

## How Planck calculated $h$ and $k$

Max Planck used two experimental results to calculate  $h$  and  $k$  in his 1901 paper “On the Law of Distribution of Energy in the Normal Spectrum.” Although the quantum of action  $h$  is well known as Planck’s constant, the use of  $k$  for Boltzmann’s constant is also due to Planck. In addition, Planck was the first to compute a numerical value for  $k$ .

One of the experimental results Planck used was the difference  $S_{100} - S_0$  determined by Ferdinand Kurlbaum in 1898 where  $S_t$  is the power radiated by a black body at  $t$  degrees Celsius.

$$S_{100} - S_0 = 7.31 \times 10^5 \text{ erg cm}^{-2} \text{ s}^{-1}$$

From the radiant power formula  $S_t = (t + 273)^4 \sigma$  we have

$$S_{100} - S_0 = (100 + 273)^4 \sigma - (0 + 273)^4 \sigma = (373^4 - 273^4) \sigma$$

Hence the Stefan-Boltzmann constant  $\sigma$  can be determined from  $S_{100} - S_0$ .

$$\sigma = \frac{S_{100} - S_0}{373^4 - 273^4}$$

The Stefan-Boltzmann law is the relation between energy density and temperature  $\theta$ .

$$\text{“energy per unit volume”} = \frac{4\sigma\theta^4}{c}$$

The use of  $\theta$  for temperature looks strange but that is what scientists used at the time.

Using the Stefan-Boltzmann law and Kurlbaum’s measurement, Planck calculated energy density for temperature  $\theta = 1$ .

$$\frac{4}{c} \times \frac{S_{100} - S_0}{373^4 - 273^4} = \frac{4}{3 \times 10^{10}} \times \frac{7.31 \times 10^5}{373^4 - 273^4} = 7.061 \times 10^{-15} \text{ erg cm}^{-3}$$

Planck’s 1901 paper has the following formula (Equation 12) for energy distribution  $u$  as a function of frequency  $\nu$  and temperature  $\theta$ .

$$u = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/k\theta} - 1}$$

The integral of  $u$  over all frequencies yields the total energy density  $u^*$ .

$$u^* = \int_0^\infty u d\nu = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3}{e^{h\nu/k\theta} - 1} d\nu$$

Planck used a series expansion to solve the integral for  $\theta = 1$ . However, we will use the following identity.

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$$

By the change of variable  $x = h\nu/k$  we have

$$u^* = \frac{8\pi h}{c^3} \left(\frac{k}{h}\right)^4 \frac{\pi^4}{15}$$

Planck then set  $u^*$  equal to the result from the Stefan-Boltzmann law.

$$\frac{8\pi h}{c^3} \left(\frac{k}{h}\right)^4 \frac{\pi^4}{15} = 7.061 \times 10^{-15}$$

Hence

$$\frac{k^4}{h^3} = 7.061 \times 10^{-15} \times \frac{15c^3}{8\pi^5} = 1.1682 \times 10^{15}$$

The second experimental result Planck used was  $\lambda_m\theta = 0.294$  obtained in 1900 by Otto Lummer and Ernst Pringsheim. Symbol  $\lambda_m$  is the wavelength in centimeters of peak radiant energy for a black body at temperature  $\theta$  in Kelvin.

Planck's 1901 paper has the following formula (Equation 13) for energy distribution  $E$  as a function of wavelength  $\lambda$  and temperature  $\theta$ .

$$E = \frac{8\pi ch}{\lambda^5} \frac{1}{e^{ch/k\lambda\theta} - 1}$$

Planck solves  $dE/d\lambda = 0$  to obtain  $\lambda_m$  which we will now do step by step. First, compute  $dE/d\lambda$ .

$$\frac{dE}{d\lambda} = \frac{8\pi c^2 h^2}{k\lambda^7 \theta} \frac{e^{ch/k\lambda\theta}}{(e^{ch/k\lambda\theta} - 1)^2} - \frac{40\pi ch}{\lambda^6} \frac{1}{e^{ch/k\lambda\theta} - 1}$$

Set  $dE/d\lambda = 0$  to obtain

$$\frac{8\pi c^2 h^2}{k\lambda^7 \theta} \frac{e^{ch/k\lambda\theta}}{(e^{ch/k\lambda\theta} - 1)^2} = \frac{40\pi ch}{\lambda^6} \frac{1}{e^{ch/k\lambda\theta} - 1}$$

Then by cancellation of terms

$$\frac{ch}{5k\lambda\theta} \frac{e^{ch/k\lambda\theta}}{e^{ch/k\lambda\theta} - 1} = 1$$

Multiply both sides by  $e^{ch/k\lambda\theta} - 1$ .

$$\frac{ch}{5k\lambda\theta} e^{ch/k\lambda\theta} = e^{ch/k\lambda\theta} - 1$$

Subtract  $e^{ch/k\lambda\theta}$  from both sides.

$$\left(\frac{ch}{5k\lambda\theta} - 1\right) e^{ch/k\lambda\theta} = -1$$

Multiply both sides by  $-1$  to obtain Planck's result.

$$\left(1 - \frac{ch}{5k\lambda\theta}\right) e^{ch/k\lambda\theta} = 1$$

Planck then provides the following numerical solution.

$$\frac{ch}{k\lambda\theta} = 4.9651$$

Then using  $c = 3 \times 10^{10}$  and  $\lambda\theta = 0.294$  Planck calculates

$$\frac{h}{k} = 4.9651 \times \frac{\lambda\theta}{c} = 4.9651 \times \frac{0.294}{3 \times 10^{10}} = 4.866 \times 10^{-11}$$

Planck then solves for  $k$ . Plug  $h/k = 4.866 \times 10^{-11}$  into the formula for  $k^4/h^3$  to obtain

$$k = 1.1682 \times 10^{15} \times \frac{h^3}{k^3} = 1.1682 \times 10^{15} \times (4.866 \times 10^{-11})^3 = 1.346 \times 10^{-16} \text{ erg K}^{-1}$$

Then calculate  $h$  directly from  $k$ .

$$h = k \times 4.866 \times 10^{-11} = 6.55 \times 10^{-27} \text{ erg s}$$