# Angular momentum 3

The cross product in $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ demands rectangular coordinates. Hence for a wavefunction $\psi$ in spherical coordinates, vectors $\mathbf{r}$ and $\mathbf{p}$ must be transformed in $\mathbf{L}\psi$.

Vector $\mathbf{r}$ transforms as

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \sin\theta \cos\phi \\ r \sin\theta \sin\phi \\ r \cos\theta \end{pmatrix}$$

To transform $\mathbf{p}$ we have by the chain rule

$$\frac{\partial}{\partial x} = \sin\theta \cos\phi \frac{\partial}{\partial r} + \frac{\cos\theta \cos\phi}{r} \frac{\partial}{\partial \theta} - \frac{\sin\phi}{r \sin\theta} \frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \sin\theta \sin\phi \frac{\partial}{\partial r} + \frac{\cos\theta \sin\phi}{r} \frac{\partial}{\partial \theta} + \frac{\cos\phi}{r \sin\theta} \frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial z} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta}$$

Hence

$$p_x = -i\hbar \frac{\partial}{\partial x} = -i\hbar \left( \sin\theta \cos\phi \frac{\partial}{\partial r} + \frac{\cos\theta \cos\phi}{r} \frac{\partial}{\partial \theta} - \frac{\sin\phi}{r \sin\theta} \frac{\partial}{\partial \phi} \right)$$

$$p_y = -i\hbar \frac{\partial}{\partial y} = -i\hbar \left( \sin\theta \sin\phi \frac{\partial}{\partial r} + \frac{\cos\theta \sin\phi}{r} \frac{\partial}{\partial \theta} + \frac{\cos\phi}{r \sin\theta} \frac{\partial}{\partial \phi} \right)$$

$$p_z = -i\hbar \frac{\partial}{\partial z} = -i\hbar \left( \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \right)$$

Using the transformed coordinates

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

we have in spherical coordinates

$$L_x = yp_z - zp_y = i\hbar \left( \sin\phi \frac{\partial}{\partial \theta} + \frac{\cos\theta \cos\phi}{\sin\theta} \frac{\partial}{\partial \phi} \right)$$

$$L_y = zp_x - xp_z = i\hbar \left( -\cos\phi \frac{\partial}{\partial \theta} + \frac{\cos\theta \sin\phi}{\sin\theta} \frac{\partial}{\partial \phi} \right)$$

$$L_z = xp_y - yp_x = -i\hbar \frac{\partial}{\partial \phi}$$

and

$$L^2 = L_x{}^2 + L_y{}^2 + L_z{}^2 = -\hbar^2 \left( \frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right)$$

Eigenmath script