

## Bell's theorem

The following theorem of independent random variables is the key to understanding Bell's theorem. If two random variables  $A$  and  $B$  are independent (uncorrelated) then

$$\langle A \rangle \langle B \rangle = \langle AB \rangle$$

Consider two machines  $A$  and  $B$  that measure spin. Each machine can be set in one of two orientations labeled 0 and 1. Assuming the measurements are uncorrelated we have the following table of expectation values and a clever formula.

$\langle A_0 \rangle$	$\langle A_1 \rangle$	$\langle B_0 \rangle$	$\langle B_1 \rangle$	$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle$
1	1	1	1	2
1	1	1	-1	2
1	1	-1	1	-2
1	1	-1	-1	-2
1	-1	1	1	2
1	-1	1	-1	-2
1	-1	-1	1	2
1	-1	-1	-1	-2
-1	1	1	1	-2
-1	1	1	-1	2
-1	1	-1	1	-2
-1	1	-1	-1	2
-1	-1	1	1	-2
-1	-1	1	-1	-2
-1	-1	-1	1	2
-1	-1	-1	-1	2

Since the table is for all minimum and maximum values we have by inspection

$$-2 \leq \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2 \quad (1)$$

Now suppose a third machine generates two spins in the following entangled state.

$$|s\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

One spin is sent to  $A$  and the other is sent to  $B$ .

Let

$$A_0 = \sigma_z, \quad A_1 = \sigma_x, \quad B_0 = -\frac{\sigma_x + \sigma_z}{\sqrt{2}}, \quad B_1 = \frac{\sigma_x - \sigma_z}{\sqrt{2}}$$

Then for the entangled state  $|s\rangle$  we have

$$\langle A_0 B_0 \rangle = \frac{1}{\sqrt{2}}, \quad \langle A_0 B_1 \rangle = \frac{1}{\sqrt{2}}, \quad \langle A_1 B_0 \rangle = \frac{1}{\sqrt{2}}, \quad \langle A_1 B_1 \rangle = -\frac{1}{\sqrt{2}}$$

Hence

$$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle = 2\sqrt{2} \quad (2)$$

The result in (2) conflicts with (1) because for an entangled state the random variables are not independent. Any theory that asserts  $A$  and  $B$  are independent for all states is constrained by (1) and falsified by (2). Hence Bell's theorem that no hidden variable theory can explain entanglement.

### Exercises

1. Verify equation (2).
2. Verify that for the singlet state  $|s\rangle$  given above we have

$$\langle A_0 \rangle = 0, \quad \langle A_1 \rangle = 0, \quad \langle B_0 \rangle = 0, \quad \langle B_1 \rangle = 0.$$

Hence  $\langle A \rangle \langle B \rangle \neq \langle AB \rangle$  for the singlet state.

3. There are three additional entangled states.

$$|s_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad |s_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad |s_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Verify that  $A$  and  $B$  are correlated for all entangled states.