Feynman and Hibbs problem 2-2

This is the Lagrangian for a harmonic oscillator.

$$L = \frac{m}{2}(\dot{x}^2 - \omega^2 x^2)$$

Let $T = t_b - t_a$. Show that the classical action is

$$S_{cl} = \frac{m\omega}{2\sin(\omega T)}\left((x_b^2 + x_a^2)\cos(\omega T) - 2x_b x_a\right)$$

From the above Lagrangian we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

and

$$\frac{\partial L}{\partial x} = -m\omega^2 x$$

By equation (2.7) which is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

we have

$$\ddot{x} = -\omega^2 x \tag{1}$$

The well-known solution to (1) is

$$x(t) = A\sin(\omega t) + B\cos(\omega t)$$

We have the following boundary conditions.

$$x(0) = x_a$$
$$x(T) = x_b$$

Solve for B.

$$x(0) = B = x_a$$

For $x(T)$ we have

$$x(T) = A\sin(\omega T) + B\cos(\omega T)$$

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Solve for A to obtain

$$A = \frac{x(T) - B\cos(\omega T)}{\sin(\omega T)} = \frac{x_b - x_a \cos(\omega T)}{\sin(\omega T)}$$

Hence the equation of motion is

$$x(t) = \frac{x_b - x_a \cos(\omega T)}{\sin(\omega T)} \sin(\omega t) + x_a \cos(\omega t)$$

Differentiate $x(t)$ to obtain velocity $\dot{x}(t)$.

$$\dot{x}(t) = \frac{d}{dt}x(t) = \frac{\omega(x_b - x_a)\cos(\omega T)}{\sin(\omega T)}\cos(\omega t) - \omega x_a \sin(\omega t)$$

Using the action integral

$$S = \int_0^T L\,dt$$

we have for the classical action

$$S_{cl} = \frac{m}{2}\int_0^T (\dot{x}^2 - \omega^2 x^2)\,dt$$

$$= \frac{m}{2}\left(\int_0^T \dot{x}^2\,dt - \int_0^T \omega^2 x^2\,dt\right)$$

Let

$$u = \dot{x}$$
$$v = \dot{x}$$

Use integration by parts to obtain

$$\int_0^T \dot{x}^2\,dt = \int_0^T uv\,dt$$

$$= \left(u\int v\,dt\right)\Big|_0^T - \int_0^T \dot{u}\left(\int v\,dt\right)dt$$

$$= \dot{x}x\Big|_0^T - \int_0^T \ddot{x}x\,dt$$

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Hence
$$S_{cl} = \frac{m}{2} \left(\left. \dot{x}x \right|_0^T - \int_0^T \ddot{x}x \, dt - \int_0^T \omega^2 x^2 \, dt \right)$$

From (1) we have
$$\ddot{x}x = -\omega^2 x^2$$

Hence the remaining integrals cancel and we have
$$S_{cl} = \frac{m}{2} \left. \dot{x}x \right|_0^T$$
$$= \frac{m}{2} \left(\dot{x}(T)x(T) - \dot{x}(0)x(0) \right)$$

Then by substitution
$$S_{cl} = \frac{m\omega}{2\sin(\omega T)} \left((x_b^2 + x_a^2)\cos(\omega T) - 2x_b x_a \right)$$