Let A_{nm} be the transition rate for the process $\psi_n \to \psi_m$ where $E_n > E_m$. From Heisenberg we have

$$A_{nm} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3}\,\omega_{nm}^3\,|\langle r_{nm}\rangle|^2$$

Bohr's frequency condition gives

$$\omega_{nm} = \frac{1}{\hbar}(E_n - E_m)$$

The radial density is

$$|\langle r_{nm}\rangle|^2 = |\langle x_{nm}\rangle|^2 + |\langle y_{nm}\rangle|^2 + |\langle z_{nm}\rangle|^2$$

where

$$\langle x_{nm}\rangle = \int \psi_m^*\,(r\sin\theta\cos\phi)\,\psi_n\,dV$$

$$\langle y_{nm}\rangle = \int \psi_m^*\,(r\sin\theta\sin\phi)\,\psi_n\,dV$$

$$\langle z_{nm}\rangle = \int \psi_m^*\,(r\cos\theta)\,\psi_n\,dV$$

Let us compute A_{21} for hydrogen. The energy levels for hydrogen are

$$E_n = -\frac{\mu}{2n^2}\left(\frac{e^2}{4\pi\varepsilon_0\hbar}\right)^2$$

where μ is reduced electron mass.

For $n = 2$ there are four eigenstates.

n	ℓ	m_ℓ
2	1	1
2	1	-1
2	1	0
2	0	0

The following table shows the radial density for every possible transition.

	$\psi_{2,1,1} \to \psi_{1,0,0}$	$\psi_{2,1,-1} \to \psi_{1,0,0}$	$\psi_{2,1,0} \to \psi_{1,0,0}$	$\psi_{2,0,0} \to \psi_{1,0,0}$
$\langle x_{21}\rangle =$	$-\frac{128}{243}\,a_0$	$\frac{128}{243}\,a_0$	0	0
$\langle y_{21}\rangle =$	$-\frac{128}{243}i\,a_0$	$-\frac{128}{243}i\,a_0$	0	0
$\langle z_{21}\rangle =$	0	0	$\frac{128}{243}\sqrt{2}\,a_0$	0
$\|\langle r_{21}\rangle\|^2 =$	$\frac{32768}{59049}\,a_0^2$	$\frac{32768}{59049}\,a_0^2$	$\frac{32768}{59049}\,a_0^2$	0

Note that the transition rate of $\psi_{2,0,0} \to \psi_{1,0,0}$ is zero. For the allowed transitions, the radial density $|\langle r_{21}\rangle|^2$ is independent of m_ℓ.

This is the Bohr radius for reduced electron mass μ.

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{e^2\mu} = 5.29 \times 10^{-11}\,\text{meter}$$

For the transition frequency we have

$$\omega_{21} = \frac{1}{\hbar}(E_2 - E_1) = 1.55 \times 10^{16}\,\text{second}^{-1}$$

Hence

$$A_{21} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3} \times \omega_{21}^3 \times \tfrac{32768}{59049}\, a_0^2 = 6.26 \times 10^8\,\text{second}^{-1}$$

It is interesting to work out A_{nm} symbolically and see how high the powers get.

$$A_{21} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3} \times \underbrace{\left(\frac{3e^4\mu}{128\pi^2\varepsilon_0^2\hbar^3}\right)^3}_{\omega_{21}^3} \times \underbrace{\frac{32768}{59049}\left(\frac{4\pi\varepsilon_0\hbar^2}{e^2\mu}\right)^2}_{|\langle r_{21}\rangle|^2} = \frac{e^{10}\mu}{26244\,\pi^5\varepsilon_0^5\hbar^6 c^3}$$

The parameters $n = 2$ and $m = 1$ contribute the following numerical factor to A_{21}.

$$\underbrace{\left(-\frac{1}{2^2} + \frac{1}{1^2}\right)^3}_{\text{from } (E_2 - E_1)^3} \times \underbrace{\frac{32768}{59049}}_{\text{from } |\langle r_{21}\rangle|^2} = \frac{512}{2187} = \frac{2^9}{3^7}$$

Multiplying out numerical factors yields the numerical factor shown above in A_{21}.

$$\frac{1}{3} \times \underbrace{\left(\frac{1}{32}\right)^3}_{\text{from } (E_n - E_m)^3} \times \underbrace{4^2}_{\text{from } a_0^2} \times \frac{512}{2187} = \frac{1}{26244} = \frac{1}{2^2 3^8}$$

Let us analyze the units involved in computing A_{nm}. For the coefficient of A_{nm} we have

$$\frac{e^2}{3\pi\varepsilon_0\hbar c^3} \propto \frac{\underbrace{\text{ampere}^2\,\text{second}^2}_{e^2}}{\underbrace{\left(\frac{\text{ampere}^2\,\text{second}^4}{\text{kilogram meter}^3}\right)}_{\varepsilon_0}\underbrace{\left(\frac{\text{kilogram meter}^2}{\text{second}}\right)}_{\hbar}\underbrace{\left(\frac{\text{meter}^3}{\text{second}^3}\right)}_{c^3}} = \frac{\text{second}^2}{\text{meter}^2}$$

For the transition frequency we have

$$\omega_{21} = \frac{3e^4\mu}{128\pi^2\varepsilon_0^2\hbar^3} \propto \frac{\underbrace{\left(\text{ampere}^4\,\text{second}^4\right)}_{e^4}\,\underbrace{\text{kilogram}}_{\mu}}{\underbrace{\left(\frac{\text{ampere}^4\,\text{second}^8}{\text{kilogram}^2\,\text{meter}^6}\right)}_{\varepsilon_0^2}\underbrace{\left(\frac{\text{kilogram}^3\,\text{meter}^6}{\text{second}^3}\right)}_{\hbar^3}} = \text{second}^{-1}$$

For the Bohr radius we have

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{e^2\mu} \propto \frac{\underbrace{\left(\frac{\text{ampere}^2\,\text{second}^4}{\text{kilogram meter}^3}\right)}_{\varepsilon_0}\underbrace{\left(\frac{\text{kilogram}^2\,\text{meter}^4}{\text{second}^2}\right)}_{\hbar^2}}{\underbrace{\left(\text{ampere}^2\,\text{second}^2\right)}_{e^2}\,\underbrace{\text{kilogram}}_{\mu}} = \text{meter}$$

Hence

$$A_{nm} \propto \frac{\text{second}^2}{\text{meter}^2} \times \underbrace{\text{second}^{-3}}_{\omega_{nm}^3} \times \underbrace{\text{meter}^2}_{a_0^2} = \text{second}^{-1}$$