

Atomic transitions 2

Start with the perturbing Hamiltonian where E_0 is the amplitude of the electric field.

$$H_1(\mathbf{r}, t) = -\frac{eE_0}{m\omega} \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \boldsymbol{\epsilon} \cdot \mathbf{p}$$

In exponential form

$$H_1(\mathbf{r}, t) = -\frac{eE_0}{m\omega} \left(\frac{1}{2} \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t) + \frac{1}{2} \exp(-i\mathbf{k} \cdot \mathbf{r} + i\omega t) \right) \boldsymbol{\epsilon} \cdot \mathbf{p}$$

Given the initial condition $c_b(0) = 0$ the first-order approximation for $c_b(t)$ is

$$c_b(t) = -\frac{i}{\hbar} \int_0^t \langle \psi_b | H_1(\mathbf{r}, t') | \psi_a \rangle \exp(i\omega_0 t') dt', \quad \omega_0 = \frac{E_b - E_a}{\hbar}$$

Factor the integrand.

$$\begin{aligned} c_b(t) = & \frac{ieE_0}{2m\hbar\omega} \langle \psi_b | \exp(i\mathbf{k} \cdot \mathbf{r}) \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \int_0^t \exp(-i\omega t') \exp(i\omega_0 t') dt' \\ & + \frac{ieE_0}{2m\hbar\omega} \langle \psi_b | \exp(-i\mathbf{k} \cdot \mathbf{r}) \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \int_0^t \exp(i\omega t') \exp(i\omega_0 t') dt' \end{aligned}$$

Solve the integrals to obtain

$$\begin{aligned} c_b(t) = & \frac{eE_0}{2m\hbar\omega} \langle \psi_b | \exp(i\mathbf{k} \cdot \mathbf{r}) \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \frac{\exp(i(\omega_0 - \omega)t) - 1}{\omega_0 - \omega} \\ & + \frac{eE_0}{2m\hbar\omega} \langle \psi_b | \exp(-i\mathbf{k} \cdot \mathbf{r}) \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \frac{\exp(i(\omega_0 + \omega)t) - 1}{\omega_0 + \omega} \quad (1) \end{aligned}$$

As an approximation, discard the second term since the first term dominates for $\omega \approx \omega_0$.

$$c_b(t) = \frac{eE_0}{2m\hbar\omega} \langle \psi_b | \exp(i\mathbf{k} \cdot \mathbf{r}) \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \frac{\exp(i(\omega_0 - \omega)t) - 1}{\omega_0 - \omega}$$

Rewrite $c_b(t)$ in the form of a sine function.

$$c_b(t) = \frac{ieE_0}{m\hbar\omega} \langle \psi_b | \exp(i\mathbf{k} \cdot \mathbf{r}) \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \frac{\sin\left(\frac{1}{2}(\omega_0 - \omega)t\right)}{\omega_0 - \omega} \exp\left(\frac{i}{2}(\omega_0 - \omega)t\right) \quad (2)$$

Verify dimensions.

$$\begin{aligned} H_1(\mathbf{r}, t) \propto & \frac{\frac{e}{\text{C}} \frac{E_0}{\text{N C}^{-1}}}{\frac{m}{\text{kg}} \frac{\omega}{\text{s}^{-1}}} \times \frac{\boldsymbol{\epsilon} \cdot \mathbf{p}}{\text{kg m s}^{-1}} = \text{N m} = \text{J} \\ c_b(t) \propto & \frac{\frac{e}{\text{C}} \frac{E_0}{\text{N C}^{-1}}}{\frac{m}{\text{kg}} \frac{\hbar}{\text{J s}} \frac{\omega}{\text{s}^{-1}}} \times \frac{\langle \psi_b | \exp(i\mathbf{k} \cdot \mathbf{r}) \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle}{\frac{\text{kg m s}^{-1}}{\text{s}^{-1}}} = \frac{\text{N m}}{\text{J}} = 1 \end{aligned}$$

Wave functions ψ_a and ψ_b have dimension meter^{-1/2} hence they cancel with $dx \propto$ meter in the integral leaving units of momentum due to \mathbf{p} .