

Spin direction vector

Expectation of spin operators is a projection of spin state $|s\rangle$ onto Euclidean space.

$$\langle x \rangle = \langle s | \sigma_x | s \rangle, \quad \langle y \rangle = \langle s | \sigma_y | s \rangle, \quad \langle z \rangle = \langle s | \sigma_z | s \rangle$$

Hence the spin direction vector is

$$\mathbf{u} = \begin{pmatrix} \langle x \rangle \\ \langle y \rangle \\ \langle z \rangle \end{pmatrix} = \langle s | \boldsymbol{\sigma} | s \rangle, \quad \boldsymbol{\sigma} = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}$$

Let

$$|s\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \exp(i\phi) \end{pmatrix}$$

Then

$$\langle x \rangle = \sin \theta \cos \phi, \quad \langle y \rangle = \sin \theta \sin \phi, \quad \langle z \rangle = \cos \theta$$

and

$$\cos(\theta/2) = \sqrt{\frac{\langle z \rangle + 1}{2}}, \quad \sin(\theta/2) \exp(i\phi) = \sqrt{\frac{1 - \langle z \rangle}{2}} \frac{\langle x \rangle + i\langle y \rangle}{\sqrt{\langle x \rangle^2 + \langle y \rangle^2}}$$

Example. Let

$$|s\rangle = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$$

Then

$$\mathbf{u} = \begin{pmatrix} \langle x \rangle \\ \langle y \rangle \\ \langle z \rangle \end{pmatrix} = \begin{pmatrix} \frac{4}{9} \\ \frac{8}{9} \\ \frac{1}{9} \end{pmatrix}$$

and

$$\cos(\theta/2) = \sqrt{\frac{\langle z \rangle + 1}{2}} = \frac{\sqrt{5}}{3}, \quad \sin(\theta/2) \exp(i\phi) = \sqrt{\frac{1 - \langle z \rangle}{2}} \frac{\langle x \rangle + i\langle y \rangle}{\sqrt{\langle x \rangle^2 + \langle y \rangle^2}} = \frac{2 + 4i}{3\sqrt{5}}$$

The last result differs from $|s\rangle$ but does represent the same state as $|s\rangle$.

The spin eigenstates are

$$\begin{array}{lll} |x_+\rangle = \frac{1}{\sqrt{2}}(1, 1) & |y_+\rangle = \frac{1}{\sqrt{2}}(1, i) & |z_+\rangle = (1, 0) \\ |x_-\rangle = \frac{1}{\sqrt{2}}(1, -1) & |y_-\rangle = \frac{1}{\sqrt{2}}(1, -i) & |z_-\rangle = (0, 1) \end{array}$$

Hence for

$$|s\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \exp(i\phi) \end{pmatrix}$$

we have

$$\begin{aligned}\Pr\left(S_x = +\frac{\hbar}{2}\right) &= |\langle x_+ | s \rangle|^2 = \frac{1}{2} + \frac{1}{2} \sin \theta \cos \phi \\ \Pr\left(S_x = -\frac{\hbar}{2}\right) &= |\langle x_- | s \rangle|^2 = \frac{1}{2} - \frac{1}{2} \sin \theta \cos \phi\end{aligned}$$

$$\begin{aligned}\Pr\left(S_y = +\frac{\hbar}{2}\right) &= |\langle y_+ | s \rangle|^2 = \frac{1}{2} + \frac{1}{2} \sin \theta \sin \phi \\ \Pr\left(S_y = -\frac{\hbar}{2}\right) &= |\langle y_- | s \rangle|^2 = \frac{1}{2} - \frac{1}{2} \sin \theta \sin \phi\end{aligned}$$

$$\begin{aligned}\Pr\left(S_z = +\frac{\hbar}{2}\right) &= |\langle z_+ | s \rangle|^2 = \cos^2(\theta/2) \\ \Pr\left(S_z = -\frac{\hbar}{2}\right) &= |\langle z_- | s \rangle|^2 = \sin^2(\theta/2)\end{aligned}$$