

Feynman and Hibbs problem 4-2

For a particle of charge  $e$  in a magnetic field the Lagrangian is

$$L(\dot{\mathbf{x}}, \mathbf{x}) = \frac{m}{2} \dot{\mathbf{x}}^2 + \frac{e}{c} \dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}, t) - e\phi(\mathbf{x}, t)$$

where  $\dot{\mathbf{x}}$  is the velocity vector,  $c$  is the velocity of light, and  $\mathbf{A}$  and  $\phi$  are the vector and scalar potentials. Show that the corresponding Schrodinger equation is

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} \left( \frac{1}{2m} \left( \frac{\hbar}{i} \nabla - \frac{e}{c} \mathbf{A} \right) \cdot \left( \frac{\hbar}{i} \nabla - \frac{e}{c} \mathbf{A} \right) \psi + e\phi \psi \right)$$

From equation (4.3)

$$\psi(\mathbf{x}, t + \epsilon) = \frac{1}{A} \int_{\mathbb{R}^3} \exp \left( \frac{i\epsilon}{\hbar} L \left( \frac{\mathbf{x} - \mathbf{y}}{\epsilon}, \frac{\mathbf{x} + \mathbf{y}}{2} \right) \right) \psi(\mathbf{y}, t) dy_1 dy_2 dy_3 \quad (1)$$

By substitution of the given Lagrangian

$$\begin{aligned} L \left( \frac{\mathbf{x} - \mathbf{y}}{\epsilon}, \frac{\mathbf{x} + \mathbf{y}}{2} \right) \\ = \frac{m}{2\epsilon^2} (\mathbf{x} - \mathbf{y})^2 + \frac{e}{c\epsilon} (\mathbf{x} - \mathbf{y}) \cdot \mathbf{A} \left( \frac{\mathbf{x} + \mathbf{y}}{2}, t \right) - e\phi \left( \frac{\mathbf{x} + \mathbf{y}}{2}, t \right) \end{aligned}$$

Then from equation (1)

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) &= \frac{1}{A} \int_{\mathbb{R}^3} \\ &\exp \left( \frac{im}{2\hbar\epsilon} (\mathbf{x} - \mathbf{y})^2 + \frac{ie}{\hbar c} (\mathbf{x} - \mathbf{y}) \cdot \mathbf{A} \left( \frac{\mathbf{x} + \mathbf{y}}{2}, t \right) - \frac{ie\epsilon}{\hbar} \phi \left( \frac{\mathbf{x} + \mathbf{y}}{2}, t \right) \right) \\ &\times \psi(\mathbf{y}, t) dy_1 dy_2 dy_3 \end{aligned}$$

Let

$$\mathbf{y} = \mathbf{x} + \boldsymbol{\eta}$$

Then

$$\mathbf{x} - \mathbf{y} = -\boldsymbol{\eta}, \quad \frac{\mathbf{x} + \mathbf{y}}{2} = \mathbf{x} + \frac{1}{2}\boldsymbol{\eta}, \quad dy_1 dy_2 dy_3 = d\eta_1 d\eta_2 d\eta_3$$

Hence

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) = \frac{1}{A} \int_{\mathbb{R}^3} \exp \left( \frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 + \frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A} \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) - \frac{ie\epsilon}{\hbar} \phi \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \\ \times \psi(\mathbf{x} + \boldsymbol{\eta}, t) d\eta_1 d\eta_2 d\eta_3 \end{aligned}$$

Factor the exponential.

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) = \frac{1}{A} \int_{\mathbb{R}^3} \exp \left( \frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 \right) \exp \left( \frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A} \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \exp \left( -\frac{ie\epsilon}{\hbar} \phi \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \\ \times \psi(\mathbf{x} + \boldsymbol{\eta}, t) d\eta_1 d\eta_2 d\eta_3 \end{aligned} \quad (2)$$

From the identity  $\exp(i\theta) = \cos(\theta) + i \sin(\theta)$  we have

$$\begin{aligned} \exp \left( -\frac{ie\epsilon}{\hbar} \phi \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \\ = \cos \left( -\frac{e\epsilon}{\hbar} \phi \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) + i \sin \left( -\frac{e\epsilon}{\hbar} \phi \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \end{aligned}$$

