Rutherford scattering 2

Find the cross section for Rutherford scattering with the following potential.

$$V(r) = -\frac{Ze^2}{r}\exp\left(-\frac{r}{a}\right)$$

Start with

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2\epsilon_0^2}\left(\frac{mQ}{2\pi\hbar^2}\right)^2, \quad Q = \int \exp\left(\frac{i\mathbf{p}\cdot\mathbf{r}}{\hbar}\right)V(\mathbf{r})\,d^3\mathbf{r}$$

Convert Q to polar coordinates.

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\exp\left(\frac{ipr\cos\theta}{\hbar}\right)V(r,\theta,\phi)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

Substitute the shielded Coulomb potential for $V(r,\theta,\phi)$ and note r^2 becomes r.

$$Q = -Ze^2\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\exp\left(\frac{ipr\cos\theta}{\hbar}\right)\exp\left(-\frac{r}{a}\right)\,r\sin\theta\,dr\,d\theta\,d\phi$$

Integrate over ϕ.

$$Q = -2\pi Ze^2\int_0^{\pi}\int_0^{\infty}\exp\left(\frac{ipr\cos\theta}{\hbar}\right)\exp\left(-\frac{r}{a}\right)\,r\sin\theta\,dr\,d\theta$$

Change the complex exponential to rectangular form.

$$Q = -2\pi Ze^2\int_0^{\pi}\int_0^{\infty}\left[\cos\left(\frac{pr\cos\theta}{\hbar}\right)+i\sin\left(\frac{pr\cos\theta}{\hbar}\right)\right]\exp\left(-\frac{r}{a}\right)\,r\sin\theta\,dr\,d\theta$$

By the definite integrals

$$\int_0^{\pi}\cos(a\cos\theta)\sin\theta\,d\theta = \frac{2\sin a}{a}, \qquad \int_0^{\pi}\sin(a\cos\theta)\sin\theta\,d\theta = 0$$

we have for the integral over θ (note r in the integrand is canceled)

$$Q = -\frac{4\pi Ze^2\hbar}{p}\int_0^{\infty}\sin\left(\frac{pr}{\hbar}\right)\exp\left(-\frac{r}{a}\right)\,dr$$

By the definite integral

$$\int_0^{\pi}\frac{\sin\theta}{\left(\sin^2(\theta/2)+a\right)^2}\,d\theta = \frac{2}{a^2+a}$$

we have for the integral over r

$$Q = -\frac{4\pi Ze^2\hbar}{p}\times\frac{p/\hbar}{(p/\hbar)^2+(1/a)^2} = -\frac{4\pi Ze^2}{(p/\hbar)^2+(1/a)^2}$$

The cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2\epsilon_0^2}\left(\frac{mQ}{2\pi\hbar^2}\right)^2 = \frac{1}{64\pi^2\epsilon_0^2} \times \frac{4m^2Z^2e^4}{\left[p^2 + (\hbar/a)^2\right]^2}$$

Substitute $(4\pi\epsilon_0\alpha\hbar c)^2$ for e^4.

$$\frac{d\sigma}{d\Omega} = \frac{m^2Z^2\alpha^2(\hbar c)^2}{\left[p^2 + (\hbar/a)^2\right]^2}$$

Symbol p is momentum transfer $|\mathbf{p}_i| - |\mathbf{p}_f|$ such that

$$p^2 = 2mE(1 - \cos\theta)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{m^2Z^2\alpha^2(\hbar c)^2}{\left[2mE(1 - \cos\theta) + (\hbar/a)^2\right]^2}$$

Cancel m^2 in the numerator.

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2(\hbar c)^2}{\left[2E(1 - \cos\theta) + \frac{1}{m}(\hbar/a)^2\right]^2} \tag{1}$$

Let $a \to \infty$ to obtain the ordinary Rutherford cross section.