

Perturbation example

Let

$$\begin{aligned}\dot{c}_a &= -\frac{i}{\hbar} c_b H_{ab}(t) e^{-i\omega t} \\ \dot{c}_b &= -\frac{i}{\hbar} c_a H_{ba}(t) e^{i\omega t}\end{aligned}$$

Find second order solutions $c_a^{(2)}(t)$ and $c_b^{(2)}(t)$ for $c_a(0) = a$ and $c_b(0) = b$. (See Griffiths and Schroeter problem 11.6.)

First order:

$$\begin{aligned}\dot{c}_a^{(1)} &= -\frac{i}{\hbar} c_b H_{ab}(t) e^{-i\omega t} \Big|_{c_b=b} = -\frac{ib}{\hbar} H_{ab}(t) e^{-i\omega t} \\ \dot{c}_b^{(1)} &= -\frac{i}{\hbar} c_a H_{ba}(t) e^{i\omega t} \Big|_{c_a=a} = -\frac{ia}{\hbar} H_{ba}(t) e^{i\omega t}\end{aligned}\tag{1}$$

Hence

$$\begin{aligned}c_a^{(1)}(t) &= a + \int_0^t \dot{c}_a^{(1)}(t') dt' = a - \frac{ib}{\hbar} \int_0^t H_{ab}(t') e^{-i\omega t'} dt' \\ c_b^{(1)}(t) &= b + \int_0^t \dot{c}_b^{(1)}(t') dt' = b - \frac{ia}{\hbar} \int_0^t H_{ba}(t') e^{i\omega t'} dt'\end{aligned}\tag{2}$$

Second order:

$$\begin{aligned}\dot{c}_a^{(2)} &= -\frac{i}{\hbar} c_b H_{ab}(t) e^{-i\omega t} \Big|_{c_b=c_b^{(1)}} = -\frac{ib}{\hbar} H_{ab}(t) e^{-i\omega t} - \frac{a}{\hbar^2} H_{ab}(t) e^{-i\omega t} \int_0^t H_{ba}(t') e^{i\omega t'} dt' \\ \dot{c}_b^{(2)} &= -\frac{i}{\hbar} c_a H_{ba}(t) e^{i\omega t} \Big|_{c_a=c_a^{(1)}} = -\frac{ia}{\hbar} H_{ba}(t) e^{i\omega t} - \frac{b}{\hbar^2} H_{ba}(t) e^{i\omega t} \int_0^t H_{ab}(t') e^{-i\omega t'} dt'\end{aligned}\tag{3}$$

Hence

$$\begin{aligned}c_a^{(2)}(t) &= a + \int_0^t \dot{c}_a^{(2)}(t') dt' \\ &= a - \frac{ib}{\hbar} \int_0^t H_{ab}(t') e^{-i\omega t'} dt' - \frac{a}{\hbar^2} \int_0^t H_{ab}(t') e^{-i\omega t'} \left[\int_0^{t'} H_{ba}(t'') e^{i\omega t''} dt'' \right] dt' \quad (4)\end{aligned}$$

and

$$\begin{aligned}c_b^{(2)}(t) &= b + \int_0^t \dot{c}_b^{(2)}(t') dt' \\ &= b - \frac{ia}{\hbar} \int_0^t H_{ba}(t') e^{i\omega t'} dt' - \frac{b}{\hbar^2} \int_0^t H_{ba}(t') e^{i\omega t'} \left[\int_0^{t'} H_{ab}(t'') e^{-i\omega t''} dt'' \right] dt' \quad (5)\end{aligned}$$