9-4. In section 2-1 we discussed the mechanisms for obtaining the mechanical equations of motion from the form of the action S by obtaining the extremum S_{cl} under the conditions $\delta S = 0$ for variations of the coordinates, $\delta \mathbf{x}$. Show how Maxwell's equations can be derived from the action S defined in equation (9.23) by requiring δS for first-order variations of $\mathbf{A}$ and ϕ.

$$S = S_1 + S_2 + S_3 \tag{9.23}$$

Since S_1 does not depend on $\mathbf{A}$ or ϕ, we only need S_2 and S_3.

$$S_2 = -\sum_i e_i \int \left(\phi(\mathbf{x}_i(t), t) - \frac{1}{c}\dot{\mathbf{x}}_i(t) \cdot \mathbf{A}(\mathbf{x}_i(t), t) \right) dt \tag{9.25}$$

$$S_3 = \frac{1}{8\pi} \int\int \left(\left| -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \right|^2 - |\nabla \times \mathbf{A}|^2 \right) d^3\mathbf{r}\, dt \tag{9.26}$$

Consider equation (2.7), the classical Lagrangian equation of motion.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \tag{2.7}$$

Extend (2.7) to three dimensions.

$$\frac{d}{dt}\dot{\nabla} L = \nabla L \tag{1}$$

where

$$\dot{\nabla} = \mathbf{i}\frac{\partial}{\partial \dot{x}} + \mathbf{j}\frac{\partial}{\partial \dot{y}} + \mathbf{k}\frac{\partial}{\partial \dot{z}} \qquad \nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

and

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

From equation (9.25) for a single particle, let

$$L = \phi - \frac{1}{c}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z)$$

Then

$$\frac{d}{dt}\dot{\nabla}L = -\frac{1}{c}\frac{d}{dt}(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})$$
$$= -\frac{1}{c}\frac{d}{dt}\mathbf{A} \tag{2}$$

and

$$\nabla L = \nabla\phi - \frac{1}{c}\left(\dot{x}\frac{\partial A_x}{\partial x}\mathbf{i} + \dot{y}\frac{\partial A_y}{\partial y}\mathbf{j} + \dot{z}\frac{\partial A_z}{\partial z}\mathbf{k}\right)$$
$$= \nabla\phi - \frac{1}{c}\nabla(\dot{\mathbf{x}}\cdot\mathbf{A}) \tag{3}$$

Hence by equations (1), (2), and (3)

$$-\frac{1}{c}\frac{d}{dt}\mathbf{A} = \nabla\phi - \frac{1}{c}\nabla(\dot{\mathbf{x}}\cdot\mathbf{A})$$