

2-3. Find  $S_{cl}$  for a particle under a constant force  $f$ , that is,

$$L = \frac{m}{2}\dot{x}^2 + fx$$

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We will need equation (2.7) to determine the classical motion  $x(t)$ .

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \quad (2.7)$$

For the Lagrangian  $L$  given in problem 2-3 we have

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = m\ddot{x} \quad \frac{\partial L}{\partial x} = f$$

By equation (2.7) we have for the classical acceleration  $\ddot{x}(t)$

$$\ddot{x}(t) = \frac{f}{m}$$

Hence  $x(t)$  must have the following quadratic form.

$$x(t) = \frac{f}{2m}t^2 + Bt + C \quad (1)$$

We have the following boundary conditions.

$$x(t_a) = x_a \quad x(t_b) = x_b$$

Subtract boundary conditions to solve for  $B$ .

$$x(t_b) - x(t_a) = x_b - x_a = \frac{f}{2m}(t_b^2 - t_a^2) + B(t_b - t_a)$$

Solve for  $B$ .

$$B = \frac{x_b - x_a}{t_b - t_a} - \frac{f}{2m} \left( \frac{t_b^2 - t_a^2}{t_b - t_a} \right) \quad (2)$$

Solve for  $C$ .

$$\begin{aligned} C &= x_a - \frac{f}{2m}t_a^2 - Bt_a \\ &= \frac{f}{2m} \left( \frac{t_a t_b^2 - t_a^2 t_b}{t_b - t_a} \right) + \frac{t_b x_a - t_a x_b}{t_b - t_a} \end{aligned} \quad (3)$$

Substitute (2) and (3) into (1) to obtain

$$x(t) = \frac{ft^2}{2m} + \left( \frac{x_b - x_a}{t_b - t_a} - \frac{f}{2m} \frac{t_b^2 - t_a^2}{t_b - t_a} \right) t + \frac{f}{2m} \frac{t_a t_b^2 - t_a^2 t_b}{t_b - t_a} + \frac{t_b x_a - t_a x_b}{t_b - t_a} \quad (4)$$

Equation (4) is too complicated to continue so translate the time coordinate as follows.

$$t_a = 0 \quad t_b = T$$

We now have

$$x(t) = \frac{ft^2}{2m} + \left( \frac{x_b - x_a}{T} - \frac{fT}{2m} \right) t + x_a \quad (5)$$

Differentiate  $x(t)$  to obtain velocity  $\dot{x}(t)$ .

$$\dot{x}(t) = \frac{d}{dt}x(t) = \frac{ft}{m} - \frac{fT}{2m} + \frac{x_b - x_a}{T} \quad (6)$$

Substitute (5) and (6) into  $L$ .

$$\begin{aligned} L &= \frac{m}{2} \dot{x}^2 + fx \\ &= \frac{f^2 t^2}{m} + \left( \frac{2f(x_b - x_a)}{T} - \frac{f^2 T}{m} \right) t + \frac{f^2 T^2}{8m} + \frac{f(3x_a - x_b)}{2} + \frac{m(x_b - x_a)^2}{2T^2} \end{aligned} \quad (7)$$

Integrate (7) to obtain  $S_{cl}$ .

$$S_{cl} = \int_0^T L dt = \frac{m(x_b - x_a)^2}{2T} + \frac{fT(x_b + x_a)}{2} - \frac{f^2 T^3}{24m} \quad (8)$$

where  $T = t_b - t_a$ .