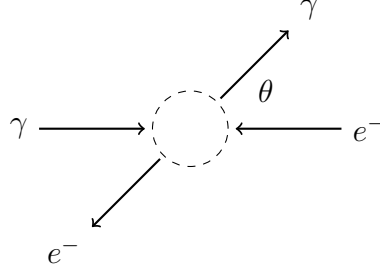


# Compton scattering

Compton scattering is the interaction  $e^- + \gamma \rightarrow e^- + \gamma$ .



In the center-of-mass frame we have the following momentum vectors where  $\omega$  is incident energy and  $E$  is total energy  $E = \sqrt{\omega^2 + m^2}$ . Polar angle  $\theta$  is the observed scattering angle. Azimuth angle  $\phi$  cancels out in scattering calculations.

$$p_1 = \begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix}, \quad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -\omega \end{pmatrix}, \quad p_3 = \begin{pmatrix} \omega \\ \omega \sin \theta \cos \phi \\ \omega \sin \theta \sin \phi \\ \omega \cos \theta \end{pmatrix}, \quad p_4 = \begin{pmatrix} E \\ -\omega \sin \theta \cos \phi \\ -\omega \sin \theta \sin \phi \\ -\omega \cos \theta \end{pmatrix}$$

inbound photon                  inbound electron                  outbound photon                  outbound electron

Spinors for the inbound electron.

$$u_{21} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ -\omega \\ 0 \end{pmatrix}, \quad u_{22} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ 0 \\ \omega \end{pmatrix}$$

inbound electron spin up                  inbound electron spin down

Spinors for the outbound electron.

$$u_{41} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix}, \quad u_{42} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}$$

outbound electron spin up                  outbound electron spin down

Let  $a$  be the spin state of the inbound electron and let  $b$  be the spin state of the outbound electron such that subscript  $ba \in \{11, 12, 21, 22\}$ .

The probability amplitude  $\mathcal{M}_{ba}$  for spin state  $ba$  is

$$\mathcal{M}_{ba} = \mathcal{M}_{1ba} + \mathcal{M}_{2ba}$$

where

$$\mathcal{M}_{1ba} = \frac{\bar{u}_{4b}(-ie\gamma^\mu)(\not{p}_1 + m)(-ie\gamma^\nu)u_{2a}}{(p_1 + p_2)^2 - m^2}, \quad \mathcal{M}_{2ba} = \frac{\bar{u}_{4b}(-ie\gamma^\nu)(\not{p}_2 + m)(-ie\gamma^\mu)u_{2a}}{(p_1 - p_4)^2 - m^2}$$

and

$$\not{q}_1 = (p_1 + p_2)^\alpha g_{\alpha\beta} \gamma^\beta, \quad \not{q}_2 = (p_4 - p_1)^\alpha g_{\alpha\beta} \gamma^\beta$$

The expected probability  $\langle |\mathcal{M}|^2 \rangle$  is the average probability for all four spin states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{a=1}^2 \sum_{b=1}^2 |\mathcal{M}_{ba}|^2$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{a=1}^2 \sum_{b=1}^2 \left( \frac{\mathcal{M}_{1ba} \mathcal{M}_{1ba}^*}{(s - m^2)^2} + \frac{\mathcal{M}_{1ba} \mathcal{M}_{2ba}^* + \mathcal{M}_{2ba} \mathcal{M}_{1ba}^*}{(s - m^2)(u - m^2)} + \frac{\mathcal{M}_{2ba} \mathcal{M}_{2ba}^*}{(u - m^2)^2} \right)$$

where  $s$  and  $u$  are Mandelstam variables

$$s = (p_1 + p_2)^2, \quad u = (p_1 - p_4)^2$$

To understand how  $\mathcal{M}_{1ba} \mathcal{M}_{1ba}^*$  is calculated, write  $\mathcal{M}_{1ba}$  in component form.

$$(\mathcal{M}_{1ba})^{\mu\nu} = \frac{(\bar{u}_{4b})_\alpha (-ie\gamma^{\mu\alpha}) (\not{q}_1 + m)^\beta_\rho (-ie\gamma^{\nu\rho}) (u_{2a})^\sigma}{s - m^2}$$

