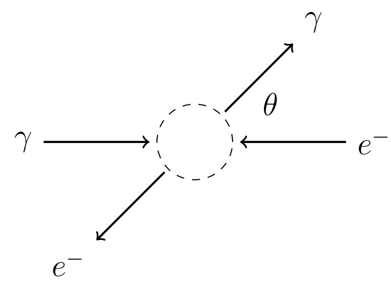
Klein-Nishina formula

The Klein-Nishina formula is the lab frame cross section for $\gamma + e^- \to \gamma + e^-$.



It is easy to derive the Klein-Nishina formula from Dirac's equation by starting out in the center-of-mass frame and then boosting to the lab frame. The following center-of-mass momentum vectors have $E = \sqrt{\omega^2 + m^2}$.

$$
p_1 = \underset{\gamma \longrightarrow}{\begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix}} \qquad
p_2 = \underset{\longleftarrow e^-}{\begin{pmatrix} E \\ 0 \\ 0 \\ -\omega \end{pmatrix}} \qquad
p_3 = \underset{\gamma \nearrow}{\begin{pmatrix} \omega \\ \omega \sin\theta \cos\phi \\ \omega \sin\theta \sin\phi \\ \omega \cos\theta \end{pmatrix}} \qquad
p_4 = \underset{e^- \swarrow}{\begin{pmatrix} E \\ -\omega \sin\theta \cos\phi \\ -\omega \sin\theta \sin\phi \\ -\omega \cos\theta \end{pmatrix}}
$$

Spinors for p_2.

$$
u_{21} = \frac{1}{\sqrt{E+m}} \underset{\text{spin up}}{\begin{pmatrix} E+m \\ 0 \\ -\omega \\ 0 \end{pmatrix}} \qquad
u_{22} = \frac{1}{\sqrt{E+m}} \underset{\text{spin down}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ \omega \end{pmatrix}}
$$

Spinors for p_4.

$$
u_{41} = \frac{1}{\sqrt{E+m}} \underset{\text{spin up}}{\begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix}} \qquad
u_{42} = \frac{1}{\sqrt{E+m}} \underset{\text{spin down}}{\begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}}
$$

The scattering amplitude ${\mathcal{M}_{ab}}^{\mu\nu}$ for spin ab and polarization $\mu\nu$ is

$$
{\mathcal{M}_{ab}}^{\mu\nu} = {\mathcal{M}_{1ab}}^{\mu\nu} + {\mathcal{M}_{2ab}}^{\nu\mu}
$$

where

$$
\begin{aligned}
{\mathcal{M}_{1ab}}^{\mu\nu} &= \frac{\bar{u}_{4b}(-ie\gamma^\mu)(\not{q}_1 + m)(-ie\gamma^\nu)u_{2a}}{s - m^2} \\
{\mathcal{M}_{2ab}}^{\nu\mu} &= \frac{\bar{u}_{4b}(-ie\gamma^\nu)(\not{q}_2 + m)(-ie\gamma^\mu)u_{2a}}{u - m^2}
\end{aligned}
$$

Matrices $\not{q}_1$ and $\not{q}_2$ represent momentum transfer.

$$\not{q}_1 = (p_1 + p_2)^\alpha g_{\alpha\beta}\gamma^\beta$$
$$\not{q}_2 = (p_4 - p_1)^\alpha g_{\alpha\beta}\gamma^\beta$$

Scalars s and u are Mandelstam variables.

$$s = (p_1 + p_2)^2$$
$$u = (p_1 - p_4)^2$$

In component form

$$\mathcal{M}_{1ab}{}^{\mu\nu} = \frac{(\bar{u}_{4b})_\alpha(-ie\gamma^{\mu\alpha}{}_\beta)(\not{q}_1 + m)^\beta{}_\rho(-ie\gamma^{\nu\rho}{}_\sigma)(u_{2a})^\sigma}{s - m^2}$$
$$\mathcal{M}_{2ab}{}^{\nu\mu} = \frac{(\bar{u}_{4b})_\alpha(-ie\gamma^{\nu\alpha}{}_\beta)(\not{q}_2 + m)^\beta{}_\rho(-ie\gamma^{\mu\rho}{}_\sigma)(u_{2a})^\sigma}{u - m^2}$$

Expected probability density $\langle|\mathcal{M}|^2\rangle$ is the sum over squared amplitudes divided by the number of inbound states.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{ab}\sum_{\mu\nu}\left|\mathcal{M}_{ab}{}^{\mu\nu}\right|^2$$

Summing over $\mu\nu$ requires $g_{\mu\nu}$ to lower indices.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{ab}\mathcal{M}_{ab}{}^{\mu\nu}\left(g_{\mu\alpha}\mathcal{M}_{ab}{}^{\alpha\beta}g_{\beta\nu}\right)^*$$

Expand the summand and label the terms. By positivity $\boxed{2} = \boxed{3}$.

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{ab}\Big[&\underbrace{\mathcal{M}_{1ab}{}^{\mu\nu}\left(g_{\mu\alpha}\mathcal{M}_{1ab}{}^{\alpha\beta}g_{\beta\nu}\right)^*}_{\boxed{1}} + \underbrace{\mathcal{M}_{1ab}{}^{\mu\nu}\left(g_{\nu\alpha}\mathcal{M}_{2ab}{}^{\alpha\beta}g_{\beta\mu}\right)^*}_{\boxed{2}} \\ &+ \underbrace{\mathcal{M}_{2ab}{}^{\nu\mu}\left(g_{\mu\alpha}\mathcal{M}_{1ab}{}^{\alpha\beta}g_{\beta\nu}\right)^*}_{\boxed{3}} + \underbrace{\mathcal{M}_{2ab}{}^{\nu\mu}\left(g_{\nu\alpha}\mathcal{M}_{2ab}{}^{\alpha\beta}g_{\beta\mu}\right)^*}_{\boxed{4}}\Big]\end{aligned}$$