Then for small  $\epsilon$

$$\exp \left( -\frac{ie\epsilon}{\hbar} \phi \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \approx 1 - \frac{ie\epsilon}{\hbar} \phi \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right)$$

The authors write that the  $\boldsymbol{\eta}$  term can be dropped “because the error is of higher order than  $\epsilon$ .” Hence

$$\exp \left( -\frac{ie\epsilon}{\hbar} \phi \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \approx 1 - \frac{ie\epsilon}{\hbar} \phi(\mathbf{x}, t) \quad (3)$$

Substitute (3) into (2).

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) = \frac{1}{A} \int_{\mathbb{R}^3} \exp \left( \frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 \right) \exp \left( \frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A} \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \left( 1 - \frac{ie\epsilon}{\hbar} \phi(\mathbf{x}, t) \right) \\ \times \psi(\mathbf{x} + \boldsymbol{\eta}, t) d\eta_1 d\eta_2 d\eta_3 \end{aligned}$$

Factor out the term involving  $\phi$ .

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) &= \frac{1}{A} \left( 1 - \frac{ie\epsilon}{\hbar} \phi(\mathbf{x}, t) \right) \int_{\mathbb{R}^3} \\ &\quad \exp \left( \frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 \right) \exp \left( \frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A} \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \psi(\mathbf{x} + \boldsymbol{\eta}, t) d\eta_1 d\eta_2 d\eta_3 \end{aligned}$$

Let

$$T = \frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{x}, t)$$

Use the following approximation.

$$\exp \left( \frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A} \left( \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \approx 1 + T + \frac{1}{2} T^2$$

Hence

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) &= \frac{1}{A} \left( 1 - \frac{ie\epsilon}{\hbar} \phi(\mathbf{x}, t) \right) \\ &\quad \times \int_{\mathbb{R}^3} \exp \left( \frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 \right) \left( 1 + T + \frac{1}{2} T^2 \right) \psi(\mathbf{x} + \boldsymbol{\eta}, t) d\eta_1 d\eta_2 d\eta_3 \quad (4) \end{aligned}$$

Next we will use the following Taylor series approximations.

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) &\approx \psi(\mathbf{x}, t) + \epsilon \frac{\partial \psi}{\partial t} \\ \psi(\mathbf{x} + \boldsymbol{\eta}, t) &\approx \psi(\mathbf{x}, t) + \boldsymbol{\eta} \cdot \nabla \psi + \frac{1}{2} \boldsymbol{\eta} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \psi) \end{aligned} \quad (5)$$

Note: In component notation

$$\boldsymbol{\eta} \cdot \nabla \psi = \eta_1 \frac{\partial \psi}{\partial x_1} + \eta_2 \frac{\partial \psi}{\partial x_2} + \eta_3 \frac{\partial \psi}{\partial x_3}$$

and

$$\begin{aligned} \boldsymbol{\eta} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \psi) &= \eta_1^2 \frac{\partial^2 \psi}{\partial x_1^2} + \eta_2^2 \frac{\partial^2 \psi}{\partial x_2^2} + \eta_3^2 \frac{\partial^2 \psi}{\partial x_3^2} \\ &\quad + 2\eta_1 \eta_2 \frac{\partial^2 \psi}{\partial x_1 \partial x_2} + 2\eta_1 \eta_3 \frac{\partial^2 \psi}{\partial x_1 \partial x_3} + 2\eta_2 \eta_3 \frac{\partial^2 \psi}{\partial x_2 \partial x_3} \end{aligned}$$

Substitute the approximations (5) into (4).

$$\begin{aligned} \psi(\mathbf{x}, t) + \epsilon \frac{\partial \psi}{\partial t} &= \frac{1}{A} \left( 1 - \frac{i\epsilon}{\hbar} \phi(\mathbf{x}, t) \right) \int_{\mathbb{R}^3} \exp \left( \frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 \right) \\ &\times \left( 1 + T + \frac{1}{2} T^2 \right) \left( \psi(\mathbf{x}, t) + \boldsymbol{\eta} \cdot \nabla \psi + \frac{1}{2} \boldsymbol{\eta} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \psi) \right) d\eta_1 d\eta_2 d\eta_3 \quad (6) \end{aligned}$$

