

Angular momentum 2

The cross product in $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ demands rectangular coordinates. Hence for a wavefunction ψ in spherical coordinates, vectors \mathbf{r} and \mathbf{p} must be transformed in $\mathbf{L}\psi$.

Vector \mathbf{r} transforms as

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \sin \theta \cos \phi \\ r \sin \theta \sin \phi \\ r \cos \theta \end{pmatrix}$$

To transform \mathbf{p} we have by the chain rule

$$\begin{aligned} \frac{\partial}{\partial x} &= \sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{\cos \theta \cos \phi}{r} \frac{\partial}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial y} &= \sin \theta \sin \phi \frac{\partial}{\partial r} + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} + \frac{\cos \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z} &= \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \end{aligned}$$

Hence

$$\begin{aligned} p_x &= -i\hbar \frac{\partial}{\partial x} = -i\hbar \left(\sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{\cos \theta \cos \phi}{r} \frac{\partial}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \\ p_y &= -i\hbar \frac{\partial}{\partial y} = -i\hbar \left(\sin \theta \sin \phi \frac{\partial}{\partial r} + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} + \frac{\cos \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \\ p_z &= -i\hbar \frac{\partial}{\partial z} = -i\hbar \left(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) \end{aligned}$$

Using the transformed coordinates

$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta \end{aligned}$$

we have in spherical coordinates

$$\begin{aligned} L_x &= yp_z - zp_y = i\hbar \left(\sin \phi \frac{\partial}{\partial \theta} + \frac{\cos \theta \cos \phi}{\sin \theta} \frac{\partial}{\partial \phi} \right) \\ L_y &= zp_x - xp_z = i\hbar \left(-\cos \phi \frac{\partial}{\partial \theta} + \frac{\cos \theta \sin \phi}{\sin \theta} \frac{\partial}{\partial \phi} \right) \\ L_z &= xp_y - yp_x = -i\hbar \frac{\partial}{\partial \phi} \end{aligned}$$

and

$$L^2 = L_x^2 + L_y^2 + L_z^2 = -\hbar^2 \left(\frac{\partial^2}{\partial \theta^2} + \frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right)$$

Eigenmath script