Two interacting particles

This is the kernel for two interacting particles.

$$K(x_b, y_b, t_b, x_a, y_a, t_a) = \int_{y_a}^{y_b} \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} \frac{m\dot{x}^2}{2}\,dt\right) \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} \frac{M\dot{y}^2}{2}\,dt\right) \times \exp\left(-\frac{i}{\hbar}\int_{t_a}^{t_b} V\big(x(t_c), y(t_c)\big)\,dt_c\right) \mathscr{D}x(t)\,\mathscr{D}y(t)$$

This is the power series expansion of the exponential of V.

$$\exp\left(-\frac{i}{\hbar}\int_{t_a}^{t_b} V\big(x(t_c), y(t_c)\big)\,dt_c\right) = 1 - \frac{i}{\hbar}\int_{t_a}^{t_b} V\big(x(t_c), y(t_c)\big)\,dt_c + \cdots$$

Hence the perturbation expansion of K is

$$K(b,a) = K_0(b,a) + K_1(b,a) + \cdots$$

where $K_0(b,a)$ is the free particle propagator and

$$K_1(b,a) = -\frac{i}{\hbar}\int_{y_a}^{y_b} \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} \frac{m\dot{x}^2}{2}\,dt\right) \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} \frac{M\dot{y}^2}{2}\,dt\right) \times \int_{t_a}^{t_b} V\big(x(t_c), y(t_c)\big)\,dt_c\,\mathscr{D}x(t)\,\mathscr{D}y(t)$$

Interchange the order of the integrals.

$$K_1(b,a) = -\frac{i}{\hbar}\int_{t_a}^{t_b}\int_{y_a}^{y_b} \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} \frac{m\dot{x}^2}{2}\,dt\right) \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b} \frac{M\dot{y}^2}{2}\,dt\right) \times V\big(x(t_c), y(t_c)\big)\,\mathscr{D}x(t)\,\mathscr{D}y(t)\,dt_c$$

Factor the exponentials.

$$K_1(b,a) = -\frac{i}{\hbar}\int_{t_a}^{t_b}\int_{y_a}^{y_b} \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_c} \frac{m\dot{x}^2}{2}\,dt\right) \exp\left(\frac{i}{\hbar}\int_{t_c}^{t_b} \frac{m\dot{x}^2}{2}\,dt\right) \times \exp\left(\frac{i}{\hbar}\int_{t_a}^{t_c} \frac{M\dot{y}^2}{2}\,dt\right) \exp\left(\frac{i}{\hbar}\int_{t_c}^{t_b} \frac{M\dot{y}^2}{2}\,dt\right) V\big(x(t_c), y(t_c)\big)\,\mathscr{D}x(t)\,\mathscr{D}y(t)\,dt_c$$

The exponentials are free particle propagators.

$$\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_c} \frac{m\dot{x}^2}{2}\,dt\right) = K_0\big(x(t_c), t_c, x_a, t_a, m\big) \quad m \text{ propagates from } x_a, t_a \text{ to } x(t_c), t_c$$

$$\exp\left(\frac{i}{\hbar}\int_{t_c}^{t_b} \frac{m\dot{x}^2}{2}\,dt\right) = K_0\big(x_b, t_b, x(t_c), t_c, m\big) \quad m \text{ propagates from } x(t_c), t_c \text{ to } x_b, t_b$$

$$\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_c} \frac{M\dot{y}^2}{2}\,dt\right) = K_0\big(y(t_c), t_c, y_a, t_a, M\big) \quad M \text{ propagates from } y_a, t_a \text{ to } y(t_c), t_c$$

$$\exp\left(\frac{i}{\hbar}\int_{t_c}^{t_b} \frac{M\dot{y}^2}{2}\,dt\right) = K_0\big(y_b, t_b, y(t_c), t_c, M\big) \quad M \text{ propagates from } y(t_c), t_c \text{ to } y_b, t_b$$

