Bell's theorem

The key to understanding Bell's theorem is the following property of independent random variables. If two random variables A and B are independent then

$$\langle A\rangle\langle B\rangle = \langle AB\rangle$$

Consider two machines A and B that measure spin. Each machine can be set in one of two orientations labeled 0 and 1. When a spin is measured the result is either 1 or -1. The expectation value (average) for a machine can be 1, -1, or something in between. Assuming that A and B are independent we have the following relationship for all combinations of extremal expectation values.

$\langle A_0\rangle$	$\langle A_1\rangle$	$\langle B_0\rangle$	$\langle B_1\rangle$	$\langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle$
1	1	1	1	2
1	1	1	-1	2
1	1	-1	1	-2
1	1	-1	-1	-2
1	-1	1	1	2
1	-1	1	-1	-2
1	-1	-1	1	2
1	-1	-1	-1	-2
-1	1	1	1	-2
-1	1	1	-1	2
-1	1	-1	1	-2
-1	1	-1	-1	2
-1	-1	1	1	-2
-1	-1	1	-1	-2
-1	-1	-1	1	2
-1	-1	-1	-1	2

Since spin expectation values are all in the range -1 to $+1$ we have

$$-2 \le \langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle \le 2 \tag{1}$$

Now suppose we have a machine that generates two spins in the following entangled state.

$$|s\rangle = \frac{|ud\rangle - |du\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

One spin is sent to A and the other is sent to B.

Let

$$A_0 = \sigma_z, \quad A_1 = \sigma_x, \quad B_0 = -\frac{\sigma_x + \sigma_z}{\sqrt{2}}, \quad B_1 = \frac{\sigma_x - \sigma_z}{\sqrt{2}}$$

Then for state $|s\rangle$ we have

$$\langle A_0 B_0\rangle = \frac{1}{\sqrt{2}}, \quad \langle A_0 B_1\rangle = \frac{1}{\sqrt{2}}, \quad \langle A_1 B_0\rangle = \frac{1}{\sqrt{2}}, \quad \langle A_1 B_1\rangle = -\frac{1}{\sqrt{2}}$$

Hence

$$\langle A_0 B_0\rangle + \langle A_0 B_1\rangle + \langle A_1 B_0\rangle - \langle A_1 B_1\rangle = 2\sqrt{2} \tag{2}$$

The result in (2) conflicts with (1) because for an entangled state the random variables are not independent. Any theory that asserts A and B are independent is constrained by (1) and falsified by (2). Hence Bell's theorem: No local theory can explain quantum mechanics. (A local theory asserts that A and B are independent.)

Exercises

1. Verify equation (2) for state $|s\rangle$.

2. Verify the following expectation values for state $|s\rangle$.

   $$\langle A_0\rangle = 0, \quad \langle A_1\rangle = 0, \quad \langle B_0\rangle = 0, \quad \langle B_1\rangle = 0$$

   Hence $\langle A\rangle\langle B\rangle \neq \langle AB\rangle$ for the singlet state.

3. There are three additional entangled states.

   $$|s_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \quad |s_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \quad |s_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

   Verify that A and B are correlated for all entangled states.