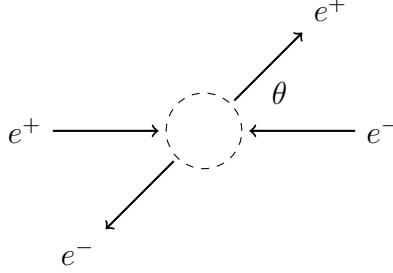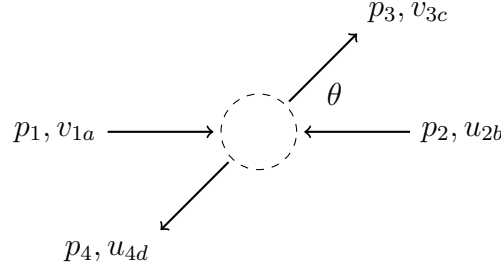# Bhabha scattering

The following diagram shows the geometry of a Bhabha scattering experiment.

Here is the same diagram with momentum and spinor labels.

In a typical collider experiment the momentum vectors are

$$
p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \qquad
p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} \qquad
p_3 = \begin{pmatrix} E \\ p\sin\theta\cos\phi \\ p\sin\theta\sin\phi \\ p\cos\theta \end{pmatrix} \qquad
p_4 = \begin{pmatrix} E \\ -p\sin\theta\cos\phi \\ -p\sin\theta\sin\phi \\ -p\cos\theta \end{pmatrix}
$$
$$
\text{inbound positron} \qquad \text{inbound electron} \qquad \text{outbound positron} \qquad \text{outbound electron}
$$

Symbol p is incident momentum. Symbol E is total energy $E = \sqrt{p^2 + m^2}$ where m is electron mass. Polar angle θ is the observed scattering angle. Azimuth angle ϕ cancels out in scattering calculations.

The spinors are

$$
v_{11} = \begin{pmatrix} p \\ 0 \\ E+m \\ 0 \end{pmatrix} \qquad
u_{21} = \begin{pmatrix} E+m \\ 0 \\ -p \\ 0 \end{pmatrix} \qquad
v_{31} = \begin{pmatrix} p_3^z \\ p_3^x + ip_3^y \\ E+m \\ 0 \end{pmatrix} \qquad
u_{41} = \begin{pmatrix} E+m \\ 0 \\ p_4^z \\ p_4^x + ip_4^y \end{pmatrix}
$$
$$
\begin{array}{cccc}
\text{inbound positron} & \text{inbound electron} & \text{outbound positron} & \text{outbound electron} \\
\text{spin up} & \text{spin up} & \text{spin up} & \text{spin up}
\end{array}
$$

$$
v_{12} = \begin{pmatrix} 0 \\ -p \\ 0 \\ E+m \end{pmatrix} \qquad
u_{22} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ p \end{pmatrix} \qquad
v_{32} = \begin{pmatrix} p_3^x - ip_3^y \\ -p_3^z \\ 0 \\ E+m \end{pmatrix} \qquad
u_{42} = \begin{pmatrix} 0 \\ E+m \\ p_4^x - ip_4^y \\ -p_4^z \end{pmatrix}
$$
$$
\begin{array}{cccc}
\text{inbound positron} & \text{inbound electron} & \text{outbound positron} & \text{outbound electron} \\
\text{spin down} & \text{spin down} & \text{spin down} & \text{spin down}
\end{array}
$$

Spinor subscripts have 1 for spin up and 2 for spin down. The spinors are not individually normalized. Instead, a combined spinor normalization constant $N = (E + m)^4$ will be used.

This is the probability density for spin state $abcd$. The formula is derived from Feynman diagrams for Bhabha scattering.

$$|\mathcal{M}_{abcd}|^2 = \frac{e^4}{N} \left| -\frac{1}{t}(\bar{v}_{1a}\gamma^\mu v_{3c})(\bar{u}_{4d}\gamma_\mu u_{2b}) + \frac{1}{s}(\bar{v}_{1a}\gamma^\nu u_{2b})(\bar{u}_{4d}\gamma_\nu v_{3c}) \right|^2$$

Symbol e is electron charge. Symbols s and t are Mandelstam variables

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$

Let

$$a_1 = (\bar{v}_{1a}\gamma^\mu v_{3c})(\bar{u}_{4d}\gamma_\mu u_{2b}), \quad a_2 = (\bar{v}_{1a}\gamma^\nu u_{2b})(\bar{u}_{4d}\gamma_\nu v_{3c})$$

