Problem 1.3. Verify (1.33).

This is equation (1.33).

$$\begin{aligned} \boldsymbol{\sigma} \cdot \boldsymbol{\pi}\ \boldsymbol{\sigma} \cdot \boldsymbol{\pi} &= \boldsymbol{\pi}^2 + i\boldsymbol{\sigma} \cdot \boldsymbol{\pi} \times \boldsymbol{\pi} \\ &= \boldsymbol{\pi}^2 - \frac{e\hbar}{c}\boldsymbol{\sigma} \cdot \mathbf{B} \end{aligned} \tag{1.33}$$

Recall that $\boldsymbol{\sigma} \cdot \boldsymbol{\pi}$ is an operator so $\boldsymbol{\sigma} \cdot \boldsymbol{\pi}\ \boldsymbol{\sigma} \cdot \boldsymbol{\pi}$ means apply the operator twice.

The first equality is an identity hence the important part is to show that

$$i\boldsymbol{\sigma} \cdot \boldsymbol{\pi} \times \boldsymbol{\pi} = -\frac{e\hbar}{c}\boldsymbol{\sigma} \cdot \mathbf{B}$$

The $\boldsymbol{\sigma}$ cancels leaving

$$i\boldsymbol{\pi} \times \boldsymbol{\pi} = -\frac{e\hbar}{c}\mathbf{B}$$

We are given

$$\boldsymbol{\pi} = \mathbf{p} - (e/c)\mathbf{A}, \quad \mathbf{p} = -i\hbar \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}, \quad \mathbf{B} = \operatorname{curl} \mathbf{A}$$

The cross product is distributive hence

$$\boldsymbol{\pi} \times \boldsymbol{\pi} = \mathbf{p} \times \mathbf{p} - \mathbf{p} \times (e/c)\mathbf{A} - (e/c)\mathbf{A} \times \mathbf{p} + (e/c)^2\mathbf{A} \times \mathbf{A}$$

The terms $\mathbf{p} \times \mathbf{p}$ and $\mathbf{A} \times \mathbf{A}$ vanish leaving

$$\boldsymbol{\pi} \times \boldsymbol{\pi} = -\frac{e}{c}(\mathbf{p} \times \mathbf{A} + \mathbf{A} \times \mathbf{p})$$

It is useful at this point to apply the operators to a wave function and write

$$(\boldsymbol{\pi} \times \boldsymbol{\pi})\psi = -\frac{e}{c}(\mathbf{p} \times \mathbf{A} + \mathbf{A} \times \mathbf{p})\psi$$

It can be shown that

$$\mathbf{p} \times (\mathbf{A}\psi) = (\mathbf{p} \times \mathbf{A})\psi + \mathbf{p}\psi \times \mathbf{A} \tag{1}$$

Hence

$$(\boldsymbol{\pi} \times \boldsymbol{\pi})\psi = -\frac{e}{c}\big((\mathbf{p} \times \mathbf{A})\psi + \mathbf{p}\psi \times \mathbf{A} + \mathbf{A} \times \mathbf{p}\psi\big)$$

The last two terms cancel by antisymmetry leaving

$$(\boldsymbol{\pi} \times \boldsymbol{\pi})\psi = -\frac{e}{c}(\mathbf{p} \times \mathbf{A})\psi \tag{2}$$

Note that

$$\mathbf{p} \times \mathbf{A} = -i\hbar \begin{pmatrix} \frac{\partial}{\partial_y} A_z - \frac{\partial}{\partial_z} A_y \\ \frac{\partial}{\partial_z} A_x - \frac{\partial}{\partial_x} A_z \\ \frac{\partial}{\partial_x} A_y - \frac{\partial}{\partial_y} A_x \end{pmatrix} = -i\hbar \operatorname{curl} \mathbf{A} = -i\hbar \mathbf{B} \tag{3}$$

Hence by (2) and (3) we have

$$\boldsymbol{\pi} \times \boldsymbol{\pi} = i\frac{e\hbar}{c}\mathbf{B}$$

Multiply both sides by i to complete the proof.

$$i\boldsymbol{\pi} \times \boldsymbol{\pi} = -\frac{e\hbar}{c}\mathbf{B}$$