

Muon decay

The following muon data is from Particle Data Group.

μ MEAN LIFE τ

Measurements with an error $> 0.001 \times 10^{-6}$ s have been omitted.

VALUE (10^{-6} s)	DOCUMENT ID	TECN	CHG	COMMENT
2.1969811\pm0.0000022 OUR AVERAGE				
2.1969803 \pm 0.0000021 \pm 0.0000007 ¹	TISHCHENKO 13	CNTR	+	Surface μ^+ at PSI
2.197083 \pm 0.000032 \pm 0.000015	BARCZYK 08	CNTR	+	Muons from π^+ decay at rest
2.197013 \pm 0.000021 \pm 0.000011	CHITWOOD 07	CNTR	+	Surface μ^+ at PSI
2.197078 \pm 0.000073	BARDIN 84	CNTR	+	
2.197025 \pm 0.000155	BARDIN 84	CNTR	–	
2.19695 \pm 0.00006	GIOVANETTI 84	CNTR	+	
2.19711 \pm 0.00008	BALANDIN 74	CNTR	+	
2.1973 \pm 0.0003	DUCLOS 73	CNTR	+	
• • • We do not use the following data for averages, fits, limits, etc. • • •				
2.1969803 \pm 0.0000022	WEBBER 11	CNTR	+	Surface μ^+ at PSI
¹ TISHCHENKO 13 uses 1.6×10^{12} μ^+ events and supersedes WEBBER 11.				

From “V minus A” theory we have the following formula for muon lifetime τ .

$$\tau = \frac{96\pi^2 h}{G_F^2 (m_\mu c^2)^5}$$

Symbol G_F is Fermi coupling constant, m_μ is muon mass.

From NIST we have

$$G_F = 1.1663787 \times 10^{-5} \text{ GeV}^{-2}$$

$$m_\mu = 1.883531627 \times 10^{-28} \text{ kilogram}$$

$$h = 6.62607015 \times 10^{-34} \text{ joule second (exact)}$$

$$c = 299792458 \text{ meter second}^{-1} \text{ (exact)}$$

$$1 \text{ eV} = 1.602176634 \times 10^{-19} \text{ joule (exact)}$$

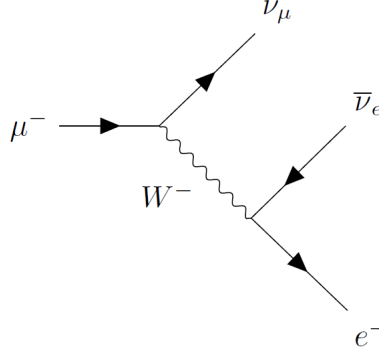
Hence

$$\tau = \frac{96\pi^2 h}{G_F^2 (m_\mu c^2)^5} = 2.18735 \times 10^{-6} \text{ second}$$

The result is a bit smaller than the observed value from Particle Data Group.

$$\frac{\tau}{\text{observed value}} = \frac{2.18735 \times 10^{-6} \text{ second}}{2.19698 \times 10^{-6} \text{ second}} = 0.9956$$

As the following diagram shows, a muon decays into a muon neutrino, an electron anti-neutrino, and an electron.



Particle	Symbol	Momentum	Spinor (up)	Spinor (down)
Muon	μ^-	p_1	u_{11}	u_{12}
Muon neutrino	ν_μ	p_2	u_{21}	u_{22}
Electron anti-neutrino	$\bar{\nu}_e$	p_3	v_{31}	v_{32}
Electron	e^-	p_4	u_{41}	u_{42}

We have the following momentum vectors for the muon decay process.

$$p_1 = \begin{pmatrix} E_1 \\ p_{1x} \\ p_{1y} \\ p_{1z} \end{pmatrix}_{\mu^-} \quad p_2 = \begin{pmatrix} E_2 \\ p_{2x} \\ p_{2y} \\ p_{2z} \end{pmatrix}_{\nu_\mu} \quad p_3 = \begin{pmatrix} E_3 \\ p_{3x} \\ p_{3y} \\ p_{3z} \end{pmatrix}_{\bar{\nu}_e} \quad p_4 = \begin{pmatrix} E_4 \\ p_{4x} \\ p_{4y} \\ p_{4z} \end{pmatrix}_{e^-}$$

Spinors for the muon.

