

Barrier potential

Let $V(x)$ be the following barrier potential.

$$V(x) = \begin{cases} 0 & x < -a \\ V_0 & -a \leq x \leq a \\ 0 & x > a \end{cases}$$

Find the transmission coefficient for this barrier potential.

Start with Schrodinger equations, one for each region of $V(x)$.

$$\begin{aligned} -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_1 &= E\psi_1, & x \leq -a \\ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_2 + V_0\psi_2 &= E\psi_2, & -a \leq x \leq a \\ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_3 &= E\psi_3, & x \geq a \end{aligned}$$

The solutions are

$$\begin{aligned} \psi_1(x) &= A \exp(ikx) + B \exp(-ikx) \\ \psi_2(x) &= C \exp(i\kappa x) + D \exp(-i\kappa x) \\ \psi_3(x) &= F \exp(ikx) \end{aligned}$$

where

$$k = \frac{\sqrt{2mE}}{\hbar}, \quad \kappa = \frac{\sqrt{2m(E - V_0)}}{\hbar}$$

Boundary conditions for ψ_1 and ψ_2 .

$$\psi_1(-a) = \psi_2(-a), \quad \left. \frac{d\psi_1}{dx} \right|_{x=-a} = \left. \frac{d\psi_2}{dx} \right|_{x=-a}$$

Boundary conditions for ψ_2 and ψ_3 .

$$\psi_2(a) = \psi_3(a), \quad \left. \frac{d\psi_2}{dx} \right|_{x=a} = \left. \frac{d\psi_3}{dx} \right|_{x=a}$$

From boundary conditions for ψ_2 and ψ_3 we have

$$C \exp(i\kappa a) + D \exp(-i\kappa a) = F \exp(ikx) \quad (1)$$

and

$$i\kappa C \exp(i\kappa a) - i\kappa D \exp(-i\kappa a) = ikF \exp(ikx) \quad (2)$$

Add $i\kappa$ times (1) to (2) to obtain

$$2i\kappa C \exp(i\kappa a) = (i\kappa + ik)F \exp(ika)$$

Hence

$$C = \frac{(\kappa + k)F \exp(ika - i\kappa a)}{2\kappa}$$

Add minus $i\kappa$ times (1) to (2) to obtain

$$-2i\kappa D \exp(-i\kappa a) = (-i\kappa + ik)F \exp(ika)$$

Hence

$$D = \frac{(\kappa - k)F \exp(ika + i\kappa a)}{2\kappa}$$

From boundary conditions for ψ_1 and ψ_2 we have

$$A \exp(-ika) + B \exp(ika) = C \exp(-i\kappa a) + D \exp(i\kappa a) \quad (3)$$

and

$$ikA \exp(-ika) - ikB \exp(ika) = i\kappa C \exp(-i\kappa a) - i\kappa D \exp(i\kappa a) \quad (4)$$

Add ik times (3) to (4) to obtain

$$2ikA \exp(-ika) = (ik + i\kappa)C \exp(-i\kappa a) + (ik - i\kappa)D \exp(i\kappa a)$$

Hence

$$A = \frac{(k + \kappa)C \exp(ika - i\kappa a)}{2k} + \frac{(k - \kappa)D \exp(ika + i\kappa a)}{2k}$$

Add minus ik times (3) to (4) to obtain

$$-2ikB \exp(ika) = (-ik + i\kappa)C \exp(-i\kappa a) + (-ik - i\kappa)D \exp(i\kappa a)$$

Hence

$$B = \frac{(k - \kappa)C \exp(-ika - i\kappa a)}{2k} + \frac{(k + \kappa)D \exp(-ika + i\kappa a)}{2k}$$

For transmission coefficient T we have

$$T^{-1} = \frac{A}{F} \left(\frac{A}{F} \right)^* = 1 + \frac{1}{8} \left(\frac{E}{V_0 - E} + \frac{V_0}{E} + 1 \right) \left[\cos \left(\frac{4a}{\hbar} \sqrt{2m(E - V_0)} \right) - 1 \right] \quad (5)$$

Equivalently

$$T^{-1} = 1 + \frac{1}{4} \left(\frac{E}{V_0 - E} + \frac{V_0}{E} + 1 \right) \sinh^2 \left(\frac{2ia}{\hbar} \sqrt{2m(E - V_0)} \right) \quad (6)$$

Cancel the imaginary unit.

$$T^{-1} = 1 + \frac{1}{4} \left(\frac{E}{V_0 - E} + \frac{V_0}{E} + 1 \right) \sinh^2 \left(\frac{2a}{\hbar} \sqrt{2m(V_0 - E)} \right)$$

Note that

$$\frac{E}{V_0 - E} + \frac{V_0}{E} + 1 = \frac{V_0^2}{E(V_0 - E)} \quad (7)$$