

Stern-Gerlach experiment 1

From problem 4.74 of Griffiths and Schroeter.

The Hamiltonian for a Stern-Gerlach experiment is

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{e}{m}\mathbf{B} \cdot \mathbf{S}$$

Given vectors

$$\mathbf{B} = -\alpha x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (B_0 + \alpha z) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{S} = \frac{\hbar}{2} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}$$

we have the dot product

$$\frac{e}{m}\mathbf{B} \cdot \mathbf{S} = -\frac{e\hbar}{2m}\alpha x\sigma_x + \frac{e\hbar}{2m}(B_0 + \alpha z)\sigma_z$$

Let Ψ be the spin state vector

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

where ψ_1 and ψ_2 are wavefunctions for spin up and down.

Plug H and Ψ into the Schrödinger equation to obtain

$$-\frac{\hbar^2}{2m}\nabla^2\Psi - \frac{e\hbar}{2m}\alpha x\sigma_x\Psi + \frac{e\hbar}{2m}(B_0 + \alpha z)\sigma_z\Psi = i\hbar\frac{\partial}{\partial t}\Psi$$

Noting that

$$\sigma_x\Psi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi = \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix}$$

and

$$\sigma_z\Psi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi = \begin{pmatrix} \psi_1 \\ -\psi_2 \end{pmatrix}$$

we have

$$-\frac{\hbar^2}{2m}\nabla^2 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} - \frac{e\hbar}{2m}\alpha x \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix} + \frac{e\hbar}{2m}(B_0 + \alpha z) \begin{pmatrix} \psi_1 \\ -\psi_2 \end{pmatrix} = i\hbar\frac{\partial}{\partial t} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

In component form

$$\begin{aligned} -\frac{\hbar^2}{2m}\nabla^2\psi_1 - \boxed{\frac{e\hbar}{2m}\alpha x\psi_2} + \frac{e\hbar}{2m}(B_0 + \alpha z)\psi_1 &= i\hbar\frac{\partial}{\partial t}\psi_1 \\ -\frac{\hbar^2}{2m}\nabla^2\psi_2 - \boxed{\frac{e\hbar}{2m}\alpha x\psi_1} - \frac{e\hbar}{2m}(B_0 + \alpha z)\psi_2 &= i\hbar\frac{\partial}{\partial t}\psi_2 \end{aligned} \tag{1}$$

We now seek to eliminate the cross terms shown in boxes.

Let

$$\begin{aligned}\psi_1 &= \exp\left(-\frac{ieB_0t}{2m}\right)\phi_1 \\ \psi_2 &= \exp\left(+\frac{ieB_0t}{2m}\right)\phi_2\end{aligned}$$

Then

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\psi_1 &= \exp\left(-\frac{ieB_0t}{2m}\right)i\hbar\frac{\partial}{\partial t}\phi_1 + \frac{e\hbar}{2m}B_0\psi_1 \\ i\hbar\frac{\partial}{\partial t}\psi_2 &= \exp\left(+\frac{ieB_0t}{2m}\right)i\hbar\frac{\partial}{\partial t}\phi_2 - \frac{e\hbar}{2m}B_0\psi_2\end{aligned}\tag{2}$$

Substitute into (1) to obtain

$$\begin{aligned}-\frac{\hbar^2}{2m}\nabla^2\psi_1 - \frac{e\hbar}{2m}\alpha x\psi_2 + \frac{e\hbar}{2m}(B_0 + \alpha z)\psi_1 &= \exp\left(-\frac{ieB_0t}{2m}\right)i\hbar\frac{\partial}{\partial t}\phi_1 + \frac{e\hbar}{2m}B_0\psi_1 \\ -\frac{\hbar^2}{2m}\nabla^2\psi_2 - \frac{e\hbar}{2m}\alpha x\psi_1 - \frac{e\hbar}{2m}(B_0 + \alpha z)\psi_2 &= \exp\left(+\frac{ieB_0t}{2m}\right)i\hbar\frac{\partial}{\partial t}\phi_2 - \frac{e\hbar}{2m}B_0\psi_2\end{aligned}$$

The B_0 terms cancel.

$$\begin{aligned}-\frac{\hbar^2}{2m}\nabla^2\psi_1 - \frac{e\hbar}{2m}\alpha x\psi_2 + \frac{e\hbar}{2m}\alpha z\psi_1 &= \exp\left(-\frac{ieB_0t}{2m}\right)i\hbar\frac{\partial}{\partial t}\phi_1 \\ -\frac{\hbar^2}{2m}\nabla^2\psi_2 - \frac{e\hbar}{2m}\alpha x\psi_1 - \frac{e\hbar}{2m}\alpha z\psi_2 &= \exp\left(+\frac{ieB_0t}{2m}\right)i\hbar\frac{\partial}{\partial t}\phi_2\end{aligned}$$

