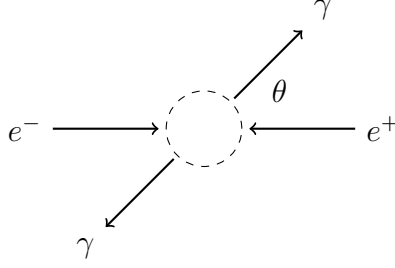


Annihilation

Annihilation is the interaction $e^- + e^+ \rightarrow \gamma + \gamma$.



In the center-of-mass frame we have the following momentum vectors where $E = \sqrt{p^2 + m^2}$.

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \quad \text{inbound electron} \quad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} \quad \text{inbound positron} \quad p_3 = \begin{pmatrix} E \\ E \sin \theta \cos \phi \\ E \sin \theta \sin \phi \\ E \cos \theta \end{pmatrix} \quad \text{outbound photon} \quad p_4 = \begin{pmatrix} E \\ -E \sin \theta \cos \phi \\ -E \sin \theta \sin \phi \\ -E \cos \theta \end{pmatrix} \quad \text{outbound photon}$$

Spinors for the inbound electron.

$$u_{11} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix} \quad \text{inbound electron spin up} \quad u_{12} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix} \quad \text{inbound electron spin down}$$

Spinors for the inbound positron.

$$v_{21} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} -p \\ 0 \\ E+m \\ 0 \end{pmatrix} \quad \text{inbound positron spin up} \quad v_{22} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ p \\ 0 \\ E+m \end{pmatrix} \quad \text{inbound positron spin down}$$

The probability amplitude \mathcal{M}_{ab} for spin state ab is

$$\mathcal{M}_{ab} = \mathcal{M}_{1ab} + \mathcal{M}_{2ab}$$

where

$$\mathcal{M}_{1ab} = \frac{\bar{v}_{2b}(-ie\gamma^\mu)(\not{q}_1 + m)(-ie\gamma^\nu)u_{1a}}{t - m^2}, \quad \mathcal{M}_{2ab} = \frac{\bar{v}_{2b}(-ie\gamma^\nu)(\not{q}_2 + m)(-ie\gamma^\mu)u_{1a}}{u - m^2}$$

Symbols q_1 and q_2 are momentum transfer.

$$q_1 = (p_1 - p_3)^\alpha g_{\alpha\beta} \gamma^\beta$$

$$q_2 = (p_1 - p_4)^\alpha g_{\alpha\beta} \gamma^\beta$$

Symbols t and u are Mandelstam variables.

$$\begin{aligned} t &= (p_1 - p_3)^2 \\ u &= (p_1 - p_4)^2 \end{aligned}$$

In component form (note that indices μ and ν are interchanged for \mathcal{M}_{2ab})

$$\begin{aligned} (\mathcal{M}_{1ab})^{\mu\nu} &= \frac{(\bar{v}_{2b})_\alpha (-ie\gamma^{\mu\alpha}_\beta)(\not{q}_1 + m)^\beta_\rho (-ie\gamma^{\nu\rho}_\sigma)(u_{1a})^\sigma}{t - m^2} \\ (\mathcal{M}_{2ab})^{\nu\mu} &= \frac{(\bar{v}_{2b})_\alpha (-ie\gamma^{\nu\alpha}_\beta)(\not{q}_2 + m)^\beta_\rho (-ie\gamma^{\mu\rho}_\sigma)(u_{1a})^\sigma}{u - m^2} \end{aligned}$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is the average of spin states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{a=1}^2 \sum_{b=1}^2 |\mathcal{M}_{ab}|^2$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{a=1}^2 \sum_{b=1}^2 (\mathcal{M}_{1ab} \mathcal{M}_{1ab}^* + \mathcal{M}_{1ab} \mathcal{M}_{2ab}^* + \mathcal{M}_{2ab} \mathcal{M}_{1ab}^* + \mathcal{M}_{2ab} \mathcal{M}_{2ab}^*)$$

Metric tensor $g_{\mu\nu}$ is required to sum over indices μ and ν .

