

15.7.1. Consider a set of modes of the electromagnetic field in which the electric field is polarized along the x direction and the magnetic field is polarized in the y direction. Restricting consideration only to those modes, find the simplest expression you can for the commutation relation $[\hat{E}_x, \hat{B}_y]$ for this multimode field.

Consider the following formulas.

$$\hat{\mathbf{E}}(\mathbf{r}, t) = i \sum_{\lambda} \left(\hat{a}_{\lambda} - \hat{a}_{\lambda}^{\dagger} \right) \sqrt{\frac{\hbar \omega_{\lambda}}{2 \epsilon_0}} \mathbf{u}_{\lambda}(\mathbf{r}) \quad (15.132)$$

$$\hat{\mathbf{B}}(\mathbf{r}, t) = \sum_{\lambda} \left(\hat{a}_{\lambda} + \hat{a}_{\lambda}^{\dagger} \right) \sqrt{\frac{\hbar \omega_{\lambda} \mu_0}{2}} \mathbf{v}_{\lambda}(\mathbf{r}) \quad (15.135)$$

By hypothesis we have for the polarization vectors

$$\mathbf{u}_{\lambda}(\mathbf{r}) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_{\lambda}(\mathbf{r}) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Hence

$$\begin{aligned} \hat{E}_x(\mathbf{r}, t) &= i \sum_{\lambda} \left(\hat{a}_{\lambda} - \hat{a}_{\lambda}^{\dagger} \right) \sqrt{\frac{\hbar \omega_{\lambda}}{2 \epsilon_0}} \\ \hat{B}_y(\mathbf{r}, t) &= \sum_{\lambda} \left(\hat{a}_{\lambda} + \hat{a}_{\lambda}^{\dagger} \right) \sqrt{\frac{\hbar \omega_{\lambda} \mu_0}{2}} \end{aligned}$$

It follows that

$$\hat{E}_x \hat{B}_y = i \frac{\hbar}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} \sum_{\lambda} \omega_{\lambda} \left(\hat{a}_{\lambda} - \hat{a}_{\lambda}^{\dagger} \right) \left(\hat{a}_{\lambda} + \hat{a}_{\lambda}^{\dagger} \right) \quad (1)$$

$$\hat{B}_y \hat{E}_x = i \frac{\hbar}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} \sum_{\lambda} \omega_{\lambda} \left(\hat{a}_{\lambda} + \hat{a}_{\lambda}^{\dagger} \right) \left(\hat{a}_{\lambda} - \hat{a}_{\lambda}^{\dagger} \right) \quad (2)$$

Subtract (2) from (1) to obtain the commutator.

$$[\hat{E}_x, \hat{B}_y] = i \frac{\hbar}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} \sum_{\lambda} \omega_{\lambda} \left(2 \hat{a}_{\lambda} \hat{a}_{\lambda}^{\dagger} - 2 \hat{a}_{\lambda}^{\dagger} \hat{a}_{\lambda} \right)$$

Recalling that $\hat{a}_\lambda \hat{a}_\lambda^\dagger - \hat{a}_\lambda^\dagger \hat{a}_\lambda = 1$ we have

$$[\hat{E}_x, \hat{B}_y] = i\hbar \sqrt{\frac{\mu_0}{\epsilon_0}} \sum_\lambda \omega_\lambda$$