

9-3. Prove that the relation $\phi_{\mathbf{k}} = 4\pi\rho_{\mathbf{k}}/k^2$ simply means that $\phi_{\mathbf{k}}$ at any instant is the Coulomb potential from the charges at that instant, so that, for example, if ρ comes from a number of charges q_i at distances R_i from a point, the potential at the point is $\phi = \sum_i q_i/R_i$.

Integrate using polar coordinates.

$$\int_0^{2\pi} \int_0^\pi \int_0^\infty \frac{4\pi\rho_{\mathbf{k}}(t)}{k^2} \exp(ikR) k^2 \sin\theta dk d\theta d\phi = \frac{16\pi^2 i\rho_{\mathbf{k}}(t)}{R} \quad (1)$$

The result is a Coulomb potential for $\rho_{\mathbf{k}}(t) \propto -iq$.

The following integrals show how (1) is obtained.

$$\int_0^\infty \exp(-ax) dx = \frac{1}{a} \quad \int_0^\pi \sin\theta d\theta = 2 \quad \int_0^{2\pi} d\phi = 2\pi$$

Note: For multiple charges q_i we have

$$\begin{aligned} \rho(\mathbf{r}, t) &= \sum_i q_i \delta(R_i) \\ \phi(\mathbf{r}, t) &= \sum_i \frac{q_i}{R_i} \end{aligned}$$