Spherical harmonics

Verify

$$r^2 \nabla^2 Y_{lm}(\theta, \phi) = -l(l+1) Y_{lm}(\theta, \phi)$$

where $Y_{lm}(\theta, \phi)$ are spherical harmonic functions

$$Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) \exp(im\phi)$$

See arxiv.org/abs/1805.12125 for the following form of $P_l^m(\cos\theta)$.

$$P_l^m(\cos\theta) = \begin{cases} \left(\frac{\sin\theta}{2}\right)^m \sum_{k=0}^{l-m} (-1)^k \frac{(l+m+k)!}{(l-m-k)!(m+k)!k!} \left(\frac{1-\cos\theta}{2}\right)^k, & m \geq 0 \\ (-1)^m \frac{(l+m)!}{(l-m)!} P_l^{|m|}(\cos\theta), & m < 0 \end{cases}$$

Function $Y_{lm}(\theta, \phi)$ is independent of r hence

$$r^2 \nabla^2 Y_{lm}(\theta, \phi) = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left[\sin\theta \frac{\partial}{\partial\theta} Y_{lm}(\theta, \phi) \right] + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} Y_{lm}(\theta, \phi)$$