(13.1) (a) Show that the conserved charge in eqn 13.16 may be written

$$\hat{\mathbf{Q}}_{N_c} = \int d^3p\, \hat{\mathbf{A}}^\dagger_{\mathbf{p}} \mathbf{J} \hat{\mathbf{A}}_{\mathbf{p}} \tag{13.41}$$

where $\hat{\mathbf{A}}_{\mathbf{p}} = (\hat{a}_{1\mathbf{p}}, \hat{a}_{2\mathbf{p}}, \hat{a}_{3\mathbf{p}})$ and $\mathbf{J} = (J_x, J_y, J_z)$ are the spin-1 angular momentum matrices from Chapter 9.

(b) Use the transformations from Exercise 3.3 to find the form of the angular momentum matrices appropriate to express the charge as $\hat{\mathbf{Q}}_{N_c} = \int d^3p\, \hat{\mathbf{B}}^\dagger_{\mathbf{p}} \mathbf{J} \hat{\mathbf{B}}_{\mathbf{p}}$ where $\hat{\mathbf{B}}_{\mathbf{p}} = (\hat{b}_{1\mathbf{p}}, \hat{b}_{0\mathbf{p}}, \hat{b}_{-1\mathbf{p}})$.

(a) Here is equation (13.16).

$$\mathbf{Q}_{N_c} = \int d^3x\, (\mathbf{\Phi} \times \partial_0 \mathbf{\Phi}) \quad \text{and} \quad \hat{Q}^a_{N_c} = -i \int d^3p\, \varepsilon^{abc} \hat{a}^\dagger_{b\mathbf{p}} \hat{a}_{c\mathbf{p}} \tag{13.16}$$

Recall that ε^{abc} is the Levi-Civita symbol

$$\varepsilon^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \varepsilon^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \varepsilon^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Scalars commute with operators hence ε^{abc} and $\hat{a}^\dagger_{b\mathbf{p}}$ can be interchanged.

$$\hat{Q}^a_{N_c} = -i \int d^3p\, \hat{a}^\dagger_{b\mathbf{p}} \varepsilon^{abc} \hat{a}_{c\mathbf{p}}$$

The sum over b and sum over c are inner products hence we can write

$$\hat{Q}^a_{N_c} = -i \int d^3p\, \hat{\mathbf{A}}^\dagger_{\mathbf{p}} \varepsilon^a \hat{\mathbf{A}}_{\mathbf{p}}$$

Let

$$J_x = -i\varepsilon^1, \quad J_y = -i\varepsilon^2, \quad J_z = -i\varepsilon^3$$

Then

$$\hat{\mathbf{Q}}_{N_c} = \begin{pmatrix} Q^1_{N_c} \\ Q^2_{N_c} \\ Q^3_{N_c} \end{pmatrix} = \int d^3p\, \hat{\mathbf{A}}^\dagger_{\mathbf{p}} \begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} \hat{\mathbf{A}}_{\mathbf{p}} = \int d^3p\, \hat{\mathbf{A}}^\dagger_{\mathbf{p}} \mathbf{J} \hat{\mathbf{A}}_{\mathbf{p}}$$

(b) From Exercise 3.3 we have

$$\hat{b}_{1\mathbf{p}} = \frac{1}{\sqrt{2}}\big(-\hat{a}_{1\mathbf{p}} + i\hat{a}_{2\mathbf{p}}\big)$$
$$\hat{b}_{0\mathbf{p}} = \hat{a}_{3\mathbf{p}}$$
$$\hat{b}_{-1\mathbf{p}} = \frac{1}{\sqrt{2}}\big(\hat{a}_{1\mathbf{p}} + i\hat{a}_{2\mathbf{p}}\big)$$

We need a unitary matrix U such that

$$\hat{\mathbf{B}}_{\mathbf{p}} = \begin{pmatrix} \frac{1}{\sqrt{2}}\big(-\hat{a}_{1\mathbf{p}} + i\hat{a}_{2\mathbf{p}}\big) \\ \hat{a}_{3\mathbf{p}} \\ \frac{1}{\sqrt{2}}\big(\hat{a}_{1\mathbf{p}} + i\hat{a}_{2\mathbf{p}}\big) \end{pmatrix} = U \begin{pmatrix} \hat{a}_{1\mathbf{p}} \\ \hat{a}_{2\mathbf{p}} \\ \hat{a}_{3\mathbf{p}} \end{pmatrix} = U\hat{\mathbf{A}}_{\mathbf{p}}$$

Hence

$$U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

It follows that

$$\hat{\mathbf{B}}_{\mathbf{p}}^{\dagger}\mathbf{J}'\hat{\mathbf{B}}_{\mathbf{p}} = \big(\hat{\mathbf{A}}_{\mathbf{p}}^{\dagger}U^{\dagger}\big)\mathbf{J}'\big(U\hat{\mathbf{A}}_{\mathbf{p}}\big) = \hat{\mathbf{A}}_{\mathbf{p}}^{\dagger}\big(U^{\dagger}\mathbf{J}'U\big)\hat{\mathbf{A}}_{\mathbf{p}}$$

By (13.41) and hypothesis in (b) we have

$$U^{\dagger}\mathbf{J}'U = \mathbf{J}$$

Then from $U^{-1} = U^{\dagger}$ we have

$$\mathbf{J}' = U\mathbf{J}U^{\dagger}$$

Hence the components of $\mathbf{J}'$ are

$$J'_x = UJ_xU^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$J'_y = UJ_yU^{\dagger} = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$J'_z = UJ_zU^{\dagger} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$