

Stern-Gerlach 2

From the previous section we have the Schrödinger equations

$$\begin{aligned} -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} \psi_1 + \mu_B(B_0 + \alpha z)\psi_1 &= i\hbar \frac{\partial}{\partial t} \psi_1 \\ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} \psi_2 - \mu_B(B_0 + \alpha z)\psi_2 &= i\hbar \frac{\partial}{\partial t} \psi_2 \end{aligned}$$

We now seek eigenfunctions for ψ_1 and ψ_2 .

The paper “Construction of Exact Solutions for the Stern-Gerlach Effect” provides the following solutions.

$$\begin{aligned} \psi_1 &= \text{Ai} \left[\left(\frac{\mu_B \alpha m}{\hbar^2} \right)^{1/3} \left(\frac{\mu_B \alpha}{4m} \right) t^2 + \left(\frac{\mu_B \alpha m}{\hbar^2} \right)^{1/3} z \right] \exp \left(-\frac{i\alpha \mu_B z t}{2\hbar} \right) \exp \left(-\frac{\mu_B B_0 t}{\hbar} \right) \\ \psi_2 &= \text{Ai} \left[\left(\frac{\mu_B \alpha m}{\hbar^2} \right)^{1/3} \left(\frac{\mu_B \alpha}{4m} \right) t^2 - \left(\frac{\mu_B \alpha m}{\hbar^2} \right)^{1/3} z \right] \exp \left(+\frac{i\alpha \mu_B z t}{2\hbar} \right) \exp \left(+\frac{\mu_B B_0 t}{\hbar} \right) \end{aligned}$$

Let us construct polynomial approximations for ψ_1 and ψ_2 .

For the Airy function we have

$$\text{Ai}(x) \approx \sum_{k=0}^n a_k x^k$$

where

$$\begin{aligned} a_0 &= \frac{1}{3^{2/3} \Gamma(2/3)} \\ a_1 &= -\frac{1}{3^{1/3} \Gamma(1/3)} \\ a_2 &= 0 \\ a_{n+3} &= \frac{a_n}{(n+3)(n+2)} \end{aligned}$$

For $n = 9$ we obtain

$$\begin{aligned} \text{Ai}(x) &\approx 2.73941 \times 10^{-5} x^9 - 0.000513531 x^7 + 0.00197238 x^6 \\ &\quad - 0.0215683 x^4 + 0.0591713 x^3 - 0.258819 x + 0.355028 \end{aligned}$$

Now calculate ψ_1 and ψ_2 using the $\text{Ai}(x)$ approximation.

Calculate departures from equality of the Schrödinger equations for ψ_1 and ψ_2 .

$$\begin{aligned} \epsilon_1 &= -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} \psi_1 + \mu_B(B_0 + \alpha z)\psi_1 - i\hbar \frac{\partial}{\partial t} \psi_1 \\ \epsilon_2 &= -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} \psi_2 - \mu_B(B_0 + \alpha z)\psi_2 - i\hbar \frac{\partial}{\partial t} \psi_2 \end{aligned}$$

After calculating ϵ_1 and ϵ_2 , cancel exponentials and zero out factors t^n and z^m where $n \geq 8$ and $m \geq 5$.

The results are

$$\begin{aligned}\epsilon_1 = & \boxed{2.1684 \times 10^{-19}} \frac{\alpha^{16/3} \mu_B^{16/3} t^6 z^2}{\hbar^{8/3} m^{5/3}} + \boxed{2.1684 \times 10^{-19}} \frac{\alpha^5 \mu_B^5 t^6 z}{\hbar^2 m^2} \\ & + \boxed{8.67362 \times 10^{-19}} \frac{\alpha^{13/3} \mu_B^{13/3} t^4 z^3}{\hbar^{8/3} m^{2/3}} + \boxed{1.73472 \times 10^{-18}} \frac{\alpha^4 \mu_B^4 t^4 z^2}{\hbar^2 m} \\ & - \boxed{1.73472 \times 10^{-18}} \frac{\alpha^{10/3} \mu_B^{10/3} t^4}{\hbar^{2/3} m^{5/3}} - \boxed{1.38778 \times 10^{-17}} \frac{\alpha^{7/3} \mu_B^{7/3} t^2 z}{\hbar^{2/3} m^{2/3}} \\ & - \boxed{2.77556 \times 10^{-17}} \frac{\alpha^{4/3} m^{1/3} \mu_B^{4/3} z^2}{\hbar^{2/3}} + \boxed{8.67362 \times 10^{-19}} \frac{i \alpha^4 \mu_B^4 t^3 z^4}{\hbar^3}\end{aligned}$$

and

$$\begin{aligned}\epsilon_2 = & \boxed{2.1684 \times 10^{-19}} \frac{\alpha^{16/3} \mu_B^{16/3} t^6 z^2}{\hbar^{8/3} m^{5/3}} - \boxed{2.1684 \times 10^{-19}} \frac{\alpha^5 \mu_B^5 t^6 z}{\hbar^2 m^2} \\ & - \boxed{8.67362 \times 10^{-19}} \frac{\alpha^{13/3} \mu_B^{13/3} t^4 z^3}{\hbar^{8/3} m^{2/3}} + \boxed{1.73472 \times 10^{-18}} \frac{\alpha^4 \mu_B^4 t^4 z^2}{\hbar^2 m} \\ & - \boxed{1.73472 \times 10^{-18}} \frac{\alpha^{10/3} \mu_B^{10/3} t^4}{\hbar^{2/3} m^{5/3}} + \boxed{1.38778 \times 10^{-17}} \frac{\alpha^{7/3} \mu_B^{7/3} t^2 z}{\hbar^{2/3} m^{2/3}} \\ & - \boxed{2.77556 \times 10^{-17}} \frac{\alpha^{4/3} m^{1/3} \mu_B^{4/3} z^2}{\hbar^{2/3}} + \boxed{8.67362 \times 10^{-19}} \frac{i \alpha^4 \mu_B^4 t^3 z^4}{\hbar^3}\end{aligned}$$

All of the numerical values are small relative to the coefficients in $\text{Ai}(x)$ hence ψ_1 and ψ_2 are confirmed as solutions.