

9-5. The momentum in the field is given by

$$\frac{1}{4\pi c} \int \mathbf{E} \times \mathbf{B} d^3\mathbf{r}$$

In the absence of matter (so $\phi_{\mathbf{k}} = 0$), show that this is

$$i \int \mathbf{k} (\mathbf{a}_{\mathbf{k}}^* \cdot \dot{\mathbf{a}}_{\mathbf{k}}) \frac{d^3\mathbf{k}}{(2\pi)^3}$$

From equations (9.26) and (9.27) with $\phi = 0$ we have

$$\mathbf{E} d^3\mathbf{r} = \sqrt{4\pi} \dot{\mathbf{a}}_{\mathbf{k}} \frac{d^3\mathbf{k}}{(2\pi)^3} \quad \mathbf{B} d^3\mathbf{r} = \sqrt{4\pi} i c \mathbf{k} \times \mathbf{a}_{\mathbf{k}} \frac{d^3\mathbf{k}}{(2\pi)^3}$$

Hence

$$\begin{aligned} \mathbf{E} \times \mathbf{B} d^3\mathbf{r} &= \sqrt{4\pi} \dot{\mathbf{a}}_{\mathbf{k}} \times \left(\sqrt{4\pi} i c \mathbf{k} \times \mathbf{a}_{\mathbf{k}} \right) \frac{d^3\mathbf{k}}{(2\pi)^3} \\ &= 4\pi i c (\dot{\mathbf{a}}_{\mathbf{k}} \cdot \mathbf{a}_{\mathbf{k}}) \mathbf{k} - 4\pi i c (\dot{\mathbf{a}}_{\mathbf{k}} \cdot \mathbf{k}) \mathbf{a}_{\mathbf{k}} \frac{d^3\mathbf{k}}{(2\pi)^3} \end{aligned}$$

By orthogonality of \mathbf{E} and \mathbf{k} (see problem 9-1), $\dot{\mathbf{a}}_{\mathbf{k}} \cdot \mathbf{k} = 0$ hence

$$\mathbf{E} \times \mathbf{B} d^3\mathbf{r} = 4\pi i c (\dot{\mathbf{a}}_{\mathbf{k}} \cdot \mathbf{a}_{\mathbf{k}}) \mathbf{k} \frac{d^3\mathbf{k}}{(2\pi)^3}$$