4-12. Carry out the integral in equation (4.64) by completing the square. Show that the correct free particle kernel (i.e., the three dimensional version of equation 3.3) results.

$$K_0(\mathbf{x}_b, t_b, \mathbf{x}_a, t_a) \\ = \int \exp\left(\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{x}_b - \mathbf{x}_a)\right)\exp\left(-\frac{i}{\hbar}\frac{\mathbf{p}^2}{2m}(t_b - t_a)\right)\frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \tag{4.64}$$

Rewrite equation (4.64) in component notation.

$$K_0(\mathbf{x}_b, t_b, \mathbf{x}_a, t_a) = \frac{1}{(2\pi\hbar)^3}\int_{\mathbb{R}^3}\exp\left(\frac{i}{\hbar}p_1 X_1 - \frac{i}{\hbar}\frac{p_1^2}{2m}T\right) \\ \times \exp\left(\frac{i}{\hbar}p_2 X_2 - \frac{i}{\hbar}\frac{p_2^2}{2m}T\right)\exp\left(\frac{i}{\hbar}p_3 X_3 - \frac{i}{\hbar}\frac{p_3^2}{2m}T\right)dp_1\,dp_2\,dp_3$$

where

$$T = t_b - t_a, \qquad (X_1, X_2, X_3) = \mathbf{x}_b - \mathbf{x}_a$$

Let

$$A = -\frac{i}{2m\hbar}T$$
$$B = \frac{i}{\hbar}X_k$$

Then

$$\int_{-\infty}^{\infty}\exp(Ap_k^2 + Bp_k)\,dp_k = \left(-\frac{\pi}{A}\right)^{1/2}\exp\left(-\frac{B^2}{4A}\right) \\ = \left(-\frac{2\pi i m\hbar}{T}\right)^{1/2}\exp\left(\frac{imX_k^2}{2\hbar T}\right) \tag{1}$$

Hence

$$K_0(\mathbf{x}_b, t_b, \mathbf{x}_a, t_a) = \frac{1}{(2\pi\hbar)^3}\prod_{k=1}^{3}\left(-\frac{2\pi i m\hbar}{T}\right)^{1/2}\exp\left(\frac{imX_k^2}{2\hbar T}\right)$$

Combine coefficients and convert to vector notation.

$$K_0(\mathbf{x}_b, t_b, \mathbf{x}_a, t_a) = \left(-\frac{im}{2\pi\hbar(t_b - t_a)}\right)^{3/2}\exp\left(\frac{im(\mathbf{x}_b - \mathbf{x}_a)^2}{2\hbar(t_b - t_a)}\right) \tag{2}$$