9-5. The momentum in the field is given by

$$\frac{1}{4\pi c}\int \mathbf{E}\times\mathbf{B}\,d^3\mathbf{r}$$

In the absence of matter (so $\phi_{\mathbf{k}} = 0$), show that this is

$$i\int \mathbf{k}\,(\mathbf{a}_{\mathbf{k}}^*\cdot\dot{\mathbf{a}}_{\mathbf{k}})\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

Let

$$D_{\mathbf{k}} = \exp(i\mathbf{k}\cdot\mathbf{r})\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

Consider equation (9.14).

$$\mathbf{A} = \sqrt{4\pi}c\int \mathbf{a}_{\mathbf{k}}\,D_{\mathbf{k}} \tag{9.14}$$

By equation (9.9) with $\phi = 0$ and (9.14)

$$\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} = -\sqrt{4\pi}\int \dot{\mathbf{a}}_{\mathbf{k}}\,D_{\mathbf{k}}$$

By equation (9.7) and (9.14)

$$\mathbf{B} = \nabla\times\mathbf{A} = \sqrt{4\pi}ic\int \mathbf{k}\times\mathbf{a}_{\mathbf{k}}\,D_{\mathbf{k}}$$

Hence

$$\mathbf{E}\times\mathbf{B} = -4\pi ic\int \dot{\mathbf{a}}_{\mathbf{k}}\times(\mathbf{k}\times\mathbf{a}_{\mathbf{k}})\;D_{\mathbf{k}}$$

By the triple product formula

$$\mathbf{E}\times\mathbf{B} = -4\pi ic\int \big((\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{a}_{\mathbf{k}})\mathbf{k} - (\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{k})\mathbf{a}_{\mathbf{k}}\big)\,D_{\mathbf{k}}$$

By orthogonality of $\mathbf{E}$ and $\mathbf{k}$ (see problem 9-1), $\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{k} = 0$ hence

$$\mathbf{E}\times\mathbf{B} = -4\pi ic\int (\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{a}_{\mathbf{k}})\mathbf{k}\,D_{\mathbf{k}}$$

Integrate over all space.

$$\int \mathbf{E}\times\mathbf{B}\,d^3\mathbf{r} = -4\pi ic\int (\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{a}_{\mathbf{k}})\mathbf{k}\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

Note: Integrating over all $\mathbf{r}$ is the same as integrating over all $\mathbf{k}$, hence the exponential in $D_{\mathbf{k}}$ is cancelled.