Line integral

There are two kinds of line integrals, one for scalar fields and one for vector fields. The following table shows how both are based on the calculation of arc length.

	Abstract form	Computable form
Arc length	$\int_C ds$	$\int_a^b \lvert g'(t)\rvert \, dt$
Line integral, scalar field	$\int_C f \, ds$	$\int_a^b f(g(t)) \, \lvert g'(t)\rvert \, dt$
Line integral, vector field	$\int_C (F \cdot u) \, ds$	$\int_a^b F(g(t)) \cdot g'(t) \, dt$

Note that for the measure ds we have

$$ds = |g'(t)| \, dt$$

For vector fields, symbol u is the unit tangent vector

$$u = \frac{g'(t)}{|g'(t)|}$$

Note that u cancels with ds as follows.

$$\int_C (F \cdot u) \, ds = \int_a^b \left(F(g(t)) \cdot \frac{g'(t)}{|g'(t)|} \right) |g'(t)| \, dt = \int_a^b F(g(t)) \cdot g'(t) \, dt$$

Example 1. Evaluate

$$\int_C x \, ds \quad \text{and} \quad \int_C x \, dx$$

where C is a straight line from $(0,0)$ to $(1,1)$.

Although the integrals appear similar, the first is over a scalar field and the second is over a vector field.

For $\int_C x \, ds$ we have

```
x = t
y = t
g = (x,y)
defint(x abs(d(g,t)), t, 0, 1)
```

$$\frac{1}{2^{1/2}}$$

For $\int_C x\,dx$ we have $x\,dx = (F \cdot u)\,ds$ hence

```
x = t
y = t
g = (x,y)
F = (x,0)
defint(dot(F,d(g,t)), t, 0, 1)
```

$\frac{1}{2}$

The following line integral problems are from *Advanced Calculus, Fifth Edition* by Wilfred Kaplan.

Example 2. Evaluate $\int y^2\,dx$ along the straight line from $(0,0)$ to $(2,2)$.

The following solution parametrizes x and y so that the endpoint $(2,2)$ corresponds to $t = 1$.

```
x = 2 t
y = 2 t
g = (x,y)
F = (y^2,0)
defint(dot(F,d(g,t)), t, 0, 1)
```

$\frac{8}{3}$

Example 3. Evaluate $\int z\,dx + x\,dy + y\,dz$ along the path $x = 2t+1$, $y = t^2$, $z = 1 + t^3$, $0 \le t \le 1$.

```
x = 2 t + 1
y = t^2
z = 1 + t^3
g = (x,y,z)
F = (z,x,y)
defint(dot(F,d(g,t)), t, 0, 1)
```

$\frac{163}{30}$