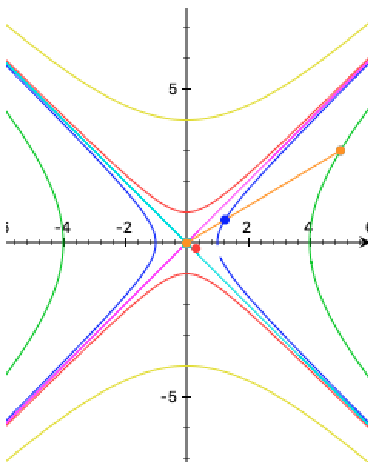
Exploring Math with EIGENMATH

Geometric Algebra *Interactive*! with Eigenmath

Complex, Hyperbolic and Geometric Algebra Numbers

$\mathbb{C} \quad \mathbb{H} \quad \mathbb{G}$



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About this Booklet

This is part 5 of a series of booklets, which want to introduce the reader to some topics of elementary Linear Algebra and at the same time into the use of CAS EIGENMATH.

EIGENMATH

The considerations in this script would be difficult to elementarize without the use of a computer algebra system like EIGENMATH, because heavy calculations of products and inverses of matrices occur in the conceptual constructions. Therefore, in EIGENMATH laboratories we explore decisive phenomena or verify or falsify hypotheses and would like to encourage ongoing dialogical practice in CAS language communication skills with the EIGENMATH assistance.
The accompanying colloquial comments are deliberately short. If possible, all CAS dialog sequences - which are shown in `typewriter font` - should be performed live on the computer. We give therefore many lively links to invocable EIGENMATH scripts. The EIGENMATH routines, which are written for this Part 5, are collected in the toolbox `craBox.txt` for the convenience of the user and are invoked by the command `run("craBox.txt")` in a running EIGENMATH Online[1] session. In this way you can s(t)imulate this communication process at the EIGENMATH prompt region in the input ("Run") window and allow a dynamic interactive 'reading act' with spontaneous deviations, additional inquiries or ad hoc explorations, which would otherwise be not possible.
EIGENMATH is a computer algebra system that can be used to solve problems in mathematics and the natural and engineering sciences. It is a personal resource for students, teachers and scientists. EIGENMATH is small, compact, capable and free. It runs on WindowsOS, MacOS, Android and online in a browser. It is in the opinion and experience of the author very well suited for doing linear algebra from the viewpoint of APOS theory.

To use this booklet interactively
... *you do not need to install any software to do the calculations*! The CAS EIGENMATH works directly out of this text, on any operating system, on every hardware (Smartphone, iPhone, tablet, PC, etc.), at any place: you only must be online and click on a link like ▷*Click here to invoke* EIGENMATH (◁ please click here! Really!). From this point on you can run a given script or fork with own computations.
... *you do not need to install any software to produce quality plots interactively*! You only must be online to press a link like `CalcPlot3D` (◁ please click here! Really!) in this script. At this point you can make a 2D/3D–plot to visualize a concept or to make a calculation visually evident.

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Hinweis auf B E and EVA2 !!! Dank.

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[1] Running the EIGENMATH app on the iMac this command has to be substituted trough `run("downloads/craBox.txt")`. The file `craBox.txt` has therefore to be copied to the 'downloads' folder.

I thank George Weigt for his friendly support, hints and help regarding his Eigenmath. So it was a real pleasure to write down these notes.

Any feedback from the user is very welcome.

PS: Being retired and no native speaker, I have no support from colleges at high school or university anymore, therefore the reader may excuse me for my grammatical and spelling mistakes.

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1 ℂ – the complex numbers

It is well known that the solution set $\mathbb{L}$ of a *singular homogeneous* 3×3 linear system is a *straight line* or a *plane through the origin*. The solution set $\mathbb{L}$ is not just a *subset* of the surrounding space $\mathbb{R}^n$, but also has a *linear structure*: with each two solution vectors $\vec{v}$ or $\vec{w}$ in $\mathbb{L}$ there are also all linear combinations $r \cdot \vec{v} + s \cdot \vec{w}$ (with $r, s \in \mathbb{R}$) solutions again.[2] Therefore, this property is particularly emphasized in a central concept of linear algebra.

1.1 ℂ as vectorspace

We already know the complex numbers $\mathbb{C}$ partially: the arithmetical playground (the "underlying set") of $\mathbb{C}$ is the well known Euclidean plane $\mathbb{R}^2$ with the two operations of addition and stretching ('multiples') of column/row vectors i.e. $\mathbb{C} \sim \mathbb{R}^2$ or more precisely

$$\mathbb{C} \simeq (\mathbb{R}^2, +, \cdot)$$

with $(a,b)+(c,d) \overset{def}{=} (a+c, b+d)$ and $r \cdot (a,b) \overset{def}{=} (r \cdot a, r \cdot b)$ for arbitrary $a,b,c,d,r \in \mathbb{R}$. For example $(1,2)+(3,4) = (4,6)$ and $0.5(-2,2) = (-1,1)$. Visualized:
Equipped with these two operations the set $\mathbb{C}$ is an "*2–dimensional vector space over the reals*", i.e. the operations + and · respect the following rules of an abstract vector space.

Definition. Let V be a set on which there are defined two operations, one called *addition* ('+') and the other called *multiplication by scalars* ('·), such that the following 10 calculation rules ('laws', 'axioms') holds:

For all $\vec{u}, \vec{v}, \vec{w} \in V$ and $r, s \in \mathbb{R}$ we have

$$\begin{array}{lrcll}
\textbf{(}\uplus\textbf{)} & \vec{v}+\vec{w} & \in & V & \\
\textbf{(C)} & \vec{v}+\vec{w} & = & \vec{w}+\vec{v} & \\
\textbf{(A)} & (\vec{u}+\vec{v})+\vec{w} & = & \vec{u}+(\vec{v}+\vec{w}) & \\
\textbf{(N)} & \vec{v}+\vec{0} & = & \vec{v} & \text{there exist such an element } \vec{0} \in V \\
\textbf{(I)} & \vec{v}+(-\vec{v}) & = & \vec{0} & \text{there exist such an element } -\vec{v} \in V \text{ for every } \vec{v} \\
 & & & & \\
(\dot{\cup}) & r \cdot \vec{v} & \in & V & \\
\textbf{(1)} & r \cdot (\vec{v}+\vec{w}) & = & r \cdot \vec{v} + r \cdot \vec{w} & \\
\textbf{(2)} & (r+s) \cdot \vec{v} & = & r \cdot \vec{v} + s \cdot \vec{v} & \\
\textbf{(3)} & r \cdot (s \cdot \vec{v}) & = & (rs) \cdot \vec{v} & \\
\textbf{(4)} & 1 \cdot \vec{v} & = & \vec{v} &
\end{array}$$

Exercise 1.1. **Mental model of a vectorspace.**

[2]This property did not apply to inhomogeous linear system!

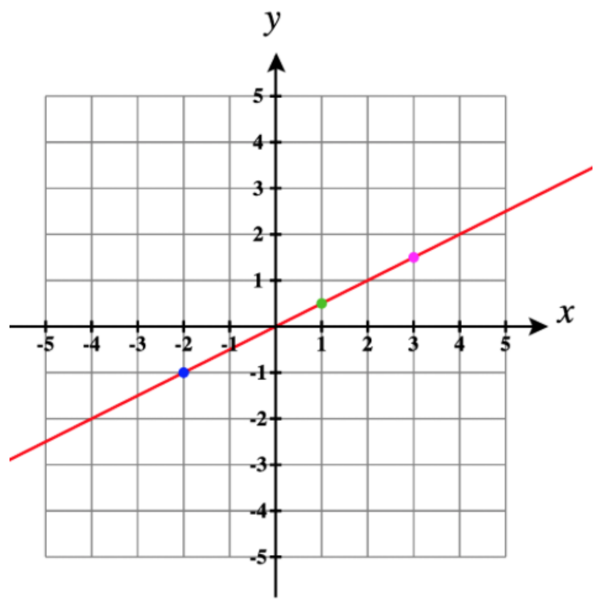


Figure 1: Red: the vectorspace $\mathbb{L}$ of solutions of $0.5x - y = 0$.
Blue: the vector $\vec{u} = [-2, -1]$.
Green: the vector $\vec{v} = [1, 0.5]$.
Magenta: the vector $w = -1 \cdot \vec{u} + 1 \cdot \vec{v} = [3, 1.5]$.

a. Verify that $\vec{u}, \vec{v}$ and $\vec{u} + \vec{v}$ from Fig.1 are solutions of $0.5x - y = 0$, i.e. $\vec{u}, \vec{v} \in \mathbb{L}$.
b. Verify that $\vec{b} = [2t, t], t \in \mathbb{R}$ is a general solution vector in $\mathbb{L}$.
c. Verify that arbitrary multiples of $\vec{b}$ are in $\mathbb{R}$, i.e. $r \cdot \vec{b} \in \mathbb{L}$ for arbitray $r \in \mathbb{R}$.
d. Verify: The set $\mathbb{L}^{(\cdot,+)}$ is a (1–dimensional) vectorspace over $\mathbb{R}$.
That is: the 10 vectorspace conditions $\uplus$**CANI**$\dot{\cup}$**1234** are fulfilled for $\mathbb{L}$.
If it is tidy to do these tests by paper'n pencil, please use EIGENMATH▷*Click here.*
♡ Keep e.g. this model in mind when thinking at the definition above..

Remark.

1. The first group of rules **(C)**, **(A)**, **(N)**, **(I)** for the *vector addition* are the **C***ommutat ive* law, the **A***ssociativ* law, the law of the existence of a **N***eutral* element and the law of the existence of **I***nvers* elements. ($\uplus$) resp. ($\dot{\cup}$) is the so-called *closeness* of the addition resp. multiple forming, i.e. with each pair of vectors there sum resp. multiple lies in the vectorspace again.

   (N) and **(I)** do not go without saying:

   ∘ **(N)** says more precisely: there is a certain element in V - which is denoted by $\vec{0}$ and called *zero vector* - with the property that $\vec{v} + \vec{0} = \vec{v}$ applies to any $\vec{v}$.

   ∘ **(I)** says more precisely: for every arbitrary $\vec{v}$ from V there is an element - which is denoted by $-\vec{v}$ and *opposite vector* or *inverse element* is called - in V with the property: $\vec{v} + (-\vec{v}) = \vec{0}$.

2. The second group of calculation rules **(1)**, **(2)**, **(3)**, **(4)** describes the formation of multiples of vectors, i.e. the multiplication of vectors with real numbers. These rules describe the distribution of numbers on vectors under '$\cdot$' and therefore are called the four *distributive laws*.

3. The 10 rules $\uplus$**CANI**$\dot{\cup}$**1234** are also called the *axioms* of a vector space.

Exercise 1.2. The arithmetic rules of ℂ.
Verify the 10 vectorspace axioms $\uplus$**CANI**$\dot{\cup}$**1234** for $\mathbb{C} \simeq (\mathbb{R}^2, +, \cdot)$
a. by paper'n pencil
b. using EIGENMATH. Here is a start ▷*Click here.*

1.2 ℂ as algebra

To construct the complex numbers ℂ in full flavor, we enhance the arithmetical playground $\mathbb{R}^2$ with a third operation – a special extraordinary version of an multiplication '$\star$' of column/row vectors in $\mathbb{R}^2$, called *multiplication of complex numbers* via the new rule

$$\boxed{(a, b) \star (c, d) \overset{def}{=} (a \cdot c - b \cdot d, a \cdot d + b \cdot c) \qquad (1.1)}$$

If we speak of the complex numbers we now think at the 2–dimensional number plane $\mathbb{R}^2$ equipped with the three operations $(+, \cdot, \star)$ and write

$$\mathbb{C} \equiv (\mathbb{R}^2, +, \cdot, \star)$$

We will motivate this strange operation $\star$ very soon.

1.2.1 ℂ as 2D algebra over the reals ℝ

For the new ℂ–typical operation $\star$ the following rules hold for arbitrary $u, v, w \in \mathbb{R}^2$:

$$\begin{array}{llcll}
(\overset{\star}{\cup}) & v \star w & \in & \mathbb{C} & \\
(\mathbf{C}^\star) & v \star w & = & w \star v & \\
(\mathbf{A}^\star) & (u \star v) \star w & = & u \star (v \star w) & \\
(\mathbf{N}^\star) & z \star e_1 & = & z & \text{for } e_1 \overset{def}{=} (1, 0) \\
(\mathbf{I}^\star) & z \star z^{-1} & = & e_1 & \text{for } z^{-1} \overset{def}{=} \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)
\end{array}$$

∘ The complex number $e_1 = (1, 0)$ in rule $(\mathrm{N}^\star)$ is called the *unit* in ℂ. It is ℂ's neutral element with respect to the new multiplication $\star$.
∘ The complex number z^{-1} in rule $(\mathrm{I}^\star)$ is called the *inverse of z* in ℂ.

Exercise 1.3. **a.** Verify the the above rules for $\star$ by paper'n pencil..
b. Verify the the above rules by EIGENMATH.

Solution:

```
-- C as algebra
-- define the new multiplication * for 2D row
vectors

star(u,v)= (u[1]*v[1]-u[2]*v[2],
u[1]*v[2]+u[2]*v[1])

-- rule C*
do( u=(a,b), v=(c,d))
star(v,u)
star(u,v)
star(u,v)==star(v,u)

-- rule N*
e1=(1,0)
z =(x,y)
star(z,e1)
star(z,e1)==z

--rule I*
zinv = (x/(x^2+y^2), -y/(x^2+y^2))

star(z, zinv)
simplify(last)
```

$$\begin{bmatrix} a\,c - b\,d \\ a\,d + b\,c \end{bmatrix}$$

$$\begin{bmatrix} a\,c - b\,d \\ a\,d + b\,c \end{bmatrix}$$

1

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

1

$$\begin{bmatrix} \dfrac{x^2}{x^2+y^2} + \dfrac{y^2}{x^2+y^2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

▷ *Click here to start the solution.*

Exercise 1.4. Check with Eigenmath, that the following rules also hold for arbitray $r, s \in$ ℝ and $u, v, w \in \mathbb{C} = (\mathbb{R}^2, +, \cdot, \star)$

a. $(r \cdot u + s \cdot v) \star w = r \cdot (u \star w) + s \cdot (v \star w)$

b. $u \star (r \cdot v + s \cdot w) = r \cdot (u \star v) + s \cdot (u \star w)$

c. $r \cdot (u \star v) = (r \cdot u) \star v = u \star (r \cdot v)$

▷ *Click here to invoke* Eigenmath

Remark. With both laws a. & b. of *distribution*, the operation $\star$ is compatible with the structure of the vector space ℂ. A vector space together with a 3rd operation $\star$, for which the above *rules of distribution* a. & b. hold, is called an ℝ–*algebra*. $\star$ itself is called *the multiplication of the algebra* ℂ.

