15.6.3. Consider the uncertainty relation for position and momentum in the coherent state of Eq. (15.94).

(i) Deduce the uncertainty relation for the operators $\hat{\xi}_\lambda$ and $\hat{\pi}_\lambda$.

(ii) By evaluating $\langle\hat{\xi}^2_\lambda\rangle - \bar{\xi}^2_\lambda$ for the coherent state, show that the standard deviation of the width of the resulting probability distribution for the "position" ξ is $1/\sqrt{2}$, independent of time.

(iii) Repeat the calculation in (ii) for "momentum", with operator $\hat{\pi}_\lambda$, instead of "position".

(iv) Deduce that the coherent state is a "minimum uncertainty" state, i.e., it has the minimum possible product of the standard deviations of "position" and "momentum".

[Hints: use the results of Probs. 15.6.1 and 15.6.2, and the general relations for uncertainty principles in Chapter 5.]

(i) We will need the following formulas.

$$[\hat{A}, \hat{B}] = i\hat{C} \tag{5.4}$$

$$(\Delta\hat{A})^2(\Delta\hat{B})^2 \geq \frac{\langle\hat{C}\rangle^2}{4} \tag{5.23}$$

From problem 15.5.1 we have

$$[\hat{\xi}_\lambda, \hat{\pi}_\lambda] = i$$

Hence by (5.4) and (5.23) with $\hat{C} = 1$ we have

$$(\Delta\hat{\xi})^2(\Delta\hat{\pi})^2 \geq \frac{1}{4}$$

Take the square root to obtain the uncertainty in terms of standard deviation.

$$\Delta\hat{\xi}\Delta\hat{\pi} \geq \frac{1}{2}$$

(ii) We have for the coherent state

$$\langle\hat{\xi}^2_\lambda\rangle = \langle\Psi_{\lambda\bar{n}}|\hat{\xi}_\lambda\hat{\xi}_\lambda|\Psi_{\lambda\bar{n}}\rangle = \frac{1}{2}\langle\Psi_{\lambda\bar{n}}| \left(\hat{a}_\lambda + \hat{a}^\dagger_\lambda\right) \left(\hat{a}_\lambda + \hat{a}^\dagger_\lambda\right) |\Psi_{\lambda\bar{n}}\rangle$$

From problem 15.6.2 we have

$$\begin{aligned}
\langle\Psi_{\lambda\bar{n}}|\hat{a}_\lambda\hat{a}_\lambda|\Psi_{\lambda\bar{n}}\rangle &= \bar{n}\exp(-2i\omega_\lambda t)\\
\langle\Psi_{\lambda\bar{n}}|\hat{a}_\lambda^\dagger\hat{a}_\lambda^\dagger|\Psi_{\lambda\bar{n}}\rangle &= \bar{n}\exp(2i\omega_\lambda t)\\
\langle\Psi_{\lambda\bar{n}}|\hat{a}_\lambda^\dagger\hat{a}_\lambda|\Psi_{\lambda\bar{n}}\rangle &= \bar{n}\\
\langle\Psi_{\lambda\bar{n}}|\hat{a}_\lambda\hat{a}_\lambda^\dagger|\Psi_{\lambda\bar{n}}\rangle &= \langle\Psi_{\lambda\bar{n}}|\left(1+\hat{a}_\lambda^\dagger\hat{a}_\lambda\right)|\Psi_{\lambda\bar{n}}\rangle = 1+\bar{n}
\end{aligned}$$

Hence

$$\langle\hat{\xi}_\lambda^2\rangle = \frac{\bar{n}}{2}\exp(-2i\omega_\lambda t) + \frac{\bar{n}}{2}\exp(2i\omega_\lambda t) + \bar{n} + \frac{1}{2}$$

From problem 15.6.2 we have

$$\langle\hat{\xi}_\lambda\rangle = \sqrt{\frac{\bar{n}}{2}}\exp(-i\omega_\lambda t) + \sqrt{\frac{\bar{n}}{2}}\exp(i\omega_\lambda t)$$

Hence the variance is

$$\langle\hat{\xi}_\lambda^2\rangle - \langle\hat{\xi}_\lambda\rangle^2 = \frac{1}{2}$$

The standard deviation $\Delta\hat{\xi}$ is

$$\Delta\hat{\xi} = \sqrt{\langle\hat{\xi}_\lambda^2\rangle - \langle\hat{\xi}_\lambda\rangle^2} = \frac{1}{\sqrt{2}}$$

(iii) We have for the coherent state

$$\langle\hat{\pi}_\lambda^2\rangle = \langle\Psi_{\lambda\bar{n}}|\hat{\pi}_\lambda\hat{\pi}_\lambda|\Psi_{\lambda\bar{n}}\rangle = -\frac{1}{2}\langle\Psi_{\lambda\bar{n}}|\left(\hat{a}_\lambda^\dagger-\hat{a}_\lambda\right)\left(\hat{a}_\lambda^\dagger-\hat{a}_\lambda\right)|\Psi_{\lambda\bar{n}}\rangle$$

From the results in part (ii) we have

$$\langle\hat{\pi}_\lambda^2\rangle = \bar{n} - \frac{\bar{n}}{2}\exp(-2i\omega_\lambda t) - \frac{\bar{n}}{2}\exp(2i\omega_\lambda t) + \frac{1}{2}$$

For the expectation $\langle\hat{\pi}_\lambda\rangle$ we have

$$\begin{aligned}
\langle\hat{\pi}_\lambda\rangle &= \langle\Psi_{\lambda\bar{n}}|\hat{\pi}_\lambda|\Psi_{\lambda\bar{n}}\rangle\\
&= \frac{i}{\sqrt{2}}\langle\Psi_{\lambda\bar{n}}|\left(\hat{a}_\lambda^\dagger-\hat{a}_\lambda\right)|\Psi_{\lambda\bar{n}}\rangle\\
&= i\sqrt{\frac{\bar{n}}{2}}\exp(i\omega_\lambda t) - i\sqrt{\frac{\bar{n}}{2}}\exp(-i\omega_\lambda t)
\end{aligned}$$

It follows that

$$\langle \hat{\pi}_\lambda \rangle^2 = \bar{n} - \frac{n}{2}\exp(2i\omega_\lambda t) - \frac{n}{2}\exp(-2i\omega_\lambda t)$$

Hence the variance is

$$\langle \hat{\pi}_\lambda^2 \rangle - \langle \hat{\pi}_\lambda \rangle^2 = \frac{1}{2}$$

The standard deviation $\Delta\hat{\pi}$ is

$$\Delta\hat{\pi} = \sqrt{\langle \hat{\pi}_\lambda^2 \rangle - \langle \hat{\pi}_\lambda \rangle^2} = \frac{1}{\sqrt{2}}$$

(iv) From part (ii) we have

$$\Delta\hat{\xi} = \sqrt{\langle \hat{\xi}_\lambda^2 \rangle - \langle \hat{\xi}_\lambda \rangle^2} = \frac{1}{\sqrt{2}}$$

and from part (iii) we have

$$\Delta\hat{\pi} = \sqrt{\langle \hat{\pi}_\lambda^2 \rangle - \langle \hat{\pi}_\lambda \rangle^2} = \frac{1}{\sqrt{2}}$$

Hence

$$\Delta\hat{\xi}\Delta\hat{\pi} = \frac{1}{2}$$

which is the minimum uncertainty from part (i).