9-2. Explain why the charge density corresponding to a single charge q located at the point $\mathbf{x}(t) = (x(t), y(t), z(t))$ at time t is

$$\rho(\mathbf{r}, t) = q\delta(r_x - x(t))\delta(r_y - y(t))\delta(r_z - z(t)) = q\delta^3(\mathbf{r} - \mathbf{x}(t))$$

For a point charge q at $\mathbf{x}(t)$ the charge density is

$$\rho(\mathbf{r}, t) = \begin{cases} q & \mathbf{r} = \mathbf{x}(t) \\ 0 & \mathbf{r} \neq \mathbf{x}(t) \end{cases}$$

Hence

$$\rho(\mathbf{r}, t) = q\delta^3(\mathbf{r} - \mathbf{x}(t)) \tag{1}$$

(9-2 cont'd) Show that

$$\rho_{\mathbf{k}}(t) = q \exp(-i\mathbf{k} \cdot \mathbf{x}(t)) \tag{2}$$

From equation (9.14)

$$\rho(\mathbf{r}, t) = \int \rho_{\mathbf{k}}(t) \exp(i\mathbf{k} \cdot \mathbf{r}) \frac{d^3\mathbf{k}}{(2\pi)^3} \tag{3}$$

Substitute (2) into (3).

$$\rho(\mathbf{r}, t) = \frac{q}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\big(i\mathbf{k} \cdot (\mathbf{r} - \mathbf{x}(t))\big)\, dk_x\, dk_y\, dk_z$$

Recall the definition of a delta function.

$$\delta(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(iax)\, dx$$

Hence

$$\rho(\mathbf{r}, t) = q\delta^3(\mathbf{r} - \mathbf{x}(t))$$

By equation (1) this is the correct result, hence (2) is correct.

(9-2 cont'd) Explain why the current density is

$$\mathbf{j}(\mathbf{r}, t) = q\dot{\mathbf{x}}(t)\delta^3(\mathbf{r} - \mathbf{x}(t))$$

(9-2 cont'd) If we have a number of charges q_i located at $\mathbf{x}_i(t)$, the values $\rho_{\mathbf{k}}$ and $\mathbf{j}_{\mathbf{k}}$ are

$$\rho_{\mathbf{k}} = \sum_i q_i \exp(-i\mathbf{k}\cdot\mathbf{x}_i(t)) \qquad \mathbf{j}_{\mathbf{k}} = \sum_i q_i \dot{\mathbf{x}}_i(t) \exp(-\mathbf{k}\cdot\mathbf{x}_i(t)) \tag{9.16}$$

From equation (9.14)

$$\mathbf{j}(\mathbf{r},t) = \int \mathbf{j}_{\mathbf{k}}(t) \exp(i\mathbf{k}\cdot\mathbf{r}) \frac{d^3\mathbf{k}}{(2\pi)^3} \tag{4}$$

Substitute (9.16) into (4).

$$\mathbf{j}(\mathbf{r},t) = \frac{q}{(2\pi)^3}\dot{\mathbf{x}}(t) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\big(i\mathbf{k}\cdot(\mathbf{r}-\mathbf{x}(t))\big)\, dk_x\, dk_y\, dk_z$$

Hence

$$\mathbf{j}(\mathbf{r},t) = q\dot{\mathbf{x}}(t)\delta^3(\mathbf{r}-\mathbf{x}(t))$$