

3-12. If the wave function for a harmonic oscillator is (at $t = 0$)

$$\psi(x, 0) = \exp\left(-\frac{m\omega}{2\hbar}(x - a)^2\right) \quad (3.67)$$

then using equation (3.42) and the results of problem 3-8, show that

$$\begin{aligned} \psi(x, T) = & \\ & \exp\left(-\frac{i\omega T}{2} - \frac{m\omega}{2\hbar}(x^2 - 2ax \exp(-i\omega T) + a^2 \cos(\omega T) \exp(-i\omega T))\right) \end{aligned} \quad (3.68)$$

and find the probability distribution $|\psi|^2$.

From problem 3-8

$$\begin{aligned} K(x, T, x_c, 0) = & \\ & \left(\frac{m\omega}{2\pi i \hbar \sin(\omega T)}\right)^{1/2} \exp\left(\frac{im\omega}{2\hbar \sin(\omega T)}((x^2 + x_c^2) \cos(\omega T) - 2xx_c)\right) \end{aligned} \quad (2)$$

From equation (3.42)

$$\psi(x, T) = \int_{-\infty}^{\infty} K(x, T, x_c, 0) \psi(x_c, 0) dx_c \quad (3)$$

Substitute (3.68) and (2) into (3) to obtain

$$\begin{aligned} \psi(x, T) = & \left(\frac{m\omega}{2\pi i \hbar \sin(\omega T)}\right)^{1/2} \\ & \times \int_{-\infty}^{\infty} \exp\left(\frac{im\omega}{2\hbar \sin(\omega T)}((x^2 + x_c^2) \cos(\omega T) - 2xx_c)\right) \\ & \times \exp\left(-\frac{m\omega}{2\hbar}(x_c - a)^2\right) dx_c \end{aligned}$$

Rewrite as

$$\psi(x, T) = \left(\frac{m\omega}{2\pi i \hbar \sin(\omega T)}\right)^{1/2} \int_{-\infty}^{\infty} \exp(Ax_c^2 + Bx_c + C) dx_c \quad (4)$$

where

$$\begin{aligned} A &= \frac{m\omega}{2\hbar} \left(\frac{i \cos(\omega T)}{\sin(\omega T)} - 1 \right) = \frac{im\omega \exp(i\omega T)}{2\hbar \sin(\omega T)} \\ B &= \frac{m\omega}{\hbar} \left(a - \frac{ix}{\sin(\omega T)} \right) \\ C &= \frac{m\omega}{2\hbar} \left(\frac{ix^2 \cos(\omega T)}{\sin(\omega T)} - a^2 \right) \end{aligned}$$

Solve the integral.

$$\int_{-\infty}^{\infty} \exp(Ax_c^2 + Bx_c + C) = \left(-\frac{\pi}{A} \right)^{1/2} \exp \left(-\frac{B^2}{4A} + C \right) \quad (5)$$

where

$$-\frac{\pi}{A} = \frac{2\pi i\hbar \sin(\omega T)}{m\omega \exp(i\omega T)} \quad (6)$$

and

$$-\frac{B^2}{4A} + C = -\frac{m\omega}{2\hbar} (x^2 - 2ax \exp(-i\omega T) + a^2 \cos(\omega T) \exp(-i\omega T)) \quad (7)$$

Note that

$$\frac{m\omega}{2\pi i\hbar \sin(\omega T)} \times \frac{2\pi i\hbar \sin(\omega T)}{m\omega \exp(i\omega T)} = \exp(-i\omega T) \quad (8)$$

from equation (4) from equation (6)

Substitute the solved integral (5) into (3) to obtain

$$\begin{aligned} \psi(x, T) &= \exp \left(-\frac{i\omega T}{2} \right) \\ &\times \exp \left(-\frac{m\omega}{2\hbar} (x^2 - 2ax \exp(-i\omega T) + a^2 \cos(\omega T) \exp(-i\omega T)) \right) \end{aligned}$$