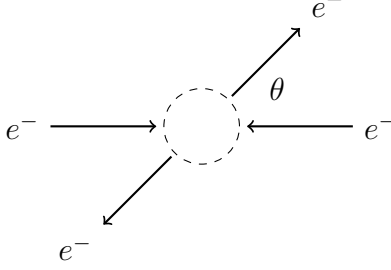
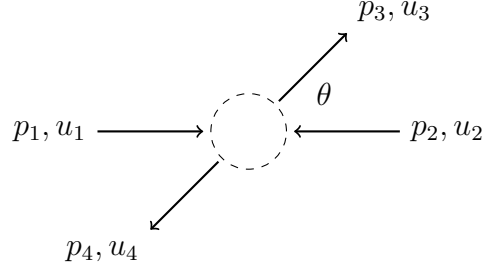


# MOLLER SCATTERING

Moller scattering is the interaction of two electrons.



Here is the same diagram with momentum and spinor labels.



In center of mass coordinates the momentum vectors are

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \quad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} \quad p_3 = \begin{pmatrix} E \\ p \sin \theta \cos \phi \\ p \sin \theta \sin \phi \\ p \cos \theta \end{pmatrix} \quad p_4 = \begin{pmatrix} E \\ -p \sin \theta \cos \phi \\ -p \sin \theta \sin \phi \\ -p \cos \theta \end{pmatrix}$$

where  $p = \sqrt{E^2 - m^2}$ . The spinors are

$$\begin{aligned} u_{11} &= \begin{pmatrix} E + m \\ 0 \\ p \\ 0 \end{pmatrix} & u_{21} &= \begin{pmatrix} E + m \\ 0 \\ -p \\ 0 \end{pmatrix} & u_{31} &= \begin{pmatrix} E + m \\ 0 \\ p_3^z \\ p_3^x + ip_3^y \end{pmatrix} & u_{41} &= \begin{pmatrix} E + m \\ 0 \\ p_4^z \\ p_4^x + ip_4^y \end{pmatrix} \\ u_{12} &= \begin{pmatrix} 0 \\ E + m \\ 0 \\ -p \end{pmatrix} & u_{22} &= \begin{pmatrix} 0 \\ E + m \\ 0 \\ p \end{pmatrix} & u_{32} &= \begin{pmatrix} 0 \\ E + m \\ p_3^x - ip_3^y \\ -p_3^z \end{pmatrix} & u_{42} &= \begin{pmatrix} 0 \\ E + m \\ p_4^x - ip_4^y \\ -p_4^z \end{pmatrix} \end{aligned}$$

The last digit in a spinor subscript is 1 for spin up and 2 for spin down. Note that the spinors are not individually normalized. Instead, a combined spinor normalization constant  $N = (E + m)^4$  will be used where needed.

This is the probability density for Moller scattering. The formula is from Feynman diagrams.

$$|\mathcal{M}(s_1, s_2, s_3, s_4)|^2 = \frac{e^4}{N} \left| \frac{1}{t} (\bar{u}_3 \gamma^\mu u_1) (\bar{u}_4 \gamma_\mu u_2) - \frac{1}{u} (\bar{u}_4 \gamma^\nu u_1) (\bar{u}_3 \gamma_\nu u_2) \right|^2$$

Symbol  $s_j$  selects the spin (up or down) of spinor  $j$ . Symbol  $e$  is electron charge. Symbols  $t$  and  $u$  are Mandelstam variables  $t = (p_1 - p_3)^2$  and  $u = (p_1 - p_4)^2$ .

Let

$$a_1 = (\bar{u}_3 \gamma^\mu u_1) (\bar{u}_4 \gamma_\mu u_2) \quad a_2 = (\bar{u}_4 \gamma^\nu u_1) (\bar{u}_3 \gamma_\nu u_2)$$

Then

$$\begin{aligned} |\mathcal{M}(s_1, s_2, s_3, s_4)|^2 &= \frac{e^4}{N} \left| \frac{a_1}{t} - \frac{a_2}{u} \right|^2 \\ &= \frac{e^4}{N} \left( \frac{a_1}{t} - \frac{a_2}{u} \right) \left( \frac{a_1}{t} - \frac{a_2}{u} \right)^* \\ &= \frac{e^4}{N} \left( \frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{tu} - \frac{a_1^* a_2}{tu} + \frac{a_2 a_2^*}{u^2} \right) \end{aligned}$$

The expected probability density  $\langle |\mathcal{M}|^2 \rangle$  is computed by summing  $|\mathcal{M}|^2$  over all spin states and dividing by the number of inbound states. There are four inbound states.

