Feynman and Hibbs problem 2-2

This is the Lagrangian for a harmonic oscillator.

$$L = \frac{m}{2}(\dot{x}^2 - \omega^2 x^2)$$

Let $T = t_b - t_a$. Show that the classical action is

$$S_{cl} = \frac{m\omega}{2\sin(\omega T)}\left((x_b^2 + x_a^2)\cos(\omega T) - 2x_b x_a\right)$$

From the above Lagrangian we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

and

$$\frac{\partial L}{\partial x} = -m\omega^2 x$$

By equation (2.7) which is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

we have

$$\ddot{x} = -\omega^2 x \tag{1}$$

The well-known solution to (1) is

$$x(t) = A\sin(\omega t) + B\cos(\omega t)$$

We have the following boundary conditions.

$$x(0) = x_a$$
$$x(T) = x_b$$

Solve for B.

$$x(0) = B = x_a$$

For $x(T)$ we have

$$x(T) = A\sin(\omega T) + B\cos(\omega T)$$

Solve for A.

$$A = \frac{x(T) - B\cos(\omega T)}{\sin(\omega T)} = \frac{x_b - x_a\cos(\omega T)}{\sin(\omega T)}$$

Hence the equation of motion is

$$x(t) = \frac{x_b - x_a\cos(\omega T)}{\sin(\omega T)}\sin(\omega t) + x_a\cos(\omega t) \tag{2}$$

Differentiate $x(t)$ to obtain velocity $\dot{x}(t)$.

$$\dot{x}(t) = \frac{d}{dt}x(t) = \omega\left(\frac{x_b - x_a\cos(\omega T)}{\sin(\omega T)}\cos(\omega t) - x_a\sin(\omega t)\right) \tag{3}$$

Using the action integral

$$S = \int_0^T L\,dt$$

we have for the classical action

$$\begin{aligned}
S_{cl} &= \frac{m}{2}\int_0^T (\dot{x}^2 - \omega^2 x^2)\,dt \\
&= \frac{m}{2}\left(\int_0^T \dot{x}^2\,dt - \int_0^T \omega^2 x^2\,dt\right)
\end{aligned}$$

Let $u = v = \dot{x}$ and note that

$$\begin{aligned}
\dot{u} &= \ddot{x} \\
\int v\,dt &= x
\end{aligned}$$

Apply integration by parts to the integral of $\dot{x}^2$.

$$\begin{aligned}
\int_0^T \dot{x}^2\,dt &= \int_0^T uv\,dt \\
&= \left(u\int v\,dt\right)\Big|_0^T - \int_0^T \dot{u}\left(\int v\,dt\right)\,dt \\
&= \dot{x}x\Big|_0^T - \int_0^T \ddot{x}x\,dt
\end{aligned}$$

Hence

$$S_{cl} = \frac{m}{2}\left(\dot{x}x\Big|_0^T - \int_0^T \ddot{x}x\,dt - \int_0^T \omega^2 x^2\,dt\right)$$

The integrals cancel by equation (1) which affirms that

$$\ddot{x}x = -\omega^2 x^2$$

We now have

$$\begin{aligned} S_{cl} &= \frac{m}{2}\dot{x}x\Big|_0^T \\ &= \frac{m}{2}\Big(\dot{x}(T)x(T) - \dot{x}(0)x(0)\Big) \end{aligned} \tag{4}$$

By substitution and simplification

$$S_{cl} = \frac{m\omega}{2\sin(\omega T)}\Big((x_b^2 + x_a^2)\cos(\omega T) - 2x_b x_a\Big) \tag{5}$$