

8-5. A transition element which employs the same wave function as both the initial and final states is called an expectation value. Thus the expectation value of F for the ground state Φ_0 of equation (8.83) is

$$\langle \Phi_0 | F | \Phi_0 \rangle = \int \cdots \int \Phi_0^* F \Phi_0 dQ_1 dQ_2 \cdots dQ_{N-1} \quad (8.84)$$

(The integral over complex variables is defined as equal to the corresponding integral over real normal coordinates Q_α^c and Q_α^s .) Show that the following expectation values are correct (for $\alpha \neq \beta$).

$$\begin{aligned} \langle \Phi_0^* | Q_\alpha | \Phi_0 \rangle &= \langle \Phi_0^* | Q_\alpha^* | \Phi_0 \rangle = 0 \\ \langle \Phi_0^* | Q_\alpha^2 | \Phi_0 \rangle &= \langle \Phi_0^* | Q_\alpha^{*2} | \Phi_0 \rangle = 0 \\ \langle \Phi_0^* | Q_\alpha^* Q_\alpha | \Phi_0 \rangle &= \frac{\hbar}{2\omega_\alpha} \langle \Phi_0^* | 1 | \Phi_0 \rangle \\ \langle \Phi_0^* | Q_\alpha^* Q_\beta | \Phi_0 \rangle &= 0, \quad \alpha \neq \beta \end{aligned} \quad (8.85)$$

From problem 8-4,

$$\Phi_0^* \Phi_0 = \exp \left(-\frac{1}{\hbar} \sum_{\alpha=1}^{N-1} \omega_\alpha ((Q_\alpha^c)^2 + (Q_\alpha^s)^2) \right)$$

For brevity in the following formulas, let

$$u = \Phi_0^* \Phi_0$$

We will use the following integrals.

$$\int_{-\infty}^{\infty} \exp(-ax^2 + b) dx = \sqrt{\frac{\pi}{a}} \exp(b) \quad (1)$$

$$\int_{-\infty}^{\infty} x \exp(-ax^2 + b) dx = 0 \quad (2)$$

$$\int_{-\infty}^{\infty} x^2 \exp(-ax^2 + b) dx = \frac{1}{2a} \sqrt{\frac{\pi}{a}} \exp(b) \quad (3)$$

Here are some specific examples with $a = \omega_1/\hbar$.

$$\int_{-\infty}^{\infty} u dQ_1^c = \left(\frac{\pi\hbar}{\omega_1}\right)^{1/2} u \exp\left(\frac{\omega_1}{\hbar}(Q_1^c)^2\right) \quad (4)$$

$$\int_{-\infty}^{\infty} Q_1^c u dQ_1^c = 0 \quad (5)$$

$$\int_{-\infty}^{\infty} (Q_1^c)^2 u dQ_1^c = \frac{\hbar}{2\omega_1} \left(\frac{\pi\hbar}{\omega_1}\right)^{1/2} u \exp\left(\frac{\omega_1}{\hbar}(Q_1^c)^2\right) \quad (6)$$

Note that multiplying u by an exponential cancels a factor, i.e.,

$$\exp(b) = u \exp\left(\frac{\omega_1}{\hbar}(Q_1^c)^2\right)$$

Compute the expectation value for Q_1 . Let

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{Q_1^c - iQ_1^s}{\sqrt{2}}\right) u dQ_1^c dQ_1^s$$

By integral (2), $I = 0$ hence by equation (8.84)

$$\langle \Phi_0^* | Q_1 | \Phi_0 \rangle = 0$$

Compute the expectation value for Q_1^* . Let

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{Q_1^c + iQ_1^s}{\sqrt{2}}\right) u dQ_1^c dQ_1^s$$

As above, $I = 0$ hence

$$\langle \Phi_0^* | Q_1^* | \Phi_0 \rangle = 0$$

Compute the expectation value for Q_1^2 . Let

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{Q_1^c - iQ_1^s}{\sqrt{2}}\right)^2 u dQ_1^c dQ_1^s$$

