Atomic transitions 3

From the previous section

$$c_b(t) = \frac{ieE_0}{m\hbar\omega}\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\frac{\sin\left(\frac{1}{2}(\omega_0-\omega)t\right)}{\omega_0-\omega}\exp\left(\frac{i}{2}(\omega_0-\omega)t\right)$$

The probability to go from state a to state b is

$$\Pr_{a\to b}(t) = |c_b(t)|^2 = \frac{e^2E_0^2}{m^2\hbar^2\omega^2}|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle|^2\frac{\sin^2\left(\frac{1}{2}(\omega_0-\omega)t\right)}{(\omega_0-\omega)^2}$$

We are now going to pivot from E_0 to a full spectrum electric field. Let u be energy density such that

$$E_0^2 = \frac{2u}{\varepsilon_0}$$

By substitution

$$\Pr_{a\to b}(t) = \frac{2u}{\varepsilon_0}\frac{e^2}{m^2\hbar^2\omega^2}|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle|^2\frac{\sin^2\left(\frac{1}{2}(\omega_0-\omega)t\right)}{(\omega_0-\omega)^2}$$

For incoherent radiation

$$u = \int_{-\infty}^{\infty}\rho(\omega)\,d\omega$$

where $\rho(\omega)$ has dimensions energy per unit volume per hertz. By substitution

$$\Pr_{a\to b}(t) = \frac{2}{\varepsilon_0}\frac{e^2}{m^2\hbar^2}|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle|^2\int_{-\infty}^{\infty}\frac{\rho(\omega)}{\omega^2}\frac{\sin^2\left(\frac{1}{2}(\omega_0-\omega)t\right)}{(\omega_0-\omega)^2}\,d\omega$$

The integrand is sharply peaked at $\omega = \omega_0$ so we use the approximation

$$\int_{-\infty}^{\infty}\frac{\rho(\omega)}{\omega^2}\cdots d\omega \approx \frac{\rho(\omega_0)}{\omega_0^2}\int_{-\infty}^{\infty}\cdots d\omega$$

Using change of variable

$$y = \tfrac{1}{2}(\omega-\omega_0)t, \quad dy = \frac{t}{2}\,d\omega, \quad \omega-\omega_0 = \frac{2y}{t}$$

we have

$$\int_{-\infty}^{\infty}\frac{\sin^2\left(\frac{1}{2}(\omega_0-\omega)t\right)}{(\omega_0-\omega)^2}\,d\omega = \frac{t}{2}\int_{-\infty}^{\infty}\frac{\sin^2(-y)}{y^2}\,dy = \frac{\pi}{2}t$$

Hence

$$\Pr_{a\to b}(t) = \frac{\pi e^2\rho(\omega_0)}{\varepsilon_0 m^2\hbar^2\omega_0^2}|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle|^2 t$$

Verify dimensions.

$$\Pr_{a\to b}(t) \propto \frac{\overset{e^2}{\underset{C^2}{}}\overset{\rho(\omega_0)}{\underset{J\,m^{-3}\,s}{}}}{\underset{\frac{\varepsilon_0}{C^2\,J^{-1}\,m^{-1}}}{}\underset{\frac{m^2}{kg^2}}{}\underset{\frac{\hbar^2}{J^2\,s^2}}{}\underset{\frac{\omega_0^2}{s^{-2}}}{}} \times \underset{kg^2\,m^2\,s^{-2}}{|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle|^2} \times \underset{s}{t} = 1$$