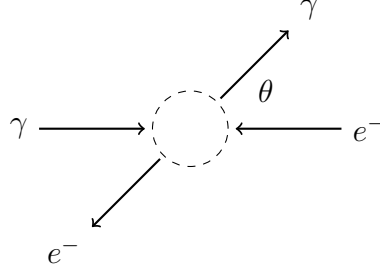


Compton scattering

Compton scattering is the interaction $e^- + \gamma \rightarrow e^- + \gamma$.



In the center-of-mass frame we have the following momentum vectors where ω is incident energy and E is total energy $E = \sqrt{\omega^2 + m^2}$. Polar angle θ is the observed scattering angle. Azimuth angle ϕ cancels out in scattering calculations.

$$p_1 = \begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix}, \quad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -\omega \end{pmatrix}, \quad p_3 = \begin{pmatrix} \omega \\ \omega \sin \theta \cos \phi \\ \omega \sin \theta \sin \phi \\ \omega \cos \theta \end{pmatrix}, \quad p_4 = \begin{pmatrix} E \\ -\omega \sin \theta \cos \phi \\ -\omega \sin \theta \sin \phi \\ -\omega \cos \theta \end{pmatrix}$$

inbound photon inbound electron outbound photon outbound electron

Spinors for the inbound electron.

$$u_{21} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ -\omega \\ 0 \end{pmatrix}, \quad u_{22} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ 0 \\ \omega \end{pmatrix}$$

inbound electron spin up inbound electron spin down

Spinors for the outbound electron.

$$u_{41} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix}, \quad u_{42} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}$$

outbound electron spin up outbound electron spin down

Let a be the spin state of the inbound electron and let b be the spin state of the outbound electron so that subscript $ba \in \{11, 12, 21, 22\}$.

The probability amplitude \mathcal{M}_{ba} for spin state ba is

$$\mathcal{M}_{ba} = \mathcal{M}_{1ba} + \mathcal{M}_{2ba}$$

where

$$\mathcal{M}_{1ba} = \frac{\bar{u}_{4b}(-ie\gamma^\mu)(\not{p}_1 + m)(-ie\gamma^\nu)u_{2a}}{(p_1 + p_2)^2 - m^2}, \quad \mathcal{M}_{2ba} = \frac{\bar{u}_{4b}(-ie\gamma^\nu)(\not{p}_2 + m)(-ie\gamma^\mu)u_{2a}}{(p_1 - p_4)^2 - m^2}$$

and

$$\not{q}_1 = (p_1 + p_2)^\alpha g_{\alpha\beta} \gamma^\beta, \quad \not{q}_2 = (p_4 - p_1)^\alpha g_{\alpha\beta} \gamma^\beta$$

The expected probability $\langle |\mathcal{M}|^2 \rangle$ is the average over all four spin states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{a=1}^2 \sum_{b=1}^2 \mathcal{M}_{ba} \mathcal{M}_{ba}^*$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{a=1}^2 \sum_{b=1}^2 \left(\frac{\mathcal{M}_{1ba} \mathcal{M}_{1ba}^*}{(s - m^2)^2} + \frac{\mathcal{M}_{1ba} \mathcal{M}_{2ba}^* + \mathcal{M}_{2ba} \mathcal{M}_{1ba}^*}{(s - m^2)(u - m^2)} + \frac{\mathcal{M}_{2ba} \mathcal{M}_{2ba}^*}{(u - m^2)^2} \right)$$

where s and u are the Mandelstam variables

$$s = (p_1 + p_2)^2, \quad u = (p_1 - p_4)^2$$

To understand how $\mathcal{M}_{1ba} \mathcal{M}_{1ba}^*$ is calculated, write \mathcal{M}_{1ba} in component form.

$$(\mathcal{M}_{1ba})^{\mu\nu} = \frac{(\bar{u}_{4b})_\alpha (-ie\gamma^{\mu\alpha}) (\not{q}_1 + m)^\beta_\rho (-ie\gamma^{\nu\rho}) (u_{2a})^\sigma}{s - m^2}$$

To sum over μ and ν of the product, $g_{\mu\nu}$ is needed to lower indices.

