

9-8. For the state for which there is just one photon present in level 1,  $\mathbf{k}$ , all of the factors in the wave function are  $\phi_0$  except one, which is  $\phi_1$ . But for an oscillator  $\phi_1(x) = \sqrt{2}x\phi_0(x)$ . The wave function representing an excited running wave is a linear superposition of the state with the cosine mode excited and  $i$  times the state with the sine wave excited, so show that the unnormalized wave function for just one photon present in 1,  $\mathbf{k}$  is  $\bar{a}_{1,\mathbf{k}}^*\Phi_0$ . The normalization is  $\int \Phi_0^* \bar{a}_{1,\mathbf{k}} \bar{a}_{1,\mathbf{k}}^* \Phi_0 d\bar{a}$ , or the expectation of  $\bar{a}_{1,\mathbf{k}} \bar{a}_{1,\mathbf{k}}^*$  for the vacuum, which we have seen in the preceding problem is  $\hbar/2kc$ . Hence the normalized one-photon state is  $\sqrt{2kc/\hbar} \bar{a}_{1,\mathbf{k}}^* \Phi_0$ .

This is a state with the cosine mode excited.

$$\bar{a}_{1,\mathbf{k}}^c \Phi_0$$

This is a state with the sine mode excited.

$$\bar{a}_{1,\mathbf{k}}^s \Phi_0$$

This is a superposition of the cosine state and  $i$  times the sine state. The factor  $1/\sqrt{2}$  is for normalization, i.e.,  $|1 + i| = \sqrt{2}$ .

$$\frac{1}{\sqrt{2}} (\bar{a}_{1,\mathbf{k}}^c \Phi_0 + i \bar{a}_{1,\mathbf{k}}^s \Phi_0) = \bar{a}_{1,\mathbf{k}}^* \Phi_0$$

Here are some additional results.

From equation (9.43) and problem 9-6, let

$$\Phi_0 = \exp \left( -\frac{kc}{4\hbar} (\bar{a}_{1,\mathbf{k}}^c)^2 - \frac{kc}{4\hbar} (\bar{a}_{1,\mathbf{k}}^s)^2 - \frac{kc}{4\hbar} (\bar{a}_{2,\mathbf{k}}^c)^2 - \frac{kc}{4\hbar} (\bar{a}_{2,\mathbf{k}}^s)^2 \right)$$

It follows that

$$\Phi_0^* \Phi_0 = \exp \left( -\frac{kc}{2\hbar} (\bar{a}_{1,\mathbf{k}}^c)^2 - \frac{kc}{2\hbar} (\bar{a}_{1,\mathbf{k}}^s)^2 - \frac{kc}{2\hbar} (\bar{a}_{2,\mathbf{k}}^c)^2 - \frac{kc}{2\hbar} (\bar{a}_{2,\mathbf{k}}^s)^2 \right)$$

For simplicity of notation, let

$$d\bar{a} = d\bar{a}_{1,\mathbf{k}}^c d\bar{a}_{1,\mathbf{k}}^s d\bar{a}_{2,\mathbf{k}}^c d\bar{a}_{2,\mathbf{k}}^s$$

The expectation of  $\Phi_0$  is

$$\langle \Phi_0 \rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_0^* \Phi_0 d\bar{a} = \left( \frac{2\pi\hbar}{kc} \right)^2 \quad (1)$$

Let

$$\Phi_1 = \bar{a}_{1,\mathbf{k}}^* \Phi_0$$

Then

$$\Phi_1^* \Phi_1 = \Phi_0^* \left( \frac{(\bar{a}_{1,\mathbf{k}}^c)^2 + (\bar{a}_{1,\mathbf{k}}^s)^2}{2} \right) \Phi_0$$

The expectation of  $\Phi_1$  is

$$\langle \Phi_1 \rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_1^* \Phi_1 d\bar{a} = \frac{\hbar}{kc} \left( \frac{2\pi\hbar}{kc} \right)^2 \quad (2)$$

The expectation for  $n$  photons is

$$\langle \Phi_n \rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_0^* \left( \frac{(\bar{a}_{1,\mathbf{k}}^c)^2 + (\bar{a}_{1,\mathbf{k}}^s)^2}{2} \right)^n \Phi_0 d\bar{a}$$

By a result from problem 9-7

$$\langle \Phi_n \rangle = n! \left( \frac{\hbar}{kc} \right)^n \left( \frac{2\pi\hbar}{kc} \right)^2 = n! \left( \frac{\hbar}{kc} \right)^n \langle \Phi_0 \rangle \quad (3)$$