Then

$$|\mathcal{M}_{abcd}|^2 = \frac{e^4}{N} \left| -\frac{a_1}{t} + \frac{a_2}{s} \right|^2$$
$$= \frac{e^4}{N} \left(-\frac{a_1}{t} + \frac{a_2}{s} \right) \left(-\frac{a_1}{t} + \frac{a_2}{s} \right)^*$$
$$= \frac{e^4}{N} \left(\frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{st} - \frac{a_1^* a_2}{st} + \frac{a_2 a_2^*}{s^2} \right)$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is computed by summing $|\mathcal{M}_{abcd}|^2$ over all spin states and then dividing by the number of inbound states. There are four inbound states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{a=1}^{2} \sum_{b=1}^{2} \sum_{c=1}^{2} \sum_{d=1}^{2} |\mathcal{M}_{abcd}|^2$$
$$= \frac{e^4}{4N} \sum_{a=1}^{2} \sum_{b=1}^{2} \sum_{c=1}^{2} \sum_{d=1}^{2} \left(\frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{st} - \frac{a_1^* a_2}{st} + \frac{a_2 a_2^*}{s^2} \right)$$

The Casimir trick uses matrix products to compute sums over spins.

$$f_{11} = \frac{1}{N} \sum_{abcd} a_1 a_1^* = \text{Tr}\left((\not{p}_1 - m)\gamma^\mu (\not{p}_3 - m)\gamma^\nu \right) \text{Tr}\left((\not{p}_4 + m)\gamma_\mu (\not{p}_2 + m)\gamma_\nu \right)$$

$$f_{12} = \frac{1}{N} \sum_{abcd} a_1 a_2^* = \text{Tr}\left((\not{p}_1 - m)\gamma^\mu (\not{p}_2 + m)\gamma^\nu (\not{p}_4 + m)\gamma_\mu (\not{p}_3 - m)\gamma_\nu \right)$$

$$f_{22} = \frac{1}{N} \sum_{abcd} a_2 a_2^* = \text{Tr}\left((\not{p}_1 - m)\gamma^\mu (\not{p}_2 + m)\gamma^\nu \right) \text{Tr}\left((\not{p}_4 + m)\gamma_\mu (\not{p}_3 - m)\gamma_\nu \right)$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left(\frac{f_{11}}{t^2} - \frac{f_{12}}{st} - \frac{f_{12}^*}{st} + \frac{f_{22}}{s^2} \right)$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$f_{11} = 32(p_1 \cdot p_2)^2 + 32(p_1 \cdot p_4)^2 - 64m^2(p_1 \cdot p_3) + 64m^4$$
$$f_{12} = -32(p_1 \cdot p_4)^2 - 32m^2(p_1 \cdot p_2) + 32m^2(p_1 \cdot p_3) - 32m^2(p_1 \cdot p_4) - 32m^4$$
$$f_{22} = 32(p_1 \cdot p_3)^2 + 32(p_1 \cdot p_4)^2 + 64m^2(p_1 \cdot p_2) + 64m^4$$

Using Mandelstam variables

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$
$$u = (p_1 - p_4)^2$$

the formulas are

$$f_{11} = 8s^2 + 8u^2 - 64sm^2 - 64um^2 + 192m^4$$
$$f_{12} = -8u^2 + 64um^2 - 96m^4$$
$$f_{22} = 8t^2 + 8u^2 - 64tm^2 - 64um^2 + 192m^4$$

High energy approximation

When $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$f_{11} = 8s^2 + 8u^2$$
$$f_{12} = -8u^2$$
$$f_{22} = 8t^2 + 8u^2$$

Hence

$$
\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4}\left(\frac{f_{11}}{t^2} - \frac{f_{12}}{st} - \frac{f_{12}^*}{st} + \frac{f_{22}}{s^2} \right) \\
&= \frac{e^4}{4}\left(\frac{8s^2 + 8u^2}{t^2} - \frac{-8u^2}{st} - \frac{-8u^2}{st} + \frac{8t^2 + 8u^2}{s^2} \right) \\
&= 2e^4\left(\frac{s^2 + u^2}{t^2} + \frac{2u^2}{st} + \frac{t^2 + u^2}{s^2} \right)
\end{aligned}
$$

Combine terms so $\langle |\mathcal{M}|^2 \rangle$ has a common denominator.

$$\langle |\mathcal{M}|^2 \rangle = 2e^4\left(\frac{s^2(s^2 + u^2) + 2stu^2 + t^2(t^2 + u^2)}{s^2 t^2} \right)$$

