9-7. Show, for the vacuum state, the expectation value of $\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}}$ is $(\hbar/2kc)\delta_{\mathbf{k},\mathbf{q}}$ and that of $\bar{a}_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}}$ is $(\hbar/2kc)\delta_{-\mathbf{k},\mathbf{q}}$.

We will use the following table of integrals.

$$\int_{-\infty}^{\infty} \exp(-ax^2+b)\,dx = \sqrt{\frac{\pi}{a}}\exp(b) \tag{1}$$

$$\int_{-\infty}^{\infty} x\exp(-ax^2+b)\,dx = 0 \tag{2}$$

$$\int_{-\infty}^{\infty} x^2\exp(-ax^2+b)\,dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}\exp(b) \tag{3}$$

For simplicity of notation, let

$$A = \bar{a}^c_{1,\mathbf{k}} \qquad B = \bar{a}^s_{1,\mathbf{k}} \qquad C = \bar{a}^c_{1,\mathbf{q}} \qquad D = \bar{a}^s_{1,\mathbf{q}}$$

From problem 9-6

$$\begin{aligned}\bar{a}_{1,\mathbf{k}} &= \frac{1}{\sqrt{2}}(A-iB)\\ \bar{a}_{1,\mathbf{q}} &= \frac{1}{\sqrt{2}}(C-iD)\end{aligned} \tag{4}$$

Adapted from equation (8.84).

$$\langle\Phi_0|f|\Phi_0\rangle = \int\limits_{-\infty}^{\infty}\cdots\int\limits_{-\infty}^{\infty} \Phi_0^* f \Phi_0\,dA\,dB\,dC\,dD$$

Adapted from problem 9-6 with q as a mode number (not electric charge).

$$\Phi_0 = \exp\left(-\frac{kc}{4\hbar}A^2 - \frac{kc}{4\hbar}B^2 - \frac{qc}{4\hbar}C^2 - \frac{qc}{4\hbar}D^2\right)$$

It follows that

$$\Phi_0^*\Phi_0 = \exp\left(-\frac{kc}{2\hbar}A^2 - \frac{kc}{2\hbar}B^2 - \frac{qc}{2\hbar}C^2 - \frac{qc}{2\hbar}D^2\right)$$

Compute the normalization constant K.

$$K = \langle\Phi_0|1|\Phi_0\rangle = \int\limits_{-\infty}^{\infty}\cdots\int\limits_{-\infty}^{\infty} \Phi_0^*\Phi_0\,dA\,dB\,dC\,dD$$

By integral (1) for each factor in the measure (see problem 8-5)

$$K = \left(\frac{2\pi\hbar}{kc}\right)^{1/2} \left(\frac{2\pi\hbar}{kc}\right)^{1/2} \left(\frac{2\pi\hbar}{qc}\right)^{1/2} \left(\frac{2\pi\hbar}{qc}\right)^{1/2}$$

Compute the expectation of $\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{k}}$. From (4) we have

$$\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{k}} = \frac{A^2 + B^2}{2}$$

Hence

$$\langle\Phi_0|\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{k}}|\Phi_0\rangle = \frac{1}{K}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Phi_0^*\Phi_0\frac{A^2+B^2}{2}\,dA\,dB\,dC\,dD$$

Rewrite as

$$\langle\Phi_0|\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{k}}|\Phi_0\rangle = \frac{1}{2K}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Phi_0^*\Phi_0 A^2\,dA\,dB\,dC\,dD + \frac{1}{2K}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Phi_0^*\Phi_0 B^2\,dA\,dB\,dC\,dD$$

By integrals (1) and (3) we have

$$\langle\Phi_0|\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{k}}|\Phi_0\rangle = \frac{1}{K}\frac{\hbar}{kc}\left(\frac{2\pi\hbar}{kc}\right)^{1/2} \left(\frac{2\pi\hbar}{kc}\right)^{1/2} \left(\frac{2\pi\hbar}{qc}\right)^{1/2} \left(\frac{2\pi\hbar}{qc}\right)^{1/2}$$

