

Coherent state

Let $|\Psi\rangle$ be the following “coherent” state where \bar{n} is the mean number of photons and $|n\rangle$ is the state with exactly n photons.

$$|\Psi\rangle = \sum_{n=0}^{\infty} \sqrt{\frac{\bar{n}^n \exp(-\bar{n})}{n!}} \exp\left(-i\left(n + \frac{1}{2}\right)\omega t\right) |n\rangle$$

Let \hat{a} be the following “lowering” operator.

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

Apply operator \hat{a} to coherent state $|\Psi\rangle$ to obtain (see derivation below)

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}} \exp(-i\omega t) |\Psi\rangle$$

and

$$\langle\Psi|\hat{a}^\dagger = (\hat{a}|\Psi\rangle)^\dagger = \sqrt{\bar{n}} \exp(i\omega t) \langle\Psi|$$

Let \hat{E} be the following electric field operator.

$$\hat{E} = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}(\hat{a} - \hat{a}^\dagger)$$

Note that \hat{E} is Hermitian.

$$\hat{E} = \hat{E}^\dagger$$

Hermitian operators have real eigenvalues, hence \hat{E} corresponds to an observable quantity.

The expected electric field is

$$\langle\hat{E}\rangle = \langle\Psi|\hat{E}|\Psi\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}} \langle\Psi|(\hat{a} - \hat{a}^\dagger)|\Psi\rangle$$

By distributive law

$$\langle\hat{E}\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}} (\langle\Psi|\hat{a}|\Psi\rangle - \langle\Psi|\hat{a}^\dagger|\Psi\rangle)$$

Substitute eigenvalues for operators.

$$\langle\hat{E}\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}} (\sqrt{\bar{n}} \exp(-i\omega t) \langle\Psi|\Psi\rangle - \sqrt{\bar{n}} \exp(i\omega t) \langle\Psi|\Psi\rangle)$$

By $\langle\Psi|\Psi\rangle = 1$ we have

$$\langle\hat{E}\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}} (\sqrt{\bar{n}} \exp(-i\omega t) - \sqrt{\bar{n}} \exp(i\omega t))$$

Recalling that

$$2 \sin(\omega t) = i \exp(-i\omega t) - i \exp(i\omega t)$$

we have

$$\langle \hat{E} \rangle = \sqrt{\frac{2\bar{n}\hbar\omega}{\epsilon_0}} \sin(\omega t)$$

Let \hat{B} be the following magnetic field operator.

$$\hat{B} = \sqrt{\frac{\hbar\omega\mu_0}{2}}(\hat{a} + \hat{a}^\dagger)$$

By deduction similar to that for $\langle \hat{E} \rangle$ we obtain

$$\langle \hat{B} \rangle = \sqrt{2\bar{n}\hbar\omega\mu_0} \cos(\omega t)$$

The energy of an electromagnetic wave is

$$U = \frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{1}{2\mu_0} |\mathbf{B}|^2$$

For linear polarization there exists a rotation matrix R such that

$$R\mathbf{E} = \begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix}, \quad R\mathbf{B} = \begin{pmatrix} 0 \\ B \\ 0 \end{pmatrix}$$

Hence in the rotated frame

$$U = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2$$

For a quantum field we have

$$U = \frac{\epsilon_0}{2} \langle \hat{E}^2 \rangle + \frac{1}{2\mu_0} \langle \hat{B}^2 \rangle$$

where

$$\begin{aligned} \langle \hat{E}^2 \rangle &= \langle \Psi | \hat{E} \hat{E} | \Psi \rangle = -\frac{\hbar\omega}{2\epsilon_0} \langle \Psi | (\hat{a} - \hat{a}^\dagger)(\hat{a} - \hat{a}^\dagger) | \Psi \rangle \\ \langle \hat{B}^2 \rangle &= \langle \Psi | \hat{B} \hat{B} | \Psi \rangle = \frac{\hbar\omega\mu_0}{2} \langle \Psi | (\hat{a} + \hat{a}^\dagger)(\hat{a} + \hat{a}^\dagger) | \Psi \rangle \end{aligned}$$

