

This is the Dirac equation.

$$i\hbar \left(\gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z} \right) \psi = mc\psi$$

The following set of gamma matrices are known as the “Dirac representation.”

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Let

$$\phi(x, y, z, t) = p_x x + p_y y + p_z z - \frac{Et}{c}$$

where

$$E = \sqrt{p_x^2 c^2 + p_y^2 c^2 + p_z^2 c^2 + m^2 c^4}$$

The four positive wave solutions to the Dirac equation are

$$\begin{aligned} \psi_1 &= \begin{pmatrix} E + mc^2 \\ 0 \\ p_z c \\ p_x c + ip_y c \end{pmatrix} \exp\left(\frac{i\phi}{\hbar}\right) & \psi_2 &= \begin{pmatrix} 0 \\ E + mc^2 \\ p_x c - ip_y c \\ -p_z c \end{pmatrix} \exp\left(\frac{i\phi}{\hbar}\right) \\ \psi_3 &= \begin{pmatrix} p_z c \\ p_x c + ip_y c \\ E - mc^2 \\ 0 \end{pmatrix} \exp\left(\frac{i\phi}{\hbar}\right) & \psi_4 &= \begin{pmatrix} p_x c - ip_y c \\ -p_z c \\ 0 \\ E - mc^2 \end{pmatrix} \exp\left(\frac{i\phi}{\hbar}\right) \end{aligned}$$

The four negative wave solutions are

$$\begin{aligned} \psi_5 &= \begin{pmatrix} E - mc^2 \\ 0 \\ p_z c \\ p_x c + ip_y c \end{pmatrix} \exp\left(-\frac{i\phi}{\hbar}\right) & \psi_6 &= \begin{pmatrix} 0 \\ E - mc^2 \\ p_x c - ip_y c \\ -p_z c \end{pmatrix} \exp\left(-\frac{i\phi}{\hbar}\right) \\ \psi_7 &= \begin{pmatrix} p_z c \\ p_x c + ip_y c \\ E + mc^2 \\ 0 \end{pmatrix} \exp\left(-\frac{i\phi}{\hbar}\right) & \psi_8 &= \begin{pmatrix} p_x c - ip_y c \\ -p_z c \\ 0 \\ E + mc^2 \end{pmatrix} \exp\left(-\frac{i\phi}{\hbar}\right) \end{aligned}$$

The negative wave solutions flip the sign of the mc^2 term.

The following solutions are used by quantum electrodynamics.

$$\begin{aligned} \psi_1 & \text{ Fermion, spin up} \\ \psi_2 & \text{ Fermion, spin down} \\ \psi_7 & \text{ Anti-fermion, spin up} \\ \psi_8 & \text{ Anti-fermion, spin down} \end{aligned}$$