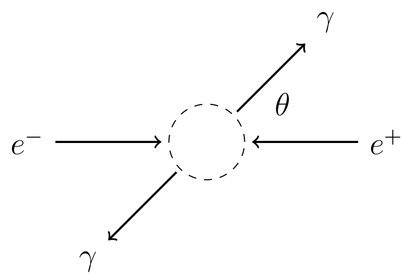
Annihilation

Annihilation is the process $e^- + e^+ \to \gamma + \gamma$.



The following center-of-mass momentum vectors have $E = \sqrt{p^2 + m^2}$.

$$p_1 = \underset{e^- \longrightarrow}{\begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}} \qquad p_2 = \underset{\longleftarrow e^+}{\begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix}} \qquad p_3 = \underset{\gamma \nearrow}{\begin{pmatrix} E \\ E\sin\theta\cos\phi \\ E\sin\theta\sin\phi \\ E\cos\theta \end{pmatrix}} \qquad p_4 = \underset{\swarrow \gamma}{\begin{pmatrix} E \\ -E\sin\theta\cos\phi \\ -E\sin\theta\sin\phi \\ -E\cos\theta \end{pmatrix}}$$

Spinors for p_1.

$$u_{11} = \frac{1}{\sqrt{E+m}} \underset{\text{spin up}}{\begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix}} \qquad u_{12} = \frac{1}{\sqrt{E+m}} \underset{\text{spin down}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix}}$$

Spinors for p_2.

$$v_{21} = \frac{1}{\sqrt{E+m}} \underset{\text{spin up}}{\begin{pmatrix} -p \\ 0 \\ E+m \\ 0 \end{pmatrix}} \qquad v_{22} = \frac{1}{\sqrt{E+m}} \underset{\text{spin down}}{\begin{pmatrix} 0 \\ p \\ 0 \\ E+m \end{pmatrix}}$$

The scattering amplitude ${\mathcal{M}_{ab}}^{\mu\nu}$ for spin ab and polarization $\mu\nu$ is

$${\mathcal{M}_{ab}}^{\mu\nu} = {\mathcal{M}_{1ab}}^{\mu\nu} + {\mathcal{M}_{2ab}}^{\nu\mu}$$

where

$${\mathcal{M}_{1ab}}^{\mu\nu} = \frac{\bar{v}_{2b}(-ie\gamma^\mu)(\not{q}_1 + m)(-ie\gamma^\nu)u_{1a}}{t - m^2}$$

$${\mathcal{M}_{2ab}}^{\nu\mu} = \frac{\bar{v}_{2b}(-ie\gamma^\nu)(\not{q}_2 + m)(-ie\gamma^\mu)u_{1a}}{u - m^2}$$

Matrices $\not{q}_1$ and $\not{q}_2$ represent momentum transfer.

$$\not{q}_1 = (p_1 - p_3)^\alpha g_{\alpha\beta}\gamma^\beta$$
$$\not{q}_2 = (p_1 - p_4)^\alpha g_{\alpha\beta}\gamma^\beta$$

Scalars t and u are Mandelstam variables.

$$t = (p_1 - p_3)^2$$
$$u = (p_1 - p_4)^2$$

In component form

$$\mathcal{M}_{1ab}{}^{\mu\nu} = \frac{(\bar{v}_{2b})_\alpha(-ie\gamma^{\mu\alpha}{}_\beta)(\not{q}_1 + m)^\beta{}_\rho(-ie\gamma^{\nu\rho}{}_\sigma)(u_{1a})^\sigma}{t - m^2}$$
$$\mathcal{M}_{2ab}{}^{\nu\mu} = \frac{(\bar{v}_{2b})_\alpha(-ie\gamma^{\nu\alpha}{}_\beta)(\not{q}_2 + m)^\beta{}_\rho(-ie\gamma^{\mu\rho}{}_\sigma)(u_{1a})^\sigma}{u - m^2}$$

Expected probability density $\langle|\mathcal{M}|^2\rangle$ is the sum over squared amplitudes divided by the number of inbound states.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{ab}\sum_{\mu\nu}\left|\mathcal{M}_{ab}{}^{\mu\nu}\right|^2$$

Summing over $\mu\nu$ requires $g_{\mu\nu}$ to lower indices.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{ab}\mathcal{M}_{ab}{}^{\mu\nu}\left(g_{\mu\alpha}\mathcal{M}_{ab}{}^{\alpha\beta}g_{\beta\nu}\right)^*$$

