Feynman and Hibbs problem 4-2

For a particle of charge e in a magnetic field the Lagrangian is

$$L(\dot{\mathbf{x}}, \mathbf{x}) = \frac{m}{2}\dot{\mathbf{x}}^2 + \frac{e}{c}\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}, t) - e\phi(\mathbf{x}, t)$$

where $\dot{\mathbf{x}}$ is the velocity vector, c is the velocity of light, and $\mathbf{A}$ and ϕ are the vector and scalar potentials. Show that the corresponding Schrodinger equation is

$$\frac{\partial\psi}{\partial t} = -\frac{i}{\hbar}\left(\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right) \cdot \left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right)\psi + e\phi\psi\right) \tag{4.18}$$

From equation (4.3) with a minor correction of $y - x$ instead of $x - y$.

$$\psi(\mathbf{x}, t + \epsilon) = \frac{1}{A}\int_{\mathbb{R}^3} \exp\left(\frac{i\epsilon}{\hbar} L\left(\frac{\mathbf{y} - \mathbf{x}}{\epsilon}, \frac{\mathbf{x} + \mathbf{y}}{2}\right)\right)\psi(\mathbf{y}, t)\, dy_1\, dy_2\, dy_3 \tag{1}$$

This is the Lagrangian with arguments from (1).

$$\begin{aligned} &L\left(\frac{\mathbf{y} - \mathbf{x}}{\epsilon}, \frac{\mathbf{x} + \mathbf{y}}{2}\right) \\ &\qquad = \frac{m}{2\epsilon^2}(\mathbf{x} - \mathbf{y})^2 - \frac{e}{c\epsilon}(\mathbf{x} - \mathbf{y}) \cdot \mathbf{A}\left(\frac{\mathbf{x} + \mathbf{y}}{2}, t\right) - e\phi\left(\frac{\mathbf{x} + \mathbf{y}}{2}, t\right) \end{aligned}$$

Hence

$$\begin{aligned} &\psi(\mathbf{x}, t + \epsilon) = \frac{1}{A}\int_{\mathbb{R}^3} \\ &\quad \exp\left(\frac{im}{2\hbar\epsilon}(\mathbf{x} - \mathbf{y})^2 - \frac{ie}{\hbar c}(\mathbf{x} - \mathbf{y}) \cdot \mathbf{A}\left(\frac{\mathbf{x} + \mathbf{y}}{2}, t\right) - \frac{ie\epsilon}{\hbar}\phi\left(\frac{\mathbf{x} + \mathbf{y}}{2}, t\right)\right) \\ &\quad \times \psi(\mathbf{y}, t)\, dy_1\, dy_2\, dy_3 \end{aligned}$$

Let

$$\mathbf{y} = \mathbf{x} + \boldsymbol{\eta}$$

Then

$$\mathbf{x} - \mathbf{y} = \boldsymbol{\eta}, \quad \frac{\mathbf{x} + \mathbf{y}}{2} = \mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, \quad dy_1\, dy_2\, dy_3 = d\eta_1\, d\eta_2\, d\eta_3$$

Hence

$$\psi(\mathbf{x},t+\epsilon)=\frac{1}{A}\int_{\mathbb{R}^3}\exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2-\frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)\\ \times\psi(\mathbf{x}+\boldsymbol{\eta},t)\,d\eta_1\,d\eta_2\,d\eta_3$$

Factor the exponential.

$$\psi(\mathbf{x},t+\epsilon)=\frac{1}{A}\int_{\mathbb{R}^3}\\ \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\exp\left(-\frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)\exp\left(-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)\\ \times\psi(\mathbf{x}+\boldsymbol{\eta},t)\,d\eta_1\,d\eta_2\,d\eta_3 \tag{2}$$

From the identity $\exp(i\theta)=\cos(\theta)+i\sin(\theta)$ we have

$$\exp\left(-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)\\ =\cos\left(-\frac{e\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)+i\sin\left(-\frac{e\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)$$

Then for small ϵ

$$\exp\left(-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)\approx 1-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)$$