For $m = 0$ the Mandelstam variables are

$$s = 4E^2$$
$$t = 2E^2(\cos\theta - 1)$$
$$u = -2E^2(\cos\theta + 1)$$

Hence

$$\langle|\mathcal{M}|^2\rangle = 2e^4 \left(\frac{32E^8 \cos^4\theta + 192E^8 \cos^2\theta + 288E^8}{64E^8(\cos\theta - 1)^2} \right)$$

$$= e^4 \left(\frac{\cos^4\theta + 6\cos^2\theta + 9}{(\cos\theta - 1)^2} \right)$$

$$= e^4 \left(\frac{\cos^2\theta + 3}{\cos\theta - 1} \right)^2$$

Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{64\pi^2 s} = \frac{e^4}{64\pi^2 s} \left(\frac{\cos^2\theta + 3}{\cos\theta - 1} \right)^2, \quad s \gg m$$

Substituting $e^4 = 16\pi^2\alpha^2$ yields

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} \left(\frac{\cos^2\theta + 3}{\cos\theta - 1} \right)^2$$

We can integrate $d\sigma$ to obtain a cumulative distribution function. Recall that

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

Hence

$$d\sigma = \frac{\alpha^2}{4s} \left(\frac{\cos^2\theta + 3}{\cos\theta - 1} \right)^2 \sin\theta \, d\theta \, d\phi$$

Let $I(\theta)$ be the following integral of $d\sigma$.

$$I(\theta) = \int \left(\frac{\cos^2\theta + 3}{\cos\theta - 1} \right)^2 \sin\theta \, d\theta$$

The solution is

$$I(\theta) = \frac{16}{\cos\theta - 1} - \frac{\cos^3\theta}{3} - \cos^2\theta - 9\cos\theta - 16\log(1 - \cos\theta)$$

The cumulative distribution function is

$$F(\theta) = \frac{I(\theta) - I(a)}{I(\pi) - I(a)}, \quad a \le \theta \le \pi$$

Angular support is limited to an arbitrary $a > 0$ because $I(0)$ is undefined.

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

Data from SLAC SPEAR experiment

The following Bhabha scattering data is adapted from SLAC-PUB-1501.

	Bin	x_k, x_{k+1}	y
(Smallest θ)	$k = 1$	$0.6, 0.5$	4432
	2	$0.5, 0.4$	2841
	3	$0.4, 0.3$	2045
	4	$0.3, 0.2$	1420
	5	$0.2, 0.1$	1136
	6	$0.1, 0.0$	852
	7	$0.0, -0.1$	656
	8	$-0.1, -0.2$	625
	9	$-0.2, -0.3$	511
	10	$-0.3, -0.4$	455
	11	$-0.4, -0.5$	402
(Largest θ)	12	$-0.5, -0.6$	398

Data column x is $\cos\theta$ of scattering angle θ, data column y is number of scattering events.

To compute predicted values $\hat{y}$, start by integrating the probability density function over each bin.

$$I_k = \int_{\arccos(x_k)}^{\arccos(x_{k+1})} \left(\frac{\cos^2\theta + 3}{\cos\theta - 1} \right)^2 \sin\theta \, d\theta$$

Normalize I_k to obtain scattering probability P_k.

$$P_k = \frac{I_k}{N}, \quad N = \sum_{j=1}^{12} I_j$$

Multiply P_k by total counts to obtain $\hat{y}_k$.

$$\hat{y}_k = P_k T, \quad T = \sum_{j=1}^{12} y_j = 15773$$

Bin	x_k, x_{k+1}	y	$\hat{y}$
1	$0.6, 0.5$	4432	4598
2	$0.5, 0.4$	2841	2880
3	$0.4, 0.3$	2045	1955
4	$0.3, 0.2$	1420	1410
5	$0.2, 0.1$	1136	1068
6	$0.1, 0.0$	852	843
7	$0.0, -0.1$	656	689
8	$-0.1, -0.2$	625	582
9	$-0.2, -0.3$	511	505
10	$-0.3, -0.4$	455	450
11	$-0.4, -0.5$	402	411
12	$-0.5, -0.6$	398	382

The coefficient of determination R^2 measures how well predicted values fit the real data.

$$R^2 = 1 - \frac{\sum(y - \hat{y})^2}{\sum(y - \bar{y})^2} = 0.997$$

The result indicates that the model $d\sigma$ explains 99.7% of the variance in the data.