Therefore the title of this section.

Exercise 1.5. Calculate with/without Eigenmath:

a. $(1, 2) \star (3, 4)$

b. For which $w \in \mathbb{C}$ is $(1, 2) \star z = (1, 0)$?

c. $2 \cdot (3, 4) \star (-1, 1)$

Exercise 1.6. (How to motivate the construction of $\star$?) The 2D vectorspace $\mathbb{R}^2$ over the scalar field ℝ has the canonical basis vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$. We want to have the new multiplication $\star$ to work such, that

(1) e_1 should be the unit element i.e. should fulfill rule (N$\star$) and
(2) e_2 should be *chosen so, that its square results in the negative unit* i.e.

$$e_2^2 = (0,1)^2 \overset{!}{=} -(0,1) = -e_1$$

Therefore for arbitrary $u = (x_1, y_1), v = (x_2, y_2) \in \mathbb{C}$ we have

$$\begin{aligned}
(x_1, y_1) \star (x_2, y_2) &= (x_1 \cdot (1,0) + y_1 \cdot (0,1)) \star (x_2 \cdot (1,0) + y_2 \cdot (0,1)) \\
&\overset{(1.2)}{=} x_1 \cdot x_2 \cdot (1,0) + (x_1 \cdot y_2 + y_1 \cdot x_2) \cdot (0,1) + y_1 \cdot y_2 \cdot (0,1)^2 \\
&\overset{!}{=} x_1 \cdot x_2 \cdot (1,0) + (x_1 \cdot y_2 + y_1 \cdot x_2) \cdot (0,1) - y_1 \cdot y_2 \cdot (0,1) \\
&= (x_1 \cdot x_2 - y_1 \cdot y_2) \cdot (1,0) + (x_1 \cdot y_2 + y_1 \cdot x_2) \cdot (0,1) \\
&= (x_1 \cdot x_2 - y_1 \cdot y_2, x_1 \cdot y_2 + y_1 \cdot x_2)
\end{aligned}$$

Explain each line for yourself.

1.2.2 Introducing the imaginary unit i.

To emphasize that we calculate in $\mathbb{R}^2$ using the new multiplication rule $\star$ one traditionally writes

$$i \overset{def}{=} (0,1) = e_2 \in \mathbb{C}$$

and name i the *imaginary unit*. In this context the unit e_1 is identified with the number 1, i.e. we have $1 \equiv (1,0) = e_1$. Therefore, per definition we have the facts:

$$i^2 = -1 \tag{1.2}$$

$$z = (x,y) = (x,0) + (0,1) \star (y,0) \overset{(1.2)}{\equiv} x + iy \in \mathbb{C} \tag{1.3}$$

∘ *Beware*: with the standart notation $x + iy$ the use of the new multiplication $\star$ in (1.3) is shadowed behind the symbol i!
∘ Fact (1.2) is equivalent expressed as $\boxed{i = \sqrt{-1}}$. While $\sqrt{a}$ exists in $\mathbb{R}$ only for $a \geq 0$, we have now constructed a number system in which roots of negative real numbers exists.
∘ We have the following important definitions:

The ℂ LEXICON I:	*Math*	EIGENMATH
complex number $z \in \mathbb{C}$:	$z = (x,y) = x + iy$	`z = x + iy`
the *real* part of z	$Re(z) = x$	`real(z)`
the *imaginary* part of z	$Im(z) = x$	`imag(z)`
the *magnitude* (length) of z	$\lvert z\rvert \overset{def}{=} \sqrt{x^2 + y^2}$	`mag(z)`
the *conjugate* of z	$\bar{z} \overset{def}{=} x - iy$	`conj(z)`

Exercise 1.7. Let $u = 1 + 2i, v = -3 - i, w = 1 + i$. Calculate with paper'n pencil
a. real and imaginary part of u
b. the magnitudes of u, v, w
c. the conjugates of all three complex numbers
d. Draw a qualtity plot with CalcPlot3D of u, $|u|$, $Re(u), Im(u), \bar{u}$.
Check the plausibility of the results using the plot.
e. Check the calculations using Eigenmath.
▷*Click here to invoke* Eigenmath

Exercise 1.8. (*Quotient of complex numbers*)
a. Calculate $\frac{1+i}{3-4i}$.
b. Prove: Let $z_1 = x_1 + y_1 i \in \mathbb{C}$ and $z_2 = x_2 + y_2 i \in \mathbb{C}$ with $x_2^2 + y_2^2 \neq 0_{\mathbb{R}}$. Then

$$\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \cdot \frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}$$

Exercise 1.9. (*Arithmetic with complex numbers*)
Let $u = 2 - 5i, v = 4 + i \in \mathbb{C}$.
a. Calculate $u + v, u - v, u \star v, u/v$ by paper'n pencil.
b. Determine $Re(u), Im(v), \bar{u}, |u|$.
c. Check the results of a. and b. by Eigenmath.

```
# EIGENMATH solution to a):
trace=1         -- trace=1=ON shows results in black
do( u=2-5i, v=4+i)
u + v
u - v
u-v
u*v             --  *-product of complex numbers
u/v             --  quotient resp. * of complex numbers
u*1/v           --  quotient via reciprocal of v
u*v^(-1)        --  quotient via *-inverse of v
```

▷*See the solution to a. here.*

Exercise 1.10. (*Conjugate complex numbers*)
Prove with/without Eigenmath that for arbitrary $z \in \mathbb{C}$ we have
a. $Re(z) + Im(z) \in \mathbb{R}$
b. $Re(z) \cdot Im(z) \in \mathbb{R}$

1.2.3 The complex scene : the complex number $z = 4 + 3i \in \mathbb{C}$

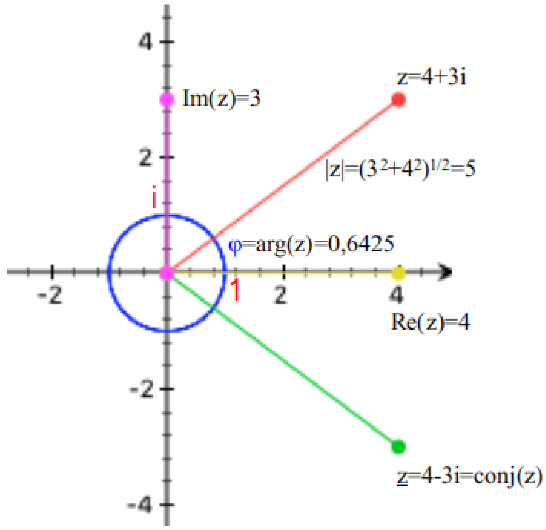


Figure 2: Red: the complex number $4+3i = (4,3) \in \mathbb{C} \equiv (\mathbb{R}^2, +, \cdot, \star)$
Green: the conjugate $\overline{4+3i} = 4 - 3i$
Yellow: the real part $Re(4+3i) = 4$
Magenta: the imaginary part $Im(4+3i) = 3$
Red: the magnitude (length) $|4+3i| = \sqrt{4^2+3^2} = 5$
Blue: the unit circle $S^1 : x^2 + y^2 = 1$.
Blue: the argument $\varphi = \measuredangle(1,z) = arctan(3/4)$='part' of S^1

Exercise 1.11.
a. Check the results in Fig.2 by a paper'n pencil calculation.
b. Check the results in Fig.2 by EIGENMATH. *Solution*:

```
# EIGENMATH
trace=1           -- trace=1=ON shows results in black
z = 4+3i
conj(4+3i)        -- conjugate of z
real(4+3i)        -- real part of z
imag(4+3i)        -- imaginary part of z
mag(4+3i)         -- magnitude (= Euclidean length) of z
phi = arg(4+3i)   -- argument (angle) of z
phi
arg(4.+3i)        -- 4.=4.0 gives back decimal approximation
                  -- phi:(2*pi)=alpha:360  -> alpha=phi*180/pi
                  -- the argument (angle) measured in degrees:
alpha=float(phi*180/pi)
alpha             --  = 36.9 deg (phi measured in radians)
```

▷ *Click here to invoke* EIGENMATH
▷ *Click here to look at the solution.*

1.2.4 The complex exponential function

Definition. We define for arbitrary $z \in \mathbb{C}$

$$e^z \equiv exp(z) \overset{def}{=} \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + ...$$

- We get a function $exp\colon \mathbb{C} \to \mathbb{C}$, because the series is absolute convergent on $\mathbb{C}$.
- Analog we define $cos, sin, ...$ via convergent series, see ▷ Calculus.

Exercise 1.12. (exp, cos, sin as complex functions)
Let $z = 1 + i$. Calculate ...
a. ... the partial sum $1 + \frac{z^1}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} \sim exp(1+i)$. Calculate $exp(1+i)$ by EIGENMATH. How many summands must the partial sum have, such that her value coincide with the first 3 decimals of $exp(1+i)$?

> ▷ *Click here to invoke* EIGENMATH *for your own calculation.*
> ▷ *Click here to look at my solution.*

b. ... the partial sum $\sum_{k=0}^{5} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$. Compare with $sin(1+i)$ using EIGENMATH.
c. ... the partial sum $\sum_{k=0}^{5} (-1)^k \frac{z^{2k}}{(2k)!}$. Compare with $cos(1+i)$ using EIGENMATH.

We remind without proof to

Theorem I. *(The* EULER *formula)*
For arbitrary $z \in \mathbb{C}$:

$$e^{iz} = cos(z) + i\, sin(z) \tag{1.4}$$

Theorem II. *(The* DE MOIVRE *formula)*
For arbitrary $n \in \mathbb{N}$, $\phi \in \mathbb{R}$ we have for $z = exp(i\phi) \in \mathbb{C}$

$$z^n = exp(i\, n\phi) \tag{1.5}$$
$$(cos(\phi) + i\, sin(\phi)^n = cos(n\phi) + i\, sin(n\phi) \quad \in \mathbb{C} \tag{1.6}$$

Exercise 1.13. Calculate
a. $exp(i \star (1+i)) \equiv e^{i\star(1+i)} =?$
b. $exp(2i)^3$
c. $(1+2i)^3$

1.2.5 The polar coordinates

Theorem III. (The polar form of a nonzero complex number)

Every $z = x + iy \in \mathbb{C} - \{0\}$ *can* uniquely *be written in the so–called* polar form

$$z \;=\; r \cdot (cos(\varphi) + i\ sin(\varphi)) = r \cdot e^{i\varphi} \tag{1.7}$$

for $0 \le \varphi \le 2\pi$, *where* $\varphi \overset{def}{=} arg(z) = tan^{-1}(\frac{y}{x})$ *and* $r \overset{def}{=} |z| = \sqrt{x^2 + y^2}$.

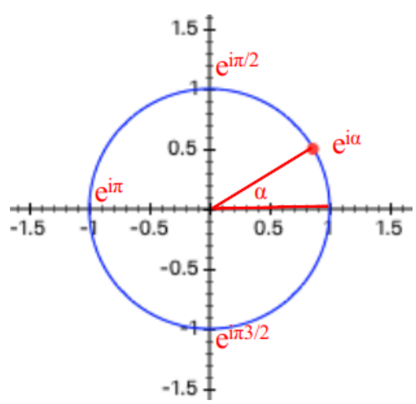
∘ The real numbers $(r, \varphi) \in \mathbb{R}^2$ are called the *polar coordinates* of $z \in \mathbb{C}$.
∘ The number $\varphi \in [0, 2\pi[$ is called the *argument* or *amplitude* of z.
∘ The real number r is the distance of z to the origin $O = (0, 0)$ and φ is the angle between the positive x–axis and the direction arrow to z, see Fig.2.
∘ Often we use the abbreviation $cis(\varphi) \overset{def}{=} (cos\varphi + i \cdot sin\varphi)$. We then have $z = cis(\varphi)$.
∘ We remind at

The ℂ LEXICON II:	*Math*	EIGENMATH
complex number z in *polar* form with ..	$z = r \cdot e^{i\varphi}$	`polar(z)`
... $r = \|z\|$ and	$= r \cdot (cos\varphi + i \cdot sin\varphi)$	
... *argument* $\varphi = arg(z) \in [0, 2\pi[$	$\varphi = \measuredangle(e_1, z) = \tan^{-1}(\frac{y}{x})$	`phi = arg(z)`
complex number z in *rectangular* form:	$z = x + i \cdot y$	`rect(z)`
the *complex root* of z	$cis(\varphi) = (cos\varphi + i \cdot sin\varphi)$	`cis(phi)=..`
the *complex power* of z	$Im(z) = x$	`real(z)`
the ν^{th} *complex unit root* of $z^n = 1$	$\zeta_\nu^n = e^{\frac{2\pi i}{n}\nu}, \nu = 0, 1, ...n - 1$	`exp(2 pi i nu/n)`

Summary: we have therefore three different shapes of a complex number

rectangular	*trigonometric*	exponentially
.. alias Cartesian		.. alias polar form
$z = x + iy =$	$r \cdot (\cos\varphi + i \cdot \sin\varphi) =$	$mag(z) \cdot \exp(i \cdot arg(z)) = r \cdot e^{i \cdot \varphi}$

∘ We visualize some polar factors $exp(i \cdot \varphi) = e^{i \cdot \varphi}$ as points at the unit circle S^1:



Exercise 1.14. (Polar vs rectangular form of a complex number)
a. What is the argument and the magnitude of $z = 3 + 2i$?
b. Give z in polar form $e^{\cdots}$. Check the equivalence of both representations.
c. Recover the rectangular form of z in a. back from its polar form in b..
d. Check your results by Eigenmath.
▷ *Click here to look at the solution.*

Exercise 1.15. (Visualization of the complex multiplication $\star$)
By means of the polar form of a complex number one can visualize the effect of the complex multiplication.

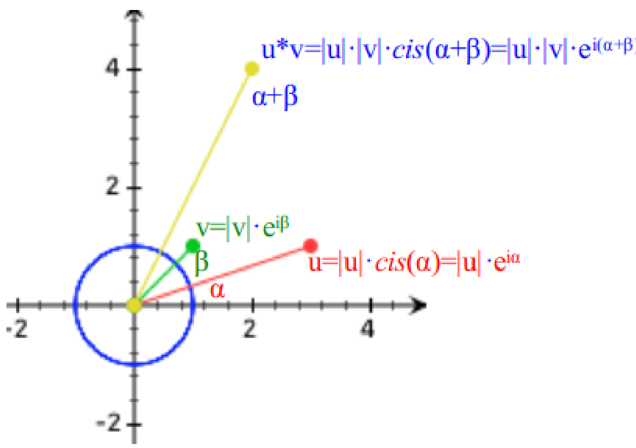


Figure 3: Red: the 1st factor u with his argument $\alpha = \measuredangle(e_1, u)$
Green: the 2nd factor v with his argument $\beta = \measuredangle(e_1, v)$
Blue: the product $u \star v$ has argument (angle) $\measuredangle(e_1, \alpha + \beta)$
The arguments (angels) are best seen as arc pieces on the unit circle $S^1 : x^2 + y^2 = 1$ starting at $e_1 = (1, 0)$.

a. In Fig.3 we have $u = 3 + i$ and $v = 1 + i$. What are the coordinates of the yellow point?
b. Transform the arguments α, β and $\alpha + \beta$ in degrees. Compare.
○ Slogan: *you get the product of two complex numbers by multiplying their magnitudes and adding their arguments (angels)*.

