

## Perturbation expansion

This is the propagator for a particle in a potential energy field  $V(x, t)$ .

$$K(b, a) = \int_{x_a}^{x_b} \exp \left[ \frac{i}{\hbar} \int_{t_a}^{t_b} \left( \frac{m\dot{x}^2}{2} - V(x(t), t) \right) dt \right] \mathcal{D}x(t)$$

Factor the exponential.

$$K(b, a) = \int_{x_a}^{x_b} \exp \left( \frac{i}{\hbar} \int_{t_a}^{t_b} \frac{m\dot{x}^2}{2} dt \right) \exp \left( -\frac{i}{\hbar} \int_{t_a}^{t_b} V(x(t), t) dt \right) \mathcal{D}x(t)$$

Expand the second exponential as a power series.

$$K(b, a) = \int_{x_a}^{x_b} \exp \left( \frac{i}{\hbar} \int_{t_a}^{t_b} \frac{m\dot{x}^2}{2} dt \right) \left( 1 - \frac{i}{\hbar} \int_{t_a}^{t_b} V(x(t), t) dt + \dots \right) \mathcal{D}x(t)$$

Hence

$$K(b, a) = K_0(b, a) + K_1(b, a) + \dots$$

where  $K_0$  is a free particle propagator and

$$K_1(b, a) = -\frac{i}{\hbar} \int_{x_a}^{x_b} \exp \left( \frac{i}{\hbar} \int_{t_a}^{t_b} \frac{m\dot{x}^2}{2} dt \right) \int_{t_a}^{t_b} V(x(t_c), t_c) dt_c \mathcal{D}x(t)$$

Interchange the order of the integrals.

$$K_1(b, a) = -\frac{i}{\hbar} \int_{t_a}^{t_b} \int_{x_a}^{x_b} \exp \left( \frac{i}{\hbar} \int_{t_a}^{t_b} \frac{m\dot{x}^2}{2} dt \right) V(x(t_c), t_c) \mathcal{D}x(t) dt_c$$

Factor the exponential with respect to  $t_c$ .

$$K_1(b, a) = -\frac{i}{\hbar} \int_{t_a}^{t_b} \int_{x_a}^{x_b} \exp \left( \frac{i}{\hbar} \int_{t_a}^{t_c} \frac{m\dot{x}^2}{2} dt \right) \exp \left( \frac{i}{\hbar} \int_{t_c}^{t_b} \frac{m\dot{x}^2}{2} dt \right) V(x(t_c), t_c) \mathcal{D}x(t) dt_c$$

The two exponentials are free particle propagators.

$$\begin{aligned} \exp \left( \frac{i}{\hbar} \int_{t_a}^{t_c} \frac{m\dot{x}^2}{2} dt \right) &= K_0(x(t_c), t_c, x_a, t_a) && \text{from } x_a, t_a \text{ to } x(t_c), t_c \\ \exp \left( \frac{i}{\hbar} \int_{t_c}^{t_b} \frac{m\dot{x}^2}{2} dt \right) &= K_0(x_b, t_b, x(t_c), t_c) && \text{from } x(t_c), t_c \text{ to } x_b, t_b \end{aligned}$$

Hence

$$K_1(b, a) = -\frac{i}{\hbar} \int_{t_a}^{t_b} \int_{x_a}^{x_b} K_0(x(t_c), t_c, x_a, t_a) K_0(x_b, t_b, x(t_c), t_c) V(x(t_c), t_c) \mathcal{D}x(t) dt_c$$

The integral is over all possible paths  $x(t)$  hence

$$-\infty < x(t_c) < \infty$$

Let  $x_c = x(t_c)$  and transform the measure from  $\mathcal{D}x(t)$  to  $dx_c$ .

$$K_1(b, a) = -\frac{i}{\hbar} \int_{t_a}^{t_b} \int_{-\infty}^{\infty} K_0(x_c, t_c, x_a, t_a) K_0(x_b, t_b, x_c, t_c) V(x_c, t_c) dx_c dt_c$$

Substitute for  $K_0$ .

$$\begin{aligned} K_1(b, a) = & -\frac{i}{\hbar} \int_{t_a}^{t_b} \int_{-\infty}^{\infty} \left( \frac{m}{2\pi i \hbar (t_c - t_a)} \right)^{\frac{1}{2}} \exp \left( \frac{im(x_c - x_a)^2}{2\hbar(t_c - t_a)} \right) \\ & \times \left( \frac{m}{2\pi i \hbar (t_b - t_c)} \right)^{\frac{1}{2}} \exp \left( \frac{im(x_b - x_c)^2}{2\hbar(t_b - t_c)} \right) V(x_c, t_c) dx_c dt_c \end{aligned}$$

Let  $t_a = 0$ ,  $t_b = T$ , and promote to three dimensions.

$$\begin{aligned} K_1(b, a) = & -\frac{i}{\hbar} \int_0^T \int \left( \frac{m}{2\pi i \hbar t_c} \right)^{\frac{3}{2}} \exp \left( \frac{im|\mathbf{x}_c - \mathbf{x}_a|^2}{2\hbar t_c} \right) \\ & \times \left( \frac{m}{2\pi i \hbar (T - t_c)} \right)^{\frac{3}{2}} \exp \left( \frac{im|\mathbf{x}_b - \mathbf{x}_c|^2}{2\hbar (T - t_c)} \right) V(\mathbf{x}_c, t_c) d^3\mathbf{x}_c dt_c \end{aligned}$$