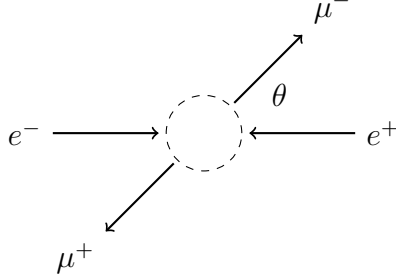


Muon pair production

Muon pair production is the interaction $e^- + e^+ \rightarrow \mu^- + \mu^+$.



In the center-of-mass frame we have the following momentum vectors where $p = \sqrt{E^2 - m^2}$ and $\rho = \sqrt{E^2 - M^2}$.

$$\begin{aligned}
 p_1 &= \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} & p_2 &= \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} & p_3 &= \begin{pmatrix} E \\ \rho \sin \theta \cos \phi \\ \rho \sin \theta \sin \phi \\ \rho \cos \theta \end{pmatrix} & p_4 &= \begin{pmatrix} E \\ -\rho \sin \theta \cos \phi \\ -\rho \sin \theta \sin \phi \\ -\rho \cos \theta \end{pmatrix} \\
 &\text{inbound } e^- & &\text{inbound } e^+ & &\text{outbound } \mu^- & &\text{outbound } \mu^+
 \end{aligned}$$

Spinors for the inbound electron.

$$\begin{aligned}
 u_{11} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix} & u_{12} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix} \\
 &\text{inbound } e^- \text{ spin up} & &\text{inbound } e^- \text{ spin down}
 \end{aligned}$$

Spinors for the inbound positron.

$$\begin{aligned}
 v_{21} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} -p \\ 0 \\ E+m \\ 0 \end{pmatrix} & v_{22} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ p \\ 0 \\ E+m \end{pmatrix} \\
 &\text{inbound } e^+ \text{ spin up} & &\text{inbound } e^+ \text{ spin down}
 \end{aligned}$$

Spinors for the outbound muon.

$$\begin{aligned}
 u_{31} &= \frac{1}{\sqrt{E+M}} \begin{pmatrix} E+M \\ 0 \\ p_{3z} \\ p_{3x} + ip_{3y} \end{pmatrix} & u_{32} &= \frac{1}{\sqrt{E+M}} \begin{pmatrix} 0 \\ E+M \\ p_{3x} - ip_{3y} \\ -p_{3z} \end{pmatrix} \\
 &\text{outbound } \mu^- \text{ spin up} & &\text{outbound } \mu^- \text{ spin down}
 \end{aligned}$$

Spinors for the outbound anti-muon.

$$v_{41} = \frac{1}{\sqrt{E+M}} \begin{pmatrix} p_{4z} \\ p_{4x} + ip_{4y} \\ E+M \\ 0 \end{pmatrix} \quad v_{42} = \frac{1}{\sqrt{E+M}} \begin{pmatrix} p_{4x} - ip_{4y} \\ -p_{4z} \\ 0 \\ E+M \end{pmatrix}$$

outbound μ^+ spin up outbound μ^+ spin down

The probability amplitude \mathcal{M}_{abcd} for spin state $abcd$ is

$$\mathcal{M}_{abcd} = \frac{e^2}{s} (\bar{u}_{3c} \gamma_\mu v_{4d}) (\bar{v}_{2b} \gamma^\mu u_{1a})$$

Symbol e is elementary charge and

$$s = (p_1 + p_2)^2 = 4E^2$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is the average of spin states.

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{1}{4} \sum_{a=1}^2 \sum_{b=1}^2 \sum_{c=1}^2 \sum_{d=1}^2 |\mathcal{M}_{abcd}|^2 \\ &= \frac{e^4}{64E^4} \sum_{a=1}^2 \sum_{b=1}^2 \sum_{c=1}^2 \sum_{d=1}^2 |(\bar{u}_{3c} \gamma_\mu v_{4d}) (\bar{v}_{2b} \gamma^\mu u_{1a})|^2 \end{aligned}$$

The Casimir trick uses matrix arithmetic to sum over spin states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{64E^4} \text{Tr} \left((\not{p}_3 + M) \gamma^\mu (\not{p}_4 - M) \gamma^\nu \right) \text{Tr} \left((\not{p}_2 - m) \gamma_\mu (\not{p}_1 + m) \gamma_\nu \right)$$

The result is

$$\langle |\mathcal{M}|^2 \rangle = e^4 \left(1 + \cos^2 \theta + \frac{m^2 + M^2}{E^2} \sin^2 \theta + \frac{m^2 M^2}{E^4} \cos^2 \theta \right)$$

For $E \gg M$ a useful approximation is

$$\langle |\mathcal{M}|^2 \rangle = e^4 (1 + \cos^2 \theta)$$

Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{4(4\pi\epsilon_0)^2 s}$$

where

$$s = (p_1 + p_2)^2 = 4E^2$$

For high energy experiments we have

$$\langle |\mathcal{M}|^2 \rangle = e^4 (1 + \cos^2 \theta)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{4(4\pi\epsilon_0)^2 s} (1 + \cos^2 \theta)$$

Noting that

$$e^2 = 4\pi\epsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{4s} (1 + \cos^2 \theta)$$

Noting that

$$d\Omega = \sin \theta d\theta d\phi$$

we also have

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{4s} (1 + \cos^2 \theta) \sin \theta d\theta d\phi$$

Let $S(\theta_1, \theta_2)$ be the following integral of $d\sigma$.