Hence

$$\langle\Phi_0|\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{k}}|\Phi_0\rangle = \frac{\hbar}{kc} \tag{5}$$

Compute the expectation of $\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}}$.

$$\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}} = \frac{AC + BD - iAD + iBC}{2}$$

Hence

$$\langle\Phi_0|\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}}|\Phi_0\rangle = \frac{1}{K}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Phi_0^*\Phi_0\frac{AC+BD-iAD+iBC}{2}\,dA\,dB\,dC\,dD$$

By integral (2) all terms are zero, hence

$$\langle \Phi_0 | \bar{a}^*_{1,\mathbf{k}} \bar{a}_{1,\mathbf{q}} | \Phi_0 \rangle = 0 \tag{6}$$

Combine (5) and (6) to obtain

$$\langle \Phi_0 | \bar{a}^*_{1,\mathbf{k}} \bar{a}_{1,\mathbf{q}} | \Phi_0 \rangle = \frac{\hbar}{kc} \delta_{\mathbf{k},\mathbf{q}}$$

Note that by equation (8.77)

$$\bar{a}^*_{1,\mathbf{k}} = \bar{a}_{1,-\mathbf{k}}$$

Hence

$$\langle \Phi_0 | \bar{a}_{1,-\mathbf{k}} \bar{a}_{1,\mathbf{q}} | \Phi_0 \rangle = \frac{\hbar}{kc} \delta_{\mathbf{k},\mathbf{q}}$$

(9-7 cont'd) Develop a formula for the expectation of $(\bar{a}^*_{1,\mathbf{k}} \bar{a}_{1,\mathbf{k}})^r$ for integral r and explain thereby how the expectation of such quantities as $(\bar{a}^*_{1,\mathbf{k}} \bar{a}_{1,\mathbf{k}})^r (\bar{a}^*_{1,\mathbf{q}} \bar{a}_{1,\mathbf{q}})^s$ can be got for $\mathbf{q} \neq \mathbf{k}$.

By the binomial theorem

$$\left(\frac{A^2 + B^2}{2} \right)^r = \frac{1}{2^r} \sum_{j=0}^{r} \binom{r}{j} A^{2j} B^{2(r-j)}$$

Hence

$$\langle \Phi_0^* | (\bar{a}^*_{1,\mathbf{k}} \bar{a}_{1,\mathbf{k}})^r | \Phi_0 \rangle = \frac{1}{2^r} \sum_{j=0}^{r} \binom{r}{j} \langle \Phi_0^* | A^{2j} B^{2(r-j)} | \Phi_0 \rangle \tag{7}$$

To evaluate (7) we need the following integral.

$$\begin{aligned} \int_{-\infty}^{\infty} x^{2n} \exp(-ax^2 + b) &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n a^n} \sqrt{\frac{\pi}{a}} \exp(b) \\ &= (2n-1)!! \, \frac{1}{2^n a^n} \sqrt{\frac{\pi}{a}} \exp(b) \end{aligned}$$

Based on the above integral, define the following function F. (The $\sqrt{\pi/a}$ factor is left out because it gets cancelled by the normalization constant K.)

$$F(n) = (2n-1)!! \left(\frac{\pi \hbar}{kc} \right)^n$$

Note that

$$F(j)F(r-j) = (2j-1)!!\,(2r-2j-1)!!\left(\frac{\pi\hbar}{kc}\right)^j\left(\frac{\pi\hbar}{kc}\right)^{r-j}$$

Hence equation (7) can be written as

$$\langle\Phi_0^*|(\bar{a}_{1,\mathbf{k}}^*\bar{a}_{1,\mathbf{k}})^r|\Phi_0\rangle = \frac{1}{2^r}\left(\frac{\pi\hbar}{kc}\right)^r\sum_{j=0}^{r}\binom{r}{j}(2j-1)!!\,(2r-2j-1)!!$$

