Atomic transitions 4

From the previous section the transition rate is

$$R_{a\to b} = \frac{\pi e^2}{\varepsilon_0 m^2 \hbar^2} \left|\langle\psi_b|\exp(i\mathbf{k}\cdot\mathbf{r})\boldsymbol{\epsilon}\cdot\mathbf{p}|\psi_a\rangle\right|^2 \frac{\rho(\omega_0)}{\omega_0^2}$$

Substitute the dipole approximation

$$\exp(i\mathbf{k}\cdot\mathbf{r}) \approx 1$$

to obtain

$$R_{a\to b} = \frac{\pi e^2}{\varepsilon_0 m^2 \hbar^2} \left|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}|\psi_a\rangle\right|^2 \frac{\rho(\omega_0)}{\omega_0^2}$$

By the identity

$$\mathbf{p} = \frac{im}{\hbar}[H_0,\mathbf{r}] \tag{1}$$

we have

$$\begin{aligned}
\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}|\psi_a\rangle &= \frac{im}{\hbar}\langle\psi_b|\boldsymbol{\epsilon}\cdot[H_0,\mathbf{r}]|\psi_a\rangle \\
&= \frac{im}{\hbar}\langle\psi_b|\boldsymbol{\epsilon}\cdot H_0\mathbf{r}|\psi_a\rangle - \frac{im}{\hbar}\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{r}H_0|\psi_a\rangle \\
&= \frac{im}{\hbar}E_b\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{r}|\psi_a\rangle - \frac{im}{\hbar}E_a\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{r}|\psi_a\rangle \\
&= \frac{im}{\hbar}(E_b - E_a)\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{r}|\psi_a\rangle \\
&= im\omega_0\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{r}|\psi_a\rangle
\end{aligned}$$

Hence

$$R_{a\to b} = \frac{\pi e^2}{\varepsilon_0 \hbar^2}\left|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{r}|\psi_a\rangle\right|^2 \rho(\omega_0)$$

Verify dimensions.

$$R_{a\to b} \propto \frac{\underset{\mathrm{C}^2}{e^2}}{\underset{\mathrm{C^2\,J^{-1}\,m^{-1}}}{\epsilon_0}\quad\underset{\mathrm{J^2\,s^2}}{\hbar^2}} \times \underset{\mathrm{m}^2}{\left|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{r}|\psi_a\rangle\right|^2} \times \underset{\mathrm{J\,m^{-3}\,s}}{\rho(\omega_0)} = \mathrm{s}^{-1}$$

Eigenmath script