

## Mott problem

Consider the emission of an  $\alpha$  particle in a cloud chamber. The quantum mechanical model of an  $\alpha$  particle is a spherical wave emanating from the origin. A spherical wave should ionize atoms throughout the cloud chamber. However, only straight tracks are observed. Nevill Mott showed that straight tracks are consistent with the Schrodinger equation.

Let  $\mathbf{R}$  be the position of the  $\alpha$  particle, let  $\mathbf{a}_1$  and  $\mathbf{a}_2$  be the positions of two atoms ionized by the  $\alpha$  particle, and let  $\mathbf{r}_1$  and  $\mathbf{r}_2$  be the positions of the free electrons. The Hamiltonian for the system is

$$\hat{H} = \hat{K}_\alpha + \hat{K}_1 + \hat{K}_2 + U_1 + U_2 + V_1 + V_2$$

where

$\hat{K}_\alpha = -\frac{\hbar^2}{2M}\nabla_\alpha^2$	kinetic energy of $\alpha$ particle
$\hat{K}_1 = -\frac{\hbar^2}{2m}\nabla_1^2$	kinetic energy of 1st electron
$\hat{K}_2 = -\frac{\hbar^2}{2m}\nabla_2^2$	kinetic energy of 2nd electron
$U_1 = -\frac{e^2}{ \mathbf{r}_1 - \mathbf{a}_1 }$	potential energy of 1st electron
$U_2 = -\frac{e^2}{ \mathbf{r}_2 - \mathbf{a}_2 }$	potential energy of 2nd electron
$V_1 = -\frac{2e^2}{ \mathbf{R} - \mathbf{r}_1 }$	potential energy of $\alpha$ and 1st electron
$V_2 = -\frac{2e^2}{ \mathbf{R} - \mathbf{r}_2 }$	potential energy of $\alpha$ and 2nd electron

Let  $\psi_1$  and  $\psi_2$  be atomic wavefunctions such that

$$\hat{H}\psi_1 = E_1\psi_1, \quad \hat{H}\psi_2 = E_2\psi_2$$

We want to find a wavefunction  $F(\mathbf{R}, \mathbf{r}_1, \mathbf{r}_2)$  such that

$$\hat{H}F = EF$$

Let

$$F = F_0 + F_1 + F_2 + \cdots$$

and let

$$\hat{H}_0 = \hat{K}_\alpha + E_1 + E_2$$

Start by finding an  $F_0$  such that

$$\hat{H}_0 F_0 = EF_0$$

The solution is

$$F_0(\mathbf{R}, \mathbf{r}_1, \mathbf{r}_2) = f_0(\mathbf{R})\psi_1(\mathbf{r}_1 - \mathbf{a}_1)\psi_2(\mathbf{r}_2 - \mathbf{a}_2) \quad (1)$$

where

$$f_0(\mathbf{R}) = \frac{1}{|\mathbf{R}|} \exp\left(\frac{ik|\mathbf{R}|}{\hbar}\right), \quad k = \sqrt{2M(E - E_1 - E_2)}$$

It follows that for the full Hamiltonian

$$\hat{H}F_0 = EF_0 + (V_1 + V_2)F_0$$

To cancel  $(V_1 + V_2)F_0$  from the full Hamiltonian, find an  $F_1$  such that

$$\hat{H}_0F_1 = EF_1 - (V_1 + V_2)F_0$$

Rewrite as

$$(\hat{H}_0 - E)F_1 = -(V_1 + V_2)F_0$$

Expand  $F_1$  and  $F_0$ .

$$(\hat{H}_0 - E)f_1(\mathbf{R})\psi_1(\mathbf{r}_1 - \mathbf{a}_1)\psi_2(\mathbf{r}_2 - \mathbf{a}_2) = -(V_1 + V_2)f_0(\mathbf{R})\psi_1(\mathbf{r}_1 - \mathbf{a}_1)\psi_2(\mathbf{r}_2 - \mathbf{a}_2)$$