$$\mathcal{M}_{1ba} \mathcal{M}_{1ba}^* = (\mathcal{M}_{1ba})^{\mu\nu} (\mathcal{M}_{1ba}^*)_{\mu\nu} = (\mathcal{M}_{1ba})^{\mu\nu} g_{\mu\alpha} (\mathcal{M}_{1ba}^*)^{\alpha\beta} g_{\beta\nu}$$

Similarly for $\mathcal{M}_{2ba} \mathcal{M}_{2ba}^*$. For \mathcal{M}_{2ba} the index order is ν followed by μ hence

$$\mathcal{M}_{1ba} \mathcal{M}_{2ba}^* = (\mathcal{M}_{1ba})^{\mu\nu} (\mathcal{M}_{2ba}^*)_{\nu\mu} = (\mathcal{M}_{1ba})^{\mu\nu} g_{\nu\beta} (\mathcal{M}_{2ba}^*)^{\beta\alpha} g_{\alpha\mu}$$

The Casimir trick uses matrix arithmetic to sum over spin states.

$$\begin{aligned} f_{11} &= \sum_{a=1}^2 \sum_{b=1}^2 \mathcal{M}_{1ba} \mathcal{M}_{1ba}^* = e^4 \text{Tr} \left((\not{p}_2 + m) \gamma^\mu (\not{q}_1 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\nu (\not{q}_1 + m) \gamma_\mu \right) \\ f_{12} &= \sum_{a=1}^2 \sum_{b=1}^2 \mathcal{M}_{1ba} \mathcal{M}_{2ba}^* = e^4 \text{Tr} \left((\not{p}_2 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\mu (\not{q}_1 + m) \gamma_\nu \right) \\ f_{22} &= \sum_{a=1}^2 \sum_{b=1}^2 \mathcal{M}_{2ba} \mathcal{M}_{2ba}^* = e^4 \text{Tr} \left((\not{p}_2 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\nu (\not{q}_2 + m) \gamma_\mu \right) \end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \left(\frac{f_{11}}{(s - m^2)^2} + \frac{f_{12} + f_{12}^*}{(s - m^2)(u - m^2)} + \frac{f_{22}}{(u - m^2)^2} \right) \quad (1)$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$\begin{aligned} f_{11} &= e^4 (32(p_1 \cdot p_2)(p_1 \cdot p_4) + 64m^2(p_1 \cdot p_2) - 32m^2(p_1 \cdot p_3) - 32m^2(p_1 \cdot p_4) + 32m^4) \\ f_{12} &= e^4 (16m^2(p_1 \cdot p_2) - 16m^2(p_1 \cdot p_4) + 32m^4) \\ f_{22} &= e^4 (32(p_1 \cdot p_2)(p_1 \cdot p_4) + 32m^2(p_1 \cdot p_2) - 32m^2(p_1 \cdot p_3) - 64m^2(p_1 \cdot p_4) + 32m^4) \end{aligned}$$

In Mandelstam variables

$$\begin{aligned} f_{11} &= e^4 (-8su + 24sm^2 + 8um^2 + 8m^4) \\ f_{12} &= e^4 (8sm^2 + 8um^2 + 16m^4) \\ f_{22} &= e^4 (-8su + 8sm^2 + 24um^2 + 8m^4) \end{aligned} \tag{2}$$

Compton scattering experiments are typically done in the lab frame where the electron is at rest. Define Lorentz boost Λ for transforming momentum vectors to the lab frame.

$$\Lambda = \begin{pmatrix} E/m & 0 & 0 & \omega/m \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega/m & 0 & 0 & E/m \end{pmatrix}$$

The electron is at rest in the lab frame.

$$\Lambda p_2 = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Mandelstam variables are invariant under a boost.

$$\begin{aligned} s &= (p_1 + p_2)^2 = (\Lambda p_1 + \Lambda p_2)^2 \\ t &= (p_1 - p_3)^2 = (\Lambda p_1 - \Lambda p_3)^2 \\ u &= (p_1 - p_4)^2 = (\Lambda p_1 - \Lambda p_4)^2 \end{aligned}$$

In the lab frame, let ω_L be the angular frequency of the incident photon and let ω'_L be the angular frequency of the scattered photon.

$$\begin{aligned} \omega_L &= \Lambda p_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2}{m} + \frac{\omega E}{m} \\ \omega'_L &= \Lambda p_3 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2 \cos \theta}{m} + \frac{\omega E}{m} \end{aligned}$$

