

# Yukawa potential

Find  $\sigma_{\text{total}}$  for the Yukawa potential

$$V(r) = -\frac{V_0 \exp(-\mu r)}{\mu r}$$

Let  $f(\mathbf{k})$  be the scattering amplitude for  $\mathbf{k} = \mathbf{k}_i - \mathbf{k}_f$ . The following formula is the Born approximation for  $f(\mathbf{k})$ .

$$f(\mathbf{k}) = \frac{m}{2\pi\hbar^2} \int \exp(i\mathbf{k} \cdot \mathbf{r}) V(\mathbf{r}) d\mathbf{r}$$

Convert to polar coordinates where  $k = |\mathbf{k}|$ .

$$f(\mathbf{k}) = \frac{m}{2\pi\hbar^2} \int_0^\infty \int_0^\pi \int_0^{2\pi} \exp(ikr \cos \theta) V(r, \theta, \phi) r^2 \sin \theta dr d\theta d\phi$$

Substitute the Yukawa potential

$$V(r, \theta, \phi) = -\frac{V_0 \exp(-\mu r)}{\mu r}$$

to obtain

$$f(\mathbf{k}) = -\frac{mV_0}{2\pi\hbar^2\mu} \int_0^\infty \int_0^\pi \int_0^{2\pi} \exp(ikr \cos \theta - \mu r) r \sin \theta dr d\theta d\phi$$

Integrate over  $\phi$  (multiply by  $2\pi$ ).

$$f(\mathbf{k}) = -\frac{mV_0}{\hbar^2\mu} \int_0^\infty \int_0^\pi \exp(ikr \cos \theta - \mu r) r \sin \theta dr d\theta$$

Let  $y = \cos \theta$  and  $dy = -\sin \theta d\theta$ . The minus sign in  $dy$  is canceled by interchanging integration limits  $\cos(0) = 1$  and  $\cos(\pi) = -1$ .

$$f(\mathbf{k}) = -\frac{mV_0}{\hbar^2\mu} \int_0^\infty \int_{-1}^1 \exp(ikry - \mu r) r dr dy$$

Solve the integral over  $y$ .

$$f(\mathbf{k}) = -\frac{mV_0}{\hbar^2\mu} \int_0^\infty \left[ \frac{1}{ikr} \exp(ikry - \mu r) \right]_{y=-1}^{y=1} r dr$$

Cancel  $r$  and evaluate the limits.

$$f(\mathbf{k}) = -\frac{mV_0}{\hbar^2\mu} \frac{1}{ik} \int_0^\infty [\exp(ikr - \mu r) - \exp(-ikr - \mu r)] dr$$

Solve the integral over  $r$ .

$$f(\mathbf{k}) = -\frac{mV_0}{\hbar^2\mu} \frac{1}{ik} \left[ \frac{1}{ik - \mu} \exp(ikr - \mu r) + \frac{1}{ik + \mu} \exp(-ikr - \mu r) \right]_{r=0}^{r=\infty}$$

Evaluate the limits. The exponentials vanish at the upper limit.

$$f(\mathbf{k}) = -\frac{mV_0}{\hbar^2\mu} \frac{1}{ik} \left[ -\frac{1}{ik - \mu} - \frac{1}{ik + \mu} \right] = -\frac{2mV_0}{\hbar^2\mu} \frac{1}{k^2 + \mu^2} \quad (1)$$

Substitute

$$k^2 = \frac{|\mathbf{p}_i - \mathbf{p}_f|^2}{\hbar^2} = \frac{4mE(1 - \cos \theta)}{\hbar^2}$$

to obtain

$$f(\theta) = -\frac{2mV_0}{\mu} \frac{1}{4mE(1 - \cos \theta) + \mu^2\hbar^2} \quad (2)$$

Hence

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left( \frac{2mV_0}{\mu} \right)^2 \frac{1}{[4mE(1 - \cos \theta) + \mu^2\hbar^2]^2}$$

For the total cross section we have

$$\sigma_{\text{total}} = \int \frac{d\sigma}{d\Omega} d\Omega = \left( \frac{2mV_0}{\mu} \right)^2 \int_0^\pi \int_0^{2\pi} \frac{1}{[4mE(1 - \cos \theta) + \mu^2\hbar^2]^2} \sin \theta d\theta d\phi$$

Integrate over  $\phi$  (multiply by  $2\pi$ ).

$$\sigma_{\text{total}} = 2\pi \left( \frac{2mV_0}{\mu} \right)^2 \int_0^\pi \frac{1}{[4mE(1 - \cos \theta) + \mu^2\hbar^2]^2} \sin \theta d\theta$$

Let  $y = 1 - \cos \theta$  and  $dy = \sin \theta d\theta$ . The limits transform as  $1 - \cos(0) = 0$  and  $1 - \cos(\pi) = 2$ .

$$\sigma_{\text{total}} = 2\pi \left( \frac{2mV_0}{\mu} \right)^2 \int_0^2 \frac{1}{[4mEy + \mu^2\hbar^2]^2} dy$$

Solve the integral.

$$\sigma_{\text{total}} = 2\pi \left( \frac{2mV_0}{\mu} \right)^2 \left[ -\frac{1}{4mE(4mEy + \mu^2\hbar^2)} \right]_{y=0}^{y=2}$$

Evaluate the limits.

$$\sigma_{\text{total}} = 2\pi \left( \frac{2mV_0}{\mu} \right)^2 \frac{2}{8mE\mu^2\hbar^2 + \mu^4\hbar^4} \quad (3)$$

For the dimensions of  $\sigma_{\text{total}}$  we have

$$\begin{aligned} mV_0 &\propto \text{mass} \times \text{energy} = \text{momentum}^2 \\ mE &\propto \text{mass} \times \text{energy} = \text{momentum}^2 \\ \mu\hbar &\propto \text{length}^{-1} \times \text{energy} \times \text{time} = \text{momentum} \end{aligned}$$

The momentum dimensions cancel leaving units of area.

$$\sigma_{\text{total}} \propto \frac{1}{\mu^2} \propto \text{length}^2$$

See exercise 10.7 of *Quantum Mechanics* by R. Fitzpatrick.