

Spin

Spin state $|s\rangle$ is a normalized vector in \mathbb{C}^2 .

$$|s\rangle = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}, \quad |c_+|^2 + |c_-|^2 = 1$$

Spin measurement probabilities are the transition probabilities from $|s\rangle$ to an eigenstate.

For spin measurements in the z direction we have

$$\begin{aligned} \Pr(S_z = +\frac{\hbar}{2}) &= |\langle z_+ | s \rangle|^2 = |c_+|^2 \\ \Pr(S_z = -\frac{\hbar}{2}) &= |\langle z_- | s \rangle|^2 = |c_-|^2 \end{aligned}$$

where the eigenstates are

$$|z_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |z_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

By definition of expectation value we have

$$\langle S_z \rangle = \frac{\hbar}{2} \Pr(S_z = +\frac{\hbar}{2}) - \frac{\hbar}{2} \Pr(S_z = -\frac{\hbar}{2})$$

Rewrite as

$$\langle S_z \rangle = \frac{\hbar}{2} |\langle z_+ | s \rangle|^2 - \frac{\hbar}{2} |\langle z_- | s \rangle|^2$$

Rewrite again as

$$\langle S_z \rangle = \frac{\hbar}{2} \langle s | z_+ \rangle \langle z_+ | s \rangle - \frac{\hbar}{2} \langle s | z_- \rangle \langle z_- | s \rangle$$

Then by

$$\langle S_z \rangle = \langle s | S_z | s \rangle$$

we have

$$S_z = \frac{\hbar}{2} |z_+\rangle \langle z_+| - \frac{\hbar}{2} |z_-\rangle \langle z_-| = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

From the commutator

$$S_+ S_- - S_- S_+ = 2\hbar S_z$$

we have

$$S_+ S_- - S_- S_+ = \hbar^2 |z_+\rangle \langle z_+| - \hbar^2 |z_-\rangle \langle z_-|$$

Rewrite as

$$S_+ S_- - S_- S_+ = \hbar^2 |z_+\rangle \langle z_- | z_- \rangle \langle z_+| - \hbar^2 |z_-\rangle \langle z_+ | z_+ \rangle \langle z_-|$$

Hence

$$\begin{aligned} S_+ &= \hbar |z_+\rangle \langle z_-| = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ S_- &= \hbar |z_-\rangle \langle z_+| = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Then by

$$\begin{aligned} S_+ &= S_x + iS_y \\ S_- &= S_x - iS_y \end{aligned}$$

we obtain

$$\begin{aligned} S_x &= \frac{S_+ + S_-}{2} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ S_y &= \frac{S_+ - S_-}{2i} = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \end{aligned}$$

By solving for the eigenstates in

$$\begin{aligned} S_x |x_{\pm}\rangle &= \pm \frac{\hbar}{2} |x_{\pm}\rangle \\ S_y |y_{\pm}\rangle &= \pm \frac{\hbar}{2} |y_{\pm}\rangle \end{aligned}$$

we obtain

$$\begin{aligned} |x_+\rangle &= \frac{|z_+\rangle + |z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ |x_-\rangle &= \frac{|z_+\rangle - |z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} |y_+\rangle &= \frac{|z_+\rangle + i|z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \\ |y_-\rangle &= \frac{|z_+\rangle - i|z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \end{aligned}$$

The expected spin direction vector is

$$\langle \mathbf{S} \rangle = \begin{pmatrix} \langle s | S_x | s \rangle \\ \langle s | S_y | s \rangle \\ \langle s | S_z | s \rangle \end{pmatrix}, \quad |\langle \mathbf{S} \rangle| = \frac{\hbar}{2}$$

To convert a direction vector to a spin state, let θ and ϕ be the polar and azimuth angles such that

$$\langle \mathbf{S} \rangle = \frac{\hbar}{2} \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix}$$

Then

$$|s\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \exp(i\phi) \end{pmatrix}$$

By the identities

$$\cos^2(\theta/2) = \frac{\cos \theta + 1}{2}, \quad \sin^2(\theta/2) = \frac{1 - \cos \theta}{2}$$

and noting that $0 \leq \theta \leq \pi$ we have

$$\cos(\theta/2) = \sqrt{\frac{\langle z \rangle + 1}{2}}, \quad \sin(\theta/2) \exp(i\phi) = \sqrt{\frac{1 - \langle z \rangle}{2}} \frac{\langle x \rangle + i\langle y \rangle}{\sqrt{\langle x \rangle^2 + \langle y \rangle^2}}$$

where

$$\begin{aligned} \langle x \rangle &= \frac{2}{\hbar} \langle S_x \rangle = \sin \theta \cos \phi \\ \langle y \rangle &= \frac{2}{\hbar} \langle S_y \rangle = \sin \theta \sin \phi \\ \langle z \rangle &= \frac{2}{\hbar} \langle S_z \rangle = \cos \theta \end{aligned}$$

