Feynman and Hibbs problem 3-9

Find the kernel for a particle in a constant field f where the Lagrangian is

$$L = \frac{m}{2}\dot{x}^2 + fx$$

From the L given above we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

and

$$\frac{\partial L}{\partial x} = f$$

By equation (2.7) which is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

we have

$$\ddot{x} = \frac{f}{m}$$

Hence velocity is constant. It follows that $x(t)$ is velocity times time.

$$x(t) = \left(\frac{x_b - x_a}{t_b - t_a}\right)(t - t_a) + x_a$$

By equation (2.1) which is

$$S = \int_{t_a}^{t_b} L\,dt$$

we have

$$S_{cl} = \int_{t_a}^{t_b} \frac{m}{2}\dot{x}(t)^2\,dt + \int_{t_a}^{t_b} fx(t)\,dt$$

Equation (2.8) provides the following solution to the first integral.

$$\int_{t_a}^{t_b} \frac{m}{2}\dot{x}(t)^2\,dt = \frac{m(x_b - x_a)^2}{2(t_b - t_a)} \tag{1}$$

Solve the second integral.

$$\int_{t_a}^{t_b} fx(t)\,dt = \frac{(x_b + x_a)(t_b - t_a)f}{2} \tag{2}$$

Action S_{cl} is the sum of (1) and (2).

$$S_{cl} = \frac{m(x_b - x_a)^2}{2(t_b - t_a)} + \frac{(x_b + x_a)(t_b - t_a)f}{2}$$

By equation (3.51) which is

$$K(b,a) = F(t_b - t_a)\exp\left(\frac{iS_{cl}}{\hbar}\right)$$

we have

$$K(b,a) = F(t_b - t_a)\exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)} + \frac{i(x_b + x_a)(t_b - t_a)f}{2\hbar}\right) \tag{3}$$

We now proceed to compute F. By equation (2.31) which is

$$K(b,a) = \int_{-\infty}^{\infty} K(b,c)K(c,a)\,dx_c$$

we have

$$\begin{aligned} K(b,a) = \int_{-\infty}^{\infty} & F(t_b - t_c)\exp\left(\frac{im(x_b - x_c)^2}{2\hbar(t_b - t_c)} + \frac{i(x_b + x_c)(t_b - t_c)f}{2\hbar}\right) \\ & \times F(t_c - t_a)\exp\left(\frac{im(x_c - x_a)^2}{2\hbar(t_c - t_a)} + \frac{i(x_c + x_a)(t_c - t_a)f}{2\hbar}\right) dx_c \end{aligned} \tag{4}$$

Rewrite equation (4) as

$$K(b,a) = F(t_b - t_c)F(t_c - t_a)\int_{-\infty}^{\infty} \exp(iAx_c^2 + iBx_c + iC) \tag{5}$$

where

$$A = \frac{m}{2\hbar}\left(\frac{1}{t_b - t_c} + \frac{1}{t_c - t_a}\right) \tag{6}$$

$$B = \frac{(t_b - t_a)f}{2\hbar} - \frac{m}{\hbar}\left(\frac{x_b}{t_b - t_c} + \frac{x_a}{t_c - t_a}\right) \tag{7}$$

$$C = \frac{f}{2\hbar}\big(x_b(t_b - t_c) + x_a(t_c - t_a)\big) + \frac{m}{2\hbar}\left(\frac{x_b^2}{t_b - t_c} + \frac{x_a^2}{t_c - t_a}\right) \tag{8}$$

From the following formula

$$\int_{-\infty}^{\infty} \exp(iAx_c^2 + iBx_c + iC)\, dx_c = \left(-\frac{\pi}{iA}\right)^{1/2} \exp\left(-\frac{iB^2}{4A} + iC\right)$$

we have

$$\int_{-\infty}^{\infty} \exp(iAx_c^2 + iBx_c + iC)\, dx_c = \left(-\frac{2\pi\hbar(t_b - t_c)(t_c - t_a)}{im(t_b - t_a)}\right)^{1/2} \\ \times \exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)} + \frac{i(x_b + x_a)(t_b - t_a)f}{2\hbar} - \frac{i(t_b - t_a)(t_b - t_c)(t_c - t_a)f^2}{8\hbar m}\right) \quad (9)$$

Substitute (9) into (5) to obtain

$$K(b,a) = F(t_b - t_c)F(t_c - t_a)\left(-\frac{2\pi\hbar(t_b - t_c)(t_c - t_a)}{im(t_b - t_a)}\right)^{1/2} \\ \times \exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)} + \frac{i(x_b + x_a)(t_b - t_a)f}{2\hbar} - \frac{i(t_b - t_a)(t_b - t_c)(t_c - t_a)f^2}{8\hbar m}\right) \quad (10)$$

Equating (3) with (10) reduces to

$$F(t_b - t_a) = F(t_b - t_c)F(t_c - t_a) \\ \times \left(-\frac{2\pi\hbar(t_b - t_c)(t_c - t_a)}{im(t_b - t_a)}\right)^{1/2} \exp\left(-\frac{i(t_b - t_a)(t_b - t_c)(t_c - t_a)f^2}{8\hbar m}\right) \quad (11)$$

From problem 3-7, let

$$F(t) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} g(t) \quad (12)$$

Substitute (12) into (11) to obtain

$$\left(\frac{m}{2\pi i\hbar(t_b - t_a)}\right)^{1/2} g(t_b - t_a) \\ = \left(\frac{m}{2\pi i\hbar(t_b - t_c)}\right)^{1/2} g(t_b - t_c) \left(\frac{m}{2\pi i\hbar(t_c - t_a)}\right)^{1/2} g(t_c - t_a) \\ \times \left(-\frac{2\pi\hbar(t_b - t_c)(t_c - t_a)}{im(t_b - t_a)}\right)^{1/2} \exp\left(-\frac{i(t_b - t_a)(t_b - t_c)(t_c - t_a)f^2}{8\hbar m}\right)$$

The coefficients cancel leaving

$$g(t_b - t_a) = g(t_b - t_c)g(t_c - t_a)\exp\left(-\frac{i(t_b - t_a)(t_b - t_c)(t_c - t_a)f^2}{8\hbar m}\right) \tag{13}$$

Note that

$$(t_b - t_a)^3 = (t_b - t_c)^3 + 3(t_b - t_a)(t_b - t_c)(t_c - t_a) + (t_c - t_a)^3 \tag{14}$$

Solve for $g(t)$ by completing the cube.

$$g(t) = \exp\left(-\frac{it^3 f^2}{24\hbar m}\right) \tag{15}$$

Substitute (15) into (12).

$$F(t) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \exp\left(-\frac{it^3 f^2}{24\hbar m}\right) \tag{16}$$

Substitute (16) into (3).

$$\begin{aligned} K(b,a) = {} & \left(\frac{m}{2\pi i\hbar(t_b - t_a)}\right)^{1/2} \\ & \times \exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)} + \frac{i(x_b + x_a)(t_b - t_a)f}{2\hbar} - \frac{i(t_b - t_a)^3 f^2}{24\hbar m}\right) \end{aligned}$$