Exercise 1.16. (Programming a `polar1` function for Eigenmath)
Eigenmath has two functions for handling polar ("$e^{\cdots}$") and rectangular ("$a + bi$") forms of complex numbers:
○ `polar(z)` awaits as input $z = a + bi$ in rectangular form and returns its *polar* form.
○ `rect(z)` awaits as input $z = e^{\cdots}$ in (exp=)polar form and returns its *rect*angular form.
Sometimes one has length r and angle $\varphi = arg(z)$ as inputs and needs the polar term. Therefore:

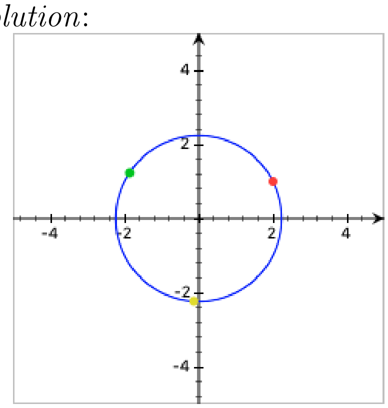
a. Write a user defined function `polar1`, which awaits (r, φ) as input and returns the polar expression $... \cdot exp(..)$ as output.
b. What is `polar1(sqrt(13), arctan(2/3))` in rectangular form? In decimals?
b. Using `polar1`, what result do you respect for the expressions

```
polar1(r,p)*polar1(s,q)
1/polar1(r,p)
polar1(r,p)^3
```

Check your guess by EIGENMATH. ▷*Click here to look at the solution.*

Exercise 1.17. (Polar form of a complex product or quotient)
Let $z, u, v \in \mathbb{C}$.
a. Determine the polar form of the complex numbers in rectangular form $1, i, -1, -i$.
b. Determine the rectangular form of $u = exp(1/3i\pi)$ and $v = \sqrt{2} \cdot exp(\frac{1}{4}i\pi)$.
c. $polar(\bar{z}) =?$
d. $polar(z^{-1}) =?$
e. Verify: $polar(u \star v) = |u| \cdot |v| \cdot e^{\varphi+\psi} = |u| \cdot |v| \cdot cis(\varphi + \psi)$
i.e. again: *you get the product of two complex numbers by multiplying their magnitudes and adding their arguments* (angels).
f. $polar(\frac{u}{v}) =?$
▷*Click here to run the solution.*

Exercise 1.18. (The 3rd roots of a complex number)[3]
We seek the complex solutions of the equation $z^3 = 2 + 11i$. We know by the so-called *Fundamental Theorem of Algebra*, that this equation must have exactly 3 solutions in $\mathbb{C}$.
a. Verify by paper'n pencil that $w1 = 2 + i$ is a solution of $z^3 = 2 + 11i$.
b. Use EIGENMATH to verify that $w2 = (2+i)\star(-\frac{1}{2} + \frac{\sqrt{3}}{2} \cdot i)$ and $w3 = (2+i)\star(-\frac{1}{2} - \frac{\sqrt{3}}{2} \cdot i)$ are solutions, too.
c. Plot the three solutions. *Solution*:



[3] see forst 29

Exercise 1.19. (Roots of complex numbers as edges of a regular polygon)
Let $a = r \cdot exp(i\psi) \in \mathbb{C}$.
a. Verify, that

$$root_k^n(a) \equiv \sqrt[n]{a}|_k \overset{def}{=} r^{1/n} \cdot \exp(i\frac{\psi + 2k\pi}{n})$$

for $k = 0, ..., n-1$ is the k^{th} of the n complex roots of a, i.e. $(\sqrt[n]{a}|_k)^n = a$.
◦ The n^{th} root of a is therefore a whole *set* of complex numbers:

$$\sqrt[n]{a}| = \{\sqrt[n]{a}|_k \in \mathbb{C} \mid k \in \{0, .., n-1\}\}$$

b. Determine $\sqrt[4]{1}|$, $\sqrt[4]{i}|$.
c. Determine $\sqrt[4]{2}|$ and visualize this root set.
d. Verify by an quality plot that the root (set) $\sqrt[n]{a}|$ of a complex number a is the edge set of a regular n-gon e.g. $\sqrt[3]{a}| = \triangle$ or $\sqrt[4]{a}| = \square$ or ...
▷ *Click here to run the solution.*

1.2.6 Inner and outer products of complex numbers

Definition. Let $u = (u_1, u_2) = u_1 + i \cdot u_2$ and $v = (v_1, v_2) = v_1 + i \cdot v_2$ be in $\mathbb{R}^2$.
◦ The *scalar alias inner product* of u and v in the real vector space $\mathbb{C} = \mathbb{R}^2$ is defined as

$$u \bullet v \overset{def}{=} u_1 \cdot v_1 + u_2 \cdot v_2$$

◦ The *outer alias wedge product* of u and v is defined as

$$u \wedge v \overset{def}{=} Im(\bar{u} \star v)$$

Exercise 1.20. Given $w = 2 + 3i, z = 3 - 5i$ and $u, v \in \mathbb{C}$.
a. Determine $w \bullet z, z \bullet z$ and $w \wedge z, w \wedge w$.
b. Verify: $u \bullet v = Re(u * \bar{v})$
c. Proof: $u \perp c \cdot u \Leftrightarrow c \in i \cdot \mathbb{R}$, i.e. if c is pure imaginary.
◦ Check your results by EIGENMATH. ▷ *Click here to look at the solution.*

1.2.7 Problems

P1. Solution of quadratic equaton - midnight formula For an first exercise,

P2. Solution of cubic equations - the Cardano formula For an first exercise, unangenehm daher eig

P3. Hoehenscnittpunkz For an first exercise,

P4. Doppelverhaetnis For an first exercise, unangenehm daher eig

P5. DV II For an first exercise,

P6. DV II For an first exercise,

P7. DV II For an first exercise,

1.3 ℂ as algebraic structure

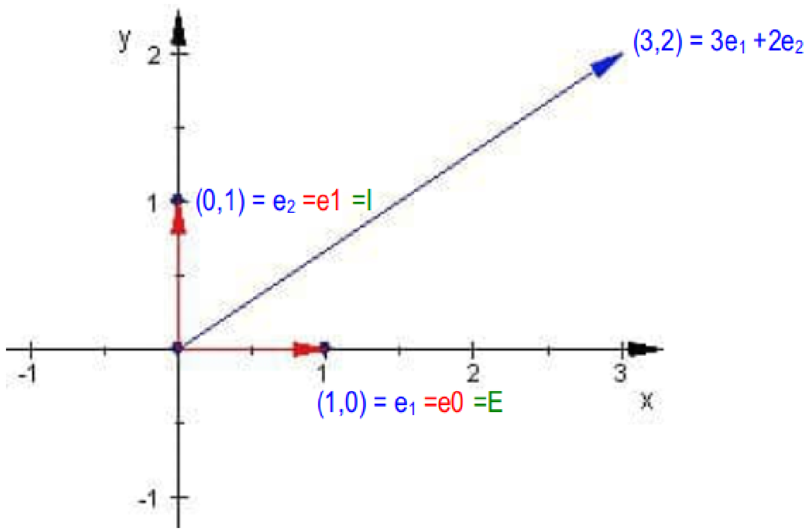


Figure 4: Blue: the point/vector $(3,2)$. The basis $\{e_1, e_2\}$ of $\mathbb{R}^2$
Red: the same basis noted $\{e_0, e_1\}$ and $(3,2) = 3 \cdot e_0 + 2 \cdot e_1$
Green: $(3,2) = 3 \cdot E + 2 \cdot I$ in basis noted $\{E, I\}$

To construct the complex numbers $\mathbb{C}$ in an alternative way, we enhance the arithmetical playground $\mathbb{R}^2$ again with a third operation – but this time by means of a 'multiplication table' for the operation (the 'algebra multiplication'), noted '$\circledast$'. This makes $\mathbb{R}^2$ into the structure $(\mathbb{R}^2, \circledast)$ of an 'algebra'[4]. One can master the algebra multiplication of the new $\mathbb{R}$–algebra by means of its property of *bilinearity*, if one knows its effect on all possible $2^2 = 4$ [5] pairs of the elements of a basis of the underlying vector space $\mathbb{R}^2$. Here we use the fact that the algebra unit $1_{\mathbb{C}} = (1,0) = 1_{\mathbb{R}^2}$ also occurs naturally in this basis.
Therefore, to construct the new multiplication rules we describe for the 2 basis vectors $e_0 = (1,0)$ and $e_1 = (0,1)$[6] of the vector space $\mathbb{R}^2$ the following results for the operation $\circledast$:

$$e_0 \circledast e_0 = e_0 \tag{1.8}$$
$$e_0 \circledast e_1 = e_1 \tag{1.9}$$
$$e_1 \circledast e_0 = e_1 \tag{1.10}$$
$$e_1 \circledast e_1 = -e_0 \tag{1.11}$$

[4]A $\mathbb{R}$-*algebra* is a pair $(A, \circledast)$, consisting of an $\mathbb{R}$-vector space and an $\mathbb{R}$-binlinear mapping $\circledast : A \times A \to A$ defined through $(a,b) \mapsto a \circledast b$

[5]in general $n^2, n = dim_{\mathbb{R}} A$, A being the algebra.

[6]Why we adopt the notation e_0, e_1 alias E, I instead of the usual notation e_1, e_2 for the two basis vectors of $\mathbb{R}^2$ will become clear later on.

Remark. If we translate the 4^{th} rule $e_1 \circledast e_1 = -e_0$ using the lexicon $\overset{e_0, e_1}{\mathbf{1, i}}$ in die language of $\mathbb{C}$, we get the desired relation $\overset{e_1 \circledast e_1 = -e_0}{i \star i = i^2 = -1}$.

Noted as a *compact multiplication table* for the algebra multiplication $\circledast$, we have:

$$\begin{array}{c|cc} \circledast & e_0 & e_1 \\ \hline e_0 & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{pmatrix} e_0 & e_1 \\ e_1 & -e_0 \end{pmatrix}$}} \\ e_1 & & \end{array}$$

If we now speak of the complex numbers we think at the 2–dimensional number plane $\mathbb{R}^2$ equipped with the three operations $(+, \cdot, \circledast)$ and write $\widetilde{\mathbb{C}} \equiv (\mathbb{R}^2, +, \cdot, \circledast)$.

Exercise 1.21. Calculate $(1+i) \star (-2+2i)$ using the multiplication table. *Solution*:

$$\begin{aligned} (1+i) \star (-2+2i) &\equiv (e_0 + e_1) \circledast (-2e_0 + 2e_1) \\ &= -2e_0 \circledast e_0 + 2e_0 \circledast e_1 - 2e_1 \circledast e_0 + 2e_1 \circledast e_1 \\ &= -2e_0 + 2e_1 - 2e_1 + 2(-e_0) \\ &= -4e_0 = (-4, 0) \equiv -4 \end{aligned}$$

1.3.1 Implementing the algebra structure $\widetilde{\mathbb{C}} \equiv (\mathbb{R}^2, +, \cdot, \circledast)$ in Eigenmath

In order to effectively calculate in the new algebraic playground $(\mathbb{R}^2, +, \cdot, \circledast)$ we have to translate the construction above into Eigenmath command language.

```
##### C alias R[i]
tty = 1

e0 = (1,0)              -- basis vectors
e1 = (0,1)

T = ((e0, e1),          -- (1) multiplication table
     (e1,-e0))

B = transpose(T,2,3)    -- (2) bilinear operation

mu(x,y) = dot(x,B,y)    -- (3) x^t*B*x

mu(e0,e0)               -- should be  e0=(1,0)
mu(e0,e1)               -- should be  e1=(0,1)
mu(e1,e0)               -- should be  e1=(0,1)
mu(e1,e1)               -- (4) should be -e0 = (-1,0)

mu( 2e0+3e1, 1e0-2e1 )   -- (5) mu_ltiplication

mu(a*e0+b*e1, c*e0+d*e1) -- (6)
```

```
T = (((1,0),(0,1)),((0,1),(-1,0)))
B = (((1,0),(0,1)),((0,-1),(1,0)))
(1,0)
(0,1)
(0,1)
(-1,0)
(8,-1)
(a c - b d,a d + b c)
```

Comment. In (1) we implement the multiplication table as a tensor, i.e. as a matrix consisting of two 2×2 matrices. Function $mu \overset{def}{=} \circledast$ defines in (2) the bilinear operation, which uses the cool possibility of Eigenmath's `dot(.)` to allow multiple inputs. (4) verifies that the construction fulfills the desired relation $e_1^2 = -(1,0) \equiv -1$. For an arbitrary input mu returns in (6) the well known formula for the complex multiplication.

▷ *Click here to run the script.*

Exercise 1.22. Verify the calculation in (5) by only using the multiplication table. □
To convince the reader that we calculate indeed in ℂ, we spend a bit syntactic sugar and set $E \stackrel{def}{=} e_0$, $I \stackrel{def}{=} e_1$ to get the usual appearance:

```
E = (1,0)            # _E_mbedding of R in C
I = (0,1)            # _I_maginary unit
                     -- C = R[i] multiplication table:
                     -- E*E = E, E*I = I, I*E = I, I*I = -E=-1

T = ((E, I), (I,-E))
B = transpose(T,2,3)

mu(x,y) = dot(x,B,y)

mu(I,I)              -- = -E = -1 i.e.  i^2=-1

mu( 2E+3I, 1E-2I)    -- complex algebra via multiplication table T
(2*1+3i)*(1*1-2i)    -- build-in complex algebra, returning  8-i = (8,-1)
```

▷ *Click here to run the script.*

1.3.2 Reengineering of some complex functions

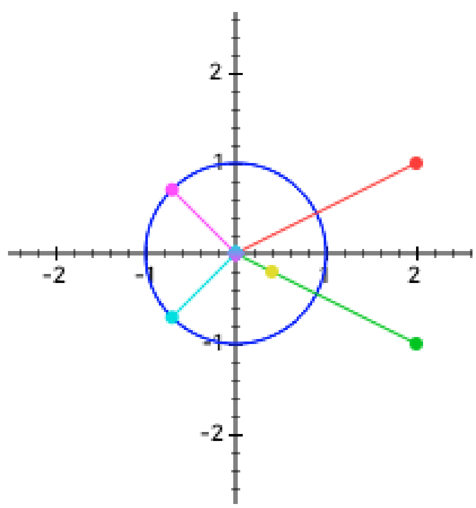


Figure 5: Blue: The unit circle $S^1 \subset \mathbb{R}^2$ with equation $x^2 + y^2 = 1$.
Red: The complex number/vector $z = 2 + 1 \cdot i = (2, 1)$
Green: ... its conjugate $\bar{z} = (2, -1) = 2 - i$
Green: ... and its inverse $1/z$.
Magenta: The complex number $w = 1_{135^o} \approx (0.707, 0.707)$
Cyan: .. and its inverse $1/w$

Exercise 1.23. Regarding Fig. 5, calculate using build–in complex functions of EIGENMATH for the complex numbers $\mathbb{C} \equiv (\mathbb{R}^2, +, \cdot, \star)$
a. the $\star$–inverse $1/z$ of $z = 2 + i$.
b. the rectangular form of $w = 1_{135^o} = r_\varphi \in S^1$.
c. the inverse of w.

