Feynman and Hibbs problem 2-3

Find S_{cl} for a particle under a constant force f, that is,

$$L = \frac{m}{2}\dot{x}^2 + fx$$

From the above Lagrangian we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

and

$$\frac{\partial L}{\partial x} = f$$

By equation (2.7) which is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

we have

$$\ddot{x} = \frac{f}{m}$$

Hence x(t) must have the following quadratic form.

$$x(t) = \frac{f}{2m}t^2 + Bt + C$$

We have the following boundary conditions.

$$x(t_a) = x_a$$
$$x(t_b) = x_b$$

Subtract to cancel C.

$$x(t_b) - x(t_a) = \frac{f}{2m} (t_b^2 - t_a^2) + B(t_b - t_a) = x_b - x_a$$

Solve for B.

$$B = \frac{x_b - x_a}{t_b - t_a} - \frac{f(t_b^2 - t_a^2)}{2m(t_b - t_a)}$$

Solve for C.

$$C = x_a - \frac{f}{2m}t_a^2 - Bt_a$$

$$= \frac{f(t_a t_b^2 - t_a^2 t_b)}{2m(t_b - t_a)} + \frac{t_b x_a - t_a x_b}{t_b - t_a}$$

Altogether we have

$$x(t) = \frac{f}{2m}t^2 + \left(\frac{x_b - x_a}{t_b - t_a} - \frac{f(t_b^2 - t_a^2)}{2m(t_b - t_a)}\right)t + \frac{f(t_a t_b^2 - t_a^2 t_b)}{2m(t_b - t_a)} + \frac{t_b x_a - t_a x_b}{t_b - t_a}$$
(1)

Equation (1) is too complicated to continue so we now translate the time coordinate as

$$t_a = 0$$
$$t_b = T$$

We now have

$$x(t) = \frac{ft^2}{2m} + \left(\frac{x_b - x_a}{T} - \frac{fT}{2m}\right)t + x_a \tag{2}$$

Differentiate x(t) to obtain $\dot{x}(t)$.

$$\dot{x}(t) = \frac{d}{dt}x(t) = \frac{ft}{m} - \frac{fT}{2m} + \frac{x_b - x_a}{T}$$
(3)

The new Lagrangian is

$$L = \frac{m}{2}\dot{x}(t)^{2} + fx(t)$$

$$= \frac{f^{2}}{m}t^{2} + \left(\frac{2f(x_{b} - x_{a})}{T} - \frac{f^{2}T}{m}\right)t + \frac{f^{2}T^{2}}{8m} + \frac{f(3x_{a} - x_{b})}{2} + \frac{m(x_{b} - x_{a})^{2}}{2T^{2}}$$
(4)

Hence

$$S_{cl} = \int_0^T L \, dt = \frac{m(x_b - x_a)^2}{2T} + \frac{fT(x_b + x_a)}{2} - \frac{f^2 T^3}{24m} \tag{5}$$

where $T = t_b - t_a$.