Stern-Gerlach 2

From the previous section we have the following Schrödinger equations for a Stern-Gerlach experiment.

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi_1 + \frac{e\hbar}{2m}(B_0+\alpha z)\psi_1 &= i\hbar\frac{\partial}{\partial t}\psi_1 \\
-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi_2 - \frac{e\hbar}{2m}(B_0+\alpha z)\psi_2 &= i\hbar\frac{\partial}{\partial t}\psi_2
\end{aligned}
$$

We now seek solutions for ψ_1 and ψ_2.

An online paper[1] provides the following solutions.

$$
\begin{aligned}
\psi_1 &= \operatorname{Ai}\left[\left(\frac{\alpha e}{2\hbar}\right)^{1/3}\frac{\alpha e\hbar}{8m^2}t^2 + \left(\frac{\alpha e}{2\hbar}\right)^{1/3} z\right]\exp\left(-\frac{ie\alpha zt}{4m}\right)\exp\left(-\frac{ieB_0 t}{2m}\right) \\
\psi_2 &= \operatorname{Ai}\left[\left(\frac{\alpha e}{2\hbar}\right)^{1/3}\frac{\alpha e\hbar}{8m^2}t^2 - \left(\frac{\alpha e}{2\hbar}\right)^{1/3} z\right]\exp\left(+\frac{ie\alpha zt}{4m}\right)\exp\left(+\frac{ieB_0 t}{2m}\right)
\end{aligned}
$$

Let us try verifying ψ_1 and ψ_2 with a Taylor series expansion of $\operatorname{Ai}(x)$.

We have

$$
\operatorname{Ai}(x) \approx \sum_{k=0}^{n} a_k x^k
$$

where

$$
\begin{aligned}
a_0 &= \frac{1}{3^{2/3}\Gamma(2/3)} \\
a_1 &= -\frac{1}{3^{1/3}\Gamma(1/3)} \\
a_2 &= 0 \\
a_{n+3} &= \frac{a_n}{(n+3)(n+2)}
\end{aligned}
$$

For $n = 9$ we obtain

$$
\begin{aligned}
\operatorname{Ai}(x) \approx 2.73941\times 10^{-5}x^9 - 0.000513531x^7 + 0.00197238x^6 \\
- 0.0215683x^4 + 0.0591713x^3 - 0.258819x + 0.355028
\end{aligned}
$$

Calculate ψ_1 and ψ_2 using the $\operatorname{Ai}(x)$ approximation.

To verify ψ_1 and ψ_2, calculate departures from equality of the Schrödinger equations.

$$
\begin{aligned}
\epsilon_1 &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi_1 + \frac{e\hbar}{2m}(B_0+\alpha z)\psi_1 - i\hbar\frac{\partial}{\partial t}\psi_1 \\
\epsilon_2 &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi_2 - \frac{e\hbar}{2m}(B_0+\alpha z)\psi_2 - i\hbar\frac{\partial}{\partial t}\psi_2
\end{aligned}
$$

[1] "Construction of Exact Solutions for the Stern-Gerlach Effect" by Bulnes and Oliveira.

After calculating ϵ_1 and ϵ_2, cancel exponentials and zero out factors t^n and z^m where $n \geq 8$ and $m \geq 5$.

The results are

$$\begin{aligned}
\epsilon_1 = {}& -1.69407 \times \boxed{10^{-20}} \frac{\alpha^{16/3} e^{16/3} \hbar^{8/3} t^6 z^2}{m^7} + 1.69407 \times \boxed{10^{-21}} \frac{\alpha^5 e^5 \hbar^3 t^6 z}{m^7} \\
& - 1.35525 \times \boxed{10^{-19}} \frac{\alpha^{13/3} e^{13/3} \hbar^{5/3} t^4 z^3}{m^5} - 4.33681 \times \boxed{10^{-19}} \frac{\alpha^{10/3} e^{10/3} \hbar^{2/3} t^2 z^4}{m^3} \\
& - 3.46945 \times \boxed{10^{-18}} \frac{\alpha^{7/3} e^{7/3} \hbar^{5/3} t^2 z}{m^3} - 1.38778 \times \boxed{10^{-17}} \frac{\alpha^{4/3} e^{4/3} \hbar^{2/3} z^2}{m} \\
& - 1.0842 \times \boxed{10^{-19}} \frac{i\alpha^{13/3} e^{13/3} \hbar^{8/3} t^5 z}{m^6} + 8.67362 \times \boxed{10^{-19}} \frac{i\alpha^{10/3} e^{10/3} \hbar^{5/3} t^3 z^2}{m^4}
\end{aligned}$$

and

$$\begin{aligned}
\epsilon_2 = {}& -1.69407 \times \boxed{10^{-20}} \frac{\alpha^{16/3} e^{16/3} \hbar^{8/3} t^6 z^2}{m^7} - 1.69407 \times \boxed{10^{-21}} \frac{\alpha^5 e^5 \hbar^3 t^6 z}{m^7} \\
& + 1.35525 \times \boxed{10^{-19}} \frac{\alpha^{13/3} e^{13/3} \hbar^{5/3} t^4 z^3}{m^5} - 4.33681 \times \boxed{10^{-19}} \frac{\alpha^{10/3} e^{10/3} \hbar^{2/3} t^2 z^4}{m^3} \\
& + 3.46945 \times \boxed{10^{-18}} \frac{\alpha^{7/3} e^{7/3} \hbar^{5/3} t^2 z}{m^3} - 1.38778 \times \boxed{10^{-17}} \frac{\alpha^{4/3} e^{4/3} \hbar^{2/3} z^2}{m} \\
& + 1.0842 \times \boxed{10^{-19}} \frac{i\alpha^{13/3} e^{13/3} \hbar^{8/3} t^5 z}{m^6} + 8.67362 \times \boxed{10^{-19}} \frac{i\alpha^{10/3} e^{10/3} \hbar^{5/3} t^3 z^2}{m^4}
\end{aligned}$$

The numerical values are round off errors hence $\epsilon_1 = \epsilon_2 = 0$ and ψ_1 and ψ_2 are confirmed as solutions.

Eigenmath script

Notes

1. Wavefunctions ψ_1 and ψ_2 given above are unnormalized and dimensionless. Recall that

   $$\int \psi^* \psi \, dx = 1$$

   Hence ψ_1 and ψ_2 must be multiplied by a normalization factor with dimension inverse square root of length to cancel with dx.

2. In SI units

   $$\begin{aligned}
   \hbar &= [\mathrm{J\,s}] \\
   e &= [\mathrm{A\,s}] \\
   B_0 &= [\mathrm{N\,A^{-1}\,m^{-1}}] \\
   \alpha &= [\mathrm{N\,A^{-1}\,m^{-2}}]
   \end{aligned}$$

Hence

$$\frac{e\hbar}{2m}(B_0 + \alpha z) = [\mathrm{J}]$$

$$\left(\frac{\alpha e}{2\hbar}\right)^{1/3} = [\mathrm{m}^{-1}]$$

$$\frac{\alpha e\hbar}{8m^2} = [\mathrm{m\,s}^{-2}]$$

As required by the Airy function, these products are dimensionless.

$$\left(\frac{\alpha e}{2\hbar}\right)^{1/3} \frac{\alpha e\hbar}{8m^2}\, t^2 = [1]$$

$$\left(\frac{\alpha e}{2\hbar}\right)^{1/3} z = [1]$$

As required by the exponential function, these products are dimensionless.

$$\frac{e\alpha z t}{4m} = [1]$$

$$\frac{eB_0 t}{2m} = [1]$$