For $\mathbf{q} \neq \mathbf{k}$

$$\begin{aligned}
&\left(\frac{A^2+B^2}{2}\right)^r\left(\frac{C^2+D^2}{2}\right)^s \\
&\quad = \left(\frac{1}{2^r}\sum_{j=0}^{r}\binom{r}{j}A^{2j}B^{2(r-j)}\right)\left(\frac{1}{2^s}\sum_{k=0}^{s}\binom{s}{k}C^{2k}D^{2(r-k)}\right) \\
&\quad = \frac{1}{2^{r+s}}\sum_{j=0}^{r}\sum_{k=0}^{s}\binom{r}{j}\binom{s}{k}A^{2j}B^{2(r-j)}C^{2k}D^{2(s-k)}
\end{aligned}$$

Hence

$$\begin{aligned}
&\langle\Phi_0^*|(\bar{a}_{1,\mathbf{k}}^*\bar{a}_{1,\mathbf{k}})^r(\bar{a}_{1,\mathbf{q}}^*\bar{a}_{1,\mathbf{q}})^s|\Phi_0\rangle = \\
&\qquad\qquad \frac{1}{2^{r+s}}\sum_{j=0}^{r}\sum_{k=0}^{s}\binom{r}{j}\binom{s}{k}\langle\Phi_0^*|A^{2j}B^{2(r-j)}C^{2k}D^{2(s-k)}|\Phi_0\rangle
\end{aligned}$$

We need to add the following to our table of integrals.

$$\begin{aligned}
\int_{-\infty}^{\infty} x^{2n}\exp(-ax^2+b) &= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n a^n}\sqrt{\frac{\pi}{a}}\exp(b) \\
&= (2n-1)!!\frac{1}{2^n a^n}\sqrt{\frac{\pi}{a}}\exp(b)
\end{aligned}$$

Based on the above integral, define the following function F. (The $\sqrt{\pi/a}$ factor is left out because it gets cancelled by the normalization constant K.)

$$F(n) = (2n-1)!!\left(\frac{\pi\hbar}{kc}\right)^n$$

Note that

$$F(j)F(r-j) = (2j-1)!!(2r-2j-1)!! \left(\frac{\pi\hbar}{kc}\right)^j \left(\frac{\pi\hbar}{kc}\right)^{r-j}$$

Hence equation (7) can be written as

$$\langle \Phi_0^* | (\bar{a}^*_{1,\mathbf{k}} \bar{a}_{1,\mathbf{k}})^r | \Phi_0 \rangle = \frac{1}{2^r} \left(\frac{\pi\hbar}{kc}\right)^r \sum_{j=0}^{r} \binom{r}{j} (2j-1)!!(2r-2j-1)!!$$

(9-7 cont'd) Show that the expectation of $(\bar{a}_{1,\mathbf{k}})^2$ or $(\bar{a}^*_{1,\mathbf{k}})^2$ vanishes.

We have

$$(\bar{a}_{1,\mathbf{k}})^2 = \frac{A^2 - B^2}{2} - iAB \qquad (\bar{a}^*_{1,\mathbf{k}})^2 = \frac{A^2 - B^2}{2} + iAB$$

The integrals of A^2 and $-B^2$ cancel each other. The integral of AB vanishes by integral (2).

(9-7 cont'd) Show that the expectation of the product of any odd number of $\bar{a}$'s is zero and that you can compute the expectation value of any product of $\bar{a}$'s or $\bar{a}^*$'s for the vacuum state.

Isn't the expectation of *any* number of $\bar{a}$'s zero? It is shown above that the expectation of $(\bar{a}_{1,\mathbf{k}})^2$ vanishes.

FIXME: The result $\hbar/kc$ is wrong by a factor of 2 everywhere it appears. Should be $\hbar/2kc$.