

This is the transition rate  $A_{nm}$  for a spontaneous emission process  $\psi_n \rightarrow \psi_m$ .

$$A_{nm} = \frac{e^2}{3\pi\epsilon_0\hbar c^3} \omega_{nm}^3 |\langle r_{nm} \rangle|^2$$

The transition frequency is

$$\omega_{nm} = \frac{1}{\hbar}(E_n - E_m)$$

The energy levels for hydrogen are

$$E_n = -\frac{\mu}{2n^2} \left( \frac{e^2}{4\pi\epsilon_0\hbar} \right)^2$$

where  $\mu$  is reduced electron mass.

The radial density is

$$|\langle r_{nm} \rangle|^2 = |\langle x_{nm} \rangle|^2 + |\langle y_{nm} \rangle|^2 + |\langle z_{nm} \rangle|^2$$

where

$$\begin{aligned} \langle x_{nm} \rangle &= \int \psi_m^* (r \sin \theta \cos \phi) \psi_n dV \\ \langle y_{nm} \rangle &= \int \psi_m^* (r \sin \theta \sin \phi) \psi_n dV \\ \langle z_{nm} \rangle &= \int \psi_m^* (r \cos \theta) \psi_n dV \end{aligned}$$

Let us compute  $A_{21}$  for a hydrogen atom. For  $n = 2$  there are four possible states.

$n$	$\ell$	$m$
2	1	1
2	1	-1
2	1	0
2	0	0

The following table shows the radial density for every possible transition.

	$\psi_{2,1,1} \rightarrow \psi_{1,0,0}$	$\psi_{2,1,-1} \rightarrow \psi_{1,0,0}$	$\psi_{2,1,0} \rightarrow \psi_{1,0,0}$	$\psi_{2,0,0} \rightarrow \psi_{1,0,0}$
$\langle x_{21} \rangle =$	$-\frac{128}{243} a_0$	$\frac{128}{243} a_0$	0	0
$\langle y_{21} \rangle =$	$-\frac{128}{243} i a_0$	$-\frac{128}{243} i a_0$	0	0
$\langle z_{21} \rangle =$	0	0	$\frac{128}{243} \sqrt{2} a_0$	0
$ \langle r_{21} \rangle ^2 =$	$\frac{32768}{59049} a_0^2$	$\frac{32768}{59049} a_0^2$	$\frac{32768}{59049} a_0^2$	0

Note that the transition rate of  $\psi_{2,0,0} \rightarrow \psi_{1,0,0}$  is zero. For the allowed transitions, the radial density  $|\langle r_{21} \rangle|^2$  is independent of  $\ell$  and  $m$ .

This is the Bohr radius for reduced electron mass  $\mu$ .

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{e^2\mu} = 5.29 \times 10^{-11} \text{ meter}$$

For the transition frequency we have

$$\omega_{21} = \frac{1}{\hbar}(E_2 - E_1) = \frac{3e^4\mu}{128\pi^2\varepsilon_0^2\hbar^3} = 1.55 \times 10^{16} \text{ second}^{-1}$$

Hence

$$A_{21} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3} \times \omega_{21}^3 \times \frac{32768}{59049} a_0^2 = 6.26 \times 10^8 \text{ second}^{-1}$$

It is interesting to work out  $A_{nm}$  symbolically and see how high the powers get.

$$A_{21} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3} \times \underbrace{\left(\frac{3e^4\mu}{128\pi^2\varepsilon_0^2\hbar^3}\right)^3}_{\omega_{21}^3} \times \frac{32768}{59049} \underbrace{\left(\frac{4\pi\varepsilon_0\hbar^2}{e^2\mu}\right)^2}_{|\langle r_{21} \rangle|^2} = \frac{e^{10}\mu}{26244\pi^5\varepsilon_0^5\hbar^6 c^3}$$

The parameters  $n = 2$  and  $m = 1$  contribute the following numerical factor to  $A_{21}$ .

$$\underbrace{\left(-\frac{1}{2^2} + \frac{1}{1^2}\right)^3}_{\text{from } (E_2 - E_1)^3} \times \underbrace{\frac{32768}{59049}}_{\text{from } |\langle r_{21} \rangle|^2} = \frac{512}{2187} = \frac{2^9}{3^7}$$

Multiplying out numerical factors yields the numerical factor shown above in  $A_{21}$ .

$$\frac{1}{3} \times \underbrace{\left(\frac{1}{32}\right)^3}_{\text{from } (E_n - E_m)^3} \times \underbrace{4^2}_{\text{from } a_0^2} \times \frac{512}{2187} = \frac{1}{26244} = \frac{1}{2^2 3^8}$$

Let us analyze the units involved in computing  $A_{nm}$ . For the coefficient of  $A_{nm}$  we have

$$\frac{e^2}{3\pi\varepsilon_0\hbar c^3} \propto \frac{\text{ampere}^2 \text{ second}^2}{\frac{e^2}{\varepsilon_0} \left(\frac{\text{kilogram meter}^2}{\text{second}}\right) \left(\frac{\text{meter}^3}{\text{second}^3}\right)} = \frac{\text{second}^2}{\text{meter}^2}$$

For the transition frequency we have

$$\omega_{21} = \frac{3e^4\mu}{128\pi^2\varepsilon_0^2\hbar^3} \propto \frac{\frac{(\text{ampere}^4 \text{ second}^4) \text{ kilogram}}{e^4 \mu}}{\frac{(\text{ampere}^4 \text{ second}^8)}{\varepsilon_0^2} \left(\frac{\text{kilogram}^3 \text{ meter}^6}{\text{second}^3}\right) \hbar^3} = \text{second}^{-1}$$

For the Bohr radius we have

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{e^2\mu} \propto \frac{\frac{(\text{ampere}^2 \text{ second}^4)}{\varepsilon_0} \left(\frac{\text{kilogram}^2 \text{ meter}^4}{\text{second}^2}\right) \hbar^2}{\frac{(\text{ampere}^2 \text{ second}^2) \text{ kilogram}}{e^2 \mu}} = \text{meter}$$

Hence

$$A_{nm} \propto \frac{\text{second}^2}{\text{meter}^2} \times \underbrace{\text{second}^{-3}}_{\omega_{nm}^3} \times \underbrace{\text{meter}^2}_{a_0^2} = \text{second}^{-1}$$