The following commutator was used to derive S_x and S_y .

$$S_+ S_- - S_- S_+ = 2\hbar S_z$$

The commutator is a consequence of the following wave equation for spin.

$$\hat{\mathbf{S}}\psi = (\mathbf{r} \times \hat{\mathbf{p}})\psi, \quad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \hat{\mathbf{p}} = -i\hbar \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

Rewrite in component form.

$$\hat{S}_x \psi = -i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \psi$$

$$\hat{S}_y \psi = -i\hbar \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) \psi$$

$$\hat{S}_z \psi = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \psi$$

By computer algebra we have

$$(\hat{S}_y \hat{S}_z - \hat{S}_z \hat{S}_y) \psi = i\hbar \hat{S}_x \psi$$

$$(\hat{S}_z \hat{S}_x - \hat{S}_x \hat{S}_z) \psi = i\hbar \hat{S}_y \psi$$

$$(\hat{S}_x \hat{S}_y - \hat{S}_y \hat{S}_x) \psi = i\hbar \hat{S}_z \psi$$

Let

$$\hat{S}_+ = \hat{S}_x + i\hat{S}_y$$

$$\hat{S}_- = \hat{S}_x - i\hat{S}_y$$

By computer algebra

$$(\hat{S}_+ \hat{S}_- - \hat{S}_- \hat{S}_+) \psi = 2\hbar \hat{S}_z \psi$$

1. Verify that

$$\begin{aligned}S_x &= \frac{\hbar}{2}(|x_+\rangle\langle x_+| - |x_-\rangle\langle x_-|) \\S_y &= \frac{\hbar}{2}(|y_+\rangle\langle y_+| - |y_-\rangle\langle y_-|) \\S_z &= \frac{\hbar}{2}(|z_+\rangle\langle z_+| - |z_-\rangle\langle z_-|)\end{aligned}$$

```
zp = (1,0)
zm = (0,1)

xp = (zp + zm) / sqrt(2)
xm = (zp - zm) / sqrt(2)

yp = (zp + i zm) / sqrt(2)
ym = (zp - i zm) / sqrt(2)

Sx = hbar / 2 ((0,1),(1,0))
Sy = hbar / 2 ((0,-i),(i,0))
Sz = hbar / 2 ((1,0),(0,-1))

check(Sx == hbar / 2 (outer(xp,conj(xp)) - outer(xm,conj(xm))))
check(Sy == hbar / 2 (outer(yp,conj(yp)) - outer(ym,conj(ym))))
check(Sz == hbar / 2 (outer(zp,conj(zp)) - outer(zm,conj(zm))))
```

2. Let $|s\rangle$ be the following spin state.

$$|s\rangle = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$$

Verify that $|s\rangle$ is normalized and that

$$\langle \mathbf{S} \rangle = \langle s | \mathbf{S} | s \rangle = \frac{\hbar}{2} \begin{pmatrix} \frac{4}{9} \\ \frac{8}{9} \\ \frac{1}{9} \end{pmatrix}$$

where

$$\mathbf{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$$

Note: In component form we have

$$\langle s | \mathbf{S} | s \rangle = s_\beta^* S^{\alpha\beta}{}_\gamma s^\gamma$$

Eigenmath requires a transpose so that the β indices are adjacent.

$$\langle s | \mathbf{S} | s \rangle = s_\beta^* S^{\beta\alpha}{}_\gamma s^\gamma$$

```
s = (1/3 - 2/3 i, 2/3)
```

```
check(dot(conj(s),s) == 1)
```

```
Sx = hbar / 2 ((0,1),(1,0))
```

```
Sy = hbar / 2 ((0,-i),(i,0))
```

```
Sz = hbar / 2 ((1,0),(0,-1))
```

```
S = (Sx,Sy,Sz)
```

```
check(dot(conj(s),transpose(S),s) == hbar / 2 (4/9, 8/9, 1/9))
```

3. Let $|s\rangle$ be the following spin state.

$$|s\rangle = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$$

Verify the following measurement probabilities for $|s\rangle$.