$$\begin{aligned} \mathcal{M}_{1ab} \mathcal{M}_{1ab}^* &= (\mathcal{M}_{1ab})^{\mu\nu} (\mathcal{M}_{1ab}^*)_{\mu\nu} = (\mathcal{M}_{1ab})^{\mu\nu} [g_{\mu\alpha} (\mathcal{M}_{1ab}^*)^{\alpha\beta} g_{\beta\nu}] \\ \mathcal{M}_{1ab} \mathcal{M}_{2ab}^* &= (\mathcal{M}_{1ab})^{\mu\nu} (\mathcal{M}_{2ab}^*)_{\nu\mu} = (\mathcal{M}_{1ab})^{\mu\nu} [g_{\nu\alpha} (\mathcal{M}_{2ab}^*)^{\alpha\beta} g_{\beta\mu}] \\ \mathcal{M}_{2ab} \mathcal{M}_{2ab}^* &= (\mathcal{M}_{2ab})^{\nu\mu} (\mathcal{M}_{2ab}^*)_{\nu\mu} = (\mathcal{M}_{2ab})^{\nu\mu} [g_{\nu\alpha} (\mathcal{M}_{2ab}^*)^{\alpha\beta} g_{\beta\mu}] \end{aligned}$$

The Casimir trick uses matrix arithmetic to sum over spin states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left(\frac{f_{11}}{(t - m^2)^2} + \frac{2f_{12}}{(t - m^2)(u - m^2)} + \frac{f_{22}}{(u - m^2)^2} \right)$$

where

$$\begin{aligned} f_{11} &= \text{Tr} \left((\not{p}_1 + m) \gamma^\mu (\not{q}_1 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\nu (\not{q}_1 + m) \gamma_\mu \right) \\ f_{12} &= \text{Tr} \left((\not{p}_1 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\mu (\not{q}_1 + m) \gamma_\nu \right) \\ f_{22} &= \text{Tr} \left((\not{p}_1 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\nu (\not{q}_2 + m) \gamma_\mu \right) \end{aligned}$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$\begin{aligned} f_{11} &= 32(p_1 \cdot p_3)(p_1 \cdot p_4) - 32m^2(p_1 \cdot p_2) + 64m^2(p_1 \cdot p_3) + 32m^2(p_1 \cdot p_4) - 64m^4 \\ f_{12} &= 16m^2(p_1 \cdot p_3) + 16m^2(p_1 \cdot p_4) - 32m^4 \\ f_{22} &= 32(p_1 \cdot p_3)(p_1 \cdot p_4) - 32m^2(p_1 \cdot p_2) + 32m^2(p_1 \cdot p_3) + 64m^2(p_1 \cdot p_4) - 64m^4 \end{aligned}$$

In Mandelstam variables

$$\begin{aligned}f_{11} &= 8tu - 24tm^2 - 8um^2 - 8m^4 \\f_{12} &= 8sm^2 - 32m^4 \\f_{22} &= 8tu - 8tm^2 - 24um^2 - 8m^4\end{aligned}$$

For $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$\begin{aligned}f_{11} &= 8tu \\f_{12} &= 0 \\f_{22} &= 8tu\end{aligned}$$

Hence

$$\begin{aligned}\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left(\frac{f_{11}}{(t-m^2)^2} + \frac{2f_{12}}{(t-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2} \right) \\&= \frac{e^4}{4} \left(\frac{8tu}{t^2} + \frac{8tu}{u^2} \right) \\&= 2e^4 \left(\frac{u}{t} + \frac{t}{u} \right)\end{aligned}$$

For $m = 0$ the Mandelstam variables are

$$\begin{aligned}t &= -2E^2(1 - \cos \theta) \\u &= -2E^2(1 + \cos \theta)\end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right)$$

Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{4(4\pi\epsilon_0)^2 s}$$

where

$$s = (p_1 + p_2)^2 = 4E^2$$

For high energy experiments we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\epsilon_0)^2 s} \left(\frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right)$$

Noting that

$$e^2 = 4\pi\varepsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s} \left(\frac{1 + \cos\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{1 + \cos\theta} \right)$$

Noting that

$$d\Omega = \sin\theta d\theta d\phi$$

we also have

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{2s} \left(\frac{1 + \cos\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{1 + \cos\theta} \right) \sin\theta d\theta d\phi$$

Let $S(\theta_1, \theta_2)$ be the following integral of $d\sigma$.

$$S(\theta_1, \theta_2) = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{\pi\alpha^2(\hbar c)^2}{s} [I(\theta_2) - I(\theta_1)]$$

where

$$I(\theta) = 2\cos\theta + 2\log(1 - \cos\theta) - 2\log(1 + \cos\theta)$$

The cumulative distribution function is

$$F(\theta) = \frac{S(a, \theta)}{S(a, \pi - a)} = \frac{I(\theta) - I(a)}{I(\pi - a) - I(a)}, \quad a \leq \theta \leq \pi - a$$

Angular support is reduced by an arbitrary angle $a > 0$ because $I(0)$ and $I(\pi)$ are undefined.

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 < \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi - a) - I(a)} \left(\frac{1 + \cos\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{1 + \cos\theta} \right) \sin\theta$$