Stern-Gerlach 2

From the previous section we have the Schrödinger equations

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi_1 + \mu_B(B_0 + \alpha z)\psi_1 = i\hbar\frac{\partial}{\partial t}\psi_1$$
$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi_2 - \mu_B(B_0 + \alpha z)\psi_2 = i\hbar\frac{\partial}{\partial t}\psi_2$$

We now seek eigenfunctions for ψ_1 and ψ_2.

The paper "Construction of Exact Solutions for the Stern-Gerlach Effect" provides the following solutions.

$$\psi_1 = \operatorname{Ai}\left[\left(\frac{\mu_B\alpha m}{\hbar^2}\right)^{1/3}\left(\frac{\mu_B\alpha}{4m}\right)t^2 + \left(\frac{\mu_B\alpha m}{\hbar^2}\right)^{1/3} z\right]\exp\left(-\frac{i\alpha\mu_B z t}{2\hbar}\right)\exp\left(-\frac{\mu_B B_0 t}{\hbar}\right)$$
$$\psi_2 = \operatorname{Ai}\left[\left(\frac{\mu_B\alpha m}{\hbar^2}\right)^{1/3}\left(\frac{\mu_B\alpha}{4m}\right)t^2 - \left(\frac{\mu_B\alpha m}{\hbar^2}\right)^{1/3} z\right]\exp\left(\frac{i\alpha\mu_B z t}{2\hbar}\right)\exp\left(\frac{\mu_B B_0 t}{\hbar}\right)$$

Let us construct polynomial approximations for ψ_1 and ψ_2.

For the Airy function we have

$$\operatorname{Ai}(x) \approx \sum_{k=0}^{n} a_k x^k$$

where

$$a_0 = \frac{1}{3^{2/3}\Gamma(2/3)}$$
$$a_1 = -\frac{1}{3^{1/3}\Gamma(1/3)}$$
$$a_2 = 0$$
$$a_{n+1} = \frac{a_n}{(n+3)(n+2)}$$

For $n = 9$ we obtain

$$\operatorname{Ai}(x) \approx 2.73941 \times 10^{-5}x^9 - 0.000513531x^7 + 0.00197238x^6 - 0.0215683x^4 + 0.0591713x^3 - 0.258819x + 0.355028$$

Now calculate ψ_1 and ψ_2 using the $\operatorname{Ai}(x)$ approximation.

Calculate departures from equality of the Schrödinger equations for ψ_1 and ψ_2.

$$\epsilon_1 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi_1 + \mu_B(B_0 + \alpha z)\psi_1 - i\hbar\frac{\partial}{\partial t}\psi_1$$
$$\epsilon_2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi_2 - \mu_B(B_0 + \alpha z)\psi_2 - i\hbar\frac{\partial}{\partial t}\psi_2$$

After calculating ϵ_1 and ϵ_2, zero-out factors t^n and z^m where $n \geq 8$ and $m \geq 5$.

The results are

$$\begin{aligned}\epsilon_1 = &\boxed{2.1684 \times 10^{-19}} \frac{\alpha^{16/3}\mu_B^{16/3}t^6z^2}{\hbar^{8/3}m^{5/3}} + \boxed{2.1684 \times 10^{-19}} \frac{\alpha^5\mu_B^5t^6z}{\hbar^2m^2} \\ &+ \boxed{8.67362 \times 10^{-19}} \frac{\alpha^{13/3}\mu_B^{13/3}t^4z^3}{\hbar^{8/3}m^{2/3}} + \boxed{1.73472 \times 10^{-18}} \frac{\alpha^4\mu_B^4t^4z^2}{\hbar^2m} \\ &- \boxed{1.73472 \times 10^{-18}} \frac{\alpha^{10/3}\mu_B^{10/3}t^4}{\hbar^{2/3}m^{5/3}} - \boxed{1.38778 \times 10^{-17}} \frac{\alpha^{7/3}\mu_B^{7/3}t^2z}{\hbar^{2/3}m^{2/3}} \\ &- \boxed{2.77556 \times 10^{-17}} \frac{\alpha^{4/3}m^{1/3}\mu_B^{4/3}z^2}{\hbar^{2/3}} + \boxed{8.67362 \times 10^{-19}} \frac{i\alpha^4\mu_B^4t^3z^4}{\hbar^3}\end{aligned}$$

and

$$\begin{aligned}\epsilon_2 = &\boxed{2.1684 \times 10^{-19}} \frac{\alpha^{16/3}\mu_B^{16/3}t^6z^2}{\hbar^{8/3}m^{5/3}} - \boxed{2.1684 \times 10^{-19}} \frac{\alpha^5\mu_B^5t^6z}{\hbar^2m^2} \\ &- \boxed{8.67362 \times 10^{-19}} \frac{\alpha^{13/3}\mu_B^{13/3}t^4z^3}{\hbar^{8/3}m^{2/3}} + \boxed{1.73472 \times 10^{-18}} \frac{\alpha^4\mu_B^4t^4z^2}{\hbar^2m} \\ &- \boxed{1.73472 \times 10^{-18}} \frac{\alpha^{10/3}\mu_B^{10/3}t^4}{\hbar^{2/3}m^{5/3}} + \boxed{1.38778 \times 10^{-17}} \frac{\alpha^{7/3}\mu_B^{7/3}t^2z}{\hbar^{2/3}m^{2/3}} \\ &- \boxed{2.77556 \times 10^{-17}} \frac{\alpha^{4/3}m^{1/3}\mu_B^{4/3}z^2}{\hbar^{2/3}} + \boxed{8.67362 \times 10^{-19}} \frac{i\alpha^4\mu_B^4t^3z^4}{\hbar^3}\end{aligned}$$

All of the numerical values are small relative to the coefficients in $\mathrm{Ai}(x)$ hence ψ_1 and ψ_2 are confirmed.