Spin flip

From exercise 5.8 of "Quantum Mechanics" by Richard Fitzpatrick.

Consider an electron at rest in the following magnetic field.

$$\mathbf{B} = B_0 \cos(\omega t) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Find the minimum B_0 such that $\langle S_x \rangle$ ranges from $-\frac{\hbar}{2}$ to $+\frac{\hbar}{2}$.

The Hamiltonian for an electron in a magnetic field is

$$H = \frac{e}{m} \mathbf{B} \cdot \mathbf{S}$$

Hence by hypothesis

$$H = \frac{e}{m} B_0 \cos(\omega t) S_z$$

Let $|s\rangle$ be the following spin state.

$$|s\rangle = \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$$

Consider the Schrodinger equation for spin state $|s\rangle$.

$$i\hbar \frac{\partial}{\partial t} |s\rangle = H|s\rangle$$

By the Schrodinger equation we have for $c_1(t)$ and $c_2(t)$

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} c_1(t) &= \frac{e\hbar}{2m} B_0 \cos(\omega t) c_1(t) \\ i\hbar \frac{\partial}{\partial t} c_2(t) &= -\frac{e\hbar}{2m} B_0 \cos(\omega t) c_2(t) \end{aligned}$$

Solve for $c_1(t)$ and $c_2(t)$.

$$\begin{aligned} c_1(t) &= \frac{1}{\sqrt{2}} \exp\left(-\frac{ie}{2m\omega} B_0 \sin(\omega t) \right) \\ c_2(t) &= \frac{1}{\sqrt{2}} \exp\left(\frac{ie}{2m\omega} B_0 \sin(\omega t) \right) \end{aligned} \tag{1}$$

Hence for $\langle S_x \rangle$ we have

$$\langle S_x \rangle = \langle s|S_x|s\rangle = \frac{\hbar}{2} \cos\left[\frac{e}{m\omega} B_0 \sin(\omega t) \right] \tag{2}$$

At time $t = 0$ we have $\sin(\omega t) = 0$ and

$$\langle S_x \rangle = \frac{\hbar}{2}$$

To obtain

$$\langle S_x \rangle = -\frac{\hbar}{2}$$

we must have

$$\frac{e}{m\omega} B_0 \sin(\omega t) = \pi$$

Taking $\sin(\omega t) = 1$ we have

$$B_0 = \frac{\pi m \omega}{e}$$