

(36.1) (a) Show that the Dirac equation can be recast in the form

$$i\frac{\partial\psi}{\partial t} = \hat{H}_D\psi \quad (36.33)$$

where  $\hat{H}_D = \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m$  and find  $\boldsymbol{\alpha}$  and  $\beta$  in terms of the  $\gamma$  matrices.

(b) Evaluate  $\hat{H}_D^2$  and show that for a Klein-Gordon dispersion to result we must have:

- (i) that the  $\alpha^i$  and  $\beta$  objects all anticommute with each other; and
- (ii)  $(\alpha^i)^2 = (\beta)^2 = 1$ .

(c) Prove the following commutation relations

- (i)  $[\hat{H}, \hat{L}^i] = i(\hat{\mathbf{p}} \times \boldsymbol{\alpha})^i$  where  $\hat{\mathbf{L}} = \hat{\mathbf{x}} \times \hat{\mathbf{p}}$ .
- (ii)  $[\hat{H}, \hat{S}^i] = -i(\hat{\mathbf{p}} \times \boldsymbol{\alpha})^i$  where  $\hat{\mathbf{S}} = \frac{1}{2}\boldsymbol{\Sigma}$  and we define  $\boldsymbol{\Sigma} = \frac{i}{2}\boldsymbol{\gamma} \times \boldsymbol{\gamma}$ .

(a) Consider the following form of the Dirac equation.

$$i\left(\gamma^0\frac{\partial}{\partial t} + \gamma^1\frac{\partial}{\partial x} + \gamma^2\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z}\right)\psi = m\psi$$

Rewrite as

$$i\gamma^0\frac{\partial}{\partial t}\psi = -i\left(\gamma^1\frac{\partial}{\partial x} + \gamma^2\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z}\right)\psi + m\psi$$

Noting that  $\gamma^0\gamma^0 = I$ , multiply both sides by  $\gamma^0$  to obtain

$$i\frac{\partial}{\partial t}\psi = -i\gamma^0\left(\gamma^1\frac{\partial}{\partial x} + \gamma^2\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z}\right)\psi + m\gamma^0\psi$$

Hence for  $\hat{\mathbf{p}} = -i\nabla$  we have

$$\boldsymbol{\alpha} = \gamma^0 \begin{pmatrix} \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{pmatrix}, \quad \beta = \gamma^0$$

(b) The dispersion relation is

$$\hat{H}_D^2 = \hat{\mathbf{p}}^2 + m^2$$

Squaring  $\hat{H}_D$  we have

$$\begin{aligned}\hat{H}_D^2 &= (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m)(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m) \\ &= (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}) + (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\beta m + \beta m(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}) + \beta^2 m^2\end{aligned}$$

The middle terms must cancel, that is

$$(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\beta m + \beta m(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}) = 0$$

Hence

$$\alpha^i \beta = -\beta \alpha^i$$

Cross terms must cancel, that is

$$\left(-i\alpha^1 \frac{\partial}{\partial x} - i\alpha^2 \frac{\partial}{\partial y} - i\alpha^3 \frac{\partial}{\partial z}\right)^2 = -(\alpha^1)^2 \frac{\partial^2}{\partial x^2} - (\alpha^2)^2 \frac{\partial^2}{\partial y^2} - (\alpha^3)^2 \frac{\partial^2}{\partial z^2}$$

Hence

$$\alpha^i \alpha^j = -\alpha^j \alpha^i, \quad i \neq j$$

With the above anticommutation relations we now have

$$\hat{H}_D^2 = -(\alpha^1)^2 \frac{\partial^2}{\partial x^2} - (\alpha^2)^2 \frac{\partial^2}{\partial y^2} - (\alpha^3)^2 \frac{\partial^2}{\partial z^2} + \beta^2 m^2 = \hat{\mathbf{p}}^2 + m^2$$

Hence

$$(\alpha^i)^2 = I \quad \text{and} \quad \beta^2 = I$$

(c) We have

$$\hat{\mathbf{L}}\psi = \hat{\mathbf{x}} \times \hat{\mathbf{p}}\psi$$

It follows that

$$\begin{aligned}[\hat{H}_D, \hat{\mathbf{L}}]\psi &= \hat{H}_D \hat{\mathbf{L}}\psi - \hat{\mathbf{L}} \hat{H}_D \psi \\ &= (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m)(\hat{\mathbf{x}} \times \hat{\mathbf{p}}\psi) - \hat{\mathbf{x}} \times \hat{\mathbf{p}}(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m)\psi\end{aligned}$$

The  $\beta$  terms cancel because  $\beta$  commutes with  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{p}}$ .

$$\beta \hat{\mathbf{x}} \times \hat{\mathbf{p}} = \hat{\mathbf{x}} \times \hat{\mathbf{p}} \beta$$

Hence

$$[\hat{H}_D, \hat{\mathbf{L}}]\psi = (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})(\hat{\mathbf{x}} \times \hat{\mathbf{p}}\psi) - \hat{\mathbf{x}} \times \hat{\mathbf{p}}(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\psi \quad (1)$$

By the product rule for differentiation

$$(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})(\hat{\mathbf{x}} \times \hat{\mathbf{p}}\psi) = (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\hat{\mathbf{x}} \times \hat{\mathbf{p}}\psi + \hat{\mathbf{x}} \times (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\hat{\mathbf{p}}\psi \quad (2)$$

Substitute (2) into (1).

$$\begin{aligned} [\hat{H}_D, \hat{\mathbf{L}}]\psi &= (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\hat{\mathbf{x}} \times \hat{\mathbf{p}}\psi + \hat{\mathbf{x}} \times (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\hat{\mathbf{p}}\psi - \hat{\mathbf{x}} \times \hat{\mathbf{p}}(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\psi \\ &= (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\hat{\mathbf{x}} \times \hat{\mathbf{p}}\psi \end{aligned}$$

