

8-4. Show that the ground-state wave function for the Lagrangian of equation (8.78) can be written

$$\Phi_0 = A \exp \left(-\frac{1}{2\hbar} \sum_{\alpha=1}^{N-1} \omega_{\alpha} Q_{\alpha}^* Q_{\alpha} \right) \quad (8.83)$$

(where A is a constant) by starting with the wave function in terms of the real variables Q_{α}^c and Q_{α}^s .

$$L = \frac{1}{2} \sum_{\alpha=0}^{N-1} \left(\dot{Q}_{\alpha}^* \dot{Q}_{\alpha} - \omega_{\alpha}^2 Q_{\alpha}^* Q_{\alpha} \right) \quad (8.78)$$

$$Q_{\alpha}^c = \frac{1}{\sqrt{2}} (Q_{\alpha} + Q_{\alpha}^*) \quad (8.79)$$

$$Q_{\alpha}^s = \frac{i}{\sqrt{2}} (Q_{\alpha} - Q_{\alpha}^*) \quad (8.80)$$

Rewrite (8.78) as

$$L = \frac{1}{2} \dot{Q}_0^* \dot{Q}_0 - \frac{1}{2} \omega_0^2 Q_0^* Q_0 + \frac{1}{2} \sum_{\alpha=1}^{N-1} \left(\dot{Q}_{\alpha}^* \dot{Q}_{\alpha} - \omega_{\alpha}^2 Q_{\alpha}^* Q_{\alpha} \right)$$

Note that

$$\begin{aligned} Q_{\alpha}^* Q_{\alpha} &= \frac{1}{2} (Q_{\alpha}^c + iQ_{\alpha}^s)(Q_{\alpha}^c - iQ_{\alpha}^s) \\ &= \frac{1}{2} (Q_{\alpha}^c)^2 + \frac{1}{2} (Q_{\alpha}^s)^2 \end{aligned}$$

and, because Q_0 is real,

$$Q_0^s = 0$$

Hence

$$L = \frac{1}{4} (\dot{Q}_0^c)^2 - \frac{1}{4} \omega^2 (Q_0^c)^2 + \frac{1}{4} \sum_{\alpha=1}^{N-1} \left((\dot{Q}_{\alpha}^c)^2 + (\dot{Q}_{\alpha}^s)^2 - \omega_{\alpha}^2 (Q_{\alpha}^c)^2 - \omega_{\alpha}^2 (Q_{\alpha}^s)^2 \right)$$

For N odd the summation interval can be halved because $Q_{\alpha} = Q_{N-\alpha}$.

$$L = \frac{1}{4} (\dot{Q}_0^c)^2 - \frac{1}{4} \omega^2 (Q_0^c)^2 + \frac{1}{2} \sum_{\alpha=1}^{(N-1)/2} \left((\dot{Q}_{\alpha}^c)^2 + (\dot{Q}_{\alpha}^s)^2 - \omega_{\alpha}^2 (Q_{\alpha}^c)^2 - \omega_{\alpha}^2 (Q_{\alpha}^s)^2 \right)$$

The Q_0 term can be dropped because it is a spatial translation.

$$L = \frac{1}{4}(\dot{Q}_0^c)^2 + \frac{1}{2} \sum_{\alpha=1}^{(N-1)/2} \left((\dot{Q}_\alpha^c)^2 + (\dot{Q}_\alpha^s)^2 - \omega_\alpha^2 (Q_\alpha^c)^2 - \omega_\alpha^2 (Q_\alpha^s)^2 \right)$$