Metric tensor  $g_{\mu\nu}$  is required to sum over indices  $\mu$  and  $\nu$ .

$$\mathcal{M}_{1ba} \mathcal{M}_{1ba}^* = (\mathcal{M}_{1ba})^{\mu\nu} (\mathcal{M}_{1ba}^*)_{\mu\nu} = (\mathcal{M}_{1ba})^{\mu\nu} g_{\mu\alpha} (\mathcal{M}_{1ba}^*)^{\alpha\beta} g_{\beta\nu}$$

Similarly for  $\mathcal{M}_{2ba} \mathcal{M}_{2ba}^*$ . For  $\mathcal{M}_{2ba}$  the index order is  $\nu$  followed by  $\mu$  hence

$$\mathcal{M}_{1ba} \mathcal{M}_{2ba}^* = (\mathcal{M}_{1ba})^{\mu\nu} (\mathcal{M}_{2ba}^*)_{\nu\mu} = (\mathcal{M}_{1ba})^{\mu\nu} g_{\nu\beta} (\mathcal{M}_{2ba}^*)^{\beta\alpha} g_{\alpha\mu}$$

The Casimir trick uses matrix arithmetic to sum over spin states.

$$\begin{aligned} f_{11} &= \sum_{a=1}^2 \sum_{b=1}^2 \mathcal{M}_{1ba} \mathcal{M}_{1ba}^* = e^4 \text{Tr} \left( (\not{p}_2 + m) \gamma^\mu (\not{q}_1 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\nu (\not{q}_1 + m) \gamma_\mu \right) \\ f_{12} &= \sum_{a=1}^2 \sum_{b=1}^2 \mathcal{M}_{1ba} \mathcal{M}_{2ba}^* = e^4 \text{Tr} \left( (\not{p}_2 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\mu (\not{q}_1 + m) \gamma_\nu \right) \\ f_{22} &= \sum_{a=1}^2 \sum_{b=1}^2 \mathcal{M}_{2ba} \mathcal{M}_{2ba}^* = e^4 \text{Tr} \left( (\not{p}_2 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\nu (\not{q}_2 + m) \gamma_\mu \right) \end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \left( \frac{f_{11}}{(s - m^2)^2} + \frac{2f_{12}}{(s - m^2)(u - m^2)} + \frac{f_{22}}{(u - m^2)^2} \right) \quad (1)$$

The following formulas are equivalent to the Casimir trick. (Recall that  $a \cdot b = a^\mu g_{\mu\nu} b^\nu$ )

$$\begin{aligned} f_{11} &= e^4 (32(p_1 \cdot p_2)(p_1 \cdot p_4) + 64m^2(p_1 \cdot p_2) - 32m^2(p_1 \cdot p_3) - 32m^2(p_1 \cdot p_4) + 32m^4) \\ f_{12} &= e^4 (16m^2(p_1 \cdot p_2) - 16m^2(p_1 \cdot p_4) + 32m^4) \\ f_{22} &= e^4 (32(p_1 \cdot p_2)(p_1 \cdot p_4) + 32m^2(p_1 \cdot p_2) - 32m^2(p_1 \cdot p_3) - 64m^2(p_1 \cdot p_4) + 32m^4) \end{aligned}$$

In Mandelstam variables

$$\begin{aligned} f_{11} &= e^4 (-8su + 24sm^2 + 8um^2 + 8m^4) \\ f_{12} &= e^4 (8sm^2 + 8um^2 + 16m^4) \\ f_{22} &= e^4 (-8su + 8sm^2 + 24um^2 + 8m^4) \end{aligned} \tag{2}$$

Compton scattering experiments are typically done in the lab frame where the electron is at rest. Define Lorentz boost  $\Lambda$  for transforming momentum vectors to the lab frame.

$$\Lambda = \begin{pmatrix} E/m & 0 & 0 & \omega/m \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega/m & 0 & 0 & E/m \end{pmatrix}$$

The electron is at rest in the lab frame.

$$\Lambda p_2 = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Mandelstam variables are invariant under a boost.