Rewrite as

$$\begin{aligned} I &= -i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_1^c Q_1^s u dQ_1^c dQ_1^s \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (Q_1^c)^2 u dQ_1^c dQ_1^s - \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (Q_1^s)^2 u dQ_1^c dQ_1^s \end{aligned}$$

The first integral vanishes by (2). The remaining integrals cancel hence

$$\langle \Phi_0^* | Q_1^2 | \Phi_0 \rangle = 0$$

Compute the expectation value for Q_1^{*2} . (As above except for a sign change.)

$$\begin{aligned} I = i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_1^c Q_1^s u \, dQ_1^c \, dQ_1^s \\ + \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (Q_1^c)^2 u \, dQ_1^c \, dQ_1^s - \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (Q_1^s)^2 u \, dQ_1^c \, dQ_1^s \end{aligned}$$

By the same arguments as Q_1^2

$$\langle \Phi_0^* | Q_1^{*2} | \Phi_0 \rangle = 0$$

Compute the expectation value for $Q_1^* Q_1$. Let

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{(Q_1^c)^2 + (Q_1^s)^2}{2} \right) u \, dQ_1^c \, dQ_1^s$$

Rewrite as

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} (Q_1^c)^2 u \, dQ_1^c \right) dQ_1^s + \frac{1}{2} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} (Q_1^s)^2 u \, dQ_1^s \right) dQ_1^c$$

By integral (3) with $a = \omega_1/\hbar$,

$$\begin{aligned} I = \frac{\hbar}{4\omega_1} \left(\frac{\pi\hbar}{\omega_1} \right)^{1/2} \exp \left(\frac{\omega_1}{\hbar} (Q_1^c)^2 \right) \int_{-\infty}^{\infty} u \, dQ_1^s \\ + \frac{\hbar}{4\omega_1} \left(\frac{\pi\hbar}{\omega_1} \right)^{1/2} \exp \left(\frac{\omega_1}{\hbar} (Q_1^s)^2 \right) \int_{-\infty}^{\infty} u \, dQ_1^c \end{aligned}$$

By integral (1),

$$\begin{aligned} I = \frac{\hbar}{4\omega_1} \frac{\pi\hbar}{\omega_1} \exp \left(\frac{\omega_1}{\hbar} (Q_1^c)^2 \right) \exp \left(\frac{\omega_1}{\hbar} (Q_1^s)^2 \right) u \\ + \frac{\hbar}{4\omega_1} \frac{\pi\hbar}{\omega_1} \exp \left(\frac{\omega_1}{\hbar} (Q_1^s)^2 \right) \exp \left(\frac{\omega_1}{\hbar} (Q_1^c)^2 \right) u \end{aligned}$$

Integrate over the remaining measure as in (8.84).

$$\begin{aligned} \langle \Phi_0^* | Q_1^* Q_1 | \Phi_0 \rangle &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} I dQ_2^c dQ_2^s \cdots dQ_{N-1}^c dQ_{N-1}^s \\ &= \frac{\hbar}{2\omega_1} \frac{\pi\hbar}{\omega_1} \prod_{k=2}^{N-1} \frac{\pi\hbar}{\omega_k} \quad (7) \end{aligned}$$

Note that the exponentials cancel with u .

Compute the normalization constant. By integral (1),

$$\langle \Phi_0^* | 1 | \Phi_0 \rangle = \prod_{k=1}^{N-1} \frac{\pi\hbar}{\omega_k} \quad (8)$$

Hence by (7) and (8),

$$\langle \Phi_0^* | Q_1^* Q_1 | \Phi_0 \rangle = \frac{\hbar}{2\omega_1} \langle \Phi_0^* | 1 | \Phi_0 \rangle$$

Compute the expectation value for $Q_1^* Q_2$. Let

$$I = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\frac{Q_1^c Q_2^c - Q_1^s Q_2^s - iQ_1^c Q_2^s - iQ_1^s Q_2^c}{2} \right) u dQ_1^c dQ_1^s dQ_2^c dQ_2^s$$

By integral (2) we have $I = 0$, hence by equation (8.84)

$$\langle \Phi_0^* | Q_1^* Q_2 | \Phi_0 \rangle = 0$$