Schrodinger from Lagrangian

Derive the Schrodinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi$$

from the Lagrangian

$$L(\dot{x}, x, t) = \frac{m\dot{x}^2}{2} - V(x, t)$$

Start with the path integral for an action S.

$$\psi(x_b, t_b) = C\int_{-\infty}^{\infty} \exp\left(\frac{i}{\hbar}S(b, a)\right)\psi(x_a, t_a)\,dx_a$$

For a small time interval $\epsilon = t_b - t_a$ we can use the approximation

$$S = \epsilon L$$

and write the path integral as

$$\psi(x_b, t+\epsilon) = C\int_{-\infty}^{\infty} \exp\left[\frac{i}{\hbar}\epsilon L\left(\frac{x_b - x_a}{\epsilon}, \frac{x_b + x_a}{2}, t\right)\right]\psi(x_a, t)\,dx_a$$

Substitute for L.

$$\psi(x_b, t+\epsilon) = C\int_{-\infty}^{\infty} \exp\left[\frac{im(x_b - x_a)^2}{2\hbar\epsilon} - \frac{i}{\hbar}\epsilon V\left(\frac{x_b + x_a}{2}, t\right)\right]\psi(x_a, t)\,dx_a$$

Let

$$x_a = x_b + \eta, \quad dx_a = d\eta$$

and write

$$\psi(x_b, t+\epsilon) = C\int_{-\infty}^{\infty} \exp\left[\frac{im\eta^2}{2\hbar\epsilon} - \frac{i}{\hbar}\epsilon V\left(x_b + \frac{\eta}{2}, t\right)\right]\psi(x_b + \eta, t)\,d\eta$$

Because the exponential is highly oscillatory for large η, most of the contribution to the integral is from small η. Hence use the approximation $x_b + \frac{1}{2}\eta \approx x_b$ for small η.

$$\psi(x_b, t+\epsilon) = C\int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\epsilon} - \frac{i}{\hbar}\epsilon V(x_b, t)\right)\psi(x_b + \eta, t)\,d\eta$$

Use the approximation $\exp(y) \approx 1 + y$ for the exponential of V.

$$\psi(x_b, t+\epsilon) = C\int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\epsilon}\right)\left(1 - \frac{i}{\hbar}\epsilon V(x_b, t)\right)\psi(x_b + \eta, t)\,d\eta$$

Expand ψ as the power series

$$\psi(x_b + \eta, t) \approx \psi(x_b, t) + \eta\frac{\partial\psi}{\partial x} + \frac{\eta^2}{2}\frac{\partial^2\psi}{\partial x^2}$$

to obtain

$$\psi(x_b, t+\epsilon) = C\int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\epsilon}\right)\left(1 - \frac{i}{\hbar}\epsilon V(x_b,t)\right)\left(\psi(x_b,t) + \eta\frac{\partial\psi}{\partial x} + \frac{\eta^2}{2}\frac{\partial^2\psi}{\partial x^2}\right)d\eta$$

Expand products of sums as

$$\psi(x_b, t+\epsilon) = C(I_1 + I_2 + I_3 + I_4 + I_5 + I_6)$$

where

$$\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\epsilon}\right)\psi\,d\eta &&= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{\frac{1}{2}}\psi \\
I_2 &= \int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\epsilon}\right)\eta\frac{\partial\psi}{\partial x}\,d\eta &&= 0 \\
I_3 &= \int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\epsilon}\right)\frac{\eta^2}{2}\frac{\partial^2\psi}{\partial x^2}\,d\eta &&= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{\frac{1}{2}}\frac{i\hbar\epsilon}{2m}\frac{\partial^2\psi}{\partial x^2} \\
I_4 &= \int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\epsilon}\right)\left(-\frac{i}{\hbar}\epsilon V\right)\psi\,d\eta &&= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{\frac{1}{2}}\left(-\frac{i}{\hbar}\epsilon V\right)\psi \\
I_5 &= \int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\epsilon}\right)\left(-\frac{i}{\hbar}\epsilon V\right)\eta\frac{\partial\psi}{\partial x}\,d\eta &&= 0 \\
I_6 &= \int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\epsilon}\right)\left(-\frac{i}{\hbar}\epsilon V\right)\frac{\eta^2}{2}\frac{\partial^2\psi}{\partial x^2}\,d\eta &&= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{\frac{1}{2}}\frac{i\hbar\epsilon}{2m}\left(-\frac{i}{\hbar}\epsilon V\right)\frac{\partial^2\psi}{\partial x^2}
\end{aligned}$$

Integrals I_1 and I_4 are solved by the identity

$$\int_{-\infty}^{\infty} \exp(ay^2)\,dy = \left(-\frac{\pi}{a}\right)^{\frac{1}{2}}$$

Integrals I_2 and I_5 vanish by the identity

$$\int_{-\infty}^{\infty} y\exp(ay^2)\,dy = 0$$

Integrals I_3 and I_6 are solved by the identity

$$\int_{-\infty}^{\infty} y^2\exp(ay^2)\,dy = \left(-\frac{\pi}{a}\right)^{\frac{1}{2}}\left(-\frac{1}{2a}\right)$$

Let

$$C = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-\frac{1}{2}}$$

and discard I_6 as proportional to ϵ^2 to obtain

$$\psi(x_b, t+\epsilon) = \psi(x_b,t) + \frac{i\hbar\epsilon}{2m}\frac{\partial^2}{\partial x^2}\psi(x_b,t) - \frac{i}{\hbar}\epsilon V\psi(x_b,t)$$

Expand ψ on the left as a power series.

$$\psi(x_b,t)+\epsilon\frac{\partial}{\partial t}\psi(x_b,t)=\psi(x_b,t)+\frac{i\hbar\epsilon}{2m}\frac{\partial^2}{\partial x^2}\psi(x_b,t)-\frac{i}{\hbar}\epsilon V\psi(x_b,t)$$

Cancel ψ and multiply both sides by $i\hbar/\epsilon$.

$$i\hbar\frac{\partial}{\partial t}\psi(x_b,t)=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x_b,t)+V\psi(x_b,t)$$

See Feynman and Hibbs section 4-1.