QED field momentum

For a free field show that

$$\frac{1}{4\pi c}\int \mathbf{E}\times\mathbf{B}\,d^3\mathbf{r} = i\int \mathbf{k}\,(\mathbf{a}^*_{\mathbf{k}}\cdot\dot{\mathbf{a}}_{\mathbf{k}})\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

This is the electric field vector.

$$\mathbf{E}(\mathbf{r},t) = \int \left(-i\mathbf{k}\phi_{\mathbf{k}} - \sqrt{4\pi}\,\dot{\mathbf{a}}_{\mathbf{k}}\right)\exp(i\mathbf{k}\cdot\mathbf{r})\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

Set $\phi_{\mathbf{k}} = 0$ for no charges.

$$\mathbf{E}(\mathbf{r},t) = -\sqrt{4\pi}\int \dot{\mathbf{a}}_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{r})\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

This is the magnetic field vector.

$$\mathbf{B}(\mathbf{r},t) = \sqrt{4\pi}ic\int (\mathbf{k}\times\mathbf{a}_{\mathbf{k}})\exp(i\mathbf{k}\cdot\mathbf{r})\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

Take the complex conjugate of $\mathbf{B}$.

$$\mathbf{B}(\mathbf{r},t) = -\sqrt{4\pi}ic\int (\mathbf{k}\times\mathbf{a}^*_{\mathbf{k}})\exp(-i\mathbf{k}\cdot\mathbf{r})\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

Hence

$$\mathbf{E}\times\mathbf{B} = 4\pi ic\iint \dot{\mathbf{a}}_{\mathbf{k}}\times(\mathbf{k}'\times\mathbf{a}^*_{\mathbf{k}'})\exp\bigl(i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}\bigr)\,\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{d^3\mathbf{k}'}{(2\pi)^3}$$

Integrate over $\mathbf{r}$ to change the exponential to a delta function.

$$\int \mathbf{E}\times\mathbf{B}\,d^3\mathbf{r} = 4\pi ic\iint \dot{\mathbf{a}}_{\mathbf{k}}\times(\mathbf{k}'\times\mathbf{a}^*_{\mathbf{k}'})(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\,\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{d^3\mathbf{k}'}{(2\pi)^3}$$

The delta function vanishes except for $\mathbf{k}=\mathbf{k}'$.

$$\int \mathbf{E}\times\mathbf{B}\,d^3\mathbf{r} = 4\pi ic\int \dot{\mathbf{a}}_{\mathbf{k}}\times(\mathbf{k}\times\mathbf{a}_{\mathbf{k}})\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

By the triple cross product formula

$$\int \mathbf{E}\times\mathbf{B}\,d\mathbf{r} = 4\pi ic\int \bigl((\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{a}^*_{\mathbf{k}})\mathbf{k} - (\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{k})\mathbf{a}^*_{\mathbf{k}}\bigr)\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

By orthogonality of $\mathbf{E}$ and $\mathbf{k}$ the $\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{k}$ term vanishes hence

$$\int \mathbf{E}\times\mathbf{B}\,d\mathbf{r} = 4\pi ic\int (\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{a}^*_{\mathbf{k}})\mathbf{k}\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$