$$-1<\theta<1 \quad \Rightarrow \quad \sum_{k=0}^{\infty}\theta^k=\frac{1}{1-\theta} \qquad \sum_{k=1}^{\infty}\theta^k=\frac{\theta}{1-\theta} \qquad \frac{d}{dx}\log x=\frac{1}{x}$$

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad \text{Binomial Thm } (x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$$

$$\mathrm{Cov}(X,Y)=E[(X-EX)(Y-EY)]=E(XY)-E(X)E(Y)$$

$$\gamma(h)=\mathrm{Cov}(X_{t+h},X_t)=E(X_{t+h}X_t) \text{ when } EX=0 \qquad \gamma(0)=\mathrm{Var}\,X=\sigma_X^2$$

$$\hat{\gamma}(h)=\frac{1}{n}\sum_{t=1}^{n-h}(x_{t+h}-\bar{x})(x_t-\bar{x}) \qquad \nabla_{12}X_t=X_t-X_{t-12} \qquad \nabla^d=(1-B)^d$$

$$\cos A\cos B+\sin A\sin B=\cos(A-B)=\cos(B-A)$$

$$\cos 2\lambda=2\cos^2\lambda-1 \qquad 2\cos\theta=e^{i\theta}+e^{-i\theta} \qquad 2i\sin\theta=e^{i\theta}-e^{-i\theta} \qquad (-1)^{|h|}=\cos(\pi h)$$

The noise band is $\pm 1.96/\sqrt{n}$ Prediction bounds $\sigma_n^2(h)\approx\sigma^2\sum_{j=0}^{h-1}\psi_j^2$

$\boxed{\text{Estimation of } \mu}$

$$\sqrt{n}\,(\bar{X}-\mu)\sim N\left(0,\sum_{h=-\infty}^{\infty}\gamma(h)\right) \qquad \Rightarrow \qquad \bar{X}\sim N\left(\mu,\frac{1}{n}\sum_{h=-\infty}^{\infty}\gamma(h)\right)$$

The 95% confidence interval for μ is $\bar{x}\pm 1.96\sqrt{v/n}$ where $v=\sum_{h=-\infty}^{\infty}\gamma(h)$

$$\mathrm{AR}(1) \quad \Rightarrow \quad v=\frac{\sigma^2}{(1-\phi)^2} \qquad \mathrm{MA}(1) \quad \Rightarrow \quad v=\sigma^2(1+2\theta+\theta^2)$$

$\boxed{\text{Best Linear Predictor}}$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}=\begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix}^{-1}\begin{pmatrix} \gamma(1) \\ \gamma(2) \end{pmatrix} \qquad a_0=\mu(1-a_1-a_2)$$

$$P_nX_{n+h}=\mu+a_1(X_n-\mu)+\cdots+a_n(X_1-\mu)=a_0+a_1X_n+\cdots+a_nX_1$$

$$E(X_{n+h}-P_nX_{n+h})^2=\gamma(0)-\mathbf{a}_n^T\gamma_n(h)$$

$$E(X_{n+h}-P_nX_{n+h})=0$$

$$E[(X_{n+h}-P_nX_{n+h})X_j]=0,\ j=1,\ldots,n$$

$$E(X_{n+1}-\hat{X}_{n+1})^2=\sigma^2 r_n=v_n$$

$\boxed{\text{Innovations}}$

$$v_0=E[(X_1-\hat{X}_1)^2]=\gamma(0) \qquad \theta_{11}=\frac{\gamma(1)}{\gamma(0)}$$

$$v_1=E[(X_2-\hat{X}_2)^2]=\gamma(0)-\frac{\gamma(1)^2}{\gamma(0)} \qquad \theta_{22}=\frac{\gamma(2)}{\gamma(0)} \qquad \theta_{21}=\frac{\gamma(1)[\gamma(0)-\gamma(2)]}{\gamma(0)^2-\gamma(1)^2} \qquad \theta_{33}=\frac{\gamma(3)}{\gamma(0)}$$

$\boxed{\text{MA}(\infty)}$ $\quad X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ $\quad \boxed{\text{AR}(\infty)}$ $\quad Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$

$\boxed{\text{AR}(1)}$ $\quad X_t - \phi X_{t-1} = Z_t \quad \{Z_t\} \sim \text{WN}(0, \sigma^2) \quad \gamma_X(h) = \dfrac{\sigma^2 \phi^{|h|}}{1-\phi^2} \quad \rho_X(h) = \phi^{|h|}$

$$\hat{\phi} = \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)} \qquad \hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\gamma}(1)\hat{\phi} \qquad \bar{x} \pm 1.96 \frac{\sigma}{(1-\phi)\sqrt{n}} \qquad E(X_{n+h} - P_n X_{n+h})^2 = \frac{\sigma^2(1-\phi^{2h})}{1-\phi^2}$$

$$|\phi| < 1 \Rightarrow \quad X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j} \qquad |\phi| > 1 \Rightarrow \quad X_t = -\sum_{j=1}^{\infty} \phi^{-j} Z_{t+j} \qquad \sum_{h=-\infty}^{\infty} \gamma_X(h) = \frac{\sigma^2}{(1-\phi)^2}$$

$$f(\lambda) = \frac{\sigma^2}{2\pi} \left(1 - 2\phi \cos\lambda + \phi^2\right)^{-1}$$

