3-12. If the wave function for a harmonic oscillator is (at $t = 0$)

$$\psi(x,0) = \exp\left(-\frac{m\omega}{2\hbar}(x-a)^2\right) \tag{3.67}$$

then using equation (3.42) and the results of problem 3-8, show that

$$\psi(x,T) = \exp\left(-\frac{i\omega T}{2} - \frac{m\omega}{2\hbar}\left(x^2 - 2ax\exp(-i\omega T) + a^2\cos(\omega T)\exp(-i\omega T)\right)\right) \tag{3.68}$$

and find the probability distribution $|\psi|^2$.

From problem 3-8

$$K(x,T,x_c,0) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega T)}\right)^{1/2} \exp\left(\frac{im\omega}{2\hbar\sin(\omega T)}\left((x^2+x_c^2)\cos(\omega T) - 2xx_c\right)\right) \tag{2}$$

From equation (3.42)

$$\psi(x,T) = \int_{-\infty}^{\infty} K(x,T,x_c,0)\psi(x_c,0)\,dx_c \tag{3}$$

Substitute (3.68) and (2) into (3) to obtain

$$\begin{aligned}\psi(x,T) = &\left(\frac{m\omega}{2\pi i\hbar\sin(\omega T)}\right)^{1/2} \\ &\times \int_{-\infty}^{\infty}\exp\left(\frac{im\omega}{2\hbar\sin(\omega T)}\left((x^2+x_c^2)\cos(\omega T) - 2xx_c\right)\right) \\ &\times \exp\left(-\frac{m\omega}{2\hbar}(x_c-a)^2\right)\,dx_c\end{aligned}$$

Rewrite as

$$\psi(x,T) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega T)}\right)^{1/2} \int_{-\infty}^{\infty}\exp(Ax_c^2 + Bx_c + C)\,dx_c \tag{4}$$

where

$$
\begin{aligned}
A &= \frac{m\omega}{2\hbar}\left(\frac{i\cos(\omega T)}{\sin(\omega T)} - 1\right) = \frac{im\omega\exp(i\omega T)}{2\hbar\sin(\omega T)} \\
B &= \frac{m\omega}{\hbar}\left(a - \frac{ix}{\sin(\omega T)}\right) \\
C &= \frac{m\omega}{2\hbar}\left(\frac{ix^2\cos(\omega T)}{\sin(\omega T)} - a^2\right)
\end{aligned}
$$

Solve the integral.

$$
\int_{-\infty}^{\infty} \exp(Ax_c^2 + Bx_c + C) = \left(-\frac{\pi}{A}\right)^{1/2} \exp\left(-\frac{B^2}{4A} + C\right) \tag{5}
$$

where

$$
-\frac{\pi}{A} = \frac{2\pi i\hbar\sin(\omega T)}{m\omega\exp(i\omega T)} \tag{6}
$$

and

$$
-\frac{B^2}{4A} + C = -\frac{m\omega}{2\hbar}\left(x^2 - 2ax\exp(-i\omega T) + a^2\cos(\omega T)\exp(-i\omega T)\right) \tag{7}
$$

Note that

$$
\underbrace{\frac{m\omega}{2\pi i\hbar\sin(\omega T)}}_{\text{from equation (4)}} \times \underbrace{\frac{2\pi i\hbar\sin(\omega T)}{m\omega\exp(i\omega T)}}_{\text{from equation (6)}} = \exp(-i\omega T) \tag{8}
$$

Substitute the solved integral (5) into (3) to obtain

$$
\begin{aligned}
\psi(x,T) &= \exp\left(-\frac{i\omega T}{2}\right) \\
&\quad \times \exp\left(-\frac{m\omega}{2\hbar}\left(x^2 - 2ax\exp(-i\omega T) + a^2\cos(\omega T)\exp(-i\omega T)\right)\right)
\end{aligned}
$$