6-6. Suppose the potential is that of a central force. Thus $V(\mathbf{r}) = V(r)$. Show that $v(\breve{\mathbf{p}})$ can be written as

$$v(\breve{\mathbf{p}}) = v(\breve{p}) = \frac{4\pi\hbar}{\breve{p}} \int_0^\infty \sin\left(\frac{\breve{p}r}{\hbar}\right) V(r) r\, dr \tag{6.45}$$

Suppose $v(r)$ is the Coulomb potential $-Ze^2/r$. In this case the integral for $v(\breve{p})$ is oscillatory at the upper limit. But convergence of the integral can be artificially forced by introducing the factor $e^{-\epsilon r}$ and then taking the limit of the result as $\epsilon \to 0$. Following through this calculation, show that the cross section corresponds to the Rutherford cross section

$$\frac{d\sigma_{\text{Ruth}}}{d\Omega} = \frac{4m^2 Z^2 e^4}{\breve{p}^4} \tag{6.46}$$

Consider equation (6.39).

$$v(\breve{\mathbf{p}}) = \int \exp\left(\frac{i\breve{\mathbf{p}}\cdot\mathbf{r}}{\hbar}\right) V(\mathbf{r}) d^3\mathbf{r} \tag{6.39}$$

Convert (6.39) to polar coordinates.

$$v(\breve{\mathbf{p}}) = \int_0^{2\pi} \int_0^{\pi} \int_0^\infty \exp\left(\frac{i\breve{p}r\cos\theta}{\hbar}\right) V(r) r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Integrate over ϕ.

$$v(\breve{\mathbf{p}}) = 2\pi \int_0^{\pi} \int_0^\infty \exp\left(\frac{i\breve{p}r\cos\theta}{\hbar}\right) V(r) r^2 \sin\theta\, dr\, d\theta$$

Convert the exponential to rectangular form.

$$v(\breve{\mathbf{p}}) = 2\pi \int_0^{\pi} \int_0^\infty \left(\cos\left(\frac{\breve{p}r\cos\theta}{\hbar}\right) + i\sin\left(\frac{\breve{p}r\cos\theta}{\hbar}\right)\right) V(r) r^2 \sin\theta\, dr\, d\theta$$

Integrate over θ.

$$v(\breve{\mathbf{p}}) = \frac{4\pi\hbar}{\breve{p}} \int_0^\infty \sin\left(\frac{\breve{p}r}{\hbar}\right) V(r) r\, dr$$

The above result is due to the integrals

$$\int_0^\pi \cos(A\cos\theta)\sin\theta\,d\theta = \frac{2\sin A}{A}$$
$$\int_0^\pi \sin(A\cos\theta)\sin\theta\,d\theta = 0$$

Given the Coulomb potential and integration trick we have

$$v(\breve{p}) = -\frac{4\pi\hbar Ze^2}{\breve{p}}\int_0^\infty \sin\left(\frac{\breve{p}r}{\hbar}\right)\exp(-\epsilon r)\,dr$$

Solve the integral.

$$v(\breve{p}) = -\frac{4\pi\hbar Ze^2}{\breve{p}}\frac{\breve{p}/\hbar}{(\breve{p}/\hbar)^2+\epsilon^2}$$

Let $\epsilon \to 0$.

$$v(\breve{p}) = -\frac{4\pi\hbar^2 Ze^2}{\breve{p}^2}$$

By equation (6.44)

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |v(\breve{p})|^2 = \frac{4m^2Z^2e^4}{\breve{p}^4}$$