$$S(\theta_1, \theta_2) = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{2\pi\alpha^2(\hbar c)^2}{4s} [I(\theta_2) - I(\theta_1)]$$

where

$$I(\theta) = -\frac{\cos^3 \theta}{3} - \cos \theta$$

The cumulative distribution function is

$$F(\theta) = \frac{S(0, \theta)}{S(0, \pi)} = \frac{I(\theta) - I(0)}{I(\pi) - I(0)} = -\frac{\cos^3 \theta}{8} - \frac{3 \cos \theta}{8} + \frac{1}{2}, \quad 0 \leq \theta \leq \pi$$

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 \leq \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{3}{8} (1 + \cos^2 \theta) \sin \theta$$

Data from SLAC PEP experiment

See www.hepdata.net/record/ins216031, Table 1, $s = (29.0 \text{ GeV})^2$.

x	y
-0.925	67.08
-0.85	58.67
-0.75	54.66
-0.65	51.72
-0.55	43.70
-0.45	41.12
-0.35	39.71
-0.25	35.34
-0.15	33.35
-0.05	34.69
0.05	34.05
0.15	34.48
0.25	34.66
0.35	35.23
0.45	35.60
0.55	40.13
0.65	42.56
0.75	46.37
0.85	49.28
0.925	55.70

Data x and y have the following relationship with the differential cross section formula.

$$x = \cos \theta, \quad y = s \frac{d\sigma}{d \cos \theta} = 2\pi s \frac{d\sigma}{d\Omega}$$

The cross section formula is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} (1 + \cos^2 \theta) \times (\hbar c)^2$$

To compute predicted values \hat{y} , multiply by 10^{37} to convert square meters to nanobarns.

$$\hat{y} = 2\pi s \frac{d\sigma}{d\Omega} = \frac{\pi\alpha^2}{2} (1 + x^2) \times (\hbar c)^2 \times 10^{37}$$

The following table shows predicted values \hat{y} .

x	y	\hat{y}
-0.925	67.08	60.44
-0.85	58.67	56.10
-0.75	54.66	50.89
-0.65	51.72	46.33
-0.55	43.70	42.42
-0.45	41.12	39.17
-0.35	39.71	36.56
-0.25	35.34	34.61
-0.15	33.35	33.30
-0.05	34.69	32.65
0.05	34.05	32.65
0.15	34.48	33.30
0.25	34.66	34.61
0.35	35.23	36.56
0.45	35.60	39.17
0.55	40.13	42.42
0.65	42.56	46.33
0.75	46.37	50.89
0.85	49.28	56.10
0.925	55.70	60.44

The coefficient of determination R^2 measures how well predicted values fit the data.

$$R^2 = 1 - \frac{\sum(y - \hat{y})^2}{\sum(y - \bar{y})^2} = 0.87$$

The result indicates that the model $d\sigma$ explains 87% of the variance in the data.

Electroweak model

The following differential cross section formula from electroweak theory results in a better fit to the data.¹

$$\frac{d\sigma}{d\Omega} = F(s)(1 + \cos^2 \theta) + G(s) \cos \theta$$

where

$$F(s) = \frac{\alpha^2}{4s} \left(1 + \frac{g_V^2}{\sqrt{2}\pi} \left(\frac{m_Z^2}{s - m_Z^2} \right) \left(\frac{sG}{\alpha} \right) + \frac{(g_A^2 + g_V^2)^2}{8\pi^2} \left(\frac{m_Z^2}{s - m_Z^2} \right)^2 \left(\frac{sG}{\alpha} \right)^2 \right)$$

$$G(s) = \frac{\alpha^2}{4s} \left(\frac{\sqrt{2}g_A^2}{\pi} \left(\frac{m_Z^2}{s - m_Z^2} \right) \left(\frac{sG}{\alpha} \right) + \frac{g_A^2 g_V^2}{\pi^2} \left(\frac{m_Z^2}{s - m_Z^2} \right)^2 \left(\frac{sG}{\alpha} \right)^2 \right)$$

¹See *Quantum Field Theory Revised Edition* by Mandl and Shaw, p. 316.

and

$$\begin{aligned}
g_A &= -0.5 \\
g_V &= -0.0348 \\
m_Z &= 91.17 \text{ GeV} \\
G &= 1.166 \times 10^{-5} \text{ GeV}^{-2}
\end{aligned}$$

The corresponding formula for \hat{y} is

$$\hat{y} = 2\pi [F(s)(1 + x^2) + G(s)x] \times (\hbar c)^2 \times 10^{37}$$

where $\sqrt{s} = 29 \text{ GeV}$ is the center of mass collision energy. Here are the predicted values \hat{y} based on the above formula.

x	y	\hat{y}
-0.925	67.08	65.59
-0.85	58.67	60.84
-0.75	54.66	55.07
-0.65	51.72	49.96
-0.55	43.70	45.49
-0.45	41.12	41.69
-0.35	39.71	38.53
-0.25	35.34	36.02
-0.15	33.35	34.17
-0.05	34.69	32.97
0.05	34.05	32.42
0.15	34.48	32.53
0.25	34.66	33.28
0.35	35.23	34.69
0.45	35.60	36.75
0.55	40.13	39.47
0.65	42.56	42.83
0.75	46.37	46.85
0.85	49.28	51.52
0.925	55.70	55.45

The coefficient of determination R^2 is

$$R^2 = 1 - \frac{\sum (y - \hat{y})^2}{\sum (y - \bar{y})^2} = 0.98$$

The result indicates that electroweak theory explains 98% of the variance in the data.