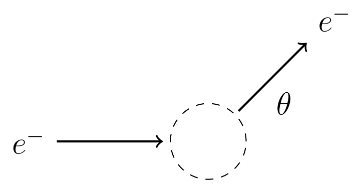
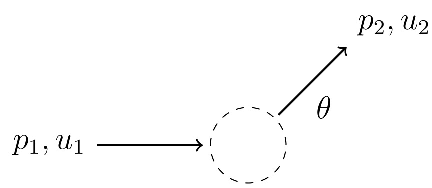
Consider an electron scattered by an atomic nucleus.



Here is the same diagram with momentum and spinor labels.



The path of the incident electron can be modeled as the z axis, resulting in the following momentum vectors.

$$\underset{\text{inbound electron}}{p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}} \qquad \underset{\text{outbound electron}}{p_2 = \begin{pmatrix} E \\ p\sin\theta\cos\phi \\ p\sin\theta\sin\phi \\ p\cos\theta \end{pmatrix}}$$

Symbol p is electron momentum and E is total energy $E = \sqrt{p^2 + m^2}$ where m is electron mass. Polar angle θ is the observed scattering angle. Azimuth angle ϕ cancels out in scattering calculations.

The spinors are

$$\underset{\text{inbound electron, spin up}}{u_{11} = \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix}} \qquad \underset{\text{inbound electron, spin down}}{u_{12} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix}} \qquad \underset{\text{outbound electron, spin up}}{u_{21} = \begin{pmatrix} E+m \\ 0 \\ p_{2z} \\ p_{2x} + ip_{2y} \end{pmatrix}} \qquad \underset{\text{outbound electron, spin down}}{u_{22} = \begin{pmatrix} 0 \\ E+m \\ p_{2x} - ip_{2y} \\ -p_{2z} \end{pmatrix}}$$

The spinors shown above are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^2$ will be used.

The following formula computes a probability density $|\mathcal{M}_{jk}|^2$ for Rutherford scattering where j is the spin state of the inbound electron and k is the spin state of the outbound electron.

$$|\mathcal{M}_{jk}|^2 = \frac{Z^2 e^4}{q^4} \frac{1}{N} \left| \bar{u}_{2k} \gamma^0 u_{1j} \right|^2$$

Symbol Z is the atomic number of the nucleus, e is electron charge, and $q = p_1 - p_2$ is momentum transfer.

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is computed by summing $|\mathcal{M}_{jk}|^2$ over all four spin states and then dividing by the number of inbound states. There are two inbound states.

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2}|\mathcal{M}_{jk}|^2 \\
&= \frac{Z^2e^4}{2q^4}\frac{1}{N}\sum_{j=1}^{2}\sum_{k=1}^{2}\left|\bar{u}_{2k}\gamma^0 u_{1j}\right|^2 \\
&= \frac{Z^2e^4}{2q^4}\operatorname{Tr}\Big((\not{p}_1+m)\gamma^0(\not{p}_2+m)\gamma^0\Big) \\
&= \frac{2Z^2e^4}{q^4}\left(E^2+m^2+p^2\cos\theta\right)
\end{aligned}$$

Run “rutherford-scattering-1.txt” to verify the following formulas.

$$\frac{1}{N}\sum_{j=1}^{2}\sum_{k=1}^{2}\left|\bar{u}_{2k}\gamma^0 u_{1j}\right|^2 = \operatorname{Tr}\Big((\not{p}_1+m)\gamma^0(\not{p}_2+m)\gamma^0\Big) = 4(E^2+m^2+p^2\cos\theta)$$

$$q^4 = (p_1-p_2)^4 = 16p^4\sin^4(\theta/2) = 4p^4(\cos\theta-1)^2$$

Low energy approximation

For low energy electrons such that $p \ll m$ we can use the following approximation.

$$E^2+m^2+p^2\cos\theta \approx 2m^2$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{4m^2Z^2e^4}{q^4}$$

Substituting $e^2 = 4\pi\alpha$ and $q^4 = 4p^4(\cos\theta-1)^2$ we have

$$\langle|\mathcal{M}|^2\rangle = \frac{16\pi^2m^2Z^2\alpha^2}{p^4(\cos\theta-1)^2}$$

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{16\pi^2} = \frac{m^2Z^2\alpha^2}{p^4(\cos\theta-1)^2}$$

We can integrate $d\sigma$ to obtain a cumulative distribution function. Recall that

$$d\Omega = \sin\theta\,d\theta\,d\phi$$

Hence

$$d\sigma = \frac{m^2Z^2\alpha^2}{p^4(\cos\theta-1)^2}\sin\theta\,d\theta\,d\phi$$

Let $I(\xi)$ be the following definite integral.

$$\begin{aligned}
I(\xi) &= \frac{p^4}{2\pi m^2 Z^2 \alpha^2} \int_0^{2\pi} \int_a^{\xi} d\sigma \\
&= \int_a^{\xi} \frac{1}{(\cos\theta - 1)^2} \sin\theta \, d\theta \\
&= \left(\frac{1}{\cos\theta - 1} \right) \Bigg|_a^{\xi} \\
&= \frac{1}{\cos\xi - 1} - \frac{1}{\cos a - 1}, \qquad a \le \xi \le \pi
\end{aligned}$$

A lower bound of $a > 0$ is required because $I(0)$ is undefined.

Let C be the normalization constant $C = I(\pi)$. Then the cumulative distribution function $F(\theta)$ is

$$F(\theta) = C^{-1} I(\theta), \qquad a \le \theta \le \pi$$

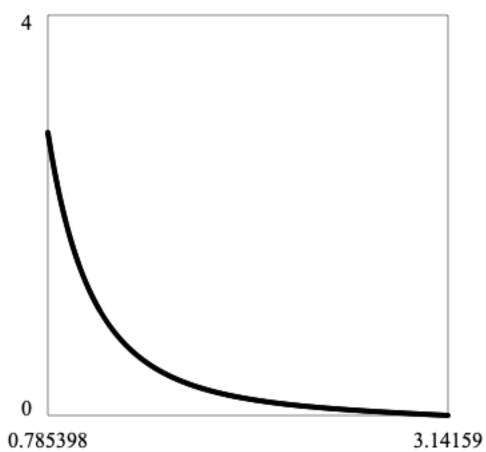
The probability of observing scattering events in the interval θ_1 to θ_2 can now be computed.

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

Probability density function $f(\theta)$ is the derivative of $F(\theta)$.

$$f(\theta) = \frac{dF(\theta)}{d\theta} = C^{-1} \frac{dI(\theta)}{d\theta} = C^{-1} \frac{\sin\theta}{(\cos\theta - 1)^2}$$

Run “rutherford-scattering-3.txt” to draw a graph of $f(\theta)$ for $a = \pi/4 = 45°$.



The following table shows the corresponding probability distribution for three bins.

θ_1	θ_2	$P(\theta_1 \le \theta \le \theta_2)$
0°	45°	–
45°	90°	0.83
90°	135°	0.14
135°	180°	0.03

Note: The original Rutherford scattering experiment in 1911 used alpha particles, not electrons. However, scattering of any charged particle by Coulomb interaction is now known as Rutherford scattering. The first Rutherford scattering experiment using electrons appears to have been done by F. L. Arnot, then a student of Rutherford, in 1929.