> We now want to reconstruct the results of Ex.1.22 using the new $\mathbb{R}$–algebra $\widetilde{\mathbb{C}} \stackrel{def}{=} (\mathbb{R}^2, +, \cdot, \circledast)$. Therefore we have to write functions e.g. to compute the *conjugate*, the real and *imaginary part*, the *length* (alias norm), the *reciprocal* (inverse) and the *quotient* resp. *the table based multiplication operation* $\circledast$.

Exercise 1.24. (Functions for the algebra $\widetilde{\mathbb{C}} \equiv (\mathbb{R}^2, +, \cdot, \circledast)$)
Let $w = (w[1], w[2])$ be a arbitrary 'new complex' number, i.e. $w \in \mathbb{R}^2 \equiv \widetilde{\mathbb{C}}$.
Use the EIGENMATH playground to solve the following tasks.
▷*Click here to open the playground.*
a. Copy/Write the following functions on the playground:

```
im(w) = w[2]
re(w) = w[1]
cj(w) = (w[1],-w[2])              -- conjugate of w
iv(w) = 1/(w[1]^2+w[2]^2)*cj(w) -- inverse of w

A = 2E+3I
B = 1E-2I
```

and test these functions on the $\widetilde{\mathbb{C}}$–numbers A and B.
b. Redo Ex.1.23 using the functions and notations from a..
c. Here is a definition to compute the *quotient* u/v of two new–complex numbers:

```
qu(u,v)= 1/(v[1]^2+v[2]^2) *
            (u[1]*v[1] + u[2]*v[2],
             v[1]*u[2] - u[1]*v[2])
```

- Calculate $qu(A, B)$. Check the result using build–in functions of EIGENMATH.
- Give an alternative definition of qu using the inverse function iv.
- Give an alternative definition of iv using the quotient function qu.

d. What does `mu(A,iv(A))` test? Write this expression in math language.
e. Collect the functions of this exercise in a toolbox named `cBox.txt` for later use.
▷*Click here to see the solution.*

1.3.3 Problems.

P8. The norm of a number $w \in \widetilde{\mathbb{C}} \equiv (\mathbb{R}^2, +, \cdot, \circledast$.
Let w be a arbitrary 'new complex' numbers of $\widetilde{\mathbb{C}}$.
a. Define a function `no(w)` to calculate the *norm* alias the length of w.
b. Determine the norm of all 5 points in Fig. 5.
c. What is the length of $A = 2E + 3I$ and $B = 1E - 2I$?
d. Check the results by interpreting and writing A and B as 'usual' complex numbers. Use paper'n pencil and EIGENMATH.
▷ *Click here to see the solution.*

P9. The inner and outer products in $\widetilde{\mathbb{C}}$.
Let $U, V \in \widetilde{\mathbb{C}}$ be two arbitrary 'new complex' numbers.
a. Define the two functions `ip(U,V)` and `op(U,V)` to compute the *inner* resp. *outer product* of complex numbers through

```
ip(U,V) = inner(U,V)              -- inner product alias scalar product
op(U,V) = U[1]*V[2] - V[1]*U[2]   -- outer product
```

- Calculate the inner and outer product of $U = 3E - 4I$ and $W = -4E + 3I$.
- Calculate the inner and outer product of $A = 2E + 3I$ and $B = 1E - 2I$.
- Calculate the inner and outer product of z and $\bar{z}$ of Fig. 5.

b. Prove: `ip(U,W) = re(mu(cj(U),W))`.
Formulate this formula in mathematical language.
c. Formulate and prove a similar formula for the outer product.
d. Verify the results of Ex.1.20 by arithmetic in $\widetilde{\mathbb{C}}$.
▷ *Click here to see the solution.*

$\circledast$

Summary: We have constructed a new algebra $\widetilde{\mathbb{C}}$ in the Euclidean plane $\mathbb{R}^2$ by means of a multiplication table for the basis vectors $span_{\mathbb{R}}\{e_1, e_2\}$. This construction was totally independent of the 'old' complex numbers build–in in EIGENMATH. Nevertheless we get also the desired relation $I^2 = -1$ to have a root of $\sqrt{-1}$. We were able to define the crucial $\mathbb{C}$–typical functions like conjugate, imaginary part, reciprocal, norm etc. in this setting, too.

1.4 ℂ as CLIFFORD algebra $c\ell(2,0)$

In this section we reconstruct the complex numbers ℂ using a universal construct, which we will use later to implement the *hyperbolic numbers* in §2.2, the *quaternions* and the *2D/3D geometry* with enhanced insights: the Geometric algebra "GA".

Now we will use the EIGENMATH package `EVA2.txt`[7] for the first time. We will use it without to say e.g. what a 'graded algebra' is. Later in Chapter 4 we have to say more about this, telling the motivation behind the construction. But first we should have some easy experiences in the mere using of `EVA` as another possibility to calculate with complex numbers ...

1.4.1 A first look at the 4D-CLIFFORD algebra $c\ell(2,0)$

Here is our new algebraic playground:

```
run("downloads/EVA2.txt") # load package EVA
cl(2)                     # (1) specify the Clifford Algebra
tty=1                     # compact output setting

e0                    -- (2) the 4 basis vectors e0,e1,e2,e12
e1
e2
e12

U = 1e0+2e1+3e2+4e12  -- (3) a 4D vector as lin.combi.
U
V = -3e1+4e2
V
U+V                    -- (4) usual 4D addition
U-V                    -- (5) .. and subtraction
2U+3V                  -- (5) a scalar multiple
magnitude(V)           -- (6) the length of V
Vn=normalize(V)        -- (7) unit vector in direction V
Vn
inp(Vn,Vn)             -- (8) the inner/scalar product
inp(U,V)               -- feel at home

gp(e0,e0)              -- (9) the gp = GeometricProduct
gp(e1,e0)              --     as new algebra multiplication
gp(e12,e12)            -- (10) a kind of imaginary unit
```

$$\begin{bmatrix} + \\ + \end{bmatrix}$$

```
e0 = (1,0,0,0)
e1 = (0,1,0,0)
e2 = (0,0,1,0)
e12 = (0,0,0,1)
U = (1,2,3,4)
V = (0,-3,4,0)
(1,-1,7,4)
(1,5,-1,4)
(2,-5,18,8)
5.0
Vn = (0.0,-0.6,0.8,0.0)
(1.0,0,0,0)
(6.0,13.0,16.0,0)
(1.0,0.0,0.0,0.0)
(0.0,1.0,0.0,0.0)
(-1.0,0.0,0.0,0.0)
```

▷ *Click here to invoke this script* and to experiment a bit.

Comment. The call $cl(2)$ alias $cl(2,0)$ of the function $cl(..)$ of the EVA package give the output $\left[\begin{smallmatrix} + \\ + \end{smallmatrix}\right]$. This means that the norm has the *signature* $(+,+)$, i.e. $\sqrt{+x^2+y^2}$. In line (2) we list the basis vectors $span_{\mathbb{R}}\{e0, e1, e2, e12\}$, which here have other names than the usual e_1, e_2, e_3, e_4. Why? Wait.
But nevertheless we feel immediately at home in this 4D vector space $c\ell(2)$ when studiying and looking at lines (3) until (8). Here *magnitude*, *normalize*, *inp*, and *gp* are functions of the package EVA, which are not available outside of this package.

[7] `EVA2` is an abbreviation for 'Euclidian Vector Algebra' version 2. We have to thank Bernard E for this wonderful piece of software. It is by far the biggest collection of user defined objects in EIGENMATH.

Line (10) is crucial: it remembers at the characteristic feature $i^2 = -1$ of the imaginary unit $i \in \mathbb{C}$ of the complex numbers, i.e.

$$gp(e12, e12) = \text{"}\ (e12)^2\ \text{"} = (-1, 0, 0, 0) \equiv -1$$

∘ This observation will lead to an realization of ℂ inside the CLIFFORD algebra $c\ell(2,0)$.
∘ For the moment we may think of the geometric product gp as given through a 4×4 =16 entry multiplication table a la ⊛ for the algebra $\widetilde{\mathbb{C}}$ in the last section.

Exercise 1.25.
a. Find two vectors $U, V \in c\ell(2)$ which are orthogonal resp. the scalar product *inp*.
b. Do some more free experiments in the 4D algebra $c\ell(2)$.

1.4.2 ℂ as part of the CLIFFORD algebra $c\ell(2,0)$

Here is our realization of the complex numbers ℂ as a 2D sub-algebra $\widehat{\mathbb{C}} \stackrel{def}{=} (\mathbb{R}^4, +, \cdot, gp)$. By sub-algebra we mean that we will only use linear combinations of the *two* basis vectors $e0, e12$, i.e. with the alias $E \stackrel{def}{=} e0$, $J \stackrel{def}{=} e12$ we have $\widehat{\mathbb{C}} = (span_{\mathbb{R}}\{E, J\}, +, \cdot, gp)$.

Remark. The CLIFFORD algebra multiplication, noted `gp(a,b)` in EIGENMATH package EVA, is often noted in mathematical texts as `ab` – *without any separating multiplication sign between the factors*. We do not recommend that use for the beginner. Instead use a notation e.g. $a \circledcirc b$ or $a \boxdot b$ or $a \circ b$ for `ab=gp(a,b)`.

Lexicon:	*Math*	EIGENMATH
	$A \circ B$	`gp(A,B)`

```
run("downloads/EVA2.txt")
cl(2)                  -- invoke Clifford Algebra (2,0)
tty=1
                       -- We give some syntactic sugar ..
E = e0                 -- to Embed the real numbers R^1
J = e12                -- to have usual name for Jmaginary unit
gp(J,J)                -- output: (-1,0,0,0) i.e. J^2 = -1

a =  1e0 + 2e12
b = -2e0 + 3e12
b                      -- output: b=(-2,0,0,3) == -2+3i
-- is now noted as

a =  1*E + 2*J
b = -2*E + 3*J
b                      -- output: b=(-2,0,0,3) == -2+3i
```

▷ *Click here to invoke this script.*

1.4.3 Clifford algebra cheatsheet for EVA2

Here is a cheatsheet of the main functions of the package EVA2 for future use:

	Math	Eigenmath EVA2
geometric product	A B	gp(A,B)
inner/scalar product	$A \bullet B$	inp(A,B)
outer product	$A \wedge B$	outp(A,B)
Clifford conjugation	$\bar{B}$	cj(B)
inverse	$1/B$	inverse(B)
magnitude	$\lvert B \rvert$	magnitude(B)
normalize	$\frac{B}{\lvert B \rvert}$	normalize(B)

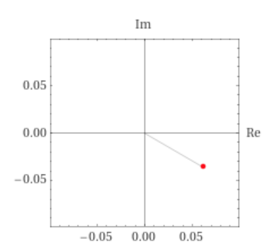
• There are also the Clifford algebra versions[8] for the regular build-in functions of the complex domain, always noted with an ending 1 to distinct it from the ℂ–functions:

imag1, real1, polar1, rect1, exp1, log1, sqrt1, power1, sin1, cos1, tan1, sinh1, cosh1, tanh1, asin1, acos1, atan1, asinh1, acosh1, atanh1, ..

Exercise 1.26. (Wolfram|*alpha* for complex numbers)
Wolfram|*alpha* works with complex numbers: ▷*Click here to invoke* Wolfram*'s page.*
Check their examples and results using the EVA package. E.g.
a. Calculate $1/(12 + 7i) \in \mathbb{C}$ inside $c\ell(2)$ using EVA. *Example* solution:

```
run("downloads/EVA2.txt")
cl(2)
inverse(12E+7J)        -- complex arithmetic in cl(2) with EVA
1/(12+7i)              -- complex arithmetic in EIGENMATH
```

Visualize the result,
loc. cit. Wolfram|*alpha*:



b. Do the other calculations from that page.
▷*Click here to invoke the script.*
c. Redo some of the exercises 1.5, 1.7–1.14, 1.17 and 1.20 using Eigenmath EVA.

[8] Most of these functions are implemented using partial Taylor sums, therefore giving 'only' approximate decimal values.

2 IH – the hyperbolic numbers

> "The hyperbolic numbers are blood relatives of the popular complex mumbers and deserve to be taught alongside the latter. They serve not only to put Lorentzian geometry on an equal mathematical footing with Euclidean geometry, but also help students develop algebraic skills and concepts necessary in higher mathematics. Whereas the complex numbers extend the read numbers to include a new number $i = \sqrt{-1}$, the hyperbolic numbers extend the real numbers to include a *new* square root $u = \sqrt{+1}$, where $u \neq \pm 1$. [**?**, p. 2]

We name this new number u the *unipotent*. This u solves the equation $x^2 = 1$, but has the algebraic properties $+1 \notin \mathbb{R}$ and $-1 \notin \mathbb{R}$ and $u \notin \mathbb{R}$! Using the same pattern of the construction of the complex numbers $\mathbb{C}$ in the last section, we build the *hyperbolic numbers* IH now in two different ways: first by means of a special multiplication (table) for the 2D vector space $\mathbb{R}^2$ and second using the CLIFFORD algebra $c\ell(1,1)$.

