

Schrodinger from Lagrangian

Derive the Schrodinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi$$

from the Lagrangian

$$L(\dot{x}, x, t) = \frac{m\dot{x}^2}{2} - V(x, t)$$

Start with the path integral for an action S .

$$\psi(x_b, t_b) = C \int_{-\infty}^{\infty} \exp\left(\frac{i}{\hbar} S(b, a)\right) \psi(x_a, t_a) dx_a$$

For a small time interval $\epsilon = t_b - t_a$ we can use the approximation

$$S = \epsilon L$$

and write the path integral as

$$\psi(x_b, t + \epsilon) = C \int_{-\infty}^{\infty} \exp\left[\frac{i}{\hbar} \epsilon L\left(\frac{x_b - x_a}{\epsilon}, \frac{x_b + x_a}{2}, t\right)\right] \psi(x_a, t) dx_a$$

Substitute for L .

$$\psi(x_b, t + \epsilon) = C \int_{-\infty}^{\infty} \exp\left[\frac{im(x_b - x_a)^2}{2\hbar\epsilon} - \frac{i}{\hbar} \epsilon V\left(\frac{x_b + x_a}{2}, t\right)\right] \psi(x_a, t) dx_a$$

Let

$$x_a = x_b + \eta, \quad dx_a = d\eta$$

and write

$$\psi(x_b, t + \epsilon) = C \int_{-\infty}^{\infty} \exp\left[\frac{im\eta^2}{2\hbar\epsilon} - \frac{i}{\hbar} \epsilon V\left(x_b + \frac{\eta}{2}, t\right)\right] \psi(x_b + \eta, t) d\eta$$

Substitute x for x_b .

$$\psi(x, t + \epsilon) = C \int_{-\infty}^{\infty} \exp\left[\frac{im\eta^2}{2\hbar\epsilon} - \frac{i}{\hbar} \epsilon V\left(x + \frac{\eta}{2}, t\right)\right] \psi(x + \eta, t) d\eta$$

Because the exponential is highly oscillatory for large η , most of the contribution to the integral is from small η . Hence use the approximation $x + \frac{1}{2}\eta \approx x$ for small η .

$$\psi(x, t + \epsilon) = C \int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\epsilon} - \frac{i}{\hbar} \epsilon V(x, t)\right) \psi(x + \eta, t) d\eta$$

Use the approximation $\exp(y) \approx 1 + y$ for the exponential of V .

$$\psi(x, t + \epsilon) = C \int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\epsilon}\right) \left(1 - \frac{i}{\hbar} \epsilon V(x, t)\right) \psi(x + \eta, t) d\eta$$

Expand ψ as the power series

$$\psi(x + \eta, t) \approx \psi(x, t) + \eta \frac{\partial \psi}{\partial x} + \frac{\eta^2}{2} \frac{\partial^2 \psi}{\partial x^2}$$

to obtain

$$\psi(x, t + \epsilon) = C \int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\epsilon}\right) \left(1 - \frac{i}{\hbar}\epsilon V(x, t)\right) \left(\psi(x, t) + \eta \frac{\partial \psi}{\partial x} + \frac{\eta^2}{2} \frac{\partial^2 \psi}{\partial x^2}\right) d\eta$$

Rewrite as

$$\psi(x, t + \epsilon) = C \left(1 - \frac{i}{\hbar}\epsilon V(x, t)\right) (I_1 + I_2 + I_3)$$

where

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\epsilon}\right) \psi d\eta &= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{\frac{1}{2}} \psi \\ I_2 &= \int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\epsilon}\right) \eta \frac{\partial \psi}{\partial x} d\eta &= 0 \\ I_3 &= \int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\epsilon}\right) \frac{\eta^2}{2} \frac{\partial^2 \psi}{\partial x^2} d\eta &= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{\frac{1}{2}} \frac{i\hbar\epsilon}{2m} \frac{\partial^2 \psi}{\partial x^2} \end{aligned}$$

Integral I_1 is solved by the identity

$$\int_{-\infty}^{\infty} \exp(ay^2) dy = \left(-\frac{\pi}{a}\right)^{\frac{1}{2}}$$

Integral I_2 vanishes by the identity

$$\int_{-\infty}^{\infty} y \exp(ay^2) dy = 0$$

Integral I_3 is solved by the identity

$$\int_{-\infty}^{\infty} y^2 \exp(ay^2) dy = \left(-\frac{\pi}{a}\right)^{\frac{1}{2}} \left(-\frac{1}{2a}\right)$$

Let

$$C = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-\frac{1}{2}}$$

and substitute the solved integrals to obtain

$$\psi(x, t + \epsilon) = \left(1 - \frac{i}{\hbar}\epsilon V\right) \left(\psi + \frac{i\hbar\epsilon}{2m} \frac{\partial^2 \psi}{\partial x^2}\right)$$

Expand the product.

$$\psi(x, t + \epsilon) = \psi + \frac{i\hbar\epsilon}{2m} \frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar}\epsilon V\psi + \frac{\epsilon^2}{2m} V \frac{\partial^2 \psi}{\partial x^2}$$

Expand ψ on the left as a power series.

$$\psi + \epsilon \frac{\partial \psi}{\partial t} = \psi + \frac{i\hbar\epsilon}{2m} \frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} \epsilon V \psi + \frac{\epsilon^2}{2m} V \frac{\partial^2 \psi}{\partial x^2}$$

Cancel ψ and multiply both sides by $i\hbar/\epsilon$.

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi + \frac{i\hbar\epsilon}{2m} V \frac{\partial^2 \psi}{\partial x^2}$$

The term on the right vanishes for $\epsilon = 0$ hence

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi$$

See Feynman and Hibbs section 4-1.