$\boxed{\text{MA}(1)}$ $\quad X_t = Z_t + \theta Z_{t-1} \quad \{Z_t\} \sim \text{WN}(0, \sigma^2) \quad \gamma_X(h) = \begin{cases} \sigma^2(1+\theta^2) & h = 0 \\ \sigma^2\theta & h = \pm 1 \\ 0 & |h| > 1 \end{cases}$

$$\bar{x}_n \pm 1.96\sqrt{v/n} \qquad v = \sum_{h=-\infty}^{\infty} \gamma(h) = \sigma^2(1 + 2\theta + \theta^2) \qquad f(\lambda) = \frac{\sigma^2}{2\pi}\left(1 + 2\theta\cos\lambda + \theta^2\right)$$

$\boxed{\text{ARMA(1,1)}}$

$\psi_0 = 1 \qquad \psi_j = (\phi + \theta)\phi^{j-1},\ j = 1, 2, \ldots$

$\pi_0 = 1 \qquad \pi_j = -(\phi + \theta)(-\theta)^{j-1},\ j = 1, 2, \ldots$

$$\gamma(0) = \sigma^2\left[1 + \frac{(\theta+\phi)^2}{1-\phi^2}\right] \qquad \gamma(1) = \sigma^2\left[\theta + \phi + \frac{(\theta+\phi)^2\phi}{1-\phi^2}\right] \qquad \gamma(h) = \phi^{h-1}\gamma(1),\ h \geq 2$$

$\boxed{\text{ARMA}(p,q)}$

$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$ Roots of $\phi(z)$ greater than 1 implies causality.

$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$ Roots of $\theta(z)$ greater than 1 implies invertibility.

$$\psi(z) = \frac{\theta(z)}{\phi(z)} = 1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \cdots$$

$$\psi_j = \theta_j + \sum_{k=1}^{p} \phi_k \psi_{j-k} \qquad \pi_j = -\phi_j - \sum_{k=1}^{q} \theta_k \pi_{j-k} \qquad f(\lambda) = \frac{\sigma^2}{2\pi} \frac{\left|\theta\left(e^{-i\lambda}\right)\right|^2}{\left|\phi\left(e^{-i\lambda}\right)\right|^2} \qquad -\pi \leq \lambda \leq \pi$$

$\boxed{\text{Yule-Walker}}$ $\quad \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix} = \begin{pmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) \end{pmatrix}^{-1} \begin{pmatrix} \hat{\gamma}(1) \\ \hat{\gamma}(2) \end{pmatrix} \quad \hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\gamma}(1)\hat{\phi}_1 - \hat{\gamma}(2)\hat{\phi}_2$

3.2.1 Calculation of the ACVF

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j},\ \{Z_t\} \sim \mathrm{WN}(0,\sigma^2) \quad \Rightarrow \quad \gamma_X(h) = E(X_{t+h}X_t) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}$$

$$\gamma_X(0) = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2 \qquad \mathrm{MA}(q) \Rightarrow \quad \gamma_X(h) = \begin{cases} \sigma^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|} & |h| \le q \\ 0 & |h| > q \end{cases}$$

4.1 Spectral Densities

$$\mathrm{WN} \Rightarrow f(\lambda) = \frac{\sigma^2}{2\pi}$$

$\lambda =$ radians $\quad \omega =$ radians/sec $\quad$ period $= 2\pi$ radians$/\omega =$ seconds

$f(\lambda) =$ spectral density function $\quad F(\lambda) =$ spectral distribution function

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma(h) = \frac{1}{2\pi}\left[\gamma(0) + \sum_{h=1}^{\infty}(e^{-ih\lambda} + e^{ih\lambda})\gamma(h)\right],\ -\infty < \lambda < \infty$$

$$f(\lambda) \ge 0 \qquad f(-\lambda) = f(\lambda) \qquad f(\lambda + 2\pi) = f(\lambda) \qquad \int_{-\pi}^{\pi} f(\lambda)\, d\lambda = \gamma(0)$$

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)\, d\lambda = \int_{-\pi}^{\pi} \cos(h\lambda) f(\lambda)\, d\lambda$$

$$F(\lambda) = \int_{-\pi}^{\lambda} f(y)\, dy \qquad F(-\pi) = 0 \qquad F(\pi) = \gamma(0) \qquad d\,F(\lambda) = f(\lambda)\, d\lambda$$

$$X_t = A\cos(\omega t) + B\sin(\omega t);\ A, B \sim (0, \nu^2);\ \gamma_X(h) = \nu^2 \cos(\omega h);\ F_X(\lambda) = \begin{cases} 0 & \lambda < -\omega \\ \nu^2/2 & -\omega \le \lambda < \omega \\ \nu^2 & \lambda \ge \omega \end{cases}$$

