(14.1) Fill in the algebra leading to eqn 14.32.

This is equation (14.32).

$$\hat{H} = \int d^3p \sum_{\lambda=1}^{2} E_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}\lambda} \hat{a}_{\mathbf{p}\lambda} \tag{14.32}$$

Consider equation (14.31).

$$\hat{A}^\mu(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_p)^{\frac{1}{2}}} \sum_{\lambda=1}^{2} \left(\epsilon^\mu_\lambda(p) \hat{a}_{p\lambda} \exp(-ip\cdot x) + \epsilon^{\mu *}_\lambda(p) \hat{a}^\dagger_{p\lambda} \exp(ip\cdot x) \right) \tag{14.31}$$

Let

$$\hat{A}^\mu = \begin{pmatrix} \hat{A}_0 \\ \hat{A}_1 \\ \hat{A}_2 \\ \hat{A}_3 \end{pmatrix}, \qquad \epsilon^\mu_\lambda(p) = \begin{pmatrix} \epsilon_{0,p\lambda} \\ \epsilon_{1,p\lambda} \\ \epsilon_{2,p\lambda} \\ \epsilon_{3,p\lambda} \end{pmatrix}$$

Then in vector form we have

$$\begin{pmatrix} \hat{A}_0 \\ \hat{A}_1 \\ \hat{A}_2 \\ \hat{A}_3 \end{pmatrix} = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{\hat{a}}{(2E_p)^{\frac{1}{2}}}$$

where

$$\begin{aligned} \hat{a} = {} & \begin{pmatrix} \epsilon_{0,p1} \\ \epsilon_{1,p1} \\ \epsilon_{2,p1} \\ \epsilon_{3,p1} \end{pmatrix} \hat{a}_{p1} \exp(-ip\cdot x) + \begin{pmatrix} \epsilon^*_{0,p1} \\ \epsilon^*_{1,p1} \\ \epsilon^*_{2,p1} \\ \epsilon^*_{3,p1} \end{pmatrix} \hat{a}^\dagger_{p1} \exp(ip\cdot x) \\ & + \begin{pmatrix} \epsilon_{0,p2} \\ \epsilon_{1,p2} \\ \epsilon_{2,p2} \\ \epsilon_{3,p2} \end{pmatrix} \hat{a}_{p2} \exp(-ip\cdot x) + \begin{pmatrix} \epsilon^*_{0,p2} \\ \epsilon^*_{1,p2} \\ \epsilon^*_{2,p2} \\ \epsilon^*_{3,p2} \end{pmatrix} \hat{a}^\dagger_{p2} \exp(ip\cdot x) \end{aligned}$$

Hence

$$\hat{A}_\nu = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_p)^{\frac{1}{2}}}$$
$$\times \left[(\epsilon_{\nu,p1}\hat{a}_{p1} + \epsilon_{\nu,p2}\hat{a}_{p2}) \exp(-ip \cdot x) + (\epsilon^*_{\nu,p1}\hat{a}^\dagger_{p1} + \epsilon^*_{\nu,p2}\hat{a}^\dagger_{p2}) \exp(ip \cdot x) \right]$$

Recall that

$$p \cdot x = E_p t - p_1 x_1 - p_2 x_2 - p_3 x_3$$

Hence the partial derivative of $\hat{A}_\nu$ is

$$\partial_\mu \hat{A}_\nu = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_p)^{\frac{1}{2}}}$$
$$\times \left[-ip_\mu (\epsilon_{\nu,p1}\hat{a}_{p1} + \epsilon_{\nu,p2}\hat{a}_{p2}) \exp(-ip \cdot x) + ip_\mu (\epsilon^*_{\nu,p1}\hat{a}^\dagger_{p1} + \epsilon^*_{\nu,p2}\hat{a}^\dagger_{p2}) \exp(ip \cdot x) \right]$$

Consider equation (14.28).

$$\mathcal{H} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \tag{14.28}$$

We need to integrate Hamiltonian density $\mathcal{H}$ to obtain $\hat{H}$, that is,

$$\hat{H} = \int d^3x \, \mathcal{H} \tag{1}$$

Substitute (14.28) into (1).

$$\hat{H} = \frac{1}{2} \int d^3x \, \left(E_1^2 + E_2^2 + E_3^2 + B_1^2 + B_2^2 + B_3^2 \right)$$

From equations (5.28) and (5.29) we have

$$\mathbf{E} = \begin{pmatrix} \partial_0 A_1 - \partial_1 A_0 \\ \partial_0 A_2 - \partial_2 A_0 \\ \partial_0 A_3 - \partial_3 A_0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} \partial_3 A_2 - \partial_2 A_3 \\ \partial_1 A_3 - \partial_3 A_1 \\ \partial_2 A_1 - \partial_1 A_2 \end{pmatrix}$$

Start with E_1^2.

$$E_1^2 = (\partial_0 \hat{A}_1)^2 - (\partial_0 \hat{A}_1)(\partial_1 \hat{A}_0) - (\partial_1 \hat{A}_0)(\partial_0 \hat{A}_1) + (\partial_1 \hat{A}_0)^2$$

Let

$$I_1 = \frac{1}{2}\int d^3x\,(\partial_0 \hat{A}_1)^2$$

Hence

$$\begin{aligned}
I_1 = \frac{1}{2}\int &\frac{d^3x\,d^3p\,d^3q}{(2\pi)^3}\frac{1}{(2E_p)^{\frac{1}{2}}(2E_q)^{\frac{1}{2}}} \\
&\times \sum_{\lambda=1}^{2}\Big(-iE_p\epsilon_\lambda^1(p)\hat{a}_{p\lambda}\exp(-ip\cdot x) + iE_p\epsilon_\lambda^{1*}(p)\hat{a}_{p\lambda}^\dagger\exp(ip\cdot x)\Big) \\
&\times \sum_{\lambda=1}^{2}\Big(-iE_q\epsilon_\lambda^1(q)\hat{a}_{q\lambda}\exp(-iq\cdot x) + iE_q\epsilon_\lambda^{1*}(q)\hat{a}_{q\lambda}^\dagger\exp(iq\cdot x)\Big)
\end{aligned}$$

Using the integral

$$\int d^3x\,\exp(ip\cdot x) = (2\pi)^3\exp(iE_pt)\delta^{(3)}(\mathbf{p})$$

we have

$$\begin{aligned}
I_1 = \frac{1}{2}\int &\frac{d^3p\,d^3q}{(2E_p)^{\frac{1}{2}}(2E_q)^{\frac{1}{2}}} \\
&\times \delta^{(3)}(\mathbf{p})\sum_{\lambda=1}^{2}\Big(-iE_p\epsilon_\lambda^1(p)\hat{a}_{p\lambda}\exp(-iE_pt) + iE_p\epsilon_\lambda^{1*}(p)\hat{a}_{p\lambda}^\dagger\exp(iE_pt)\Big) \\
&\times \delta^{(3)}(\mathbf{q})\sum_{\lambda=1}^{2}\Big(-iE_q\epsilon_\lambda^1(q)\hat{a}_{q\lambda}\exp(-iE_qt) + iE_q\epsilon_\lambda^{1*}(q)\hat{a}_{q\lambda}^\dagger\exp(iE_qt)\Big)
\end{aligned}$$