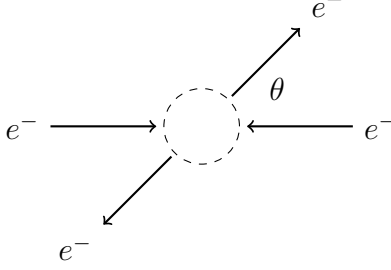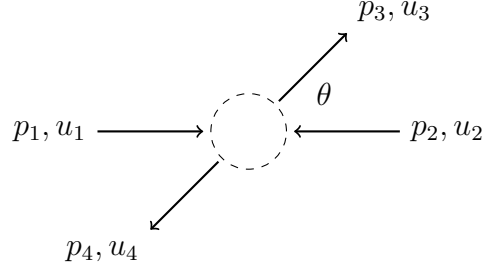Moller scattering is the interaction of two electrons.

Here is the same diagram with momentum and spinor labels.

In center of mass coordinates the momentum vectors are

$$
p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \quad
p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} \quad
p_3 = \begin{pmatrix} E \\ p\sin\theta\cos\phi \\ p\sin\theta\sin\phi \\ p\cos\theta \end{pmatrix} \quad
p_4 = \begin{pmatrix} E \\ -p\sin\theta\cos\phi \\ -p\sin\theta\sin\phi \\ -p\cos\theta \end{pmatrix}
$$

where $p = \sqrt{E^2 - m^2}$. The spinors are

$$
u_{11} = \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix} \quad
u_{21} = \begin{pmatrix} E+m \\ 0 \\ -p \\ 0 \end{pmatrix} \quad
u_{31} = \begin{pmatrix} E+m \\ 0 \\ p_3^z \\ p_3^x + ip_3^y \end{pmatrix} \quad
u_{41} = \begin{pmatrix} E+m \\ 0 \\ p_4^z \\ p_4^x + ip_4^y \end{pmatrix}
$$

$$
u_{12} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix} \quad
u_{22} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ p \end{pmatrix} \quad
u_{32} = \begin{pmatrix} 0 \\ E+m \\ p_3^x - ip_3^y \\ -p_3^z \end{pmatrix} \quad
u_{42} = \begin{pmatrix} 0 \\ E+m \\ p_4^x - ip_4^y \\ -p_4^z \end{pmatrix}
$$

The last digit in a spinor subscript is 1 for spin up and 2 for spin down. Note that the spinors are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^4$ will be used where needed.

This is the probability density for Moller scattering. The formula is from Feynman diagrams.

$$|\mathcal{M}(s_1, s_2, s_3, s_4)|^2 = \frac{e^4}{N} \left| \frac{1}{t}(\bar{u}_3\gamma^\mu u_1)(\bar{u}_4\gamma_\mu u_2) - \frac{1}{u}(\bar{u}_4\gamma^\nu u_1)(\bar{u}_3\gamma_\nu u_2) \right|^2$$

Symbol s_j selects the spin (up or down) of spinor j. Symbol e is electron charge. Symbols t and u are Mandelstam variables $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$.

Let

$$a_1 = (\bar{u}_3\gamma^\mu u_1)(\bar{u}_4\gamma_\mu u_2) \qquad a_2 = (\bar{u}_4\gamma^\nu u_1)(\bar{u}_3\gamma_\nu u_2)$$

Then

$$\begin{aligned}
|\mathcal{M}(s_1, s_2, s_3, s_4)|^2 &= \frac{e^4}{N} \left| \frac{a_1}{t} - \frac{a_2}{u} \right|^2 \\
&= \frac{e^4}{N} \left(\frac{a_1}{t} - \frac{a_2}{u} \right) \left(\frac{a_1}{t} - \frac{a_2}{u} \right)^* \\
&= \frac{e^4}{N} \left(\frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{tu} - \frac{a_1^* a_2}{tu} + \frac{a_2 a_2^*}{u^2} \right)
\end{aligned}$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is computed by summing $|\mathcal{M}|^2$ over all spin states and dividing by the number of inbound states. There are four inbound states.

