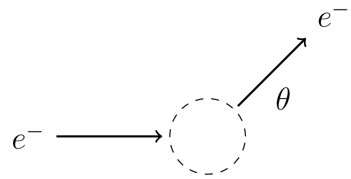
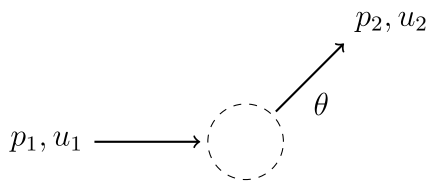
RUTHERFORD SCATTERING

Consider an electron scattered by an atomic nucleus.[1]



Here is the same diagram with momentum and spinor labels.



For a typical scattering experiment the momentum vectors are

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \qquad p_2 = \begin{pmatrix} E \\ p\sin\theta\cos\phi \\ p\sin\theta\sin\phi \\ p\cos\theta \end{pmatrix}$$

where $E = \sqrt{p^2 + m^2}$. Symbol p is electron momentum and m is electron mass. The spinors are

$$u_{11} = \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix} \qquad u_{12} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix} \qquad u_{21} = \begin{pmatrix} E+m \\ 0 \\ p_{2z} \\ p_{2x} + ip_{2y} \end{pmatrix} \qquad u_{22} = \begin{pmatrix} 0 \\ E+m \\ p_{2x} - ip_{2y} \\ -p_{2z} \end{pmatrix}$$

The second digit in a spinor subscript is the spin state. The spinors are not normalized. Instead, a combined spinor normalization constant $N = (E+m)^2$ is used.

The probability density for Rutherford scattering is

$$|\mathcal{M}(s_1, s_2)|^2 = \frac{Z^2 e^4}{q^4} \frac{1}{N} \left| \bar{u}_2 \gamma^0 u_1 \right|^2$$

Symbol s_j selects the spin of spinor j, Z is the atomic number of the nucleus, e is electron charge, and q is momentum transfer such that $q^4 = (p_1 - p_2)^4 = 16p^4 \sin^4(\theta/2)$.

[1] The original Rutherford scattering experiment in 1911 used alpha particles, not electrons. However, scattering of any charged particle by Coulomb interaction is now known as Rutherford scattering. The first Rutherford scattering experiment using electrons appears to have been done by F. L. Arnot, then a student of Rutherford, in 1929.

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is computed by summing $|\mathcal{M}|^2$ over all four spin states and then dividing by the number of inbound states. There are two inbound states.

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{2}\sum_{s_1=1}^{2}\sum_{s_2=1}^{2}|\mathcal{M}(s_1,s_2)|^2 \\
&= \frac{Z^2e^4}{2q^4}\frac{1}{N}\sum_{s_1=1}^{2}\sum_{s_2=1}^{2}\left|\bar{u}_2\gamma^0 u_1\right|^2 \\
&= \frac{Z^2e^4}{2q^4}\operatorname{Tr}\Big((\not{p}_1+m)\gamma^0(\not{p}_2+m)\gamma^0\Big) \\
&= \frac{2Z^2e^4}{q^4}\left(E^2+m^2+p^2\cos\theta\right)
\end{aligned}$$

Run "rutherford-scattering-1.txt" to verify that

$$\frac{1}{N}\sum_{s_1=1}^{2}\sum_{s_2=1}^{2}\left|\bar{u}_2\gamma^0 u_1\right|^2 = \operatorname{Tr}\Big((\not{p}_1+m)\gamma^0(\not{p}_2+m)\gamma^0\Big)$$

Run "rutherford-scattering-2.txt" to verify that

$$\frac{1}{2}\operatorname{Tr}\Big((\not{p}_1+m)\gamma^0(\not{p}_2+m)\gamma^0\Big) = 2(E^2+m^2+p^2\cos\theta)$$

and

$$q^4 = (p_1-p_2)^4 = 16p^4\sin^4(\theta/2)$$

For low energy electron beams such that $p \ll m$ we have

$$E^2+m^2+p^2\cos\theta \approx 2m^2$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{Z^2e^4m^2}{4p^4\sin^4(\theta/2)}$$

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{16\pi^2}$$

From $e^2 = 4\pi\alpha$ and $(\cos\theta-1)^2 = 4\sin^4(\theta/2)$ we have

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2m^2}{p^4(\cos\theta-1)^2}$$

We can integrate $d\sigma$ to obtain a cumulative distribution function. Recall that $d\Omega = \sin\theta\,d\theta\,d\phi$ hence

$$d\sigma = \frac{Z^2\alpha^2m^2}{p^4(\cos\theta-1)^2}\sin\theta\,d\theta\,d\phi$$

Let $I(\xi)$ be the following definite integral.

$$
\begin{aligned}
I(\xi) &= \frac{p^4}{Z^2\alpha^2 m^2}\int_0^{2\pi}\int_a^{\xi} d\sigma \\
&= 2\pi\int_a^{\xi}\frac{1}{(\cos\theta-1)^2}\sin\theta\,d\theta \\
&= 2\pi\left(\frac{1}{\cos\theta-1}\right)\Bigg|_a^{\xi} \\
&= 2\pi\left(\frac{1}{\cos\xi-1}-\frac{1}{\cos a-1}\right), \qquad a\le\xi\le\pi
\end{aligned}
$$

The minimum supported angle is a such that $a>0$ because $d\sigma$ is undefined for $\theta=0$.

Let C be the normalization constant $C=I(\pi)$. Then the cumulative distribution function $F(\theta)$ is

$$F(\theta)=C^{-1}I(\theta), \qquad a\le\theta\le\pi$$

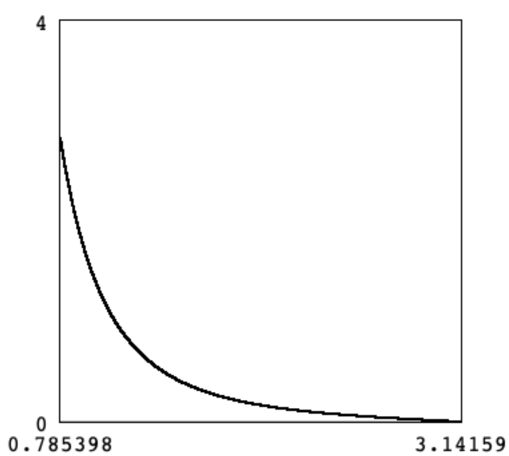
The probability of observing a scattering angle θ such that $\theta_1\le\theta\le\theta_2$ is

$$P(\theta_1\le\theta\le\theta_2)=F(\theta_2)-F(\theta_1)$$

The probability density $f(\theta)$ is the derivative of $F(\theta)$.

$$f(\theta)=\frac{dF(\theta)}{d\theta}=C^{-1}\frac{dI(\theta)}{d\theta}=C^{-1}\frac{2\pi\sin\theta}{(\cos\theta-1)^2}$$

Run “rutherford-scattering-3.txt” to draw a graph of $f(\theta)$ for $a=\pi/4=45°$.



The following table shows the corresponding probability distribution for three bins.

θ_1	θ_2	$P(\theta_1\le\theta\le\theta_2)$
0°	45°	–
45°	90°	0.83
90°	135°	0.14
135°	180°	0.03