(8.2) For the Hamiltonian

$$\hat{H} = \sum_k E_k \hat{a}_k^\dagger \hat{a}_k \tag{8.21}$$

use the Heisenberg equation of motion to show that the time dependence of the operator $\hat{a}_k^\dagger$ is given by

$$\hat{a}_k^\dagger(t) = \hat{a}_k^\dagger(0) \exp\left(\frac{iE_k t}{\hbar}\right) \tag{8.22}$$

and find a similar expression for $\hat{a}_k(t)$.

The Heisenberg equation of motion is

$$\frac{d}{dt}\hat{a}^\dagger(t) = -\frac{i}{\hbar}\left[\hat{a}^\dagger(t), \hat{H}\right] \tag{8.15}$$

Substitute (8.22) into the right-hand side of (8.15) and call it I.

$$I = -\frac{i}{\hbar} \exp\left(\frac{iE_k t}{\hbar}\right)\left(\hat{a}_k^\dagger(0)\hat{H} - \hat{H}\hat{a}_k^\dagger(0)\right) \tag{1}$$

Rewrite the Hamiltonian in equation (8.21) as

$$\hat{H} = \sum_j E_j \hat{n}_j \tag{2}$$

Substitute (2) into (1). Note that $\hat{H}$ commutes with $\hat{a}_k^\dagger(0)$ for $j \neq k$.

$$I = -\frac{i}{\hbar} E_k \exp\left(\frac{iE_k t}{\hbar}\right)\left(\hat{a}_k^\dagger(0)\hat{n}_k - \hat{n}_k\hat{a}_k^\dagger(0)\right)$$

Note that

$$\hat{n}\hat{a}^\dagger|n\rangle = (n+1)|n+1\rangle = (n+1)\hat{a}^\dagger|n\rangle$$

and

$$\hat{a}\hat{n}|n\rangle = n|n+1\rangle = n\hat{a}^\dagger|n\rangle$$

Hence

$$
\begin{aligned}
I &= -\frac{i}{\hbar} E_k \exp\left(\frac{iE_k t}{\hbar}\right)\left(\hat{n}_k \hat{a}_k^\dagger(0) - (\hat{n}_k + 1)\hat{a}_k^\dagger(0)\right) \\
&= \frac{i}{\hbar} E_k \exp\left(\frac{iE_k t}{\hbar}\right)\hat{a}_k^\dagger(0) \\
&= \frac{i}{\hbar} E_k \hat{a}_k^\dagger(t) \qquad (3)
\end{aligned}
$$

Substitute (8.22) into the left-hand side of (8.15) to obtain

$$
\frac{d}{dt}\hat{a}^\dagger(t) = \frac{i}{\hbar} E_k \hat{a}_k^\dagger(t) \qquad (4)
$$

By the equivalence of (3) and (4) we have

$$
\frac{d}{dt}\hat{a}^\dagger(t) = -\frac{i}{\hbar}\left[\hat{a}^\dagger(t), \hat{H}\right]
$$

for the $\hat{a}^\dagger(t)$ given in (8.22).

Because the number operator is time-independent we must have

$$
\hat{a}_k(t) = \hat{a}_k(0)\exp\left(-\frac{iE_k t}{\hbar}\right)
$$

so that

$$
\hat{a}_k^\dagger(t)\hat{a}_k(t) = \hat{a}_k^\dagger(0)\hat{a}_k(0)
$$