Exercise 4.6. Carry out the Schrodinger Ket recipe for a single spin. The Hamiltonian is $\mathbf{H} = \frac{\omega\hbar}{2}\sigma_z$ and the final observable is σ_x. The initial state is given as $|u\rangle$ (the state in which $\sigma_z = +1$).

After time t, an experiment is done to measure σ_y. What are the possible outcomes and what are the probabilities for those outcomes?

Note: Use $|r\rangle$ for the initial state instead of $|u\rangle$. Otherwise, the result is time-independent.

Step 1 of the Schrodinger Ket recipe is obtain $\mathbf{H}$, which we already have by hypothesis.

$$\mathbf{H} = \frac{\hbar\omega}{2}\sigma_z = \frac{\hbar\omega}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Step 2. Prepare an initial state $|\Psi(0)\rangle$. By hypothesis the initial state is $|r\rangle$.

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle \tag{2.5}$$

Hence

$$|\Psi(0)\rangle = |r\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Step 3. Find the eigenvalues and eigenvectors of $\mathbf{H}$ by solving the time-independent Schrodinger equation,

$$\mathbf{H}|E_j\rangle = E_j|E_j\rangle$$

The eigenvalues are obtained by solving the characteristic equation $\det(\mathbf{H} - E_j\mathbf{I}) = 0$.

$$\begin{aligned}
\det(\mathbf{H} - E_j\mathbf{I}) &= \begin{vmatrix} \frac{\hbar\omega}{2} - E_j & 0 \\ 0 & -\frac{\hbar\omega}{2} - E_j \end{vmatrix} \\
&= \left(\frac{\hbar\omega}{2} - E_j\right)\left(-\frac{\hbar\omega}{2} - E_j\right) \\
&= E_j^2 - \left(\frac{\hbar\omega}{2}\right)^2 = 0
\end{aligned}$$

Hence the eigenvalues are

$$E_1 = \frac{\hbar\omega}{2}, \quad E_2 = -\frac{\hbar\omega}{2}$$

The eigenvectors of $\mathbf{H}$ are $|u\rangle$ and $|d\rangle$.

$$|E_1\rangle = |u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |E_2\rangle = |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Step 4. Use the initial state vector $|\Psi(0)\rangle$, along with the eigenvectors $|E_j\rangle$ from step 3, to calculate the initial coefficients $\alpha_j(0)$:

$$\alpha_j(0) = \langle E_j|\Psi(0)\rangle$$

We have

$$\alpha_1(0) = \langle E_1|r\rangle = \frac{1}{\sqrt{2}}$$

$$\alpha_2(0) = \langle E_2|r\rangle = \frac{1}{\sqrt{2}}$$

Step 5. Rewrite $|\Psi(0)\rangle$ in terms of the eigenvectors $|E_j\rangle$ and the initial coefficients $\alpha_j(0)$:

$$|\Psi(0)\rangle = \sum_j \alpha_j(0)|E_j\rangle$$

We have

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Step 6. In the above equation, replace each $\alpha_j(0)$ with $\alpha_j(t)$ to capture its time-independence. As a result, $|\Psi(0)\rangle$ becomes $|\Psi(t)\rangle$:

$$|\Psi(t)\rangle = \sum_j \alpha_j(t)|E_j\rangle$$

We have

$$|\Psi(t)\rangle = \alpha_1(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Step 7. Using Eq. 4.30, replace each $\alpha_j(t)$ with $\alpha_j(0)\exp(-iE_jt/\hbar)$:

$$|\Psi(t)\rangle = \sum_j \alpha_j(0) \exp\left(-\frac{i}{\hbar}E_jt\right)|E_j\rangle \tag{4.34}$$

We have

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}} \exp\left(-\frac{i}{\hbar}E_1 t\right)\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}} \exp\left(-\frac{i}{\hbar}E_2 t\right)\begin{pmatrix}0\\1\end{pmatrix}$$

Hence

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}} \exp\left(-\frac{i\omega t}{2}\right)\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}} \exp\left(\frac{i\omega t}{2}\right)\begin{pmatrix}0\\1\end{pmatrix}$$

This concludes the Schrodinger Ket recipe.

Recall that

$$|i\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix}, \qquad |o\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{pmatrix}$$

The probability of measuring $\sigma_y = 1$ is

$$P(1 \mid \sigma_y) = \langle i|\Psi(t)\rangle\langle\Psi(t)|i\rangle = \frac{1+\sin\omega t}{2}$$

The probability of measuring $\sigma_y = -1$ is

$$P(-1 \mid \sigma_y) = \langle o|\Psi(t)\rangle\langle\Psi(t)|o\rangle = \frac{1-\sin\omega t}{2}$$