Spin measurements

Let $|s\rangle$ be the following spin state given polar angle θ and azimuth ϕ.

$$|s\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\exp(i\phi) \end{pmatrix}$$

Find spin measurement probabilities for directions x, y, and z.

Eigenstates for the z direction.

$$|z_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |z_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Eigenstates for the x direction.

$$|x_+\rangle = \frac{|z_+\rangle + |z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |x_-\rangle = \frac{|z_+\rangle - |z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Eigenstates for the y direction.

$$|y_+\rangle = \frac{|z_+\rangle + i|z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |y_-\rangle = \frac{|z_+\rangle - i|z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

For spin measurements in the z direction

$$\begin{aligned} \Pr\left(+\tfrac{\hbar}{2}\right) &= |\langle z_+|s\rangle|^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos\theta \\ \Pr\left(-\tfrac{\hbar}{2}\right) &= |\langle z_-|s\rangle|^2 = \tfrac{1}{2} - \tfrac{1}{2}\cos\theta \end{aligned}$$

For the x direction

$$\begin{aligned} \Pr\left(+\tfrac{\hbar}{2}\right) &= |\langle x_+|s\rangle|^2 = \tfrac{1}{2} + \tfrac{1}{2}\sin\theta\cos\phi \\ \Pr\left(-\tfrac{\hbar}{2}\right) &= |\langle x_-|s\rangle|^2 = \tfrac{1}{2} - \tfrac{1}{2}\sin\theta\cos\phi \end{aligned}$$

For the y direction

$$\begin{aligned} \Pr\left(+\tfrac{\hbar}{2}\right) &= |\langle y_+|s\rangle|^2 = \tfrac{1}{2} - \tfrac{1}{2}\sin\theta\sin\phi \\ \Pr\left(-\tfrac{\hbar}{2}\right) &= |\langle y_-|s\rangle|^2 = \tfrac{1}{2} + \tfrac{1}{2}\sin\theta\sin\phi \end{aligned}$$

Note that for each direction $\Pr\left(+\tfrac{\hbar}{2}\right) + \Pr\left(-\tfrac{\hbar}{2}\right) = 1$ as required by total probability.