

A quantum computer can be simulated by applying rotations to a unit vector  $u \in \mathbb{C}^{2^n}$  where  $\mathbb{C}$  is the set of complex numbers and  $n$  is the number of qubits. The dimension is  $2^n$  because a register with  $n$  qubits has  $2^n$  eigenstates. (Recall that an eigenstate is the output of a quantum computer.) Quantum operations are “rotations” because they preserve  $|u| = 1$ . Mathematically, a rotation of  $u$  is equivalent to the product  $Ru$  where  $R$  is a  $2^n \times 2^n$  matrix.

Eigenstates  $|j\rangle$  are represented by the following vectors. (Each vector has  $2^n$  elements.)

$$\begin{aligned} |0\rangle &= (1, 0, 0, \dots, 0) \\ |1\rangle &= (0, 1, 0, \dots, 0) \\ |2\rangle &= (0, 0, 1, \dots, 0) \\ &\vdots \\ |2^n - 1\rangle &= (0, 0, 0, \dots, 1) \end{aligned}$$

A quantum computer algorithm is a sequence of rotations applied to the initial state  $|0\rangle$ . (The sequence could be combined into a single rotation by associativity of matrix multiplication.) Let  $\psi_f$  be the final state of the quantum computer after all the rotations have been applied. Like any other state,  $\psi_f$  is a linear combination of eigenstates.

$$\psi_f = \sum_{j=0}^{2^n-1} c_j |j\rangle, \quad c_j \in \mathbb{C}, \quad |\psi_f| = 1$$

The last step is to measure  $\psi_f$  and get a result. Measurement rotates  $\psi_f$  to an eigenstate  $|j\rangle$ . The measurement result is  $|j\rangle$ . The probability  $P_j$  of getting a specific result  $|j\rangle$  is

$$P_j = |c_j|^2 = c_j c_j^*$$

Note that if  $\psi_f$  is already an eigenstate then no rotation occurs. (The probability of observing a different eigenstate is zero.) Since the measurement result is always an eigenstate, the coefficients  $c_j$  cannot be observed. However, the same calculation can be run multiple times to obtain a probability distribution of results. The probability distribution is an estimate of  $|c_j|^2$  for each  $|j\rangle$  in  $\psi_f$ .

Unlike a real quantum computer, in a simulation the final state  $\psi_f$ , or any other state, is available for inspection. Hence there is no need to simulate the measurement process. The probability distribution of the result can be computed directly as

$$P = \psi_f \psi_f^*$$

where  $\psi_f \psi_f^*$  is the Hadamard (element-wise) product of vector  $\psi_f$  and its complex conjugate. Result  $P$  is a vector such that  $P_j$  is the probability of eigenstate  $|j\rangle$  and

$$\sum_{j=0}^{2^n-1} P_j = 1$$

Note: Eigenmath index numbering begins with 1 hence  $P[1]$  is the probability of  $|0\rangle$ ,  $P[2]$  is the probability of  $|1\rangle$ , etc.

The Eigenmath function `rotate( $u, s, k, \dots$ )` rotates vector  $u$  and returns the result. Vector  $u$  is required to have  $2^n$  elements where  $n$  is an integer from 1 to 15. Arguments  $s, k, \dots$  are a sequence of rotation codes where  $s$  is an upper case letter and  $k$  is a qubit number from 0 to  $n - 1$ . Rotations are evaluated from left to right. The available rotation codes are

|              |   |
|--------------|---|
| $C, k$       | Control prefix  |
| $H, k$       | Hadamard  |
| $P, k, \phi$ | Phase modifier (use $\phi = \frac{1}{4}\pi$ for $T$ rotation) |
| $Q, k$       | Quantum Fourier transform                                     |
| $V, k$       | Inverse quantum Fourier transform                             |
| $W, k, j$    | Swap bits   |
| $X, k$       | Pauli X   |
| $Y, k$       | Pauli Y   |
| $Z, k$       | Pauli Z   |

Control prefix  $C, k$  modifies the next rotation code so that it is a controlled rotation with  $k$  as the control qubit. Use two or more prefixes to specify multiple control qubits. For example,  $C, k, C, j, X, m$  is a Toffoli rotation. Fourier rotations  $Q, k$  and  $V, k$  are applied to qubits 0 through  $k$ . ( $Q$  and  $V$  ignore any control prefix.)

List of `rotate( $u, s, k, \dots$ )` error codes:

- 1 Argument  $u$  is not a vector or does not have  $2^n$  elements where  $n = 1, 2, \dots, 15$ .
- 2 Unexpected end of argument list (i.e., missing argument).
- 3 Bit number format error or range error.
- 4 Unknown rotation code.

Example: Verify the following truth table for quantum operator CNOT where qubit 0 is the control and qubit 1 is the target. (Target is inverted when control is set.)

| Target | Control | Output |
|--------|---------|--------|
| 0      | 0       | 00     |
| 0      | 1       | 11     |
| 1      | 0       | 10     |
| 1      | 1       | 01     |

```
U(psi) = rotate(psi,C,0,X,1) -- CNOT, control 0, target 1
```

```
ket00 = (1,0,0,0)
ket01 = (0,1,0,0)
ket10 = (0,0,1,0)
ket11 = (0,0,0,1)
```

```
U(ket00) == ket00
U(ket01) == ket11
U(ket10) == ket10
U(ket11) == ket01
```

Here are some useful Eigenmath code snippets for setting up a simulation and computing the result.

1. Initialize  $\psi = |0\rangle$ .

```
n = 4          -- number of qubits (example)
N = 2^n        -- number of eigenstates
psi = zero(N)
psi[1] = 1
```

2. Compute the probability distribution for state  $\psi$ .

```
P = psi conj(psi)
```

Hence

$P[1]$  = probability that  $|0\rangle$  will be the result

$P[2]$  = probability that  $|1\rangle$  will be the result

$P[3]$  = probability that  $|2\rangle$  will be the result

$\vdots$

$P[N]$  = probability that  $|N - 1\rangle$  will be the result

3. (Only for macOS) Draw a probability distribution.

```
xrange = (0,N)
yrange = (0,1)
draw(P[ceiling(x)],x)
```

4. Compute an expectation value.

```
sum(k,1,N, (k - 1) P[k])
```

5. Make the high order qubit “don’t care.”

```
for(k,1,N/2, P[k] = P[k] + P[k + N/2])
```

Hence for  $N = 16$

$P[1]$  = probability that the result will be  $|0\rangle$  or  $|8\rangle$

$P[2]$  = probability that the result will be  $|1\rangle$  or  $|9\rangle$

$P[3]$  = probability that the result will be  $|2\rangle$  or  $|10\rangle$

$\vdots$

$P[8]$  = probability that the result will be  $|7\rangle$  or  $|15\rangle$