2.1 IH as algebraic structure

$\overset{g}{*}$ ⊙ ◯ □◊△ ⊠ . ⊡ ⊛□★ ⟲⟳ ⊚

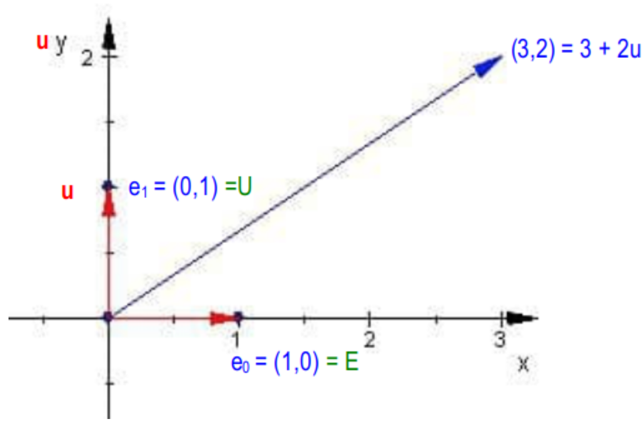


Figure 6: The hyperbolic number plane IH.
Blue: The hyperbolic number $3 + 2u$. Basis $\{e_0, e_1\}$ of $\mathbb{R}^2$.
Red: The unipotent u with $u^2 = 1$, but $u \neq \pm 1 \in \mathbb{R}$.
Green: The hyperbolic basis $\{1, u\}$ alias $\{e_0, e_1\}$ or $\{E, U\}$.

To construct the hyperbolic numbers IH as an $\mathbb{R}$–algebra, we extend the real vector space $\mathbb{R}^2$ to include the unipotent element u together with a new third operation $\boxdot\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ by means of a 'hyperbolic multiplication table' for it. Analog to the reconstruction of the complex numbers $\mathbb{C}$, we will use the *bilinearity* of $\boxdot$ and describe its action on all 4 possible pairs of basis elements.

Here are the hyperbolic multiplication rules for $\boxdot$, acting on the two basis vectors $e_0 \overset{def}{=} (1,0)$[9] and $e_1 \overset{def}{=} (0,1)$ of the vector space $\mathbb{R}^2$:

$$e_0 \boxdot e_0 = e_0 \tag{2.1}$$
$$e_0 \boxdot e_1 = e_1 \tag{2.2}$$
$$e_1 \boxdot e_0 = e_1 \tag{2.3}$$
$$e_1 \boxdot e_1 = e_0 \tag{2.4}$$

• If we translate the 4^{th} rule using the lexicon $\frac{e_0, e_1}{\mathbf{1, u}}$ into the language of $\mathbb{H}$, we get the desired relation $\frac{e_1 \boxdot e_1 = e_0}{u \boxdot u = u^2 = 1}$.

• If we put the above rules in an *hyperbolic multiplication table* for this algebra multiplication $\boxdot$ for $\mathbb{H}$, we have:

$$\begin{array}{cc} \boxdot & \begin{array}{cc} e_0 & e_1 \end{array} \\ \begin{array}{c} e_0 \\ e_1 \end{array} & \begin{pmatrix} e_0 & e_1 \\ e_1 & e_0 \end{pmatrix} \end{array}$$

Definition. The hyperbolic numbers $\mathbb{H}$ are the elements of the 2–dimensional number plane $\mathbb{R}^2$ equipped with the three operations $(+,\cdot,\boxdot)$ and the unipotent element $u \in \mathbb{H}$ with $u \neq \pm 1$, but $u^2 = 1$. We write: $\mathbb{H} \overset{def}{=} (\mathbb{R}^2, +, \cdot, \boxdot)$.

Exercise 2.1. Calculate $(1e_0 + 2e_1) \boxdot (-2e_0 + 3e_1)$ using the multiplication table for hyperbolic numbers. *Solution*:

$$\begin{aligned} (1e_0 + 2e_1) \boxdot (-2e_0 + 3e_1) &= -2e_0 \boxdot e_0 + 3e_0 \boxdot e_1 - 4e_1 \boxdot e_0 + 6e_1 \boxdot e_1 \\ &= -2e_0 + 3e_1 - 4e_1 + 6e_0 \\ &= 4e_0 - 1e_1 = 4 \cdot (1,0) - 1 \cdot (0,1) = (4,-1) \end{aligned}$$

2.1.1 Implementing ℍ alias the binarions in Eigenmath

Remark. Using the abbreviation $E := e_0, U := e_1$ the hyperbolic multiplication table reads

$$\begin{array}{cc} \boxdot & \begin{array}{cc} E & U \end{array} \\ \begin{array}{c} E \\ U \end{array} & \begin{pmatrix} E & U \\ U & E \end{pmatrix} \end{array}$$

and looks like a 2D analogue of the 4D table for the quaternions, which we discuss later on. Therefore the name '*bi*'narions for the hyperbolic numbers.

[9] For systematically reasons, which will become clear later on, we again do not use the usual notation $\{e_1, e_2\}$ for the two basis vectors.

In order to calculate in the new algebra $\mathbb{H} = (\mathbb{R}^2, +, \cdot, \boxdot)$, we have to translate the table above into EIGENMATH command language.

```
##### HYPERBOLICS H   -- alias: the Binarions
tty=1

e0 = (1,0)            -- basis of H
e1 = (0,1)

T = ((e0,e1),
     (e1,e0))         -- Hyperbolics multiplication _T_able

M(x,y) = dot(x,T,y)   -- billinear operation on H

"Checking binarions multiplication table."
check(M(e0,e0)=e0)
check(M(e0,e1)=e1)
check(M(e1,e0)=e1)
check(M(e1,e1)=e0)
"pass"                -- (0) check ok? yes!

M(e1,e1)              -- (1) with e1=u we see: u^2=1
M(1e0+2e1,-2e0+3e1)   -- (2) checking Ex.
```

```
Checking binarions multiplication table.
pass
(1,0)
(4,-1)
```

Comment. Function $M \stackrel{def}{=} \boxdot$ realize the bilinear operation, i.e. the Multiplication of hyperbolic numbers.. The checks in lines (0) verifies, that the operation M implements the values of the $\mathbb{H}$–multiplication table and especially fulfills the desired relation $U^2 = e_1^2 = (1,0) \equiv 1$. Code line (2) verifies the result of Ex.2.1.
▷ *Click here to run the script.*

Exercise 2.2. (The algebraic characteristics of the hyperbolic number multiplication $\boxdot$)
The multiplication $\boxdot$ is prescribed on its values on the finite 4 element table. Therefore it suffices to check its properties like *commutativity*, *associativity*, *distributivity* etc. on it. E.g.

```
-- define 3 arbitrary hyperbolic numbers ('binarions')
x = (x0,x1)
y = (y0,y1)
z = (z0,z1)

"Is multiplication M of hyperbolic numbers commutative?"
test( M(x,y)=M(y,x), "yes","no")

"Is multiplication M alternative?"
test( and(M(M(x,x),y)=M(x,M(x,y)),
          M(M(y,x),x)=M(y,M(x,x))),"yes","no")
```

EIGENMATH output: commutativ: yes alternativ: yes
▷ *Click here to run the script.*

a. Check the *associativity* of $M \equiv \boxdot$.
b. Check the *distributivity* of M.

Exercise 2.3. (The explicit formula for the hyperbolic number multiplication $\boxdot$)
We know the explicit formula $(a+bi)\star(c+di) = (ac-bd)+(ab-cd)i$ for the complex multiplication $\star$. Derive a similar formula for the hyperbolic multiplication $\boxdot$ by paper'n pencil and EIGENMATH.

Solution. First, let's spend a bit syntactic sugar and set $E \overset{def}{=} e_0$ and $U \overset{def}{=} e_1$ to get a similar appearance of hyperbolic numbers like the complex one's, i.e. $\begin{matrix} e_0, e_1 \\ \text{E, U} \end{matrix}$ and $\begin{matrix} \mathbb{C}\colon z = a\cdot 1 + b\cdot i \\ \mathbb{H}\colon w = a\cdot E + b\cdot U \end{matrix}$.

```
E = (1,0)              -- _E_mbedding of R in H
U = (0,1)              -- the _u_unipotent - the analogue to i in C
                       -- H  multiplication rules:
                       -- E*E=E, E*U=U, U*E=U, U*U=E == 1
T = ((E, U), (U,E))
M(x,y) = dot(x,T,y)    -- multiplication as bilinear operation on H

                       -- two arbitrary hyp.numbers:
x = a*E+b*U
y = c*E+d*U
y
M(x,y)
```

EIGENMATH output: `y=(c,d)` `M(x,y)=(a c + b d, a d + b c)`
▷ *Click here to run the script.*

Using the explicit formula for M we can forget about the construction of the algebra IH by means of a multiplication table and think of the hyperbolic number s as $(\mathbb{R}^2, +, \cdot, \boxdot)$ with

$$\boxdot\colon \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \tag{2.5}$$

$$(a,b),(c,d) \rightarrow (a,b)\boxdot(c,d) \overset{def}{=} (a\cdot c + b\cdot d, a\cdot d + b\cdot c) \tag{2.6}$$

Exercise 2.4. (The hyperbolic multiplication)
a. Write the following function in a toolbox named `hyBox.txt` for future use:

```
# multiplication of hyperbolic numbers
hymult(x,y) = (x[1]*y[1] + x[2]*y[2], x[1]*y[2] + x[2]*y[1])
hymult(e1,e1)             -- result: (1,0) == 1
```

b. Solve Ex.2.1 using `hymult(..)`.

2.1.2 Implementing specific user functions for IH

From now on we use the following lexicon for calculation in the hyperbolic number plane IH with $E = (1,0)$ and $U = (0,1)$:

	Math	EIGENMATH
standard basis	$(1, u)$	`(E,U)`
arbitrary hyperbolic number	$w = x + yu$	`w=x*E+y*U`

Definition. (*the hyperbolic length*)
The *hyperbolic norm* (modulus, length) of $w = x + y \cdot u \in \mathbb{H}$ is defined as the real number

$$|w|_h \overset{def}{=} \sqrt{|x^2 - y^2|} \tag{2.7}$$

The set $H^1 \overset{def}{=} \{w \in \mathbb{H} \mid x^2 - y^2 = 1\}$ is called the *unit hyperbola.*

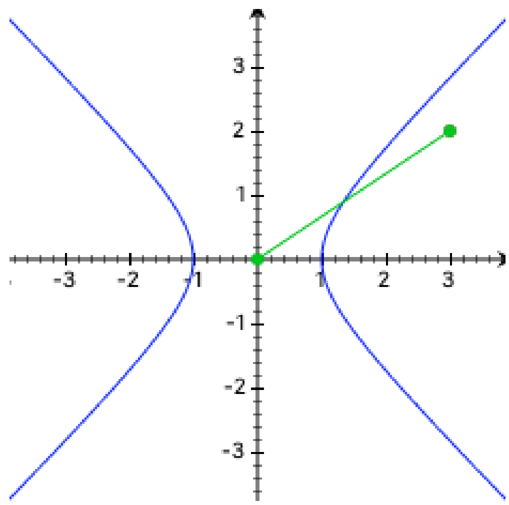


Figure 7: Blue: unit hyperbola $H^1 \subset \mathbb{R}^2$ with equation $x^2 - y^2 = 1$.
Green: the hyperbolic number/vector $w = 3 + 2 \cdot u = (3, 2)$
with hyperbolic length $|3 + 2 \cdot u|_h = \sqrt{5} \approx 2.24$.

Exercise 2.5. (The norm of a hyperbolic number)
Let $w = x + yu \in \mathbb{H}$ be a arbitrary hyperbolic number.
a. Define a function `hyno(w)` to calculate the *hy*perbolic *norm* alias the length of w as given in the definition previously.
b. Calculate the hyperbolic lenght of $w = 3 + 2u$ in Fig.7 by paper'n pencil and *hyno*(..).
c. Determine the 'hyno' of the points $P = 1 + 0u$, $Q = -1 + 0u$ and $R = -1 + 0u$.
d. What is the length of $A = 2E - 3U$ and $B = 1E - 2U$?
▷ *Click here to see the solution.*

Exercise 2.6.
a. Put the following functions for hyperbolic numbers in the toolbox `hyBox.txt`:

```
hyreal(w)   = w[1]                -- REAL part of hyperbolic number w
hyunip(w)   = w[2]                -- UNIPotent part of hyperbolic number w
hyconj(w)   = (w[1],-w[2])        -- hyperbolic CONJugate of w
hyinv(w)    = 1/(w[1]^2-w[2]^2)*hyconj(w) -- hyperbolic INVerse of w
hyquot(v,w) = hymult( v, hyinv(w))          -- QUOTient of v and w
hynorm(w)   = sqrt(abs(w[1]^2-w[2]^2))  -- hyperbolic NORM of w
```

b. Calculate the hyperbolic lenght of $w = 3 + 2u$ of Fig.7.

```
-- EIGENMATH solution
do( E=(1,0), U=(0,1) )
w = 3E+2U
hynorm(w)             -- output: 5^1/2 = 2.24
abs(w)                -- the Euclidean length of w in R^2 is 13^1/2=3.6 !
```

▷ *Click here to run the script.*

c. Determine for w the hyperbolic real part, its unipotent part, its conjugate, the inverse.

d. Calculate the hyperbolic quotient of $w = 3 + 2u$ and $v = 1 + 2u$.

e. Try to calculate the hyperbolic quotients of $w = 3+2u$ and $v = 1+1u$ resp. $w/(2-2u)$.

2.1.3 Isotropic points in IH

Ex.2.6 showed, that there are hyperbolic numbers not equal $0 \cdot E + 0 \cdot U = (0, 0)$, for which the calculation of the hyperbolic quotient ejected a '*division by zero*' error message. We observed, that the denominators lie on the diagonals of the coordinate system. Those point are called *isotropic*.

Definition. A hyperbolic number $w \neq 0$ s called *isotropic*, if its hyperbolic length is zero, i.e. $|w|_h = 0$.

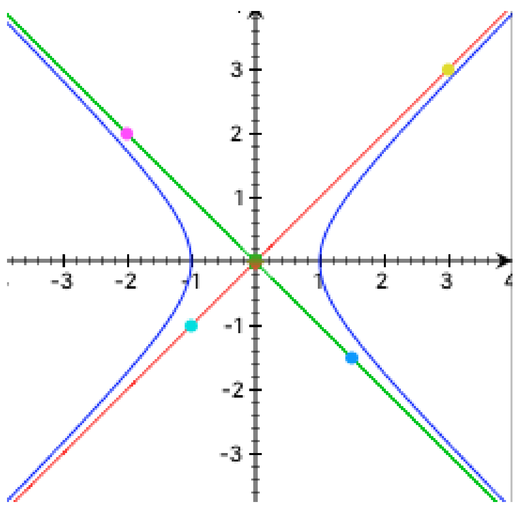


Figure 8: Blue: unit hyperbola $H^1 \subset \mathbb{R}^2$ with equation $x^2 - y^2 = 1$.
Red: the main diagonal $y = x$ as isotropic point line
Green: the second diagonal $y = -xx$ as isotropic points
Cyan ..: isotropic points $3 + 3u, -2 + 2u, -1 - u, 1.5 + 1.5u$.

Exercise 2.7. (isotropic points)

a. Verify, that the points (hyperbolic numbers) in Fig.8 are isotropic.

b. Prove: all points on the diagonals $y = \pm x$ are isotropic.

