

Harmonic oscillator propagator

Consider the harmonic oscillator eigenstate

$$\psi_n(x, t) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi \hbar} \right)^{\frac{1}{4}} H_n \left(\sqrt{\frac{m\omega}{\hbar}} x \right) \exp \left(-\frac{m\omega x^2}{2\hbar} - i \left(n + \frac{1}{2} \right) \omega t \right)$$

and the harmonic oscillator propagator

$$K(x_b, t_b, x_a, t_a) = \left(\frac{m\omega}{2\pi i \hbar \sin(\omega T)} \right)^{\frac{1}{2}} \exp \left[\frac{im\omega}{2\hbar \sin(\omega T)} (x_a^2 \cos(\omega T) - 2x_a x_b + x_b^2 \cos(\omega T)) \right]$$

where $T = t_b - t_a$.

We should have

$$\psi_n(x_b, T) = \int_{-\infty}^{\infty} K(x_b, T, x_a, 0) \psi_n(x_a, 0) dx_a$$

Try for $n = 1$.

$$\psi_1(x_a, 0) = \sqrt{2} \left(\frac{m^2 \omega^3}{\pi \hbar^3} \right)^{\frac{1}{4}} x_a \exp \left(-\frac{m\omega x_a^2}{2\hbar} \right)$$

By the identity

$$\int_{-\infty}^{\infty} y \exp(-ay^2 + by) dy = \frac{\sqrt{\pi}}{2} \frac{b}{a^{3/2}} \exp \left(\frac{b^2}{4a} \right)$$

the path integral is

$$I = \int_{-\infty}^{\infty} K(x_b, T, x_a, 0) \psi_1(x_a, 0) dx_a = \sqrt{2} \left(\frac{m^3 \omega^3}{\pi \hbar^3} \right)^{\frac{1}{4}} x_b \exp \left(-\frac{m\omega x_b^2}{2\hbar} - \frac{3}{2} i \omega T \right)$$

Hence (click here to verify)

$$I = \psi_1(x_b, T)$$

Ref. Feynman and Hibbs problem 3-12.