

8-4. Show that the ground-state wave function for the Lagrangian of equation (8.78) can be written

$$\Phi_0 = A \exp \left(-\frac{1}{2\hbar} \sum_{\alpha=1}^{N-1} \omega_{\alpha} Q_{\alpha}^* Q_{\alpha} \right) \quad (8.83)$$

(where A is a constant) by starting with the wave function in terms of the real variables Q_{α}^c and Q_{α}^s .

$$L = \frac{1}{2} \sum_{\alpha=0}^{N-1} \left(\dot{Q}_{\alpha}^* \dot{Q}_{\alpha} - \omega_{\alpha}^2 Q_{\alpha}^* Q_{\alpha} \right) \quad (8.78)$$

Consider the following equation from p. 216.

$$Q_{\alpha} = \frac{1}{\sqrt{2}} (Q_{\alpha}^c - iQ_{\alpha}^s)$$

It follows that

$$Q_{\alpha}^* Q_{\alpha} = \frac{1}{2} (Q_{\alpha}^c)^2 + \frac{1}{2} (Q_{\alpha}^s)^2 \quad (1)$$

Substitute (1) into (8.78).

$$L = \frac{1}{4} \sum_{\alpha=0}^{N-1} \left((\dot{Q}_{\alpha}^c)^2 + (\dot{Q}_{\alpha}^s)^2 - \omega_{\alpha}^2 (Q_{\alpha}^c)^2 - \omega_{\alpha}^2 (Q_{\alpha}^s)^2 \right) \quad (2)$$

Consider equation (2.7).

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{Q}} = \frac{\partial L}{\partial Q} \quad (2.7)$$

Substitute (2) into (2.7) to obtain the following equations of motion.

$$\ddot{Q}_{\alpha}^c(t) = -\omega_{\alpha}^2 Q_{\alpha}^c(t) \quad \ddot{Q}_{\alpha}^s(t) = -\omega_{\alpha}^2 Q_{\alpha}^s(t) \quad (3)$$

From equation (8.58) and the associated text on p. 210, the unnormalized ground state eigenfunction corresponding to (3) is

$$\phi_0(x_{\alpha}) = \exp \left(-\frac{\omega_{\alpha} x_{\alpha}^2}{2\hbar} \right)$$

Then by equation (8.62)

$$\Phi_0 = \prod_{\alpha=0}^{N-1} \phi_0(Q_{\alpha}^c) \phi_0(Q_{\alpha}^s) = \exp \left(-\frac{1}{2\hbar} \sum_{\alpha=0}^{N-1} \omega_{\alpha} (Q_{\alpha}^c + Q_{\alpha}^s) \right)$$