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{1}{4} \sum_{s_1=1}^{2} \sum_{s_2=1}^{2} \sum_{s_3=1}^{2} \sum_{s_4=1}^{2} |\mathcal{M}(s_1, s_2, s_3, s_4)|^2 \\
&= \frac{e^4}{4} \sum_{s_1=1}^{2} \sum_{s_2=1}^{2} \sum_{s_3=1}^{2} \sum_{s_4=1}^{2} \frac{1}{N} \left(\frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{tu} - \frac{a_1^* a_2}{tu} + \frac{a_2 a_2^*}{u^2} \right)
\end{aligned}$$

Use the Casimir trick to replace sums over spins with matrix products.

$$f_{11} = \frac{1}{N} \sum_{\text{spins}} a_1 a_1^* = \text{Tr}\left((\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu \right) \text{Tr}\left((\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu \right)$$

$$f_{12} = \frac{1}{N} \sum_{\text{spins}} a_1 a_2^* = \text{Tr}\left((\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu \right)$$

$$f_{22} = \frac{1}{N} \sum_{\text{spins}} a_2 a_2^* = \text{Tr}\left((\not{p}_4 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu \right) \text{Tr}\left((\not{p}_3 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu \right)$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left(\frac{f_{11}}{t^2} - \frac{f_{12}}{tu} - \frac{f_{12}^*}{tu} + \frac{f_{22}}{u^2} \right)$$

Run "moller-scattering-1.txt" to verify the Casimir trick.

These formulas compute probability densities from dot products. Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$.

$$f_{11} = 32(p_1 \cdot p_2)(p_3 \cdot p_4) + 32(p_1 \cdot p_4)(p_2 \cdot p_3) - 32m^2(p_1 \cdot p_3) - 32m^2(p_2 \cdot p_4) + 64m^4$$
$$f_{12} = -32(p_1 \cdot p_2)(p_3 \cdot p_4) + 16m^2(p_1 \cdot p_2) + 16m^2(p_1 \cdot p_3) + 16m^2(p_1 \cdot p_4)$$
$$+ 16m^2(p_2 \cdot p_3) + 16m^2(p_2 \cdot p_4) + 16m^2(p_3 \cdot p_4) - 32m^4$$
$$f_{22} = 32(p_1 \cdot p_2)(p_3 \cdot p_4) + 32(p_1 \cdot p_3)(p_2 \cdot p_4) - 32m^2(p_1 \cdot p_4) - 32m^2(p_2 \cdot p_3) + 64m^4$$

In Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$ the formulas are

$$f_{11} = 8s^2 + 8u^2 - 64sm^2 - 64um^2 + 192m^4$$
$$f_{12} = -8s^2 + 64sm^2 - 96m^4$$
$$f_{22} = 8s^2 + 8t^2 - 64sm^2 - 64tm^2 + 192m^4$$

When $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$f_{11} = 8s^2 + 8u^2$$
$$f_{12} = -8s^2$$
$$f_{22} = 8s^2 + 8t^2$$

For $m = 0$ the Mandelstam variables are

$$s = 4E^2$$
$$t = -2E^2(1 - \cos\theta) = -4E^2 \sin^2(\theta/2)$$
$$u = -2E^2(1 + \cos\theta) = -4E^2 \cos^2(\theta/2)$$

The corresponding expected probability density is

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left(\frac{8s^2 + 8u^2}{t^2} + \frac{16s^2}{tu} + \frac{8s^2 + 8t^2}{u^2} \right)$$
$$= 2e^4 \left(\frac{s^2 + u^2}{t^2} + \frac{2s^2}{tu} + \frac{s^2 + t^2}{u^2} \right)$$
$$= 2e^4 \left(\frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{8}{\sin^2\theta} + \frac{1 + \sin^4(\theta/2)}{\cos^4(\theta/2)} \right)$$
$$= \frac{4e^4 \left(3 + \cos^2\theta\right)^2}{\sin^4\theta}$$

Run "moller-scattering-2.txt" to verify the formulas on this page.

In the center of mass frame the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{64\pi^2 s} = \frac{\alpha^2}{8E^2} \left(\frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{8}{\sin^2\theta} + \frac{1 + \sin^4(\theta/2)}{\cos^4(\theta/2)} \right)$$

We can integrate $d\sigma$ to obtain a normalized probability density function.

Let

$$I(\xi) = 2\pi \int_\alpha^\xi \frac{d\sigma}{d\Omega} \sin\theta \, d\theta, \qquad \alpha \le \xi \le \pi - \alpha$$

for some $\alpha > 0$. The support region is restricted because $d\sigma$ is undefined for $\theta = 0$ and $\theta = \pi$.