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{1}{4} \sum_{s_1=1}^2 \sum_{s_2=1}^2 \sum_{s_3=1}^2 \sum_{s_4=1}^2 |\mathcal{M}(s_1, s_2, s_3, s_4)|^2 \\ &= \frac{e^4}{4} \sum_{s_1=1}^2 \sum_{s_2=1}^2 \sum_{s_3=1}^2 \sum_{s_4=1}^2 \frac{1}{N} \left( \frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{tu} - \frac{a_1^* a_2}{tu} + \frac{a_2 a_2^*}{u^2} \right) \end{aligned}$$

Use the Casimir trick to replace sums over spins with matrix products.

$$\begin{aligned} f_{11} &= \frac{1}{N} \sum_{\text{spins}} a_1 a_1^* = \text{Tr} \left( (\not{p}_3 + m) \gamma^\mu (\not{p}_1 + m) \gamma^\nu \right) \text{Tr} \left( (\not{p}_4 + m) \gamma_\mu (\not{p}_2 + m) \gamma_\nu \right) \\ f_{12} &= \frac{1}{N} \sum_{\text{spins}} a_1 a_2^* = \text{Tr} \left( (\not{p}_3 + m) \gamma^\mu (\not{p}_1 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\mu (\not{p}_2 + m) \gamma_\nu \right) \\ f_{22} &= \frac{1}{N} \sum_{\text{spins}} a_2 a_2^* = \text{Tr} \left( (\not{p}_4 + m) \gamma^\mu (\not{p}_1 + m) \gamma^\nu \right) \text{Tr} \left( (\not{p}_3 + m) \gamma_\mu (\not{p}_2 + m) \gamma_\nu \right) \end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left( \frac{f_{11}}{t^2} - \frac{f_{12}}{tu} - \frac{f_{12}^*}{tu} + \frac{f_{22}}{u^2} \right)$$

Run “moller-scattering-1.txt” to verify the Casimir trick.

These formulas compute probability densities from dot products. Recall that  $a \cdot b = a^\mu g_{\mu\nu} b^\nu$ .

$$\begin{aligned} f_{11} &= 32(p_1 \cdot p_2)(p_3 \cdot p_4) + 32(p_1 \cdot p_4)(p_2 \cdot p_3) - 32m^2(p_1 \cdot p_3) - 32m^2(p_2 \cdot p_4) + 64m^4 \\ f_{12} &= -32(p_1 \cdot p_2)(p_3 \cdot p_4) + 16m^2(p_1 \cdot p_2) + 16m^2(p_1 \cdot p_3) + 16m^2(p_1 \cdot p_4) \\ &\quad + 16m^2(p_2 \cdot p_3) + 16m^2(p_2 \cdot p_4) + 16m^2(p_3 \cdot p_4) - 32m^4 \\ f_{22} &= 32(p_1 \cdot p_2)(p_3 \cdot p_4) + 32(p_1 \cdot p_3)(p_2 \cdot p_4) - 32m^2(p_1 \cdot p_4) - 32m^2(p_2 \cdot p_3) + 64m^4 \end{aligned}$$

In Mandelstam variables  $s = (p_1 + p_2)^2$ ,  $t = (p_1 - p_3)^2$ ,  $u = (p_1 - p_4)^2$  the formulas are

$$\begin{aligned} f_{11} &= 8s^2 + 8u^2 - 64sm^2 - 64um^2 + 192m^4 \\ f_{12} &= -8s^2 + 64sm^2 - 96m^4 \\ f_{22} &= 8s^2 + 8t^2 - 64sm^2 - 64tm^2 + 192m^4 \end{aligned}$$

When  $E \gg m$  a useful approximation is to set  $m = 0$  and obtain

$$\begin{aligned} f_{11} &= 8s^2 + 8u^2 \\ f_{12} &= -8s^2 \\ f_{22} &= 8s^2 + 8t^2 \end{aligned}$$

