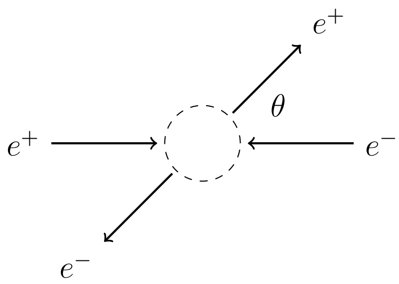
Bhabha scattering

Bhabha scattering is the process $e^- + e^+ \to e^- + e^+$.



The following center-of-mass momentum vectors have $E = \sqrt{p^2 + m^2}$.

$$p_1 = \underset{e^+\longrightarrow}{\begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}} \qquad p_2 = \underset{\longleftarrow e^-}{\begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix}} \qquad p_3 = \underset{e^+\nearrow}{\begin{pmatrix} E \\ p\sin\theta\cos\phi \\ p\sin\theta\sin\phi \\ p\cos\theta \end{pmatrix}} \qquad p_4 = \underset{\swarrow e^-}{\begin{pmatrix} E \\ -p\sin\theta\cos\phi \\ -p\sin\theta\sin\phi \\ -p\cos\theta \end{pmatrix}}$$

Spinors for p_1.

$$v_{11} = \frac{1}{\sqrt{E+m}} \underset{\text{spin up}}{\begin{pmatrix} p \\ 0 \\ E+m \\ 0 \end{pmatrix}} \qquad v_{12} = \frac{1}{\sqrt{E+m}} \underset{\text{spin down}}{\begin{pmatrix} 0 \\ -p \\ 0 \\ E+m \end{pmatrix}}$$

Spinors for p_2.

$$u_{21} = \frac{1}{\sqrt{E+m}} \underset{\text{spin up}}{\begin{pmatrix} E+m \\ 0 \\ -p \\ 0 \end{pmatrix}} \qquad u_{22} = \frac{1}{\sqrt{E+m}} \underset{\text{spin down}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ p \end{pmatrix}}$$

Spinors for p_3.

$$v_{31} = \frac{1}{\sqrt{E+m}} \underset{\text{spin up}}{\begin{pmatrix} p_{3z} \\ p_{3x} + ip_{3y} \\ E+m \\ 0 \end{pmatrix}} \qquad v_{32} = \frac{1}{\sqrt{E+m}} \underset{\text{spin down}}{\begin{pmatrix} p_{3x} - ip_{3y} \\ -p_{3z} \\ 0 \\ E+m \end{pmatrix}}$$

Spinors for p_4.

$$u_{41} = \frac{1}{\sqrt{E+m}} \underset{\text{spin up}}{\begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix}} \qquad u_{42} = \frac{1}{\sqrt{E+m}} \underset{\text{spin down}}{\begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}}$$

The scattering amplitude $\mathcal{M}_{abcd}$ for spin state $abcd$ is

$$\mathcal{M}_{abcd} = \mathcal{M}_{1abcd} + \mathcal{M}_{2abcd}$$

where

$$\mathcal{M}_{1abcd} = -\frac{e^2}{t}\underbrace{(\bar{v}_{1a}\gamma^\mu v_{3c})(\bar{u}_{4d}\gamma_\mu u_{2b})}_{\text{scattering}}, \quad \mathcal{M}_{2abcd} = \frac{e^2}{s}\underbrace{(\bar{v}_{1a}\gamma^\nu u_{2b})(\bar{u}_{4d}\gamma_\nu v_{3c})}_{\text{annihilation}}$$

Symbols s and t are Mandelstam variables.

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$

Expected probability density $\langle|\mathcal{M}|^2\rangle$ is the sum over squared amplitudes divided by the number of inbound states.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{abcd}|\mathcal{M}_{abcd}|^2$$

Expand the summand and label the terms. By positivity $\boxed{2} = \boxed{3}$.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{abcd}\big(\underbrace{\mathcal{M}_{1abcd}\mathcal{M}^*_{1abcd}}_{\boxed{1}} + \underbrace{\mathcal{M}_{1abcd}\mathcal{M}^*_{2abcd}}_{\boxed{2}} + \underbrace{\mathcal{M}_{2abcd}\mathcal{M}^*_{1abcd}}_{\boxed{3}} + \underbrace{\mathcal{M}_{2abcd}\mathcal{M}^*_{2abcd}}_{\boxed{4}}\big)$$

The following Casimir trick uses matrix arithmetic to sum over spin states.

$$\sum_{abcd}\boxed{1} = \frac{e^4}{t^2}\operatorname{Tr}\Big[(\not{p}_1 - m)\gamma^\mu(\not{p}_3 - m)\gamma^\nu\Big]\operatorname{Tr}\Big[(\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\Big]$$
$$\sum_{abcd}\boxed{2} = -\frac{e^4}{st}\operatorname{Tr}\Big[(\not{p}_1 - m)\gamma^\mu(\not{p}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{p}_3 - m)\gamma_\nu\Big]$$
$$\sum_{abcd}\boxed{4} = \frac{e^4}{s^2}\operatorname{Tr}\Big[(\not{p}_1 - m)\gamma^\mu(\not{p}_2 + m)\gamma^\nu\Big]\operatorname{Tr}\Big[(\not{p}_4 + m)\gamma_\mu(\not{p}_3 - m)\gamma_\nu\Big]$$

