

## Eigenvalues of angular momentum

In this section we will derive the eigenvalues of  $L^2$  and  $L_z$  from the commutators

$$\begin{aligned}[L_x, L_y] &= i\hbar L_z \\ [L_y, L_z] &= i\hbar L_x \\ [L_z, L_x] &= i\hbar L_y\end{aligned}$$

and

$$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0$$

Start by defining ladder operators  $L_+$  and  $L_-$ .

$$\begin{aligned}L_+ &= L_x + iL_y \\ L_- &= L_x - iL_y\end{aligned}$$

We have the following commutation relations for ladder operators.

$$\begin{aligned}[L_z, L_+] &= [L_z, L_x] + i[L_z, L_y] \\ &= i\hbar L_y + i(-i\hbar L_x) \\ &= \hbar L_+ \\ [L_z, L_-] &= [L_z, L_x] - i[L_z, L_y] \\ &= i\hbar L_y - i(-i\hbar L_x) \\ &= -\hbar L_-\end{aligned}$$

We also have

$$\begin{aligned}L_- L_+ &= (L_x - iL_y)(L_x + iL_y) \\ &= L_x^2 + L_y^2 + i[L_x, L_y] \\ &= L^2 - L_z^2 - \hbar L_z \\ L_+ L_- &= (L_x + iL_y)(L_x - iL_y) \\ &= L_x^2 + L_y^2 - i[L_x, L_y] \\ &= L^2 - L_z^2 + \hbar L_z\end{aligned}$$

Operators  $L^2$  and  $L_z$  commute hence they share eigenfunctions  $\psi$ .

Let  $\lambda$  be an eigenvalue of  $L^2$  and let  $\mu$  be an eigenvalue of  $L_z$  such that

$$L^2\psi = \lambda\psi$$

and

$$L_z\psi = \mu\psi$$

We will now show that

$$\lambda \geq \mu^2$$

By definition of  $L^2$  we have

$$L^2\psi = (L_x^2 + L_y^2 + L_z^2)\psi$$

Substitute  $\lambda$  for  $L^2$  and  $\mu$  for  $L_z$  to obtain

$$\lambda\psi = (L_x^2 + L_y^2 + \mu^2)\psi$$

Rewrite as

$$(L_x^2 + L_y^2)\psi = (\lambda - \mu^2)\psi$$

The eigenvalues of squared Hermitian operators are nonnegative hence  $\lambda - \mu^2 \geq 0$ . Hence

$$\lambda \geq \mu^2$$

The property  $\lambda \geq \mu^2$  means that  $\mu$  has an upper limit.

Let  $\mu_m$  be the maximum  $\mu$ . Then for eigenfunction  $\psi_m$  we have

$$L_z\psi_m = \mu_m\psi_m$$

Apply  $L_+$  to both sides.

$$L_+L_z\psi_m = \mu_mL_+\psi_m$$

Expand the left hand side.

$$(L_zL_+ - L_zL_+ + L_+L_z)\psi_m = \mu_mL_+\psi_m$$

Substitute  $\hbar L_+$  for  $[L_z, L_+]$ .

$$L_zL_+\psi_m - \hbar L_+\psi_m = \mu_mL_+\psi_m$$

Hence

$$L_zL_+\psi_m = (\mu_m + \hbar)L_+\psi_m$$

Because  $\mu_m$  is the maximum eigenvalue and  $\mu_m + \hbar > \mu_m$  we must have

$$L_+\psi_m = 0$$

Consequently

$$L_-L_+\psi_m = 0$$

Recalling that

$$L_-L_+ = L^2 - L_z^2 - \hbar L_z$$

we have

$$(L^2 - L_z^2 - \hbar L_z)\psi_m = (\lambda - \mu_m^2 - \hbar\mu_m)\psi_m = 0$$

Hence

$$\lambda = \mu_m^2 + \hbar\mu_m \tag{1}$$

Let  $\mu_k$  be the minimum  $\mu$ . Then for eigenfunction  $\psi_k$  we have

$$L_z\psi_k = \mu_k\psi_k$$

Apply  $L_-$  to both sides.

$$L_- L_z \psi_k = \mu_k L_- \psi_k$$

Expand the left hand side.

$$(L_z L_- - L_z L_- + L_- L_z) \psi_k = \mu_m L_- \psi_k$$

Substitute  $-\hbar L_-$  for  $[L_z, L_-]$ .

$$L_z L_- \psi_k + \hbar L_- \psi_k = \mu_k L_- \psi_k$$

Hence

$$L_z L_- \psi_k = (\mu_k - \hbar) L_- \psi_k$$

Because  $\mu_k$  is the minimum eigenvalue and  $\mu_k - \hbar < \mu_k$  we must have

$$L_- \psi_k = 0$$

Consequently

$$L_+ L_- \psi_k = 0$$

Recalling that

$$L_+ L_- = L^2 - L_z^2 + \hbar L_z$$

we have

$$(L^2 - L_z^2 + \hbar L_z) \psi_k = (\lambda - \mu_k^2 + \hbar \mu_k) \psi_k = 0$$

Hence

$$\lambda = \mu_k^2 - \hbar \mu_k \quad (2)$$

By equivalence of (1) and (2) we have

$$\mu_m^2 + \hbar \mu_m - \mu_k^2 + \hbar \mu_k = 0 \quad (3)$$

By ladder operators there is an integer  $n$  such that

$$\mu_m = \mu_k + n\hbar$$

Substitute  $\mu_k + n\hbar$  for  $\mu_m$  in (3) to obtain

$$\mu_k^2 + 2\mu_k n\hbar + n^2 \hbar^2 + \hbar \mu_k + n\hbar^2 - \mu_k^2 + \hbar \mu_k = 0$$

Cancel  $\mu_k^2$  and rewrite the remaining terms as

$$2\mu_k(n+1)\hbar + n(n+1)\hbar^2 = 0$$

Divide through by  $(n+1)\hbar$  to obtain

$$2\mu_k + n\hbar = 0$$

Hence

$$\mu_k = -\frac{n\hbar}{2}$$

and

$$\mu_m = \mu_k + n\hbar = \frac{n\hbar}{2}$$

Define quantum number  $l$  as

$$l = \frac{n}{2} = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$$

Then

$$\mu_m = l\hbar$$

By equation (1) we have

$$\lambda = \mu_m^2 + \mu_m \hbar = (l\hbar)^2 + l\hbar^2 = l(l+1)\hbar^2$$

Hence  $l(l+1)\hbar^2$  are eigenvalues of  $L^2$ .

$$L^2\psi = \lambda\psi = l(l+1)\hbar^2\psi$$

For a given  $l$ , operator  $L_z$  has eigenvalues

$$\mu = \mu_k, \dots, \mu_m = -l\hbar, (-l+1)\hbar, \dots, (l-1)\hbar, l\hbar$$

Define quantum number  $m$  as

$$m = -l, -l+1, \dots, l-1, l$$

Then

$$\mu = m\hbar$$

Hence  $m\hbar$  are eigenvalues of  $L_z$ .

$$L_z\psi = \mu\psi = m\hbar\psi$$

Eigenmath code