7-6. Show, for a particle moving in three-dimensional space x, y, z,

$$\langle (x_{k+1}-x_k)^2\rangle = \langle (y_{k+1}-y_k)^2\rangle = \langle (z_{k+1}-z_k)^2\rangle = -\frac{\hbar\epsilon}{im}\langle 1\rangle \tag{7.50}$$

$$\begin{aligned}\langle (x_{k+1}-x_k)(y_{k+1}-y_k)\rangle &= \langle (x_{k+1}-x_k)(z_{k+1}-z_k)\rangle \\ &= \langle (y_{k+1}-y_k)(z_{k+1}-z_k)\rangle = 0\end{aligned} \tag{7.51}$$

To extend (7.33) to three dimensions, change derivative to gradient, hence

$$\langle \nabla_k F\rangle = -\frac{i}{\hbar}\langle F\nabla_k S\rangle \tag{1}$$

where

$$\nabla_k = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

and

$$\mathbf{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Let

$$F = x_k + y_k + z_k$$

Then

$$\begin{aligned}\nabla_k F &= \frac{\partial F}{\partial x_k}\mathbf{i} + \frac{\partial F}{\partial y_k}\mathbf{j} + \frac{\partial F}{\partial z_k}\mathbf{k} \\ &= \mathbf{i} + \mathbf{j} + \mathbf{k}\end{aligned}$$

Hence

$$\langle \nabla_k F\rangle = \langle 1\rangle\mathbf{i} + \langle 1\rangle\mathbf{j} + \langle 1\rangle\mathbf{k} \tag{2}$$

By equation (7.43)

$$\begin{aligned}\left\langle (x_k+y_k+z_k)\frac{\partial S}{\partial x_k}\right\rangle \mathbf{i} &= \left\langle x_k\frac{\partial S}{\partial x_k}\right\rangle \mathbf{i} \\ \left\langle (x_k+y_k+z_k)\frac{\partial S}{\partial y_k}\right\rangle \mathbf{j} &= \left\langle y_k\frac{\partial S}{\partial y_k}\right\rangle \mathbf{j} \\ \left\langle (x_k+y_k+z_k)\frac{\partial S}{\partial z_k}\right\rangle \mathbf{k} &= \left\langle z_k\frac{\partial S}{\partial z_k}\right\rangle \mathbf{k}\end{aligned}$$

Hence

$$\langle F\nabla_k S\rangle = \left\langle x_k \frac{\partial S}{\partial x_k}\right\rangle \mathbf{i} + \left\langle y_k \frac{\partial S}{\partial y_k}\right\rangle \mathbf{j} + \left\langle z_k \frac{\partial S}{\partial z_k}\right\rangle \mathbf{k} \tag{3}$$

By equations (1), (2), and (3)

$$\begin{aligned}
\langle 1\rangle \mathbf{i} &= -\frac{i}{\hbar}\left\langle x_k \frac{\partial S}{\partial x_k}\right\rangle \mathbf{i} \\
\langle 1\rangle \mathbf{j} &= -\frac{i}{\hbar}\left\langle y_k \frac{\partial S}{\partial y_k}\right\rangle \mathbf{j} \\
\langle 1\rangle \mathbf{k} &= -\frac{i}{\hbar}\left\langle z_k \frac{\partial S}{\partial z_k}\right\rangle \mathbf{k}
\end{aligned}$$

Then by the same arguments in the book for equation (7.49) we have

$$\begin{aligned}
\left\langle (x_{k+1}-x_k)^2\right\rangle \mathbf{i} &= -\frac{\hbar\epsilon}{im}\langle 1\rangle \mathbf{i} \\
\left\langle (y_{k+1}-y_k)^2\right\rangle \mathbf{j} &= -\frac{\hbar\epsilon}{im}\langle 1\rangle \mathbf{j} \\
\left\langle (z_{k+1}-z_k)^2\right\rangle \mathbf{k} &= -\frac{\hbar\epsilon}{im}\langle 1\rangle \mathbf{k}
\end{aligned}$$

Hence (7.50) is shown to be true.

The action for a particle in three dimensions is

$$S = \int_{t_a}^{t_b} \left(\frac{m}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - V(x,y,z)\right)\, dt$$

Hence

$$\nabla_k S = \frac{\partial S}{\partial x_k}\mathbf{i} + \frac{\partial S}{\partial y_k}\mathbf{j} + \frac{\partial S}{\partial z_k}\mathbf{k}$$

where

$$\begin{aligned}
\frac{\partial S}{\partial x_k} &= -m\left(\frac{x_{k+1}-x_k}{\epsilon} - \frac{x_k - x_{k-1}}{\epsilon}\right) - \epsilon \frac{\partial V}{\partial x}\bigg|_{x_k,y_k,z_k} \\
\frac{\partial S}{\partial y_k} &= -m\left(\frac{y_{k+1}-y_k}{\epsilon} - \frac{y_k - y_{k-1}}{\epsilon}\right) - \epsilon \frac{\partial V}{\partial y}\bigg|_{x_k,y_k,z_k} \\
\frac{\partial S}{\partial z_k} &= -m\left(\frac{z_{k+1}-z_k}{\epsilon} - \frac{z_k - z_{k-1}}{\epsilon}\right) - \epsilon \frac{\partial V}{\partial z}\bigg|_{x_k,y_k,z_k}
\end{aligned}$$