

WKB approximation

(Adapted from “WKB approximation” at physicspages.com)

Start with the time-independent Schrodinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

Rewrite as

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E - V(x))\psi \quad (1)$$

Let ψ be composed of amplitude A and phase ϕ .

$$\psi(x) = A(x)e^{i\phi(x)} \quad (2)$$

It follows that

$$\frac{d\psi}{dx} = \left(\frac{dA}{dx} + iA \frac{d\phi}{dx} \right) e^{i\phi}$$

and

$$\frac{d^2\psi}{dx^2} = \left(\frac{d^2A}{dx^2} + 2i \frac{dA}{dx} \frac{d\phi}{dx} + iA \frac{d^2\phi}{dx^2} - A \left(\frac{d\phi}{dx} \right)^2 \right) e^{i\phi} \quad (3)$$

Substitute (2) and (3) into (1) to obtain

$$\left(\frac{d^2A}{dx^2} + 2i \frac{dA}{dx} \frac{d\phi}{dx} + iA \frac{d^2\phi}{dx^2} - A \left(\frac{d\phi}{dx} \right)^2 \right) e^{i\phi} = -\frac{2m}{\hbar^2}(E - V(x))Ae^{i\phi} \quad (4)$$

Partition (4) as a real equivalence

$$\frac{d^2A}{dx^2} - A \left(\frac{d\phi}{dx} \right)^2 = -\frac{2m}{\hbar^2}(E - V(x))A \quad (5)$$

and an imaginary equivalence

$$2i \frac{dA}{dx} \frac{d\phi}{dx} + iA \frac{d^2\phi}{dx^2} = 0 \quad (6)$$

Divide equation (5) by A to obtain

$$\frac{1}{A} \frac{d^2A}{dx^2} - \left(\frac{d\phi}{dx} \right)^2 = -\frac{2m}{\hbar^2}(E - V(x))$$

Rewrite as

$$\left(\frac{d\phi}{dx} \right)^2 = \frac{2m}{\hbar^2}(E - V(x)) + \frac{1}{A} \frac{d^2A}{dx^2}$$

For the circumstance of

$$\left| \frac{2m}{\hbar^2}(E - V(x)) \right| \gg \left| \frac{1}{A} \frac{d^2A}{dx^2} \right|$$

we can use the approximation

$$\left(\frac{d\phi}{dx}\right)^2 \approx \frac{2m}{\hbar^2}(E - V(x))$$

Hence

$$\frac{d\phi}{dx} \approx \pm \sqrt{\frac{2m}{\hbar^2}(E - V(x))} \quad (7)$$

and

$$\phi(x) \approx \pm \int \sqrt{\frac{2m}{\hbar^2}(E - V(x))} dx \quad (8)$$

We can rewrite (6) as

$$\frac{i}{A} \frac{d}{dx} \left(A^2 \frac{d\phi}{dx} \right) = 0 \quad (9)$$

by noting that

$$\frac{i}{A} \frac{d}{dx} \left(A^2 \frac{d\phi}{dx} \right) = \frac{i}{A} \left(2A \frac{dA}{dx} \frac{d\phi}{dx} + A^2 \frac{d^2\phi}{dx^2} \right) = 2i \frac{dA}{dx} \frac{d\phi}{dx} + iA \frac{d^2\phi}{dx^2}$$

Multiply by sides of (9) by $-iA$ to obtain

$$\frac{d}{dx} \left(A^2 \frac{d\phi}{dx} \right) = 0$$

By antiderivative

$$A^2 \frac{d\phi}{dx} = C^2$$

where C^2 is an arbitrary constant. Hence

$$A(x) = C \left(\frac{d\phi}{dx} \right)^{-\frac{1}{2}} \quad (10)$$

Substitute (7) into (10) to obtain

$$A(x) \approx C \left(\pm \sqrt{\frac{2m}{\hbar^2}(E - V(x))} \right)^{-\frac{1}{2}} \quad (11)$$

Substitute (8) and (11) into (2) to obtain

$$\psi(x) \approx C \left(\pm \sqrt{\frac{2m}{\hbar^2}(E - V(x))} \right)^{-\frac{1}{2}} \exp \left(\pm i \int \sqrt{\frac{2m}{\hbar^2}(E - V(x))} dx \right)$$

Rewrite as

$$\psi(x) \approx C e^{i\theta} \left(\frac{2m}{\hbar^2} |E - V(x)| \right)^{-\frac{1}{4}} \exp \left(\pm i \int \sqrt{\frac{2m}{\hbar^2}(E - V(x))} dx \right) \quad (12)$$

where

$$\theta = \begin{cases} 0 & \text{for } \left(+\sqrt{\frac{2m}{\hbar^2}(E - V(x))} \right)^{-\frac{1}{2}} \text{ and } E > V(x) \\ -\frac{\pi}{4} & \text{for } \left(+\sqrt{\frac{2m}{\hbar^2}(E - V(x))} \right)^{-\frac{1}{2}} \text{ and } E < V(x) \\ -\frac{\pi}{2} & \text{for } \left(-\sqrt{\frac{2m}{\hbar^2}(E - V(x))} \right)^{-\frac{1}{2}} \text{ and } E > V(x) \\ -\frac{3\pi}{4} & \text{for } \left(-\sqrt{\frac{2m}{\hbar^2}(E - V(x))} \right)^{-\frac{1}{2}} \text{ and } E < V(x) \end{cases}$$

The constant $e^{i\theta}$ can be discarded because it cancels in (1) hence

$$\psi(x) \approx C \left(\frac{2m}{\hbar^2} |E - V(x)| \right)^{-\frac{1}{4}} \exp \left(\pm i \int \sqrt{\frac{2m}{\hbar^2}(E - V(x))} dx \right)$$