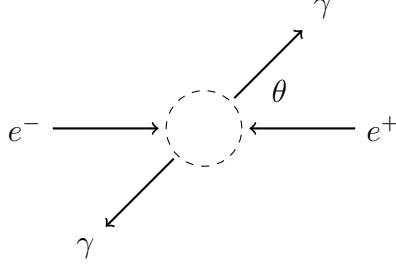


Annihilation

Annihilation is the interaction $e^- + e^+ \rightarrow \gamma + \gamma$.



In the center-of-mass frame we have the following momentum vectors where $E = \sqrt{p^2 + m^2}$.

$$\begin{aligned}
 p_1 &= \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} & p_2 &= \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} & p_3 &= \begin{pmatrix} E \\ E \sin \theta \cos \phi \\ E \sin \theta \sin \phi \\ E \cos \theta \end{pmatrix} & p_4 &= \begin{pmatrix} E \\ -E \sin \theta \cos \phi \\ -E \sin \theta \sin \phi \\ -E \cos \theta \end{pmatrix} \\
 &\text{inbound} & &\text{inbound} & &\text{outbound} & &\text{outbound} \\
 &\text{electron} & &\text{positron} & &\text{photon} & &\text{photon}
 \end{aligned}$$

Spinors for the inbound electron.

$$\begin{aligned}
 u_{11} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix} & u_{12} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix} \\
 &\text{inbound electron} & &\text{inbound electron} \\
 &\text{spin up} & &\text{spin down}
 \end{aligned}$$

Spinors for the inbound positron.

$$\begin{aligned}
 v_{21} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} -p \\ 0 \\ E+m \\ 0 \end{pmatrix} & v_{22} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ p \\ 0 \\ E+m \end{pmatrix} \\
 &\text{inbound positron} & &\text{inbound positron} \\
 &\text{spin up} & &\text{spin down}
 \end{aligned}$$

The probability amplitude \mathcal{M}_{ab} for spin state ab is

$$\mathcal{M}_{ab} = \mathcal{M}_{1ab} + \mathcal{M}_{2ab}$$

where

$$\mathcal{M}_{1ab} = \frac{\bar{v}_{2b}(-ie\gamma^\mu)(\not{p}_1 + m)(-ie\gamma^\nu)u_{1a}}{t - m^2}, \quad \mathcal{M}_{2ab} = \frac{\bar{v}_{2b}(-ie\gamma^\nu)(\not{p}_2 + m)(-ie\gamma^\mu)u_{1a}}{u - m^2}$$

Symbol e is elementary charge and

$$\begin{aligned}\not{q}_1 &= (p_1 - p_3)^\alpha g_{\alpha\beta} \gamma^\beta \\ \not{q}_2 &= (p_1 - p_4)^\alpha g_{\alpha\beta} \gamma^\beta \\ t &= (p_1 - p_3)^2 \\ u &= (p_1 - p_4)^2\end{aligned}$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is the average of spin states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{a=1}^2 \sum_{b=1}^2 |\mathcal{M}_{ab}|^2$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{a=1}^2 \sum_{b=1}^2 (\mathcal{M}_{1ab} \mathcal{M}_{1ab}^* + \mathcal{M}_{1ab} \mathcal{M}_{2ab}^* + \mathcal{M}_{2ab} \mathcal{M}_{1ab}^* + \mathcal{M}_{2ab} \mathcal{M}_{2ab}^*)$$

To understand how $\mathcal{M}_{1ab} \mathcal{M}_{1ab}^*$ is calculated, write \mathcal{M}_{1ab} in component form.

$$(\mathcal{M}_{1ab})^{\mu\nu} = \frac{(\bar{v}_{2b})_\alpha (-ie\gamma^{\mu\alpha}_\beta) (\not{q}_1 + m)^\beta_\rho (-ie\gamma^{\nu\rho}_\sigma) (u_{1a})^\sigma}{t - m^2}$$

Metric tensor $g_{\mu\nu}$ is required to sum over indices μ and ν .

$$\mathcal{M}_{1ab} \mathcal{M}_{1ab}^* = (\mathcal{M}_{1ab})^{\mu\nu} (\mathcal{M}_{1ab}^*)_{\mu\nu} = (\mathcal{M}_{1ab})^{\mu\nu} g_{\mu\alpha} (\mathcal{M}_{1ab}^*)^{\alpha\beta} g_{\beta\nu}$$

Similarly for $\mathcal{M}_{2ab} \mathcal{M}_{2ab}^*$. For \mathcal{M}_{2ab} the index order is ν followed by μ hence

$$\mathcal{M}_{1ab} \mathcal{M}_{2ab}^* = (\mathcal{M}_{1ab})^{\mu\nu} (\mathcal{M}_{2ab}^*)_{\nu\mu} = (\mathcal{M}_{1ab})^{\mu\nu} g_{\nu\beta} (\mathcal{M}_{2ab}^*)^{\beta\alpha} g_{\alpha\mu}$$

