(a)

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\Psi(t) &= i\hbar\left[\frac{\partial}{\partial t}\hat{G}+\frac{1}{2}\frac{\partial}{\partial t}(\hat{G}\hat{G})+\frac{1}{3!}\frac{\partial}{\partial t}(\hat{G}\hat{G}\hat{G})+\cdots\right]\Psi(0)\\
&= \left[\hat{H}+\frac{1}{2}(\hat{H}\hat{G}+\hat{G}\hat{H})+\frac{1}{3!}(\hat{H}\hat{G}\hat{G}+\hat{G}\hat{H}\hat{G}+\hat{G}\hat{G}\hat{H})+\cdots\right]\Psi(0)
\end{aligned}
$$

If $\hat{G}$ and $\hat{H}$ commute then $\hat{G}\hat{H}=\hat{H}\hat{G}$ and for the general case of n operators

$$
\frac{\partial}{\partial t}\hat{G}^n = n\hat{H}\hat{G}^{n-1}
$$

Hence

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\Psi(t) &= \left[\hat{H}+\hat{H}\hat{G}+\frac{1}{2}\hat{H}\hat{G}\hat{G}+\dots\right]\Psi(0)\\
&= \hat{H}\left[1+\hat{G}+\frac{1}{2}\hat{G}\hat{G}+\dots\right]\Psi(0)\\
&= \hat{H}\Psi(t)
\end{aligned}
$$

Hence $\Psi(t)$ satisfies the Schrödinger equation.

(b) For the single integral

$$
\frac{\partial}{\partial t}\int_0^t \hat{H}(t_1)\,dt_1 = \hat{H}(t)
$$

For the double integral we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_0^t \hat{H}(t_1)\left[\int_0^{t_1}\hat{H}(t_2)\,dt_2\right]dt_1\\
&\qquad = \int_0^t \frac{\partial\hat{H}(t_1)}{\partial t}\left[\int_0^{t_1}\hat{H}(t_2)\,dt_2\right]dt_1 + \int_0^t \hat{H}(t_1)\underbrace{\frac{\partial}{\partial t}\left[\int_0^{t_1}\hat{H}(t_2)\,dt_2\right]}_{\substack{\text{vanishes, integral is}\\ \text{not a function of } t}}dt_1\\
&\qquad = \hat{H}(t)\int_0^t \hat{H}(t_1)\,dt_1
\end{aligned}
$$

and similarly for the other integrals. Hence

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\Psi(t) &= i\hbar\left\{\left(-\frac{i}{\hbar}\right)\hat{H}(t)+\left(-\frac{i}{\hbar}\right)^2\hat{H}(t)\int_0^t\hat{H}(t_1)\,dt_1+\cdots\right\}\Psi(0)\\
&= \left\{\hat{H}(t)+\left(-\frac{i}{\hbar}\right)\hat{H}(t)\int_0^t\hat{H}(t_1)\,dt_1+\cdots\right\}\Psi(0)\\
&= \hat{H}(t)\left\{1+\left(-\frac{i}{\hbar}\right)\int_0^t\hat{H}(t_1)\,dt_1+\cdots\right\}\Psi(0)\\
&= \hat{H}(t)\Psi(t)
\end{aligned}
$$