Bell's theorem

The following theorem of independent random variables is the key to understanding Bell's theorem. If two random variables A and B are independent (uncorrelated) then

$$\langle A\rangle\langle B\rangle = \langle AB\rangle$$

Imagine two machines A and B that measure spin. Each machine can be set in one of two orientations labeled 0 and 1. Assuming the measurements are uncorrelated we have the following table of expectation values and a clever formula.

$\langle A_0\rangle$	$\langle A_1\rangle$	$\langle B_0\rangle$	$\langle B_1\rangle$	$\langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle$
1	1	1	1	2
1	1	1	−1	2
1	1	−1	1	−2
1	1	−1	−1	−2
1	−1	1	1	2
1	−1	1	−1	−2
1	−1	−1	1	2
1	−1	−1	−1	−2
−1	1	1	1	−2
−1	1	1	−1	2
−1	1	−1	1	−2
−1	1	−1	−1	2
−1	−1	1	1	−2
−1	−1	1	−1	−2
−1	−1	−1	1	2
−1	−1	−1	−1	2

Since the table is for all minimum and maximum values we have by inspection the range

$$-2 \le \langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle \le 2 \tag{1}$$

Now suppose a third machine generates two spins in the following singlet state.

$$|s\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

One spin is sent to A and the other is sent to B.

Let

$$A_0 = \sigma_z,\quad A_1 = \sigma_x,\quad B_0 = -\frac{\sigma_x+\sigma_z}{\sqrt{2}},\quad B_1 = \frac{\sigma_x-\sigma_z}{\sqrt{2}}$$

Then for the singlet state we have

$$\langle A_0B_0\rangle = \frac{1}{\sqrt{2}},\quad \langle A_0B_1\rangle = \frac{1}{\sqrt{2}},\quad \langle A_1B_0\rangle = \frac{1}{\sqrt{2}},\quad \langle A_1B_1\rangle = -\frac{1}{\sqrt{2}}$$

Hence

$$\langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle = 2\sqrt{2} \tag{2}$$

The result in (2) conflicts with (1) because for the singlet state the random variables are not independent. Any theory that attempts to make A and B uncorrelated for the singlet state (for example, a hidden variable theory) is constrained by (1) and falsified by (2).

Exercises

1. Verify equation (2).

2. Verify that for the singlet state

$$\langle A_0\rangle = 0, \quad \langle A_1\rangle = 0, \quad \langle B_0\rangle = 0, \quad \langle B_1\rangle = 0$$

Hence $\langle A\rangle\langle B\rangle \neq \langle AB\rangle$ for the singlet state.