$$\left(\hat{H}_0 + \epsilon \hat{V}\right)\left(\psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2\right) = \left(E_0 + \epsilon E_1 + \epsilon^2 E_2\right)\left(\psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2\right)$$

Let $\epsilon = 0$. Then

$$\hat{H}_0\psi_0 = E_0\psi_0$$

Let $\epsilon^1 = 1$ and $\epsilon^k = 0$ for $k > 1$. Then

$$\hat{H}_0(\psi_0 + \psi_1) + \hat{V}\psi_0 = E_0(\psi_0 + \psi_1) + E_1\psi_0$$

Since $\hat{H}_0\psi_0 = E_0\psi_0$ we can cancel $\hat{H}_0\psi_0$ with $E_0\psi_0$ and obtain

$$\hat{H}_0\psi_1 + \hat{V}\psi_0 = E_0\psi_1 + E_1\psi_0$$

We want to solve for E_1. Multiply both sides by $\psi_0^\dagger$ and integrate over all space.

$$\int_V \psi_0^\dagger \left(\hat{H}_0\psi_1 + \hat{V}\psi_0\right) dV = \int_V \psi_0^\dagger \left(E_0\psi_1 + E_1\psi_0\right) dV$$

Then by the distributive property

$$\int_V \psi_0^\dagger \hat{H}_0 \psi_1 \, dV + \int_V \psi_0^\dagger \hat{V}\psi_0 \, dV = E_0 \int_V \psi_0^\dagger \psi_1 \, dV + E_1 \int_V \psi_0^\dagger \psi_0 \, dV$$

Because $\hat{H}_0$ is Hermitian we have

$$\psi_0^\dagger \hat{H}_0 \psi_1 = (\hat{H}_0\psi_0)^\dagger \psi_1 = E_0 \psi_0^\dagger \psi_1$$

Hence the $\hat{H}_0$ and E_0 integrals cancel. We now have

$$\int_V \psi_0^\dagger \hat{V}\psi_0 \, dV = E_1 \int_V \psi_0^\dagger \psi_0 \, dV$$

Rearrange to solve for E_1.

$$E_1 = \frac{\int_V \psi_0^\dagger \hat{V}\psi_0 \, dV}{\int_V \psi_0^\dagger \psi_0 \, dV}$$