Barrier potential

Let $V(x)$ be the following barrier potential.

$$V(x) = \begin{cases} 0 & x < -a \\ V_0 & -a \le x \le a \\ 0 & x > a \end{cases}$$

Find the transmission coefficient for this barrier potential.

Start with Schrodinger equations, one for each region of $V(x)$.

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_1 &= E\psi_1, && x \le -a \\ -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_2 + V_0\psi_2 &= E\psi_2, && -a \le x \le a \\ -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_3 &= E\psi_3, && x \ge a \end{aligned}$$

The solutions are

$$\begin{aligned} \psi_1(x) &= A\exp(ikx) + B\exp(-ikx) \\ \psi_2(x) &= C\exp(i\kappa x) + B\exp(-i\kappa x) \\ \psi_3(x) &= F\exp(ikx) \end{aligned}$$

where

$$k = \frac{\sqrt{2mE}}{\hbar}, \quad \kappa = \frac{\sqrt{2m(E - V_0)}}{\hbar}$$

Boundary conditions for ψ_1 and ψ_2.

$$\psi_1(-a) = \psi_2(-a), \quad \left.\frac{d\psi_1}{dx}\right|_{x=-a} = \left.\frac{d\psi_2}{dx}\right|_{x=-a}$$

Boundary conditions for ψ_2 and ψ_3.

$$\psi_2(a) = \psi_3(a), \quad \left.\frac{d\psi_2}{dx}\right|_{x=a} = \left.\frac{d\psi_3}{dx}\right|_{x=a}$$

From boundary conditions for ψ_2 and ψ_3 we have

$$C\exp(i\kappa a) + D\exp(-i\kappa a) = F\exp(ikx) \tag{1}$$

and

$$i\kappa C\exp(i\kappa a) - i\kappa D\exp(-i\kappa a) = ikF\exp(ikx) \tag{2}$$

Add $i\kappa$ times (1) to (2) to obtain

$$2i\kappa C\exp(i\kappa a) = (i\kappa + ik)F\exp(ika)$$

Hence

$$C = \frac{(\kappa + k)F\exp(ika - i\kappa a)}{2\kappa}$$

Add minus $i\kappa$ times (1) to (2) to obtain

$$-2i\kappa D\exp(-i\kappa a) = (-i\kappa + ik)F\exp(ika)$$

Hence

$$D = \frac{(\kappa - k)F\exp(ika + i\kappa a)}{2\kappa}$$

From boundary conditions for ψ_1 and ψ_2 we have

$$A\exp(-ika) + B\exp(ika) = C\exp(-i\kappa a) + D\exp(i\kappa a) \tag{3}$$

and

$$ikA\exp(-ika) - ikB\exp(ika) = i\kappa C\exp(-i\kappa a) - i\kappa D\exp(i\kappa a) \tag{4}$$

Add ik times (3) to (4) to obtain

$$2ikA\exp(-ika) = (ik + i\kappa)C\exp(-i\kappa a) + (ik - i\kappa)D\exp(i\kappa a)$$

Hence

$$A = \frac{(k + \kappa)C\exp(ika - i\kappa a)}{2k} + \frac{(k - \kappa)D\exp(ika + i\kappa a)}{2k}$$

Add minus ik times (3) to (4) to obtain

$$-2ikB\exp(ika) = (-ik + i\kappa)C\exp(-i\kappa a) + (-ik - i\kappa)D\exp(i\kappa a)$$

Hence

$$B = \frac{(k - \kappa)C\exp(-ika - i\kappa a)}{2k} + \frac{(k + \kappa)D\exp(-ika + i\kappa a)}{2k}$$

For transmission coefficient T we have

$$T^{-1} = \frac{A}{F}\left(\frac{A}{F}\right)^* = 1 + \frac{1}{8}\left(\frac{E}{V_0 - E} + \frac{V_0}{E} + 1\right)\left[\cos\left(\frac{4a}{\hbar}\sqrt{2m(E - V_0)}\right) - 1\right] \tag{5}$$

Equivalently

$$T^{-1} = 1 + \frac{1}{4}\left(\frac{E}{V_0 - E} + \frac{V_0}{E} + 1\right)\sinh^2\left(\frac{2ia}{\hbar}\sqrt{2m(E - V_0)}\right) \tag{6}$$

Cancel the imaginary unit.

$$T^{-1} = 1 + \frac{1}{4}\left(\frac{E}{V_0 - E} + \frac{V_0}{E} + 1\right)\sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0 - E)}\right)$$

Note that

$$\frac{E}{V_0 - E} + \frac{V_0}{E} + 1 = \frac{V_0^2}{E(V_0 - E)} \tag{7}$$