Probability current

Let $\mathbf{J}$ be the probability current

$$\mathbf{J} = \frac{i\hbar}{2m}\left(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi\right) \tag{1}$$

Show that

$$\nabla\cdot\mathbf{J} = -\frac{\partial}{\partial t}|\Psi|^2 \tag{2}$$

Recall that divergence of gradient equals Laplacian.

$$\nabla\cdot\nabla = \nabla^2$$

By the product rule for divergence

$$\begin{aligned}
\nabla\cdot\Psi\nabla\Psi^* &= \nabla\Psi\cdot\nabla\Psi^* + \Psi\nabla^2\Psi^* \\
\nabla\cdot\Psi^*\nabla\Psi &= \nabla\Psi^*\cdot\nabla\Psi + \Psi^*\nabla^2\Psi
\end{aligned}$$

Hence the divergence of $\mathbf{J}$ in (1) is

$$\begin{aligned}
\nabla\cdot\mathbf{J} &= \frac{i\hbar}{2m}\left(\nabla\cdot\Psi\nabla\Psi^* - \nabla\cdot\Psi^*\nabla\Psi\right) \\
&= \frac{i\hbar}{2m}\left(\Psi\nabla^2\Psi^* - \Psi^*\nabla^2\Psi\right) \qquad (3)
\end{aligned}$$

For the time derivative in (2) we have

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{\partial}{\partial t}(\Psi^*\Psi) = \Psi^*\frac{\partial}{\partial t}\Psi + \Psi\frac{\partial}{\partial t}\Psi^* \tag{4}$$

Recall the Schrödinger equation

$$\frac{\partial}{\partial t}\Psi = \frac{i\hbar}{2m}\nabla^2\Psi - \frac{i}{\hbar}V\Psi$$

and

$$\frac{\partial}{\partial t}\Psi^* = \left(\frac{\partial}{\partial t}\Psi\right)^* = -\frac{i\hbar}{2m}\nabla^2\Psi^* + \frac{i}{\hbar}V\Psi^*$$

Substitute the Schrödinger equation into (4) to obtain

$$\begin{aligned}
\frac{\partial}{\partial t}|\Psi|^2 &= \Psi^*\left(\frac{i\hbar}{2m}\nabla^2\Psi - \frac{i}{\hbar}V\Psi\right) + \Psi\left(-\frac{i\hbar}{2m}\nabla^2\Psi^* + \frac{i}{\hbar}V\Psi^*\right) \\
&= \frac{i\hbar}{2m}\left(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*\right) \qquad (5)
\end{aligned}$$

Substitute (5) into (3) to obtain

$$\nabla\cdot\mathbf{J} = -\frac{\partial}{\partial t}|\Psi|^2 \tag{6}$$