(11.1) One of the criteria we had for a successful theory of a scalar field was that the commutator for space-like separations would be zero. Let's see if our scalar field has this feature. Show that

$$[\hat{\phi}(x), \hat{\phi}(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \big(\exp(-ip \cdot (x-y)) - \exp(-ip \cdot (y-x)) \big) \quad (11.51)$$

For space-like separation we are able to swap $(y-x)$ in the second term to $(x-y)$. This gives us zero, as required.

Consider equation (11.12).

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_p)^{\frac{1}{2}}} \left(\hat{a}_p \exp(-ip \cdot x) + \hat{a}_p^\dagger \exp(ip \cdot x) \right) \quad (11.12)$$

It follows from (11.12) that

$$[\hat{\phi}(x), \hat{\phi}(y)] = \frac{1}{(2\pi)^3} \int \frac{d^3p}{(2E_p)^{\frac{1}{2}}} \int \frac{d^3q}{(2E_q)^{\frac{1}{2}}} (PQ - QP) \quad (1)$$

where

$$\begin{aligned} P &= \hat{a}_p \exp(-ip \cdot x) + \hat{a}_p^\dagger \exp(ip \cdot x) \\ Q &= \hat{a}_q \exp(-iq \cdot y) + \hat{a}_q^\dagger \exp(iq \cdot y) \end{aligned}$$

Expanding the commutator in (1) we have

$$\begin{aligned} PQ - QP = {} & [\hat{a}_p, \hat{a}_q] \exp(-ip \cdot x - iq \cdot y) + [\hat{a}_p, \hat{a}_q^\dagger] \exp(-ip \cdot x + iq \cdot y) \\ & + [\hat{a}_p^\dagger, \hat{a}_q] \exp(ip \cdot x - iq \cdot y) + [\hat{a}_p^\dagger, \hat{a}_q^\dagger] \exp(ip \cdot x + iq \cdot y) \end{aligned}$$

Then from the commutation relations

$$[\hat{a}_p, \hat{a}_q] = 0 \qquad [\hat{a}_p^\dagger, \hat{a}_q^\dagger] = 0 \qquad [\hat{a}_p, \hat{a}_q^\dagger] = \delta(p-q)$$

we have

$$PQ - QP = \delta(p-q) \big(\exp(-ip \cdot x + iq \cdot y) - \exp(ip \cdot x - iq \cdot y) \big) \quad (2)$$

Substitute (2) into (1) to obtain

$$[\hat{\phi}(x), \hat{\phi}(y)] = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_p} \big(\exp(-ip \cdot (x-y)) - \exp(-ip \cdot (y-x)) \big)$$