Feynman and Hibbs problem 2-2

This is the Lagrangian for a harmonic oscillator.

$$L = \frac{m}{2}(\dot{x}^2 - \omega^2 x^2)$$

Let $T = t_b - t_a$. Show that the classical action is

$$S_{cl} = \frac{m\omega}{2\sin(\omega T)}\left((x_b^2 + x_a^2)\cos(\omega T) - 2x_b x_a\right)$$

From the above Lagrangian we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

and

$$\frac{\partial L}{\partial x} = -m\omega^2 x$$

By equation (2.7) which is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

we have

$$\ddot{x} = -\omega^2 x \tag{1}$$

The well-known solution to (1) is

$$x(t) = A\sin(\omega t) + B\cos(\omega t)$$

We have the following boundary conditions.

$$x(0) = x_a$$
$$x(T) = x_b$$

Solve for B.

$$x(0) = B = x_a$$

For $x(T)$ we have

$$x(T) = A\sin(\omega T) + B\cos(\omega T)$$

Solve for A to obtain

$$A = \frac{x(T) - B\cos(\omega T)}{\sin(\omega T)} = \frac{x_b - x_a\cos(\omega T)}{\sin(\omega T)}$$

Hence the equation of motion is

$$x(t) = \frac{x_b - x_a\cos(\omega T)}{\sin(\omega T)}\sin(\omega t) + x_a\cos(\omega t) \tag{2}$$

From the action integral

$$S = \int_0^T L\,dt$$

we have

$$\begin{aligned} S_{cl} &= \frac{m}{2}\int_0^T (\dot{x}^2 - \omega^2 x^2)\,dt \\ &= \frac{m\omega}{2\sin(\omega T)}\Big((x_b^2 + x_a^2)\cos(\omega T) - 2x_b x_a\Big) \end{aligned}$$

for the $x(t)$ given in (2).