This is the transition rate A_{nm} for a spontaneous emission process $\psi_n \to \psi_m$.

$$A_{nm} = \frac{e^2}{3\pi\varepsilon_0 \hbar c^3}\, \omega_{nm}^3 \, |\langle r_{nm}\rangle|^2$$

The transition frequency is

$$\omega_{nm} = \frac{1}{\hbar}(E_n - E_m)$$

For a hydrogen atom we have

$$E_n = -\frac{\mu}{2n^2}\left(\frac{e^2}{4\pi\varepsilon_0 \hbar}\right)^2$$

where μ is reduced electron mass.

The radial density is

$$|\langle r_{nm}\rangle|^2 = |\langle x_{nm}\rangle|^2 + |\langle y_{nm}\rangle|^2 + |\langle z_{nm}\rangle|^2$$

where

$$\langle x_{nm}\rangle = \int \psi_m^* \left(r \sin\theta \cos\phi\right) \psi_n \, dV$$

$$\langle y_{nm}\rangle = \int \psi_m^* \left(r \sin\theta \sin\phi\right) \psi_n \, dV$$

$$\langle z_{nm}\rangle = \int \psi_m^* \left(r \cos\theta\right) \psi_n \, dV$$

Let us compute A_{21} for a hydrogen atom. For $n = 2$ there are four possible states.

n	ℓ	m
2	1	1
2	1	−1
2	1	0
2	0	0

The following table shows the radial density for every possible transition.

	$\psi_{2,1,1} \to \psi_{1,0,0}$	$\psi_{2,1,-1} \to \psi_{1,0,0}$	$\psi_{2,1,0} \to \psi_{1,0,0}$	$\psi_{2,0,0} \to \psi_{1,0,0}$
$\langle x_{21}\rangle =$	$-\frac{128}{243}\, a_0$	$\frac{128}{243}\, a_0$	0	0
$\langle y_{21}\rangle =$	$-\frac{128}{243}i\, a_0$	$-\frac{128}{243}i\, a_0$	0	0
$\langle z_{21}\rangle =$	0	0	$\frac{128}{243}\sqrt{2}\, a_0$	0
$|\langle r_{21}\rangle|^2 =$	$\frac{32768}{59049}\, a_0^2$	$\frac{32768}{59049}\, a_0^2$	$\frac{32768}{59049}\, a_0^2$	0

Note that the transition rate of $\psi_{2,0,0} \to \psi_{1,0,0}$ is zero. For the allowed transitions, the radial density $|\langle r_{21}\rangle|^2$ is independent of ℓ and m.

Symbol a_0 is the Bohr radius

$$a_0 = \frac{4\pi\varepsilon_0 \hbar^2}{e^2 \mu}$$

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For the transition frequency we have

$$\omega_{21} = \frac{1}{\hbar}(E_2 - E_1) = \frac{3e^4\mu}{128\pi^2\varepsilon_0^2\hbar^3}$$

Hence

$$A_{21} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3}\underbrace{\left(\frac{3e^4\mu}{128\pi^2\varepsilon_0^2\hbar^3}\right)^3}_{\omega_{21}}\underbrace{\frac{32768}{59049}\left(\frac{4\pi\varepsilon_0\hbar^2}{e^2\mu}\right)^2}_{|\langle r_{21}\rangle|^2} = \frac{e^{10}\mu}{26244\pi^5\varepsilon_0^5\hbar^6 c^3} = 6.27\times10^8\ \text{second}^{-1}$$

Let us analyze the units involved. For the coefficient of A_{nm} we have

$$\frac{e^2}{3\pi\varepsilon_0\hbar c^3} \propto \frac{\overbrace{\text{ampere}^2\,\text{second}^2}^{e^2}}{\underbrace{\left(\frac{\text{ampere}^2\,\text{second}^4}{\text{kilogram}\,\text{meter}^3}\right)}_{\varepsilon_0}\underbrace{\left(\frac{\text{kilogram}\,\text{meter}^2}{\text{second}}\right)}_{\hbar}\underbrace{\left(\frac{\text{meter}^3}{\text{second}^3}\right)}_{c^3}} = \frac{\text{second}^2}{\text{meter}^2}$$

For the transition frequency we have

$$\omega_{21} = \frac{3e^4\mu}{128\pi^2\varepsilon_0^2\hbar^3} \propto \frac{\overbrace{\left(\text{ampere}^4\,\text{second}^4\right)}^{e^4}\overbrace{\text{kilogram}}^{\mu}}{\underbrace{\left(\frac{\text{ampere}^4\,\text{second}^8}{\text{kilogram}^2\,\text{meter}^6}\right)}_{\varepsilon_0^2}\underbrace{\left(\frac{\text{kilogram}^3\,\text{meter}^6}{\text{second}^3}\right)}_{\hbar^3}} = \text{second}^{-1}$$

For the Bohr radius we have

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{e^2\mu} \propto \frac{\underbrace{\left(\frac{\text{ampere}^2\,\text{second}^4}{\text{kilogram}\,\text{meter}^3}\right)}_{\varepsilon_0}\underbrace{\left(\frac{\text{kilogram}^2\,\text{meter}^4}{\text{second}^2}\right)}_{\hbar^2}}{\underbrace{\left(\text{ampere}^2\,\text{second}^2\right)}_{e^2}\underbrace{\text{kilogram}}_{\mu}} = \text{meter}$$

Hence

$$A_{nm} \propto \frac{\text{second}^2}{\text{meter}^2} \times \underbrace{\text{second}^{-3}}_{\omega_{nm}^3} \times \underbrace{\text{meter}^2}_{a_0^2} = \text{second}^{-1}$$