

Yukawa potential

Find σ_{total} for the Yukawa potential

$$V(r) = -\frac{V_0 \exp(-\mu r)}{\mu r}$$

Let $f(\mathbf{k})$ be the scattering amplitude for $\mathbf{k} = \mathbf{k}_i - \mathbf{k}_f$. The following formula is the Born approximation for $f(\mathbf{k})$.

$$f(\mathbf{k}) = \frac{m}{2\pi\hbar^2} \int \exp(i\mathbf{k} \cdot \mathbf{r}) V(\mathbf{r}) d\mathbf{r}$$

Convert to polar coordinates where $k = |\mathbf{k}|$.

$$f(\mathbf{k}) = \frac{m}{2\pi\hbar^2} \int_0^\infty \int_0^\pi \int_0^{2\pi} \exp(ikr \cos \theta) V(r, \theta, \phi) r^2 \sin \theta dr d\theta d\phi$$

Substitute the Yukawa potential

$$V(r, \theta, \phi) = -\frac{V_0 \exp(-\mu r)}{\mu r}$$

to obtain

$$f(\mathbf{k}) = -\frac{mV_0}{2\pi\hbar^2\mu} \int_0^\infty \int_0^\pi \int_0^{2\pi} \exp(ikr \cos \theta - \mu r) r \sin \theta dr d\theta d\phi$$

Integrate over ϕ (multiply by 2π).

$$f(\mathbf{k}) = -\frac{mV_0}{\hbar^2\mu} \int_0^\infty \int_0^\pi \exp(ikr \cos \theta - \mu r) r \sin \theta dr d\theta$$

Let $u = \cos \theta$ and $du = -\sin \theta d\theta$. The minus sign in du is canceled by interchanging integration limits $\cos 0 = 1$ and $\cos \pi = -1$.

$$f(\mathbf{k}) = -\frac{mV_0}{\hbar^2\mu} \int_0^\infty \int_{-1}^1 \exp(ikru - \mu r) r dr du$$

Solve the integral over u .

$$f(\mathbf{k}) = -\frac{mV_0}{\hbar^2\mu} \int_0^\infty \left[\frac{1}{ikr} \exp(ikru - \mu r) \right]_{u=-1}^{u=1} r dr$$

Cancel r and evaluate the limits.

$$f(\mathbf{k}) = -\frac{mV_0}{\hbar^2\mu} \int_0^\infty \frac{1}{ik} [\exp(ikr - \mu r) - \exp(-ikr - \mu r)] dr$$

Solve the integral over r .

$$f(\mathbf{k}) = -\frac{mV_0}{\hbar^2\mu} \frac{1}{ik} \left[\frac{1}{ik - \mu} \exp(ikr - \mu r) + \frac{1}{ik + \mu} \exp(-ikr - \mu r) \right]_{r=0}^{r=\infty}$$

Evaluate the limits. The exponentials vanish at the upper limit.

$$f(\mathbf{k}) = -\frac{mV_0}{\hbar^2\mu} \frac{1}{ik} \left[-\frac{1}{ik - \mu} - \frac{1}{ik + \mu} \right] = -\frac{2mV_0}{\hbar^2\mu} \frac{1}{k^2 + \mu^2} \quad (1)$$

Substitute

$$k^2 = \frac{|\mathbf{p}_i - \mathbf{p}_f|^2}{\hbar^2} = \frac{4mE(1 - \cos \theta)}{\hbar^2}$$

to obtain

$$f(\theta) = -\frac{2mV_0}{\mu} \frac{1}{4mE(1 - \cos \theta) + \mu^2\hbar^2} \quad (2)$$

Hence

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left(\frac{2mV_0}{\mu} \right)^2 \frac{1}{[4mE(1 - \cos \theta) + \mu^2\hbar^2]^2}$$

For the total cross section we have

$$\sigma_{\text{total}} = \int \frac{d\sigma}{d\Omega} d\Omega = \left(\frac{2mV_0}{\mu} \right)^2 \int_0^\pi \int_0^{2\pi} \frac{1}{[4mE(1 - \cos \theta) + \mu^2\hbar^2]^2} \sin \theta d\theta d\phi$$

Integrate over ϕ (multiply by 2π).

$$\sigma_{\text{total}} = 2\pi \left(\frac{2mV_0}{\mu} \right)^2 \int_0^\pi \frac{1}{[4mE(1 - \cos \theta) + \mu^2\hbar^2]^2} \sin \theta d\theta$$

Let $y = 1 - \cos \theta$ and $dy = \sin \theta d\theta$. The limits transform as $1 - \cos 0 = 0$ and $1 - \cos \pi = 2$.

$$\sigma_{\text{total}} = 2\pi \left(\frac{2mV_0}{\mu} \right)^2 \int_0^2 \frac{1}{[4mEy + \mu^2\hbar^2]^2} dy$$

Solve the integral.

$$\sigma_{\text{total}} = 2\pi \left(\frac{2mV_0}{\mu} \right)^2 \left[-\frac{1}{4mE(4mEy + \mu^2\hbar^2)} \right]_{y=0}^{y=2}$$

Evaluate the limits.

$$\sigma_{\text{total}} = 2\pi \left(\frac{2mV_0}{\mu} \right)^2 \frac{2}{8mE\mu^2\hbar^2 + \mu^4\hbar^4} \quad (3)$$

Dimension check:

$$\begin{aligned} mV_0 &\propto \text{mass} \times \text{energy} = \text{momentum}^2 \\ mE &\propto \text{mass} \times \text{energy} = \text{momentum}^2 \\ \mu\hbar &\propto \text{length}^{-1} \times \text{energy} \times \text{time} = \text{momentum} \end{aligned}$$

The momentum dimensions cancel leaving units of area.

$$\sigma_{\text{total}} \propto \frac{1}{\mu^2} \propto \text{length}^2$$