Data from DESY PETRA experiment

See www.hepdata.net/record/ins191231, Table 3, 14.0 GeV.

x	y
-0.73	0.10115
-0.6495	0.12235
-0.5495	0.11258
-0.4494	0.09968
-0.3493	0.14749
-0.2491	0.14017
-0.149	0.1819
-0.0488	0.22964
0.0514	0.25312
0.1516	0.30998
0.252	0.40898
0.3524	0.62695
0.4529	0.91803
0.5537	1.51743
0.6548	2.56714
0.7323	4.30279

Data x and y have the following relationship with the cross section model.

$$x = \cos\theta \qquad y = \frac{d\sigma}{d\Omega}$$

The cross section formula for Bhabha scattering is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2$$

To compute predicted values $\hat{y}$, substitute x for $\cos\theta$, multiply by $(\hbar c)^2$ to convert to SI, and multiply by 10^{37} to convert square meters to nanobarns.

$$\hat{y} = \frac{\alpha^2}{4s}\left(\frac{x^2 + 3}{x - 1}\right)^2 \times (\hbar c)^2 \times 10^{37}$$

The following table shows $\hat{y}$ for $s = (14.0\,\text{GeV})^2$.

x	y	$\hat{y}$
-0.73	0.10115	0.110296
-0.6495	0.12235	0.113816
-0.5495	0.11258	0.120101
-0.4494	0.09968	0.129075
-0.3493	0.14749	0.141592
-0.2491	0.14017	0.158934
-0.149	0.1819	0.182976
-0.0488	0.22964	0.216737
0.0514	0.25312	0.264989
0.1516	0.30998	0.335782
0.252	0.40898	0.44363
0.3524	0.62695	0.615528
0.4529	0.91803	0.9077
0.5537	1.51743	1.45175
0.6548	2.56714	2.60928
0.7323	4.30279	4.61509

The coefficient of determination R^2 measures how well predicted values fit the real data.

$$R^2 = 1 - \frac{\sum(y-\hat{y})^2}{\sum(y-\bar{y})^2} = 0.995$$

The result indicates that the model $d\sigma$ explains 99.5% of the variance in the data.

Notes

Here are a few notes about how the Eigenmath scripts work. In component notation the trace operators of the Casimir trick become sums over the repeated index α.

$$f_{11} = \left((\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_3 - m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \right) \left((\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \right)$$

$$f_{12} = (\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_3 - m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha$$

$$f_{22} = \left((\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \right) \left((\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_3 - m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \right)$$

To convert the above formulas to Eigenmath code, the γ tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply γ^μ by the metric tensor to lower the index.

$$\gamma^{\beta\mu}{}_\rho \quad \rightarrow \quad \texttt{gammaT = transpose(gamma)}$$
$$\gamma^\beta{}_{\mu\rho} \quad \rightarrow \quad \texttt{gammaL = transpose(dot(gmunu,gamma))}$$

Define the following 4×4 matrices.

$$(\not{p}_1 - m) \quad \rightarrow \quad \texttt{X1 = pslash1 - m I}$$
$$(\not{p}_2 + m) \quad \rightarrow \quad \texttt{X2 = pslash2 + m I}$$
$$(\not{p}_3 - m) \quad \rightarrow \quad \texttt{X3 = pslash3 - m I}$$
$$(\not{p}_4 + m) \quad \rightarrow \quad \texttt{X4 = pslash4 + m I}$$

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Then for f_{11} we have the following Eigenmath code. The contract function sums over α.

$$(\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_3 - m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \quad \rightarrow \quad \texttt{T1 = contract(dot(X1,gammaT,X3,gammaT),1,4)}$$

$$(\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \quad \rightarrow \quad \texttt{T2 = contract(dot(X4,gammaL,X2,gammaL),1,4)}$$

Next, multiply then sum over repeated indices. The dot function sums over ν then the contract function sums over μ. The transpose makes the ν indices adjacent as required by the dot function.

$$f_{11} = \mathrm{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu)\,\mathrm{Tr}(\cdots \gamma_\mu \cdots \gamma_\nu) \quad \rightarrow \quad \texttt{f11 = contract(dot(T1,transpose(T2)))}$$

Follow suit for f_{22}.

$$(\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \quad \rightarrow \quad \texttt{T1 = contract(dot(X1,gammaT,X2,gammaT),1,4)}$$

$$(\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_3 - m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \quad \rightarrow \quad \texttt{T2 = contract(dot(X4,gammaL,X3,gammaL),1,4)}$$

Hence

$$f_{22} = \mathrm{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu)\,\mathrm{Tr}(\cdots \gamma_\mu \cdots \gamma_\nu) \quad \rightarrow \quad \texttt{f22 = contract(dot(T1,transpose(T2)))}$$

The calculation of f_{12} begins with

$$(\not{p}_1 - m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_3 - m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha$$
$$\rightarrow \quad \texttt{T = contract(dot(X1,gammaT,X2,gammaT,X4,gammaL,X3,gammaL),1,6)}$$

Then sum over repeated indices μ and ν.

$$f_{12} = \mathrm{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu \cdots \gamma_\mu \cdots \gamma_\nu) \quad \rightarrow \quad \texttt{f12 = contract(contract(T,1,3))}$$