Stern-Gerlach 2

From the previous section we have the following Schrödinger equations for a Stern-Gerlach experiment.

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi_1 + \mu_B(B_0+\alpha z)\psi_1 &= i\hbar\frac{\partial}{\partial t}\psi_1 \\
-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi_2 - \mu_B(B_0+\alpha z)\psi_2 &= i\hbar\frac{\partial}{\partial t}\psi_2
\end{aligned}
$$

We now seek solutions for ψ_1 and ψ_2.

An online paper[1] provides the following solutions.

$$
\begin{aligned}
\psi_1 &= \operatorname{Ai}\left[\left(\frac{\mu_B\alpha m}{\hbar^2}\right)^{1/3}\left(\frac{\mu_B\alpha}{4m}\right)t^2 + \left(\frac{\mu_B\alpha m}{\hbar^2}\right)^{1/3}z\right]\exp\left(-\frac{i\alpha\mu_B z t}{2\hbar}\right)\exp\left(-\frac{\mu_B B_0 t}{\hbar}\right) \\
\psi_2 &= \operatorname{Ai}\left[\left(\frac{\mu_B\alpha m}{\hbar^2}\right)^{1/3}\left(\frac{\mu_B\alpha}{4m}\right)t^2 - \left(\frac{\mu_B\alpha m}{\hbar^2}\right)^{1/3}z\right]\exp\left(+\frac{i\alpha\mu_B z t}{2\hbar}\right)\exp\left(+\frac{\mu_B B_0 t}{\hbar}\right)
\end{aligned}
$$

Let us try verifying ψ_1 and ψ_2 with a Taylor series expansion of the Airy function $\operatorname{Ai}(x)$.

We have

$$
\operatorname{Ai}(x) \approx \sum_{k=0}^{n} a_k x^k
$$

where

$$
\begin{aligned}
a_0 &= \frac{1}{3^{2/3}\Gamma(2/3)} \\
a_1 &= -\frac{1}{3^{1/3}\Gamma(1/3)} \\
a_2 &= 0 \\
a_{n+3} &= \frac{a_n}{(n+3)(n+2)}
\end{aligned}
$$

For $n = 9$ we obtain

$$
\begin{aligned}
\operatorname{Ai}(x) \approx 2.73941\times10^{-5}x^9 - 0.000513531x^7 + 0.00197238x^6 \\
- 0.0215683x^4 + 0.0591713x^3 - 0.258819x + 0.355028
\end{aligned}
$$

Calculate ψ_1 and ψ_2 using the $\operatorname{Ai}(x)$ approximation.

To verify ψ_1 and ψ_2, calculate departures from equality of the Schrödinger equations.

$$
\begin{aligned}
\epsilon_1 &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi_1 + \mu_B(B_0+\alpha z)\psi_1 - i\hbar\frac{\partial}{\partial t}\psi_1 \\
\epsilon_2 &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi_2 - \mu_B(B_0+\alpha z)\psi_2 - i\hbar\frac{\partial}{\partial t}\psi_2
\end{aligned}
$$

[1] "Construction of Exact Solutions for the Stern-Gerlach Effect" by Bulnes and Oliveira.

After calculating ϵ_1 and ϵ_2, cancel exponentials and zero out factors t^n and z^m where $n \geq 8$ and $m \geq 5$.

The results are

$$\begin{aligned}\epsilon_1 = \boxed{-6.50521 \times 10^{-19}} \frac{\alpha^{16/3} \mu_B^{16/3} t^6 z^2}{\hbar^{8/3} m^{5/3}} + \boxed{5.42101 \times 10^{-20}} \frac{\alpha^5 \mu_B^5 t^6 z}{\hbar^2 m^2} \\ - \boxed{1.73472 \times 10^{-18}} \frac{\alpha^{13/3} \mu_B^{13/3} t^4 z^3}{\hbar^{8/3} m^{2/3}} - \boxed{3.46945 \times 10^{-18}} \frac{\alpha^{10/3} m^{1/3} \mu_B^{10/3} t^2 z^4}{\hbar^{8/3}} \\ - \boxed{2.1684 \times 10^{-19}} \frac{i \alpha^{16/3} \mu_B^{16/3} t^5 z^4}{\hbar^{11/3} m^{2/3}}\end{aligned}$$

and

$$\begin{aligned}\epsilon_2 = \boxed{-6.50521 \times 10^{-19}} \frac{\alpha^{16/3} \mu_B^{16/3} t^6 z^2}{\hbar^{8/3} m^{5/3}} - \boxed{5.42101 \times 10^{-20}} \frac{\alpha^5 \mu_B^5 t^6 z}{\hbar^2 m^2} \\ + \boxed{1.73472 \times 10^{-18}} \frac{\alpha^{13/3} \mu_B^{13/3} t^4 z^3}{\hbar^{8/3} m^{2/3}} - \boxed{3.46945 \times 10^{-18}} \frac{\alpha^{10/3} m^{1/3} \mu_B^{10/3} t^2 z^4}{\hbar^{8/3}} \\ - \boxed{2.1684 \times 10^{-19}} \frac{i \alpha^{16/3} \mu_B^{16/3} t^5 z^4}{\hbar^{11/3} m^{2/3}}\end{aligned}$$

The numerical values are negligible hence ψ_1 and ψ_2 are confirmed as solutions.

Eigenmath script