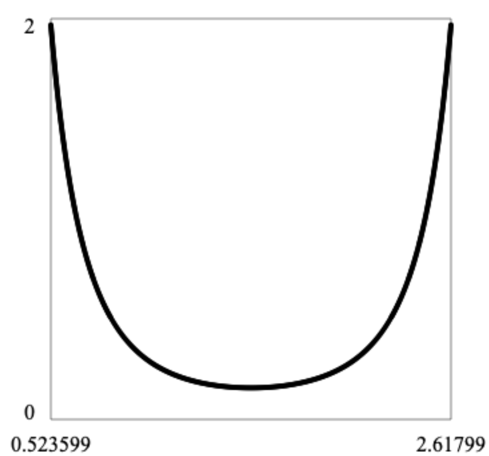
Hence

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{2\pi}{I(\pi-\alpha)}\left(\frac{d\sigma}{d\Omega}\right)\sin\theta, \quad \alpha \le \theta \le \pi-\alpha$$

Run "moller-scattering-3.txt" to draw a graph of $f(\theta)$ for $\alpha = \pi/6 = 30°$.



The following table shows the probability distribution for 30° bins ($\alpha = \pi/6 = 30°$).

θ_1	θ_2	$P(\theta_1 \le \theta \le \theta_2)$
0°	30°	–
30°	60°	0.40
60°	90°	0.10
90°	120°	0.10
120°	150°	0.40
150°	180°	–

Here are a few notes about how the scripts work.

In component notation, the trace operators of the Casimir trick become sums over a repeated index, in this case α.

$$
\begin{aligned}
f_{11} &= \left((\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha\right)\left((\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha\right)\\
f_{12} &= (\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_2 + m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha\\
f_{22} &= \left((\not{p}_4 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha\right)\left((\not{p}_3 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha\right)
\end{aligned}
$$

To convert the above formulas to Eigenmath code, the γ tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply γ^μ by the metric tensor to lower the index.

$\gamma^{\beta\mu}{}_\rho \quad \rightarrow \quad$ `gammaT = transpose(gamma)`

$\gamma^\beta{}_{\mu\rho} \quad \rightarrow \quad$ `gammaL = transpose(dot(gmunu,gamma))`

Define the following 4×4 matrices.

$(\not{p}_1 + m) \quad \rightarrow \quad$ `X1 = pslash1 + m I`

$(\not{p}_2 + m) \quad \rightarrow \quad$ `X2 = pslash2 + m I`

$(\not{p}_3 + m) \quad \rightarrow \quad$ `X3 = pslash3 + m I`

$(\not{p}_4 + m) \quad \rightarrow \quad$ `X4 = pslash4 + m I`

Then for f_{11} we have the following Eigenmath code. The contract function sums over α.

$(\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \quad \rightarrow \quad$ `T1 = contract(dot(X3,gammaT,X1,gammaT),1,4)`

$(\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \quad \rightarrow \quad$ `T2 = contract(dot(X4,gammaL,X2,gammaL),1,4)`

Next, multiply then sum over repeated indices. The dot function sums over ν then the contract function sums over μ. The transpose makes the ν indices adjacent as required by the dot function.

$f_{11} = \mathrm{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu)\,\mathrm{Tr}(\cdots\gamma_\mu\cdots\gamma_\nu) \quad \rightarrow \quad$ `contract(dot(T1,transpose(T2)))`

Follow suit for f_{22}.

$(\not{p}_4 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \quad \rightarrow \quad$ `T1 = contract(dot(X4,gammaT,X1,gammaT),1,4)`

$(\not{p}_3 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \quad \rightarrow \quad$ `T2 = contract(dot(X3,gammaL,X2,gammaL),1,4)`

Then

$f_{22} = \mathrm{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu)\,\mathrm{Tr}(\cdots\gamma_\mu\cdots\gamma_\nu) \quad \rightarrow \quad$ `contract(dot(T1,transpose(T2)))`

The calculation of f_{12} begins with

$(\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_2 + m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha$

$\rightarrow \quad$ `T = contract(dot(X3,gammaT,X1,gammaT,X4,gammaL,X2,gammaL),1,6)`

Then sum over repeated indices μ and ν.

$f_{12} = \mathrm{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu\cdots\gamma_\mu\cdots\gamma_\nu) \quad \rightarrow \quad$ `contract(contract(T,1,3))`