

Harmonic oscillator action

This is the Lagrangian for a harmonic oscillator.

$$L = \frac{m}{2}(\dot{x}^2 - \omega^2 x^2)$$

Show that

$$S = \int_0^T L dt = \frac{m\omega}{2 \sin \omega T} ((x_b^2 + x_a^2) \cos \omega T - 2x_b x_a)$$

where $T = t_b - t_a$.

The first step is to derive $x(t)$ and $\dot{x}(t)$ from L and the Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

From L we have

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = m\ddot{x}, \quad \frac{\partial L}{\partial x} = -m\omega^2 x$$

and by Euler-Lagrange

$$\ddot{x}(t) = -\omega^2 x \tag{1}$$

The well-known solution to (1) is

$$x(t) = A \sin(\omega t) + B \cos(\omega t)$$

We have the following boundary conditions.

$$x(0) = x_a, \quad x(T) = x_b \tag{2}$$

Solve for B .

$$B = x(0) = x_a$$

For $x(T)$ we have

$$x(T) = A \sin(\omega T) + B \cos(\omega T)$$

Solve for A .

$$A = \frac{x(T) - B \cos(\omega T)}{\sin(\omega T)} = \frac{x_b - x_a \cos(\omega T)}{\sin(\omega T)}$$

Hence the equation of motion is

$$\begin{aligned} x(t) &= A \sin(\omega t) + B \cos(\omega t) \\ &= \frac{x_b - x_a \cos(\omega T)}{\sin(\omega T)} \sin(\omega t) + x_a \cos(\omega t) \end{aligned} \tag{3}$$

Differentiate $x(t)$ to obtain velocity $\dot{x}(t)$.

$$\dot{x}(t) = \frac{d}{dt} x(t) = \omega \left(\frac{x_b - x_a \cos(\omega T)}{\sin(\omega T)} \cos(\omega t) - x_a \sin(\omega t) \right) \tag{4}$$

The action is

$$\begin{aligned}
S &= \int_0^T L dt \\
&= \frac{m}{2} \int_0^T (\dot{x}^2 - \omega^2 x^2) dt \\
&= \frac{m}{2} \left(\int_0^T \dot{x}^2 dt - \int_0^T \omega^2 x^2 dt \right)
\end{aligned}$$

Use integration by parts to solve the first integral. Let $u = v = \dot{x}$ so that

$$\dot{u} = \ddot{x}, \quad \int v dt = x$$

The integral transforms as

$$\begin{aligned}
\int_0^T \dot{x}^2 dt &= \int_0^T uv dt \\
&= \left(u \int v dt \right) \Big|_0^T - \int_0^T \dot{u} \left(\int v dt \right) dt \\
&= \dot{x}(T)x(T) - \dot{x}(0)x(0) - \int_0^T \ddot{x}x dt
\end{aligned}$$

Hence

$$S = \frac{m}{2} \left(\dot{x}(T)x(T) - \dot{x}(0)x(0) - \int_0^T \ddot{x}x dt - \int_0^T \omega^2 x^2 dt \right)$$

The remaining integrals cancel by $\ddot{x} = -\omega^2 x$ from equation (1) leaving

$$S = \frac{m}{2} (\dot{x}(T)x(T) - \dot{x}(0)x(0)) \quad (5)$$

From the boundary conditions (2)

$$S = \frac{m}{2} (\dot{x}(T)x_b - \dot{x}(0)x_a)$$

From equation (3)

$$\dot{x}(0) = \frac{\omega(x_b - x_a \cos(\omega T))}{\sin(\omega T)} \quad (6)$$

and

$$\dot{x}(T) = \frac{\omega(x_b \cos(\omega T) - x_a)}{\sin(\omega T)} \quad (7)$$

Hence

$$\begin{aligned}
S &= \frac{m\omega}{2 \sin(\omega T)} \left[(x_b \cos(\omega T) - x_a)x_b - (x_b - x_a \cos(\omega T))x_a \right] \\
&= \frac{m\omega}{2 \sin(\omega T)} ((x_b^2 + x_a^2) \cos(\omega T) - 2x_b x_a)
\end{aligned} \quad (8)$$