Hence

$$K_1(b,a) = -\frac{i}{\hbar}\int_{t_a}^{t_b}\int_{y_a}^{y_b}\int_{x_a}^{x_b} K_0\big(x(t_c),t_c,x_a,t_a,m\big)K_0\big(x_b,t_b,x(t_c),t_c,m\big)$$
$$\times K_0\big(y(t_c),t_c,y_a,t_a,M\big)K_0\big(y_b,t_b,y(t_c),t_c,M\big)V\big(x(t_c),y(t_c)\big)\,\mathscr{D}x(t)\,\mathscr{D}y(t)\,dt_c$$

The integral is over all possible paths $x(t)$ and $y(t)$ hence

$$-\infty < x(t_c) < \infty, \quad -\infty < y(t_c) < \infty$$

Let $x_c = x(t_c)$ and $y_c = y(t_c)$ and transform the integral into an integral over x_c and y_c.

$$K_1(b,a) = -\frac{i}{\hbar}\int_{t_a}^{t_b}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_0(x_c,t_c,x_a,t_a,m)K_0(x_b,t_b,x_c,t_c,m)$$
$$\times K_0(y_c,t_c,y_a,t_a,M)K_0(y_b,t_b,y_c,t_c,M)V(x_c,y_c)\,dx_c\,dy_c\,dt_c$$

These are the free particle propagators in three dimensions.

$$K_0(x_c,t_c,x_a,t_a,m) = \left(\frac{m}{2\pi i\hbar(t_c-t_a)}\right)^{\frac{3}{2}}\exp\left(\frac{im|\mathbf{x}_c-\mathbf{x}_a|^2}{2\hbar(t_c-t_a)}\right) \qquad m \text{ from } x_a,t_a \text{ to } x_c,t_c$$
$$K_0(x_b,t_b,x_c,t_c,m) = \left(\frac{m}{2\pi i\hbar(t_b-t_c)}\right)^{\frac{3}{2}}\exp\left(\frac{im|\mathbf{x}_b-\mathbf{x}_c|^2}{2\hbar(t_b-t_c)}\right) \qquad m \text{ from } x_c,t_c \text{ to } x_b,t_b$$
$$K_0(y_c,t_c,y_a,t_a,M) = \left(\frac{M}{2\pi i\hbar(t_c-t_a)}\right)^{\frac{3}{2}}\exp\left(\frac{iM|\mathbf{y}_c-\mathbf{y}_a|^2}{2\hbar(t_c-t_a)}\right) \qquad M \text{ from } y_a,t_a \text{ to } y_c,t_c$$
$$K_0(y_b,t_b,y_c,t_c,M) = \left(\frac{M}{2\pi i\hbar(t_b-t_c)}\right)^{\frac{3}{2}}\exp\left(\frac{iM|\mathbf{y}_b-\mathbf{y}_c|^2}{2\hbar(t_b-t_c)}\right) \qquad M \text{ from } y_c,t_c \text{ to } y_b,t_b$$

Hence

$$K_1(b,a) = -\frac{i}{\hbar}\int_{t_a}^{t_b}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{m}{2\pi i\hbar(t_c-t_a)}\right)^{\frac{3}{2}}\exp\left(\frac{im|\mathbf{x}_c-\mathbf{x}_a|^2}{2\hbar(t_c-t_a)}\right)$$
$$\times\left(\frac{m}{2\pi i\hbar(t_b-t_c)}\right)^{\frac{3}{2}}\exp\left(\frac{im|\mathbf{x}_b-\mathbf{x}_c|^2}{2\hbar(t_b-t_c)}\right)$$
$$\times\left(\frac{M}{2\pi i\hbar(t_c-t_a)}\right)^{\frac{3}{2}}\exp\left(\frac{iM|\mathbf{y}_c-\mathbf{y}_a|^2}{2\hbar(t_c-t_a)}\right)$$
$$\times\left(\frac{M}{2\pi i\hbar(t_b-t_c)}\right)^{\frac{3}{2}}\exp\left(\frac{iM|\mathbf{y}_b-\mathbf{y}_c|^2}{2\hbar(t_b-t_c)}\right)V(\mathbf{x}_c,\mathbf{y}_c)\,d\mathbf{x}_c\,d\mathbf{y}_c\,dt_c$$