

9-7. Show, for the vacuum state, the expectation value of  $\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{q}}$  is  $(\hbar/2kc)\delta_{\mathbf{k},\mathbf{q}}$  and that of  $\bar{a}_{1,\mathbf{k}} \bar{a}_{1,\mathbf{q}}$  is  $(\hbar/2kc)\delta_{-\mathbf{k},\mathbf{q}}$ . Develop a formula for the expectation of  $(\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}})^r$  for integral  $r$  and explain thereby how the expectation of such quantities as  $(\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}})^r (\bar{a}_{1,\mathbf{q}}^* \bar{a}_{1,\mathbf{q}})^s$  can be got for  $\mathbf{q} \neq \mathbf{k}$ . Show that the expectation of  $(\bar{a}_{1,\mathbf{k}})^2$  or  $(\bar{a}_{1,\mathbf{k}}^*)^2$  vanishes. Show that the expectation of  $(\bar{a}_{1,\mathbf{k}})^2$  or  $(\bar{a}_{1,\mathbf{k}}^*)^2$  vanishes. Show that the expectation of the product of any odd number of  $\bar{a}$ 's is zero and that you can compute the expectation value of any product of  $\bar{a}$ 's or  $\bar{a}^*$ 's for the vacuum state.

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We will use the following table of integrals.

$$\int_{-\infty}^{\infty} \exp(-ax^2 + b) dx = \sqrt{\frac{\pi}{a}} \exp(b) \quad (1)$$

$$\int_{-\infty}^{\infty} x \exp(-ax^2 + b) dx = 0 \quad (2)$$

$$\int_{-\infty}^{\infty} x^2 \exp(-ax^2 + b) dx = \frac{1}{2a} \sqrt{\frac{\pi}{a}} \exp(b) \quad (3)$$

For simplicity of notation, let

$$A = \bar{a}_{1,\mathbf{k}}^c \quad B = \bar{a}_{1,\mathbf{k}}^s \quad C = \bar{a}_{1,\mathbf{q}}^c \quad D = \bar{a}_{1,\mathbf{q}}^s$$

These formulas convert  $\bar{a}$  to sine and cosine modes.

$$\bar{a}_{1,\mathbf{k}} = \frac{1}{\sqrt{2}}(A - iB) \quad \bar{a}_{1,\mathbf{q}} = \frac{1}{\sqrt{2}}(C - iD) \quad (4)$$

Adapted from equation (8.84)

$$\langle \Phi_0 | f | \Phi_0 \rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f |\Phi_0|^2 dA dB dC dD$$

The following  $|\Phi_0|^2$  is adapted from equation (9.43). Symbol  $q$  is a mode (physical unit meter<sup>-1</sup>), not an electric charge. Note that we *could* include other modes in addition to  $k$  and  $q$ . However, integrals over unused modes are cancelled by the normalization constant.

$$|\Phi_0|^2 = \Phi_0^* \Phi_0 = \exp \left( -\frac{kc}{\hbar} A^2 - \frac{kc}{\hbar} B^2 - \frac{qc}{\hbar} C^2 - \frac{qc}{\hbar} D^2 \right)$$

Compute the normalization constant  $K$ .

$$K = \langle \Phi_0 | 1 | \Phi_0 \rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\Phi_0|^2 dA dB dC dD$$

By integral (1) for each factor in the measure and with  $a = kc/\hbar$

$$K = \left( \frac{\pi \hbar}{kc} \right)^{1/2} \left( \frac{\pi \hbar}{kc} \right)^{1/2} \left( \frac{\pi \hbar}{qc} \right)^{1/2} \left( \frac{\pi \hbar}{qc} \right)^{1/2}$$

Compute the expectation of  $\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}}$ . From (4) we have

$$\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} = \frac{A^2 + B^2}{2}$$

Hence

$$\langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle = \frac{1}{K} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \frac{A^2 + B^2}{2} \right) |\Phi_0|^2 dA dB dC dD$$

Rewrite as

$$\begin{aligned} \langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle &= \frac{1}{2K} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} A^2 |\Phi_0|^2 dA dB dC dD \\ &\quad + \frac{1}{2K} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} B^2 |\Phi_0|^2 dA dB dC dD \end{aligned}$$

By integrals (1) and (3) with  $a = kc/\hbar$  we have

$$\langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle = \frac{1}{K} \frac{\hbar}{2kc} \left( \frac{\pi \hbar}{kc} \right)^{1/2} \left( \frac{\pi \hbar}{kc} \right)^{1/2} \left( \frac{\pi \hbar}{qc} \right)^{1/2} \left( \frac{\pi \hbar}{qc} \right)^{1/2}$$