It can be shown that

$$\begin{aligned} s &= m^2 + 2m\omega_L \\ t &= 2m(\omega'_L - \omega_L) \\ u &= m^2 - 2m\omega'_L \end{aligned} \tag{3}$$

Then by (1), (2), and (3) we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \left(\frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1 \right)^2 - 1 \right)$$

Lab scattering angle θ_L is given by the Compton equation

$$\cos \theta_L = \frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1$$

Hence

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \cos^2 \theta_L - 1 \right) \\ &= 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} - \sin^2 \theta_L \right) \end{aligned}$$

Cross section

Now that we have derived $\langle |\mathcal{M}|^2 \rangle$ we can investigate the angular distribution of scattered photons. For simplicity let us drop the L subscript from lab variables. From now on the symbols ω , ω' , and θ will be lab frame variables.

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4(4\pi\epsilon_0)^2 s} \left(\frac{\omega'}{\omega} \right)^2 \langle |\mathcal{M}|^2 \rangle$$

where

$$s = m^2 + 2m\omega = (mc^2)^2 + 2(mc^2)(\hbar\omega)$$

and ω' is given by the Compton equation

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos \theta)}$$

For the lab frame we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

Hence in the lab frame

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\epsilon_0)^2 s} \left(\frac{\omega'}{\omega} \right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

Noting that

$$e^2 = 4\pi\epsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s} \left(\frac{\omega'}{\omega} \right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

Noting that

$$d\Omega = \sin \theta d\theta d\phi$$

we also have

$$d\sigma = \frac{\alpha^2 (\hbar c)^2}{2s} \left(\frac{\omega'}{\omega} \right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right) \sin \theta d\theta d\phi$$

Let $S(\theta_1, \theta_2)$ be the following surface integral of $d\sigma$.

$$S(\theta_1, \theta_2) = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{2\pi\alpha^2(\hbar c)^2}{2s} (I(\theta_2) - I(\theta_1))$$

where

$$I(\theta) = -\frac{\cos \theta}{R^2} + \log(1 + R(1 - \cos \theta)) \left(\frac{1}{R} - \frac{2}{R^2} - \frac{2}{R^3} \right) - \frac{1}{2R(1 + R(1 - \cos \theta))^2} + \frac{1}{1 + R(1 - \cos \theta)} \left(-\frac{2}{R^2} - \frac{1}{R^3} \right)$$

and

$$R = \frac{\hbar\omega}{mc^2}$$

The cumulative distribution function is

$$F(\theta) = \frac{S(0, \theta)}{S(0, \pi)} = \frac{I(\theta) - I(0)}{I(\pi) - I(0)}, \quad 0 \leq \theta \leq \pi$$

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 \leq \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

Let N be the total number of scattering events from an experiment. Then the number of scattering events in the interval θ_1 to θ_2 is predicted to be

$$NP(\theta_1 \leq \theta \leq \theta_2)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi) - I(0)} \left(\frac{\omega'}{\omega} \right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right) \sin \theta$$

Thomson scattering

For $\hbar\omega \ll mc^2$ we have

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2} (1 - \cos \theta)} \approx \omega$$

Hence we can use the approximations

$$\omega = \omega' \quad \text{and} \quad s = (mc^2)^2$$

to obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \hbar^2}{2m^2 c^2} (1 + \cos^2 \theta)$$

which is the formula for Thomson scattering.

High energy approximation

For $\omega \gg m$ a useful approximation is to set $m = 0$ and obtain

$$\begin{aligned}f_{11} &= e^4(-8su) \\f_{12} &= 0 \\f_{22} &= e^4(-8su)\end{aligned}$$

Hence

$$\begin{aligned}\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left(\frac{-8su}{s^2} + \frac{-8su}{u^2} \right) \\&= 2e^4 \left(-\frac{u}{s} - \frac{s}{u} \right)\end{aligned}$$

Also for $m = 0$ the Mandelstam variables s and u are

$$\begin{aligned}s &= 4\omega^2 \\u &= -2\omega^2(\cos \theta + 1)\end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{\cos \theta + 1}{2} + \frac{2}{\cos \theta + 1} \right)$$