Probability density $\langle|\mathcal{M}|^2\rangle$ can be reformulated as

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left(\frac{f_{11}}{t^2} + \frac{2f_{12}}{st} + \frac{f_{22}}{s^2}\right)$$

with Casimir trick terms

$$f_{11} = \operatorname{Tr}\Big[(\not{p}_1 - m)\gamma^\mu(\not{p}_3 - m)\gamma^\nu\Big]\operatorname{Tr}\Big[(\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\Big]$$
$$f_{12} = -\operatorname{Tr}\Big[(\not{p}_1 - m)\gamma^\mu(\not{p}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{p}_3 - m)\gamma_\nu\Big]$$
$$f_{22} = \operatorname{Tr}\Big[(\not{p}_1 - m)\gamma^\mu(\not{p}_2 + m)\gamma^\nu\Big]\operatorname{Tr}\Big[(\not{p}_4 + m)\gamma_\mu(\not{p}_3 - m)\gamma_\nu\Big]$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$f_{11} = 32(p_1 \cdot p_2)^2 + 32(p_1 \cdot p_4)^2 - 64(p_1 \cdot p_2)m^2 + 64(p_1 \cdot p_4)m^2$$
$$f_{12} = 32(p_1 \cdot p_4)^2 + 64(p_1 \cdot p_4)m^2$$
$$f_{22} = 32(p_1 \cdot p_3)^2 + 32(p_1 \cdot p_4)^2 + 64(p_1 \cdot p_3)m^2 + 64(p_1 \cdot p_4)m^2$$

In Mandelstam variables

$$
\begin{aligned}
f_{11} &= 8u^2 + 8s^2 - 64um^2 - 64sm^2 + 192m^4 \\
f_{12} &= 8u^2 - 64um^2 + 96m^4 \\
f_{22} &= 8u^2 + 8t^2 - 64um^2 - 64tm^2 + 192m^4
\end{aligned}
$$

For $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$
\begin{aligned}
f_{11} &= 8u^2 + 8s^2 \\
f_{12} &= 8u^2 \\
f_{22} &= 8u^2 + 8t^2
\end{aligned}
$$

For $m = 0$ the Mandelstam variables are

$$
\begin{aligned}
s &= 4E^2 \\
t &= -2E^2(1 - \cos\theta) \\
u &= -2E^2(1 + \cos\theta)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4}\left(\frac{f_{11}}{t^2} + \frac{2f_{12}}{st} + \frac{f_{22}}{s^2}\right) \\
&= 2e^4\left(\frac{u^2+s^2}{t^2} + \frac{2u^2}{st} + \frac{u^2+t^2}{s^2}\right) \\
&= e^4\left(\underbrace{\frac{2(1+\cos\theta)^2+8}{(1-\cos\theta)^2}}_{\text{scattering}} - \underbrace{\frac{2(1+\cos\theta)^2}{1-\cos\theta}}_{\text{interference}} + \underbrace{1+\cos^2\theta}_{\text{annihilation}}\right)
\end{aligned}
$$

The expected probability density can be written more compactly as

$$
\langle |\mathcal{M}|^2 \rangle = e^4\left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2
$$

Cross section

The differential cross section is

$$
\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{4(4\pi\varepsilon_0)^2 s}
$$

where

$$
s = (p_1 + p_2)^2 = 4E^2
$$

For high energy experiments we have

$$
\langle |\mathcal{M}|^2 \rangle = e^4\left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2
$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{4(4\pi\varepsilon_0)^2 s}\left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2$$

Noting that

$$e^2 = 4\pi\varepsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{4s}\left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2$$

Noting that

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

we also have

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{4s}\left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2 \sin\theta\, d\theta\, d\phi$$

Let $S(\theta_1, \theta_2)$ be the following integral of $d\sigma$.

$$S(\theta_1, \theta_2) = \int_0^{2\pi}\int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{\pi\alpha^2(\hbar c)^2}{2s}\left[I(\theta_2) - I(\theta_1)\right]$$

where

$$I(\theta) = \frac{16}{\cos\theta - 1} - \frac{\cos^3\theta}{3} - \cos^2\theta - 9\cos\theta - 16\log(1 - \cos\theta)$$

The cumulative distribution function is

$$F(\theta) = \frac{S(a,\theta)}{S(a,\pi)} = \frac{I(\theta) - I(a)}{I(\pi) - I(a)}, \quad a \le \theta \le \pi$$

Angular support is reduced by an arbitrary angle $a > 0$ because $I(0)$ is undefined.

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 < \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi) - I(a)}\left(\frac{\cos^2\theta + 3}{\cos\theta - 1}\right)^2 \sin\theta$$