Chapter 6

Start with equation (6.1).

$$K_V(b, a) = \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} \left(\tfrac{1}{2}m\dot{x}^2 - V(x(t), t)\right) dt\right) \mathcal{D}x(t) \qquad (6.1)$$

Partition the integral.

$$K_V(b, a) = \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2 \, dt - \frac{i}{\hbar} \int_{t_a}^{t_b} V(x(t), t) \, dt\right) \mathcal{D}x(t)$$

Factor the exponential.

$$K_V(b, a) = \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2 \, dt\right) \exp\left(-\frac{i}{\hbar} \int_{t_a}^{t_b} V(x(t), t) \, dt\right) \mathcal{D}x(t)$$

Use t_c for the measure in the second integral.

$$K_V(b, a) = \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2 \, dt\right) \exp\left(-\frac{i}{\hbar} \int_{t_a}^{t_b} V(x(t_c), t_c) \, dt_c\right) \mathcal{D}x(t)$$

Make the second exponential a power series.

$$K_V(b, a) = \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2 \, dt\right) \times$$

$$\left(1 - \frac{i}{\hbar} \int_{t_a}^{t_b} V(x(t_c), t_c) \, dt_c + \frac{1}{2}\left(-\frac{i}{\hbar} \int_{t_a}^{t_b} V(x(t_c), t_c) \, dt_c\right)^2 + \cdots\right) \mathcal{D}x(t)$$

Expand the product.

$$K_V(b, a) = \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2 \, dt\right) \mathcal{D}x(t)$$

$$- \frac{i}{\hbar} \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2 \, dt\right) \left(\int_{t_a}^{t_b} V(x(t_c), t_c) \, dt_c\right) \mathcal{D}x(t)$$

$$- \frac{1}{2\hbar^2} \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2 \, dt\right) \left(\int_{t_a}^{t_b} V(x(t_c), t_c) \, dt_c\right)^2 \mathcal{D}x(t) + \cdots$$

Let

$$K_0(b, a) = \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2 \, dt\right) \mathcal{D}x(t) \tag{6.5}$$

$$K^{(1)}(b, a) = -\frac{i}{\hbar} \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2 \, dt\right) \left(\int_{t_a}^{t_b} V(x(t_c), t_c) \, dt_c\right) \mathcal{D}x(t)$$

$$K^{(2)}(b, a) = -\frac{1}{2\hbar^2} \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2 \, dt\right) \left(\int_{t_a}^{t_b} V(x(t_c), t_c) \, dt_c\right)^2 \mathcal{D}x(t)$$

Then equation (6.4) follows.

$$K_V(b, a) = K_0(b, a) + K^{(1)}(b, a) + K^{(2)}(b, a) + \cdots \tag{6.4}$$

Let us take a closer look at $K^{(1)}$. By the distributive law we can change the order of integration and obtain the following.

$$K^{(1)}(b, a) = -\frac{i}{\hbar} \int_{t_a}^{t_b} \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2 \, dt\right) V(x(t_c), t_c) \, \mathcal{D}x(t) \, dt_c$$

Let

$$I(t_c) = \int_{x_a}^{x_b} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} \tfrac{1}{2}m\dot{x}^2 \, dt\right) V(x(t_c), t_c) \, \mathcal{D}x(t)$$

so that

$$K^{(1)}(b, a) = -\frac{i}{\hbar} \int_{t_a}^{t_b} I(t_c) \, dt_c$$

We want to rewrite $I(t_c)$ as an integral over $x(t_c)$.

Let $x_c = x(t_c)$ and note that x_c can take on any value. In other words, for any $x_c \in (-\infty, \infty)$ there is a path from x_a to x_b that goes through x_c. Since $V(x_c, t_c)$ is a function of c only, the kernel for the path is a free particle from a to c and from c to b. Hence

$$I(t_c) = \int_{-\infty}^{\infty} K_0(x_b, t_b; x_c, t_c) V(x_c, t_c) K_0(x_c, t_c; x_a, t_a) \, dx_c$$

Or more compactly

$$I(t_c) = \int_{-\infty}^{\infty} K_0(b, c) V(c) K_0(c, a) \, dx_c$$

Hence

$$K^{(1)}(b, a) = -\frac{i}{\hbar} \int_{t_a}^{t_b} \int_{-\infty}^{\infty} K_0(b, c) V(c) K_0(c, a) \, dx_c \, dt_c$$

2

7-1

"Thus a transition element will be written as $\langle F \rangle_S$ instead of $\langle \chi | F | \psi \rangle_S$."

7-2

δF is the differential of F.

Consider equation (7.29).

$$\langle F \rangle_S = \int F\big(x(t) + \eta(t)\big) \exp\left(\frac{i}{\hbar} S\big(x(t) + \eta(t)\big)\right) \mathcal{D}x(t)$$

Consider equation (7.31).

$$\int \frac{\partial F}{\partial x_k} \exp\left(\frac{i}{\hbar} S\big(x(t)\big)\right) \mathcal{D}x(t) \tag{7.31}$$

Integrate by parts. Let

$$u = \exp\left(\frac{i}{\hbar} S\big(x(t)\big)\right)$$

$$dv = \frac{\partial F}{\partial x_k} \mathcal{D}x(t)$$

Then

$$\int u\, dv = uv - \int v\, du$$

The uv part vanishes, why?