$$\begin{aligned} s &= (p_1 + p_2)^2 = (\Lambda p_1 + \Lambda p_2)^2 \\ t &= (p_1 - p_3)^2 = (\Lambda p_1 - \Lambda p_3)^2 \\ u &= (p_1 - p_4)^2 = (\Lambda p_1 - \Lambda p_4)^2 \end{aligned}$$

In the lab frame, let  $\omega_L$  be the angular frequency of the incident photon and let  $\omega'_L$  be the angular frequency of the scattered photon.

$$\begin{aligned} \omega_L &= \Lambda p_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2}{m} + \frac{\omega E}{m} \\ \omega'_L &= \Lambda p_3 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2 \cos \theta}{m} + \frac{\omega E}{m} \end{aligned}$$

It can be shown that

$$\begin{aligned} s &= m^2 + 2m\omega_L \\ t &= 2m(\omega'_L - \omega_L) \\ u &= m^2 - 2m\omega'_L \end{aligned} \tag{3}$$

Then by (1), (2), and (3) we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left( \frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \left( \frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1 \right)^2 - 1 \right)$$

Lab scattering angle  $\theta_L$  is given by the Compton equation

$$\cos \theta_L = \frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1$$

Hence

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= 2e^4 \left( \frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \cos^2 \theta_L - 1 \right) \\ &= 2e^4 \left( \frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} - \sin^2 \theta_L \right) \end{aligned}$$

### Cross section

Now that we have derived  $\langle |\mathcal{M}|^2 \rangle$  we can investigate the angular distribution of scattered photons. For simplicity let us drop the  $L$  subscript from lab variables. From now on the symbols  $\omega$ ,  $\omega'$ , and  $\theta$  will be lab frame variables.

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4(4\pi\epsilon_0)^2 s} \left( \frac{\omega'}{\omega} \right)^2 \langle |\mathcal{M}|^2 \rangle$$

where

$$s = m^2 + 2m\omega = (mc^2)^2 + 2(mc^2)(\hbar\omega)$$

and  $\omega'$  is given by the Compton equation

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos \theta)}$$

For the lab frame we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

Hence in the lab frame

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\epsilon_0)^2 s} \left( \frac{\omega'}{\omega} \right)^2 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

Noting that

$$e^2 = 4\pi\epsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s} \left( \frac{\omega'}{\omega} \right)^2 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

Noting that

$$d\Omega = \sin \theta d\theta d\phi$$

we also have

$$d\sigma = \frac{\alpha^2 (\hbar c)^2}{2s} \left( \frac{\omega'}{\omega} \right)^2 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right) \sin \theta d\theta d\phi$$

Let  $S(\theta_1, \theta_2)$  be the following surface integral of  $d\sigma$ .

$$S(\theta_1, \theta_2) = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{\pi \alpha^2 (\hbar c)^2}{s} (I(\theta_2) - I(\theta_1))$$

where

$$I(\theta) = -\frac{\cos \theta}{R^2} + \log(1 + R(1 - \cos \theta)) \left( \frac{1}{R} - \frac{2}{R^2} - \frac{2}{R^3} \right) - \frac{1}{2R(1 + R(1 - \cos \theta))^2} + \frac{1}{1 + R(1 - \cos \theta)} \left( -\frac{2}{R^2} - \frac{1}{R^3} \right)$$

and

$$R = \frac{\hbar \omega}{mc^2}$$

The cumulative distribution function is

$$F(\theta) = \frac{S(0, \theta)}{S(0, \pi)} = \frac{I(\theta) - I(0)}{I(\pi) - I(0)}, \quad 0 \leq \theta \leq \pi$$

The probability of observing scattering events in the interval  $\theta_1$  to  $\theta_2$  is

$$P(\theta_1 \leq \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

Let  $N$  be the total number of scattering events from an experiment. Then the number of scattering events in the interval  $\theta_1$  to  $\theta_2$  is predicted to be

$$NP(\theta_1 \leq \theta \leq \theta_2)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi) - I(0)} \left( \frac{\omega'}{\omega} \right)^2 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right) \sin \theta$$

### Thomson scattering

For  $\hbar \omega \ll mc^2$  we have

$$\omega' = \frac{\omega}{1 + \frac{\hbar \omega}{mc^2} (1 - \cos \theta)} \approx \omega$$

Hence we can use the approximations

$$\omega = \omega' \quad \text{and} \quad s = (mc^2)^2$$

to obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \hbar^2}{2m^2 c^2} (1 + \cos^2 \theta)$$

which is the formula for Thomson scattering.

