Harmonic oscillator propagator

Consider the harmonic oscillator eigenstate

$$\psi_n(x,t) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right) \exp\left(-\frac{m\omega x^2}{2\hbar} - i\left(n+\tfrac{1}{2}\right)\omega t\right)$$

and the harmonic oscillator propagator

$$K(x_b,t_b,x_a,t_a) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega T)}\right)^{\frac{1}{2}} \exp\left[\frac{im\omega}{2\hbar\sin(\omega T)}\left(x_a^2\cos(\omega T) - 2x_a x_b + x_b^2\cos(\omega T)\right)\right]$$

where $T = t_b - t_a$.

We should have

$$\psi_n(x_b,t_b) = \int_{-\infty}^{\infty} K(x_b,t_b,x_a,t_a)\psi_n(x_a,t_a)\,dx_a$$

Try for $n = 1$.

$$\psi_1(x_a,t_a) = \sqrt{2}\left(\frac{m^2\omega^3}{\pi\hbar^3}\right)^{\frac{1}{4}} x_a \exp\left(-\frac{m\omega x_a^2}{2\hbar} - \tfrac{3}{2}i\omega t_a\right)$$

By the identity

$$\int_{-\infty}^{\infty} y\exp\left(-ay^2+by\right)\,dy = \frac{\sqrt{\pi}}{2}\frac{b}{a^{3/2}}\exp\left(\frac{b^2}{4a}\right)$$

the path integral is

$$\begin{aligned} I = \int_{-\infty}^{\infty} K(x_b,t_b,x_a,t_a)\psi_1(x_a,t_a)\,dx_a &= \frac{1}{2}\left(-\frac{m^9\omega^9}{\pi\hbar^9}\right)^{\frac{1}{4}}\left(\frac{m\omega}{2\hbar}\sin(\omega T) - \frac{im\omega}{2\hbar}\cos(\omega T)\right)^{-\frac{3}{2}} \\ &\times x_b \exp\left(-\frac{m\omega x_b^2}{2\hbar\left(\sin(\omega T)^2 - i\sin(\omega T)\cos(\omega T)\right)} + \frac{im\omega x_b^2\cos(\omega T)}{2\hbar\sin(\omega T)} - \tfrac{3}{2}i\omega t_a\right) \end{aligned}$$

Substitute $t_b - T$ for t_a and substitute $-\pi/(2\omega)$ for T. The result is

$$I = \psi_1(x_b,t_b)$$

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