Note that

$$(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\hat{\mathbf{x}} = \boldsymbol{\alpha} \cdot (-i\nabla\hat{\mathbf{x}}) = -i\boldsymbol{\alpha}$$

Hence

$$[\hat{H}_D, \hat{\mathbf{L}}]\psi = -i\boldsymbol{\alpha} \times \hat{\mathbf{p}}\psi = i(\hat{\mathbf{p}} \times \boldsymbol{\alpha})\psi$$

For part (ii) we have

$$\hat{\mathbf{S}}\psi = \frac{i}{4}\boldsymbol{\gamma} \times \boldsymbol{\gamma}\psi = \frac{i}{4}\gamma^0\boldsymbol{\alpha} \times \gamma^0\boldsymbol{\alpha}\psi = \frac{i}{4}\gamma^0\boldsymbol{\alpha}\gamma^0 \times \boldsymbol{\alpha}\psi = -\frac{i}{4}\boldsymbol{\alpha} \times \boldsymbol{\alpha}\psi$$

It follows that

$$\begin{aligned} [\hat{H}_D, \hat{\mathbf{S}}]\psi &= \hat{H}_D\hat{\mathbf{S}}\psi - \hat{\mathbf{S}}\hat{H}_D\psi \\ &= -\frac{i}{4}(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m)(\boldsymbol{\alpha} \times \boldsymbol{\alpha}\psi) + \frac{i}{4}\boldsymbol{\alpha} \times \boldsymbol{\alpha}(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m)\psi \end{aligned}$$

The  $\beta$  terms cancel by the following identity. (Recall  $\beta\boldsymbol{\alpha} = -\boldsymbol{\alpha}\beta$ .)

$$\beta\boldsymbol{\alpha} \times \boldsymbol{\alpha} = -\boldsymbol{\alpha}\beta \times \boldsymbol{\alpha} = \boldsymbol{\alpha} \times \boldsymbol{\alpha}\beta$$

Hence

$$[\hat{H}_D, \hat{\mathbf{S}}]\psi = -\frac{i}{4}(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})(\boldsymbol{\alpha} \times \boldsymbol{\alpha}\psi) + \frac{i}{4}\boldsymbol{\alpha} \times \boldsymbol{\alpha}(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\psi$$

Consider the following component forms.

$$\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} = \alpha^1\hat{p}^1 + \alpha^2\hat{p}^2 + \alpha^3\hat{p}^3, \quad \boldsymbol{\alpha} \times \boldsymbol{\alpha} = \begin{pmatrix} \alpha^2\alpha^3 - \alpha^3\alpha^2 \\ \alpha^3\alpha^1 - \alpha^1\alpha^3 \\ \alpha^1\alpha^2 - \alpha^2\alpha^1 \end{pmatrix} = 2 \begin{pmatrix} \alpha^2\alpha^3 \\ \alpha^3\alpha^1 \\ \alpha^1\alpha^2 \end{pmatrix}$$

It follows that

$$\begin{aligned} -\frac{i}{4}(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})(\boldsymbol{\alpha} \times \boldsymbol{\alpha}\psi) &= \frac{i}{2} \begin{pmatrix} -\alpha^1\alpha^2\alpha^3\hat{p}^1 - \alpha^3\hat{p}^2 + \alpha^2\hat{p}^3 \\ \alpha^3\hat{p}^1 - \alpha^2\alpha^3\alpha^1\hat{p}^2 - \alpha^1\hat{p}^3 \\ -\alpha^2\hat{p}^1 + \alpha^1\hat{p}^2 - \alpha^3\alpha^1\alpha^2\hat{p}^3 \end{pmatrix} \psi = \frac{i}{2}(\boldsymbol{\alpha} \times \hat{\mathbf{p}} - \epsilon)\psi \\ \frac{i}{4}(\boldsymbol{\alpha} \times \boldsymbol{\alpha})(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\psi &= \frac{i}{2} \begin{pmatrix} \alpha^2\alpha^3\alpha^1\hat{p}^1 - \alpha^3\hat{p}^2 + \alpha^2\hat{p}^3 \\ \alpha^3\hat{p}^1 + \alpha^3\alpha^1\alpha^2\hat{p}^2 - \alpha^1\hat{p}^3 \\ -\alpha^2\hat{p}^1 + \alpha^1\hat{p}^2 + \alpha^1\alpha^2\alpha^3\hat{p}^3 \end{pmatrix} \psi = \frac{i}{2}(\boldsymbol{\alpha} \times \hat{\mathbf{p}} + \epsilon)\psi \end{aligned}$$

where

$$\epsilon = \begin{pmatrix} \alpha^1 \alpha^2 \alpha^3 \hat{p}^1 \\ \alpha^2 \alpha^3 \alpha^1 \hat{p}^2 \\ \alpha^3 \alpha^1 \alpha^2 \hat{p}^3 \end{pmatrix} = \begin{pmatrix} \alpha^2 \alpha^3 \alpha^1 \hat{p}^1 \\ \alpha^3 \alpha^1 \alpha^2 \hat{p}^2 \\ \alpha^1 \alpha^2 \alpha^3 \hat{p}^3 \end{pmatrix}$$

Hence

$$[\hat{H}_D, \hat{\mathbf{S}}]\psi = \frac{i}{2}(\boldsymbol{\alpha} \times \hat{\mathbf{p}} - \epsilon)\psi + \frac{i}{2}(\boldsymbol{\alpha} \times \hat{\mathbf{p}} + \epsilon)\psi = i\boldsymbol{\alpha} \times \hat{\mathbf{p}}\psi = -i(\hat{\mathbf{p}} \times \boldsymbol{\alpha})\psi$$