

Rutherford scattering 1

Find the scattering cross section for Coulomb potential $V(r)$.

$$V(r) = -\frac{Ze^2}{4\pi\epsilon_0 r}$$

Start with the Born approximation.

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |Q|^2, \quad Q = \int \exp\left(\frac{i\mathbf{p} \cdot \mathbf{r}}{\hbar}\right) V(\mathbf{r}) d\mathbf{r}^3$$

Convert Q to polar coordinates.

$$Q = \int_0^{2\pi} \int_0^\pi \int_0^\infty \exp\left(\frac{ipr \cos \theta}{\hbar}\right) V(r, \theta, \phi) r^2 \sin \theta dr d\theta d\phi$$

Substitute the Coulomb potential for $V(r, \theta, \phi)$ and note r^2 becomes r .

$$Q = -\frac{Ze^2}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^\pi \int_0^\infty \exp\left(\frac{ipr \cos \theta}{\hbar}\right) r \sin \theta dr d\theta d\phi$$

Integrate over ϕ (multiplies Q by 2π).

$$Q = -\frac{Ze^2}{2\epsilon_0} \int_0^\pi \int_0^\infty \exp\left(\frac{ipr \cos \theta}{\hbar}\right) r \sin \theta dr d\theta$$

Transform the integral over θ to an integral over y where $y = \cos \theta$ and $dy = -\sin \theta d\theta$. The minus sign in dy is canceled by interchanging integration limits $\cos 0 = 1$ and $\cos \pi = -1$.

$$Q = -\frac{Ze^2}{2\epsilon_0} \int_{-1}^1 \int_0^\infty \exp\left(\frac{ipry}{\hbar}\right) r dr dy$$

Solve the integral over y (note r in the integrand cancels).

$$Q = -\frac{Ze^2}{2\epsilon_0} \int_0^\infty \frac{\hbar}{ip} \left[\exp\left(\frac{ipr}{\hbar}\right) - \exp\left(-\frac{ipr}{\hbar}\right) \right] dr$$

Solve the integral over r .

$$Q = -\frac{Ze^2}{2\epsilon_0} \frac{\hbar}{ip} \left[\frac{\hbar}{ip} \exp\left(\frac{ipr}{\hbar}\right) + \frac{\hbar}{ip} \exp\left(-\frac{ipr}{\hbar}\right) \right]_0^\infty$$

The first exponential is a problem so go back and multiply the integrand by $\exp(-\epsilon r)$.

$$Q = -\frac{Ze^2}{2\epsilon_0} \int_0^\infty \frac{\hbar}{ip} \left[\exp\left(\frac{ipr}{\hbar} - \epsilon r\right) - \exp\left(-\frac{ipr}{\hbar} - \epsilon r\right) \right] dr$$

Solve the integral.

$$Q = -\frac{Ze^2}{2\epsilon_0} \frac{\hbar}{ip} \left[\frac{1}{ip/\hbar - \epsilon} \exp\left(\frac{ipr}{\hbar} - \epsilon r\right) + \frac{1}{ip/\hbar + \epsilon} \exp\left(-\frac{ipr}{\hbar} - \epsilon r\right) \right]_0^\infty$$

Evaluate the limits.

$$Q = -\frac{Ze^2}{2\varepsilon_0} \frac{\hbar}{ip} \left(-\frac{1}{ip/\hbar - \epsilon} - \frac{1}{ip/\hbar + \epsilon} \right) = -\frac{Ze^2}{2\varepsilon_0} \frac{2}{(p/\hbar)^2 + \epsilon^2} \quad (1)$$

Set $\epsilon = 0$ to obtain

$$Q = -\frac{Ze^2\hbar^2}{\varepsilon_0 p^2}$$

Calculate the cross section.

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2} \right)^2 |Q|^2 = \frac{m^2 Z^2 e^4}{4\pi^2 \varepsilon_0^2 p^4} \quad (2)$$

Substitute $16\pi^2 \varepsilon_0^2 \alpha^2 \hbar^2 c^2$ for e^4 .

$$\frac{d\sigma}{d\Omega} = \frac{4m^2 Z^2 \alpha^2 \hbar^2 c^2}{p^4}$$

Symbol p is momentum transfer $|\mathbf{p}_i| - |\mathbf{p}_f|$ such that

$$p^2 = 4mE(1 - \cos \theta)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{Z^2 \alpha^2 \hbar^2 c^2}{4E^2 (1 - \cos \theta)^2} \quad (3)$$

Noting that

$$4 \sin^4 \frac{\theta}{2} = (1 - \cos \theta)^2$$

we have the alternative form of (3)

$$\frac{d\sigma}{d\Omega} = \frac{Z^2 \alpha^2 \hbar^2 c^2}{16E^2 \sin^4(\theta/2)}$$

Experimental data

The following data is from Geiger and Marsden's 1913 paper where y is the number of scattering events.

| θ | y |
|----------|--------|
| 150 | 22.2 |
| 135 | 27.4 |
| 120 | 33.0 |
| 105 | 47.3 |
| 75 | 136 |
| 60 | 320 |
| 45 | 989 |
| 37.5 | 1760 |
| 30 | 5260 |
| 22.5 | 20300 |
| 15 | 105400 |

Let x be the momentum transfer part of $d\sigma$.

$$x_i = \frac{1}{(1 - \cos \theta_i)^2}$$

The scattering probability for angle θ_i is x_i normalized by $\sum x = 1132.19$.

$$\text{Pr}(\theta_i) = \frac{x_i}{1132.19}$$

Predicted values \hat{y}_i are $\text{Pr}(\theta_i)$ times total scattering events $\sum y = 134295$.

$$\hat{y}_i = \text{Pr}(\theta_i) \times 134295$$

The following table shows the predicted values \hat{y} .

| θ | y | \hat{y} |
|----------|--------|-----------|
| 150 | 22.2 | 34.1 |
| 135 | 27.4 | 40.7 |
| 120 | 33.0 | 52.7 |
| 105 | 47.3 | 74.9 |
| 75 | 136 | 216 |
| 60 | 320 | 474 |
| 45 | 989 | 1383 |
| 37.5 | 1760 | 2778 |
| 30 | 5260 | 6608 |
| 22.5 | 20300 | 20471 |
| 15 | 105400 | 102162 |

The coefficient of determination R^2 measures how well predicted values fit the data.

$$R^2 = 1 - \frac{\sum_i (y_i - \hat{y}_i)^2}{\sum_i (y_i - \bar{y})^2} = 0.999$$

The result indicates that $d\sigma$ explains 99.9% of the variance in the data.