

16.1.1. Consider a system that has two possible single fermion states, 1 and 2, and can have anywhere from zero to two particles in it. There are therefore four possible states of this system:  $|0_1, 0_2\rangle$  (the state with no particles in either single-fermion state, a state we could also write as the empty state  $|0\rangle$ ),  $|1_1, 0_2\rangle$ ,  $|0_1, 1_2\rangle$ , and  $|1_1, 1_2\rangle$ . (We will also choose the standard ordering of the states to be in the order 1, 2.) Any state of the system could be described as a linear combination of these four basis states, i.e.,

$$|\Psi\rangle = c_1|0_1, 0_2\rangle + c_2|1_1, 0_2\rangle + c_3|0_1, 1_2\rangle + c_4|1_1, 1_2\rangle$$

which we could also choose to write as a vector

$$|\Psi\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$$

- (i) Construct  $4 \times 4$  matrices for each of the operators  $\hat{b}_1^\dagger$ ,  $\hat{b}_1$ ,  $\hat{b}_2^\dagger$ , and  $\hat{b}_2$ .
- (ii) Explicitly verify by matrix multiplication the anticommutation relations

$$\begin{aligned}\hat{b}_1^\dagger \hat{b}_1 + \hat{b}_1 \hat{b}_1^\dagger &= 1 \\ \hat{b}_2^\dagger \hat{b}_2 + \hat{b}_2 \hat{b}_2^\dagger &= 1 \\ \hat{b}_1^\dagger \hat{b}_2^\dagger + \hat{b}_2^\dagger \hat{b}_1^\dagger &= 0 \\ \hat{b}_1^\dagger \hat{b}_1^\dagger + \hat{b}_1^\dagger \hat{b}_1^\dagger &= 0\end{aligned}$$

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(i) Let

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |10\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |01\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

For  $\hat{b}_1$  we require

$$\hat{b}_1|10\rangle = |00\rangle \quad \text{and} \quad \hat{b}_1|11\rangle = -|01\rangle$$

where the minus sign is due to antisymmetry. It follows that

$$\hat{b}_1 = |00\rangle \otimes |10\rangle - |01\rangle \otimes |11\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where  $\otimes$  indicates an outer product.

For  $\hat{b}_2$  we require

$$\hat{b}_2|01\rangle = |00\rangle \quad \text{and} \quad \hat{b}_2|11\rangle = |10\rangle$$

Hence

$$\hat{b}_2 = |00\rangle \otimes |01\rangle + |10\rangle \otimes |11\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For the creation operators we have

$$\hat{b}_1^\dagger = (\hat{b}_1)^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and

$$\hat{b}_2^\dagger = (\hat{b}_2)^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(ii) See Eigenmath demo.