Laplace transform example

Solve for $c_a(t)$ and $c_b(t)$ where $c_a(0) = 1$, $c_b(0) = 0$, and

$$
\begin{aligned}
i\hbar\dot{c}_a(t) &= Ae^{-i\omega t}c_b(t) \\
i\hbar\dot{c}_b(t) &= Be^{i\omega t}c_a(t)
\end{aligned}
$$

Start with Laplace transforms.

$$
\begin{aligned}
i\hbar s C_a(s) - i\hbar c_a(0) &= AC_b(s + i\omega) \\
i\hbar s C_b(s) - i\hbar c_b(0) &= BC_a(s - i\omega)
\end{aligned} \tag{1}
$$

Solve for $C_a(s)$ with $c_a(0) = 1$.

$$
C_a(s) = \frac{AC_b(s + i\omega) + i\hbar}{i\hbar s}
$$

Hence

$$
C_a(s - i\omega) = -\frac{iAC_b(s)}{\hbar(s - i\omega)} + \frac{1}{s - i\omega} \tag{2}
$$

Substitute (2) into (1) with $c_b(0) = 0$ to obtain

$$
i\hbar s C_b(s) = -\frac{iABC_b(s)}{\hbar(s - i\omega)} + \frac{B}{s - i\omega}
$$

Rearrange as

$$
i\hbar s C_b(s) + \frac{iABC_b(s)}{\hbar(s - i\omega)} = \frac{B}{s - i\omega}
$$

Rearrange again as

$$
C_b(s)\left[i\hbar s + \frac{iAB}{\hbar(s - i\omega)}\right] = \frac{B}{s - i\omega}
$$

Hence

$$
C_b(s) = \frac{\dfrac{B}{s - i\omega}}{i\hbar s + \dfrac{iAB}{\hbar(s - i\omega)}} = \frac{-iB/\hbar}{s^2 - i\omega s + AB/\hbar^2}
$$

Inverse Laplace transform:

$$
\frac{1}{s^2 + as + b} \quad \Rightarrow \quad \frac{2}{k}\sin\left(\frac{kt}{2}\right)\exp\left(-\frac{at}{2}\right), \quad k = \sqrt{4b - a^2}
$$

Hence for $a = -i\omega$ and $b = AB/\hbar^2$ we have

$$
c_b(t) = -\frac{2iB}{\hbar k}\sin\left(\frac{kt}{2}\right)\exp\left(\frac{i\omega t}{2}\right), \quad k = \sqrt{\frac{4AB}{\hbar^2} + \omega^2}
$$

Solve for $c_a(t)$.

$$
c_a(t) = \frac{i\hbar\dot{c}_b(t)}{Be^{i\omega t}} = \cos\left(\frac{kt}{2}\right)\exp\left(-\frac{i\omega t}{2}\right) + \frac{i\omega}{k}\sin\left(\frac{kt}{2}\right)\exp\left(-\frac{i\omega t}{2}\right)
$$