Muon decay

From the Particle Data Group website[1]

<div align="center">

μ MEAN LIFE τ

Measurements with an error > 0.001×10^{-6} s have been omitted.

</div>

VALUE (10^{-6} s)	DOCUMENT ID		TECN	CHG	COMMENT
2.1969811±0.0000022 OUR AVERAGE					
2.1969803±0.0000021±0.0000007 [1]	TISHCHENKO	13	CNTR	+	Surface μ^+ at PSI
2.197083 ±0.000032 ±0.000015	BARCZYK	08	CNTR	+	Muons from π^+ decay at rest
2.197013 ±0.000021 ±0.000011	CHITWOOD	07	CNTR	+	Surface μ^+ at PSI
2.197078 ±0.000073	BARDIN	84	CNTR	+	
2.197025 ±0.000155	BARDIN	84	CNTR	−	
2.19695 ±0.00006	GIOVANETTI	84	CNTR	+	
2.19711 ±0.00008	BALANDIN	74	CNTR	+	
2.1973 ±0.0003	DUCLOS	73	CNTR	+	
• • • We do not use the following data for averages, fits, limits, etc. • • •					
2.1969803±0.0000022	WEBBER	11	CNTR	+	Surface μ^+ at PSI

[1] TISHCHENKO 13 uses 1.6×10^{12} μ^+ events and supersedes WEBBER 11.

From "V minus A" theory we have the following formula for muon lifetime τ.

$$\tau = \frac{96\pi^2 h}{G_F^2 \left(m_\mu c^2\right)^5}$$

Symbol G_F is Fermi coupling constant, m_μ is muon mass.

From NIST we have

$$G_F = 1.1663787 \times 10^{-5} \text{ GeV}^{-2}$$
$$m_\mu = 1.883531627 \times 10^{-28} \text{ kilogram}$$
$$h = 6.62607015 \times 10^{-34} \text{ joule second (exact)}$$
$$c = 299792458 \text{ meter second}^{-1} \text{ (exact)}$$
$$1\,\text{eV} = 1.602176634 \times 10^{-19} \text{ joule (exact)}$$
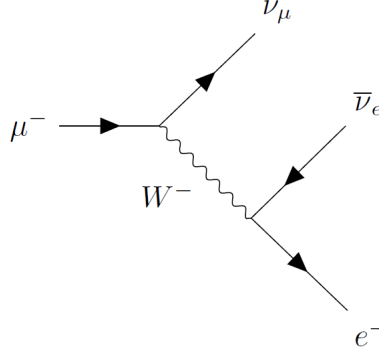
Hence

$$\tau = \frac{96\pi^2 h}{G_F^2 \left(m_\mu c^2\right)^5} = 2.18735 \times 10^{-6} \text{ second}$$

The result is a bit smaller than the observed value from Particle Data Group.

$$\frac{\tau}{\text{observed value}} = \frac{2.18735 \times 10^{-6} \text{ second}}{2.19698 \times 10^{-6} \text{ second}} = 0.9956$$

As the following diagram shows, a muon decays into a muon neutrino, an electron anti-neutrino, and an electron.

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[1] https://pdg.lbl.gov/2020/listings/rpp2020-list-muon.pdf

Particle	Momentum	Mass	Spin up	Spin down
Muon μ^-	p_1	m_1	u_{11}	u_{12}
Muon neutrino ν_μ	p_2	m_2	u_{21}	u_{22}
Electron anti-neutrino $\bar{\nu}_e$	p_3	m_3	v_{31}	v_{32}
Electron e^-	p_4	m_4	u_{41}	u_{42}

For $E_n = \sqrt{\mathbf{p}_n{}^2 + m_n{}^2}$ we have

$$
p_1 = \begin{pmatrix} E_1 \\ p_{1x} \\ p_{1y} \\ p_{1z} \end{pmatrix}_{\mu^-}
\qquad
p_2 = \begin{pmatrix} E_2 \\ p_{2x} \\ p_{2y} \\ p_{2z} \end{pmatrix}_{\nu_\mu}
\qquad
p_3 = \begin{pmatrix} E_3 \\ p_{3x} \\ p_{3y} \\ p_{3z} \end{pmatrix}_{\bar{\nu}_e}
\qquad
p_4 = \begin{pmatrix} E_4 \\ p_{4x} \\ p_{4y} \\ p_{4z} \end{pmatrix}_{e^-}
$$

Spinors for the muon.

