

9-9. For a complicated system moving nonrelativistically

$$(j_{1,\mathbf{k}})_{NM} = \sum_i (q_i \mathbf{e}_1 \cdot \dot{\mathbf{x}}_i \exp(-i\mathbf{k} \cdot \mathbf{x}_i))_{NM}$$

where \mathbf{e}_1 is a unit vector in the direction of the polarization of the light and q_i and \mathbf{x}_i are the charge and position of the i th particle. Assume the wavelength of the light is very large compared with the size of the atom, i.e., that the absolute square of the wave function describing the position of the i th electron falls to zero over a distance small compared with $1/k$. Show that we can then approximate $\exp(-i\mathbf{k} \cdot \mathbf{x}_i)$ by unity and write the matrix element as

$$(j_{1,\mathbf{k}})_{NM} = i\omega \mathbf{e}_1 \cdot \boldsymbol{\mu}_{NM} \quad (9.57)$$

where

$$\boldsymbol{\mu}_{NM} = \sum_i (q_i \mathbf{x}_i)_{NM} \quad (9.58)$$

The function $\boldsymbol{\mu}_{NM}$ is called the *matrix element of the electric dipole moment* of the atom, and the approximation used to derive equation (9.57) is called the *dipole approximation*. Show that the probability to emit light in any direction per unit time is

$$\frac{dP}{dt} = \frac{4\omega^3}{3\hbar c^3} |\boldsymbol{\mu}_{NM}|^2 \quad (9.59)$$

(Integrate equation (9.54) over all directions, remembering that \mathbf{e}_1 is perpendicular to \mathbf{k} and that there are two possible directions of polarization.)

$$\frac{dP}{dt} = \frac{\omega}{2\pi\hbar c^3} |j_{1,\mathbf{q}}|_{NM}^2 d\Omega \quad (9.54)$$

Adapted from problem 7-12,

$$\dot{\mathbf{x}}_i = -\frac{i}{\hbar}(\mathbf{x}_i \hat{H} - \hat{H} \mathbf{x}_i) = i\omega \mathbf{x}_i$$

The squared magnitude of $i\omega$ in (9.57) is ω^2 . It follows that

$$\int \frac{\omega}{2\pi\hbar c^3} (|j_{1,\mathbf{k}}|_{NM}^2 + |j_{2,\mathbf{k}}|_{NM}^2) d\Omega = 2 |\boldsymbol{\mu}_{NM}|^2 \int_0^{2\pi} \int_0^\pi \frac{\omega^3}{2\pi\hbar c^3} \sin \theta d\theta d\phi$$

From the following integrals

$$\int_0^\pi \sin \theta \, d\theta = 2 \qquad \int_0^{2\pi} d\phi = 2\pi$$

the combined multiplier is 4π hence

$$\frac{dP}{dt} = \frac{4\omega^3}{\hbar c^3} |\boldsymbol{\mu}_{NM}|^2$$