For  $m = 0$  the Mandelstam variables are

$$\begin{aligned} s &= 4E^2 \\ t &= -2E^2(1 - \cos \theta) = -4E^2 \sin^2(\theta/2) \\ u &= -2E^2(1 + \cos \theta) = -4E^2 \cos^2(\theta/2) \end{aligned}$$

The corresponding expected probability density is

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left( \frac{8s^2 + 8u^2}{t^2} + \frac{16s^2}{tu} + \frac{8s^2 + 8t^2}{u^2} \right) \\ &= 2e^4 \left( \frac{s^2 + u^2}{t^2} + \frac{2s^2}{tu} + \frac{s^2 + t^2}{u^2} \right) \\ &= 2e^4 \left( \frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{8}{\sin^2 \theta} + \frac{1 + \sin^4(\theta/2)}{\cos^4(\theta/2)} \right) \\ &= \frac{4e^4 (3 + \cos^2 \theta)^2}{\sin^4 \theta} \end{aligned}$$

Run “moller-scattering-2.txt” to verify the formulas on this page.

In the center of mass frame the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{64\pi^2 s} = \frac{e^4}{64\pi^2 E^2} \frac{(3 + \cos^2 \theta)^2}{\sin^4 \theta}$$

Substituting  $e^2 = 4\pi\alpha$  yields

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2} \frac{(3 + \cos^2 \theta)^2}{\sin^4 \theta}$$

We can integrate  $d\sigma$  to obtain a normalized probability density function. Recall that  $d\Omega = \sin \theta d\theta d\phi$  hence

$$d\sigma = \frac{\alpha^2}{4E^2} \frac{(3 + \cos^2 \theta)^2}{\sin^4 \theta} \sin \theta d\theta d\phi$$

Let  $I(\xi)$  be the following definite integral.

$$\begin{aligned} I(\xi) &= \frac{4E^2}{2\pi\alpha^2} \int_0^{2\pi} \int_a^\xi d\sigma \\ &= \int_a^\xi \frac{(3 + \cos^2 \theta)^2}{\sin^4 \theta} \sin \theta d\theta \\ &= \left( -\cos \theta - \frac{8 \cos \theta}{\sin^2 \theta} \right) \Big|_a^\xi \\ &= \cos a + \frac{8 \cos a}{\sin^2 a} - \cos \xi - \frac{8 \cos \xi}{\sin^2 \xi}, \quad a \leq \xi \leq \pi - a \end{aligned}$$

Support is limited to  $a > 0$  because  $I(0)$  and  $I(\pi)$  are undefined.

Let  $C$  be the normalization constant  $C = I(\pi - a)$ . Then the cumulative distribution function  $F(\theta)$  is

$$F(\theta) = C^{-1} I(\theta), \quad a \leq \theta \leq \pi - a$$

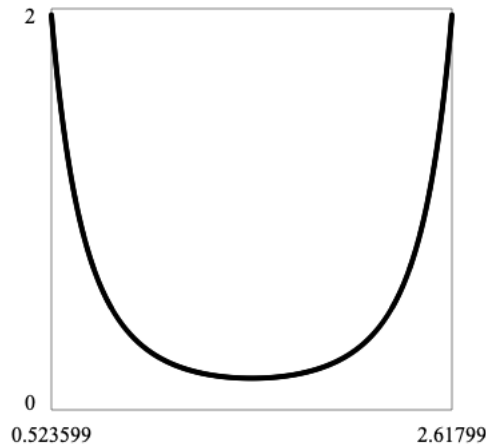
The probability of observing scattering events in the interval  $\theta_1$  to  $\theta_2$  can now be computed.

$$P(\theta_1 \leq \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

Probability density function  $f(\theta)$  is the derivative of  $F(\theta)$ .

$$f(\theta) = \frac{dF(\theta)}{d\theta} = C^{-1} \frac{dI(\theta)}{d\theta} = C^{-1} \frac{(3 + \cos^2 \theta)^2}{\sin^3 \theta}$$

Run “moller-scattering-3.txt” to draw a graph of  $f(\theta)$  for  $a = \pi/6 = 30^\circ$ .



