Mott problem

Consider the emission of an α particle in a cloud chamber. The quantum mechanical model for the α particle is a spherical wave emanating from the origin. A spherical wave should ionize atoms throughout the cloud chamber. However, only straight tracks are observed. Nevill Mott showed that the Schrodinger equation is consistent with the phenomenon of straight tracks.

Let $\mathbf{R}$ be the position of the α particle, let $\mathbf{a}_1$ and $\mathbf{a}_2$ be the positions of two atoms ionized by the α particle, and let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the positions of the free electrons. The Hamiltonian for the system is

$$\hat{H} = \hat{K}_\alpha + \hat{K}_1 + \hat{K}_2 + V_1 + V_2 + U_1 + U_2$$

where

$\hat{K}_\alpha = -\dfrac{\hbar^2}{2M}\nabla_\alpha^2$	kinetic energy of α particle
$\hat{K}_1 = -\dfrac{\hbar^2}{2m}\nabla_1^2$	kinetic energy of 1st electron
$\hat{K}_2 = -\dfrac{\hbar^2}{2m}\nabla_2^2$	kinetic energy of 2nd electron
$V_1 = -\dfrac{e^2}{\lvert\mathbf{r}_1 - \mathbf{a}_1\rvert}$	potential energy of 1st electron
$V_2 = -\dfrac{e^2}{\lvert\mathbf{r}_2 - \mathbf{a}_2\rvert}$	potential energy of 2nd electron
$U_1 = -\dfrac{2e^2}{\lvert\mathbf{R} - \mathbf{r}_1\rvert}$	potential energy of α and 1st electron
$U_2 = -\dfrac{2e^2}{\lvert\mathbf{R} - \mathbf{r}_2\rvert}$	potential energy of α and 2nd electron

Let ψ_1 and ψ_2 be atomic wavefunctions such that

$$\hat{H}\psi_1 = E_1\psi_1, \quad \hat{H}\psi_2 = E_2\psi_2$$

We want to find a wavefunction $F(\mathbf{R}, \mathbf{r}_1, \mathbf{r}_2)$ such that

$$\hat{H}F = EF$$

Let

$$F = F_0 + F_1 + F_2 + \cdots$$

and let

$$\hat{H}_0 = \hat{K}_\alpha + E_1 + E_2$$

Start by finding an F_0 such that

$$\hat{H}_0 F_0 = EF_0$$

The solution is

$$F_0(\mathbf{R}, \mathbf{r}_1, \mathbf{r}_2) = f_0(\mathbf{R})\psi_1(\mathbf{r}_1 - \mathbf{a}_1)\psi_2(\mathbf{r}_2 - \mathbf{a}_2) \tag{1}$$

where

$$f_0(\mathbf{R}) = \frac{1}{|\mathbf{R}|}\exp\left(\frac{ik|\mathbf{R}|}{\hbar}\right), \quad k = \sqrt{2M(E - E_1 - E_2)}$$

It follows that for the full Hamiltonian

$$\hat{H}F_0 = EF_0 + (U_1 + U_2)F_0$$

To cancel $(U_1 + U_2)F_0$ from the full Hamiltonian, find an F_1 such that

$$\hat{H}_0 F_1 = EF_1 - (U_1 + U_2)F_0$$

Rewrite as

$$\left(\hat{H}_0 - E\right) F_1 = -(U_1 + U_2)F_0$$

Expand F_1 and F_0.

$$\left(\hat{H}_0 - E\right) f_1(\mathbf{R})\psi_1(\mathbf{r}_1 - \mathbf{a}_1)\psi_2(\mathbf{r}_2 - \mathbf{a}_2) = -(U_1 + U_2)f_0(\mathbf{R})\psi_1(\mathbf{r}_1 - \mathbf{a}_1)\psi_2(\mathbf{r}_2 - \mathbf{a}_2)$$

To solve for $f_1(\mathbf{R})$ multiply both sides by

$$\psi_1^*(\mathbf{r}_1 - \mathbf{a}_1)\psi_2^*(\mathbf{r}_2 - \mathbf{a}_2)$$

and integrate over $\mathbf{r}_1$ and $\mathbf{r}_2$ to obtain

$$\left(\hat{H}_0 - E\right) f_1(\mathbf{R}) = V(\mathbf{R})f_0(\mathbf{R}) \tag{2}$$

where

$$V(\mathbf{R}) = 2e^2 \int \frac{|\psi_1(\mathbf{r}_1 - \mathbf{a}_1)|^2 \, |\psi_2(\mathbf{r}_2 - \mathbf{a}_2)|^2}{|\mathbf{R} - \mathbf{r}_1|} \, d\mathbf{r}_1 \, d\mathbf{r}_2 + 2e^2 \int \frac{|\psi_1(\mathbf{r}_1 - \mathbf{a}_1)|^2 \, |\psi_2(\mathbf{r}_2 - \mathbf{a}_2)|^2}{|\mathbf{R} - \mathbf{r}_2|} \, d\mathbf{r}_1 \, d\mathbf{r}_2$$

