Cumulative distribution for hydrogen atom

Start with the ground state wave function.

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a_0^3}} \exp\left(-\frac{r}{a_0}\right)$$

The cumulative distribution function $\Pr(r < a)$ is obtained by integrating $|\psi_{100}|^2$ over the volume element $r^2 \sin\theta \, dr \, d\theta \, d\phi$.

$$\Pr(r < a) = \frac{1}{\pi a_0^3} \int_0^a \int_0^\pi \int_0^{2\pi} \exp\left(-\frac{2r}{a_0}\right) r^2 \sin\theta \, dr \, d\theta \, d\phi$$

Integrate over ϕ (multiply by 2π).

$$\Pr(r < a) = \frac{2}{a_0^3} \int_0^a \int_0^\pi \exp\left(-\frac{2r}{a_0}\right) r^2 \sin\theta \, dr \, d\theta$$

Transform the integral over θ to an integral over y where $y = \cos\theta$ and $dy = -\sin\theta \, d\theta$. The minus sign in dy is canceled by interchanging integration limits $\cos 0 = 1$ and $\cos\pi = -1$.

$$\Pr(r < a) = \frac{2}{a_0^3} \int_0^a \int_{-1}^1 \exp\left(-\frac{2r}{a_0}\right) r^2 \, dr \, dy$$

Integrate over y (multiply by 2).

$$\Pr(r < a) = \frac{4}{a_0^3} \int_0^a \exp\left(-\frac{2r}{a_0}\right) r^2 \, dr$$

Solve the integral over r.

$$\Pr(r < a) = 1 - \left(\frac{2a^2}{a_0^2} + \frac{2a}{a_0} + 1\right) \exp\left(-\frac{2a}{a_0}\right) \tag{1}$$

For $a = a_0$ we have

$$\Pr(r < a_0) = 0.32$$

Hence the probability of finding the electron inside the Bohr radius is 32%.