The Casimir trick uses matrix arithmetic to sum over spin states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left(\frac{f_{11}}{(t - m^2)^2} + \frac{2f_{12}}{(t - m^2)(u - m^2)} + \frac{f_{22}}{(u - m^2)^2} \right)$$

where

$$\begin{aligned}f_{11} &= \text{Tr} \left((\not{p}_1 + m) \gamma^\mu (\not{q}_1 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\nu (\not{q}_1 + m) \gamma_\mu \right) \\ f_{12} &= \text{Tr} \left((\not{p}_1 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\mu (\not{q}_1 + m) \gamma_\nu \right) \\ f_{22} &= \text{Tr} \left((\not{p}_1 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\nu (\not{q}_2 + m) \gamma_\mu \right)\end{aligned}$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$\begin{aligned}f_{11} &= 32(p_1 \cdot p_3)(p_1 \cdot p_4) - 32m^2(p_1 \cdot p_2) + 64m^2(p_1 \cdot p_3) + 32m^2(p_1 \cdot p_4) - 64m^4 \\ f_{12} &= 16m^2(p_1 \cdot p_3) + 16m^2(p_1 \cdot p_4) - 32m^4 \\ f_{22} &= 32(p_1 \cdot p_3)(p_1 \cdot p_4) - 32m^2(p_1 \cdot p_2) + 32m^2(p_1 \cdot p_3) + 64m^2(p_1 \cdot p_4) - 64m^4\end{aligned}$$

In Mandelstam variables

$$\begin{aligned}f_{11} &= 8tu - 24tm^2 - 8um^2 - 8m^4 \\f_{12} &= 8sm^2 - 32m^4 \\f_{22} &= 8tu - 8tm^2 - 24um^2 - 8m^4\end{aligned}$$

For high energy experiments such that $E \gg m$, let $m = 0$ and obtain

$$\begin{aligned}f_{11} &= 8tu \\f_{12} &= 0 \\f_{22} &= 8tu\end{aligned}$$

Hence

$$\begin{aligned}\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left(\frac{8tu}{t^2} + \frac{8tu}{u^2} \right) \\&= 2e^4 \left(\frac{u}{t} + \frac{t}{u} \right)\end{aligned}$$

For $m = 0$ the Mandelstam variables are

$$\begin{aligned}t &= -2E^2(1 - \cos \theta) \\u &= -2E^2(1 + \cos \theta)\end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right)$$

Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{4(4\pi\epsilon_0)^2 s}$$

where

$$s = (p_1 + p_2)^2 = 4E^2$$

For high energy experiments we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\epsilon_0)^2 s} \left(\frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right)$$

Noting that

$$e^2 = 4\pi\epsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s} \left(\frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right)$$

Noting that

$$d\Omega = \sin \theta d\theta d\phi$$

we also have

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{2s} \left(\frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right) \sin \theta d\theta d\phi$$

Let $S(\theta_1, \theta_2)$ be the following integral of $d\sigma$.

$$S(\theta_1, \theta_2) = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{\pi \alpha^2 (\hbar c)^2}{s} [I(\theta_2) - I(\theta_1)]$$

where

$$I(\theta) = 2 \cos \theta + 2 \log(1 - \cos \theta) - 2 \log(1 + \cos \theta)$$

The cumulative distribution function is

$$F(\theta) = \frac{S(a, \theta)}{S(a, \pi - a)} = \frac{I(\theta) - I(a)}{I(\pi - a) - I(a)}, \quad a \leq \theta \leq \pi - a$$

Angular support is reduced by an arbitrary angle $a > 0$ because $I(0)$ and $I(\pi)$ are undefined.

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 < \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi - a) - I(a)} \left(\frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right) \sin \theta$$

Data from DESY PETRA experiment

See www.hepdata.net/record/ins191231, Table 2, 14.0 GeV.

x	y
0.0502	0.09983
0.1505	0.10791
0.2509	0.12026
0.3512	0.13002
0.4516	0.17681
0.5521	0.19570
0.6526	0.27900
0.7312	0.33204

Data x and y have the following relationship with the differential cross section formula.

$$x = \cos \theta, \quad y = \frac{d\sigma}{d\Omega}$$

The cross section formula is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s} \left(\frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right) \times (\hbar c)^2$$

To compute predicted values \hat{y} , multiply by 10^{37} to convert square meters to nanobarns.

$$\hat{y} = \frac{\alpha^2}{2s} \left(\frac{1 + x}{1 - x} + \frac{1 - x}{1 + x} \right) \times (\hbar c)^2 \times 10^{37}$$

The following table shows predicted values \hat{y} for $s = (14.0 \text{ GeV})^2$.

x	y	\hat{y}
0.0502	0.09983	0.106325
0.1505	0.10791	0.110694
0.2509	0.12026	0.120005
0.3512	0.13002	0.135559
0.4516	0.17681	0.159996
0.5521	0.19570	0.198562
0.6526	0.27900	0.262745
0.7312	0.33204	0.348884

The coefficient of determination R^2 measures how well predicted values fit the data.

$$R^2 = 1 - \frac{\sum (y - \hat{y})^2}{\sum (y - \bar{y})^2} = 0.98$$

The result indicates that the model $d\sigma$ explains 98% of the variance in the data.