Bohr radius

Let $E(r)$ be total electron energy for a hydrogen atom (kinetic energy plus potential energy).

$$E(r) = \frac{\hbar^2}{2m_e r^2} - \frac{e^2}{4\pi\varepsilon_0 r}$$

Energy is minimized for r such that

$$\frac{dE(r)}{dr} = -\frac{\hbar^2}{m_e r^3} + \frac{e^2}{4\pi\varepsilon_0 r^2} = 0$$

Multiply both sides by r^2.

$$-\frac{\hbar^2}{m_e r} + \frac{e^2}{4\pi\varepsilon_0} = 0$$

Rewrite as

$$\frac{m_e r}{\hbar^2} = \frac{4\pi\varepsilon_0}{e^2}$$

Hence energy is minimized for

$$r = \frac{4\pi\varepsilon_0\hbar^2}{e^2 m_e} \equiv a_0$$