Free particle propagator

The propagator $K(x_b, t_b, x_a, t_a)$ is the amplitude for a particle to go from x_a at time t_a to x_b at time t_b.

Show that for a free particle

$$K(x_b, t_b, x_a, t_a) = \left(\frac{m}{2\pi i\hbar(t_b - t_a)}\right)^{\frac{1}{2}} \exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)}\right)$$

Start with the amplitude

$$K(x_b, t_b, x_a, t_a) = \langle x_b| \exp\left(-\frac{i\hat{H}(t_b - t_a)}{\hbar}\right) |x_a\rangle \tag{1}$$

where $\hat{H}$ is the free particle Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} \tag{2}$$

Substitute (2) into (1) to obtain

$$K(x_b, t_b, x_a, t_a) = \langle x_b| \exp\left(-\frac{i\hat{p}^2(t_b - t_a)}{2m\hbar}\right) |x_a\rangle$$

By the identity

$$\int_{-\infty}^{\infty} |p\rangle\langle p|\, dp = 1$$

we can write

$$K(x_b, t_b, x_a, t_a) = \int_{-\infty}^{\infty} \langle x_b| \exp\left(-\frac{i\hat{p}^2(t_b - t_a)}{2m\hbar}\right) |p\rangle\langle p|x_a\rangle\, dp$$

By the identity $\hat{p}|p\rangle = p|p\rangle$ replace operator $\hat{p}$ with its eigenvalue p.

$$K(x_b, t_b, x_a, t_a) = \int \langle x_b| \exp\left(-\frac{ip^2(t_b - t_a)}{2m\hbar}\right) |p\rangle\langle p|x_a\rangle\, dp$$

Rearrange as

$$K(x_b, t_b, x_a, t_a) = \int_{-\infty}^{\infty} \exp\left(-\frac{ip^2(t_b - t_a)}{2m\hbar}\right) \langle x_b|p\rangle\langle p|x_a\rangle\, dp$$

For a free particle

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right)$$

Hence

$$K(x_b, t_b, x_a, t_a) = \int_{-\infty}^{\infty} \exp\left(-\frac{ip^2(t_b - t_a)}{2m\hbar}\right) \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx_b}{\hbar}\right) \frac{1}{\sqrt{2\pi\hbar}} \exp\left(-\frac{ipx_a}{\hbar}\right)\, dp$$

Simplify as

$$K(x_b, t_b, x_a, t_a) = \frac{1}{2m\hbar} \int_{-\infty}^{\infty} \exp\left(-\frac{ip^2(t_b - t_a)}{2m\hbar} + \frac{ip(x_b - x_a)}{\hbar} \right) \, dp$$

Let

$$a = \frac{i(t_b - t_a)}{2m\hbar}, \quad b = \frac{i(x_b - x_a)}{\hbar}$$

so that

$$K(x_b, t_b, x_a, t_a) = \frac{1}{2m\hbar} \int_{-\infty}^{\infty} \exp\left(-ap^2 + bp \right) \, dp$$

Solve the integral.

$$\begin{aligned} K(x_b, t_b, x_a, t_a) &= \frac{1}{2m\hbar} \left(\frac{\pi}{a} \right)^{\frac{1}{2}} \exp\left(\frac{b^2}{4a} \right) \\ &= \frac{1}{2\pi\hbar} \left(\frac{2\pi m\hbar}{i(t_b - t_a)} \right)^{\frac{1}{2}} \exp\left(-\frac{m(x_b - x_a)^2}{2i\hbar(t_b - t_a)} \right) \end{aligned}$$

Rewrite as

$$K(x_b, t_b, x_a, t_a) = \left(\frac{m}{2\pi i\hbar(t_b - t_a)} \right)^{\frac{1}{2}} \exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)} \right) \tag{3}$$