Feynman and Hibbs problem 3-9

Find the kernel for a particle in a constant field f where the Lagrangian is

$$L = \frac{m}{2}\dot{x}^2 + fx$$

From problem 2-3 we have

$$S(b,a) = \frac{m(x_b - x_a)^2}{2(t_b - t_a)} + \frac{f\,(t_b - t_a)(x_b + x_a)}{2} - \frac{f^2(t_b - t_a)^3}{24m}$$

By equation (3.51) which is

$$K(b,a) = F(t_b - t_a)\exp\left(\frac{iS(b,a)}{\hbar}\right)$$

we have

$$\begin{aligned} &K(b,a) = \\ &F(t_b - t_a)\exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)} + \frac{if\,(t_b - t_a)(x_b + x_a)}{2\hbar} - \frac{if^2(t_b - t_a)^3}{24\hbar m}\right) \end{aligned} \tag{1}$$

We now proceed to compute F. By equation (2.31) which is

$$K(b,a) = \int_{-\infty}^{\infty} K(b,c)K(c,a)\,dx_c$$

we have

$$K(b,a) = F(t_b - t_c)F(t_c - t_a)\int_{-\infty}^{\infty}\exp\left(\frac{iS(b,c)}{\hbar} + \frac{iS(c,a)}{\hbar}\right)\,dx_c$$

Reorganize the exponential as quadratic in x_c to obtain

$$\begin{aligned} K(b,a) = F(t_b - t_c)F(t_c - t_a)\exp\left(-\frac{if^2(t_b - t_c)^3}{24\hbar m} - \frac{if^2(t_c - t_a)^3}{24\hbar m}\right) \\ \times\int_{-\infty}^{\infty}\exp\left(iAx_c^2 + iBx_c + iC\right) \end{aligned} \tag{2}$$

where

$$A = \frac{m}{2\hbar}\left(\frac{1}{t_b - t_c} + \frac{1}{t_c - t_a}\right) \tag{3}$$

$$B = \frac{(t_b - t_a)f}{2\hbar} - \frac{m}{\hbar}\left(\frac{x_b}{t_b - t_c} + \frac{x_a}{t_c - t_a}\right) \tag{4}$$

$$C = \frac{f}{2\hbar}\big(x_b(t_b - t_c) + x_a(t_c - t_a)\big) + \frac{m}{2\hbar}\left(\frac{x_b^2}{t_b - t_c} + \frac{x_a^2}{t_c - t_a}\right) \tag{5}$$

Note that the exponential moved outside the integral is independent of x_c.

From the following formula

$$\int_{-\infty}^{\infty} \exp(iAx_c^2 + iBx_c + iC)\, dx_c = \left(-\frac{\pi}{iA}\right)^{1/2} \exp\left(-\frac{iB^2}{4A} + iC\right)$$

we have

$$\begin{aligned}\int_{-\infty}^{\infty} \exp(iAx_c^2 + iBx_c + iC)\, dx_c &= \left(-\frac{2\pi\hbar(t_b - t_c)(t_c - t_a)}{im(t_b - t_a)}\right)^{1/2} \\ &\times \exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)} + \frac{if\,(t_b - t_a)(x_b + x_a)}{2\hbar} - \frac{if^2(t_b - t_a)(t_b - t_c)(t_c - t_a)}{8\hbar m}\right)\end{aligned} \tag{6}$$

Substitute (6) into (2) to obtain

$$\begin{aligned}K(b,a) &= F(t_b - t_c)F(t_c - t_a)\exp\left(-\frac{if^2(t_b - t_c)^3}{24\hbar m} - \frac{if^2(t_c - t_a)^3}{24\hbar m}\right) \\ &\times \left(-\frac{2\pi\hbar(t_b - t_c)(t_c - t_a)}{im(t_b - t_a)}\right)^{1/2} \\ &\times \exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)} + \frac{if\,(t_b - t_a)(x_b + x_a)}{2\hbar} - \frac{if^2(t_b - t_a)(t_b - t_c)(t_c - t_a)}{8\hbar m}\right)\end{aligned}$$

Note that

$$(t_b - t_a)^3 = (t_b - t_c)^3 + (t_c - t_a)^3 + 3(t_b - t_a)(t_b - t_c)(t_c - t_a) \tag{7}$$

Using (7) we can combine exponentials to obtain

$$\begin{aligned}K(b,a) &= F(t_b - t_c)F(t_c - t_a)\left(-\frac{2\pi\hbar(t_b - t_c)(t_c - t_a)}{im(t_b - t_a)}\right)^{1/2} \\ &\times \exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)} + \frac{if\,(t_b - t_a)(x_b + x_a)}{2\hbar} - \frac{if^2(t_b - t_a)^3}{24\hbar m}\right)\end{aligned} \tag{8}$$

Equating (1) with (8) cancels the exponentials and leaves

$$F(t_b - t_a) = F(t_b - t_c)F(t_c - t_a)\left(-\frac{2\pi\hbar(t_b - t_c)(t_c - t_a)}{im(t_b - t_a)}\right)^{1/2} \tag{9}$$

From problem 3-7, let

$$F(t) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} g(t) \tag{10}$$

Substitute (10) into (9) to obtain

$$\begin{aligned}\left(\frac{m}{2\pi i\hbar(t_b - t_a)}\right)^{1/2} g(t_b - t_a) &= \left(\frac{m}{2\pi i\hbar(t_b - t_c)}\right)^{1/2} g(t_b - t_c)\\ &\times \left(\frac{m}{2\pi i\hbar(t_c - t_a)}\right)^{1/2} g(t_c - t_a)\left(-\frac{2\pi\hbar(t_b - t_c)(t_c - t_a)}{im(t_b - t_a)}\right)^{1/2}\end{aligned}$$

The coefficients cancel leaving

$$g(t_b - t_a) = g(t_b - t_c)g(t_c - t_a) \tag{11}$$

Hence

$$g(t) = 1$$

and

$$F(t_b - t_a) = \left(\frac{m}{2\pi i\hbar(t_b - t_a)}\right)^{1/2} \tag{12}$$

Substitute (12) into (1).

$$\begin{aligned}K(b, a) = &\left(\frac{m}{2\pi i\hbar(t_b - t_a)}\right)^{1/2}\\ &\times \exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)} + \frac{if\,(t_b - t_a)(x_b + x_a)}{2\hbar} - \frac{if^2(t_b - t_a)^3}{24\hbar m}\right)\end{aligned}$$