

# Spin

Spin state  $|s\rangle$  is a normalized vector in  $\mathbb{C}^2$ .

$$|s\rangle = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}, \quad |c_+|^2 + |c_-|^2 = 1$$

Spin measurement probabilities are the transition probabilities from  $|s\rangle$  to an eigenstate.

For spin measurements in the  $z$  direction we have

$$\begin{aligned} \Pr(S_z = +\frac{\hbar}{2}) &= |\langle z_+ | s \rangle|^2 = |c_+|^2 \\ \Pr(S_z = -\frac{\hbar}{2}) &= |\langle z_- | s \rangle|^2 = |c_-|^2 \end{aligned}$$

where the eigenstates are

$$|z_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |z_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

By definition of expectation value we have

$$\langle S_z \rangle = \frac{\hbar}{2} \Pr(S_z = +\frac{\hbar}{2}) - \frac{\hbar}{2} \Pr(S_z = -\frac{\hbar}{2})$$

Rewrite as

$$\langle S_z \rangle = \frac{\hbar}{2} |\langle z_+ | s \rangle|^2 - \frac{\hbar}{2} |\langle z_- | s \rangle|^2$$

Rewrite again as

$$\langle S_z \rangle = \frac{\hbar}{2} \langle s | z_+ \rangle \langle z_+ | s \rangle - \frac{\hbar}{2} \langle s | z_- \rangle \langle z_- | s \rangle$$

Then by

$$\langle S_z \rangle = \langle s | S_z | s \rangle$$

we have

$$S_z = \frac{\hbar}{2} |z_+\rangle \langle z_+| - \frac{\hbar}{2} |z_-\rangle \langle z_-| = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

From the commutator

$$S_+ S_- - S_- S_+ = 2\hbar S_z$$

we have

$$S_+ S_- - S_- S_+ = \hbar^2 |z_+\rangle \langle z_+| - \hbar^2 |z_-\rangle \langle z_-|$$

Rewrite as

$$S_+ S_- - S_- S_+ = \hbar^2 |z_+\rangle \langle z_- | z_- \rangle \langle z_+| - \hbar^2 |z_-\rangle \langle z_+ | z_+ \rangle \langle z_-|$$

Hence

$$\begin{aligned} S_+ &= \hbar |z_+\rangle \langle z_-| = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ S_- &= \hbar |z_-\rangle \langle z_+| = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Then by

$$\begin{aligned} S_+ &= S_x + iS_y \\ S_- &= S_x - iS_y \end{aligned}$$

we obtain

$$\begin{aligned} S_x &= \frac{S_+ + S_-}{2} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ S_y &= \frac{S_+ - S_-}{2i} = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \end{aligned}$$

By solving for the eigenstates in

$$\begin{aligned} S_x |x_{\pm}\rangle &= \pm \frac{\hbar}{2} |x_{\pm}\rangle \\ S_y |y_{\pm}\rangle &= \pm \frac{\hbar}{2} |y_{\pm}\rangle \end{aligned}$$

we obtain

$$\begin{aligned} |x_+\rangle &= \frac{|z_+\rangle + |z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ |x_-\rangle &= \frac{|z_+\rangle - |z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} |y_+\rangle &= \frac{|z_+\rangle + i|z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \\ |y_-\rangle &= \frac{|z_+\rangle - i|z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \end{aligned}$$

Expectation values for spin operators.

$$\begin{aligned} \langle S_x \rangle &= \langle s | S_x | s \rangle = \frac{\hbar}{2} (c_+ c_-^* + c_+^* c_-) \\ \langle S_y \rangle &= \langle s | S_y | s \rangle = \frac{i\hbar}{2} (c_+ c_-^* - c_+^* c_-) \\ \langle S_z \rangle &= \langle s | S_z | s \rangle = \frac{\hbar}{2} (c_+ c_+^* - c_- c_-^*) \end{aligned}$$

Expected spin vector.

$$\langle \mathbf{S} \rangle = \begin{pmatrix} \langle S_x \rangle \\ \langle S_y \rangle \\ \langle S_z \rangle \end{pmatrix}$$

Consider the case of having determined  $\langle \mathbf{S} \rangle$  by experiment. To convert  $\langle \mathbf{S} \rangle$  to a spin state  $|s\rangle$ , let

$$\begin{aligned} x &= \frac{2}{\hbar} \langle S_x \rangle = \sin \theta \cos \phi \\ y &= \frac{2}{\hbar} \langle S_y \rangle = \sin \theta \sin \phi \\ z &= \frac{2}{\hbar} \langle S_z \rangle = \cos \theta \end{aligned}$$