Expand the summand and label the terms. By positivity $\boxed{2} = \boxed{3}$.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{ab}\Big[\underbrace{\mathcal{M}_{1ab}{}^{\mu\nu}\left(g_{\mu\alpha}\mathcal{M}_{1ab}{}^{\alpha\beta}g_{\beta\nu}\right)^*}_{\boxed{1}} + \underbrace{\mathcal{M}_{1ab}{}^{\mu\nu}\left(g_{\nu\alpha}\mathcal{M}_{2ab}{}^{\alpha\beta}g_{\beta\mu}\right)^*}_{\boxed{2}}$$
$$+ \underbrace{\mathcal{M}_{2ab}{}^{\nu\mu}\left(g_{\mu\alpha}\mathcal{M}_{1ab}{}^{\alpha\beta}g_{\beta\nu}\right)^*}_{\boxed{3}} + \underbrace{\mathcal{M}_{2ab}{}^{\nu\mu}\left(g_{\nu\alpha}\mathcal{M}_{2ab}{}^{\alpha\beta}g_{\beta\mu}\right)^*}_{\boxed{4}}\Big]$$

The following Casimir trick uses matrix arithmetic to sum over spin and polarization states.

$$\sum_{ab}\boxed{1} = \frac{e^4}{(t - m^2)^2}\operatorname{Tr}\left[(\not{p}_1 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\right]$$
$$\sum_{ab}\boxed{2} = \frac{e^4}{(t - m^2)(u - m^2)}\operatorname{Tr}\left[(\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\right]$$
$$\sum_{ab}\boxed{4} = \frac{e^4}{(u - m^2)^2}\operatorname{Tr}\left[(\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_2 + m)\gamma_\mu\right]$$

Probability density $\langle|\mathcal{M}|^2\rangle$ can be reformulated as

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left[\frac{f_{11}}{(t - m^2)^2} + \frac{2f_{12}}{(t - m^2)(u - m^2)} + \frac{f_{22}}{(u - m^2)^2}\right]$$

with Casimir trick terms

$$
\begin{aligned}
f_{11} &= \mathrm{Tr}\left[(\not{p}_1 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\right] \\
f_{12} &= \mathrm{Tr}\left[(\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\right] \\
f_{22} &= \mathrm{Tr}\left[(\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_2 + m)\gamma_\mu\right]
\end{aligned}
$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$
\begin{aligned}
f_{11} &= 32(p_1 \cdot p_3)(p_1 \cdot p_4) + 32(p_1 \cdot p_3)m^2 - 32m^4 \\
f_{12} &= 16(p_1 \cdot p_2)m^2 - 16m^4 \\
f_{22} &= 32(p_1 \cdot p_3)(p_1 \cdot p_4) + 32(p_1 \cdot p_4)m^2 - 32m^4
\end{aligned}
$$

In Mandelstam variables

$$
\begin{aligned}
f_{11} &= 8tu - 24tm^2 - 8um^2 - 8m^4 \\
f_{12} &= 8sm^2 - 32m^4 \\
f_{22} &= 8tu - 8tm^2 - 24um^2 - 8m^4
\end{aligned}
$$

For $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$
\begin{aligned}
f_{11} &= 8tu \\
f_{12} &= 0 \\
f_{22} &= 8tu
\end{aligned}
$$

Hence

$$
\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{e^4}{4}\left(\frac{f_{11}}{(t-m^2)^2} + \frac{2f_{12}}{(t-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2}\right) \\
&= \frac{e^4}{4}\left(\frac{8tu}{t^2} + \frac{8tu}{u^2}\right) \\
&= 2e^4\left(\frac{u}{t} + \frac{t}{u}\right)
\end{aligned}
$$

For $m = 0$ the Mandelstam variables are

$$
\begin{aligned}
t &= -2E^2(1 - \cos\theta) \\
u &= -2E^2(1 + \cos\theta)
\end{aligned}
$$

Hence

$$
\langle|\mathcal{M}|^2\rangle = 2e^4\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)
$$

Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{4(4\pi\varepsilon_0)^2 s}$$

where

$$s = (p_1 + p_2)^2 = 4E^2$$

For high energy experiments we have

$$\langle|\mathcal{M}|^2\rangle = 2e^4 \left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\varepsilon_0)^2 s} \left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)$$

Noting that

$$e^2 = 4\pi\varepsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s} \left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)$$

Noting that

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

we also have

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{2s} \left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right) \sin\theta\, d\theta\, d\phi$$

Let $S(\theta_1, \theta_2)$ be the following integral of $d\sigma$.

$$S(\theta_1, \theta_2) = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{\pi\alpha^2(\hbar c)^2}{s} [I(\theta_2) - I(\theta_1)]$$

where

$$I(\theta) = 2\cos\theta + 2\log(1-\cos\theta) - 2\log(1+\cos\theta)$$

The cumulative distribution function is

$$F(\theta) = \frac{S(a,\theta)}{S(a,\pi-a)} = \frac{I(\theta) - I(a)}{I(\pi-a) - I(a)}, \quad a \le \theta \le \pi - a$$

Angular support is reduced by an arbitrary angle $a > 0$ because $I(0)$ and $I(\pi)$ are undefined.

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 < \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi-a) - I(a)} \left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right) \sin\theta$$