Remark. This phenomenon of the existence of isotropic subvector spaces with respect to the hyperbolic norm leads to a new non-Euclidean geometry, the LORENTZian geometry on $\mathbb{R}^2$. It plays a great role in Special Relativity.

2.1.4 Problems.

P10. A scene for the hyperbolic number plane.

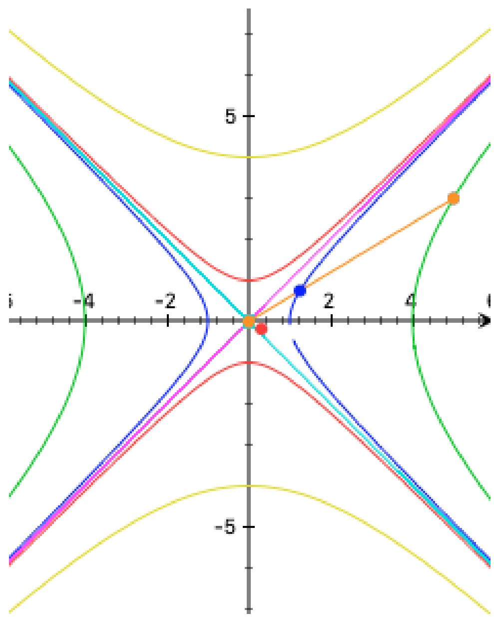


Figure 9: Blue: unit hyperbola $H^1_+ \subset \mathbb{R}^2$ with equation $x^2 - y^2 = +1$.
Red: unit hyperbola $H^1_- \subset \mathbb{R}^2$ with equation $x^2 - y^2 = -1$.
Green: hyperbola $H^4_+ \subset \mathbb{R}^2$ with equation $x^2 - y^2 = +16$.
Yellow: hyperbola $H^4_- \subset \mathbb{R}^2$ with equation $x^2 - y^2 = -16$.
Cyan: isotropic line $y = -x$. Magenta: isotropic line $y = x$.
$w = 5 + 3u \in \mathbb{H}$, Blue: $w/|w|_h$ Red: w_h^{-1} =`hyinv(w)`.

a. Calculate the coordinates of the blue point $w/|w|_h \in H^1_+$ by paper'n pencil. Check your result with EIGENMATH.
b. Determine the hyperbolic distance and the Euclidean distance of w from the origin.
c. Determine the coordinates of the red point $w^o := w/|w|_h \in H^1_+$ by paper'n pencil. Check your result with EIGENMATH. Determine the hyperbolic distance of w^o from the origin and from its 'father' w.
c. Determine the coordinates of the hyperbolic inverse w_h^{-1} of w by paper'n pencil. How long is the distance of this inverse to w? Check your results with EIGENMATH.
d. Verify the result of c. by calculating the hyperbolic product $w_h^{-1} \boxdot w$.

P11. Alternativ formula for the hyperbolic conjugate.
a. Argue, why the following function `Cj(.)` calculates the hyperbolic inverse.

```
-- EIGENMATH
do( E=(1,0), U=(0,1) )
w = 3E+2U
Cj(x) = 2*dot(x,E)*E - x
```

b. Calculate the hyperbolic conjugates of the three points (hyperbolic numbers) of Fig. 9. Check your results with *hyconj*(.).
▷ *Click here to open the playground.*
▷ *Click here to open the playground.*

P12. The inner and outer product in ℍ.
Let $U, V \in \mathbb{H}$ be two arbitrary hyperbolic numbers.
a. Define the functions `hyinp(U,V)` and `hyoutp(U,V)` to compute the *inner* resp. *outer product* of two hyperbolic numbers $U, V \in \mathbb{H}$ through

```
hyinp(U,V)  = U[1]*V[1] - U[2]*V[2]   -- inner product alias scalar product
hyoutp(U,V) = U[1]*V[2] - U[2]*V[1]   -- outer product
```

- Calculate the inner and outer product of $U = 3E - 4U$ and $W = -4E + 3U$.
- Calculate the inner and outer product of $A = 2E + 3U$ and $B = 1E - 2U$.
- Calculate the inner and outer product of w and $\bar{w}$ of Fig. 59.

b. Prove: `hyinp(U,W) = hyre( hymult( hyconj(U),W))`.
Formulate this formula in mathematical language.
c. Formulate and prove a similar formula for the outer product.
d. Find a hyperbolic number $w^{\perp}$, which is hyperbolic orthogonal to w.
▷ *Click here to see the solution.*

P13. Realization of ℍ as matrix algebra.
Using the correspondence

$$\mathbb{H} \longrightarrow \mathbb{R}^{2\times 2}_{sym} \tag{2.8}$$

$$x + y \cdot u \mapsto \begin{bmatrix} x & y \\ y & x \end{bmatrix} \tag{2.9}$$

the hyperbolic numbers can be identified with the symmetric 2×2 matrices with equal diagonal entries.
a. Why is this assignment an isomorphism?
b. The hyperbolic numbers $w1 = 2 + 3u$ and $w2 = 3 - 5u$ are represented via the isomorphism (2.9) through $W1 = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ and $W2 = \begin{bmatrix} 3 & -5 \\ -5 & 3 \end{bmatrix}$.
Calculate the values of $w1 + w2, w1 - w2, 2 \cdot w1$ by paper'n pencil and EIGENMATH.
Partial *solution*:

```
W1=((2,3),(3,2))
W2=((3,-5),(-5,3))
W1+W2
W1-W2
2*W1
```

c. Verify, that the hyperbolic multiplication $\boxdot$ alias $hymult()|_{Eigenmath}$ corresponds to the usual matrix multiplication $\star$ of symmetric 2×2 matrices, i.e.

$$hymult((2,3),(3,-5)) \overset{Math}{=} W1 \star W2 \overset{Eigenmath}{=} dot(W1, W2)$$

Example:

```
W1=((2,3),(3,2))
W2=((3,-5),(-5,3))
dot(W1,W2)                   -- corresponds to (2+3u) hymult (3-5u)
```

d. Write corresponding EIGENMATH functions for the hyperbolic real part, unipotent part, the hyperbolic multiplication, quotient, conjugate, norm and the hyperbolic inverse of a hyperbolic number. Here is a start: ▷*Click here to start the start.*

```
Hconj(z)=((z[1,1],-z[1,2]),(-z[2,1],z[2,2]))
Hnorm(z)=sqrt(abs(z[1,1]^2-z[1,2]^2))
Hinv(z)=1/(z[1,1]^2-z[1,2]^2)*Hconj(z)

# TEST
W1=((2,3),(3,2))
W2=((3,-5),(-5,3))
Hconj(W1)
Hinv(W1)
```

P14. More functions for IH.
Let $U, V \in \mathbb{H}$ be two arbitrary hyperbolic numbers.
If you like it: derive functions for the hyperbolic polar form, the hyperbolic angle (argument) etc.

$\boxdot$

Summary: We have constructed a new algebra IH inside the Euclidean plane $\mathbb{R}^2$ by means of a special multiplication table for the basis vectors $span_{\mathbb{R}}\{e_0, e_1\}$ alias $span_{\mathbb{R}}\{E, U\}$. With this hyperbolic multiplication we got the desired relation $U^2 = 1$ to have a root of $\sqrt{+1}$, not being $\pm 1 \in \mathbb{R}$. We were able to define the crucial IH–typical functions like hyperbolic conjugate, hyperbolic imaginary part, hyperbolic reciprocal, hyperbolic norm etc. in this setting, too.
We did not give EIGENMATH formulas e.g. for the hyperbolic polar form, because we show these constructs in a more general setting – viewing IH as a special CLIFFORD algebra. This is the done in the next section.

2.2 IH as split-complex numbers – the Clifford algebra $c\ell(1,1)$

In this section we reconstruct the hyperbolic numbers IH using the same universal construction, which we used to represent the algebra $\mathbb{C}$ and which is a special case of an Geometric algebra "GA".

Here we use the Eigenmath package `EVA2.txt` for the second time. We want to broaden our experience in the use of `EVA` as another possibility to calculate with hyperbolic numbers in a more straight way.

2.2.1 A look at the 4D-Clifford algebra $c\ell(1,1)$

Let's look at $c\ell(1,1)$ as our new algebraic modelling of the hyperbolic numbers IH:

```
run("downloads/EVA2.txt")  -- (1)
tty=1
cl(1,1)                    -- (2)

do( print(e0),print(e1),print(e2),print(e12))  -- (3)

E = e0       -- (4) E mbedding of R in cl(1,1), i.e. first entry
E
U = e12      -- (5) U will play the role of the unipotent u
U

gp(E,E)
gp(U,U)      -- (6) U is indeed unipotent resp gp
```

```
(+,-)
e0 = (1,0,0,0)
e1 = (0,1,0,0)
e2 = (0,0,1,0)
e12 = (0,0,0,1)
E = (1,0,0,0)
U = (0,0,0,1)
(1.0,0.0,0.0,0.0)
(1.0,0.0,0.0,0.0)
```

Comment. At first glance, all looks similar to the $c\ell(2,0)$ construction of $\mathbb{C}$ in §1.4.1. That's good, because we do not have to learn a new vocabulary and may use the same notations, that we're used to. *But watch*: The call $cl(1,1)$ in code line (2) of the constructor function $cl(..)$ of the `EVA2` package give the output $(+,-)$! This means, that the norm of $cl(1,1)$ has the *signature* $(+,-)$, i.e. the norm has now the term $\sqrt{+x^2-y^2}$. Therefore the name 'split-complex'. In line (3) we list the basis vectors $span_{\mathbb{R}}\{e0,e1,e2,e12\}$ of $c\ell(1,1)\sim\mathbb{R}^4$, which have the expected canonical coordinates of the 4D vector space $\mathbb{R}^4$. In line (4) we embed the real number line $\mathbb{R}$ by means of E and his multiples into $c\ell(1,1)$.
Line (6) is crucial and shows, why we chose $c\ell(1,1)$ as model for IH: it verifies the characteristic feature $u^2=1$ of the unipotent element $u\in$ IH of the hyperbolic numbers is fulfilled with respect to the geometric product of $c\ell(1,1)$, i.e.

$$gp(U,U)=U^2=(1,0,0,0)\equiv 1$$

◦ This observation (6) lead to an realization of IH inside the Clifford algebra $c\ell(1,1)$.
◦ For the moment we may think of the geometric product gp as given through the 4×4 $=16$ entry multiplication table for the hyperbolic multiplication $\boxdot$ for the algebra IH in the last section or as a re–construction of the function $hymult(.)$ inside $c\ell(1,1)$.
▷ *Click here to invoke cl(1,1).*

Exercise 2.8.
Do some free experiments in the 4D algebra $c\ell(1,1)$. – A possible *Solution*.

```
run("downloads/EVA2.txt") # load package EVA
cl(1,1)                   # specify the Clifford Algebra
tty=1                     # line oriented output
do( E=e0, U=e12 )         # set (E,U) 2D sub-algebra

A = 1e0+2e1+3e2+4e12   -- an element in full cl(1,1), but not in H
B = 1E+2e1+3e2+4U      -- the same in other notation
A
B

a =  3E+2U             -- an element in H
b = -2E+U

a+b                    -- normal 4D addition in H
a-b                    -- normal 4D subtraction in H
2a+3b                  -- usual linear combination
magnitude(a)           -- hyperbolic length, see Fig.2
abs(a)                 -- Euclidean length

inp(U,U)               -- feel at home
```

▷ *Click here to invoke this script.*

2.2.2 Doing algebra in IH as part of the CLIFFORD algebra $c\ell(1,1)$

After Ex.2.8 we feel immediately at home in this 4D vector space $c\ell(1,1)$. The functions *magnitude*, *normalize*, *inp*, and *gp* (geometric product) are available by means of the package EVA2 and work as expected. This realizes the hyperbolic number plane IH as a 2D sub-algebra $\mathbb{H} := (\mathbb{R}^4, +, \cdot, gp)$ of $c\ell(1,1)$

Math IH	EIGENMATH EVA2 $c\ell(1,1)$
$A \boxdot B$	gp(A,B)

.. and we can use the same EVA2–functions as for the complex numbers:

	Math	EIGENMATH EVA2
geometric product	A B	gp(A,B)
inner/scalar product	$A \bullet B$	inp(A,B)
outer product	$A \wedge B$	outp(A,B)
Clifford conjugation	$\bar{B}$	cj(B)
inverse	$1/B$	inverse(B)
magnitude	$\|B\|$	magnitude(B)
normalize	$\frac{B}{\|B\|}$	normalize(B)

• The CLIFFORD algebra functions are usable also for the hyperbolic domain, they are noted with an ending 1 to distinct them from the EIGENMATH build–in functions for the complex domain:

`imag1, real1, polar1, rect1, exp1, log1, sqrt1, power1, sin1, cos1, tan1, sinh1, cosh1, tanh1, asin1, acos1, atan1, asinh1, acosh1, atanh1`

Exercise 2.9. (Using $c\ell(1,1)$ for arithmetic with hyperbolic numbers)
a. Re–do Ex.2.5 and Ex.2.6 calculating in the CLIFFORD algebra $c\ell(1,1)$ using `EVA2`.
b. Re–do problems P.10 and P.12 calculating in the CLIFFORD algebra $c\ell(1,1)$ using `EVA2`.
▷ *Click here to invoke the script.*

2.2.3 The hyperbolic polar form in $\mathbb{H} \sim c\ell(1,1)$

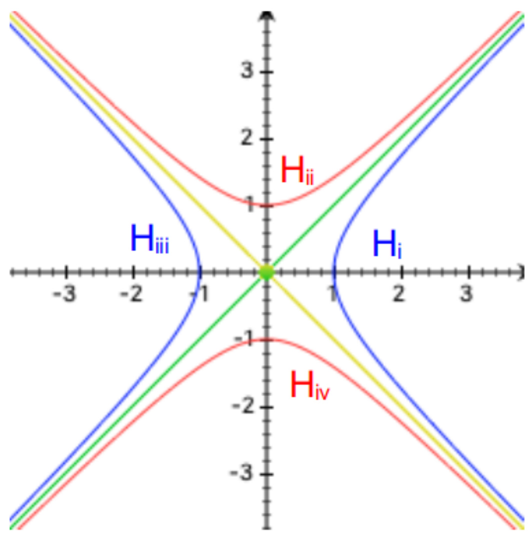


Figure 10: Blue: *unit* hyperbola $H^1_+ \subset \mathbb{R}^2$ with equation $x^2-y^2=+1$.
Red: *conjugate* hyperbola H^1_- with equation $x^2-y^2=-1$.
Green: first asymptote with equation $y=x$.
Yellow: second asymptote with equation $y=-x$.
Four hyperbolic quadrants $H_i, H_{ii}, H_{iii}, H_{iv}$, demarcated by the two asymptotes..

