Spontaneous emission rate

Find the spontaneous emission rate for hydrogen state $2p$.

The wave function for hydrogen is

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi)$$

where

$$R_{nl}(r) = \frac{2}{n^2}\sqrt{\frac{(n-l-1)!}{(n+l)!}}\left(\frac{2r}{na_0}\right)^l L_{n-l-1}^{2l+1}\left(\frac{2r}{na_0}\right)\exp\left(-\frac{r}{na_0}\right)a_0^{-3/2}$$

$$L_n^m(x) = (n+m)!\sum_{k=0}^{n}\frac{(-x)^k}{(n-k)!(m+k)!k!}$$

$$Y_{lm}(\theta,\phi) = (-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\exp(im\phi)$$

$$P_l^m(\cos\theta) = \begin{cases}\left(\frac{\sin\theta}{2}\right)^m\sum_{k=0}^{l-m}(-1)^k\frac{(l+m+k)!}{(l-m-k)!(m+k)!k!}\left(\frac{1-\cos\theta}{2}\right)^k, & m\geq 0 \\[2mm] (-1)^m\frac{(l+m)!}{(l-m)!}P_l^{|m|}(\cos\theta), & m<0\end{cases}$$

State $2p$ is shorthand for $n=2$ and $l=1$. For $l=1$ there are three ways to choose m hence all of the following processes correspond to the transition $2p \to 1s$. It turns out that all three processes have the same transition rate.

$$\left.\begin{array}{c}\psi_{2,1,1}\\ \psi_{2,1,0}\\ \psi_{2,1,-1}\end{array}\right\} \to \psi_{100} + \text{photon}$$

The spontaneous emission rate is

$$A_{21} = \frac{4\alpha\omega_{21}^3|r_{21}|^2}{3c^2} \tag{1}$$

where

$$\omega_{21} = \frac{E_2 - E_1}{\hbar}, \quad E_n = -\frac{\alpha^2\mu c^2}{2n^2}$$

$$|r_{21}|^2 = |x_{21}|^2 + |y_{21}|^2 + |z_{21}|^2$$

$$x_{21} = \int_0^\infty\int_0^\pi\int_0^{2\pi} xf_{21}\,dV, \quad y_{21} = \int_0^\infty\int_0^\pi\int_0^{2\pi} yf_{21}\,dV, \quad z_{21} = \int_0^\infty\int_0^\pi\int_0^{2\pi} zf_{21}\,dV$$

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and

$$f_{21} = \psi_{100}^* \psi_{210}$$

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

$$dV = r^2\sin\theta\,dr\,d\theta\,d\phi$$

The integrals work out to be

$$x_{21} = 0, \quad y_{21} = 0, \quad z_{21} = \frac{2^7}{3^5}\sqrt{2}a_0$$

hence

$$|r_{21}|^2 = |z_{21}|^2 = \frac{2^{15}}{3^{10}}a_0^2 = \frac{32768}{59049}a_0^2$$

By equation (1) the spontaneous emission rate is

$$A_{21} = 6.26 \times 10^8 \text{ second}^{-1}$$

The mean interval is

$$\frac{1}{A_{21}} = 1.60 \times 10^{-9} \text{ second}$$