$$u_{11} = \frac{1}{\sqrt{E_1 + m_1}} \begin{pmatrix} E_1 + m_1 \\ 0 \\ p_{1z} \\ p_{1x} + ip_{1y} \end{pmatrix}_{\mu^- \text{ spin up}} \quad u_{12} = \frac{1}{\sqrt{E_1 + m_1}} \begin{pmatrix} 0 \\ E_1 + m_1 \\ p_{1x} - ip_{1y} \\ -p_{1z} \end{pmatrix}_{\mu^- \text{ spin down}}$$

Spinors for the muon neutrino.

$$u_{21} = \frac{1}{\sqrt{E_2 + m_2}} \begin{pmatrix} E_2 + m_2 \\ 0 \\ p_{2z} \\ p_{2x} + ip_{2y} \end{pmatrix}_{\nu_\mu \text{ spin up}} \quad u_{22} = \frac{1}{\sqrt{E_2 + m_2}} \begin{pmatrix} 0 \\ E_2 + m_2 \\ p_{2x} - ip_{2y} \\ -p_{2z} \end{pmatrix}_{\nu_\mu \text{ spin down}}$$

Spinors for the electron anti-neutrino.

$$v_{31} = \frac{1}{\sqrt{E_3 + m_3}} \begin{pmatrix} p_{3z} \\ p_{3x} + ip_{3y} \\ E_3 + m_3 \\ 0 \end{pmatrix}_{\bar{\nu}_e \text{ spin up}} \quad v_{32} = \frac{1}{\sqrt{E_3 + m_3}} \begin{pmatrix} p_{3x} - ip_{3y} \\ -p_{3z} \\ 0 \\ E_3 + m_3 \end{pmatrix}_{\bar{\nu}_e \text{ spin down}}$$

Spinors for the electron.

$$u_{41} = \frac{1}{\sqrt{E_4 + m_4}} \begin{pmatrix} E_4 + m_4 \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix} \quad u_{42} = \frac{1}{\sqrt{E_4 + m_4}} \begin{pmatrix} 0 \\ E_4 + m_4 \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}$$

e^- spin up e^- spin down

The probability amplitude \mathcal{M}_{abcd} for spin state $abcd$ is

$$\mathcal{M}_{abcd} = \frac{G_F}{\sqrt{2}} (\bar{u}_{4d} \gamma^\mu (1 - \gamma^5) v_{3c}) (\bar{u}_{2b} \gamma_\mu (1 - \gamma^5) u_{1a})$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is the average of spin states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{2} \sum_{a=1}^2 \sum_{b=1}^2 \sum_{c=1}^2 \sum_{d=1}^2 |\mathcal{M}_{abcd}|^2$$

The Casimir trick uses matrix arithmetic to sum over spin states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{G_F^2}{4} \text{Tr} \left(\not{p}_4 \gamma^\mu (1 - \gamma^5) \not{p}_3 \gamma^\nu (1 - \gamma^5) \right) \text{Tr} \left(\not{p}_2 \gamma_\mu (1 - \gamma^5) \not{p}_1 \gamma_\nu (1 - \gamma^5) \right)$$

The result is a simple formula.

$$\langle |\mathcal{M}|^2 \rangle = 64 G_F^2 (p_1 \cdot p_3) (p_2 \cdot p_4) \quad (1)$$

In component notation

$$\langle |\mathcal{M}|^2 \rangle = 64 G_F^2 (p_1^\mu g_{\mu\nu} p_3^\nu) (p_2^\rho g_{\rho\sigma} p_4^\sigma)$$

where

$$g_{\mu\nu} = g_{\rho\sigma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

In the muon rest frame p_1 is fixed at $p_1 = (m_\mu, 0, 0, 0)$. The remaining momenta p_2 , p_3 , and p_4 can have any values that conserve energy and momentum. Muon decay rate Γ is the expectation value for all possible decay momenta. By Fermi's golden rule

$$\Gamma = \frac{1}{512\pi^5 m_\mu} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \langle |\mathcal{M}|^2 \rangle \delta(p_1 - p_2 - p_3 - p_4) \frac{d^3 p_2}{E_2} \frac{d^3 p_3}{E_3} \frac{d^3 p_4}{E_4}$$

Altogether there are nine integrals, three for each of p_2 , p_3 , and p_4 . The delta function restricts the integration space to values that conserve energy and momentum.

It can be shown that

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

Muon lifetime τ is the inverse of decay rate.

$$\tau = \frac{1}{\Gamma} = \frac{192\pi^3}{G_F^2 m_\mu^5}$$

Change natural units to \hbar and c .

$$\tau = \frac{96\pi^2 \hbar}{G_F^2 (m_\mu c^2)^5}$$