For the use of the *hyperbolic* polar form `polar1`, we divide the hyperbolic plane $\mathbb{H}$ in 4 hyperbolic quadrants $H_i, H_{ii}, H_{iii}, H_{iv}$, see Fig.10, *with the asymptotes as axes.* The set of all points $w \in \mathbb{H}$ in the hyperbolic plane that fulfill the relation $\|B\| = |w|_h = \rho$ for an hyperbolic radius $\rho > 0$ is a four branched hyperbola. For a hyperbolic number $w = x + yu$ we therefore have

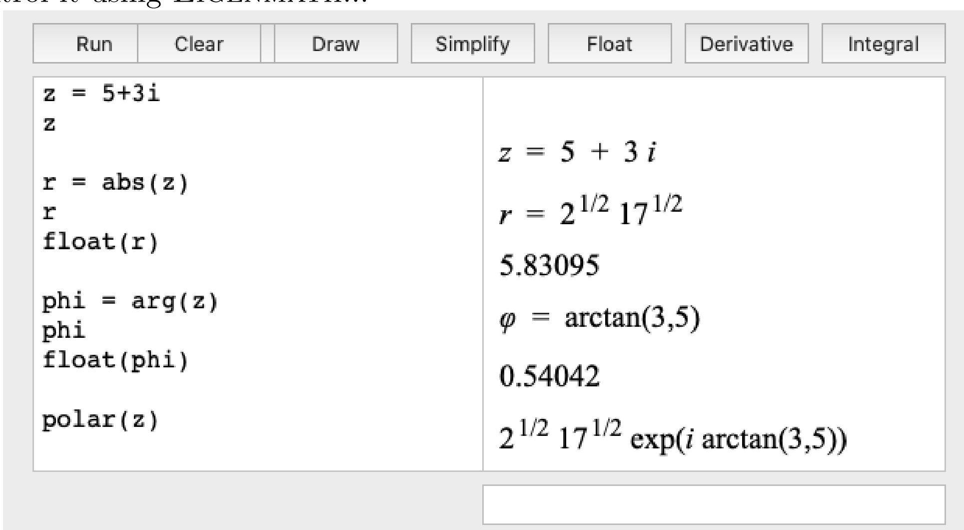
$$polar1(w) = \begin{cases} +\rho \cdot exp1(\phi \cdot u) & : \text{ for } w \text{ in } H_i \quad \text{(pol1)} \\ -\rho \cdot exp1(\phi \cdot u) & : \text{ for } w \text{ in } H_{iii} \quad \text{(pol3)} \end{cases}$$

resp.

$$polar1(w) = \begin{cases} +\rho u \boxdot exp1(\phi \cdot u) & : \text{ for } w \text{ in } H_{ii} \quad \text{(pol2)} \\ -\rho u \boxdot exp1(\phi \cdot u) & : \text{ for } w \text{ in } H_{iv} \quad \text{(pol4)} \end{cases}.$$

Example. (the hyperbolic polar form of $w = 5 + 3u \in \mathbb{H}$, see [10, p. 6]) For analogy and contrast we look at the point $(5, 3) \in \mathbb{R}^2$ of the Euclidean plane from the viewpoints of $\mathbb{C}$, i.e. $z = (5, 3) = 5 + 3i$ and $\mathbb{H}$, i.e. $w = (5, 3) = 5 + 3u$.

$(5, 3) = z = 5 + 3i \in \mathbb{C}$: First we calculate the polar form of w seen as complex number.
The radius is $r = \sqrt{+5^2 + 3^2} = \sqrt{34} \approx 5.83$.
The argument (angle) is $\varphi = \arctan(3/5) =\approx 0.54042$, i.e. $\varphi \approx 31^o$.
Therefore $polar(z) = \sqrt{34} \cdot exp(0.54042 \cdot i)$.
Let's control it using EIGENMATH...



.. and by means of a plot:

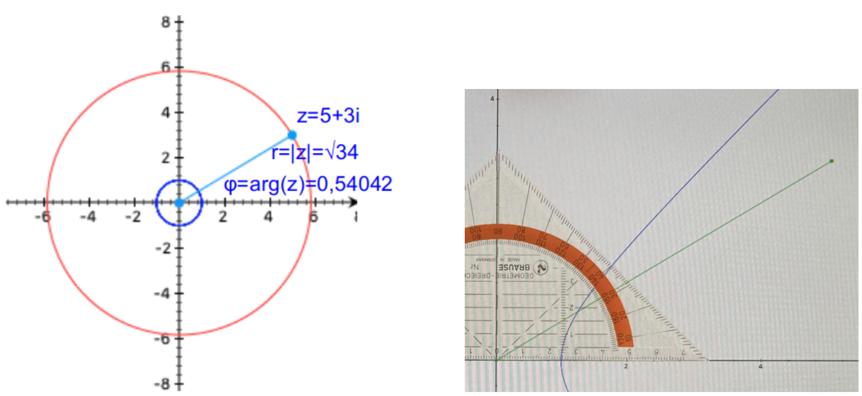
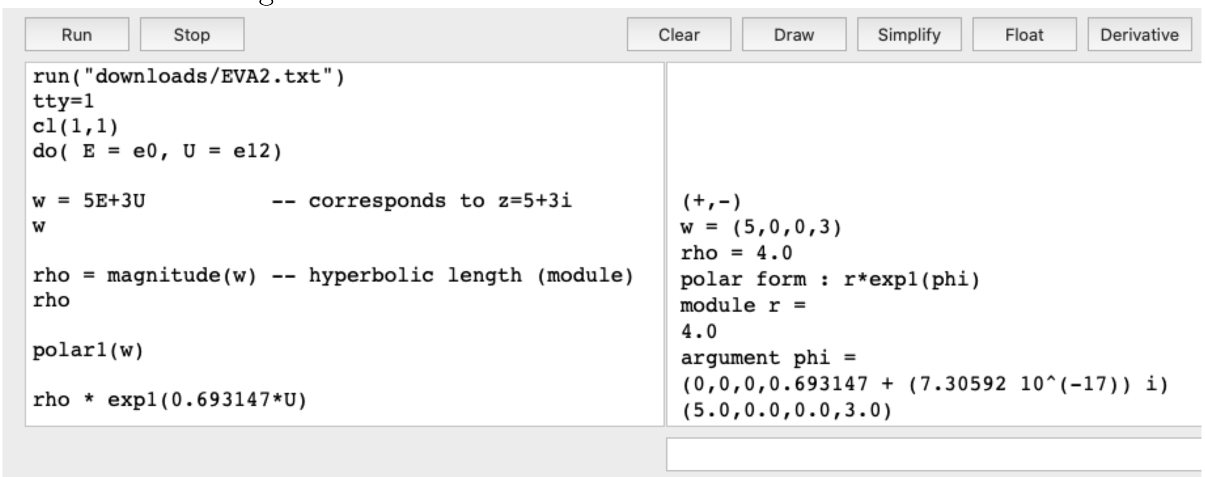


Figure 11: Figure left:
Blue: unit circle S^1 with equation $x^2 + y^2 = 1$.
Red: circle S^r of radius $r = \sqrt{34} \approx 5.83$
Cyan: the complex number $z = 5 + 3i$ with $\varphi = \measuredangle = 31^o$.
Figure right: hardware measured $\varphi = \measuredangle = 31^o$

$(5,3) = w = 5 + 3u \in \mathbb{H}$: Now we calculate the polar form of w as a hyperbolic number.
The hyperbolic radius is $\rho = \sqrt{+5^2 - 3^2} = \sqrt{16} = 4$.
The hyperbolic argument (angle) is $\phi \overset{H_i}{=} arctanh(3/5) \approx 0.6931$. No degree!
Therefore $polar1(z) \overset{H_i}{=} 4 \cdot exp(0.6931 \cdot u)$.
Let's control it using EIGENMATH:




```
run("downloads/EVA2.txt")
tty=1
cl(1,1)
do( E = e0, U = e12)

w = 5E+3U           -- corresponds to z=5+3i
w

rho = magnitude(w) -- hyperbolic length (module)
rho

polar1(w)

rho * exp1(0.693147*U)
```

```
(+,-)
w = (5,0,0,3)
rho = 4.0
polar form : r*exp1(phi)
module r =
4.0
argument phi =
(0,0,0,0.693147 + (7.30592 10^(-17)) i)
(5.0,0.0,0.0,3.0)
```

Comment. The hyperbolic number $w = 5 + 3u$ is represented in $\mathbb{H} \sim c\ell(1,1)$ as a 4D vector, where only the 1^{st} and the 4^{th} entry is used, therefore giving a 2D sub-algebra. The EVA function *magnitude* returns the hyperbolic length of w alias the *hyperbolic radius*. The complete polar form of w is calculated by the EVA function *polar1*, which returns the hyperbolic angle (alias the *hyperbolic argument*) as the real part 0.693147 of the 4th entry. Because for $w \in H_i$ we verify the result using formula (pol1) and get back the rectangular form of w. Ok.
Let's look at the geometric situation.

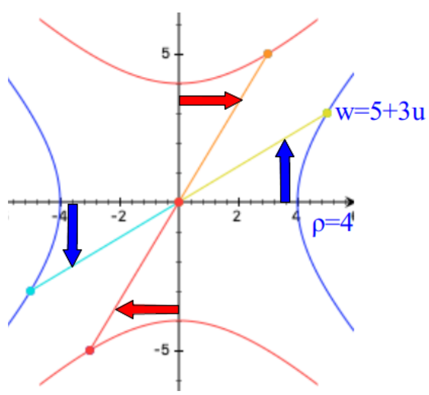


Figure 12: Blue: hyperbola H_i^4, H_{iii}^4: $x^2 - y^2 = 16$. w is a point on it.
Red: conjugate hyperbola $H_i^4 i, H_{iv}^4$: $x^2 - y^2 = -16$
Cyan: the hyperbolic number $w = -5 - 3u$ and its three branch "brothers" with same hyperbolic angle (argument).

Exercise 2.10. (branch points and their hyperbolic polar forms)
a. Verify, that the hyperbolic conjugate of $w = 5+3u$ is $cj(w) = 5-3u$. Use EIGENMATH. Calculate its hyperbolic polar form.
b. There are 3 more points on the 4 branched hyperbola $H^{\rho=4}$, to be seen in Fig. 12. Give their polar1 forms and their rectangular forms. *Hint*: use symmetry. Here is a start.

```
run("downloads/EVA2.txt")
tty=1
cl(1,1)
do( E = e0, U = e12)
w = 5E+3U

w3 = -magnitude(w)*exp1(0.693147 U)      -- use formula (pol3) with ..
w3                                       -- .. same hyperbolic angle !

w2 = +gp(magnitude(w)*U, exp1(0.693147 U))
w2                            -- use (pol2), because w2 on 2nd branch
```

▷ *Click here to invoke this script.*

Exercise 2.11. (The hyperbolic polar form)
a. Express each of the following the hyperbolic numbers (points) in hyperbolic polar form:
$A = 2E + \sqrt{12}U, \quad B = -5E + 5U, \quad C = -\sqrt{6}E - \sqrt{6}U, \quad D = -3U$
and plot them on the hyperbolic number plane. Use EIGENMATH.
b. Interpret the points of Ex.a. as complex numbers, using their real and imaginary parts. Plot the points on the *complex* number plane. Use paper'n pencil and/or EIGENMATH.

Exercise 2.12. (An EIGENMATH function for the hyperbolic polar formulas)
Bundle the four branched separated hyperbolic polar formulas $(pol1), (pol2), (pol3), (pol4)$ in one function EIGENMATH `polH(w)`, which checks beforehand to which branch the hyperbolic number w belongs and than choses the appropriate formula (pol$i$).
Check your function on the 4 points of Ex.2.11.a.

> In section 2.2.3 we have read of the value of the hyperbolic argument of a hyperbolic number at the output of `EVA2`–function `polar1(.)`, see the screenshot of the EIGENMATH session before Fig.12. This hyperbolic angle was 'hidden' in real part of the complex number of the 4th coordinate entry of the result. We therefore will give some possibilities of a direct calculation of the hyperbolic argument (angle). This will need a little knowledge from calculus, e.g. [8, p. 500 ff].

2.2.4 The hyperbolic angle (argument) in $\mathbb{H} \sim c\ell(1,1)$

For analogy and contrast we look again at the point $(5,3) \in \mathbb{R}^2$ of the Euclidean plane from the viewpoints of $\mathbb{C}$, i.e. $z = (5,3) = 5 + 3i$ and $\mathbb{H}$, i.e. $w = (5,3) = 5 + 3u$.

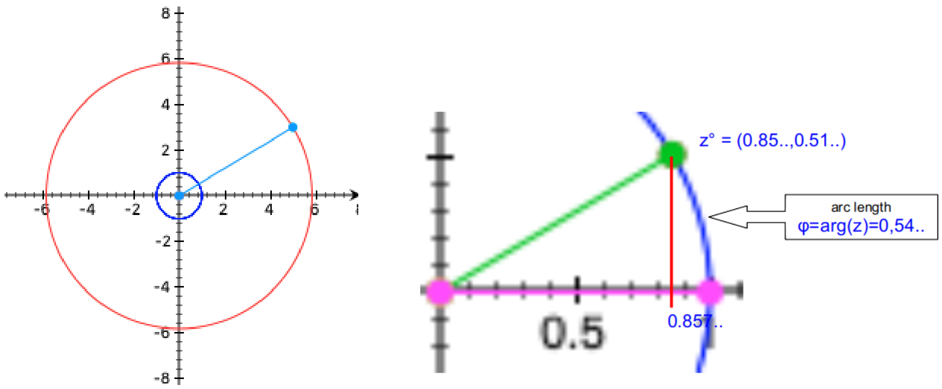
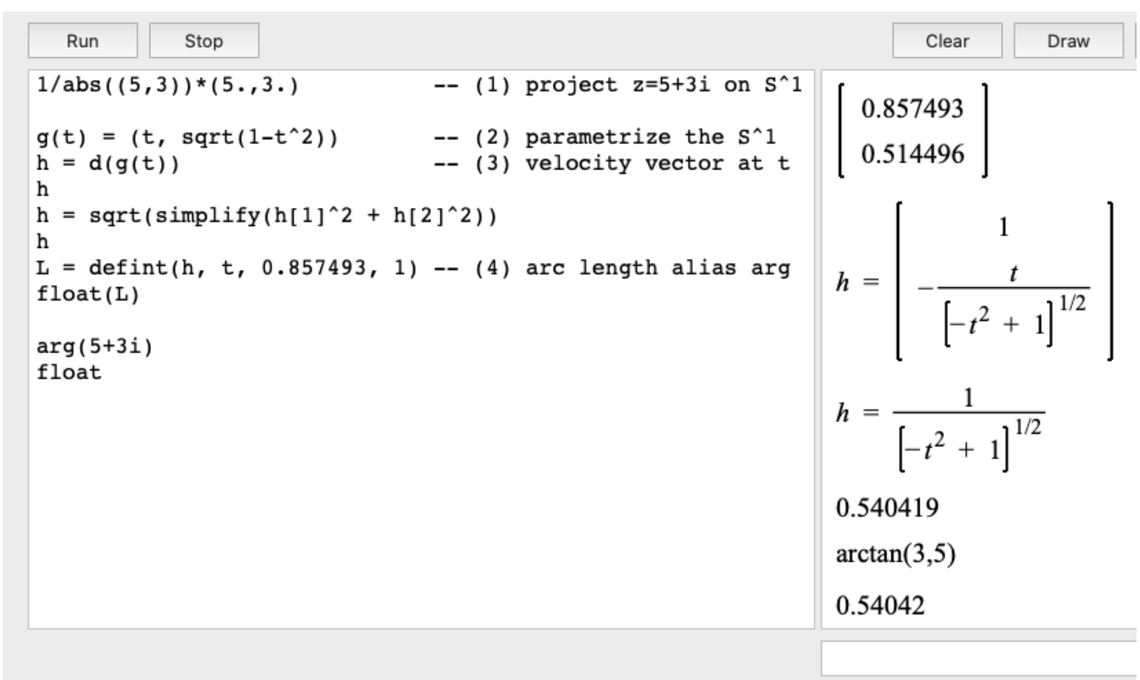


Figure 13: Blue: unit circle $S^1 \colon x^2 + y^2 = 1$.
Cyan: complex number $z = 5 + 3i$.
Right: a microscopic view at the complex number angle.
Best mental image as length of the arc from • ⌒ • in rad.