To solve the above integral, we will use the following formulas provided by the authors.

$$\begin{aligned} I_k &= \int_{-\infty}^{\infty} \exp \left( \frac{im\eta_k^2}{2\hbar\epsilon} \right) d\eta_k = \left( \frac{2\pi i\hbar\epsilon}{m} \right)^{1/2} \\ J_k &= \int_{-\infty}^{\infty} \eta_k \exp \left( \frac{im\eta_k^2}{2\hbar\epsilon} \right) d\eta_k = 0 \\ K_k &= \int_{-\infty}^{\infty} \eta_k^2 \exp \left( \frac{im\eta_k^2}{2\hbar\epsilon} \right) d\eta_k = \frac{i\hbar\epsilon}{m} \left( \frac{2\pi i\hbar\epsilon}{m} \right)^{1/2} \end{aligned}$$

Hence

$$\begin{aligned} \int_{\mathbb{R}^3} \exp \left( \frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 \right) \psi(\mathbf{x}, t) d\eta_1 d\eta_2 d\eta_3 \\ = I_1 I_2 I_3 \psi(\mathbf{x}, t) = \left( \frac{2\pi i\hbar\epsilon}{m} \right)^{3/2} \psi(\mathbf{x}, t) \\ \int_{\mathbb{R}^3} \exp \left( \frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 \right) \boldsymbol{\eta} \cdot \nabla \psi(\mathbf{x}, t) d\eta_1 d\eta_2 d\eta_3 \\ = J_1 I_2 I_3 \frac{\partial \psi}{\partial x_1} + I_1 J_2 I_3 \frac{\partial \psi}{\partial x_2} + I_1 I_2 J_3 \frac{\partial \psi}{\partial x_3} \\ = 0 \quad (8) \end{aligned}$$

$$\begin{aligned} \int_{\mathbb{R}^3} \exp \left( \frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 \right) \frac{1}{2} \boldsymbol{\eta} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \psi) d\eta_1 d\eta_2 d\eta_3 \\ = \frac{1}{2} K_1 I_2 I_3 \frac{\partial^2 \psi}{\partial x_1^2} + \frac{1}{2} I_1 K_2 I_3 \frac{\partial^2 \psi}{\partial x_2^2} + \frac{1}{2} I_1 I_2 K_3 \frac{\partial^2 \psi}{\partial x_3^2} \\ + J_1 J_2 I_1 \frac{\partial^2 \psi}{\partial x_1 x_2} + J_1 I_2 J_3 \frac{\partial^2 \psi}{\partial x_1 x_3} + I_1 J_2 J_3 \frac{\partial^2 \psi}{\partial x_2 x_3} = \frac{i\hbar\epsilon}{2m} \left( \frac{2\pi i\hbar\epsilon}{m} \right)^{3/2} \nabla^2 \psi \quad (9) \end{aligned}$$

$$\begin{aligned}
& \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \left(\frac{ie}{\hbar c}\boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{x}, t)\right) \psi(\mathbf{x}, t) d\eta_1 d\eta_2 d\eta_3 \\
&= J_1 I_2 I_3 \frac{ie}{\hbar c} A_1 \psi + I_1 J_2 I_3 \frac{ie}{\hbar c} A_2 \psi + I_1 I_2 J_3 \frac{ie}{\hbar c} A_3 \psi \\
&= 0
\end{aligned} \tag{10}$$