The $\boldsymbol{\eta}$ term can be discarded because the integral is Gaussian. (Contributions to the integral are small for $\boldsymbol{\eta}^2>2\hbar\epsilon/m$.)

$$\exp\left(-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)\approx 1-\frac{ie\epsilon}{\hbar}\phi(\mathbf{x},t) \tag{3}$$

Substitute (3) into (2).

$$\psi(\mathbf{x},t+\epsilon)=\frac{1}{A}\left(1-\frac{ie\epsilon}{\hbar}\phi\right)\int_{\mathbb{R}^3}\\ \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\exp\left(-\frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)\psi(\mathbf{x}+\boldsymbol{\eta},t)\,d\eta_1\,d\eta_2\,d\eta_3$$

Approximate the exponential involving $\mathbf{A}$ with a Taylor series.

$$\exp\left(-\frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)\approx\exp\left(-\frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}(\mathbf{x})-\frac{ie}{2\hbar c}\boldsymbol{\eta}\cdot(\boldsymbol{\eta}\cdot\nabla\mathbf{A}(\mathbf{x}))\right)$$

Expand the right-hand side as a power series.

$$\exp\left(-\frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)\approx\left(1+T+\tfrac{1}{2}T^2\right)$$

where

$$T=-\frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}(\mathbf{x})-\frac{ie}{2\hbar c}\boldsymbol{\eta}\cdot(\boldsymbol{\eta}\cdot\nabla\mathbf{A}(\mathbf{x}))$$

Discard high order terms.

$$\begin{aligned}&\exp\left(-\frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta},t\right)\right)\\&\qquad\approx 1-\frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}(\mathbf{x})-\frac{ie}{2\hbar c}\boldsymbol{\eta}\cdot(\boldsymbol{\eta}\cdot\nabla\mathbf{A}(\mathbf{x}))+\frac{1}{2}\left(-\frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}(\mathbf{x})\right)^2\end{aligned}$$

Hence

$$\begin{aligned}\psi(\mathbf{x},t+\epsilon)&=\frac{1}{A}\left(1-\frac{ie\epsilon}{\hbar}\phi\right)\int_{\mathbb{R}^3}\exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\\&\times\left(1-\frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}-\frac{ie}{2\hbar c}\boldsymbol{\eta}\cdot(\boldsymbol{\eta}\cdot\nabla\mathbf{A})+\frac{1}{2}\left(-\frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\right)^2\right)\\&\times\psi(\mathbf{x}+\boldsymbol{\eta},t)\,d\eta_1\,d\eta_2\,d\eta_3\end{aligned}\tag{4}$$

Next we will use the following Taylor series approximations.

$$\begin{aligned}\psi(\mathbf{x},t+\epsilon)&\approx\psi+\epsilon\frac{\partial\psi}{\partial t}\\\psi(\mathbf{x}+\boldsymbol{\eta},t)&\approx\psi+\boldsymbol{\eta}\cdot\nabla\psi+\tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi)\end{aligned}\tag{5}$$

Substitute the approximations (5) into (4).

$$\begin{aligned}\psi+\epsilon\frac{\partial\psi}{\partial t}&=\frac{1}{A}\left(1-\frac{ie\epsilon}{\hbar}\phi\right)\int_{\mathbb{R}^3}\exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\\&\times\left(1-\frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}-\frac{ie}{2\hbar c}\boldsymbol{\eta}\cdot(\boldsymbol{\eta}\cdot\nabla\mathbf{A})+\frac{1}{2}\left(-\frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\right)^2\right)\\&\times\left(\psi+\boldsymbol{\eta}\cdot\nabla\psi+\tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi)\right)\,d\eta_1\,d\eta_2\,d\eta_3\end{aligned}\tag{6}$$

To solve the above integral, we will use the following formulas provided by the authors.

