Feynman and Hibbs problem 4-2

For a particle of charge $e$ in a magnetic field the Lagrangian is

$$L(\dot{\mathbf{x}}, \mathbf{x}) = \frac{m}{2}\dot{\mathbf{x}}^2 + \frac{e}{c}\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}, t) - e\phi(\mathbf{x}, t)$$

where $\dot{\mathbf{x}}$ is the velocity vector, $c$ is the velocity of light, and $\mathbf{A}$ and $\phi$ are the vector and scalar potentials. Show that the corresponding Schrodinger equation is

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar}\left(\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right) \cdot \left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right)\psi + e\phi\psi\right)$$

From equation (4.3)

$$\psi(\mathbf{x}, t + \epsilon) = \frac{1}{A}\int_{\mathbb{R}^3} \exp\left(\frac{i\epsilon}{\hbar} L\left(\frac{\mathbf{x} - \mathbf{y}}{\epsilon}, \frac{\mathbf{x} + \mathbf{y}}{2}\right)\right) \psi(\mathbf{y}, t)\, dy_1\, dy_2\, dy_3 \tag{1}$$

By substitution of the given Lagrangian

$$L\left(\frac{\mathbf{x} - \mathbf{y}}{\epsilon}, \frac{\mathbf{x} + \mathbf{y}}{2}\right) = \frac{m}{2\epsilon^2}(\mathbf{x} - \mathbf{y})^2 + \frac{e}{c\epsilon}(\mathbf{x} - \mathbf{y}) \cdot \mathbf{A}\left(\frac{\mathbf{x} + \mathbf{y}}{2}, t\right) - e\phi\left(\frac{\mathbf{x} + \mathbf{y}}{2}, t\right)$$

Then from equation (1)

$$\psi(\mathbf{x}, t + \epsilon) = \frac{1}{A}\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}(\mathbf{x} - \mathbf{y})^2 + \frac{ie}{\hbar c}(\mathbf{x} - \mathbf{y}) \cdot \mathbf{A}\left(\frac{\mathbf{x} + \mathbf{y}}{2}, t\right) - \frac{ie\epsilon}{\hbar}\phi\left(\frac{\mathbf{x} + \mathbf{y}}{2}, t\right)\right) \times \psi(\mathbf{y}, t)\, dy_1\, dy_2\, dy_3$$

Let

$$\mathbf{y} = \mathbf{x} + \boldsymbol{\eta}$$

Then

$$\mathbf{x} - \mathbf{y} = \boldsymbol{\eta}, \quad \frac{\mathbf{x} + \mathbf{y}}{2} = \mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, \quad dy_1\, dy_2\, dy_3 = d\eta_1\, d\eta_2\, d\eta_3$$

Hence

$$\psi(\mathbf{x}, t+\epsilon) = \frac{1}{A}\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2 + \frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right) - \frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right)\right) \\ \times \psi(\mathbf{x}+\boldsymbol{\eta}, t)\, d\eta_1\, d\eta_2\, d\eta_3$$

Factor the exponential.

$$\psi(\mathbf{x}, t+\epsilon) = \frac{1}{A}\int_{\mathbb{R}^3} \\ \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2 + \frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right)\right)\exp\left(-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right)\right) \\ \times \psi(\mathbf{x}+\boldsymbol{\eta}, t)\, d\eta_1\, d\eta_2\, d\eta_3 \tag{2}$$

From the identity $\exp(i\theta) = \cos(\theta) + i\sin(\theta)$ we have

$$\exp\left(-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right)\right) \\ = \cos\left(-\frac{e\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right)\right) + i\sin\left(-\frac{e\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right)\right)$$

Then for small $\epsilon$

$$\exp\left(-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right)\right) \approx 1 - \frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right)$$

The authors write that the $\boldsymbol{\eta}$ term can be dropped "because the error is of higher order than $\epsilon$." Hence

$$\exp\left(-\frac{ie\epsilon}{\hbar}\phi\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right)\right) \approx 1 - \frac{ie\epsilon}{\hbar}\phi(\mathbf{x}, t) \tag{3}$$

Substitute (3) into (2).

$$\psi(\mathbf{x}, t+\epsilon) = \frac{1}{A}\int_{\mathbb{R}^3} \\ \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2 + \frac{ie}{\hbar c}\boldsymbol{\eta}\cdot\mathbf{A}\left(\mathbf{x}+\tfrac{1}{2}\boldsymbol{\eta}, t\right)\right)\left(1 - \frac{ie\epsilon}{\hbar}\phi(\mathbf{x}, t)\right) \\ \times \psi(\mathbf{x}+\boldsymbol{\eta}, t)\, d\eta_1\, d\eta_2\, d\eta_3 \tag{4}$$

Next we will use the following Taylor series approximations.

$$
\begin{aligned}
\psi(\mathbf{x}, t+\epsilon) &\approx \psi(\mathbf{x}, t) + \epsilon \frac{\partial \psi}{\partial t} \\
\psi(\mathbf{x}+\boldsymbol{\eta}, t) &\approx \psi(\mathbf{x}, t) + \boldsymbol{\eta} \cdot \nabla \psi + \tfrac{1}{2} \boldsymbol{\eta} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \psi)
\end{aligned} \tag{5}
$$

Note: In component notation

$$
\boldsymbol{\eta} \cdot \nabla \psi = \eta_1 \frac{\partial \psi}{\partial x_1} + \eta_2 \frac{\partial \psi}{\partial x_2} + \eta_2 \frac{\partial \psi}{\partial x_2}
$$

and

$$
\begin{aligned}
\boldsymbol{\eta} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \psi) = \eta_1^2 \frac{\partial^2 \psi}{\partial x_1^2} + \eta_2^2 \frac{\partial^2 \psi}{\partial x_2^2} + \eta_3^2 \frac{\partial^2 \psi}{\partial x_3^2} \\
+ 2\eta_1 \eta_2 \frac{\partial^2 \psi}{\partial x_1 \partial x_2} + 2\eta_1 \eta_3 \frac{\partial^2 \psi}{\partial x_1 \partial x_3} + 2\eta_2 \eta_3 \frac{\partial^2 \psi}{\partial x_2 \partial x_3}
\end{aligned}
$$

Substitute the approximations (5) into (4).

$$
\begin{aligned}
\psi(\mathbf{x}, t) + \epsilon \frac{\partial \psi}{\partial t} &= \frac{1}{A} \int_{\mathbb{R}^3} \\
\exp\left( \frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 + \frac{ie}{\hbar c} \boldsymbol{\eta} \cdot \mathbf{A}\left(\mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, t\right) \right) & \left( 1 - \frac{ie\epsilon}{\hbar} \phi(\mathbf{x}, t) \right) \\
\times \left( \psi(\mathbf{x}, t) + \boldsymbol{\eta} \cdot \nabla \psi + \tfrac{1}{2} \boldsymbol{\eta} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \psi) \right) & \, d\eta_1 \, d\eta_2 \, d\eta_3
\end{aligned} \tag{6}
$$