Path integral step by step

① Let H be the following Hamiltonian where $V \equiv V(x,t)$.

$$H = \frac{\hat{p}^2}{2m} + V$$

② Let K_j be the amplitude to go from x_j to x_{j+1} in time δt where $\delta t = T/N$.

$$K_j = \langle x_{j+1}| \exp\left(-\frac{i}{\hbar} H \delta t\right) |x_j\rangle$$

③ By the identity

$$\int |p\rangle\langle p|\, dp = 1$$

we can write

$$K_j = \int \langle x_{j+1}| \exp\left(-\frac{i}{\hbar} H \delta t\right) |p\rangle\langle p|x_j\rangle\, dp$$

④ Replace operator $\hat{p}$ with its eigenvalue p.

$$K_j = \int \exp\left(-\frac{i}{\hbar}\left(\frac{p^2}{2m} + V\right)\delta t\right) \langle x_{j+1}|p\rangle\langle p|x_j\rangle\, dp$$

⑤ By the identity

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right)$$

we can write

$$K_j = \frac{1}{2\pi\hbar} \int \left(-\frac{i}{\hbar}\left(\frac{p^2}{2m} + V\right)\delta t\right) \exp\left(\frac{i}{\hbar} p x_{j+1}\right) \exp\left(-\frac{i}{\hbar} p x_j\right)\, dp$$

⑥ Combine exponentials.

$$K_j = \frac{1}{2\pi\hbar} \int \exp\left(-\frac{i}{\hbar}\left(\frac{p^2}{2m} + V\right)\delta t + \frac{i}{\hbar} p(x_{j+1} - x_j)\right)\, dp$$

⑦ The V term does not depend on p so factor it out of the integrand.

$$K_j = \frac{1}{2\pi\hbar} \exp\left(-\frac{i}{\hbar} V \delta t\right) \int \exp\left(-\frac{i}{\hbar}\frac{p^2}{2m}\delta t + \frac{i}{\hbar} p(x_{j+1} - x_j)\right)\, dp$$

⑧ To solve the integral, use the identity

$$\int_{-\infty}^{\infty} \exp(-ay^2 + by)\, dy = \left(\frac{\pi}{a}\right)^{1/2} \exp\left(\frac{b^2}{4a}\right)$$

with

$$a = \frac{i}{\hbar}\frac{\delta t}{2m}, \quad b = \frac{i}{\hbar}(x_{j+1} - x_j)$$

to obtain

$$K_j = \frac{1}{2\pi\hbar} \exp\left(-\frac{i}{\hbar} V \delta t\right) \left(\frac{2\pi m\hbar}{i\delta t}\right)^{1/2} \exp\left(\frac{i}{\hbar}\frac{m}{2}\frac{(x_{j+1}-x_j)^2}{\delta t}\right)$$

⑨ Combine coefficients and exponentials.

$$K_j = \left(\frac{m}{2\pi i\hbar\delta t}\right)^{1/2} \exp\left(\frac{i}{\hbar}\frac{m}{2}\frac{(x_{j+1}-x_j)^2}{\delta t} - \frac{i}{\hbar} V \delta t\right)$$

⑩ Factor out $i/\hbar$ and δt.

$$K_j = \left(\frac{m}{2\pi i\hbar\delta t}\right)^{1/2} \exp\left[\frac{i}{\hbar}\left(\frac{m}{2}\left(\frac{x_{j+1}-x_j}{\delta t}\right)^2 - V\right)\delta t\right]$$

⑪ Let $K(b,a)$ be the amplitude to go from x_a at time zero to x_b at time T where $x_0 \equiv x_a$ and $x_N \equiv x_b$.

$$K(b,a) = \int dx_{N-1} \cdots \int dx_2 \int dx_1\, K_{N-1} \cdots K_2 K_1 K_0$$

⑫ Substitute for the K_j.

$$K(b,a) = \int dx_{N-1} \cdots \int dx_2 \int dx_1 \left(\frac{m}{2\pi i\hbar\delta t}\right)^{N/2} \exp\left[\frac{i}{\hbar}\sum_{j=0}^{N-1}\left(\frac{m}{2}\left(\frac{x_{j+1}-x_j}{\delta t}\right)^2 - V\right)\delta t\right]$$

⑬ In the limit we have

$$\lim_{N\to\infty}\sum_{j=0}^{N-1}\left(\frac{m}{2}\left(\frac{x_{j+1}-x_j}{\delta t}\right)^2 - V\right)\delta t = \int_0^T \left(\frac{m}{2}\left(\frac{d}{dt}x(t)\right)^2 - V\right) dt$$

Hence

$$K(b,a) = \int dx_{N-1} \cdots \int dx_2 \int dx_1 \left(\frac{m}{2\pi i\hbar\delta t}\right)^{N/2} \exp\left[\frac{i}{\hbar}\int_0^T \left(\frac{m}{2}\left(\frac{d}{dt}x(t)\right)^2 - V\right) dt\right]$$

⑭ Define

$$\int Dx(t) = \lim_{N\to\infty} \int dx_{N-1} \cdots \int dx_2 \int dx_1 \left(\frac{m}{2\pi i\hbar\delta t}\right)^{N/2}$$

and write

$$K(b,a) = \int Dx(t)\, \exp\left[\frac{i}{\hbar}\int_0^T \left(\frac{m}{2}\left(\frac{d}{dt}x(t)\right)^2 - V\right) dt\right]$$