Two spins

The spin state $|s\rangle$ for two spins is a unit vector in $\mathbb{C}^4$.

$$|s\rangle = \begin{pmatrix} c_{++} \\ c_{+-} \\ c_{-+} \\ c_{--} \end{pmatrix}, \quad |c_{++}|^2 + |c_{+-}|^2 + |c_{-+}|^2 + |c_{--}|^2 = 1$$

Spin measurement probabilities are the transition probabilities from $|s\rangle$ to an eigenstate.

For spin measurements in the z direction we have

$$\begin{aligned}
\Pr\left(S_{1z} = +\tfrac{\hbar}{2} \text{ and } S_{2z} = +\tfrac{\hbar}{2}\right) &= |\langle z_{++}|s\rangle|^2 = |c_{++}|^2 \\
\Pr\left(S_{1z} = +\tfrac{\hbar}{2} \text{ and } S_{2z} = -\tfrac{\hbar}{2}\right) &= |\langle z_{+-}|s\rangle|^2 = |c_{+-}|^2 \\
\Pr\left(S_{1z} = -\tfrac{\hbar}{2} \text{ and } S_{2z} = +\tfrac{\hbar}{2}\right) &= |\langle z_{-+}|s\rangle|^2 = |c_{-+}|^2 \\
\Pr\left(S_{1z} = -\tfrac{\hbar}{2} \text{ and } S_{2z} = -\tfrac{\hbar}{2}\right) &= |\langle z_{--}|s\rangle|^2 = |c_{--}|^2
\end{aligned}$$

where the eigenstates are

$$z_{++} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad z_{+-} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad z_{-+} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad z_{--} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Spin operators and expectation values for the first spin ($\otimes$ is kronecker product).

$$S_{1x} = \frac{\hbar}{2}\sigma_x \otimes I = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \langle S_{1x}\rangle = \langle s|S_{1x}|s\rangle$$

$$S_{1y} = \frac{\hbar}{2}\sigma_y \otimes I = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \qquad \langle S_{1y}\rangle = \langle s|S_{1y}|s\rangle$$

$$S_{1z} = \frac{\hbar}{2}\sigma_z \otimes I = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \langle S_{1z}\rangle = \langle s|S_{1z}|s\rangle$$

Spin operators and expectation values for the second spin.

$$S_{2x} = \frac{\hbar}{2} I \otimes \sigma_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \langle S_{2x} \rangle = \langle s | S_{2x} | s \rangle$$

$$S_{2y} = \frac{\hbar}{2} I \otimes \sigma_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \qquad \langle S_{2y} \rangle = \langle s | S_{2y} | s \rangle$$

$$S_{2z} = \frac{\hbar}{2} I \otimes \sigma_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \langle S_{2z} \rangle = \langle s | S_{2z} | s \rangle$$

By definition of expectation value we have for the z direction

$$\langle S_{1z} \rangle = \tfrac{\hbar}{2} \left(|c_{++}|^2 + |c_{+-}|^2 - |c_{-+}|^2 - |c_{--}|^2 \right)$$
$$\langle S_{2z} \rangle = \tfrac{\hbar}{2} \left(|c_{++}|^2 - |c_{+-}|^2 + |c_{-+}|^2 - |c_{--}|^2 \right)$$

1. Verify spin operators for two spins.

```
sigmax = ((0,1),(1,0))
sigmay = ((0,-i),(i,0))
sigmaz = ((1,0),(0,-1))

I = ((1,0),(0,1))

S1x = 1/2 hbar kronecker(sigmax,I)
S1y = 1/2 hbar kronecker(sigmay,I)
S1z = 1/2 hbar kronecker(sigmaz,I)

S2x = 1/2 hbar kronecker(I,sigmax)
S2y = 1/2 hbar kronecker(I,sigmay)
S2z = 1/2 hbar kronecker(I,sigmaz)

check(S1x == 1/2 hbar ((0,0,1,0),(0,0,0,1),(1,0,0,0),(0,1,0,0)))
check(S1y == 1/2 hbar ((0,0,-i,0),(0,0,0,-i),(i,0,0,0),(0,i,0,0)))
check(S1z == 1/2 hbar ((1,0,0,0),(0,1,0,0),(0,0,-1,0),(0,0,0,-1)))

check(S2x == 1/2 hbar ((0,1,0,0),(1,0,0,0),(0,0,0,1),(0,0,1,0)))
check(S2y == 1/2 hbar ((0,-i,0,0),(i,0,0,0),(0,0,0,-i),(0,0,i,0)))
check(S2z == 1/2 hbar ((1,0,0,0),(0,-1,0,0),(0,0,1,0),(0,0,0,-1)))
```

2. Verify expectation values for the z direction.

```
sigmax = ((0,1),(1,0))
sigmay = ((0,-i),(i,0))
sigmaz = ((1,0),(0,-1))

I = ((1,0),(0,1))

S1x = 1/2 hbar kronecker(sigmax,I)
S1y = 1/2 hbar kronecker(sigmay,I)
S1z = 1/2 hbar kronecker(sigmaz,I)

S2x = 1/2 hbar kronecker(I,sigmax)
S2y = 1/2 hbar kronecker(I,sigmay)
S2z = 1/2 hbar kronecker(I,sigmaz)

c1 = x1 + i y1
c2 = x2 + i y2
c3 = x3 + i y3
c4 = x4 + i y4

s = (c1,c2,c3,c4)

check(dot(conj(s),S1z,s) ==
1/2 hbar (conj(c1) c1 + conj(c2) c2 - conj(c3) c3 - conj(c4) c4))

check(dot(conj(s),S2z,s) ==
1/2 hbar (conj(c1) c1 - conj(c2) c2 + conj(c3) c3 - conj(c4) c4))
```