The radicals are cancelled by the normalization constant  $K$ , hence

$$\langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} | \Phi_0 \rangle = \frac{\hbar}{2kc} \quad (5)$$

Compute the expectation of  $\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{q}}$ .

$$\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{q}} = \frac{AC + BD - iAD + iBC}{2}$$

Hence

$$\begin{aligned} \langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{q}} | \Phi_0 \rangle = \\ \frac{1}{K} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \frac{AC + BD - iAD + iBC}{2} \right) |\Phi_0|^2 dA dB dC dD \end{aligned}$$

By integral (2) all terms are zero, hence

$$\langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{q}} | \Phi_0 \rangle = 0 \quad (6)$$

Combine (5) and (6) to obtain

$$\langle \Phi_0 | \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{q}} | \Phi_0 \rangle = \frac{\hbar}{2kc} \delta_{\mathbf{k},\mathbf{q}}$$

By equation (8.77)

$$\bar{a}_{1,\mathbf{k}}^* = \bar{a}_{1,-\mathbf{k}}$$

Hence

$$\langle \Phi_0 | \bar{a}_{1,-\mathbf{k}} \bar{a}_{1,\mathbf{q}} | \Phi_0 \rangle = \frac{\hbar}{2kc} \delta_{\mathbf{k},\mathbf{q}}$$

(9-7 cont'd) Develop a formula for the expectation of  $(\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}})^r$  for integral  $r$  and explain thereby how the expectation of such quantities as  $(\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}})^r (\bar{a}_{1,\mathbf{q}}^* \bar{a}_{1,\mathbf{q}})^s$  can be got for  $\mathbf{q} \neq \mathbf{k}$ .

By the binomial theorem

$$\left( \frac{A^2 + B^2}{2} \right)^r = \frac{1}{2^r} \sum_{j=0}^r \binom{r}{j} A^{2j} B^{2(r-j)} \quad (7)$$

To compute the expectation of (7) we need the following integral.

$$\begin{aligned} \int_{-\infty}^{\infty} x^{2n} \exp(-ax^2 + b) dx &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n a^n} \sqrt{\frac{\pi}{a}} \exp(b) \\ &= (2n-1)!! \left( \frac{1}{2a} \right)^n \sqrt{\frac{\pi}{a}} \exp(b) \end{aligned} \quad (8)$$

From equation (8), define the following function  $F$ . (The  $\sqrt{\pi/a}$  factor is left out because it gets cancelled by the normalization constant  $K$ .)

$$F(n) = (2n - 1)!! \left( \frac{\hbar}{2kc} \right)^n$$

Note that

$$F(j)F(r-j) = (2j-1)!! (2r-2j-1)!! \left( \frac{\hbar}{2kc} \right)^j \left( \frac{\hbar}{2kc} \right)^{r-j}$$

It turns out that

$$\frac{1}{2^r} \sum_{j=0}^r \binom{r}{j} (2j-1)!! (2r-2j-1)!! = r!$$

Hence

$$\langle \Phi_0^* | (\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}})^r | \Phi_0 \rangle = r! \left( \frac{\hbar}{2kc} \right)^r$$

Regarding the  $\mathbf{q} \neq \mathbf{k}$  part of the problem, we have

$$\begin{aligned} \left( \frac{A^2 + B^2}{2} \right)^r \left( \frac{C^2 + D^2}{2} \right)^s = \\ \left( \frac{1}{2^r} \sum_{j=0}^r \binom{r}{j} A^{2j} B^{2(r-j)} \right) \left( \frac{1}{2^s} \sum_{k=0}^s \binom{s}{k} C^{2k} D^{2(s-k)} \right) \end{aligned}$$

Hence

$$\langle \Phi_0^* | (\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}})^r (\bar{a}_{1,\mathbf{q}}^* \bar{a}_{1,\mathbf{q}})^s | \Phi_0 \rangle = r! \left( \frac{\hbar}{2kc} \right)^r s! \left( \frac{\hbar}{2qc} \right)^s$$

(9-7 cont'd) Show that the expectation of  $(\bar{a}_{1,\mathbf{k}})^2$  or  $(\bar{a}_{1,\mathbf{k}}^*)^2$  vanishes.

We have

$$(\bar{a}_{1,\mathbf{k}})^2 = \frac{A^2 - B^2}{2} - iAB \quad (\bar{a}_{1,\mathbf{k}}^*)^2 = \frac{A^2 - B^2}{2} + iAB$$

The integrals of  $A^2$  and  $-B^2$  cancel each other. The integral of  $AB$  vanishes by integral (2).

(9-7 cont'd) Show that the expectation of the product of any odd number of  $\bar{a}$ 's is zero and that you can compute the expectation value of any product of  $\bar{a}$ 's or  $\bar{a}^*$ 's for the vacuum state.