Legendre polynomials

Verify that

$$(1-x^2)\frac{d^2}{dx^2}P_{lm}(x) - 2x\frac{d}{dx}P_{lm}(x) + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_{lm}(x) = 0 \tag{1}$$

where $P_{lm}(x)$ are associated Legendre polynomials

$$P_{lm}(x) = \begin{cases} \dfrac{(-1)^m}{2^l l!}(1-x^2)^{\frac{m}{2}}\dfrac{d^{l+m}}{dx^{l+m}}(x^2-1)^l, & m \geq 0 \\[2ex] (-1)^m\dfrac{(l+m)!}{(l-m)!}P_{l|m|}(x), & m < 0 \end{cases}$$

Legendre polynomials are needed for spherical harmonic functions $Y_{lm}(\theta, \phi)$.

$$Y_{lm}(\theta,\phi) = (-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_{lm}(\cos\theta)\exp(im\phi)$$

See arxiv.org/abs/1805.12125 for the following form of $P_{lm}(\cos\theta)$.

$$P_{lm}(\cos\theta) = \begin{cases} \left(\dfrac{\sin\theta}{2}\right)^m \displaystyle\sum_{k=0}^{l-m}(-1)^k\frac{(l+m+k)!}{(l-m-k)!(m+k)!k!}\left(\frac{1-\cos\theta}{2}\right)^k, & m \geq 0 \\[2ex] (-1)^m\dfrac{(l+m)!}{(l-m)!}P_{l|m|}(\cos\theta), & m < 0 \end{cases}$$