Feynman and Hibbs problem 3-12

If the wave function for a harmonic oscillator (at $t=0$) is

$$\psi(x,0)=\exp\left(-\frac{m\omega}{2\hbar}(x-a)^2\right)$$

then, using equation (3.42) and the results of problem 3-8, show that

$$\begin{aligned}&\psi(x,T)=\\&\qquad\exp\left(-\frac{i\omega T}{2}-\frac{m\omega}{2\hbar}\left(x^2-2ax\exp(-i\omega T)+a^2\cos(\omega T)\exp(-i\omega T)\right)\right)\end{aligned}$$

and find the probability density $|\psi|^2$.

Adapted from equation (3.42)

$$\psi(x,T)=\int_{-\infty}^{\infty}K(x,T;x_c,0)\psi(x_c,0)\,dx_c$$

Adapted from problem 3-8

$$K=\left(\frac{m\omega}{2\pi i\hbar\sin(\omega T)}\right)^{1/2}\exp\left(\frac{im\omega}{2\hbar\sin(\omega T)}\left((x^2+x_c^2)\cos(\omega T)-2xx_c\right)\right)$$

Hence

$$\begin{aligned}\psi(x,T)=&\left(\frac{m\omega}{2\pi i\hbar\sin(\omega T)}\right)^{1/2}\\&\times\int_{-\infty}^{\infty}\exp\left(\frac{im\omega}{2\hbar\sin(\omega T)}\left((x^2+x_c^2)\cos(\omega T)-2xx_c\right)\right)\\&\qquad\times\exp\left(-\frac{m\omega}{2\hbar}(x_c-a)^2\right)\,dx_c\end{aligned}$$

Rewrite as

$$\psi(x,T)=\left(\frac{m\omega}{2\pi i\hbar\sin(\omega T)}\right)^{1/2}\int_{-\infty}^{\infty}\exp(Ax_c^2+Bx_c+C)\tag{1}$$

where

$$A = \frac{m\omega}{2\hbar}\left(\frac{i\cos(\omega T)}{\sin(\omega T)} - 1\right) \tag{2}$$

$$B = \frac{m\omega}{\hbar}\left(a - \frac{ix}{\sin(\omega T)}\right) \tag{3}$$

$$C = \frac{m\omega}{2\hbar}\left(\frac{ix^2\cos(\omega T)}{\sin(\omega T)} - a^2\right) \tag{4}$$

Solve the integral.

$$\int_{-\infty}^{\infty} \exp(Ax_c^2 + Bx_c + C) = \left(-\frac{\pi}{A}\right)^{1/2} \exp\left(-\frac{B^2}{4A} + C\right)$$

$$-\frac{\pi}{A} = -\frac{2\pi\hbar\sin(\omega T)}{im\omega\cos(\omega T) - m\omega\sin(\omega T)} \tag{5}$$

$$-\frac{B^2}{4A} + C = -\frac{m\omega}{2\hbar}\left(x^2 - 2ax\exp(-i\omega T) + a^2\cos(\omega T)\exp(-i\omega T)\right) \tag{6}$$

It can be shown that

$$\frac{m\omega}{2\pi i\hbar\sin(\omega T)} \times \left(-\frac{\pi}{A}\right) = \exp(-i\omega T) \tag{7}$$

from equation (1)

Hence from equation (1)

$$\psi(x,T) = \exp\left(-\frac{i\omega T}{2}\right) \times \exp\left(-\frac{m\omega}{2\hbar}\left(x^2 - 2ax\exp(-i\omega T) + a^2\cos(\omega T)\exp(-i\omega T)\right)\right)$$