Fun trick

Show that

$$[p^2, \mathbf{r}] = -2i\hbar\mathbf{p}$$

where

$$\mathbf{r} = \otimes(x, y, z), \quad \mathbf{p} = -i\hbar\nabla, \quad p^2 = \mathbf{p}\cdot\mathbf{p} = -\hbar^2\nabla^2$$

We have

$$\begin{aligned}
[p^2, \mathbf{r}] &= p^2\mathbf{r} - \mathbf{r}p^2 \\
&= \mathbf{p}\cdot\mathbf{p}\mathbf{r} - \mathbf{r}\mathbf{p}\cdot\mathbf{p} \\
&= [\mathbf{p}\mathbf{p}\mathbf{r} - \mathbf{p}\mathbf{r}\mathbf{p}]_{1,2} + [\mathbf{p}\mathbf{r}\mathbf{p} - \mathbf{r}\mathbf{p}\mathbf{p}]_{2,3} && \text{trick!} \\
&= [\mathbf{p}(\mathbf{p}\mathbf{r} - \mathbf{r}\mathbf{p})]_{1,2} + [(\mathbf{p}\mathbf{r} - \mathbf{r}\mathbf{p})\mathbf{p}]_{2,3} \\
&= [\mathbf{p}(-i\hbar\mathbf{I})]_{1,2} + [(-i\hbar\mathbf{I})\mathbf{p}]_{2,3} \\
&= \mathbf{p}(-i\hbar) + (-i\hbar)\mathbf{p} \\
&= -2i\hbar\mathbf{p}
\end{aligned}$$

where $[\,]_{i,j}$ means contract on indices i and j and $\mathbf{I}$ is the 3×3 identity matrix.

Verify the following formulas.

$$[p^2, \mathbf{r}] = -2i\hbar\mathbf{p} \tag{1}$$

$$[p^2, \mathbf{r}] = [\mathbf{p}\mathbf{p}\mathbf{r} - \mathbf{p}\mathbf{r}\mathbf{p}]_{1,2} + [\mathbf{p}\mathbf{r}\mathbf{p} - \mathbf{r}\mathbf{p}\mathbf{p}]_{2,3} \tag{2}$$

$$\mathbf{p}\mathbf{r} - \mathbf{r}\mathbf{p} = -i\hbar\mathbf{I} \tag{3}$$

$$\mathbf{p}\cdot\mathbf{p} = [\mathbf{p}\mathbf{p}]_{1,2} \tag{4}$$