Spin part 2

From the previous section we have for the z direction

$$\Pr(+) = |\langle z_+|s\rangle|^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos\theta$$
$$\Pr(-) = |\langle z_-|s\rangle|^2 = \tfrac{1}{2} - \tfrac{1}{2}\cos\theta$$

If $|s\rangle$ is not an eigenstate then the result of measuring $|s\rangle$ is a random value. For example, let $\theta = \pi/3$. Then $\cos\theta = \frac{1}{2}$ and the probabilities are

$$\Pr(+) = \tfrac{3}{4}$$
$$\Pr(-) = \tfrac{1}{4}$$

A useful statistic for analyzing random data is expected value. From theory a "plus" result indicates a spin value of $+\frac{\hbar}{2}$ and a "minus" result indicates a spin value of $-\frac{\hbar}{2}$. Hence the expected value is

$$\left(+\frac{\hbar}{2}\right)\Pr(+) + \left(-\frac{\hbar}{2}\right)\Pr(-)$$

For state $|s\rangle$ such that $\theta = \pi/3$ the expected value in the z direction is

$$\left(+\frac{\hbar}{2}\right)\frac{3}{4} + \left(-\frac{\hbar}{2}\right)\frac{1}{4} = \frac{\hbar}{4}$$

Expected values can be computed directly from $|s\rangle$ by introducing the following matrices.

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then

$$\langle S_x\rangle = \langle s|S_x|s\rangle \qquad \langle S_y\rangle = \langle s|S_y|s\rangle \qquad \langle S_z\rangle = \langle s|S_z|s\rangle$$

Returning to the example $\theta = \pi/3$ we have

$$|s\rangle = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2}e^{i\phi} \end{pmatrix}$$

Hence the expected value in the z direction is

$$\langle S_z\rangle = \langle s|S_z|s\rangle = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2}e^{i\phi} \end{pmatrix}^\dagger \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2}e^{i\phi} \end{pmatrix} = \frac{\hbar}{4}$$