

# Addition of angular momentum

Let  $\mathbf{J}$  be the sum of orbital angular momentum  $\mathbf{L}$  and spin angular momentum  $\mathbf{S}$ .

$$\mathbf{J} = \mathbf{L} + \mathbf{S}$$

Recall that

$$\begin{aligned} L_x \psi &= -i\hbar \left( y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \psi & S_x \chi &= \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \chi \\ L_y \psi &= -i\hbar \left( z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) \psi & S_y \chi &= \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \chi \\ L_z \psi &= -i\hbar \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \psi & S_z \chi &= \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \chi \end{aligned}$$

Let  $\phi$  be the product of wave function  $\psi$  and electron spinor  $\chi$ .

$$\phi = \psi \chi$$

Then

$$\mathbf{J}\phi = \mathbf{L}\phi + \mathbf{S}\phi$$

Let  $J^2$  be the magnitude-squared of total angular momentum.

$$J^2 = \mathbf{J} \cdot \mathbf{J} = J_x^2 + J_y^2 + J_z^2$$

Operator  $J^2$  can be decomposed as

$$J^2 = (\mathbf{L} + \mathbf{S}) \cdot (\mathbf{L} + \mathbf{S}) = L^2 + S^2 + 2\mathbf{L} \cdot \mathbf{S}$$

In Eigenmath code

$$\mathbf{L} \cdot \mathbf{S} \phi = \text{contract}(\mathbf{L}(\mathbf{S}(\phi))) \quad (1)$$

The commutation relations for  $J^2$  are

$$\begin{aligned} [J^2, L^2] &= 0 \\ [J^2, S^2] &= 0 \\ [J^2, J_x] &= 0 \\ [J^2, J_y] &= 0 \\ [J^2, J_z] &= 0 \\ [J^2, L_x] &= 2i\hbar(L_y S_z - L_z S_y) \\ [J^2, L_y] &= 2i\hbar(L_z S_x - L_x S_z) \\ [J^2, L_z] &= 2i\hbar(L_x S_y - L_y S_x) \\ [J^2, S_x] &= -2i\hbar(L_y S_z - L_z S_y) \\ [J^2, S_y] &= -2i\hbar(L_z S_x - L_x S_z) \\ [J^2, S_z] &= -2i\hbar(L_x S_y - L_y S_x) \end{aligned}$$