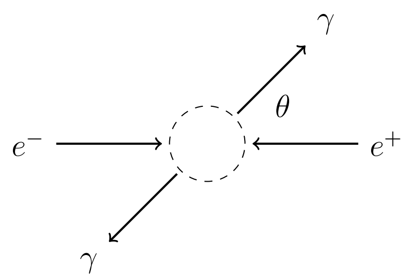
Annihilation

Annihilation is the interaction $e^- + e^+ \rightarrow \gamma + \gamma$.



In the center-of-mass frame we have the following momentum vectors where $E = \sqrt{p^2 + m^2}$.

$$p_1 = \underset{\substack{\text{inbound}\\ \text{electron}}}{\begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}} \qquad p_2 = \underset{\substack{\text{inbound}\\ \text{positron}}}{\begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix}} \qquad p_3 = \underset{\substack{\text{outbound}\\ \text{photon}}}{\begin{pmatrix} E \\ E\sin\theta\cos\phi \\ E\sin\theta\sin\phi \\ E\cos\theta \end{pmatrix}} \qquad p_4 = \underset{\substack{\text{outbound}\\ \text{photon}}}{\begin{pmatrix} E \\ -E\sin\theta\cos\phi \\ -E\sin\theta\sin\phi \\ -E\cos\theta \end{pmatrix}}$$

Spinors for the inbound electron.

$$u_{11} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound electron}\\ \text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix}} \qquad u_{12} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound electron}\\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix}}$$

Spinors for the inbound positron.

$$v_{21} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound positron}\\ \text{spin up}}}{\begin{pmatrix} -p \\ 0 \\ E+m \\ 0 \end{pmatrix}} \qquad v_{22} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound positron}\\ \text{spin down}}}{\begin{pmatrix} 0 \\ p \\ 0 \\ E+m \end{pmatrix}}$$

The scattering amplitude $\mathcal{M}_{ab}{}^{\mu\nu}$ for spin ab and polarization $\mu\nu$ is

$$\mathcal{M}_{ab}{}^{\mu\nu} = \mathcal{M}_{1ab}{}^{\mu\nu} + \mathcal{M}_{2ab}{}^{\nu\mu}$$

where

$$\mathcal{M}_{1ab}{}^{\mu\nu} = \frac{\bar{v}_{2b}(-ie\gamma^\mu)(\not{q}_1 + m)(-ie\gamma^\nu)u_{1a}}{t - m^2}$$
$$\mathcal{M}_{2ab}{}^{\nu\mu} = \frac{\bar{v}_{2b}(-ie\gamma^\nu)(\not{q}_2 + m)(-ie\gamma^\mu)u_{1a}}{u - m^2}$$

Matrices $\not{q}_1$ and $\not{q}_2$ represent momentum transfer.

$$\begin{aligned}\not{q}_1 &= (p_1 - p_3)^\alpha g_{\alpha\beta}\gamma^\beta \\ \not{q}_2 &= (p_1 - p_4)^\alpha g_{\alpha\beta}\gamma^\beta\end{aligned}$$

Scalars t and u are Mandelstam variables.

$$\begin{aligned}t &= (p_1 - p_3)^2 \\ u &= (p_1 - p_4)^2\end{aligned}$$

In component form (note that indices μ and ν are interchanged for $\mathcal{M}_{2ab}$)

$$\begin{aligned}\mathcal{M}_{1ab}{}^{\mu\nu} &= \frac{(\bar{v}_{2b})_\alpha(-ie\gamma^{\mu\alpha}{}_\beta)(\not{q}_1 + m)^\beta{}_\rho(-ie\gamma^{\nu\rho}{}_\sigma)(u_{1a})^\sigma}{t - m^2} \\ \mathcal{M}_{2ab}{}^{\nu\mu} &= \frac{(\bar{v}_{2b})_\alpha(-ie\gamma^{\nu\alpha}{}_\beta)(\not{q}_2 + m)^\beta{}_\rho(-ie\gamma^{\mu\rho}{}_\sigma)(u_{1a})^\sigma}{u - m^2}\end{aligned}$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is the average over spin and polarization states.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{a,b}\sum_{\mu,\nu}\left|\mathcal{M}_{ab}{}^{\mu\nu}\right|^2$$

Summing over μ and ν requires $g_{\mu\nu}$ to lower indices.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{a,b}\mathcal{M}_{ab}{}^{\mu\nu}\left(g_{\mu\alpha}\mathcal{M}_{ab}{}^{\alpha\beta}g_{\beta\nu}\right)^*$$

Substitute $\mathcal{M}_{1ab} + \mathcal{M}_{2ab}$ for $\mathcal{M}_{ab}$. (Note that $P_{12} = P_{21}^*$ hence by the property that probabilities are real we have $P_{12} = P_{21}$.)

