Feynman and Hibbs problem 4-3

Show that the complex conjugate function $\psi^*$, defined as the function $\psi$ with every $i$ changed to $-i$, satisfies

$$\frac{\partial \psi^*}{\partial t} = +\frac{i}{\hbar}(H\psi)^*$$

Start with equation (4.14)

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar}H\psi$$

Conjugate both sides.

$$\left(\frac{\partial \psi}{\partial t}\right)^* = +\frac{i}{\hbar}(H\psi)^*$$

It is well known that conjugation and differentiation commute, hence

$$\left(\frac{\partial \psi}{\partial t}\right)^* = \frac{\partial \psi^*}{\partial t} = +\frac{i}{\hbar}(H\psi)^*$$

However, just for the fun of it, let us complete the proof without using the commutation rule.

Consider equation (2.22)

$$K(b,a) = \lim_{\epsilon \to 0} \frac{1}{A^N} \int \cdots \int \exp\left(\frac{i}{\hbar}S(b,a)\right)\, dx_1 \cdots dx_{N-1}$$

Differentiate with respect to $t$.

$$\frac{\partial}{\partial t}K(b,a) = \lim_{\epsilon \to 0} \frac{1}{A^N} \int \cdots \int \frac{i}{h}\frac{\partial}{\partial t}S(b,a) \exp\left(\frac{i}{\hbar}S(b,a)\right)\, dx_1 \cdots dx_{N-1}$$

Then conjugate.

$$\left(\frac{\partial}{\partial t}K(b,a)\right)^*$$
$$= \lim_{\epsilon \to 0} \frac{1}{A^N} \int \cdots \int -\frac{i}{h}\frac{\partial}{\partial t}S(b,a) \exp\left(-\frac{i}{\hbar}S(b,a)\right)\, dx_1 \cdots dx_{N-1}$$

Clearly the result is the same for conjugate first then differentiate, hence

$$\left(\frac{\partial}{\partial t}K(b,a)\right)^* = \frac{\partial}{\partial t}K^*(b,a) \tag{1}$$

Now consider this form of equation (4.2) that has $x, t$ instead of $x_b, t_b$.

$$\psi(x,t) = \int_\infty^\infty K(x,t,x_a,t_a)\psi(x_a,t_a)\,dx_a \tag{2}$$

Differentiate equation (2) then conjugate. Note that $\psi(x_a, t_a)$ is a constant with respect to $t$.

$$\left(\frac{\partial}{\partial t}\psi(x,t)\right)^* = \int_\infty^\infty \left(\frac{\partial}{\partial t}K(x,t,x_a,t_a)\right)^* \psi^*(x_a,t_a)\,dx_a$$

Conjugate (2) then differentiate.

$$\frac{\partial}{\partial t}\psi^*(x,t) = \int_\infty^\infty \frac{\partial}{\partial t}K^*(x,t,x_a,t_a)\psi^*(x_a,t_a)\,dx_a$$

Then by equation (1)

$$\left(\frac{\partial}{\partial t}\psi(x,t)\right)^* = \frac{\partial}{\partial t}\psi^*(x,t)$$