Atomic transitions 3

In the previous section we obtained

$$c_b(t) = \frac{ieE_0}{m\hbar\omega}\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\frac{\sin\left(\frac{1}{2}(\omega_0-\omega)t\right)}{\omega_0-\omega}\exp\left(\tfrac{i}{2}(\omega_0-\omega)t\right) \tag{1}$$

Use the dipole approximation

$$\exp(i\mathbf{k}\cdot\mathbf{r}) = 1 + i\mathbf{k}\cdot\mathbf{r} + \cdots \approx 1$$

to obtain

$$\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle \approx \langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}|\psi_a\rangle$$

By the identity

$$\mathbf{p} = \frac{im}{\hbar}[H_0,\mathbf{r}] \tag{2}$$

we have

$$\begin{aligned}
\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}|\psi_a\rangle &= \frac{im}{\hbar}\langle\psi_b|\boldsymbol{\epsilon}\cdot[H_0,\mathbf{r}]|\psi_a\rangle \\
&= \frac{im}{\hbar}\langle\psi_b|\boldsymbol{\epsilon}\cdot H_0\mathbf{r}|\psi_a\rangle - \frac{im}{\hbar}\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{r}H_0|\psi_a\rangle \\
&= \frac{im}{\hbar}E_b\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{r}|\psi_a\rangle - \frac{im}{\hbar}E_a\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{r}|\psi_a\rangle \\
&= \frac{im}{\hbar}(E_b-E_a)\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{r}|\psi_a\rangle \\
&= im\omega_0\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{r}|\psi_a\rangle
\end{aligned}$$

Hence

$$\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle \approx im\omega_0\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{r}|\psi_a\rangle$$

Substitute into (1) to obtain

$$c_b(t) = -\frac{eE_0}{\hbar}\frac{\omega_0}{\omega}\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{r}|\psi_a\rangle\frac{\sin\left(\frac{1}{2}(\omega_0-\omega)t\right)}{\omega_0-\omega}\exp\left(\tfrac{i}{2}(\omega_0-\omega)t\right)$$

Verify dimensions.

$$c_b(t) \propto \frac{\overset{e}{\text{coulomb}}\quad\overset{E_0}{\text{newton coulomb}^{-1}}}{\underset{\text{joule second}}{\hbar}} \times \frac{\overset{\omega_0}{\text{second}^{-1}}}{\underset{\text{second}^{-1}}{\omega}} \times \frac{\overset{\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{r}|\psi_a\rangle}{\text{meter}}}{\underset{\text{second}^{-1}}{\omega_0-\omega}} = 1$$

Note that for an experiment with $\boldsymbol{\epsilon}\cdot\mathbf{r} = z$ we have

$$c_b(t) = -\frac{eE_0}{\hbar}\frac{\omega_0}{\omega}\langle\psi_b|z|\psi_a\rangle\frac{\sin\left(\frac{1}{2}(\omega_0-\omega)t\right)}{\omega_0-\omega}\exp\left(\tfrac{i}{2}(\omega_0-\omega)t\right)$$