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{a,b}\Big[&\underbrace{\mathcal{M}_{1ab}{}^{\mu\nu}\left(g_{\mu\alpha}\mathcal{M}_{1ab}{}^{\alpha\beta}g_{\beta\nu}\right)^*}_{P_{11ab}} + \underbrace{\mathcal{M}_{1ab}{}^{\mu\nu}\left(g_{\nu\alpha}\mathcal{M}_{2ab}{}^{\alpha\beta}g_{\beta\mu}\right)^*}_{P_{12ab}} \\ &+ \underbrace{\mathcal{M}_{2ab}{}^{\nu\mu}\left(g_{\mu\alpha}\mathcal{M}_{1ab}{}^{\alpha\beta}g_{\beta\nu}\right)^*}_{P_{21ab}} + \underbrace{\mathcal{M}_{2ab}{}^{\nu\mu}\left(g_{\nu\alpha}\mathcal{M}_{2ab}{}^{\alpha\beta}g_{\beta\mu}\right)^*}_{P_{22ab}}\Big]\end{aligned}$$

The Casimir trick uses matrix arithmetic to sum over spin and polarization states:

$$\begin{aligned}\sum_{a,b} P_{11ab} &= \frac{e^4}{(t - m^2)^2}\operatorname{Tr}\left[(\not{p}_1 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\right] \\ \sum_{a,b} P_{12ab} &= \frac{e^4}{(t - m^2)(u - m^2)}\operatorname{Tr}\left[(\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\right] \\ \sum_{a,b} P_{22ab} &= \frac{e^4}{(u - m^2)^2}\operatorname{Tr}\left[(\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_2 + m)\gamma_\mu\right]\end{aligned}$$

Let

$$
\begin{aligned}
f_{11} &= \mathrm{Tr}\left[(\not{p}_1 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\right] \\
f_{12} &= \mathrm{Tr}\left[(\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\right] \\
f_{22} &= \mathrm{Tr}\left[(\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_2 + m)\gamma_\mu\right]
\end{aligned}
$$

so that

$$
\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left[\frac{f_{11}}{(t-m^2)^2} + \frac{2f_{12}}{(t-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2}\right]
$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$
\begin{aligned}
f_{11} &= 32(p_1 \cdot p_3)(p_1 \cdot p_4) - 32m^2(p_1 \cdot p_2) + 64m^2(p_1 \cdot p_3) + 32m^2(p_1 \cdot p_4) - 64m^4 \\
f_{12} &= 16m^2(p_1 \cdot p_3) + 16m^2(p_1 \cdot p_4) - 32m^4 \\
f_{22} &= 32(p_1 \cdot p_3)(p_1 \cdot p_4) - 32m^2(p_1 \cdot p_2) + 32m^2(p_1 \cdot p_3) + 64m^2(p_1 \cdot p_4) - 64m^4
\end{aligned}
$$

In Mandelstam variables

$$
\begin{aligned}
f_{11} &= 8tu - 24tm^2 - 8um^2 - 8m^4 \\
f_{12} &= 8sm^2 - 32m^4 \\
f_{22} &= 8tu - 8tm^2 - 24um^2 - 8m^4
\end{aligned}
$$

For $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$
\begin{aligned}
f_{11} &= 8tu \\
f_{12} &= 0 \\
f_{22} &= 8tu
\end{aligned}
$$

Hence

$$
\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{e^4}{4}\left(\frac{f_{11}}{(t-m^2)^2} + \frac{2f_{12}}{(t-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2}\right) \\
&= \frac{e^4}{4}\left(\frac{8tu}{t^2} + \frac{8tu}{u^2}\right) \\
&= 2e^4\left(\frac{u}{t} + \frac{t}{u}\right)
\end{aligned}
$$

For $m = 0$ the Mandelstam variables are

$$
\begin{aligned}
t &= -2E^2(1 - \cos\theta) \\
u &= -2E^2(1 + \cos\theta)
\end{aligned}
$$

Hence

$$
\langle|\mathcal{M}|^2\rangle = 2e^4\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)
$$

Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{4(4\pi\varepsilon_0)^2 s}$$

where

$$s = (p_1 + p_2)^2 = 4E^2$$

For high energy experiments we have

$$\langle|\mathcal{M}|^2\rangle = 2e^4 \left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\varepsilon_0)^2 s} \left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)$$

Noting that

$$e^2 = 4\pi\varepsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s} \left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)$$

Noting that

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

we also have

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{2s} \left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right) \sin\theta \, d\theta \, d\phi$$

Let $S(\theta_1, \theta_2)$ be the following integral of $d\sigma$.

$$S(\theta_1, \theta_2) = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{\pi\alpha^2(\hbar c)^2}{s} [I(\theta_2) - I(\theta_1)]$$

where

$$I(\theta) = 2\cos\theta + 2\log(1-\cos\theta) - 2\log(1+\cos\theta)$$

The cumulative distribution function is

$$F(\theta) = \frac{S(a,\theta)}{S(a,\pi-a)} = \frac{I(\theta) - I(a)}{I(\pi-a) - I(a)}, \quad a \le \theta \le \pi - a$$

Angular support is reduced by an arbitrary angle $a > 0$ because $I(0)$ and $I(\pi)$ are undefined.

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 < \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi-a) - I(a)} \left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right) \sin\theta$$