$$\begin{aligned}
& \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \left(\frac{ie}{\hbar c}\boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{x}, t)\right) (\boldsymbol{\eta} \cdot \nabla \psi) d\eta_1 d\eta_2 d\eta_3 \\
&= \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \\
&\quad \times \frac{ie}{\hbar c} (\eta_1 A_1 + \eta_2 A_2 + \eta_3 A_3) \left( \eta_1 \frac{\partial \psi}{\partial x_1} + \eta_2 \frac{\partial \psi}{\partial x_2} + \eta_3 \frac{\partial \psi}{\partial x_3} \right) d\eta_1 d\eta_2 d\eta_3 \\
&= K_1 I_2 I_3 \frac{ie A_1}{\hbar c} \frac{\partial \psi}{\partial x_1} + J_1 J_2 I_3 \frac{ie A_1}{\hbar c} \frac{\partial \psi}{\partial x_2} + J_1 I_2 J_3 \frac{ie A_1}{\hbar c} \frac{\partial \psi}{\partial x_3} \\
&\quad + J_1 J_2 I_3 \frac{ie A_2}{\hbar c} \frac{\partial \psi}{\partial x_1} + I_1 K_2 I_3 \frac{ie A_2}{\hbar c} \frac{\partial \psi}{\partial x_2} + I_1 J_2 J_3 \frac{ie A_2}{\hbar c} \frac{\partial \psi}{\partial x_3} \\
&\quad + J_1 I_2 J_3 \frac{ie A_3}{\hbar c} \frac{\partial \psi}{\partial x_1} + I_1 J_2 J_3 \frac{ie A_3}{\hbar c} \frac{\partial \psi}{\partial x_2} + I_1 I_2 K_3 \frac{ie A_3}{\hbar c} \frac{\partial \psi}{\partial x_3} \\
&= \frac{i\hbar\epsilon}{m} \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \frac{ie}{\hbar c} \left( A_1 \frac{\partial \psi}{\partial x_1} + A_2 \frac{\partial \psi}{\partial x_2} + A_3 \frac{\partial \psi}{\partial x_3} \right) \\
&= \frac{i\hbar\epsilon}{m} \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \frac{ie}{\hbar c} (\mathbf{A}(\mathbf{x}, t) \cdot \nabla \psi)
\end{aligned} \tag{11}$$

$$\begin{aligned}
& \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \left(\frac{ie}{\hbar c}\boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{x}, t)\right) \left(\frac{1}{2}\boldsymbol{\eta} \cdot (\boldsymbol{\eta} \cdot \nabla \psi)\right) d\eta_1 d\eta_2 d\eta_3 \\
&= \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \frac{ie}{\hbar c} (\eta_1 A_1 + \eta_2 A_2 + \eta_3 A_3) \\
&\quad \times \left( \frac{1}{2}\eta_1^2 \frac{\partial^2 \psi}{\partial x_1^2} + \frac{1}{2}\eta_2^2 \frac{\partial^2 \psi}{\partial x_2^2} + \frac{1}{2}\eta_3^2 \frac{\partial^2 \psi}{\partial x_3^2} + \eta_1 \eta_2 \frac{\partial^2 \psi}{\partial x_1 \partial x_2} + \eta_1 \eta_3 \frac{\partial^2 \psi}{\partial x_1 \partial x_3} + \eta_2 \eta_3 \frac{\partial^2 \psi}{\partial x_2 \partial x_3} \right) \\
&= 0
\end{aligned} \tag{12}$$

$$\begin{aligned}
& \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \frac{1}{2} \left(\frac{ie}{\hbar c}\boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{x}, t)\right)^2 \psi(\mathbf{x}, t) d\eta_1 d\eta_2 d\eta_3 \\
&= \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \left(-\frac{e^2}{2\hbar^2 c^2}\right) (\eta_1 A_1 + \eta_2 A_2 + \eta_3 A_3)^2 \psi(\mathbf{x}, t) d\eta_1 d\eta_2 d\eta_3 \\
&= \frac{i\hbar\epsilon}{m} \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \left(-\frac{e^2}{2\hbar^2 c^2}\right) (A_1^2 + A_2^2 + A_3^2) \psi(\mathbf{x}, t) \\
&= \frac{i\hbar\epsilon}{m} \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \left(-\frac{e^2}{2\hbar^2 c^2}\right) (\mathbf{A} \cdot \mathbf{A}) \psi(\mathbf{x}, t)
\end{aligned} \tag{13}$$