9-5. The momentum in the field is given by

$$\frac{1}{4\pi c}\int \mathbf{E}\times\mathbf{B}\,d^3\mathbf{r}$$

In the absence of matter (so $\phi_{\mathbf{k}} = 0$), show that this is

$$i\int \mathbf{k}\,(\mathbf{a}^*_{\mathbf{k}}\cdot\dot{\mathbf{a}}_{\mathbf{k}})\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

By equation (9.14)

$$\mathbf{A}\,d^3\mathbf{r} = \sqrt{4\pi}c\,\mathbf{a}_{\mathbf{k}}\,\frac{d^3\mathbf{k}}{(2\pi)^3} \tag{1}$$

By equation (1) and (9.9) with $\phi = 0$

$$\mathbf{E}\,d^3\mathbf{r} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} = -\sqrt{4\pi}\,\dot{\mathbf{a}}_{\mathbf{k}}\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

By equation (1) and (9.7)

$$\mathbf{B}\,d^3\mathbf{r} = \nabla\times\mathbf{A}\,d^3\mathbf{r} = \sqrt{4\pi}ic\,\mathbf{k}\times\mathbf{a}_{\mathbf{k}}\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

Hence

$$\begin{aligned}\mathbf{E}\times\mathbf{B}\,d^3\mathbf{r} &= -\sqrt{4\pi}\,\dot{\mathbf{a}}_{\mathbf{k}}\times\left(\sqrt{4\pi}ic\,\mathbf{k}\times\mathbf{a}_{\mathbf{k}}\right)\,\frac{d^3\mathbf{k}}{(2\pi)^3}\\ &= -4\pi ic\,(\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{a}_{\mathbf{k}})\mathbf{k}\,\frac{d^3\mathbf{k}}{(2\pi)^3} + 4\pi ic\,(\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{k})\mathbf{a}_{\mathbf{k}}\,\frac{d^3\mathbf{k}}{(2\pi)^3}\end{aligned}$$

By orthogonality of $\mathbf{E}$ and $\mathbf{k}$ (see problem 9-1), $\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{k} = 0$ hence

$$\mathbf{E}\times\mathbf{B}\,d^3\mathbf{r} = -4\pi ic\,(\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{a}_{\mathbf{k}})\mathbf{k}\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$