For the coherent state

$$\begin{aligned} \langle \Psi | \hat{a} \hat{a} | \Psi \rangle &= (\sqrt{\bar{n}} \exp(-i\omega t))^2 &= \bar{n} \exp(-2i\omega t) \\ \langle \Psi | \hat{a}^\dagger \hat{a} | \Psi \rangle &= (\sqrt{\bar{n}} \exp(i\omega t)) (\sqrt{\bar{n}} \exp(-i\omega t)) &= \bar{n} \\ \langle \Psi | \hat{a} \hat{a}^\dagger | \Psi \rangle &= \langle \Psi | (\hat{a}^\dagger \hat{a} + 1) | \Psi \rangle = \langle \Psi | \hat{a}^\dagger \hat{a} | \Psi \rangle + \langle \Psi | \Psi \rangle &= \bar{n} + 1 \\ \langle \Psi | \hat{a}^\dagger \hat{a}^\dagger | \Psi \rangle &= (\sqrt{\bar{n}} \exp(i\omega t))^2 &= \bar{n} \exp(2i\omega t) \end{aligned}$$

The expectation $\bar{n} + 1$ for $\hat{a} \hat{a}^\dagger$ is from the commutator

$$\hat{a} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} = 1$$

Using the expectation values derived above we now have

$$\begin{aligned}\langle \hat{E}^2 \rangle &= -\frac{\hbar\omega}{2\epsilon_0} (\bar{n} \exp(-2i\omega t) + \bar{n} \exp(2i\omega t) - 2\bar{n} - 1) \\ \langle \hat{B}^2 \rangle &= \frac{\hbar\omega\mu_0}{2} (\bar{n} \exp(-2i\omega t) + \bar{n} \exp(2i\omega t) + 2\bar{n} + 1)\end{aligned}$$

Noting that

$$\begin{aligned}-4 \sin^2(\omega t) &= \exp(-2i\omega t) + \exp(2i\omega t) - 2 \\ 4 \cos^2(\omega t) &= \exp(-2i\omega t) + \exp(2i\omega t) + 2\end{aligned}$$

we have

$$\begin{aligned}\langle \hat{E}^2 \rangle &= -\frac{\hbar\omega}{2\epsilon_0} (-4\bar{n} \sin^2(\omega t) - 1) \\ \langle \hat{B}^2 \rangle &= \frac{\hbar\omega\mu_0}{2} (4\bar{n} \cos^2(\omega t) + 1)\end{aligned}$$

Rewrite as

$$\frac{\epsilon_0}{2} \langle \hat{E}^2 \rangle = \hbar\omega \left(\bar{n} \sin^2(\omega t) + \frac{1}{4} \right) \quad (1)$$

$$\frac{1}{2\mu_0} \langle \hat{B}^2 \rangle = \hbar\omega \left(\bar{n} \cos^2(\omega t) + \frac{1}{4} \right) \quad (2)$$

The total energy per unit volume is the sum of (1) and (2).

$$U = \frac{\epsilon_0}{2} \langle \hat{E}^2 \rangle + \frac{1}{2\mu_0} \langle \hat{B}^2 \rangle = \hbar\omega \left(\bar{n} + \frac{1}{2} \right)$$

Check units.

$$\hbar\omega = h\nu \propto \text{joule second} \times \frac{1}{\text{second}} = \text{joule}$$

We will now show that

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}} \exp(-i\omega t)|\Psi\rangle$$

Let

$$c_n = \sqrt{\frac{\bar{n}^n \exp(-\bar{n})}{n!}} \exp\left(-i\left(n + \frac{1}{2}\right)\omega t\right)$$

It follows that

$$c_n = \sqrt{\frac{\bar{n}}{n}} \exp(-i\omega t) c_{n-1}$$

Hence

$$\hat{a}(c_n|n\rangle) = c_n \sqrt{n} |n-1\rangle = \sqrt{\bar{n}} \exp(-i\omega t) c_{n-1} |n-1\rangle \quad (3)$$

Noting that $\hat{a}|0\rangle = 0$ we can write the summation starting from $n = 1$.

$$\hat{a}|\Psi\rangle = \hat{a} \sum_{n=1}^{\infty} c_n |n\rangle$$

By equation (3) we have

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}} \exp(-i\omega t) \sum_{n=1}^{\infty} c_{n-1} |n-1\rangle = \sqrt{\bar{n}} \exp(-i\omega t) |\Psi\rangle$$