

Spin part 2

From the previous section we have for the z direction

$$\Pr(+) = |\langle z_+ | s \rangle|^2 = \frac{1}{2} + \frac{1}{2} \cos \theta$$

$$\Pr(-) = |\langle z_- | s \rangle|^2 = \frac{1}{2} - \frac{1}{2} \cos \theta$$

If $|s\rangle$ is not an eigenstate then the result of measuring $|s\rangle$ is random. For example, let $\theta = \pi/3$. Then $\cos \theta = \frac{1}{2}$ and the probabilities are

$$\Pr(+) = \frac{3}{4}$$

$$\Pr(-) = \frac{1}{4}$$

When results are unpredictable, a useful statistic is the expected value. From theory a “plus” result indicates a spin value of $+\frac{\hbar}{2}$ and a “minus” result indicates a spin value of $-\frac{\hbar}{2}$. Hence the expected value is

$$\left(+\frac{\hbar}{2}\right) \Pr(+) + \left(-\frac{\hbar}{2}\right) \Pr(-)$$

For state $|s\rangle$ such that $\theta = \pi/3$ the expected value in the z direction is

$$\left(+\frac{\hbar}{2}\right) \frac{3}{4} + \left(-\frac{\hbar}{2}\right) \frac{1}{4} = \frac{\hbar}{4}$$

Expected values can be computed directly from $|s\rangle$ by introducing the following matrices.

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Expected values for $|s\rangle$ are the following inner products.

$$\langle S_x \rangle = \langle s | S_x | s \rangle \quad \langle S_y \rangle = \langle s | S_y | s \rangle \quad \langle S_z \rangle = \langle s | S_z | s \rangle$$

Returning to the example $\theta = \pi/3$ we have

$$|s\rangle = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2}e^{i\phi} \end{pmatrix}$$

Hence the expected value in the z direction is

$$\langle S_z \rangle = \langle s | S_z | s \rangle = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2}e^{i\phi} \end{pmatrix}^\dagger \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2}e^{i\phi} \end{pmatrix} = \frac{\hbar}{4}$$