

4-3. Show that the complex conjugate function  $\psi^*$ , defined as the function  $\psi$  with every  $i$  changed to  $-i$ , satisfies

$$\frac{\partial \psi^*}{\partial t} = +\frac{i}{\hbar}(H\psi)^*$$

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Start with equation (4.14).

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar}H\psi \quad (4.14)$$

Conjugate both sides.

$$\left(\frac{\partial \psi}{\partial t}\right)^* = +\frac{i}{\hbar}(H\psi)^*$$

It is well known that conjugation and differentiation commute, hence

$$\left(\frac{\partial \psi}{\partial t}\right)^* = \frac{\partial \psi^*}{\partial t} = +\frac{i}{\hbar}(H\psi)^*$$

However, just for the fun of it, let us complete the proof without using the commutation rule.

Consider equation (2.22).

$$K(b, a) = \lim_{\epsilon \rightarrow 0} \frac{1}{A^N} \int \cdots \int \exp\left(\frac{i}{\hbar}S(b, a)\right) dx_1 \cdots dx_{N-1} \quad (2.22)$$

Differentiate (2.22) with respect to  $t_b$ .

$$\frac{\partial}{\partial t_b} K(b, a) = \lim_{\epsilon \rightarrow 0} \frac{1}{A^N} \int \cdots \int \frac{i}{\hbar} \frac{\partial}{\partial t_b} S(b, a) \exp\left(\frac{i}{\hbar}S(b, a)\right) dx_1 \cdots dx_{N-1}$$

Then conjugate.

$$\begin{aligned} & \left(\frac{\partial}{\partial t_b} K(b, a)\right)^* \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{A^N} \int \cdots \int -\frac{i}{\hbar} \frac{\partial}{\partial t_b} S(b, a) \exp\left(-\frac{i}{\hbar}S(b, a)\right) dx_1 \cdots dx_{N-1} \end{aligned}$$

Clearly the result is the same for conjugate first then differentiate, hence

$$\left( \frac{\partial}{\partial t_b} K(b, a) \right)^* = \frac{\partial}{\partial t_b} K^*(b, a) \quad (1)$$

Now consider this form of equation (4.2) that has  $x, t$  instead of  $x_b, t_b$ .

$$\psi(x, t) = \int_{-\infty}^{\infty} K(x, t, x_a, t_a) \psi(x_a, t_a) dx_a \quad (2)$$

Differentiate (2) with respect to  $t$  then conjugate. Note that  $\psi(x_a, t_a)$  is a constant with respect to  $t$ .

$$\left( \frac{\partial}{\partial t} \psi(x, t) \right)^* = \int_{-\infty}^{\infty} \left( \frac{\partial}{\partial t} K(x, t, x_a, t_a) \right)^* \psi^*(x_a, t_a) dx_a$$

Now do the reverse, conjugate (2) then differentiate.

$$\frac{\partial}{\partial t} \psi^*(x, t) = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} K^*(x, t, x_a, t_a) \psi^*(x_a, t_a) dx_a$$

By equation (1) the integrals are equivalent, hence

$$\left( \frac{\partial}{\partial t} \psi(x, t) \right)^* = \frac{\partial}{\partial t} \psi^*(x, t)$$