9-7. Show, for the vacuum state, the expectation value of $\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}}$ is $(\hbar/2kc)\delta_{\mathbf{k},\mathbf{q}}$ and that of $\bar{a}_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}}$ is $(\hbar/2kc)\delta_{-\mathbf{k},\mathbf{q}}$.

We will use the following table of integrals.

$$\int_{-\infty}^{\infty} \exp(-ax^2+b)\,dx = \sqrt{\frac{\pi}{a}}\exp(b) \tag{1}$$
$$\int_{-\infty}^{\infty} x\exp(-ax^2+b)\,dx = 0 \tag{2}$$
$$\int_{-\infty}^{\infty} x^2\exp(-ax^2+b)\,dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}\exp(b) \tag{3}$$

For simplicity of notation, let

$$A = \bar{a}^c_{1,\mathbf{k}} \qquad B = \bar{a}^s_{1,\mathbf{k}} \qquad C = \bar{a}^c_{1,\mathbf{q}} \qquad D = \bar{a}^s_{1,\mathbf{q}}$$

From problem 9-6 we have

$$\begin{aligned} \bar{a}_{1,\mathbf{k}} &= \frac{1}{\sqrt{2}}(A-iB) \\ \bar{a}_{1,\mathbf{q}} &= \frac{1}{\sqrt{2}}(C-iD) \end{aligned} \tag{4}$$

Adapted from equation (8.84),

$$\langle\Phi_0|f|\Phi_0\rangle = \int\limits_{-\infty}^{\infty}\cdots\int\limits_{-\infty}^{\infty}\Phi_0^* f\Phi_0\,dA\,dB\,dC\,dD$$

The following vacuum state is adapted from (9.43) and problem 9-6. Symbol q is a mode number, not an electric charge. Note that we *could* include other modes in addition to k and q. However, integrals over unused modes are cancelled by the normalization constant.

$$\Phi_0 = \exp\left(-\frac{kc}{4\hbar}A^2 - \frac{kc}{4\hbar}B^2 - \frac{qc}{4\hbar}C^2 - \frac{qc}{4\hbar}D^2\right)$$

It follows that

$$\Phi_0^*\Phi_0 = \exp\left(-\frac{kc}{2\hbar}A^2 - \frac{kc}{2\hbar}B^2 - \frac{qc}{2\hbar}C^2 - \frac{qc}{2\hbar}D^2\right)$$

Compute the normalization constant K.

$$K = \langle \Phi_0 | 1 | \Phi_0 \rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_0^* \Phi_0 \, dA \, dB \, dC \, dD$$

By integral (1) for each factor in the measure (see problem 8-5)

$$K = \left(\frac{2\pi\hbar}{kc} \right)^{1/2} \left(\frac{2\pi\hbar}{kc} \right)^{1/2} \left(\frac{2\pi\hbar}{qc} \right)^{1/2} \left(\frac{2\pi\hbar}{qc} \right)^{1/2}$$

Compute the expectation of $\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}}$. From (4) we have

$$\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} = \frac{A^2 + B^2}{2}$$

Hence

$$\langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle = \frac{1}{K} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_0^* \Phi_0 \frac{A^2 + B^2}{2} \, dA \, dB \, dC \, dD$$

Rewrite as

$$\begin{aligned} \langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle = \frac{1}{2K} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_0^* \Phi_0 A^2 \, dA \, dB \, dC \, dD \\ + \frac{1}{2K} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_0^* \Phi_0 B^2 \, dA \, dB \, dC \, dD \end{aligned}$$

By integrals (1) and (3) we have

$$\langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle = \frac{1}{K} \frac{\hbar}{kc} \left(\frac{2\pi\hbar}{kc} \right)^{1/2} \left(\frac{2\pi\hbar}{kc} \right)^{1/2} \left(\frac{2\pi\hbar}{qc} \right)^{1/2} \left(\frac{2\pi\hbar}{qc} \right)^{1/2}$$

Hence

$$\langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle = \frac{\hbar}{kc} \tag{5}$$

Compute the expectation of $\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{q}}$.

$$\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{q}} = \frac{AC + BD - iAD + iBC}{2}$$

Hence

$$\langle\Phi_0|\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}}|\Phi_0\rangle = \frac{1}{K}\int\limits_{-\infty}^{\infty}\cdots\int\limits_{-\infty}^{\infty}\Phi_0^*\Phi_0\frac{AC+BD-iAD+iBC}{2}\,dA\,dB\,dC\,dD$$

By integral (2) all terms are zero, hence

$$\langle\Phi_0|\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}}|\Phi_0\rangle = 0 \tag{6}$$

