Harmonic oscillator coherent state

A coherent state minimizes uncertainty. The ground state is a coherent state. To make the ground state more interesting, parameters r and θ are added to shift $\langle x\rangle$ and $\langle p\rangle$ from zero. Shifting $\langle x\rangle$ and $\langle p\rangle$ makes the state a function of time t.

$$\psi_{n,r,\theta}(x,t) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,(x-\langle x\rangle)\right)$$
$$\times\exp\left[-\frac{m\omega}{2\hbar}\,(x-\langle x\rangle)^2+\frac{i}{\hbar}\langle p\rangle\left(x-\frac{\langle x\rangle}{2}\right)-i\left(n+\frac{1}{2}\right)\omega t\right]$$

Parameters r and θ are polar coordinates in phase space such that

$$\langle x\rangle = \sqrt{\frac{2\hbar}{m\omega}}\,r\cos(\omega t+\theta),\quad \langle p\rangle = -\sqrt{2m\hbar\omega}\,r\sin(\omega t+\theta)$$

Note that $\psi_{0,0,\theta}(x,0)$ is equivalent to the ordinary ground state.

$$\psi_{0,0,\theta}(x,0) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

Exercises

1. Verify

$$i\hbar\frac{d}{dt}\psi(x,t) = \hat{H}\psi(x,t)$$

2. Verify

$$\int_{-\infty}^{\infty}\psi^*(x,t)\psi(x,t)\,dx = 1$$

3. For the ground state $\psi_{0,r,\theta}(x,t)$, verify that

$$\Delta x = \sqrt{\langle x^2\rangle-\langle x\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}}$$

and

$$\Delta p = \sqrt{\langle p^2\rangle-\langle p\rangle^2} = \sqrt{\frac{m\hbar\omega}{2}}$$

Hence $\Delta x\Delta p$ is the minimum allowed by the uncertainty principle.

$$\Delta x\Delta p = \frac{\hbar}{2}$$