(36.1) (a) Show that the Dirac equation can be recast in the form

$$i\frac{\partial\psi}{\partial t} = \hat{H}_D\psi \tag{36.33}$$

where $\hat{H}_D = \boldsymbol{\alpha}\cdot\hat{\mathbf{p}} + \beta m$ and find $\boldsymbol{\alpha}$ and β in terms of the γ matrices.

(b) Evaluate $\hat{H}_D^2$ and show that for a Klein-Gordon dispersion to result we must have:
(i) that the α^i and β objects all anticommute with each other; and
(ii) $(\alpha^i)^2 = (\beta)^2 = 1$.

(c) Prove the following commutation relations
(i) $[\hat{H}, \hat{L}^i] = i(\hat{\mathbf{p}}\times\boldsymbol{\alpha})^i$ where $\hat{\mathbf{L}} = \hat{\mathbf{x}}\times\hat{\mathbf{p}}$.
(ii) $[\hat{H}, \hat{S}^i] = -i(\hat{\mathbf{p}}\times\boldsymbol{\alpha})^i$ where $\hat{\mathbf{S}} = \frac{1}{2}\boldsymbol{\Sigma}$ and we define $\boldsymbol{\Sigma} = \frac{i}{2}\boldsymbol{\gamma}\times\boldsymbol{\gamma}$.

(a) Consider the following form of the Dirac equation.

$$i\left(\gamma^0\frac{\partial}{\partial t} + \gamma^1\frac{\partial}{\partial x} + \gamma^2\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z}\right)\psi = m\psi$$

Rewrite as

$$i\gamma^0\frac{\partial}{\partial t}\psi = -i\left(\gamma^1\frac{\partial}{\partial x} + \gamma^2\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z}\right)\psi + m\psi$$

Noting that $\gamma^0\gamma^0 = I$, multiply both sides by γ^0 to obtain

$$i\frac{\partial}{\partial t}\psi = -i\gamma^0\left(\gamma^1\frac{\partial}{\partial x} + \gamma^2\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z}\right)\psi + m\gamma^0\psi$$

Hence for $\hat{\mathbf{p}} = -i\nabla$ we have

$$\boldsymbol{\alpha} = \gamma^0\begin{pmatrix}\gamma^1\\ \gamma^2\\ \gamma^3\end{pmatrix}, \quad \beta = \gamma^0$$

(b) The dispersion relation is

$$\hat{H}_D^2 = \hat{\mathbf{p}}^2 + m^2$$

Squaring $\hat{H}_D$ we have

$$\begin{aligned}\hat{H}_D^2 &= (\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}+\beta m)(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}+\beta m)\\ &= (\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})+(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})\beta m+\beta m(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})+\beta^2 m^2\end{aligned}$$

(i) The middle terms must cancel, that is

$$(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})\beta+\beta(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})=0$$

Hence

$$\alpha^i\beta=-\beta\alpha^i$$

Cross terms must cancel, that is

$$\left(-i\alpha^1\frac{\partial}{\partial x}-i\alpha^2\frac{\partial}{\partial y}-i\alpha^3\frac{\partial}{\partial z}\right)^2=-(\alpha^1)^2\frac{\partial^2}{\partial x^2}-(\alpha^2)^2\frac{\partial^2}{\partial y^2}-(\alpha^3)^2\frac{\partial^2}{\partial z^2}$$

Hence

$$\alpha^i\alpha^j=-\alpha^j\alpha^i$$

(ii) We now have

$$\hat{H}_D^2=-(\alpha^1)^2\frac{\partial^2}{\partial x^2}-(\alpha^2)^2\frac{\partial^2}{\partial y^2}-(\alpha^3)^2\frac{\partial^2}{\partial z^2}+\beta^2 m=\hat{\mathbf{p}}^2+m^2$$

Hence

$$(\alpha^i)^2=I\quad\text{and}\quad\beta^2=I$$

(c) In component form we have

$$\hat{\mathbf{p}}\times\boldsymbol{\alpha}=-i\begin{pmatrix}\frac{\partial}{\partial y}\alpha^3-\frac{\partial}{\partial z}\alpha^2\\ \frac{\partial}{\partial z}\alpha^1-\frac{\partial}{\partial x}\alpha^3\\ \frac{\partial}{\partial x}\alpha^2-\frac{\partial}{\partial y}\alpha^1\end{pmatrix},\quad \hat{\mathbf{L}}=\hat{\mathbf{x}}\times\hat{\mathbf{p}}=-i\begin{pmatrix}y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\\ z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\\ x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\end{pmatrix}$$

(i) Noting that βm commutes with $\hat{\mathbf{L}}$ we have

$$[\hat{H}_D,\hat{\mathbf{L}}]=(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})(\hat{\mathbf{x}}\times\hat{\mathbf{p}})-(\hat{\mathbf{x}}\times\hat{\mathbf{p}})(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})$$

By the product rule we have for $(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})(\hat{\mathbf{x}}\times\hat{\mathbf{p}})$ that

$$(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})(\hat{\mathbf{x}}\times\hat{\mathbf{p}})=(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\hat{\mathbf{x}})\times\hat{\mathbf{p}}+\hat{\mathbf{x}}\times(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\hat{\mathbf{p}})$$

By ordinary operator arithmetic $(\hat{\mathbf{x}} \times \hat{\mathbf{p}})(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})$ transforms as

$$(\hat{\mathbf{x}} \times \hat{\mathbf{p}})(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}) = \hat{\mathbf{x}} \times (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}\hat{\mathbf{p}})$$

Hence

$$[\hat{H}_D, \hat{\mathbf{L}}] = (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}\hat{\mathbf{x}}) \times \hat{\mathbf{p}} + \hat{\mathbf{x}} \times (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}\hat{\mathbf{p}}) - \hat{\mathbf{x}} \times (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}\hat{\mathbf{p}}) = (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}\hat{\mathbf{x}}) \times \hat{\mathbf{p}}$$

Noting that

$$\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}\hat{\mathbf{x}} = \boldsymbol{\alpha} \cdot (-i\nabla\hat{\mathbf{x}}) = -i\boldsymbol{\alpha}$$

we have

$$[\hat{H}_D, \hat{\mathbf{L}}] = -i\boldsymbol{\alpha} \times \hat{\mathbf{p}} = i(\hat{\mathbf{p}} \times \boldsymbol{\alpha})$$