

3-12. Let the wave function for a harmonic oscillator at time $t = 0$ be

$$\psi(x, 0) = \exp\left(-\frac{m\omega}{2\hbar}(x - a)^2\right)$$

Using equation (3.42) and the results of problem 3-8 show that at time $t = T$ the wave function is

$$\psi(x, T) = \exp\left(-\frac{i\omega T}{2} - \frac{m\omega}{2\hbar}(x^2 - 2ax \exp(-i\omega T) + a^2 \cos(\omega T) \exp(-i\omega T))\right)$$

From equation (3.42)

$$\psi(x, T) = \int_{-\infty}^{\infty} K(x, T, x_c, 0) \psi(x_c, 0) dx_c \quad (1)$$

From problem 3-8

$$K(x, T, x_c, 0) = \left(\frac{m\omega}{2\pi i \hbar \sin(\omega T)}\right)^{1/2} \exp\left(\frac{im\omega}{2\hbar \sin(\omega T)}((x^2 + x_c^2) \cos(\omega T) - 2xx_c)\right) \quad (2)$$

Substitute (2) and $\psi(x, 0)$ into (1). We now have for $\psi(x, T)$

$$\psi(x, T) = \left(\frac{m\omega}{2\pi i \hbar \sin(\omega T)}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left(\frac{im\omega}{2\hbar \sin(\omega T)}((x^2 + x_c^2) \cos(\omega T) - 2xx_c)\right) \exp\left(-\frac{m\omega}{2\hbar}(x_c - a)^2\right) dx_c$$

Rewrite as

$$\psi(x, T) = \left(\frac{m\omega}{2\pi i \hbar \sin(\omega T)}\right)^{1/2} \int_{-\infty}^{\infty} \exp(Ax_c^2 + Bx_c + C) dx_c \quad (1)$$

where

$$\begin{aligned} A &= \frac{m\omega}{2\hbar} \left(\frac{i \cos(\omega T)}{\sin(\omega T)} - 1 \right) = \frac{im\omega \exp(i\omega T)}{2\hbar \sin(\omega T)} \\ B &= \frac{m\omega}{\hbar} \left(a - \frac{ix}{\sin(\omega T)} \right) \\ C &= \frac{m\omega}{2\hbar} \left(\frac{ix^2 \cos(\omega T)}{\sin(\omega T)} - a^2 \right) \end{aligned}$$

Solve the integral.

$$\int_{-\infty}^{\infty} \exp(Ax_c^2 + Bx_c + C) = \left(-\frac{\pi}{A}\right)^{1/2} \exp\left(-\frac{B^2}{4A} + C\right)$$

$$-\frac{\pi}{A} = \frac{2\pi i\hbar \sin(\omega T)}{m\omega \exp(i\omega T)} \quad (2)$$

$$-\frac{B^2}{4A} + C = -\frac{m\omega}{2\hbar} (x^2 - 2ax \exp(-i\omega T) + a^2 \cos(\omega T) \exp(-i\omega T)) \quad (3)$$

Note that

$$\frac{m\omega}{2\pi i\hbar \sin(\omega T)} \times \frac{2\pi i\hbar \sin(\omega T)}{m\omega \exp(i\omega T)} = \exp(-i\omega T) \quad (4)$$

from equation (1) from equation (2)

Hence from equation (1)

$$\psi(x, T) = \exp\left(-\frac{i\omega T}{2}\right) \times \exp\left(-\frac{m\omega}{2\hbar} (x^2 - 2ax \exp(-i\omega T) + a^2 \cos(\omega T) \exp(-i\omega T))\right)$$