

8-3. Show that  $Q_\alpha^c$ ,  $Q_\alpha^s$  are normal coordinates corresponding to standing wave normal modes  $\cos(2\pi\alpha j/N)$  and  $\sin(2\pi\alpha j/N)$ , in the sense that (for  $N$  odd)

$$q_j(t) = \sqrt{\frac{2}{N}} \left( \frac{1}{2} Q_0^c(t) + \sum_{\alpha=1}^{(N-1)/2} \left( Q_\alpha^c(t) \cos \frac{2\pi\alpha j}{N} + Q_\alpha^s(t) \sin \frac{2\pi\alpha j}{N} \right) \right) \quad (8.82)$$

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Consider the following equations.

$$Q_\alpha(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^N q_k(t) \left( \cos \frac{2\pi\alpha k}{N} - i \sin \frac{2\pi\alpha k}{N} \right) \quad (8.77)$$

$$Q_\alpha^c = \frac{1}{\sqrt{2}} (Q_\alpha + Q_\alpha^*) \quad (8.79)$$

$$Q_\alpha^s = \frac{i}{\sqrt{2}} (Q_\alpha - Q_\alpha^*) \quad (8.80)$$

Substitute (8.77) into (8.82).

$$q_j = \frac{1}{\sqrt{2N}} Q_0^c + \frac{1}{N} \sum_{\alpha=1}^{(N-1)/2} \sum_{k=1}^N q_k (T_1 + T_2 + T_3 + T_4) \quad (1)$$

where

$$\begin{aligned} T_1 &= \cos \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N} - i \sin \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N} \\ T_2 &= \cos \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N} + i \sin \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N} \\ T_3 &= i \cos \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N} + \sin \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N} \\ T_4 &= -i \cos \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N} + \sin \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N} \end{aligned}$$

It follows that

$$T_1 + T_2 + T_3 + T_4 = 2 \cos \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N} + 2 \sin \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N}$$

By trigonometric identities

$$T_1 + T_2 + T_3 + T_4 = 2 \cos \left( \frac{2\pi\alpha}{N}(j - k) \right) \quad (2)$$

Substitute (2) into (1) to obtain

$$q_j = \frac{1}{\sqrt{2N}} Q_0^c + \frac{2}{N} \sum_{\alpha=1}^{(N-1)/2} \sum_{k=1}^N q_k \cos \left( \frac{2\pi\alpha}{N}(j - k) \right) \quad (3)$$

By equations (8.77) and (8.79) with  $\alpha = 0$  we have

$$Q_0^c = \sqrt{\frac{2}{N}} \sum_{k=1}^N q_k \quad (4)$$

Substitute (4) into (3).

$$q_j = \frac{1}{N} \sum_{k=1}^N q_k + \frac{2}{N} \sum_{\alpha=1}^{(N-1)/2} \sum_{k=1}^N q_k \cos \left( \frac{2\pi\alpha}{N}(j - k) \right)$$

Rewrite as

$$q_j = \sum_{k=1}^N q_k \left( \frac{1}{N} + \frac{2}{N} \sum_{\alpha=1}^{(N-1)/2} \cos \left( \frac{2\pi\alpha}{N}(j - k) \right) \right) \quad (5)$$

For the sum over  $\alpha$  in (5) we have

$$\sum_{\alpha=1}^{(N-1)/2} \cos \left( \frac{2\pi\alpha}{N}(j - k) \right) = \begin{cases} (N-1)/2 & j = k \\ -1/2 & j \neq k \end{cases} \quad (\text{see proof below})$$

Hence for  $j = k$

$$\frac{1}{N} + \frac{2}{N} \sum_{\alpha=1}^{(N-1)/2} \cos \left( \frac{2\pi\alpha}{N}(j - k) \right) = 1$$

and for  $j \neq k$

$$\frac{1}{N} + \frac{2}{N} \sum_{\alpha=1}^{(N-1)/2} \cos \left( \frac{2\pi\alpha}{N}(j - k) \right) = 0$$

It follows that (5) reduces to the following tautology.

$$q_j = \sum_{k=1}^N q_k \delta(j - k) = q_j$$

Hence (8.82) is proven to be correct.

We will now show that for  $N$  odd and  $j \neq k$ ,

$$\sum_{\alpha=1}^{(N-1)/2} \cos\left(\frac{2\pi\alpha}{N}(j - k)\right) = -\frac{1}{2}$$

Consider the following geometric series formula.

$$\sum_{k=0}^{N-1} z^k = \frac{1 - z^N}{1 - z}, \quad |z| < 1 \quad (6)$$

Let

$$z = \exp\left(\frac{2\pi in}{N}\right)$$

where  $n$  is an integer such that  $2n < N$  so that  $|z| < 1$ . Then

$$z^N = \exp(2\pi in) = 1$$

Hence by (6)

$$\sum_{k=0}^{N-1} z^k = 0 \quad (7)$$

Noting that

$$z^k = \exp\left(\frac{2\pi ink}{N}\right) = \cos\left(\frac{2\pi nk}{N}\right) + i \sin\left(\frac{2\pi nk}{N}\right)$$

we have by (7)

$$\sum_{k=0}^{N-1} \cos\left(\frac{2\pi nk}{N}\right) + i \sum_{k=0}^{N-1} \sin\left(\frac{2\pi nk}{N}\right) = 0$$

Hence

$$\sum_{k=0}^{N-1} \cos\left(\frac{2\pi nk}{N}\right) = 0 \quad (8)$$

We need to remove the restriction  $2n < N$ . Consider the following cosine function.

$$\cos\left(\frac{2\pi(N-n)k}{N}\right) \quad (9)$$

By trigonometric identities, (9) is equivalent to

$$\cos\left(\frac{2\pi nk}{N}\right) \cos(2\pi k) + \sin\left(\frac{2\pi nk}{N}\right) \sin(2\pi k)$$

By  $\cos(2\pi k) = 1$  and  $\sin(2\pi k) = 0$  we have

$$\cos\left(\frac{2\pi(N-n)k}{N}\right) = \cos\left(\frac{2\pi nk}{N}\right) \quad (10)$$

Hence (8) is valid for  $0 < n < N$ .

Returning to (8) and starting the summation at  $k = 1$ , we have

$$\sum_{k=1}^{N-1} \cos\left(\frac{2\pi nk}{N}\right) = -1$$

By (10) we have for  $N$  odd

$$\sum_{k=1}^{(N-1)/2} \cos\left(\frac{2\pi nk}{N}\right) = -\frac{1}{2}$$

Finally, replace  $k$  with  $\alpha$  and  $n$  with  $j - k$ . (The sign of  $j - k$  doesn't matter.)

$$\sum_{\alpha=1}^{(N-1)/2} \cos\left(\frac{2\pi\alpha}{N}(j - k)\right) = -\frac{1}{2}$$