(13.1) (a) Show that the conserved charge in eqn 13.16 may be written

$$\hat{\mathbf{Q}}_{N_c} = \int d^3 p \, \hat{\mathbf{A}}_{\mathbf{p}}^{\dagger} \mathbf{J} \hat{\mathbf{A}}_{\mathbf{p}} \tag{13.41}$$

where  $\hat{\mathbf{A}}_{\mathbf{p}} = (\hat{a}_{1\mathbf{p}}, \hat{a}_{2\mathbf{p}}, \hat{a}_{3\mathbf{p}})$  and  $\mathbf{J} = (J_x, J_y, J_z)$  are the spin-1 angular momentum matrices from Chapter 9.

(b) Use the transformations from Exercise 3.3 to find the form of the angular momentum matrices appropriate to express the charge as  $\hat{\mathbf{Q}}_{N_c} = \int d^3p \, \hat{\mathbf{B}}_{\mathbf{p}}^{\dagger} \mathbf{J} \hat{\mathbf{B}}_{\mathbf{p}}$  where  $\hat{\mathbf{B}}_{\mathbf{p}} = (\hat{b}_{1\mathbf{p}}, \hat{b}_{0\mathbf{p}}, \hat{b}_{-1\mathbf{p}})$ .

(a) Here is equation (13.16).

$$\mathbf{Q}_{N_c} = \int d^3x \left( \mathbf{\Phi} \times \partial_0 \mathbf{\Phi} \right) \quad \text{and} \quad \hat{Q}_{N_c}^a = -i \int d^3p \, \varepsilon^{abc} \hat{a}_{b\mathbf{p}}^{\dagger} \hat{a}_{c\mathbf{p}}$$
 (13.16)

Recall that  $\varepsilon^{abc}$  is the Levi-Civita symbol

$$\varepsilon^{1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \varepsilon^{2} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \varepsilon^{3} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Scalars commute with operators hence  $\varepsilon^{abc}$  and  $\hat{a}_{b\mathbf{p}}^{\dagger}$  can be interchanged.

$$\hat{Q}_{N_c}^a = -i \int d^3 p \, \hat{a}_{b\mathbf{p}}^{\dagger} \varepsilon^{abc} \hat{a}_{c\mathbf{p}}$$

The sum over b and sum over c are inner products hence we can write

$$\hat{Q}_{N_c}^a = -i \int d^3 p \, \hat{\mathbf{A}}_{\mathbf{p}}^{\dagger} \varepsilon^a \hat{\mathbf{A}}_{\mathbf{p}}$$

Let

$$J_x = -i\varepsilon^1$$
,  $J_y = -i\varepsilon^2$ ,  $J_z = -i\varepsilon^3$ 

Then

$$\hat{\mathbf{Q}}_{N_c} = \begin{pmatrix} Q_{N_c}^1 \\ Q_{N_c}^2 \\ Q_{N_c}^3 \end{pmatrix} = \int d^3 p \, \hat{\mathbf{A}}_{\mathbf{p}}^{\dagger} \begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} \hat{\mathbf{A}}_{\mathbf{p}} = \int d^3 p \, \hat{\mathbf{A}}_{\mathbf{p}}^{\dagger} \mathbf{J} \hat{\mathbf{A}}_{\mathbf{p}}$$

(b) From Exercise 3.3 we have

$$\hat{b}_{1\mathbf{p}} = \frac{1}{\sqrt{2}} \left( -\hat{a}_{1\mathbf{p}} + i\hat{a}_{2\mathbf{p}} \right)$$

$$\hat{b}_{0\mathbf{p}} = \hat{a}_{3\mathbf{p}}$$

$$\hat{b}_{-1\mathbf{p}} = \frac{1}{\sqrt{2}} \left( \hat{a}_{1\mathbf{p}} + i\hat{a}_{2\mathbf{p}} \right)$$

We need a unitary matrix U such that

$$\hat{\mathbf{B}}_{\mathbf{p}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \left( -\hat{a}_{1\mathbf{p}} + i\hat{a}_{2\mathbf{p}} \right) \\ \hat{a}_{3\mathbf{p}} \\ \frac{1}{\sqrt{2}} \left( \hat{a}_{1\mathbf{p}} + i\hat{a}_{2\mathbf{p}} \right) \end{pmatrix} = U \begin{pmatrix} \hat{a}_{1\mathbf{p}} \\ \hat{a}_{2\mathbf{p}} \\ \hat{a}_{3\mathbf{p}} \end{pmatrix} = U \hat{\mathbf{A}}_{\mathbf{p}}$$

Hence

$$U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0\\ 0 & 0 & 1\\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

It follows that

$$\hat{\mathbf{B}}_{\mathbf{p}}^{\dagger} \mathbf{J}' \hat{\mathbf{B}}_{\mathbf{p}} = (\hat{\mathbf{A}}_{\mathbf{p}}^{\dagger} U^{\dagger}) \mathbf{J}' (U \hat{\mathbf{A}}_{\mathbf{p}}) = \hat{\mathbf{A}}_{\mathbf{p}}^{\dagger} (U^{\dagger} \mathbf{J}' U) \hat{\mathbf{A}}_{\mathbf{p}}$$

By equivalence of (13.41) with hypothesis in (b) we have

$$U^{\dagger}\mathbf{J}'U=\mathbf{J}$$

Then by unitarity property  $U^{-1} = U^{\dagger}$  we can write

$$\mathbf{J}' = U\mathbf{J}U^{\dagger}$$

Hence the components of  $\mathbf{J}'$  are

$$J'_{x} = UJ_{x}U^{\dagger} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$J'_{y} = UJ_{y}U^{\dagger} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$J'_{z} = UJ_{z}U^{\dagger} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$