To solve for  $f_1(\mathbf{R})$  multiply both sides by

$$\psi_1^*(\mathbf{r}_1 - \mathbf{a}_1)\psi_2^*(\mathbf{r}_2 - \mathbf{a}_2)$$

and integrate over  $\mathbf{r}_1$  and  $\mathbf{r}_2$  to obtain

$$(\hat{H}_0 - E)f_1(\mathbf{R}) = V_1(\mathbf{R})f_0(\mathbf{R}) + V_2(\mathbf{R})f_0(\mathbf{R}) \quad (2)$$

where

$$V_1(\mathbf{R}) = 2e^2 \int \frac{|\psi_1(\mathbf{r})|^2}{|\mathbf{R} - \mathbf{a}_1 - \mathbf{r}|} d\mathbf{r}, \quad V_2(\mathbf{R}) = 2e^2 \int \frac{|\psi_2(\mathbf{r})|^2}{|\mathbf{R} - \mathbf{a}_2 - \mathbf{r}|} d\mathbf{r}$$

Per Mott the solution to (2) is

$$f_1(\mathbf{R}) = \frac{M}{2\pi\hbar^2} \int \frac{V_1(\mathbf{r})f_0(\mathbf{r})}{|\mathbf{R} - \mathbf{r}|} \exp\left(\frac{ik|\mathbf{R} - \mathbf{r}|}{\hbar}\right) d\mathbf{r} + \frac{M}{2\pi\hbar^2} \int \frac{V_2(\mathbf{r})f_0(\mathbf{r})}{|\mathbf{R} - \mathbf{r}|} \exp\left(\frac{ik|\mathbf{R} - \mathbf{r}|}{\hbar}\right) d\mathbf{r}$$

Let  $I_1$  be the first integral.

$$I_1(\mathbf{R}) = \frac{M}{2\pi\hbar^2} \int \frac{V_1(\mathbf{r})f_0(\mathbf{r})}{|\mathbf{R} - \mathbf{r}|} \exp\left(\frac{ik|\mathbf{R} - \mathbf{r}|}{\hbar}\right) d\mathbf{r}$$

Substitute for  $f_0(\mathbf{r})$  to obtain

$$I_1(\mathbf{R}) = \frac{M}{2\pi\hbar^2} \int \frac{V_1(\mathbf{r})}{|\mathbf{R} - \mathbf{r}||\mathbf{r}|} \exp\left(\frac{ik|\mathbf{R} - \mathbf{r}|}{\hbar} + \frac{ik|\mathbf{r}|}{\hbar}\right) d\mathbf{r}$$

Change of variable  $\mathbf{r} \rightarrow \mathbf{r} + \mathbf{a}_1$

$$I_1(\mathbf{R}) = \frac{M}{2\pi\hbar^2} \int \frac{V_1(\mathbf{r} + \mathbf{a}_1)}{|\mathbf{R} - \mathbf{r} - \mathbf{a}_1||\mathbf{r} + \mathbf{a}_1|} \exp\left(\frac{ik|\mathbf{R} - \mathbf{r} - \mathbf{a}_1|}{\hbar} + \frac{ik|\mathbf{r} + \mathbf{a}_1|}{\hbar}\right) d\mathbf{r}$$

Per Mott (see also Figari and Teta)

$$I_1(\mathbf{R}) \approx \frac{1}{|\mathbf{R} - \mathbf{a}_1|} \exp\left(\frac{ik|\mathbf{R} - \mathbf{a}_1|}{\hbar}\right) \times \frac{M}{2\pi\hbar^2} \int \frac{V_1(\mathbf{r} + \mathbf{a}_1)}{|\mathbf{r} + \mathbf{a}_1|} \exp\left(-\frac{ik\mathbf{u}_1(\mathbf{R}) \cdot \mathbf{r}}{\hbar} + \frac{ik|\mathbf{r} + \mathbf{a}_1|}{\hbar}\right) d\mathbf{r}$$

where

$$\mathbf{u}_1(\mathbf{R}) = \frac{\mathbf{R} - \mathbf{a}_1}{|\mathbf{R} - \mathbf{a}_1|}$$

The condition for stationary phase is

$$g' = \frac{d}{d\mathbf{r}} (-\mathbf{u}_1(\mathbf{R}) \cdot \mathbf{r} + |\mathbf{r} + \mathbf{a}_1|) = -\mathbf{u}_1(\mathbf{R}) + \frac{\mathbf{r} + \mathbf{a}_1}{|\mathbf{r} + \mathbf{a}_1|} = 0$$

