(36.1) *An illustration of the reason for anticommutation and spin*

(a) Show that the Dirac equation can be recast in the form

$$i\frac{\partial \psi}{\partial t} = \hat{H}_D \psi \tag{36.33}$$

where $\hat{H}_D = \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m$ and find $\boldsymbol{\alpha}$ and β in terms of the γ matrices.

(b) Evaluate $\hat{H}_D^2$ and show that for a Klein-Gordon dispersion to result we must have:
(i) that the α^i and β objects all anticommute with each other; and
(ii) $(\alpha^i)^2 = (\beta)^2 = 1$.
This provides some justification for the anticommutation relations we imposed on the γs.

(a) Consider the following form of the Dirac equation.

$$i\left(\gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z}\right)\psi = m\psi$$

Rewrite as

$$i\gamma^0 \frac{\partial}{\partial t}\psi = -i\left(\gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z}\right)\psi + m\psi$$

Noting that $\gamma^0\gamma^0 = I$, multiply both sides by γ^0 to obtain

$$i\frac{\partial}{\partial t}\psi = -i\gamma^0\left(\gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z}\right)\psi + m\gamma^0\psi$$

Hence for $\hat{\mathbf{p}} = -i\nabla$ we have

$$\boldsymbol{\alpha} = \gamma^0 \begin{pmatrix} \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{pmatrix}, \quad \beta = \gamma^0$$