

Hydrogen transition 1

Start with the perturbing Hamiltonian

$$H_1(\mathbf{r}, t) = -\frac{eA_0\boldsymbol{\epsilon} \cdot \mathbf{p}}{m_e} \cos(-\mathbf{k} \cdot \mathbf{r} + \omega t)$$

In exponential form

$$H_1(\mathbf{r}, t) = -\frac{eA_0\boldsymbol{\epsilon} \cdot \mathbf{p}}{m_e} \left[\frac{1}{2} \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t) + \frac{1}{2} \exp(-i\mathbf{k} \cdot \mathbf{r} + i\omega t) \right]$$

Given the initial condition $c_b(0) = 0$ the first-order approximation for $c_b(t)$ is

$$c_b(t) = -\frac{i}{\hbar} \int_0^t \langle \psi_b | H_1 | \psi_a \rangle \exp(i\omega_0 t') dt', \quad \omega_0 = \frac{E_b - E_a}{\hbar}$$

Solve the integral to obtain

$$c_b(t) = \frac{eA_0}{2\hbar m_e} \langle \psi_a | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_b \rangle \frac{\exp(i(\omega_0 - \omega)t) - 1}{\omega_0 - \omega} + \frac{eA_0}{2\hbar m_e} \langle \psi_a | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(-i\mathbf{k} \cdot \mathbf{r}) | \psi_b \rangle \frac{\exp(i(\omega_0 + \omega)t) - 1}{\omega_0 + \omega} \quad (1)$$

As an approximation, discard the second term since the first term dominates for $\omega \approx \omega_0$.

$$c_b(t) = \frac{eA_0}{2\hbar m_e} \langle \psi_a | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_b \rangle \frac{\exp(i(\omega_0 - \omega)t) - 1}{\omega_0 - \omega}$$

Rewrite $c_b(t)$ in the form of a sine function.

$$c_b(t) = \frac{ieA_0}{\hbar m_e} \langle \psi_a | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_b \rangle \frac{\sin(\frac{1}{2}(\omega_0 - \omega)t)}{\omega_0 - \omega} \exp\left(\frac{i}{2}(\omega_0 - \omega)t\right) \quad (2)$$

Hence the transition probability is

$$P_{a \rightarrow b}(t) = |c_b(t)|^2 = \frac{e^2 |A_0|^2}{\hbar^2 m_e^2} \left| \langle \psi_a | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_b \rangle \right|^2 \frac{\sin^2(\frac{1}{2}(\omega_0 - \omega)t)}{(\omega_0 - \omega)^2}$$