Then

$$|s\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \exp(i\phi) \end{pmatrix}$$

where

$$\begin{aligned} \cos(\theta/2) &= \sqrt{\frac{\cos \theta + 1}{2}} = \sqrt{\frac{z + 1}{2}} \\ \sin(\theta/2) &= \sqrt{\frac{1 - \cos \theta}{2}} = \sqrt{\frac{1 - z}{2}} \end{aligned}$$

and

$$\exp(i\phi) = \cos \phi + i \sin \phi = \frac{x + iy}{\sqrt{x^2 + y^2}}$$

The following commutator was used to derive  $S_x$  and  $S_y$ .

$$S_+ S_- - S_- S_+ = 2\hbar S_z$$

The commutator is a consequence of the following wave equation for spin.

$$\hat{\mathbf{S}}\psi = (\mathbf{r} \times \hat{\mathbf{p}})\psi, \quad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \hat{\mathbf{p}} = -i\hbar \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

Rewrite in component form.

$$\begin{aligned} \hat{S}_x \psi &= -i\hbar \left( y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \psi \\ \hat{S}_y \psi &= -i\hbar \left( z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) \psi \\ \hat{S}_z \psi &= -i\hbar \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \psi \end{aligned}$$

By computer algebra we have

$$\begin{aligned} (\hat{S}_y \hat{S}_z - \hat{S}_z \hat{S}_y) \psi &= i\hbar \hat{S}_x \psi \\ (\hat{S}_z \hat{S}_x - \hat{S}_x \hat{S}_z) \psi &= i\hbar \hat{S}_y \psi \\ (\hat{S}_x \hat{S}_y - \hat{S}_y \hat{S}_x) \psi &= i\hbar \hat{S}_z \psi \end{aligned}$$

Let

$$\begin{aligned} \hat{S}_+ &= \hat{S}_x + i\hat{S}_y \\ \hat{S}_- &= \hat{S}_x - i\hat{S}_y \end{aligned}$$

By computer algebra

$$(\hat{S}_+ \hat{S}_- - \hat{S}_- \hat{S}_+) \psi = 2\hbar \hat{S}_z \psi$$

## Exercises

1. Verify that

$$\begin{aligned} S_x &= \frac{\hbar}{2} (|x_+\rangle\langle x_+| - |x_-\rangle\langle x_-|) \\ S_y &= \frac{\hbar}{2} (|y_+\rangle\langle y_+| - |y_-\rangle\langle y_-|) \\ S_z &= \frac{\hbar}{2} (|z_+\rangle\langle z_+| - |z_-\rangle\langle z_-|) \end{aligned}$$

2. Let  $|s\rangle$  be the following spin state.

$$|s\rangle = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$$

Verify that  $|s\rangle$  is normalized and that

$$\langle \mathbf{S} \rangle = \langle s | \mathbf{S} | s \rangle = \frac{\hbar}{2} \begin{pmatrix} \frac{4}{9} \\ \frac{8}{9} \\ \frac{1}{9} \end{pmatrix}$$

where

$$\mathbf{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$$

Note: In component form we have

$$\langle s | \mathbf{S} | s \rangle = s_\beta^* S^{\alpha\beta}{}_\gamma s^\gamma$$

Eigenmath requires a transpose so that the  $\beta$  indices are adjacent.

$$\langle s | \mathbf{S} | s \rangle = s_\beta^* S^{\beta\alpha}{}_\gamma s^\gamma$$

3. Let  $|s\rangle$  be the following spin state.

$$|s\rangle = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$$

Verify the following measurement probabilities for  $|s\rangle$ .

$$\begin{aligned} \Pr(S_x = +\frac{\hbar}{2}) &= |\langle x_+ | s \rangle|^2 = \frac{13}{18} \\ \Pr(S_x = -\frac{\hbar}{2}) &= |\langle x_- | s \rangle|^2 = \frac{5}{18} \\ \Pr(S_y = +\frac{\hbar}{2}) &= |\langle y_+ | s \rangle|^2 = \frac{17}{18} \\ \Pr(S_y = -\frac{\hbar}{2}) &= |\langle y_- | s \rangle|^2 = \frac{1}{18} \\ \Pr(S_z = +\frac{\hbar}{2}) &= |\langle z_+ | s \rangle|^2 = \frac{5}{9} \\ \Pr(S_z = -\frac{\hbar}{2}) &= |\langle z_- | s \rangle|^2 = \frac{4}{9} \end{aligned}$$

4. Let  $|s\rangle$  be the following spin state.

$$|s\rangle = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$$

Verify that the following spin state  $|\chi\rangle$  is indistinguishable from  $|s\rangle$ .

$$|\chi\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \exp(i\phi) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{5}}{3} \\ \frac{2+4i}{3\sqrt{5}} \end{pmatrix}$$

5. Verify the following commutators for  $\mathbf{S}\psi = (\mathbf{r} \times \mathbf{p})\psi$ .

$$[S_y, S_z] = i\hbar S_x$$

$$[S_z, S_x] = i\hbar S_y$$

$$[S_x, S_y] = i\hbar S_z$$

$$[S_+, S_-] = 2\hbar S_z$$