

Exercise 5.2.

1) Show that $\Delta \mathbf{A}^2 = \langle \bar{\mathbf{A}}^2 \rangle$ and $\Delta \mathbf{B}^2 = \langle \bar{\mathbf{B}}^2 \rangle$.

2) Show that $[\bar{\mathbf{A}}, \bar{\mathbf{B}}] = [\mathbf{A}, \mathbf{B}]$.

3) Using these relations, show that

$$\Delta \mathbf{A} \Delta \mathbf{B} \geq \frac{1}{2} |\langle \Psi | [\mathbf{A}, \mathbf{B}] | \Psi \rangle|$$

1) Recall that $\Delta \mathbf{A}$ is standard deviation and $(\Delta \mathbf{A})^2$ is variance.

From page 147 we have

$$\bar{\mathbf{A}} = \mathbf{A} - \langle \mathbf{A} \rangle \tag{1}$$

Hence

$$\bar{\mathbf{A}}^2 = \mathbf{A}^2 - 2\langle \mathbf{A} \rangle \mathbf{A} + \langle \mathbf{A} \rangle^2$$

It follows that the expectation of $\bar{\mathbf{A}}^2$ is

$$\langle \bar{\mathbf{A}}^2 \rangle = \langle \mathbf{A}^2 \rangle - \langle \mathbf{A} \rangle^2$$

Noting that $\langle \mathbf{A}^2 \rangle - \langle \mathbf{A} \rangle^2$ is the variance of \mathbf{A} , we have

$$(\Delta \mathbf{A})^2 = \langle \mathbf{A}^2 \rangle - \langle \mathbf{A} \rangle^2 = \langle \bar{\mathbf{A}}^2 \rangle$$

By the same argument

$$(\Delta \mathbf{B})^2 = \langle \mathbf{B}^2 \rangle - \langle \mathbf{B} \rangle^2 = \langle \bar{\mathbf{B}}^2 \rangle$$

2) We have

$$[\bar{\mathbf{A}}, \bar{\mathbf{B}}] = \bar{\mathbf{A}}\bar{\mathbf{B}} - \bar{\mathbf{B}}\bar{\mathbf{A}}$$

By substitution from (1)

$$[\bar{\mathbf{A}}, \bar{\mathbf{B}}] = (\mathbf{A} - \langle \mathbf{A} \rangle)(\mathbf{B} - \langle \mathbf{B} \rangle) - (\mathbf{B} - \langle \mathbf{B} \rangle)(\mathbf{A} - \langle \mathbf{A} \rangle)$$

Expand products of sums.

$$\begin{aligned}
 [\bar{\mathbf{A}}, \bar{\mathbf{B}}] &= \mathbf{A}\mathbf{B} - \underset{\text{cancel a}}{\mathbf{A}\langle\mathbf{B}\rangle} - \underset{\text{cancel b}}{\langle\mathbf{A}\rangle\mathbf{B}} + \underset{\text{cancel c}}{\langle\mathbf{A}\rangle\langle\mathbf{B}\rangle} \\
 &\quad - \mathbf{B}\mathbf{A} + \underset{\text{term b}}{\mathbf{B}\langle\mathbf{A}\rangle} + \underset{\text{term a}}{\langle\mathbf{B}\rangle\mathbf{A}} - \underset{\text{term c}}{\langle\mathbf{B}\rangle\langle\mathbf{A}\rangle}
 \end{aligned}$$

Hence

$$[\bar{\mathbf{A}}, \bar{\mathbf{B}}] = \mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A} = [\mathbf{A}, \mathbf{B}]$$

3) By simple algebra we have

$$\Delta\mathbf{A} \Delta\mathbf{B} = \sqrt{(\Delta\mathbf{A})^2(\Delta\mathbf{B})^2}$$

By the result from step 1 we can write

$$\Delta\mathbf{A} \Delta\mathbf{B} = \sqrt{\langle\bar{\mathbf{A}}^2\rangle\langle\bar{\mathbf{B}}^2\rangle}$$

Consider equation (5.12).

$$2\sqrt{\langle\mathbf{A}^2\rangle\langle\mathbf{B}^2\rangle} \geq |\langle\Psi|[\mathbf{A}, \mathbf{B}]|\Psi\rangle| \quad (5.12)$$

Equation (5.12) is valid for any linear operator hence we can bar \mathbf{A} and \mathbf{B} and obtain

$$2\sqrt{\langle\bar{\mathbf{A}}^2\rangle\langle\bar{\mathbf{B}}^2\rangle} \geq |\langle\Psi|[\bar{\mathbf{A}}, \bar{\mathbf{B}}]|\Psi\rangle| \quad (2)$$

By equation (2) and the result from step 2 we have

$$2\sqrt{\langle\bar{\mathbf{A}}^2\rangle\langle\bar{\mathbf{B}}^2\rangle} \geq |\langle\Psi|[\mathbf{A}, \mathbf{B}]|\Psi\rangle|$$

Hence

$$2\Delta\mathbf{A} \Delta\mathbf{B} \geq |\langle\Psi|[\mathbf{A}, \mathbf{B}]|\Psi\rangle|$$