The following table shows the probability distribution for  $30^\circ$  bins ( $a = \pi/6 = 30^\circ$ ).

$\theta_1$	$\theta_2$	$P(\theta_1 \leq \theta \leq \theta_2)$
$0^\circ$	$30^\circ$	—
$30^\circ$	$60^\circ$	0.40
$60^\circ$	$90^\circ$	0.10
$90^\circ$	$120^\circ$	0.10
$120^\circ$	$150^\circ$	0.40
$150^\circ$	$180^\circ$	—

Here are a few notes about how the scripts work.

In component notation, the trace operators of the Casimir trick become sums over a repeated index, in this case  $\alpha$ .

$$\begin{aligned} f_{11} &= \left( (\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \right) \left( (\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \right) \\ f_{12} &= (\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_2 + m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha \\ f_{22} &= \left( (\not{p}_4 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \right) \left( (\not{p}_3 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \right) \end{aligned}$$

To convert the above formulas to Eigenmath code, the  $\gamma$  tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply  $\gamma^\mu$  by the metric tensor to lower the index.

$$\begin{aligned} \gamma^{\beta\mu}{}_\rho &\rightarrow \text{gammaT} = \text{transpose}(\text{gamma}) \\ \gamma^\beta{}_{\mu\rho} &\rightarrow \text{gammaL} = \text{transpose}(\text{dot}(\text{gmunu}, \text{gamma})) \end{aligned}$$

Define the following  $4 \times 4$  matrices.

$$\begin{aligned} (\not{p}_1 + m) &\rightarrow \text{X1} = \text{pslash1} + \text{m I} \\ (\not{p}_2 + m) &\rightarrow \text{X2} = \text{pslash2} + \text{m I} \\ (\not{p}_3 + m) &\rightarrow \text{X3} = \text{pslash3} + \text{m I} \\ (\not{p}_4 + m) &\rightarrow \text{X4} = \text{pslash4} + \text{m I} \end{aligned}$$

Then for  $f_{11}$  we have the following Eigenmath code. The contract function sums over  $\alpha$ .

$$\begin{aligned} (\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha &\rightarrow \text{T1} = \text{contract}(\text{dot}(\text{X3}, \text{gammaT}, \text{X1}, \text{gammaT}), 1, 4) \\ (\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha &\rightarrow \text{T2} = \text{contract}(\text{dot}(\text{X4}, \text{gammaL}, \text{X2}, \text{gammaL}), 1, 4) \end{aligned}$$

Next, multiply then sum over repeated indices. The dot function sums over  $\nu$  then the contract function sums over  $\mu$ . The transpose makes the  $\nu$  indices adjacent as required by the dot function.

$$f_{11} = \text{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu) \text{Tr}(\cdots \gamma_\mu \cdots \gamma_\nu) \rightarrow \text{contract}(\text{dot}(\text{T1}, \text{transpose}(\text{T2})))$$

Follow suit for  $f_{22}$ .

$$\begin{aligned} (\not{p}_4 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha &\rightarrow \text{T1} = \text{contract}(\text{dot}(\text{X4}, \text{gammaT}, \text{X1}, \text{gammaT}), 1, 4) \\ (\not{p}_3 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha &\rightarrow \text{T2} = \text{contract}(\text{dot}(\text{X3}, \text{gammaL}, \text{X2}, \text{gammaL}), 1, 4) \end{aligned}$$

Then

$$f_{22} = \text{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu) \text{Tr}(\cdots \gamma_\mu \cdots \gamma_\nu) \rightarrow \text{contract}(\text{dot}(\text{T1}, \text{transpose}(\text{T2})))$$

The calculation of  $f_{12}$  begins with

$$\begin{aligned} (\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_2 + m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha \\ \rightarrow \text{T} = \text{contract}(\text{dot}(\text{X3}, \text{gammaT}, \text{X1}, \text{gammaT}, \text{X4}, \text{gammaL}, \text{X2}, \text{gammaL}), 1, 6) \end{aligned}$$

Then sum over repeated indices  $\mu$  and  $\nu$ .

$$f_{12} = \text{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu \cdots \gamma_\mu \cdots \gamma_\nu) \rightarrow \text{contract}(\text{contract}(\text{T}, 1, 3))$$