Addition of angular momentum 2

In spherical coordinates

$$L_x = i\hbar \left(\sin\phi \frac{\partial}{\partial\theta} + \cot\theta \cos\phi \frac{\partial}{\partial\phi} \right)$$

$$L_y = i\hbar \left(-\cos\phi \frac{\partial}{\partial\theta} + \cot\theta \sin\phi \frac{\partial}{\partial\phi} \right)$$

$$L_z = -i\hbar \frac{\partial}{\partial\phi}$$

and

$$L^2 = -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right)$$

Recall that

$$L_z Y_{lm} = m\hbar Y_{lm}$$

Hence for $\Psi = Y_{lm}\chi_\pm$ we have

$$\begin{aligned}
J_z \Psi &= L_z \Psi + S_z \Psi \\
&= m\hbar\Psi \pm \tfrac{1}{2}\hbar\Psi \\
&= \left(m \pm \tfrac{1}{2} \right) \hbar\Psi
\end{aligned}$$