Multiply through by the conjugate of the respective exponential (note $\psi \rightarrow \phi$).

$$\begin{aligned}-\frac{\hbar^2}{2m}\nabla^2\phi_1 - \exp\left(+\frac{ieB_0t}{m}\right)\frac{e\hbar}{2m}\alpha x\phi_2 + \frac{e\hbar}{2m}\alpha z\phi_1 &= i\hbar\frac{\partial}{\partial t}\phi_1 \\ -\frac{\hbar^2}{2m}\nabla^2\phi_2 - \exp\left(-\frac{ieB_0t}{m}\right)\frac{e\hbar}{2m}\alpha x\phi_1 - \frac{e\hbar}{2m}\alpha z\phi_2 &= i\hbar\frac{\partial}{\partial t}\phi_2\end{aligned}$$

We now simply discard the cross terms with the argument that the exponential factors oscillate causing the cross terms to vanish on average.

$$\begin{aligned}-\frac{\hbar^2}{2m}\nabla^2\phi_1 + \frac{e\hbar}{2m}\alpha z\phi_1 &= i\hbar\frac{\partial}{\partial t}\phi_1 \\ -\frac{\hbar^2}{2m}\nabla^2\phi_2 - \frac{e\hbar}{2m}\alpha z\phi_2 &= i\hbar\frac{\partial}{\partial t}\phi_2\end{aligned}\tag{3}$$

Now do the reverse. Let

$$\begin{aligned}\phi_1 &= \exp\left(+\frac{ieB_0t}{2m}\right)\psi_1 \\ \phi_2 &= \exp\left(-\frac{ieB_0t}{2m}\right)\psi_2\end{aligned}$$

Then

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \phi_1 &= \exp\left(+\frac{ieB_0 t}{2m}\right) i\hbar \frac{\partial}{\partial t} \psi_1 - \frac{e\hbar}{2m} B_0 \phi_1 \\ i\hbar \frac{\partial}{\partial t} \phi_2 &= \exp\left(-\frac{ieB_0 t}{2m}\right) i\hbar \frac{\partial}{\partial t} \psi_2 + \frac{e\hbar}{2m} B_0 \phi_2 \end{aligned}$$

Substitute into (3) to obtain

$$\begin{aligned} -\frac{\hbar^2}{2m} \nabla^2 \phi_1 + \frac{e\hbar}{2m} \alpha z \phi_1 &= \exp\left(+\frac{ieB_0 t}{2m}\right) i\hbar \frac{\partial}{\partial t} \psi_1 - \frac{e\hbar}{2m} B_0 \phi_1 \\ -\frac{\hbar^2}{2m} \nabla^2 \phi_2 - \frac{e\hbar}{2m} \alpha z \phi_2 &= \exp\left(-\frac{ieB_0 t}{2m}\right) i\hbar \frac{\partial}{\partial t} \psi_2 + \frac{e\hbar}{2m} B_0 \phi_2 \end{aligned}$$

Multiply through by the conjugate of the respective exponential (note $\phi \rightarrow \psi$).

$$\begin{aligned} -\frac{\hbar^2}{2m} \nabla^2 \psi_1 + \frac{e\hbar}{2m} \alpha z \psi_1 &= i\hbar \frac{\partial}{\partial t} \psi_1 - \frac{e\hbar}{2m} B_0 \psi_1 \\ -\frac{\hbar^2}{2m} \nabla^2 \psi_2 - \frac{e\hbar}{2m} \alpha z \psi_2 &= i\hbar \frac{\partial}{\partial t} \psi_2 + \frac{e\hbar}{2m} B_0 \psi_2 \end{aligned}$$

Rewrite as

$$\begin{aligned} -\frac{\hbar^2}{2m} \nabla^2 \psi_1 + \frac{e\hbar}{2m} (B_0 + \alpha z) \psi_1 &= i\hbar \frac{\partial}{\partial t} \psi_1 \\ -\frac{\hbar^2}{2m} \nabla^2 \psi_2 - \frac{e\hbar}{2m} (B_0 + \alpha z) \psi_2 &= i\hbar \frac{\partial}{\partial t} \psi_2 \end{aligned}$$

The result is identical to (1) with the boxed terms discarded.

Eigenmath script