

## Surface integral

A surface integral is like adding up all the wind on a sail. In other words, we want to compute

$$\iint \mathbf{F} \cdot \mathbf{n} dA$$

where  $\mathbf{F} \cdot \mathbf{n}$  is the amount of wind normal to a tiny parallelogram  $dA$ . The integral sums over the entire area of the sail. Let  $S$  be the surface of the sail parameterized by  $x$  and  $y$ . (In this model, the  $z$  direction points downwind.) By the properties of the cross product we have the following for the unit normal  $\mathbf{n}$  and for  $dA$ .

$$\mathbf{n} = \frac{\frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y}}{\left| \frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} \right|} \quad dA = \left| \frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} \right| dx dy$$

Hence

$$\iint \mathbf{F} \cdot \mathbf{n} dA = \iint \mathbf{F} \cdot \left( \frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} \right) dx dy$$

The following exercise is from *Advanced Calculus* by Wilfred Kaplan, p. 313. Evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma$$

where  $\mathbf{F} = xy^2z\mathbf{i} - 2x^3\mathbf{j} + yz^2\mathbf{k}$ ,  $S$  is the surface  $z = 1 - x^2 - y^2$ ,  $x^2 + y^2 \leq 1$  and  $\mathbf{n}$  is upper.

Note that the surface intersects the  $xy$  plane in a circle. By the right hand rule, crossing  $x$  into  $y$  yields  $\mathbf{n}$  pointing upwards hence

$$\mathbf{n} d\sigma = \left( \frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} \right) dx dy$$

The following code computes the surface integral. The symbols  $f$  and  $h$  are used as temporary variables.

```
z = 1 - x^2 - y^2
F = (x y^2 z, -2 x^3, y z^2)
S = (x,y,z)
f = dot(F,cross(d(S,x),d(S,y)))
h = sqrt(1 - x^2)
defint(f, y, -h, h, x, -1, 1)
```

$$\frac{1}{48}\pi$$