Harmonic oscillator propagator 1

Show that for a harmonic oscillator

$$K(b,a)=\left(\frac{m\omega}{2\pi i\hbar\sin(\omega t)}\right)^{\frac{1}{2}}\exp\left(\frac{im\omega(x_b^2+x_a^2)\cot(\omega t)}{2\hbar}-\frac{im\omega x_b x_a}{\hbar\sin(\omega t)}\right),\quad t=t_b-t_a$$

Start with the amplitude

$$K(b,a)=\langle x_b|\exp\left(-\frac{i\hat{H}t}{\hbar}\right)|x_a\rangle \tag{1}$$

where $\hat{H}$ is the harmonic oscillator Hamiltonian

$$\hat{H}=\frac{\hat{p}^2}{2m}+\frac{m\omega^2\hat{x}^2}{2}$$

This is van Kortryk's identity.

$$\exp\left(-\frac{i\hat{H}t}{\hbar}\right)=\exp\left(-\frac{im\omega\hat{x}^2}{2\hbar}\tan\frac{\omega t}{2}\right)\exp\left(-\frac{i\hat{p}^2\sin(\omega t)}{2m\hbar\omega}\right)\exp\left(-\frac{im\omega\hat{x}^2}{2\hbar}\tan\frac{\omega t}{2}\right) \tag{2}$$

Substitute (2) into (1) to obtain

$$K(b,a)=\langle x_b|\exp\left(-\frac{im\omega\hat{x}^2}{2\hbar}\tan\frac{\omega t}{2}\right)\exp\left(-\frac{i\hat{p}^2\sin(\omega t)}{2m\hbar\omega}\right)\exp\left(-\frac{im\omega\hat{x}^2}{2\hbar}\tan\frac{\omega t}{2}\right)|x_a\rangle$$

Replace operators $\hat{x}^2$ with eigenvalues x_b^2 and x_a^2.

$$K(b,a)=\langle x_b|\exp\left(-\frac{im\omega x_b^2}{2\hbar}\tan\frac{\omega t}{2}\right)\exp\left(-\frac{i\hat{p}^2\sin(\omega t)}{2m\hbar\omega}\right)\exp\left(-\frac{im\omega x_a^2}{2\hbar}\tan\frac{\omega t}{2}\right)|x_a\rangle$$

With the $\hat{x}^2$ operators eliminated, states $\langle x_b|$ and $|x_a\rangle$ can be moved inwards.

$$K(b,a)=\exp\left(-\frac{im\omega x_b^2}{2\hbar}\tan\frac{\omega t}{2}\right)\langle x_b|\exp\left(-\frac{i\hat{p}^2\sin(\omega t)}{2m\hbar\omega}\right)|x_a\rangle\exp\left(-\frac{im\omega x_a^2}{2\hbar}\tan\frac{\omega t}{2}\right)$$

Combine exponentials.

$$K(b,a)=\exp\left(-\frac{im\omega(x_b^2+x_a^2)}{2\hbar}\tan\frac{\omega t}{2}\right)\langle x_b|\exp\left(-\frac{i\hat{p}^2\sin(\omega t)}{2m\hbar\omega}\right)|x_a\rangle$$

Rewrite as

$$K(b,a)=\exp\left(-\frac{im\omega(x_b^2+x_a^2)}{2\hbar}\tan\frac{\omega t}{2}\right)K_0(b,a) \tag{3}$$

where

$$K_0(b,a)=\langle x_b|\exp\left(-\frac{i\hat{p}^2\sin(\omega t)}{2m\hbar\omega}\right)|x_a\rangle$$

Noting that K_0 is a free particle propagator with $t_b - t_a = \sin(\omega t)/\omega$ we have

$$K_0(b,a) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega t)}\right)^{\frac{1}{2}} \exp\left(\frac{im\omega(x_b - x_a)^2}{2\hbar\sin(\omega t)}\right) \tag{4}$$

Substitute (4) into (3) to obtain

$$K(b,a) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega t)}\right)^{\frac{1}{2}} \exp\left(-\frac{im\omega(x_b^2 + x_a^2)}{2\hbar}\tan\frac{\omega t}{2} + \frac{im\omega(x_b - x_a)^2}{2\hbar\sin(\omega t)}\right) \tag{5}$$

Using the identity

$$\cot\alpha = -\tan\frac{\alpha}{2} + \frac{1}{\sin\alpha}$$

rewrite (5) as

$$K(b,a) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega t)}\right)^{\frac{1}{2}} \exp\left(\frac{im\omega(x_b^2 + x_a^2)\cot(\omega t)}{2\hbar} - \frac{im\omega x_b x_a}{\hbar\sin(\omega t)}\right) \tag{6}$$

Equation (6) can also be written as

$$K(b,a) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega t)}\right)^{\frac{1}{2}} \exp\left(\frac{im\omega}{2\hbar\sin(\omega t)}\big((x_b^2 + x_a^2)\cos(\omega t) - 2x_b x_a\big)\right)$$