### High energy approximation

For  $\omega \gg m$  a useful approximation is to set  $m = 0$  and obtain

$$\begin{aligned}f_{11} &= e^4(-8su) \\f_{12} &= 0 \\f_{22} &= e^4(-8su)\end{aligned}$$

Hence

$$\begin{aligned}\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left( \frac{-8su}{s^2} + \frac{-8su}{u^2} \right) \\&= 2e^4 \left( -\frac{u}{s} - \frac{s}{u} \right)\end{aligned}$$

Also for  $m = 0$  the Mandelstam variables  $s$  and  $u$  are

$$\begin{aligned}s &= 4\omega^2 \\u &= -2\omega^2(\cos \theta + 1)\end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left( \frac{\cos \theta + 1}{2} + \frac{2}{\cos \theta + 1} \right)$$

### Data from a CERN LEP experiment

See “Compton Scattering of Quasi-Real Virtual Photons at LEP,” [arxiv.org/abs/hep-ex/0504012](https://arxiv.org/abs/hep-ex/0504012).

| $x$   | $y$   |
|-------|-------|
| -0.74 | 13380 |
| -0.60 | 7720  |
| -0.47 | 6360  |
| -0.34 | 4600  |
| -0.20 | 4310  |
| -0.07 | 3700  |
| 0.06  | 3640  |
| 0.20  | 3340  |
| 0.33  | 3500  |
| 0.46  | 3010  |
| 0.60  | 3310  |
| 0.73  | 3330  |

The data are for the center of mass frame and have the following relationship with the differential cross section formula.

$$x = \cos \theta, \quad y = \frac{d\sigma}{d \cos \theta} = 2\pi \frac{d\sigma}{d\Omega}$$

For the high energy approximation we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left( \frac{\cos \theta + 1}{2} + \frac{2}{\cos \theta + 1} \right)$$

The corresponding cross section formula is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{64\pi^2 s} = \frac{e^4}{32\pi^2 s} \left( \frac{\cos \theta + 1}{2} + \frac{2}{\cos \theta + 1} \right), \quad s \gg m$$

Substituting  $e^4 = 16\pi^2 \alpha^2$  yields

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s} \left( \frac{\cos \theta + 1}{2} + \frac{2}{\cos \theta + 1} \right)$$

Multiply by  $2\pi$  to obtain

$$\frac{d\sigma}{d\cos \theta} = \frac{\pi \alpha^2}{s} \left( \frac{\cos \theta + 1}{2} + \frac{2}{\cos \theta + 1} \right)$$

To compute predicted values  $\hat{y}$  from the above formula, multiply by  $(hc)^2$  to convert to SI and multiply by  $10^{40}$  to convert square meters to picobarns.

$$\hat{y} = \frac{\pi \alpha^2}{s} \left( \frac{x + 1}{2} + \frac{2}{x + 1} \right) \times (hc)^2 \times 10^{40}$$

The following table shows  $\hat{y}$  for  $s = (40 \text{ GeV})^2$ .

| $x$   | $y$   | $\hat{y}$ |
|-------|-------|-----------|
| -0.74 | 13380 | 12573     |
| -0.60 | 7720  | 8358      |
| -0.47 | 6360  | 6491      |
| -0.34 | 4600  | 5401      |
| -0.20 | 4310  | 4661      |
| -0.07 | 3700  | 4204      |
| 0.06  | 3640  | 3884      |
| 0.20  | 3340  | 3643      |
| 0.33  | 3500  | 3486      |
| 0.46  | 3010  | 3375      |
| 0.60  | 3310  | 3295      |
| 0.73  | 3330  | 3248      |

The coefficient of determination  $R^2$  measures how well predicted values fit the data.

$$R^2 = 1 - \frac{\sum (y - \hat{y})^2}{\sum (y - \bar{y})^2} = 0.97$$

The result indicates that the model  $d\sigma$  explains 97% of the variance in the data.