16.2.1. Consider the two particle wavefunction operator

$$\hat{\psi}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}} \sum_{j,n} \hat{b}_n \hat{b}_j \phi_j(\mathbf{r}_1) \phi_n(\mathbf{r}_2)$$

and a state $|\ldots, 1_k, \ldots, 1_m, \ldots\rangle \equiv \hat{b}_k^\dagger \hat{b}_m^\dagger |0\rangle$ that has one fermion in single-particle state k and an identical fermion in single-particle state m. Show that

$$\hat{\psi}(\mathbf{r}_1, \mathbf{r}_2)|\ldots, 1_k, \ldots, 1_m, \ldots\rangle = \frac{1}{\sqrt{2}} \left[\phi_k(\mathbf{r}_1)\phi_m(\mathbf{r}_2) - \phi_k(\mathbf{r}_2)\phi_m(\mathbf{r}_1)\right] |0\rangle$$

(i.e., this operator correctly constructs the combination of wavefunction products that is antisymmetric with respect to exchange of identical prticles.)

By substitution we have

$$\hat{\psi}(\mathbf{r}_1, \mathbf{r}_2)|\ldots, 1_k, \ldots, 1_m, \ldots\rangle = \frac{1}{\sqrt{2}} \sum_{j,n} \hat{b}_n \hat{b}_j \phi_j(\mathbf{r}_1) \phi_n(\mathbf{r}_2) \hat{b}_k^\dagger \hat{b}_m^\dagger |0\rangle$$

Noting that

$$\begin{aligned}
\hat{b}_n \hat{b}_j \hat{b}_k^\dagger \hat{b}_m^\dagger &= \hat{b}_n (\delta_{jk} - \hat{b}_k^\dagger \hat{b}_j) \hat{b}_m^\dagger \\
&= \delta_{jk} \hat{b}_n \hat{b}_m^\dagger - \hat{b}_n \hat{b}_k^\dagger \hat{b}_j \hat{b}_m^\dagger \\
&= \delta_{jk} (\delta_{nm} - \hat{b}_m^\dagger \hat{b}_n) - (\delta_{nk} - \hat{b}_k^\dagger \hat{b}_n)(\delta_{jm} - \hat{b}_m^\dagger \hat{b}_j)
\end{aligned}$$

we have

$$\hat{b}_n \hat{b}_j \hat{b}_k^\dagger \hat{b}_m^\dagger |0\rangle = (\delta_{jk}\delta_{nm} - \delta_{nk}\delta_{jm})|0\rangle$$

Hence

$$\hat{\psi}(\mathbf{r}_1, \mathbf{r}_2)|\ldots, 1_k, \ldots, 1_m, \ldots\rangle = \frac{1}{\sqrt{2}} \sum_{j,n} \phi_j(\mathbf{r}_1)\phi_n(\mathbf{r}_2)(\delta_{jk}\delta_{nm} - \delta_{nk}\delta_{jm})|0\rangle$$

All terms vanishes except for $j, n = k, m$ and $j, n = m, k$ hence

$$\hat{\psi}(\mathbf{r}_1, \mathbf{r}_2)|\ldots, 1_k, \ldots, 1_m, \ldots\rangle = \frac{1}{\sqrt{2}} \left[\phi_k(\mathbf{r}_1)\phi_m(\mathbf{r}_2) - \phi_m(\mathbf{r}_1)\phi_k(\mathbf{r}_2)\right] |0\rangle$$