

Feynman and Hibbs problem 2-2

This is the Lagrangian for a harmonic oscillator.

$$L = \frac{m}{2}(\dot{x}^2 - \omega^2 x^2)$$

Let $T = t_b - t_a$. Show that the classical action is

$$S_{cl} = \frac{m\omega}{2\sin(\omega T)} \left((x_b^2 + x_a^2) \cos(\omega T) - 2x_b x_a \right)$$

From the above Lagrangian we have

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = m\ddot{x}$$

and

$$\frac{\partial L}{\partial x} = -m\omega^2 x$$

By equation (2.7) which is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$$

we have

$$\ddot{x} = -\omega^2 x \tag{1}$$

The well-known solution to (1) is

$$x(t) = A \sin(\omega t) + B \cos(\omega t)$$

We have the following boundary conditions.

$$\begin{aligned} x(0) &= x_a \\ x(T) &= x_b \end{aligned}$$

Solve for B .

$$x(0) = B = x_a$$

For $x(T)$ we have

$$x(T) = A \sin(\omega T) + B \cos(\omega T)$$

Solve for A to obtain

$$A = \frac{x(T) - B \cos(\omega T)}{\sin(\omega T)} = \frac{x_b - x_a \cos(\omega T)}{\sin(\omega T)}$$

Hence the equation of motion is

$$x(t) = \frac{x_b - x_a \cos(\omega T)}{\sin(\omega T)} \sin(\omega t) + x_a \cos(\omega t)$$

Differentiate $x(t)$ to obtain velocity $\dot{x}(t)$.

$$\dot{x}(t) = \frac{d}{dt}x(t) = \omega \left(\frac{x_b - x_a \cos(\omega T)}{\sin(\omega T)} \cos(\omega t) - x_a \sin(\omega t) \right)$$

Using the action integral

$$S = \int_0^T L dt$$

we have for the classical action

$$\begin{aligned} S_{cl} &= \frac{m}{2} \int_0^T (\dot{x}^2 - \omega^2 x^2) dt \\ &= \frac{m}{2} \left(\int_0^T \dot{x}^2 dt - \int_0^T \omega^2 x^2 dt \right) \end{aligned}$$

Let

$$\begin{aligned} u &= \dot{x} \\ v &= \dot{x} \end{aligned}$$

Use integration by parts to obtain

$$\begin{aligned} \int_0^T \dot{x}^2 dt &= \int_0^T uv dt \\ &= \left(u \int v dt \right)_0^T - \int_0^T \dot{u} \left(\int v dt \right) dt \\ &= \dot{x}x \Big|_0^T - \int_0^T \ddot{x}x dt \end{aligned}$$

Hence

$$S_{cl} = \frac{m}{2} \left(\dot{x}x \Big|_0^T - \int_0^T \ddot{x}x \, dt - \int_0^T \omega^2 x^2 \, dt \right)$$

From (1) we have

$$\ddot{x}x = -\omega^2 x^2$$

Hence the remaining integrals cancel leaving

$$\begin{aligned} S_{cl} &= \frac{m}{2} \dot{x}x \Big|_0^T \\ &= \frac{m}{2} \left(\dot{x}(T)x(T) - \dot{x}(0)x(0) \right) \end{aligned}$$

By substitution and simplification

$$S_{cl} = \frac{m\omega}{2 \sin(\omega T)} \left((x_b^2 + x_a^2) \cos(\omega T) - 2x_b x_a \right)$$