9-5. The momentum in the field is given by

$$\frac{1}{4\pi c}\int \mathbf{E}\times\mathbf{B}\,d^3\mathbf{r}$$

In the absence of matter (so $\phi_{\mathbf{k}}=0$), show that this is

$$i\int \mathbf{k}\,(\mathbf{a}^*_{\mathbf{k}}\cdot\dot{\mathbf{a}}_{\mathbf{k}})\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

Let

$$D_{\mathbf{k}}=\exp(i\mathbf{k}\cdot\mathbf{r})\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

By equation (9.14)

$$\mathbf{A}\,d^3\mathbf{r}=\sqrt{4\pi}c\,\mathbf{a}_{\mathbf{k}}\,D_{\mathbf{k}} \tag{1}$$

By equation (1) and (9.9) with $\phi=0$

$$\mathbf{E}\,d^3\mathbf{r}=-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}=-\sqrt{4\pi}\,\dot{\mathbf{a}}_{\mathbf{k}}\,D_{\mathbf{k}}$$

By equation (1) and (9.7)

$$\mathbf{B}\,d^3\mathbf{r}=\nabla\times\mathbf{A}\,d^3\mathbf{r}=\sqrt{4\pi}ic\,\mathbf{k}\times\mathbf{a}_{\mathbf{k}}\,D_{\mathbf{k}}$$

Hence

$$\mathbf{E}\times\mathbf{B}=-4\pi ic\int \dot{\mathbf{a}}_{\mathbf{k}}\times(\mathbf{k}\times\mathbf{a}_{\mathbf{k}})\;D_{\mathbf{k}}$$

By the triple product formula

$$\mathbf{E}\times\mathbf{B}=-4\pi ic\int \big((\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{a}_{\mathbf{k}})\mathbf{k}-(\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{k})\mathbf{a}_{\mathbf{k}}\big)\,D_{\mathbf{k}}$$

By orthogonality of $\mathbf{E}$ and $\mathbf{k}$ (see problem 9-1), $\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{k}=0$ hence

$$\mathbf{E}\times\mathbf{B}=-4\pi ic\int(\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{a}_{\mathbf{k}})\mathbf{k}\,D_{\mathbf{k}}$$

Integrate over all space.

$$\int\mathbf{E}\times\mathbf{B}\,d^3\mathbf{r}=-4\pi ic\int(\dot{\mathbf{a}}_{\mathbf{k}}\cdot\mathbf{a}_{\mathbf{k}})\mathbf{k}\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

Note: Integrating over all $\mathbf{r}$ is the same as integrating over all $\mathbf{k}$, hence the exponential in $D_{\mathbf{k}}$ is cancelled.