Note that  $V_1(\mathbf{r} + \mathbf{a}_1)$  is small except for  $\mathbf{r} \approx 0$  so we only require stationarity at the origin. Hence for  $\mathbf{r} = 0$  the integral is stationary ( $g' = 0$ ) when  $\mathbf{R}$  satisfies the condition

$$\mathbf{u}_1(\mathbf{R}) = \frac{\mathbf{a}_1}{|\mathbf{a}_1|}$$

By symmetry of the integrals,  $I_2$  is stationary when  $\mathbf{R}$  satisfies the condition

$$\mathbf{u}_2(\mathbf{R}) = \frac{\mathbf{a}_2}{|\mathbf{a}_2|}$$

Because nonstationary integrals vanish we have

$$f_1(\mathbf{R}) = \begin{cases} I_1(\mathbf{R}), & \mathbf{u}_1(\mathbf{R}) = \frac{\mathbf{a}_1}{|\mathbf{a}_1|} \quad \text{and} \quad \mathbf{u}_2(\mathbf{R}) \neq \frac{\mathbf{a}_2}{|\mathbf{a}_2|} \\ I_2(\mathbf{R}), & \mathbf{u}_1(\mathbf{R}) \neq \frac{\mathbf{a}_1}{|\mathbf{a}_1|} \quad \text{and} \quad \mathbf{u}_2(\mathbf{R}) = \frac{\mathbf{a}_2}{|\mathbf{a}_2|} \\ I_1(\mathbf{R}) + I_2(\mathbf{R}), & \mathbf{u}_1(\mathbf{R}) = \frac{\mathbf{a}_1}{|\mathbf{a}_1|} \quad \text{and} \quad \mathbf{u}_2(\mathbf{R}) = \frac{\mathbf{a}_2}{|\mathbf{a}_2|} \\ 0, & \text{otherwise} \end{cases}$$

Since both atoms are ionized, it follows that both  $V_1$  and  $V_2$  should contribute to  $f_1$ . Hence we reject the first two cases as unphysical and write

$$f_1(\mathbf{R}) = \begin{cases} I_1(\mathbf{R}) + I_2(\mathbf{R}), & \frac{\mathbf{a}_1}{|\mathbf{a}_1|} = \frac{\mathbf{a}_2}{|\mathbf{a}_2|} \\ 0, & \text{otherwise} \end{cases}$$

To satisfy the condition,  $\mathbf{a}_1$  and  $\mathbf{a}_2$  must be on the same ray emanating from the origin. Hence straight tracks are consistent with the Schrodinger equation.

## Note

The condition for stationarity

$$\mathbf{u}_1(\mathbf{R}) = \frac{\mathbf{R} - \mathbf{a}_1}{|\mathbf{R} - \mathbf{a}_1|} = \frac{\mathbf{a}_1}{|\mathbf{a}_1|}$$

is satisfied by all  $\mathbf{R}$  and constant  $c > 1$  such that

$$\mathbf{R} = c\mathbf{a}_1$$

Condition  $c > 1$  implies that  $|\mathbf{R}| > |\mathbf{a}_1|$ . So technically the condition

$$\frac{\mathbf{a}_1}{|\mathbf{a}_1|} = \frac{\mathbf{a}_2}{|\mathbf{a}_2|}$$

is less stringent than

$$\mathbf{u}_1(\mathbf{R}) = \frac{\mathbf{a}_1}{|\mathbf{a}_1|} \quad \text{and} \quad \mathbf{u}_2(\mathbf{R}) = \frac{\mathbf{a}_2}{|\mathbf{a}_2|}$$

The exact condition for stationarity of both  $I_1$  and  $I_2$  is

$$\frac{\mathbf{a}_1}{|\mathbf{a}_1|} = \frac{\mathbf{a}_2}{|\mathbf{a}_2|} \quad \text{and} \quad |\mathbf{R}| > \max(|\mathbf{a}_1|, |\mathbf{a}_2|)$$