White dwarf

The radius of a white dwarf can be estimated using the electron gas model of a solid.

The total electron energy E of a spherical electron gas is

$$E = \left(\frac{3\pi^2}{2}\right)^{\frac{1}{3}} \frac{9\hbar^2 N^{\frac{5}{3}}}{20 m_e R^2}$$

where R is the radius, N is the number of free electrons, and m_e is electron mass.

The gravitational energy U of a sphere with mass M and uniform density is

$$U = -\frac{3GM^2}{5R}$$

Minimize the total energy by finding R such that

$$\frac{d}{dR}(E + U) = 0$$

Hence

$$-\left(\frac{3\pi^2}{2}\right)^{\frac{1}{3}} \frac{9\hbar^2 N^{\frac{5}{3}}}{10 m_e R^3} + \frac{3GM^2}{5R^2} = 0$$

Multiply both sides by R^3.

$$-\left(\frac{3\pi^2}{2}\right)^{\frac{1}{3}} \frac{9\hbar^2 N^{\frac{5}{3}}}{10 m_e} + \frac{3GM^2}{5} R = 0$$

Hence

$$R = \left(\frac{3\pi^2}{2}\right)^{\frac{1}{3}} \frac{9\hbar^2 N^{\frac{5}{3}}}{10 m_e} \frac{5}{3GM^2} = \left(\frac{3\pi^2}{2}\right)^{\frac{1}{3}} \frac{3\hbar^2 N^{\frac{5}{3}}}{2 m_e G M^2} \tag{1}$$

The number of free electrons is estimated to be one-half the number of nucleons.

$$N = \frac{M}{2m_p} \tag{2}$$

Substitute (2) into (1) to obtain

$$R = \frac{3\hbar^2}{8Gm_e} \left(\frac{3\pi^2}{M m_p^5}\right)^{\frac{1}{3}} \tag{3}$$

Noting that

$$\frac{3\hbar^2}{8Gm_e} \left(\frac{3\pi^2}{m_p^5}\right)^{\frac{1}{3}} = 9.00395 \times 10^{13}\,\mathrm{km\ kg}^{\frac{1}{3}} \tag{4}$$

we can also write

$$R = \frac{9 \times 10^{13}\,\mathrm{km\ kg}^{\frac{1}{3}}}{M^{\frac{1}{3}}}$$

For one solar mass $M = M_\odot = 2 \times 10^{30}$ kg we have

$$R = 7146\,\mathrm{km}$$