Laplacian of product

Let

$$F(\mathbf{r}) = \frac{e^{ikr}}{r}$$

where $r = |\mathbf{r}|$. Show that

$$\nabla^2 F(\mathbf{r}) = -k^2 F(\mathbf{r}) - 4\pi\delta^3(\mathbf{r})$$

Recall $\nabla^2 = \nabla \cdot \nabla$ and

$$\nabla \cdot (f\mathbf{A}) = \nabla f \cdot \mathbf{A} + f\nabla\cdot\mathbf{A}$$

Hence

$$\begin{aligned}
\nabla^2 F(\mathbf{r}) &= \nabla \cdot \nabla \left(\frac{e^{ikr}}{r}\right) \\
&= \nabla \cdot \left(\frac{1}{r}\nabla e^{ikr} + e^{ikr}\nabla\frac{1}{r}\right) \\
&= \Bigg(\underbrace{\nabla\frac{1}{r}\cdot\nabla e^{ikr}}_{\text{see (2)}} + \underbrace{\frac{1}{r}\nabla^2 e^{ikr}}_{\text{see (3)}} + \underbrace{\nabla e^{ikr}\cdot\nabla\frac{1}{r}}_{\text{see (2)}} + e^{ikr}\nabla^2\frac{1}{r}\Bigg) \qquad (1)
\end{aligned}$$

In spherical coordinates

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\mathbf{e}}_\phi$$

Hence

$$\nabla\frac{1}{r}\cdot\nabla e^{ikr} = \left(-\frac{1}{r^2}\hat{\mathbf{e}}_r + 0\hat{\mathbf{e}}_\theta + 0\hat{\mathbf{e}}_\phi\right)\cdot\left(ike^{ikr}\hat{\mathbf{e}}_r + 0\hat{\mathbf{e}}_\theta + 0\hat{\mathbf{e}}_\phi\right) = -\frac{ike^{ikr}}{r^2} \qquad (2)$$

and

$$\begin{aligned}
\frac{1}{r}\nabla^2 e^{ikr} &= \frac{1}{r^2}\frac{\partial^2}{\partial r^2}(re^{ikr}) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(e^{ikr} + ikre^{ikr}\right) \\
&= \frac{1}{r^2}\left(2ike^{ikr} - k^2re^{ikr}\right) \\
&= \frac{2ike^{ikr}}{r^2} - \frac{k^2e^{ikr}}{r} \qquad (3)
\end{aligned}$$

Substitute (2) and (3) into (1) to obtain

$$\begin{aligned}
\nabla^2 F(\mathbf{r}) &= -\frac{k^2e^{ikr}}{r} + e^{ikr}\nabla^2\frac{1}{r} \\
&= -k^2F(\mathbf{r}) - 4\pi\delta^3(\mathbf{r})e^{ikr}
\end{aligned}$$

Noting that $\delta^3(\mathbf{r})$ vanishes for $r \neq 0$ and $e^{ikr} = 1$ for $r = 0$ we have

$$\nabla^2 F(\mathbf{r}) = -k^2F(\mathbf{r}) - 4\pi\delta^3(\mathbf{r}) \qquad (4)$$