

8-3. Show that Q_α^c , Q_α^s are normal coordinates corresponding to standing wave normal modes $\cos(2\pi\alpha j/N)$ and $\sin(2\pi\alpha j/N)$, in the sense that (for N odd)

$$q_j(t) = \sqrt{\frac{2}{N}} \left(\frac{1}{2} Q_0^c(t) + \sum_{\alpha=1}^{(N-1)/2} \left(Q_\alpha^c(t) \cos \frac{2\pi\alpha j}{N} + Q_\alpha^s(t) \sin \frac{2\pi\alpha j}{N} \right) \right) \quad (8.82)$$

(Note: A typo in 8.82 is fixed.)

Consider the following equations.

$$Q_\alpha(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^N q_k(t) \left(\cos \frac{2\pi\alpha k}{N} - i \sin \frac{2\pi\alpha k}{N} \right) \quad (8.77)$$

$$Q_\alpha^c = \frac{1}{\sqrt{2}} (Q_\alpha + Q_\alpha^*) \quad (8.79)$$

$$Q_\alpha^s = \frac{i}{\sqrt{2}} (Q_\alpha - Q_\alpha^*) \quad (8.80)$$

Rewrite (8.82) as

$$q_j = \frac{1}{\sqrt{2N}} Q_0^c + \frac{1}{N} \sum_{\alpha=1}^{(N-1)/2} \sum_{k=1}^N q_k (T_1 + T_2 + T_3 - T_4) \quad (1)$$

where

$$\begin{aligned} T_1 &= \cos \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N} - i \sin \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N} \\ T_2 &= \cos \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N} + i \sin \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N} \\ T_3 &= i \cos \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N} + \sin \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N} \\ T_4 &= i \cos \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N} - \sin \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N} \end{aligned}$$

It follows that

$$T_1 + T_2 + T_3 - T_4 = 2 \cos \frac{2\pi\alpha k}{N} \cos \frac{2\pi\alpha j}{N} + 2 \sin \frac{2\pi\alpha k}{N} \sin \frac{2\pi\alpha j}{N}$$

By trigonometric identities

$$T_1 + T_2 + T_3 - T_4 = 2 \cos \left(\frac{2\pi\alpha}{N}(j - k) \right) \quad (2)$$

Substitute (2) into (1) to obtain

$$q_j = \frac{1}{\sqrt{2N}} Q_0^c + \frac{2}{N} \sum_{\alpha=1}^{(N-1)/2} \sum_{k=1}^N q_k \cos \left(\frac{2\pi\alpha}{N}(j - k) \right)$$

Note that

$$Q_0^c = \sqrt{\frac{2}{N}} \sum_{k=1}^N q_k$$

Hence

$$q_j = \frac{1}{N} \sum_{k=1}^N q_k + \frac{2}{N} \sum_{\alpha=1}^{(N-1)/2} \sum_{k=1}^N q_k \cos \left(\frac{2\pi\alpha}{N}(j - k) \right)$$

Rewrite as

$$q_j = \frac{1}{N} \sum_{k=1}^N q_k \left(1 + 2 \sum_{\alpha=1}^{(N-1)/2} \cos \left(\frac{2\pi\alpha}{N}(j - k) \right) \right) \quad (3)$$

For the sum over α we have

$$\sum_{\alpha=1}^{(N-1)/2} \cos \left(\frac{2\pi\alpha}{N}(j - k) \right) = \begin{cases} (N-1)/2 & j = k \\ -1/2 & j \neq k \end{cases}$$

Hence for $j = k$

$$1 + 2 \sum_{\alpha=1}^{(N-1)/2} \cos \left(\frac{2\pi\alpha}{N}(j - k) \right) = N$$

and for $j \neq k$

$$1 + 2 \sum_{\alpha=1}^{(N-1)/2} \cos \left(\frac{2\pi\alpha}{N}(j - k) \right) = 0$$

Therefore (3) is shown to be true.