$$\int_{-\infty}^{\infty} \exp\left(\frac{im\eta_k^2}{2\hbar\epsilon}\right) d\eta_k = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{1/2} \qquad \text{FH (4.7)}$$

$$\int_{-\infty}^{\infty} \eta_k \exp\left(\frac{im\eta_k^2}{2\hbar\epsilon}\right) d\eta_k = 0 \qquad \text{FH (4.9)}$$

$$\int_{-\infty}^{\infty} \eta_k^2 \exp\left(\frac{im\eta_k^2}{2\hbar\epsilon}\right) d\eta_k = \frac{i\hbar\epsilon}{m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{1/2} \qquad \text{FH (4.10)}$$

The integrand in (6) has twelve terms. The following table summarizes the integrals $\int uv$ for each factor pair uv.

	$v = \psi$	$v = \boldsymbol{\eta} \cdot \nabla\psi$	$v = \frac{1}{2}\boldsymbol{\eta} \cdot \nabla(\boldsymbol{\eta} \cdot \nabla\psi)$
$u = 1$	I_1	0	I_3
$u = -\frac{ie}{\hbar c}\boldsymbol{\eta} \cdot \mathbf{A}$	0	I_5	0
$u = -\frac{ie}{2\hbar c}\boldsymbol{\eta} \cdot (\boldsymbol{\eta} \cdot \nabla\mathbf{A})$	I_7	0	$\propto \epsilon^2$
$u = \frac{1}{2}\left(-\frac{ie}{\hbar c}\boldsymbol{\eta} \cdot \mathbf{A}\right)^2$	I_{10}	0	$\propto \epsilon^2$

$$I_1 = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\psi$$

$$I_3 = \frac{i\hbar\epsilon}{2m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\nabla^2\psi$$

$$I_5 = \frac{i\hbar\epsilon}{m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\frac{-ie}{\hbar c}\mathbf{A}\cdot\nabla\psi$$

$$I_7 = \frac{i\hbar\epsilon}{m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\frac{-ie}{2\hbar c}\nabla\cdot\mathbf{A}\psi$$

$$I_{10} = \frac{i\hbar\epsilon}{m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\frac{1}{2}\left(\frac{-ie}{\hbar c}\right)^2\mathbf{A}^2\psi$$

Substitute the solved integrals into (6) to obtain

$$\begin{aligned}\psi + \epsilon\frac{\partial\psi}{\partial t} &= \frac{1}{A}\left(1 - \frac{ie\epsilon}{\hbar}\phi\right)\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \\ &\quad\times\left(\psi + \frac{i\hbar\epsilon}{2m}\nabla^2\psi + \frac{e\epsilon}{mc}\mathbf{A}\cdot\nabla\psi + \frac{e\epsilon}{2mc}\nabla\cdot\mathbf{A}\psi - \frac{ie^2\epsilon}{2m\hbar c^2}\mathbf{A}^2\psi\right)\end{aligned}$$

In the limit as $\epsilon \to 0$ we have

$$\psi = \frac{1}{A}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\psi$$

hence

$$A = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}$$

Cancel A.

$$\begin{aligned}\psi + \epsilon\frac{\partial\psi}{\partial t} &= \left(1 - \frac{ie\epsilon}{\hbar}\phi\right) \\ &\quad\times\left(\psi + \frac{i\hbar\epsilon}{2m}\nabla^2\psi + \frac{e\epsilon}{mc}\mathbf{A}\cdot\nabla\psi + \frac{e\epsilon}{2mc}\nabla\cdot\mathbf{A}\psi - \frac{ie^2\epsilon}{2m\hbar c^2}\mathbf{A}^2\psi\right)\end{aligned}$$

Expand the product and discard terms of order ϵ^2.

$$\psi + \epsilon\frac{\partial\psi}{\partial t}$$
$$= \psi + \frac{i\hbar\epsilon}{2m}\nabla^2\psi + \frac{e\epsilon}{mc}\mathbf{A}\cdot\nabla\psi + \frac{e\epsilon}{2mc}\nabla\cdot\mathbf{A}\psi - \frac{ie^2\epsilon}{2m\hbar c^2}\mathbf{A}^2\psi - \frac{ie\epsilon}{\hbar}\phi\psi$$