The following Casimir trick uses matrix arithmetic to sum over spin and polarization states.

$$\sum_{ab}\boxed{1} = \frac{e^4}{(s - m^2)^2}\operatorname{Tr}\left[(\not{p}_2 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\right]$$
$$\sum_{ab}\boxed{2} = \frac{e^4}{(s - m^2)(u - m^2)}\operatorname{Tr}\left[(\not{p}_2 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\right]$$
$$\sum_{ab}\boxed{4} = \frac{e^4}{(u - m^2)^2}\operatorname{Tr}\left[(\not{p}_2 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\nu(\not{q}_2 + m)\gamma_\mu\right]$$

Let

$$
\begin{aligned}
f_{11} &= \mathrm{Tr}\left[(\not p_2 + m)\gamma^\mu(\not p_1 + m)\gamma^\nu(\not p_4 + m)\gamma_\nu(\not p_1 + m)\gamma_\mu\right] \\
f_{12} &= \mathrm{Tr}\left[(\not p_2 + m)\gamma^\mu(\not p_2 + m)\gamma^\nu(\not p_4 + m)\gamma_\mu(\not p_1 + m)\gamma_\nu\right] \\
f_{22} &= \mathrm{Tr}\left[(\not p_2 + m)\gamma^\mu(\not p_2 + m)\gamma^\nu(\not p_4 + m)\gamma_\nu(\not p_2 + m)\gamma_\mu\right]
\end{aligned}
$$

so that

$$
\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left[\frac{f_{11}}{(s-m^2)^2} + \frac{2f_{12}}{(s-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2}\right]
$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$.)

$$
\begin{aligned}
f_{11} &= 32(p_1\cdot p_2)(p_1\cdot p_4) + 32(p_1\cdot p_2)m^2 + 32m^4 \\
f_{12} &= 16(p_1\cdot p_2)m^2 - 16(p_1\cdot p_4)m^2 + 32m^4 \\
f_{22} &= 32(p_1\cdot p_2)(p_1\cdot p_4) - 32(p_1\cdot p_4)m^2 + 32m^4
\end{aligned} \tag{2}
$$

In Mandelstam variables

$$
\begin{aligned}
f_{11} &= -8su + 24sm^2 + 8um^2 + 8m^4 \\
f_{12} &= 8sm^2 + 8um^2 + 16m^4 \\
f_{22} &= -8su + 8sm^2 + 24um^2 + 8m^4
\end{aligned} \tag{3}
$$

Scattering experiments are typically done in the lab frame. Define Lorentz boost Λ for transforming momentum vectors to the lab frame.

$$
\Lambda = \begin{pmatrix} E/m & 0 & 0 & \omega/m \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega/m & 0 & 0 & E/m \end{pmatrix}
$$

The electron is at rest in the lab frame.

$$
\Lambda p_2 = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

Mandelstam variables are invariant under a boost.

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = (\Lambda p_1 + \Lambda p_2)^2 \\
t &= (p_1 - p_3)^2 = (\Lambda p_1 - \Lambda p_3)^2 \\
u &= (p_1 - p_4)^2 = (\Lambda p_1 - \Lambda p_4)^2
\end{aligned} \tag{4}
$$

In the lab frame, let ω_L be the angular frequency of the incident photon and let ω_L' be the angular frequency of the scattered photon.

$$\omega_L = \Lambda p_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2}{m} + \frac{\omega E}{m}$$

$$\omega_L' = \Lambda p_3 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2 \cos\theta}{m} + \frac{\omega E}{m}$$

It can be shown that

$$\begin{aligned} s &= m^2 + 2m\omega_L \\ t &= 2m(\omega_L' - \omega_L) \\ u &= m^2 - 2m\omega_L' \end{aligned} \tag{5}$$

Then by (1), (3), and (5) we have

$$\langle|\mathcal{M}|^2\rangle = 2e^4 \left[\frac{\omega_L}{\omega_L'} + \frac{\omega_L'}{\omega_L} + \left(\frac{m}{\omega_L} - \frac{m}{\omega_L'} + 1 \right)^2 - 1 \right] \tag{6}$$

Lab scattering angle θ_L is given by the Compton equation

$$\cos\theta_L = \frac{m}{\omega_L} - \frac{m}{\omega_L'} + 1$$

Hence

$$\begin{aligned} \langle|\mathcal{M}|^2\rangle &= 2e^4 \left(\frac{\omega_L}{\omega_L'} + \frac{\omega_L'}{\omega_L} + \cos^2\theta_L - 1 \right) \\ &= 2e^4 \left(\frac{\omega_L}{\omega_L'} + \frac{\omega_L'}{\omega_L} - \sin^2\theta_L \right) \end{aligned}$$

Now that we have derived $\langle|\mathcal{M}|^2\rangle$ we can investigate the angular distribution of scattered photons. For simplicity let us drop the L subscript from lab variables. From now on the symbols ω, ω', and θ will be lab frame variables.

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4(4\pi\varepsilon_0)^2 s} \left(\frac{\omega'}{\omega} \right)^2 \langle|\mathcal{M}|^2\rangle$$

where

$$s = m^2 + 2m\omega = (mc^2)^2 + 2(mc^2)(\hbar\omega)$$

and ω' is given by the Compton equation

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos\theta)}$$

For the lab frame we have

$$\langle|\mathcal{M}|^2\rangle = 2e^4\left(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta\right)$$

Hence in the lab frame

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\varepsilon_0)^2 s}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta\right)$$

Substituting

$$e^2 = 4\pi\varepsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta\right)$$

which is the Klein-Nishina formula.