Because $|\psi|^2$ is a normalized probability density function we have

$$V(\mathbf{R}) = 2e^2 \int \frac{|\psi_1(\mathbf{r}_1 - \mathbf{a}_1)|^2}{|\mathbf{R} - \mathbf{r}_1|} \, d\mathbf{r}_1 + 2e^2 \int \frac{|\psi_2(\mathbf{r}_2 - \mathbf{a}_2)|^2}{|\mathbf{R} - \mathbf{r}_2|} \, d\mathbf{r}_2$$

Per Mott the solution to (2) is

$$f_1(\mathbf{R}) = \frac{M}{2\pi\hbar^2} \int \frac{V(\mathbf{r})f_0(\mathbf{r})}{|\mathbf{R} - \mathbf{r}|} \exp\left(\frac{ik|\mathbf{R} - \mathbf{r}|}{\hbar}\right) d\mathbf{r}, \quad k = \sqrt{2M(E - E_1 - E_2)}$$

Substitute for $f_0(\mathbf{r})$ to obtain

$$f_1(\mathbf{R}) = \frac{M}{2\pi\hbar^2} \int \frac{V(\mathbf{r})}{|\mathbf{R} - \mathbf{r}||\mathbf{r}|} \exp\left(\frac{ik|\mathbf{R} - \mathbf{r}|}{\hbar} + \frac{ik|\mathbf{r}|}{\hbar}\right) d\mathbf{r}$$

Change of variable $\mathbf{r} \to \mathbf{y} + \mathbf{a}_1$.

$$f_1(\mathbf{R}) = \frac{M}{2\pi\hbar^2} \int \frac{V(\mathbf{y}+\mathbf{a}_1)}{|\mathbf{R}-\mathbf{y}-\mathbf{a}_1||\mathbf{y}+\mathbf{a}_1|} \exp\left(\frac{ik|\mathbf{R}-\mathbf{y}-\mathbf{a}_1|}{\hbar} + \frac{ik|\mathbf{y}+\mathbf{a}_1|}{\hbar}\right) d\mathbf{y}$$

Per Mott (see also Figari and Teta)

$$f_1(\mathbf{R}) \approx \frac{\exp(ik|\mathbf{R}-\mathbf{a}_1|)}{|\mathbf{R}-\mathbf{a}_1|} \frac{M}{2\pi\hbar^2} \int \frac{V(\mathbf{y}+\mathbf{a}_1)}{|\mathbf{y}+\mathbf{a}_1|} \exp\left(-\frac{ik\mathbf{u}\cdot\mathbf{y}}{\hbar} + \frac{ik|\mathbf{y}|}{\hbar}\right) d\mathbf{y} \tag{3}$$

where

$$\mathbf{u} = \frac{\mathbf{R}-\mathbf{a}_1}{|\mathbf{R}-\mathbf{a}_1|}$$

The method of stationary phase requires that

$$\frac{d}{d\mathbf{y}}\left(-\mathbf{u}\cdot\mathbf{y} + |\mathbf{y}+\mathbf{a}_1|\right) = -\mathbf{u} + \frac{\mathbf{y}+\mathbf{a}_1}{|\mathbf{y}+\mathbf{a}_1|} = 0 \tag{4}$$

Note that $V(\mathbf{y}+\mathbf{a}_1)$ in equation (3) is insignificant except for $\mathbf{y} \approx 0$ and $\mathbf{y} \approx \mathbf{a}_2 - \mathbf{a}_1$ so we only need to consider $\mathbf{y}$ near those values. To satisfy (4) for both $\mathbf{y} = 0$ and $\mathbf{y} = \mathbf{a}_2 - \mathbf{a}_1$ we have

$$\mathbf{u} = \frac{\mathbf{a}_1}{|\mathbf{a}_1|} = \frac{\mathbf{a}_2}{|\mathbf{a}_2|}$$

Hence $\mathbf{a}_1$ and $\mathbf{a}_2$ form a line through the origin. If we keep $\mathbf{a}_1$ and choose a different $\mathbf{a}_2$ we get the same result. Hence the ionized atoms form a straight track.