(36.1) (a) Show that the Dirac equation can be recast in the form

$$i\frac{\partial \psi}{\partial t} = \hat{H}_D \psi \tag{36.33}$$

where $\hat{H}_D = \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m$ and find $\boldsymbol{\alpha}$ and β in terms of the γ matrices.

(b) Evaluate $\hat{H}_D^2$ and show that for a Klein-Gordon dispersion to result we must have:
(i) that the α^i and β objects all anticommute with each other; and
(ii) $(\alpha^i)^2 = (\beta)^2 = 1$.

(c) Prove the following commutation relations
(i) $[\hat{H}, \hat{L}^i] = i(\hat{\mathbf{p}} \times \boldsymbol{\alpha})^i$ where $\hat{\mathbf{L}} = \hat{\mathbf{x}} \times \hat{\mathbf{p}}$.
(ii) $[\hat{H}, \hat{S}^i] = -i(\hat{\mathbf{p}} \times \boldsymbol{\alpha})^i$ where $\hat{\mathbf{S}} = \frac{1}{2}\boldsymbol{\Sigma}$ and we define $\boldsymbol{\Sigma} = \frac{i}{2}\boldsymbol{\gamma} \times \boldsymbol{\gamma}$.

(a) Consider the following form of the Dirac equation.

$$i\left(\gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z}\right)\psi = m\psi$$

Rewrite as

$$i\gamma^0 \frac{\partial}{\partial t}\psi = -i\left(\gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z}\right)\psi + m\psi$$

Noting that $\gamma^0\gamma^0 = I$, multiply both sides by γ^0 to obtain

$$i\frac{\partial}{\partial t}\psi = -i\gamma^0\left(\gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z}\right)\psi + m\gamma^0\psi$$

Hence for $\hat{\mathbf{p}} = -i\nabla$ we have

$$\boldsymbol{\alpha} = \gamma^0 \begin{pmatrix} \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{pmatrix}, \quad \beta = \gamma^0$$

(b) The dispersion relation is

$$\hat{H}_D^2 = \hat{\mathbf{p}}^2 + m^2$$

Squaring $\hat{H}_D$ we have

$$\begin{aligned}\hat{H}_D^2 &= (\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}+\beta m)(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}+\beta m)\\ &= (\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})+(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})\beta m+\beta m(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})+\beta^2 m^2\end{aligned}$$

The middle terms must cancel, that is

$$(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})\beta m+\beta m(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})=0$$

Hence

$$\alpha^i\beta=-\beta\alpha^i$$

Cross terms must cancel, that is

$$\left(-i\alpha^1\frac{\partial}{\partial x}-i\alpha^2\frac{\partial}{\partial y}-i\alpha^3\frac{\partial}{\partial z}\right)^2=-(\alpha^1)^2\frac{\partial^2}{\partial x^2}-(\alpha^2)^2\frac{\partial^2}{\partial y^2}-(\alpha^3)^2\frac{\partial^2}{\partial z^2}$$

Hence

$$\alpha^i\alpha^j=-\alpha^j\alpha^i,\quad i\neq j$$

With the above anticommutation relations we now have

$$\hat{H}_D^2=-(\alpha^1)^2\frac{\partial^2}{\partial x^2}-(\alpha^2)^2\frac{\partial^2}{\partial y^2}-(\alpha^3)^2\frac{\partial^2}{\partial z^2}+\beta^2 m=\hat{\mathbf{p}}^2+m^2$$

Hence

$$(\alpha^i)^2=I\quad\text{and}\quad\beta^2=I$$

(c) We have

$$\hat{\mathbf{L}}\psi=\hat{\mathbf{x}}\times\hat{\mathbf{p}}\psi$$

If follows that

$$\begin{aligned}[\hat{H}_D,\hat{\mathbf{L}}]\psi &= \hat{H}_D\hat{\mathbf{L}}\psi-\hat{\mathbf{L}}\hat{H}_D\psi\\ &= (\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}+\beta m)(\hat{\mathbf{x}}\times\hat{\mathbf{p}}\psi)-\hat{\mathbf{x}}\times\hat{\mathbf{p}}(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}+\beta m)\psi\end{aligned}$$

The βm terms cancel by the following identity.

$$\beta m\hat{\mathbf{x}}\times\hat{\mathbf{p}}\psi=\hat{\mathbf{x}}\times\beta m\hat{\mathbf{p}}\psi=\hat{\mathbf{x}}\times\hat{\mathbf{p}}(\beta m\psi)$$

Hence

$$[\hat{H}_D,\hat{\mathbf{L}}]\psi=(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})(\hat{\mathbf{x}}\times\hat{\mathbf{p}}\psi)-\hat{\mathbf{x}}\times\hat{\mathbf{p}}(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})\psi$$

By the product rule for differentiation

$$(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})(\hat{\mathbf{x}} \times \hat{\mathbf{p}}\psi) = (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\hat{\mathbf{x}} \times \hat{\mathbf{p}}\psi + \hat{\mathbf{x}} \times (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\hat{\mathbf{p}}\psi$$

Hence

$$\begin{aligned}[\hat{H}_D, \hat{\mathbf{L}}]\psi &= (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\hat{\mathbf{x}} \times \hat{\mathbf{p}}\psi + \hat{\mathbf{x}} \times (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\hat{\mathbf{p}}\psi - \hat{\mathbf{x}} \times \hat{\mathbf{p}}(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\psi \\ &= (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\hat{\mathbf{x}} \times \hat{\mathbf{p}}\psi\end{aligned}$$

Noting that

$$(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})\hat{\mathbf{x}} = \boldsymbol{\alpha} \cdot (-i\nabla\hat{\mathbf{x}}) = -i\boldsymbol{\alpha}$$

we have

$$[\hat{H}_D, \hat{\mathbf{L}}] = -i\boldsymbol{\alpha} \times \hat{\mathbf{p}} = i(\hat{\mathbf{p}} \times \boldsymbol{\alpha})$$