Sine perturbation

Let $\Psi(\mathbf{r},t)$ be the following wave function for a two state system.

$$\Psi(\mathbf{r},t)=\psi_a(\mathbf{r})c_a(t)\exp(-\tfrac{i}{\hbar}E_a t)+\psi_b(\mathbf{r})c_b(t)\exp(-\tfrac{i}{\hbar}E_b t)$$

Let $\hat{H}(\mathbf{r},t)$ be the Hamiltonian

$$\hat{H}(\mathbf{r},t)=\hat{H}_0(\mathbf{r})+\hat{H}_1(\mathbf{r},t)$$

where

$$\hat{H}_0\psi_a=E_a\psi_a,\quad \hat{H}_0\psi_b=E_b\psi_b,\quad \hat{H}_0\Psi=(E_a+E_b)\Psi$$

It was shown that to first order

$$c_b(t)=-\frac{i}{\hbar}\int_0^t\langle\psi_b|\hat{H}_1|\psi_a\rangle\exp(i\omega_0 t')\,dt',\quad \omega_0=\frac{E_b-E_a}{\hbar}$$

Let $\hat{H}_1(\mathbf{r},t)$ be the perturbation

$$\hat{H}_1(\mathbf{r},t)=\hat{V}(\mathbf{r})\cos(\omega t)$$

Then by substitution

$$c_b(t)=-\frac{i}{\hbar}\langle\psi_b|\hat{V}|\psi_a\rangle\int_0^t\cos(\omega t')\exp(i\omega_0 t')\,dt'$$

Solve the integral.

$$\int_0^t\cos(\omega t')\exp(i\omega_0 t')\,dt'=-\frac{i}{2}\left[\frac{\exp\big(i(\omega_0-\omega)t\big)-1}{\omega_0-\omega}+\frac{\exp\big(i(\omega_0+\omega)t\big)-1}{\omega_0+\omega}\right]\tag{1}$$

Hence

$$c_b(t)=-\frac{\langle\psi_b|\hat{V}|\psi_a\rangle}{2\hbar}\left[\frac{\exp\big(i(\omega_0-\omega)t\big)-1}{\omega_0-\omega}+\frac{\exp\big(i(\omega_0+\omega)t\big)-1}{\omega_0+\omega}\right]\tag{2}$$

Discard the second term as an approximation since the first term dominates for $\omega\approx\omega_0$.

$$c_b(t)=-\frac{\langle\psi_b|\hat{V}|\psi_a\rangle}{2\hbar}\frac{\exp\big(i(\omega_0-\omega)t\big)-1}{\omega_0-\omega}$$

Coefficient $c_b(t)$ can also be written as

$$c_b(t)=-\frac{i\langle\psi_b|\hat{V}|\psi_a\rangle}{\hbar}\frac{\sin\left(\frac{1}{2}(\omega_0-\omega)t\right)}{\omega_0-\omega}\exp\left(\tfrac{i}{2}(\omega_0-\omega)t\right)\tag{3}$$

Hence the transition probability is

$$P_{a\to b}(t)=|c_b(t)|^2=\frac{|\langle\psi_b|\hat{V}|\psi_a\rangle|^2}{\hbar^2}\frac{\sin^2\left(\frac{1}{2}(\omega_0-\omega)t\right)}{(\omega_0-\omega)^2}\tag{4}$$