

Electromagnetic tensor

$$\mathbf{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}, \quad A^\mu = \begin{pmatrix} \phi \\ A_x \\ A_y \\ A_z \end{pmatrix}, \quad A_\mu = g_{\mu\nu} A^\nu = \begin{pmatrix} \phi \\ -A_x \\ -A_y \\ -A_z \end{pmatrix}$$

$$g_{\mu\nu} = ((1,0,0,0), (0,-1,0,0), (0,0,-1,0), (0,0,0,-1))$$

$$\begin{aligned} \mathbf{A} &= (A_x(), A_y(), A_z()) \\ A_\mu &= (\phi(), A_x(), A_y(), A_z()) \\ \text{Ad} &= \text{dot}(g_{\mu\nu}, A_\mu) \end{aligned}$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\begin{aligned} \mathbf{B} &= \text{curl}(\mathbf{A}) \\ \mathbf{E} &= -\text{d}(\phi(), (x,y,z)) - \text{d}(\mathbf{A}, t) \end{aligned}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$\begin{aligned} \mathbf{X} &= (t, x, y, z) \\ \text{Fdd} &= \text{d}(\text{Ad}, \mathbf{X}) \\ \text{Fdd} &= \text{transpose}(\text{Fdd}) - \text{Fdd} \end{aligned}$$

$$\begin{aligned} \mathbf{T} &= ((0, E[1], E[2], E[3]), \\ &\quad (-E[1], 0, -B[3], B[2]), \\ &\quad (-E[2], B[3], 0, -B[1]), \\ &\quad (-E[3], -B[2], B[1], 0)) \end{aligned}$$

$$\text{check}(\text{Fdd} == \mathbf{T})$$

$$F_{\mu\nu} F^{\mu\nu} = 2\mathbf{B}^2 - 2\mathbf{E}^2$$

$$\begin{aligned} \text{Fuu} &= \text{dot}(g_{\mu\nu}, \text{Fdd}, g_{\mu\nu}) \\ \mathbf{T} &= \text{contract}(\text{dot}(\text{transpose}(\text{Fdd}), \text{Fuu})) \\ \text{check}(\mathbf{T} == 2 \text{ dot}(\mathbf{B}, \mathbf{B}) - 2 \text{ dot}(\mathbf{E}, \mathbf{E})) \end{aligned}$$

$$\det(F_{\mu\nu}) = \det(F^{\mu\nu}) = (\mathbf{B} \cdot \mathbf{E})^2$$

$$\begin{aligned} \text{check}(\det(\text{Fdd}) == \text{dot}(\mathbf{B}, \mathbf{E})^2) \\ \text{check}(\det(\text{Fuu}) == \text{dot}(\mathbf{B}, \mathbf{E})^2) \end{aligned}$$