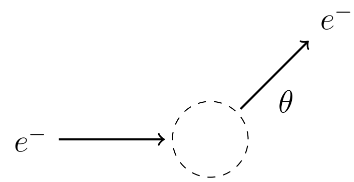
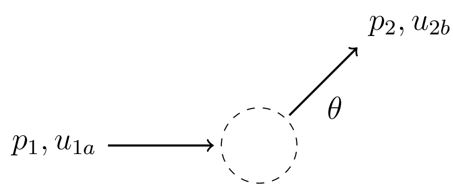
Rutherford scattering

Consider an electron scattered by an atomic nucleus.



Here is the same diagram with momentum and spinor labels.



The path of the incident electron can be modeled as the z axis, resulting in the following momentum vectors.

$$p_1 = \underbrace{\begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}}_{\text{inbound electron}} \qquad p_2 = \underbrace{\begin{pmatrix} E \\ p\sin\theta\cos\phi \\ p\sin\theta\sin\phi \\ p\cos\theta \end{pmatrix}}_{\text{outbound electron}}$$

Symbol p is incident momentum. Symbol E is total energy $E = \sqrt{p^2 + m^2}$ where m is electron mass. Polar angle θ is the observed scattering angle. Azimuth angle ϕ cancels out in scattering calculations.

The spinors are

$$u_{11} = \underbrace{\begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix}}_{\substack{\text{inbound electron} \\ \text{spin up}}} \qquad u_{21} = \underbrace{\begin{pmatrix} E+m \\ 0 \\ p_{2z} \\ p_{2x} + ip_{2y} \end{pmatrix}}_{\substack{\text{outbound electron} \\ \text{spin up}}}$$

$$u_{12} = \underbrace{\begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix}}_{\substack{\text{inbound electron} \\ \text{spin down}}} \qquad u_{22} = \underbrace{\begin{pmatrix} 0 \\ E+m \\ p_{2x} - ip_{2y} \\ -p_{2z} \end{pmatrix}}_{\substack{\text{outbound electron} \\ \text{spin down}}}$$

Spinor subscripts have 1 for spin up and 2 for spin down. The spinors are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^2$ will be used.

This is the probability density for spin state ab. Symbol Z is the atomic number of the nucleus, e is electron charge, and $q = p_1 - p_2$ is momentum transfer. The formula is derived from Feynman diagrams for Rutherford scattering.

$$|\mathcal{M}_{ab}|^2 = \frac{Z^2 e^4}{q^4} \frac{1}{N} \left| \bar{u}_{2b} \gamma^0 u_{1a} \right|^2$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is computed by summing $|\mathcal{M}_{ab}|^2$ over all four spin states and then dividing by the number of inbound states. There are two inbound states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{2} \sum_{a=1}^{2} \sum_{b=1}^{2} |\mathcal{M}_{ab}|^2$$

The Casimir trick uses matrix arithmetic to compute the sum.

$$\langle |\mathcal{M}|^2 \rangle = \frac{Z^2 e^4}{2q^4} \operatorname{Tr} \Big((\not{p}_1 + m) \gamma^0 (\not{p}_2 + m) \gamma^0 \Big)$$

The result is

$$\langle |\mathcal{M}|^2 \rangle = \frac{2 Z^2 e^4}{q^4} \left(E^2 + m^2 + p^2 \cos\theta \right)$$

Cross section

For low energy electrons such that $p \ll m$ we can use the following approximation.

$$E^2 + m^2 + p^2 \cos\theta \approx 2m^2$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{4 m^2 Z^2 e^4}{q^4}$$

Substituting $e^4 = 16\pi^2 \alpha^2$ and $q^4 = 4p^4(\cos\theta - 1)^2$ we have

$$\langle |\mathcal{M}|^2 \rangle = \frac{16\pi^2 m^2 Z^2 \alpha^2}{p^4 (\cos\theta - 1)^2}$$

The cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{16\pi^2} = \frac{m^2 Z^2 \alpha^2}{p^4 (\cos\theta - 1)^2}$$

We can integrate $d\sigma$ to obtain a cumulative distribution function. Recall that

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

Hence

$$d\sigma = \frac{m^2 Z^2 \alpha^2}{p^4 (\cos\theta - 1)^2} \sin\theta \, d\theta \, d\phi$$

Let $I(\theta)$ be the following integral of $d\sigma$.

$$I(\theta) = \int \frac{1}{(\cos\theta - 1)^2} \sin\theta \, d\theta = \frac{1}{\cos\theta - 1}$$

The cumulative distribution function is

$$F(\theta) = \frac{I(\theta) - I(a)}{I(\pi) - I(a)} = \frac{2(\cos a - \cos\theta)}{(1 + \cos a)(1 - \cos\theta)}, \quad a \le \theta \le \pi$$

Angular support is limited to an arbitrary $a > 0$ because $I(0)$ is undefined.

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

Notes

1. The original Rutherford scattering experiment in 1911 used alpha particles, not electrons. However, scattering of any charged particle by Coulomb interaction is now known as Rutherford scattering. The first Rutherford scattering experiment using electrons appears to have been done by F. L. Arnot, then a student of Rutherford, in 1929.

2. Lancaster and Blundell page 356 has

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4m^2\mathbf{v}^4 \sin^4(\theta/2)}$$

Noting that

$$\frac{1}{m^2\mathbf{v}^4} = \frac{m^2}{m^4\mathbf{v}^4} = \frac{m^2}{p^4}$$

and

$$4\sin^4(\theta/2) = (\cos\theta - 1)^2$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4m^2\mathbf{v}^4 \sin^4(\theta/2)} = \frac{m^2 Z^2\alpha^2}{p^4(\cos\theta - 1)^2}$$