9-8. For the state for which there is just one photon present in level $1, \mathbf{k}$, all of the factors in the wave function are ϕ_0 except one, which is ϕ_1. But for an oscillator $\phi_1(x) = \sqrt{2}x\phi_0(x)$. The wave function representing an excited running wave is a linear superposition of the state with the cosine mode excited and i times the state with the sine wave excited, so show that the unnormalized wave function for just one photon present in $1, \mathbf{k}$ is $\bar{a}^*_{1,\mathbf{k}}\Phi_0$. The normalization is $\int \Phi_0^* \bar{a}_{1,\mathbf{k}} \bar{a}^*_{1,\mathbf{k}} \Phi_0 \, d\bar{a}$, or the expectation of $\bar{a}_{1,\mathbf{k}}\bar{a}^*_{1,\mathbf{k}}$ for the vacuum, which we have seen in the preceding problem is $\hbar/2kc$. Hence the normalized one-photon state is $\sqrt{2kc/\hbar}\bar{a}^*_{1,\mathbf{k}}\Phi_0$.

This is a state with the cosine mode excited.

$$\bar{a}^c_{1,\mathbf{k}}\Phi_0$$

This is a state with the sine mode excited.

$$\bar{a}^s_{1,\mathbf{k}}\Phi_0$$

This is a linear superposition of the cosine state and i times the sine state. The $1/\sqrt{2}$ is for normalization, i.e., $|1+i| = \sqrt{2}$.

$$\frac{1}{\sqrt{2}}\left(\bar{a}^c_{1,\mathbf{k}}\Phi_0 + i\bar{a}^s_{1,\mathbf{k}}\Phi_0\right) = \bar{a}^*_{1,\mathbf{k}}\Phi_0$$

Here are some additional results.

From equation (9.43) and problem 9-6, let

$$\Phi_0 = \exp\left(-\frac{kc}{4\hbar}(\bar{a}^c_{1,\mathbf{k}})^2 - \frac{kc}{4\hbar}(\bar{a}^s_{1,\mathbf{k}})^2 - \frac{kc}{4\hbar}(\bar{a}^c_{2,\mathbf{k}})^2 - \frac{kc}{4\hbar}(\bar{a}^s_{2,\mathbf{k}})^2\right)$$

It follows that

$$\Phi_0^*\Phi_0 = \exp\left(-\frac{kc}{2\hbar}(\bar{a}^c_{1,\mathbf{k}})^2 - \frac{kc}{2\hbar}(\bar{a}^s_{1,\mathbf{k}})^2 - \frac{kc}{2\hbar}(\bar{a}^c_{2,\mathbf{k}})^2 - \frac{kc}{2\hbar}(\bar{a}^s_{2,\mathbf{k}})^2\right)$$

For simplicity of notation, let

$$d\bar{a} = d\bar{a}^c_{1,\mathbf{k}}\, d\bar{a}^s_{1,\mathbf{k}}\, d\bar{a}^c_{2,\mathbf{k}}\, d\bar{a}^s_{2,\mathbf{k}}$$

The expectation of Φ_0 is

$$\langle\Phi_0\rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_0^*\Phi_0 \, d\bar{a} = \left(\frac{2\pi\hbar}{kc}\right)^2 \tag{1}$$

Let

$$\Phi_1 = \bar{a}^*_{1,\mathbf{k}}\Phi_0$$

Then

$$\Phi_1^*\Phi_1 = \Phi_0^*\left(\frac{(\bar{a}^c_{1,\mathbf{k}})^2 + (\bar{a}^s_{1,\mathbf{k}})^2}{2}\right)\Phi_0$$

The expectation of Φ_1 is

$$\langle\Phi_1\rangle = \int\limits_{-\infty}^{\infty}\cdots\int\limits_{-\infty}^{\infty}\Phi_1^*\Phi_1\,d\bar{a} = \frac{\hbar}{kc}\left(\frac{2\pi\hbar}{kc}\right)^2 \tag{2}$$

The expectation for n photons is

$$\langle\Phi_n\rangle = \int\limits_{-\infty}^{\infty}\cdots\int\limits_{-\infty}^{\infty}\Phi_0^*\left(\frac{(\bar{a}^c_{1,\mathbf{k}})^2 + (\bar{a}^s_{1,\mathbf{k}})^2}{2}\right)^n\Phi_0\,d\bar{a}$$

By a result from problem 9-7

$$\langle\Phi_n\rangle = n!\left(\frac{\hbar}{kc}\right)^n\left(\frac{2\pi\hbar}{kc}\right)^2 = n!\left(\frac{\hbar}{kc}\right)^n\langle\Phi_0\rangle \tag{3}$$