Cancel leading terms ψ and divide through by ϵ.

$$\frac{\partial\psi}{\partial t} = \frac{i\hbar}{2m}\nabla^2\psi + \frac{e}{mc}\mathbf{A}\cdot\nabla\psi + \frac{e}{2mc}\nabla\cdot\mathbf{A}\psi - \frac{ie^2}{2m\hbar c^2}\mathbf{A}^2\psi - \frac{ie}{\hbar}\phi\psi \qquad (7)$$

To show that (7) is the same as (4.18), expand (4.18) step by step. First we have

$$\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right)\psi = -i\hbar\nabla\psi - \frac{e}{c}\mathbf{A}\psi$$

Next

$$\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right)^2\psi = \left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)\left(-i\hbar\nabla\psi - \frac{e}{c}\mathbf{A}\psi\right)$$
$$= -\hbar^2\nabla(\nabla\psi) + \frac{ie\hbar}{c}\nabla(\mathbf{A}\psi) + \frac{ie\hbar}{c}\mathbf{A}\cdot\nabla\psi + \frac{e^2}{c^2}\mathbf{A}\cdot\mathbf{A}\psi$$

Because $\nabla\psi$ is a vector we have

$$\nabla(\nabla\psi) = \nabla\cdot\nabla\psi = \nabla^2\psi$$

and because $\mathbf{A}$ is a vector and ψ is a scalar we have

$$\nabla(\mathbf{A}\psi) = \nabla\cdot\mathbf{A}\psi + \mathbf{A}\cdot\nabla\psi$$

Hence

$$\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right)^2\psi = -\hbar^2\nabla^2\psi + \frac{2ie\hbar}{c}\mathbf{A}\cdot\nabla\psi + \frac{ie\hbar}{c}\nabla\cdot\mathbf{A}\psi + \frac{e^2}{c^2}\mathbf{A}^2\psi$$

Divide through by $2m$.

$$\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right)^2\psi$$
$$= -\frac{\hbar^2}{2m}\nabla^2\psi + \frac{ie\hbar}{mc}\mathbf{A}\cdot\nabla\psi + \frac{ie\hbar}{2mc}\nabla\cdot\mathbf{A}\psi + \frac{e^2}{2mc^2}\mathbf{A}^2\psi$$

Add the scalar potential.

$$\begin{aligned}&\frac{1}{2m}\left(\frac{\hbar}{i}\nabla-\frac{e}{c}\mathbf{A}\right)^2\psi+e\phi\psi\\&\qquad=-\frac{\hbar^2}{2m}\nabla^2\psi+\frac{ie\hbar}{mc}\mathbf{A}\cdot\nabla\psi+\frac{ie\hbar}{2mc}\nabla\cdot\mathbf{A}\psi+\frac{e^2}{2mc^2}\mathbf{A}^2\psi+e\phi\psi\end{aligned}$$

Finally, multiply by $-i/\hbar$.

$$\begin{aligned}&-\frac{i}{\hbar}\left(\frac{1}{2m}\left(\frac{\hbar}{i}\nabla-\frac{e}{c}\mathbf{A}\right)^2\psi+e\phi\psi\right)\\&\qquad=\frac{i\hbar}{2m}\nabla^2\psi+\frac{e}{mc}\mathbf{A}\cdot\nabla\psi+\frac{e}{2mc}\nabla\cdot\mathbf{A}\psi-\frac{ie^2}{2m\hbar c^2}\mathbf{A}^2\psi-\frac{ie}{\hbar}\phi\psi\end{aligned}$$

The result is identical to (7).

$$\frac{\partial\psi}{\partial t}=\frac{i\hbar}{2m}\nabla^2\psi+\frac{e}{mc}\mathbf{A}\cdot\nabla\psi+\frac{e}{2mc}\nabla\cdot\mathbf{A}\psi-\frac{ie^2}{2m\hbar c^2}\mathbf{A}^2\psi-\frac{ie}{\hbar}\phi\psi \tag{7}$$