4.3 Time-Invariant Linear Filters

$$\text{Filter } Y_t = \sum_{j=-\infty}^{\infty} \psi_j X_{t-j} \quad \Rightarrow \quad f_Y(\lambda) = \left|\Psi\left(e^{-i\lambda}\right)\right|^2 f_X(\lambda) \quad = \Psi\left(e^{-i\lambda}\right)\Psi\left(e^{i\lambda}\right) f_X(\lambda)$$

$$\text{Transfer function } \Psi\left(e^{-i\lambda}\right) = \sum_{j=-\infty}^{\infty} \psi_j e^{-ij\lambda} \qquad \text{Power transfer function } \left|\Psi\left(e^{-i\lambda}\right)\right|^2$$

$$Y_t = \frac{1}{2q+1} \sum_{j=-q}^{q} X_{t-j} \quad \Rightarrow \quad f_Y(\lambda) = \frac{1}{2q+1}\left(\sum_{j=-q}^{q} e^{-ij\lambda}\right)^2 f_X(\lambda)$$

$$q = 1 \quad \Rightarrow \quad f_Y(\lambda) = \tfrac{1}{3}\left(e^{i\lambda} + 1 + e^{-i\lambda}\right)^2 f_X(\lambda)$$

5.3 Diagnostic Checking

$$\hat{W}_t = \frac{X_t - \hat{X}_t}{\sqrt{r_{t-1}}} \quad t = 1, \ldots, n \qquad \text{The noise band is } \pm 1.96/\sqrt{n}$$

$\boxed{\text{5.4 Forecasting}}$ $\quad P_nX_{n+1} = \phi_1 X_n + \cdots + \phi_p X_{n+1-p} + \sum_{j=1}^{q} \theta_{nj}(X_{n+1-j} - \hat{X}_{n+1-j}) \quad \theta_{nj} \approx \theta_j$

$X_{n+h} - \hat{X}_{n+h} = 0$ for $h \geq 1$ $\quad v_n = \sigma^2 r_n = E[(X_{n+1} - P_nX_{n+1})^2] = \sigma_n^2(1) \approx \sigma^2 \sum_{j=0}^{0} \psi_j^2 = \sigma^2$

$\boxed{\text{5.5 Order Selection}}$ $\quad \text{AICC} = -2\log L + 2(p+q+1)n/(n-p-q-2)$

$$L = \frac{1}{\sqrt{(2\pi)^n v_0 \cdots v_{n-1}}} \exp\left[-\frac{1}{2}\sum_{j=1}^{n} \frac{(X_j - \hat{X}_j)^2}{v_{j-1}}\right] \qquad v_n = \sigma^2 r_n$$

$$\ell = \log L = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2}\sum_{j=1}^{n}\log r_{j-1} - \frac{1}{2\sigma^2}\sum_{j=1}^{n}\frac{(X_j - \hat{X}_j)^2}{r_{j-1}}$$

Solve $\frac{\partial \ell}{\partial \sigma^2} = 0$ to obtain $\hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n}\frac{(X_j - \hat{X}_j)^2}{r_{j-1}}$

Maximize reduced log likelihood $\ell^* = -\frac{n}{2}\log\hat{\sigma}^2 - \frac{1}{2}\sum_{j=1}^{n}\log r_{j-1}$

Or multiply by $-2/n$ and minimize $\ell^* = \log\hat{\sigma}^2 + \frac{1}{n}\sum_{j=1}^{n}\log r_{j-1}$ (5.2.12) on p. 160

$$E(Y_{n+1} - \hat{\phi}_1 Y_n - \cdots - \hat{\phi}_p Y_{n+1-p})^2 \approx \sigma^2\left(1 + \frac{p}{n}\right)$$

$\boxed{\text{Classical Decomposition}}$ $\quad X_t = m_t + s_t + Y_t \quad EY_t = 0 \quad s_{t+d} = s_t \quad \sum_{j=1}^{d} s_j = 0$

No distortion $\Rightarrow$ $\quad m_t = \sum_j a_j m_{t-j}$ for all polynomials $m_t = c_0 + c_1 t + \cdots + c_k t^k$

No distortion $\Rightarrow$ $\quad \sum_j a_j = 1$ and $\sum_j j^r a_j = 0$ for $j = 1, \ldots, k$

Eliminate seasonal components $\Rightarrow$ $\quad \sum_j a_j s_{t-j} = \text{const} \times \sum_{j=1}^{d} s_j = 0$

Model	ACF	PACF
AR(p)	Decays	Zero for $h > p$
MA(q)	Zero for $h > q$	Decays
ARMA(p,q)	Decays	Decays