8-4. Show that the ground-state wave function for the Lagrangian of equation (8.78) can be written

$$\Phi_0 = A \exp\left(-\frac{1}{2\hbar}\sum_{\alpha=1}^{N-1}\omega_\alpha Q_\alpha^* Q_\alpha\right) \tag{8.83}$$

(where A is a constant) by starting with the wave function in terms of the real variables Q_α^c and Q_α^s.

$$L = \frac{1}{2}\sum_{\alpha=0}^{N-1}\left(\dot{Q}_\alpha^*\dot{Q}_\alpha - \omega_\alpha^2 Q_\alpha^* Q_\alpha\right) \tag{8.78}$$

$$Q_\alpha^c = \frac{1}{\sqrt{2}}(Q_\alpha + Q_\alpha^*) \tag{8.79}$$

$$Q_\alpha^s = \frac{i}{\sqrt{2}}(Q_\alpha - Q_\alpha^*) \tag{8.80}$$

Rewrite (8.78) as

$$L = \frac{1}{2}\dot{Q}_0^*\dot{Q}_0 - \frac{1}{2}\omega_0^2 Q_0^* Q_0 + \frac{1}{2}\sum_{\alpha=1}^{N-1}\left(\dot{Q}_\alpha^*\dot{Q}_\alpha - \omega_\alpha^2 Q_\alpha^* Q_\alpha\right)$$

Note that

$$\begin{aligned} Q_\alpha^* Q_\alpha &= \frac{1}{2}(Q_\alpha^c + iQ_\alpha^s)(Q_\alpha^c - iQ_\alpha^s) \\ &= \frac{1}{2}(Q_\alpha^c)^2 + \frac{1}{2}(Q_\alpha^s)^2 \end{aligned}$$

and, because Q_0 is real,

$$Q_0^s = 0$$

Hence

$$L = \frac{1}{4}(\dot{Q}_0^c)^2 - \frac{1}{4}\omega^2(Q_0^c)^2 + \frac{1}{4}\sum_{\alpha=1}^{N-1}\left((\dot{Q}_\alpha^c)^2 + (\dot{Q}_\alpha^s)^2 - \omega_\alpha^2(Q_\alpha^c)^2 - \omega_\alpha^2(Q_\alpha^s)^2\right)$$

For N odd the summation interval can be halved because $Q_\alpha = Q_{N-\alpha}$.

$$L = \frac{1}{4}(\dot{Q}_0^c)^2 - \frac{1}{4}\omega^2(Q_0^c)^2 + \frac{1}{2}\sum_{\alpha=1}^{(N-1)/2}\left((\dot{Q}_\alpha^c)^2 + (\dot{Q}_\alpha^s)^2 - \omega_\alpha^2(Q_\alpha^c)^2 - \omega_\alpha^2(Q_\alpha^s)^2\right)$$

The Q_0 term can be dropped because it is a spatial translation.

$$L = \frac{1}{4}(\dot{Q}_0^c)^2 + \frac{1}{2}\sum_{\alpha=1}^{(N-1)/2}\left((\dot{Q}_\alpha^c)^2 + (\dot{Q}_\alpha^s)^2 - \omega_\alpha^2(Q_\alpha^c)^2 - \omega_\alpha^2(Q_\alpha^s)^2\right)$$