Combine (5) and (6) to obtain

$$\langle\Phi_0|\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}}|\Phi_0\rangle = \frac{\hbar}{kc}\delta_{\mathbf{k},\mathbf{q}}$$

Note that by equation (8.77)

$$\bar{a}^*_{1,\mathbf{k}} = \bar{a}_{1,-\mathbf{k}}$$

Hence

$$\langle\Phi_0|\bar{a}_{1,-\mathbf{k}}\bar{a}_{1,\mathbf{q}}|\Phi_0\rangle = \frac{\hbar}{kc}\delta_{\mathbf{k},\mathbf{q}}$$

(9-7 cont'd) Develop a formula for the expectation of $(\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{k}})^r$ for integral r and explain thereby how the expectation of such quantities as $(\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{k}})^r(\bar{a}^*_{1,\mathbf{q}}\bar{a}_{1,\mathbf{q}})^s$ can be got for $\mathbf{q}\neq\mathbf{k}$.

By the binomial theorem

$$\left(\frac{A^2+B^2}{2}\right)^r = \frac{1}{2^r}\sum_{j=0}^{r}\binom{r}{j}A^{2j}B^{2(r-j)} \tag{7}$$

To compute the expectation of (7) we need the following integral.

$$\begin{aligned}\int_{-\infty}^{\infty}x^{2n}\exp(-ax^2+b) &= \frac{1\cdot3\cdot5\cdots(2n-1)}{2^na^n}\sqrt{\frac{\pi}{a}}\exp(b)\\ &= (2n-1)!!\,\frac{1}{2^na^n}\sqrt{\frac{\pi}{a}}\exp(b)\end{aligned} \tag{8}$$

Given (8), define the following function F. (The $\sqrt{\pi/a}$ factor is left out because it gets cancelled by the normalization constant K.)

$$F(n) = (2n-1)!!\left(\frac{\hbar}{kc}\right)^n$$

Note that

$$F(j)F(r-j) = (2j-1)!!\,(2r-2j-1)!!\left(\frac{\hbar}{kc}\right)^j\left(\frac{\hbar}{kc}\right)^{r-j}$$

It turns out that

$$\frac{1}{2^r}\sum_{j=0}^{r}\binom{r}{j}(2j-1)!!\,(2r-2j-1)!! = r!$$

Hence

$$\langle\Phi_0^*|(\bar{a}_{1,\mathbf{k}}^*\bar{a}_{1,\mathbf{k}})^r|\Phi_0\rangle = r!\left(\frac{\hbar}{kc}\right)^r$$

Regarding the $\mathbf{q}\neq\mathbf{k}$ part of the problem, we have

$$\left(\frac{A^2+B^2}{2}\right)^r\left(\frac{C^2+D^2}{2}\right)^s = \left(\frac{1}{2^r}\sum_{j=0}^{r}\binom{r}{j}A^{2j}B^{2(r-j)}\right)\left(\frac{1}{2^s}\sum_{k=0}^{s}\binom{s}{k}C^{2k}D^{2(r-k)}\right)$$

Hence

$$\langle\Phi_0^*|(\bar{a}_{1,\mathbf{k}}^*\bar{a}_{1,\mathbf{k}})^r(\bar{a}_{1,\mathbf{q}}^*\bar{a}_{1,\mathbf{q}})^s|\Phi_0\rangle = r!\left(\frac{\hbar}{kc}\right)^r s!\left(\frac{\hbar}{qc}\right)^s$$

(9-7 cont'd) Show that the expectation of $(\bar{a}_{1,\mathbf{k}})^2$ or $(\bar{a}_{1,\mathbf{k}}^*)^2$ vanishes.

We have

$$(\bar{a}_{1,\mathbf{k}})^2 = \frac{A^2-B^2}{2} - iAB \qquad (\bar{a}_{1,\mathbf{k}}^*)^2 = \frac{A^2-B^2}{2} + iAB$$

The integrals of A^2 and $-B^2$ cancel each other. The integral of AB vanishes by integral (2).

(9-7 cont'd) Show that the expectation of the product of any odd number of $\bar{a}$'s is zero and that you can compute the expectation value of any product of $\bar{a}$'s or $\bar{a}^*$'s for the vacuum state.

Isn't the expectation of *any* number of $\bar{a}$'s zero? It is shown above that the expectation of $(\bar{a}_{1,\mathbf{k}})^2$ vanishes.

FIXME: The result $\hbar/kc$ is wrong by a factor of 2 everywhere it appears. Should be $\hbar/2kc$ according to the problem statement.