Exercise 5.2.

1) Show that $\Delta\mathbf{A}^2 = \langle\bar{\mathbf{A}}^2\rangle$ and $\Delta\mathbf{B}^2 = \langle\bar{\mathbf{B}}^2\rangle$.

2) Show that $[\bar{\mathbf{A}}, \bar{\mathbf{B}}] = [\mathbf{A}, \mathbf{B}]$.

3) Using these relations, show that

$$\Delta\mathbf{A}\,\Delta\mathbf{B} \geq \tfrac{1}{2}|\langle\Psi|[\mathbf{A}, \mathbf{B}]|\Psi\rangle|$$

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1) Recall that $\Delta\mathbf{A}$ is standard deviation and $\Delta\mathbf{A}^2 = (\Delta\mathbf{A})^2$ is variance.

From page 147 we have

$$\bar{\mathbf{A}} = \mathbf{A} - \langle\mathbf{A}\rangle \tag{1}$$

Hence

$$\bar{\mathbf{A}}^2 = \mathbf{A}^2 - 2\langle\mathbf{A}\rangle\mathbf{A} + \langle\mathbf{A}\rangle^2$$

It follows that the expectation of $\bar{\mathbf{A}}^2$ is

$$\langle\bar{\mathbf{A}}^2\rangle = \langle\mathbf{A}^2\rangle - \langle\mathbf{A}\rangle^2$$

Noting that $\langle\mathbf{A}^2\rangle - \langle\mathbf{A}\rangle^2$ is the variance of $\mathbf{A}$, we have

$$(\Delta\mathbf{A})^2 = \langle\mathbf{A}^2\rangle - \langle\mathbf{A}\rangle^2 = \langle\bar{\mathbf{A}}^2\rangle$$

By the same argument

$$(\Delta\mathbf{B})^2 = \langle\mathbf{B}^2\rangle - \langle\mathbf{B}\rangle^2 = \langle\bar{\mathbf{B}}^2\rangle$$

2) We have

$$[\bar{\mathbf{A}}, \bar{\mathbf{B}}] = \bar{\mathbf{A}}\bar{\mathbf{B}} - \bar{\mathbf{B}}\bar{\mathbf{A}}$$

By substitution from (1)

$$[\bar{\mathbf{A}}, \bar{\mathbf{B}}] = (\mathbf{A} - \langle\mathbf{A}\rangle)(\mathbf{B} - \langle\mathbf{B}\rangle) - (\mathbf{B} - \langle\mathbf{B}\rangle)(\mathbf{A} - \langle\mathbf{A}\rangle)$$

Expand products of sums.

$$
\begin{aligned}
[\bar{\mathbf{A}}, \bar{\mathbf{B}}] = \mathbf{AB} - \underbrace{\mathbf{A}\langle\mathbf{B}\rangle}_{\text{cancel a}} - \underbrace{\langle\mathbf{A}\rangle\mathbf{B}}_{\text{cancel b}} + \underbrace{\langle\mathbf{A}\rangle\langle\mathbf{B}\rangle}_{\text{cancel c}} \\
- \mathbf{BA} + \underbrace{\mathbf{B}\langle\mathbf{A}\rangle}_{\text{term b}} + \underbrace{\langle\mathbf{B}\rangle\mathbf{A}}_{\text{term a}} - \underbrace{\langle\mathbf{B}\rangle\langle\mathbf{A}\rangle}_{\text{term c}}
\end{aligned}
$$

Hence

$$[\bar{\mathbf{A}}, \bar{\mathbf{B}}] = \mathbf{AB} - \mathbf{BA} = [\mathbf{A}, \mathbf{B}]$$

3) By simple algebra we have

$$\Delta\mathbf{A}\,\Delta\mathbf{B} = \sqrt{(\Delta\mathbf{A})^2(\Delta\mathbf{B})^2}$$

By the result from step 1 we can write this as

$$\Delta\mathbf{A}\,\Delta\mathbf{B} = \sqrt{\langle\bar{\mathbf{A}}^2\rangle\langle\bar{\mathbf{B}}^2\rangle} \tag{2}$$

Consider equation (5.12).

$$2\sqrt{\langle\mathbf{A}^2\rangle\langle\mathbf{B}^2\rangle} \geq |\langle\Psi|[\mathbf{A}, \mathbf{B}]|\Psi\rangle| \tag{5.12}$$

Equation (5.12) is valid for any linear operator, hence we can put bars over $\mathbf{A}$ and $\mathbf{B}$ and obtain

$$2\sqrt{\langle\bar{\mathbf{A}}^2\rangle\langle\bar{\mathbf{B}}^2\rangle} \geq |\langle\Psi|[\bar{\mathbf{A}}, \bar{\mathbf{B}}]|\Psi\rangle|$$

By the result from step 2 we can unbar $\mathbf{A}$ and $\mathbf{B}$ on the right hand side.

$$2\sqrt{\langle\bar{\mathbf{A}}^2\rangle\langle\bar{\mathbf{B}}^2\rangle} \geq |\langle\Psi|[\mathbf{A}, \mathbf{B}]|\Psi\rangle| \tag{3}$$

By equations (2) and (3) we have

$$2\Delta\mathbf{A}\,\Delta\mathbf{B} \geq |\langle\Psi|[\mathbf{A}, \mathbf{B}]|\Psi\rangle|$$