

Laplacian of product

Let

$$F(\mathbf{r}) = \frac{e^{ikr}}{r}$$

where $r = |\mathbf{r}|$. Show that

$$\nabla^2 F(\mathbf{r}) = -k^2 F(\mathbf{r}) - 4\pi\delta^3(\mathbf{r})$$

Recall $\nabla^2 = \nabla \cdot \nabla$ and

$$\nabla \cdot (f\mathbf{A}) = \nabla f \cdot \mathbf{A} + f\nabla \cdot \mathbf{A}$$

Hence

$$\begin{aligned} \nabla^2 F(\mathbf{r}) &= \nabla \cdot \nabla \left(\frac{e^{ikr}}{r} \right) \\ &= \nabla \cdot \left(\frac{1}{r} \nabla e^{ikr} + e^{ikr} \nabla \frac{1}{r} \right) \\ &= \left(\nabla \frac{1}{r} \cdot \nabla e^{ikr} + \frac{1}{r} \nabla^2 e^{ikr} + \nabla e^{ikr} \cdot \nabla \frac{1}{r} + e^{ikr} \nabla^2 \frac{1}{r} \right) \end{aligned} \quad (1)$$

see (2) see (3) see (2)

In spherical coordinates

$$\nabla f = \frac{\partial f}{\partial r} \hat{\mathbf{e}}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\mathbf{e}}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\mathbf{e}}_\phi$$

Hence

$$\nabla \frac{1}{r} \cdot \nabla e^{ikr} = \left(-\frac{1}{r^2} \hat{\mathbf{e}}_r + 0\hat{\mathbf{e}}_\theta + 0\hat{\mathbf{e}}_\phi \right) \cdot (ike^{ikr} \hat{\mathbf{e}}_r + 0\hat{\mathbf{e}}_\theta + 0\hat{\mathbf{e}}_\phi) = -\frac{ik e^{ikr}}{r^2} \quad (2)$$

and

$$\begin{aligned} \frac{1}{r} \nabla^2 e^{ikr} &= \frac{1}{r^2} \frac{\partial^2}{\partial r^2} (re^{ikr}) \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} (e^{ikr} + ikre^{ikr}) \\ &= \frac{1}{r^2} (2ike^{ikr} - k^2 re^{ikr}) \\ &= \frac{2ike^{ikr}}{r^2} - \frac{k^2 e^{ikr}}{r} \end{aligned} \quad (3)$$

Substitute (2) and (3) into (1) to obtain

$$\begin{aligned} \nabla^2 F(\mathbf{r}) &= -\frac{k^2 e^{ikr}}{r} + e^{ikr} \nabla^2 \frac{1}{r} \\ &= -k^2 F(\mathbf{r}) - 4\pi\delta^3(\mathbf{r}) e^{ikr} \end{aligned}$$

Noting that $\delta^3(\mathbf{r})$ vanishes for $r \neq 0$ and $e^{ikr} = 1$ for $r = 0$ we have

$$\nabla^2 F(\mathbf{r}) = -k^2 F(\mathbf{r}) - 4\pi\delta^3(\mathbf{r}) \quad (4)$$