

Dirac equation 2

Let

$$\begin{aligned}\psi_1 &= \frac{e^{-i\xi/\hbar}}{\sqrt{E/c + mc}} \begin{pmatrix} E/c + mc \\ 0 \\ p_z \\ p_x + ip_y \end{pmatrix} & \psi_2 &= \frac{e^{-i\xi/\hbar}}{\sqrt{E/c + mc}} \begin{pmatrix} 0 \\ E/c + mc \\ p_x - ip_y \\ -p_z \end{pmatrix} \\ &\text{wavefunction for fermion spin up} & & \text{wavefunction for fermion spin down} \\ \psi_3 &= \frac{e^{i\xi/\hbar}}{\sqrt{E/c + mc}} \begin{pmatrix} p_z \\ p_x + ip_y \\ E/c + mc \\ 0 \end{pmatrix} & \psi_4 &= \frac{e^{i\xi/\hbar}}{\sqrt{E/c + mc}} \begin{pmatrix} p_x - ip_y \\ -p_z \\ 0 \\ E/c + mc \end{pmatrix} \\ &\text{wavefunction for antifermion spin up} & & \text{wavefunction for antifermion spin down}\end{aligned}$$

where

$$\xi = p_\mu x^\mu = Et - p_x x - p_y y - p_z z$$

and

$$E = \sqrt{p_x^2 c^2 + p_y^2 c^2 + p_z^2 c^2 + m^2 c^4}$$

Verify that ψ_1 et cetera are solutions to the Dirac equation

$$i\hbar \left(\frac{1}{c} \gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z} \right) \psi = mc\psi$$

and that the solutions are normalized as

$$|\psi|^2 = \frac{2E}{c}$$

Gamma matrices for the “Dirac representation” are

$$\begin{aligned}\gamma^0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & \gamma^1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\ \gamma^2 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} & \gamma^3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\end{aligned}$$

Let us check physical dimensions.

For the Et term in ξ we have

$$Et \propto \frac{\text{kilogram meter}^2}{\text{second}^2} \times \text{second} = \frac{\text{kilogram meter}^2}{\text{second}}$$

The momenta p_x , p_y , and p_z have units of

$$\frac{\text{kilogram meter}}{\text{second}}$$

Hence

$$-p_x x - p_y y - p_z z \propto \frac{\text{kilogram meter}}{\text{second}} \times \text{meter} = \frac{\text{kilogram meter}^2}{\text{second}}$$

We have for the reduced Planck constant

$$\hbar \propto \frac{\text{kilogram meter}^2}{\text{second}}$$

Hence ϕ/\hbar is dimensionless as required by the exponential function:

$$\frac{Et - p_x x - p_y y - p_z z}{\hbar} \propto \frac{\text{kilogram meter}^2}{\text{second}} \times \frac{\text{second}}{\text{kilogram meter}^2} = 1$$

The derivatives introduce inverse units.

$$\frac{\partial \psi}{\partial t} \propto \frac{1}{\text{second}} \quad \frac{\partial \psi}{\partial x} \propto \frac{1}{\text{meter}}$$

Hence

$$\frac{\hbar}{c} \frac{\partial \psi}{\partial t} \propto \frac{\text{kilogram meter}}{\text{second}}$$

and

$$\hbar \frac{\partial \psi}{\partial x} \propto \frac{\text{kilogram meter}}{\text{second}}$$

The resulting units match the right-hand side of the Dirac equation.

$$mc \propto \frac{\text{kilogram meter}}{\text{second}}$$

Eigenmath script