

Tunneling probability

Consider the following potential energy function.

$$V(x) = \begin{cases} 0, & x < 0 \\ V_0, & 0 \leq x \leq L \\ 0, & x > L \end{cases}$$

Suppose a particle with mass m and energy $E < V_0$ is traveling from left to right along the x axis. The particle has a Schrodinger equation for each region of $V(x)$.

$$\begin{aligned} \frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_1 &= E\psi_1, & x < 0 \\ \frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_2 + V_0\psi_2 &= E\psi_2, & 0 \leq x \leq L \\ \frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_3 &= E\psi_3, & x > L \end{aligned}$$

Let ψ_1 and ψ_3 have the most general free-particle solutions.

$$\begin{aligned} \psi_1(x) &= A \exp\left(i\sqrt{\frac{2mE}{\hbar^2}}x\right) + B \exp\left(-i\sqrt{\frac{2mE}{\hbar^2}}x\right) \\ \psi_3(x) &= F \exp\left(i\sqrt{\frac{2mE}{\hbar^2}}x\right) + G \exp\left(-i\sqrt{\frac{2mE}{\hbar^2}}x\right) \end{aligned}$$

Use the WKB approximation to solve for ψ_2 .

$$\psi_2(x) \approx C \exp\left(i \int \sqrt{\frac{2m(E - V_0)}{\hbar^2}} dx\right) + D \exp\left(-i \int \sqrt{\frac{2m(E - V_0)}{\hbar^2}} dx\right)$$

Cancel i by swapping E and V_0 .

$$\psi_2(x) \approx C \exp\left(\int \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} dx\right) + D \exp\left(-\int \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} dx\right)$$

Substitute x for $\int dx$.

$$\psi_2(x) \approx C \exp\left(\sqrt{\frac{2m(V_0 - E)}{\hbar^2}}x\right) + D \exp\left(-\sqrt{\frac{2m(V_0 - E)}{\hbar^2}}x\right)$$

To simplify the formulas let

$$k = \sqrt{\frac{2mE}{\hbar^2}}, \quad \beta = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

and write

$$\begin{aligned}\psi_1(x) &= A \exp(ikx) + B \exp(-ikx) \\ \psi_2(x) &= C \exp(\beta x) + D \exp(-\beta x) \\ \psi_3(x) &= F \exp(ikx) + G \exp(-ikx)\end{aligned}$$

Exponentials of $-i$ represent particles moving from right to left. The B exponential represents a particle reflected from the boundary at $x = 0$. There is no particle moving right to left at $x > L$ hence $G = 0$.

Let us now solve for the coefficients using boundary conditions. Four boundary conditions are needed to ensure continuity at $x = 0$ and $x = L$.

$$\begin{aligned}\psi_1(0) &= \psi_2(0) \\ \psi'_1(0) &= \psi'_2(0) \\ \psi_2(L) &= \psi_3(L) \\ \psi'_2(L) &= \psi'_3(L)\end{aligned}$$

From the boundary condition $\psi_2(L) = \psi_3(L)$ we have

$$C \exp(\beta L) + D \exp(-\beta L) = F \exp(ikL) \quad (1)$$

From the boundary condition $\psi'_2(L) = \psi'_3(L)$ we have

$$\beta C \exp(\beta L) - \beta D \exp(-\beta L) = ikF \exp(ikL) \quad (2)$$

Add β times (1) to (2) to obtain

$$2\beta C \exp(\beta L) = (\beta + ik)F \exp(ikL)$$

Hence

$$C = \frac{(\beta + ik)F \exp(ikL - \beta L)}{2\beta} \quad (3)$$

Add minus β times (1) to (2) to obtain

$$-2\beta D \exp(-\beta L) = (-\beta + ik)F \exp(ikL)$$

Hence

$$D = \frac{(\beta - ik)F \exp(ikL + \beta L)}{2\beta} \quad (4)$$

From the boundary condition $\psi_1(0) = \psi_2(0)$ we have

$$A + B = C + D \quad (5)$$

From the boundary condition $\psi'_1(0) = \psi'_2(0)$ we have

$$ik(A - B) = \beta(C - D) \quad (6)$$

Add ik times (5) to (6) to obtain

$$2ikA = \beta(C - D) + ik(C - D)$$

Hence

$$A = \frac{\beta(C - D)}{2ik} + \frac{C + D}{2}$$

Substitute (3) and (4) for C and D to obtain

$$A = F \exp(ikL) (\cosh(\beta L) + i\gamma \sinh(\beta L)) \quad (7)$$

where

$$\gamma = \frac{1}{2} \left(\frac{\beta}{k} - \frac{k}{\beta} \right)$$

The tunneling probability T is the magnitude of the transmitted wave divided by the magnitude of the inbound wave.

$$T = \frac{|F|^2}{|A|^2} = \left| \frac{1}{\exp(ikL) (\cosh(\beta L) + i\gamma \sinh(\beta L))} \right|^2$$

Hence

$$T = \frac{1}{\cosh^2(\beta L) + \gamma^2 \sinh^2(\beta L)} \quad (8)$$

The result can also be written as

$$T = \left(1 + \frac{V_0^2 \sinh^2(\beta L)}{4E(V_0 - E)} \right)^{-1} \quad (9)$$

If T is small we can use the approximation

$$T \approx \left(\frac{V_0^2 \frac{1}{4} \exp(2\beta L)}{4E(V_0 - E)} \right)^{-1} = 16 \frac{E(V_0 - E)}{V_0^2} \exp(-2\beta L)$$

(Ref. “Quantum Tunneling of Particles through Potential Barriers” at phys.libretexts.org)