Exercise 5.1. Verify this claim.

The claim is that any 2×2 Hermitian matrix $\mathbf{L}$ can be written as

$$\mathbf{L} = a\sigma_x + b\sigma_y + c\sigma_z + dI$$

where a, b, c, and d are real numbers.

Let $\mathbf{L}$ be the Hermitian matrix

$$\mathbf{L} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \tag{1}$$

Recall from page 137 that

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It follows that

$$\mathbf{L} = \begin{pmatrix} d+c & a-ib \\ a+ib & d-c \end{pmatrix} \tag{2}$$

Then by equivalence of (1) and (2) we have

$$a = \frac{L_{12}+L_{21}}{2}, \quad b = \frac{i(L_{12}-L_{21})}{2}, \quad c = \frac{L_{11}-L_{22}}{2}, \quad d = \frac{L_{11}+L_{22}}{2}$$

By Hermiticity we have $\mathbf{L} = \mathbf{L}^\dagger$ hence

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} L_{11}^* & L_{21}^* \\ L_{12}^* & L_{22}^* \end{pmatrix}$$

Therefore L_{11} and L_{22} are real, hence c and d are real.

Also by Hermiticity, $L_{21} = L_{12}^*$ hence a and b are real.

$$a = \frac{L_{12}+L_{21}}{2} = \frac{L_{12}+L_{12}^*}{2} = \mathrm{Re}(L_{12})$$
$$b = \frac{i(L_{12}-L_{21})}{2} = \frac{i(L_{12}-L_{12}^*)}{2} = -\mathrm{Im}(L_{12})$$