$$\Pr(S_x = +\frac{\hbar}{2}) = |\langle x_+ | s \rangle|^2 = \frac{13}{18}$$

$$\Pr(S_x = -\frac{\hbar}{2}) = |\langle x_- | s \rangle|^2 = \frac{5}{18}$$

$$\Pr(S_y = +\frac{\hbar}{2}) = |\langle y_+ | s \rangle|^2 = \frac{17}{18}$$

$$\Pr(S_y = -\frac{\hbar}{2}) = |\langle y_- | s \rangle|^2 = \frac{1}{18}$$

$$\Pr(S_z = +\frac{\hbar}{2}) = |\langle z_+ | s \rangle|^2 = \frac{5}{9}$$

$$\Pr(S_z = -\frac{\hbar}{2}) = |\langle z_- | s \rangle|^2 = \frac{4}{9}$$

```
s = (1/3 - 2/3 i, 2/3)
```

```
zp = (1,0)
```

```
zm = (0,1)
```

```
xp = (zp + zm) / sqrt(2)
```

```
xm = (zp - zm) / sqrt(2)
```

```
yp = (zp + i zm) / sqrt(2)
```

```
ym = (zp - i zm) / sqrt(2)
```

```
Pr(a,b) = dot(conj(a),b) dot(conj(b),a)
```

```
check(Pr(xp,s) == 13/18)
```

```
check(Pr(xm,s) == 5/18)
```

```
check(Pr(yp,s) == 17/18)
```

```
check(Pr(ym,s) == 1/18)
```

```
check(Pr(zp,s) == 5/9)
```

```
check(Pr(zm,s) == 4/9)
```

4. Let $|s\rangle$ be the following spin state.

$$|s\rangle = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$$

Verify that the following spin state $|\chi\rangle$ is indistinguishable from $|s\rangle$.

$$|\chi\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \exp(i\phi) \end{pmatrix}$$

where

$$\cos(\theta/2) = \sqrt{\frac{\langle z \rangle + 1}{2}} = \frac{\sqrt{5}}{3}$$

and

$$\sin(\theta/2) \exp(i\phi) = \sqrt{\frac{1 - \langle z \rangle}{2}} \frac{\langle x \rangle + i\langle y \rangle}{\sqrt{\langle x \rangle^2 + \langle y \rangle^2}} = \frac{2 + 4i}{3\sqrt{5}}$$

with

$$\begin{aligned} \langle x \rangle &= \frac{2}{\hbar} \langle S_x \rangle \\ \langle y \rangle &= \frac{2}{\hbar} \langle S_y \rangle \\ \langle z \rangle &= \frac{2}{\hbar} \langle S_z \rangle \end{aligned}$$

```
s = (1/3 - 2/3 i, 2/3)
```

```
Sx = hbar / 2 ((0,1),(1,0))
Sy = hbar / 2 ((0,-i),(i,0))
Sz = hbar / 2 ((1,0),(0,-1))
```

```
S = (Sx,Sy,Sz)
```

```
x = 2 / hbar dot(conj(s),Sx,s)
y = 2 / hbar dot(conj(s),Sy,s)
z = 2 / hbar dot(conj(s),Sz,s)
```

```
cp = sqrt((z + 1) / 2)
cm = sqrt((1 - z) / 2) (x + i y) / sqrt(x^2 + y^2)
```

```
check(cp == sqrt(5) / 3)
check(cm == (2 + 4 i) / (3 sqrt(5)))
```

```
chi = (cp,cm)
```

```
check(dot(conj(s),transpose(S),s) == dot(conj(chi),transpose(S),chi))
```

5. Verify the following commutators for $\mathbf{S}\psi = (\mathbf{r} \times \mathbf{p})\psi$.

$$[S_y, S_z] = i\hbar S_x$$

$$[S_z, S_x] = i\hbar S_y$$

$$[S_x, S_y] = i\hbar S_z$$

$$[S^2, S_x] = 0$$

$$[S^2, S_y] = 0$$

$$[S^2, S_z] = 0$$

$$[S_+, S_-] = 2\hbar S_z$$

where

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

and

$$S_+ = S_x + iS_y$$

$$S_- = S_x - iS_y$$

```

Sx(psi) = -i hbar (y d(psi,z) - z d(psi,y))
Sy(psi) = -i hbar (z d(psi,x) - x d(psi,z))
Sz(psi) = -i hbar (x d(psi,y) - y d(psi,x))

psi = Psi()

check(Sy(Sz(psi)) - Sz(Sy(psi)) == i hbar Sx(psi))
check(Sz(Sx(psi)) - Sx(Sz(psi)) == i hbar Sy(psi))
check(Sx(Sy(psi)) - Sy(Sx(psi)) == i hbar Sz(psi))

S2(psi) = Sx(Sx(psi)) + Sy(Sy(psi)) + Sz(Sz(psi))

check(S2(Sx(psi)) - Sx(S2(psi)) == 0)
check(S2(Sy(psi)) - Sy(S2(psi)) == 0)
check(S2(Sz(psi)) - Sz(S2(psi)) == 0)

Sp(psi) = Sx(psi) + i Sy(psi)
Sm(psi) = Sx(psi) - i Sy(psi)

check(Sp(Sm(psi)) - Sm(Sp(psi)) == 2 hbar Sz(psi))

```