$(5,3) = z = 5 + 3i \in \mathbb{C}$: First we calculate again the **arg**(ument, angle) of w interpreted as complex number z in an alternative way *to gain an appropriate mental image of the concept* 'argument of z'. We calculate this angle as a proportional piece of the plane unit circle curve $S^1 \colon x^2 + y^2 = 1$ as seen in the microscopic view in Fig.13.right:



Comment. In the EIGENMATH realization, we first (1) calculate the normalized point $z^\circ = \frac{z}{|z|} \in S^1$, i.e. the coordinates of the green point in Fig.13. Second we define a parametrization $g\colon \mathbb{R} \to \mathbb{R}^2$of the unit circle, i.e. starting from the equation $x^2 + y^2 = 1$ we gain $y^2 = 1 - x^2$ and therefore $g(t) = (1, \sqrt{1-t^2})$. Third we calculate the argument of z realized as the arc length L of g between the magenta point $(1, 0)$ and the green point $z^\circ \approx (0.86, 0.51)$, i.e. we have the integral[10]

$$L \overset{(4)}{=} \int_{x=0.8574}^{x=1} |g'(t)|dt \approx 0.5404 \overset{def}{=} arg(z)$$

We give a second interpretation of the complex arg (angle) as *area of the sector* $\sphericalangle = ((0,0),(1,0),z^\circ)$ ('trigonometric triangle'). If we express the unit circle S^1 in polar coordinates by the equation $r = f(\theta)$, together with the rays $\theta = \alpha$ to $\theta = \beta$ we enclose a region, whose area A is given by[11]

$$A = \frac{1}{2}\int_{\theta=\alpha}^{\theta=\beta} f(\theta)^2 d\theta \overset{S^1}{=} \frac{1}{2}\int_{\theta=0}^{\theta=0.5404} 1^2 d\theta = \frac{1}{2}\cdot arg(z)$$

We have the fact:

$$arg(z) = 2\cdot\sphericalangle$$

i.e. *the double area of the sector of the unit circle with central angel θ equals $arg(z)$*.

```
# EIGENMATH
# Express the unit circle by equation r = f(theta) in polar coordinates.
f(theta) = r
r = 1
argZ = 2*1/2 * defint( f(theta)^2, theta, 0, 0.5404)
argZ        -- result 0.5404
```

▷ *Click here to invoke this script.*

In summa: besides the usual trigonometric definition of $arg(z) = arctanh(\frac{y}{x})$[12] we have two more possibilities to calculate the complex angel: first as length of an arc and second as area of a sector. This will help us to gain insight into the concept of the hyperbolic angel (argument) of an hyperbolic number.

Exercise 2.13. Use the arc length construction of the complex argument to program an alternative EIGENMATH– function `argC(z)` for the calculation of $arg(z)$.

[10] I thank G. WEIGT for a work around to calculate the intergral L with EIGENMATH.

[11] see e.g. [8, p. 502], where the authors also give a nice infinitesimal argument of this formula.

[12] cum grano salis, because one has to chose the correct order of nominator and denominator ..

$(5,3) = w = 5 + 3u \in \mathbb{H}$: Now we look at the argument (angel) of $w = x + yu$ as a hyperbolic number. Because the formal definition via analogy $argH(w) = arctanh(\frac{y}{x})$[13] gives no geometric insight we try to go the alternative ways.

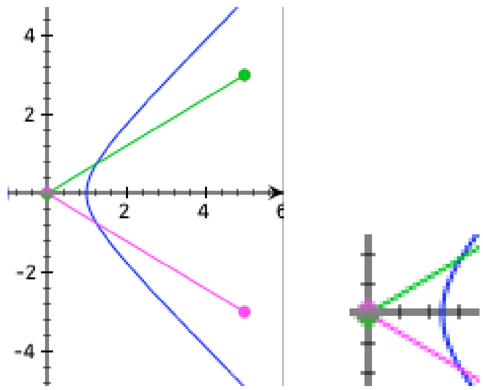
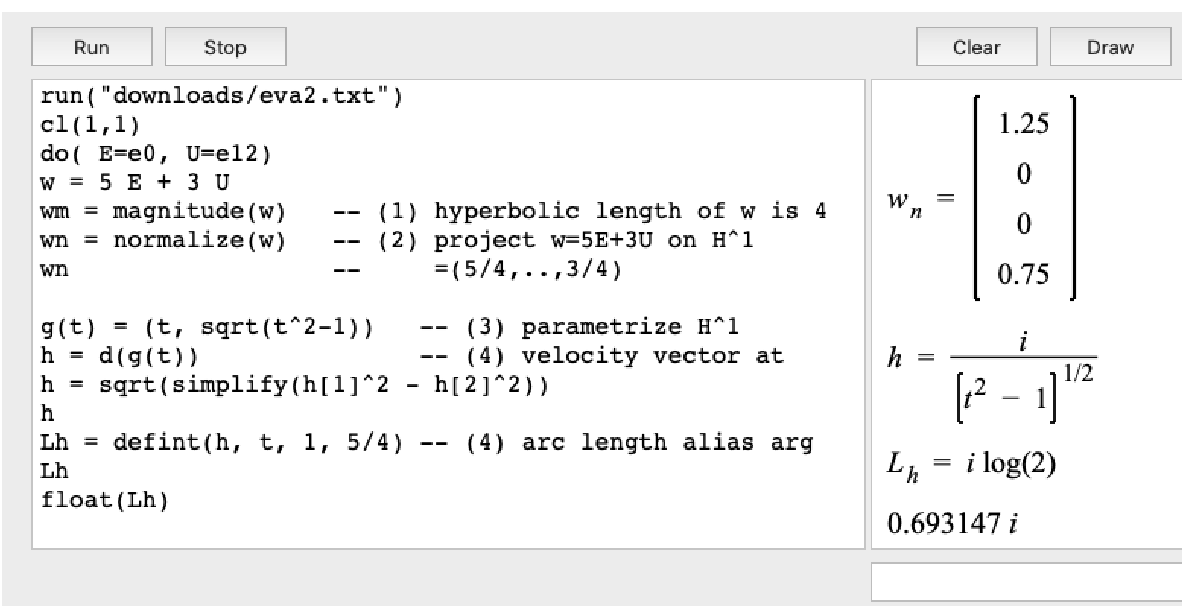


Figure 14: Blue: hyperbola $H_i^1: x^2 - y^2 = 1$.
Green: hyperbolic number $w = 5E + 3U$
and its hyperbolic conjugate $w_h^- = 5E - 3U$.
Right: a microscopic view at the hyperbolic angle of w.
Best mental image as area of the sector formed by $\circ \nearrow\searrow ($

1st: We calculate the hyperbolic arc length as length of the blue arc $\circ \nearrow_{\to} ($ on the unit hyperbola H_i^1 using EIGENMATH.




```
run("downloads/eva2.txt")
cl(1,1)
do( E=e0, U=e12)
w = 5 E + 3 U
wm = magnitude(w)    -- (1) hyperbolic length of w is 4
wn = normalize(w)    -- (2) project w=5E+3U on H^1
wn                   --      =(5/4,..,3/4)

g(t) = (t, sqrt(t^2-1))    -- (3) parametrize H^1
h = d(g(t))                -- (4) velocity vector at
h = sqrt(simplify(h[1]^2 - h[2]^2))
h
Lh = defint(h, t, 1, 5/4) -- (4) arc length alias arg
Lh
float(Lh)
```

$$w_n = \begin{bmatrix} 1.25 \\ 0 \\ 0 \\ 0.75 \end{bmatrix}$$

$$h = \frac{i}{\left[t^2 - 1\right]^{1/2}}$$

$$L_h = i\log(2)$$

$0.693147\, i$

[13]cum grano salis, because one has to chose the correct order of nominator and denominator fitting to the correct hyperbolic quadrant H_i,..

Comment. Invoking `cl(1,1)` we first beam us into the hyperbolic plane. Because the hyperbolic length (magnitude) of w is $\|w\| = 4$, the normalized hyperbolic number is $w/\|w\| = 1/4 \cdot w$ and lies on H^1, i.e. the point $(\frac{5}{4}, \frac{3}{4}) \in \mathbb{R}^2$. Second we define a parametrization $g\colon \mathbb{R} \to \mathbb{R}^2$of the unit hyperbola, i.e. starting from the equation $x^2 - y^2 = 1$ (therefore the choice of `cl(1,1)`!) we gain $y^2 = x^2 - 1$ and therefore $g(t) = (1, \sqrt{t^2 - 1})$. Now we calculate the argument (hyperbolic angle) of w realized as the arc length Lh of g of the half "(", i.e. we have the integral

$$Lh \overset{(4)}{=} \int\limits_{x=1}^{x=5/4} |g'(t)|dt = log(2) \overset{def}{=} \text{argH}(z) \approx 0.6931$$

2st: We use the fact

$$\text{argH}(z) = \text{area of sector} \circ \substack{\nearrow \\ \searrow} ($$

The area of the sector of the unit hyperbola between w° *and its conjugate* $(w_h^-)^\circ$ *equals* $argH(z)$.

```
# EIGENMATH
-- 5/4 and 3/4 are the edges of a box
-- from the normalized wn on H^1. Therefore:

Lh = 5/4*3/4 - 2*defint(sqrt(x^2-1), x,1,5/4)
Lh

Lh = mag(defint( sqrt(-1/(x^2-1)), x,1,5/4))
Lh
float
```

▷ *Click here to invoke this script.*

Exercise 2.14. Use the trigonometric definition $argH(w) = arctanh(\frac{y}{x})$ for the hyperbolic argument in the quadrant H_i to program an EIGENMATH– function `arg2(z)`, which works for all four quadrants.

Exercise 2.15. Use the arc length definition via the integral to program a function `argH(w)` for the hyperbolic argument in the quadrant H_i. Try to make it work for all four hyperbolic quadrants.

Exercise 2.16. The `polar1` function of `EVA2` often gives back the argument of its input in complex number form. If you like to have only the real part, you can try the following function. Explain.

```
# EIGENMATH
run("downloads/EVA2.txt")
cl(1,1)
```

```
do( E = e0, U = e12)

phiH(c) = arctanh( mag( magnitude(imag1(c)) / magnitude(real1(c)))

w = 5E+3U
phiH(w)
```

2.2.5 Problems.

P15. The cubic equation $x^3 + 3ax + b = 0$.
The usefulness of the complex hyperbolic numbers is shown by G. SOBCZYK in [10, p. 13 ff.]. On p. 14 there is the solved example:
∘ *find the solutions of the reduced cubic equation* $x^3 - 6x + 4 = 0$.
Calculate the solutions by EIGENMATH.

P16. The Special relativity and LORENTZian Geometry.
SOBCZYK shows [10, p. 15 ff.] the application of hyperbolic numbers IH resp. $c\ell(1,1)$ in LORENTZian Geometry. There you will see e.g. the *spacetime distance* aka. the hyperbolic norm in action. Read about it. Use EIGENMATH and its package `EVA2` as companion.

⊡

Summary: We have constructed the new algebra IH of the hyperbolic numbers in the Euclidean plane $\mathbb{R}^2$ by means of a multiplication table for the basis vectors $span_{\mathbb{R}}\{e_0, e_{12}\}$. This way we get also the desired relation $u^2 = 1$ to have a root of $\sqrt{1}$, not being an element of $\mathbb{R}$. This construction is also known as the algebra of the *binarions*.

We did a second realization of the hyperbolic numbers IH by invoking the CLIFFORD algebra $c\ell(1,1)$ of the EIGENMATH package `EVA2`. This package defines in this setting all crucial IH–typical functions like conjugate, imaginary part, reciprocal, norm, polar form etc.

Meanwhile the user should have gained a working knowledge of the hyperbolic numbers IH and the use of the package `EVA2`. We now turn to a last low dimensional special example of a CLIFFORD algebra – the famous *quaternions*.

3 $\mathbb{H}$ – the quaternions

Please: distinct the symbols $\mathrm{I\!H}$ as designation of the hyperbolic numbers and the symbol $\mathbb{H}$ for the HAMILTONian quaternions.

Math concept	notation
hyperbolic numbers	$\mathrm{I\!H}$ alias $c\ell(1,1)$
HAMILTON's quaternions	$\mathbb{H}$ alias $c\ell(1,1)$

For an algebraic construction of HAMILTON's quaternions $\mathbb{H}$ by means of a multiplication table for the basis vectors $span_{\mathrm{I\!R}}\{I, J, K, L\}$, see G. WEIGT [15][14].
Therefore we will restrict the treatment of HAMILTON's quaternions on its realization as a CLIFFORD algebra.

3.1 $c\ell(3)^+$ – the CLIFFORD algebra realization of $\mathbb{H}$

3.2 EULER angels ?

[14] This demo was an inspiration for our construction of $\mathbb{C}$ and $\mathrm{I\!H}$ via multiplication tables.

4 $c\ell$ – the 2D/3D Geometric Algebra

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url: `https://georgeweigt.github.io/quaternions.html`

[16] `https://m.youtube.com/watch?v=5bxsxM2UTb4`

[17] `http://www.math.uni-bonn.de/people/woermann/MoorePenrose.pdf`

$\bowtie$

Links checked 27.11.2020, wL

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2020