The cumulative distribution function is

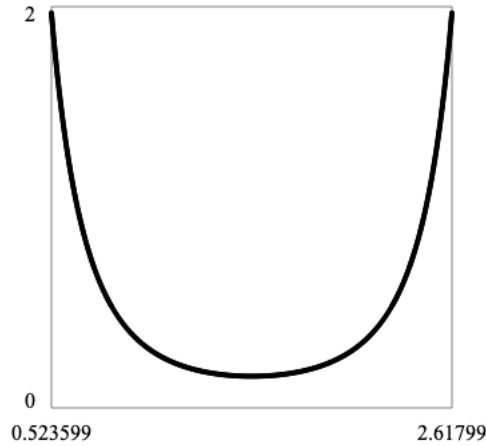$$F(\theta) = \frac{I(\theta)}{I(\pi - \alpha)}, \qquad \alpha \le \theta \le \pi - \alpha$$

Hence

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{2\pi}{I(\pi - \alpha)} \left(\frac{d\sigma}{d\Omega} \right) \sin\theta, \quad \alpha \le \theta \le \pi - \alpha$$

Run "moller-scattering-3.txt" to draw a graph of $f(\theta)$ for $\alpha = \pi/6 = 30°$.

The following table shows the probability distribution for 30° bins ($\alpha = \pi/6 = 30°$).

θ_1	θ_2	$P(\theta_1 \le \theta \le \theta_2)$
0°	30°	—
30°	60°	0.40
60°	90°	0.10
90°	120°	0.10
120°	150°	0.40
150°	180°	—

Here are a few notes about how the scripts work.

In component notation, the trace operators of the Casimir trick become sums over a repeated index, in this case α.

$$f_{11} = \left((\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \right) \left((\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \right)$$

$$f_{12} = (\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_2 + m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha$$

$$f_{22} = \left((\not{p}_4 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \right) \left((\not{p}_3 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \right)$$

To convert the above formulas to Eigenmath code, the γ tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply γ^μ by the metric tensor to lower the index.

$$\gamma^{\beta\mu}{}_\rho \quad \rightarrow \quad \texttt{gammaT = transpose(gamma)}$$
$$\gamma^\beta{}_{\mu\rho} \quad \rightarrow \quad \texttt{gammaL = transpose(dot(gmunu,gamma))}$$

Define the following 4×4 matrices.

$$(\not{p}_1 + m) \quad \rightarrow \quad \texttt{X1 = pslash1 + m I}$$
$$(\not{p}_2 + m) \quad \rightarrow \quad \texttt{X2 = pslash2 + m I}$$
$$(\not{p}_3 + m) \quad \rightarrow \quad \texttt{X3 = pslash3 + m I}$$
$$(\not{p}_4 + m) \quad \rightarrow \quad \texttt{X4 = pslash4 + m I}$$

Then for f_{11} we have the following Eigenmath code. The contract function sums over α.

$$(\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \quad \rightarrow \quad \texttt{T1 = contract(dot(X3,gammaT,X1,gammaT),1,4)}$$
$$(\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \quad \rightarrow \quad \texttt{T2 = contract(dot(X4,gammaL,X2,gammaL),1,4)}$$

Next, multiply then sum over repeated indices. The dot function sums over ν then the contract function sums over μ. The transpose makes the ν indices adjacent as required by the dot function.

$$f_{11} = \text{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu) \, \text{Tr}(\cdots \gamma_\mu \cdots \gamma_\nu) \quad \rightarrow \quad \texttt{contract(dot(T1,transpose(T2)))}$$

Follow suit for f_{22}.

$$(\not{p}_4 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \quad \rightarrow \quad \texttt{T1 = contract(dot(X4,gammaT,X1,gammaT),1,4)}$$
$$(\not{p}_3 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \quad \rightarrow \quad \texttt{T2 = contract(dot(X3,gammaL,X2,gammaL),1,4)}$$

Then

$$f_{22} = \text{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu) \, \text{Tr}(\cdots \gamma_\mu \cdots \gamma_\nu) \quad \rightarrow \quad \texttt{contract(dot(T1,transpose(T2)))}$$

The calculation of f_{12} begins with

$$(\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_2 + m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha$$
$$\rightarrow \quad \texttt{T = contract(dot(X3,gammaT,X1,gammaT,X4,gammaL,X2,gammaL),1,6)}$$

Then sum over repeated indices μ and ν.

$$f_{12} = \text{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu \cdots \gamma_\mu \cdots \gamma_\nu) \quad \rightarrow \quad \texttt{contract(contract(T,1,3))}$$