

(2.1) For the one-dimensional harmonic oscillator, show that with creation and annihilation operators defined as in eqns 2.9 and 2.10, $[\hat{a}, \hat{a}] = 0$, $[\hat{a}^\dagger, \hat{a}^\dagger] = 0$, $[\hat{a}, \hat{a}^\dagger] = 1$ and $\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)$.

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p} \right) \quad (2.9)$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega} \hat{p} \right) \quad (2.10)$$

We have by elementary algebra

$$[\hat{a}, \hat{a}] = \hat{a}\hat{a} - \hat{a}\hat{a} = 0 \quad [\hat{a}^\dagger, \hat{a}^\dagger] = \hat{a}^\dagger\hat{a}^\dagger - \hat{a}^\dagger\hat{a}^\dagger = 0$$

For the commutator $[\hat{a}, \hat{a}^\dagger]$ we have

$$\begin{aligned} \hat{a}\hat{a}^\dagger &= \frac{m\omega}{2\hbar} \left(\hat{x} \left(\hat{x} - \frac{i}{m\omega} \hat{p} \right) + \frac{i}{m\omega} \hat{p} \left(\hat{x} - \frac{i}{m\omega} \hat{p} \right) \right) \\ &= \frac{m\omega}{2\hbar} \left(\hat{x}^2 - \frac{i}{m\omega} \hat{x}\hat{p} + \frac{i}{m\omega} \hat{p}\hat{x} + \frac{1}{m^2\omega^2} \hat{p}^2 \right) \end{aligned}$$

Rewrite as

$$\hat{a}\hat{a}^\dagger = \frac{m\omega}{2\hbar} \hat{x}^2 - \frac{i}{2\hbar} [\hat{x}, \hat{p}] + \frac{1}{2\hbar m\omega} \hat{p}^2$$

From page 20 just below equation (2.8) we have

$$[\hat{x}, \hat{p}] = i\hbar$$

It follows that

$$\hat{a}\hat{a}^\dagger = \frac{m\omega}{2\hbar} \hat{x}^2 + \frac{1}{2\hbar m\omega} \hat{p}^2 + \frac{1}{2} \quad (1)$$

By similar argument

$$\hat{a}^\dagger \hat{a} = \frac{m\omega}{2\hbar} \hat{x}^2 + \frac{1}{2\hbar m\omega} \hat{p}^2 - \frac{1}{2} \quad (2)$$

Hence

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a} = \frac{1}{2} - \left(-\frac{1}{2} \right) = 1$$

Multiply equation (2) by $\omega\hbar$ to obtain

$$\omega\hbar (\hat{a}^\dagger\hat{a}) = \frac{m\omega^2}{2}\hat{x}^2 + \frac{1}{2m}\hat{p}^2 - \frac{\omega\hbar}{2}$$

Rewrite as

$$\omega\hbar (\hat{a}^\dagger\hat{a} + \tfrac{1}{2}) = \frac{m\omega^2}{2}\hat{x}^2 + \frac{1}{2m}\hat{p}^2 \quad (3)$$

Consider equation (2.6).

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \quad (2.6)$$

Substitute (3) into (2.6).

$$\hat{H} = \omega\hbar (\hat{a}^\dagger\hat{a} + \tfrac{1}{2})$$