

3-9. Find the kernel for a particle in a constant field  $f$  where the Lagrangian is

$$L = \frac{m}{2}\dot{x}^2 + fx$$

From problem 2-3 we have

$$S(b, a) = \frac{m(x_b - x_a)^2}{2T} + \frac{fT(x_b + x_a)}{2} - \frac{f^2T^3}{24m}$$

where  $T = t_b - t_a$ .

From equation (3.51)

$$K(b, a) = F(T) \exp\left(\frac{iS(b, a)}{\hbar}\right) \quad (3.51')$$

It follows that

$$K(b, a) = F(T) \exp\left(\frac{im(x_b - x_a)^2}{2\hbar T} + \frac{ifT(x_b + x_a)}{2\hbar} - \frac{if^2T^3}{24\hbar m}\right) \quad (1)$$

We now proceed to compute  $F(T)$ . By equation (2.31) which is

$$K(b, a) = \int_{-\infty}^{\infty} K(b, c)K(c, a) dx_c$$

we have

$$K(b, a) = F(t_b - t_c)F(t_c - t_a) \int_{-\infty}^{\infty} \exp\left(\frac{iS(b, c)}{\hbar} + \frac{iS(c, a)}{\hbar}\right) dx_c$$

Reorganize as powers of  $x_c$ .

$$K(b, a) = F(t_b - t_c)F(t_c - t_a) \exp\left(-\frac{if^2(t_b - t_c)^3}{24\hbar m} - \frac{if^2(t_c - t_a)^3}{24\hbar m}\right) \times \int_{-\infty}^{\infty} \exp(Ax_c^2 + Bx_c + C) dx_c \quad (2)$$

where

$$A = \frac{im}{2\hbar} \left( \frac{1}{t_b - t_c} + \frac{1}{t_c - t_a} \right) \quad (3)$$

$$B = \frac{ifT}{2\hbar} - \frac{im}{\hbar} \left( \frac{x_b}{t_b - t_c} + \frac{x_a}{t_c - t_a} \right) \quad (4)$$

$$C = \frac{if}{2\hbar} (x_b(t_b - t_c) + x_a(t_c - t_a)) + \frac{im}{2\hbar} \left( \frac{x_b^2}{t_b - t_c} + \frac{x_a^2}{t_c - t_a} \right) \quad (5)$$

Note that the exponential involving  $f^2$  is independent of  $x_c$  and is factored out of the integrand in (2) by the distributive law.

Solve the integral in (2).

$$\begin{aligned} \int_{-\infty}^{\infty} \exp(Ax_c^2 + Bx_c + C) dx_c &= \left(-\frac{\pi}{A}\right)^{1/2} \exp\left(-\frac{B^2}{4A} + C\right) \\ &= \left(-\frac{2\pi\hbar(t_b - t_c)(t_c - t_a)}{imT}\right)^{1/2} \\ &\times \exp\left(\frac{im(x_b - x_a)^2}{2\hbar T} + \frac{ifT(x_b + x_a)}{2\hbar} - \frac{if^2T(t_b - t_c)(t_c - t_a)}{8\hbar m}\right) \end{aligned} \quad (6)$$

Substitute (6) into (2) to obtain

$$\begin{aligned} K(b, a) &= F(t_b - t_c)F(t_c - t_a) \exp\left(-\frac{if^2(t_b - t_c)^3}{24\hbar m} - \frac{if^2(t_c - t_a)^3}{24\hbar m}\right) \\ &\times \left(-\frac{2\pi\hbar(t_b - t_c)(t_c - t_a)}{imT}\right)^{1/2} \\ &\times \exp\left(\frac{im(x_b - x_a)^2}{2\hbar T} + \frac{ifT(x_b + x_a)}{2\hbar} - \frac{if^2T(t_b - t_c)(t_c - t_a)}{8\hbar m}\right) \end{aligned}$$

Note that

$$T^3 = (t_b - t_c)^3 + (t_c - t_a)^3 + 3T(t_b - t_c)(t_c - t_a) \quad (7)$$

Use (7) to combine exponentials involving  $f^2$ .

$$\begin{aligned} K(b, a) &= F(t_b - t_c)F(t_c - t_a) \\ &\times \left(-\frac{2\pi\hbar(t_b - t_c)(t_c - t_a)}{imT}\right)^{1/2} \\ &\times \exp\left(\frac{im(x_b - x_a)^2}{2\hbar T} + \frac{ifT(x_b + x_a)}{2\hbar} - \frac{if^2T^3}{24\hbar m}\right) \end{aligned} \quad (8)$$

Equating (1) with (8) cancels the exponentials and leaves

$$F(T) = F(t_b - t_c)F(t_c - t_a) \left(-\frac{2\pi\hbar(t_b - t_c)(t_c - t_a)}{imT}\right)^{1/2} \quad (9)$$

From problem 3-7, let

$$F(t) = \left( \frac{m}{2\pi i \hbar t} \right)^{1/2} g(t) \quad (10)$$

Substitute (10) into (9) to obtain

$$\begin{aligned} \left( \frac{m}{2\pi i \hbar T} \right)^{1/2} g(T) &= \left( \frac{m}{2\pi i \hbar (t_b - t_c)} \right)^{1/2} g(t_b - t_c) \\ &\times \left( \frac{m}{2\pi i \hbar (t_c - t_a)} \right)^{1/2} g(t_c - t_a) \left( -\frac{2\pi \hbar (t_b - t_c)(t_c - t_a)}{imT} \right)^{1/2} \end{aligned}$$

The coefficients cancel leaving

$$g(T) = g(t_b - t_c)g(t_c - t_a) \quad (11)$$

Hence

$$g(t) = 1$$

and

$$F(T) = \left( \frac{m}{2\pi i \hbar T} \right)^{1/2} \quad (12)$$

Substitute (12) into (1).

$$K(b, a) = \left( \frac{m}{2\pi i \hbar T} \right)^{1/2} \exp \left( \frac{im(x_b - x_a)^2}{2\hbar T} + \frac{ifT(x_b + x_a)}{2\hbar} - \frac{if^2T^3}{24\hbar m} \right)$$