$$
u_{11} = \frac{1}{\sqrt{E_1 + m_1}} \begin{pmatrix} E_1 + m_1 \\ 0 \\ p_{1z} \\ p_{1x} + ip_{1y} \end{pmatrix}_{\mu^- \text{ spin up}}
\qquad
u_{12} = \frac{1}{\sqrt{E_1 + m_1}} \begin{pmatrix} 0 \\ E_1 + m_1 \\ p_{1x} - ip_{1y} \\ -p_{1z} \end{pmatrix}_{\mu^- \text{ spin down}}
$$

Spinors for the muon neutrino.

$$
u_{21} = \frac{1}{\sqrt{E_2 + m_2}} \begin{pmatrix} E_2 + m_2 \\ 0 \\ p_{2z} \\ p_{2x} + ip_{2y} \end{pmatrix}_{\nu_\mu \text{ spin up}}
\qquad
u_{22} = \frac{1}{\sqrt{E_2 + m_2}} \begin{pmatrix} 0 \\ E_2 + m_2 \\ p_{2x} - ip_{2y} \\ -p_{2z} \end{pmatrix}_{\nu_\mu \text{ spin down}}
$$

Spinors for the electron anti-neutrino.

$$
v_{31} = \frac{1}{\sqrt{E_3 + m_3}} \begin{pmatrix} p_{3z} \\ p_{3x} + ip_{3y} \\ E_3 + m_3 \\ 0 \end{pmatrix}_{\bar{\nu}_e \text{ spin up}}
\qquad
v_{32} = \frac{1}{\sqrt{E_3 + m_3}} \begin{pmatrix} p_{3x} - ip_{3y} \\ -p_{3z} \\ 0 \\ E_3 + m_3 \end{pmatrix}_{\bar{\nu}_e \text{ spin down}}
$$

Spinors for the electron.

$$u_{41} = \frac{1}{\sqrt{E_4 + m_4}} \begin{pmatrix} E_4 + m_4 \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix} \qquad u_{42} = \frac{1}{\sqrt{E_4 + m_4}} \begin{pmatrix} 0 \\ E_4 + m_4 \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}$$

$$e^- \text{ spin up} \qquad\qquad\qquad e^- \text{ spin down}$$

The probability amplitude $\mathcal{M}_{abcd}$ for spin state $abcd$ is

$$\mathcal{M}_{abcd} = \frac{G_F}{\sqrt{2}} \left(\bar{u}_{4d}\gamma^\mu(1 - \gamma^5)v_{3c} \right) \left(\bar{u}_{2b}\gamma_\mu(1 - \gamma^5)u_{1a} \right)$$

The expected probability $\langle |\mathcal{M}|^2 \rangle$ is the average of spin states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{2} \sum_{a=1}^{2} \sum_{b=1}^{2} \sum_{c=1}^{2} \sum_{d=1}^{2} |\mathcal{M}_{abcd}|^2$$

The Casimir trick uses matrix arithmetic to sum over spin states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{G_F^2}{4} \text{Tr} \left(\not{p}_4 \gamma^\mu(1 - \gamma^5)\not{p}_3\gamma^\nu(1 - \gamma^5) \right) \text{Tr} \left(\not{p}_2\gamma_\mu(1 - \gamma^5)\not{p}_1\gamma_\nu(1 - \gamma^5) \right)$$

The result is a simple formula.

$$\langle |\mathcal{M}|^2 \rangle = 64 G_F^2 (p_1 \cdot p_3)(p_2 \cdot p_4)$$

In the muon rest frame p_1 is fixed at $p_1 = (m_1, 0, 0, 0)$. The remaining momentum vectors are free to have any values that conserve energy and momentum. Muon decay rate Γ is the expectation value for all possible decay momenta. By Fermi's golden rule

$$\Gamma = \frac{1}{512\pi^5 m_\mu} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \langle |\mathcal{M}|^2 \rangle \, \delta(p_1 - p_2 - p_3 - p_4) \frac{d^3 p_2}{E_2} \frac{d^3 p_3}{E_3} \frac{d^3 p_4}{E_4}$$

Altogether there are nine integrals, three for each of p_2, p_3, and p_4. The delta function restricts the integration space to values that conserve energy and momentum.

It can be shown that

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

Muon lifetime τ is the inverse of decay rate.

$$\tau = \frac{1}{\Gamma} = \frac{192\pi^3}{G_F^2 m_\mu^5}$$

Change natural units to h and c.

$$\tau = \frac{96\pi^2 h}{G_F^2 \left(m_\mu c^2 \right)^5}$$

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