Feynman and Hibbs problem 4-1

Show that for a single particle moving in three dimensions in a potential energy $V(\mathbf{x},t)$ the Schrodinger equation is

$$\frac{\partial\psi(\mathbf{x},t)}{\partial t} = -\frac{i}{\hbar}\left(-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x},t) + V(\mathbf{x},t)\psi(\mathbf{x},t)\right)$$

This is the Lagrangian.

$$L(\dot{\mathbf{x}},\mathbf{x}) = \frac{m}{2}\dot{\mathbf{x}}^2 - V(\mathbf{x},t) \tag{1}$$

Extend equation (4.3) from one dimension to three dimensions.

$$\psi(\mathbf{x},t+\epsilon) = \frac{1}{A}\int_{\mathbb{R}^3} \exp\left(\frac{i\epsilon}{\hbar}L\left(\frac{\mathbf{x}-\mathbf{y}}{\epsilon},\frac{\mathbf{x}+\mathbf{y}}{2}\right)\right)\psi(\mathbf{y},t)\,dy_1\,dy_2\,dy_3$$

where

$$\int_{\mathbb{R}^3} \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$$

From (1) we have

$$L\left(\frac{\mathbf{x}-\mathbf{y}}{\epsilon},\frac{\mathbf{x}+\mathbf{y}}{2}\right) = \frac{m}{2\epsilon^2}(\mathbf{x}-\mathbf{y})^2 - V\left(\frac{\mathbf{x}+\mathbf{y}}{2},t\right)$$

Hence

$$\psi(\mathbf{x},t+\epsilon) = \frac{1}{A}\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}(\mathbf{x}-\mathbf{y})^2 - \frac{i\epsilon}{\hbar}V\left(\frac{\mathbf{x}+\mathbf{y}}{2},t\right)\right) \times \psi(\mathbf{y},t)\,dy_1\,dy_2\,dy_3$$

Let

$$\mathbf{y} = \mathbf{x} + \boldsymbol{\eta}$$

Then

$$(\mathbf{x}-\mathbf{y})^2 = \boldsymbol{\eta}^2$$

and

$$\frac{\mathbf{x}+\mathbf{y}}{2} = \mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}$$

Hence

$$\psi(\mathbf{x}, t+\epsilon) = \frac{1}{A} \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2 - \frac{i\epsilon}{\hbar} V\left(\mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, t\right) \right) \\ \times \psi(\mathbf{x}+\boldsymbol{\eta}, t)\, d\eta_1\, d\eta_2\, d\eta_3$$

Factor the exponential.

$$\psi(\mathbf{x}, t+\epsilon) = \frac{1}{A} \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2 \right) \exp\left(-\frac{i\epsilon}{\hbar} V\left(\mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, t\right) \right) \\ \times \psi(\mathbf{x}+\boldsymbol{\eta}, t)\, d\eta_1\, d\eta_2\, d\eta_3 \quad (2)$$

Now we are going to use an approximation for the exponential of V. From the identity $\exp(i\theta) = \cos(\theta) + i\sin(\theta)$ we have

$$\exp\left(-\frac{i\epsilon}{\hbar} V\left(\mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, t\right) \right) = \\ \cos\left(-\frac{\epsilon}{\hbar} V\left(\mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, t\right) \right) + i\sin\left(-\frac{\epsilon}{\hbar} V\left(\mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, t\right) \right)$$

For small ϵ we have the approximation

$$\exp\left(-\frac{i\epsilon}{\hbar} V\left(\mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, t\right) \right) \approx 1 - \frac{i\epsilon}{\hbar} V\left(\mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, t\right)$$

The authors write that the $\frac{1}{2}\boldsymbol{\eta}$ term can be dropped "because the error is of higher order than ϵ." Hence

$$\exp\left(-\frac{i\epsilon}{\hbar} V\left(\mathbf{x} + \tfrac{1}{2}\boldsymbol{\eta}, t\right) \right) \approx 1 - \frac{i\epsilon}{\hbar} V\left(\mathbf{x}, t\right) \quad (3)$$

Substituting (3) into (2) yields

$$\psi(\mathbf{x}, t+\epsilon) = \frac{1}{A} \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2 \right) \left(1 - \frac{i\epsilon}{\hbar} V\left(\mathbf{x}, t\right) \right) \\ \times \psi(\mathbf{x}+\boldsymbol{\eta}, t)\, d\eta_1\, d\eta_2\, d\eta_3 \quad (4)$$

Next we will use the following Taylor series approximations.

$$\begin{aligned} \psi(\mathbf{x}, t+\epsilon) &\approx \psi(\mathbf{x}, t) + \epsilon\frac{\partial\psi}{\partial t} \\ \psi(\mathbf{x}+\boldsymbol{\eta}, t) &\approx \psi(\mathbf{x}, t) + \boldsymbol{\eta}\cdot\nabla\psi + \tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi) \end{aligned} \quad (5)$$

Note: In component notation

$$\boldsymbol{\eta} \cdot \nabla\psi = \eta_1 \frac{\partial \psi}{\partial x_1} + \eta_2 \frac{\partial \psi}{\partial x_2} + \eta_2 \frac{\partial \psi}{\partial x_2}$$

and

$$\begin{aligned}\boldsymbol{\eta} \cdot \nabla(\boldsymbol{\eta} \cdot \nabla\psi) = \eta_1^2 \frac{\partial^2 \psi}{\partial x_1^2} + \eta_2^2 \frac{\partial^2 \psi}{\partial x_2^2} + \eta_3^2 \frac{\partial^2 \psi}{\partial x_3^2} \\ + 2\eta_1\eta_2 \frac{\partial^2 \psi}{\partial x_1 \partial x_2} + 2\eta_1\eta_3 \frac{\partial^2 \psi}{\partial x_1 \partial x_3} + 2\eta_2\eta_3 \frac{\partial^2 \psi}{\partial x_2 \partial x_3}\end{aligned}$$

Substitute the approximations (5) into (4).

$$\begin{aligned}\psi(\mathbf{x}, t) + \epsilon \frac{\partial \psi}{\partial t} = \frac{1}{A} \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \left(1 - \frac{i\epsilon}{\hbar} V\left(\mathbf{x}, t\right)\right) \\ \times \left(\psi(\mathbf{x}, t) + \boldsymbol{\eta} \cdot \nabla\psi + \tfrac{1}{2}\boldsymbol{\eta} \cdot \nabla(\boldsymbol{\eta} \cdot \nabla\psi)\right) d\eta_1\, d\eta_2\, d\eta_3 \quad (6)\end{aligned}$$

Expand the right-hand side.

$$\frac{\psi(\mathbf{x}, t)}{A} \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) d\eta_1\, d\eta_2\, d\eta_3 \quad (7)$$

$$+ \frac{1}{A} \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \boldsymbol{\eta} \cdot \nabla\psi\, d\eta_1\, d\eta_2\, d\eta_3 \quad (8)$$

$$+ \frac{1}{2A} \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \boldsymbol{\eta} \cdot \nabla(\boldsymbol{\eta} \cdot \nabla\psi)\, d\eta_1\, d\eta_2\, d\eta_3 \quad (9)$$

$$- \frac{i\epsilon}{A\hbar} V\left(\mathbf{x}, t\right) \psi(\mathbf{x}, t) \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) d\eta_1\, d\eta_2\, d\eta_3 \quad (10)$$

$$- \frac{i\epsilon}{A\hbar} V\left(\mathbf{x}, t\right) \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \boldsymbol{\eta} \cdot \nabla\psi\, d\eta_1\, d\eta_2\, d\eta_3 \quad (11)$$

$$- \frac{i\epsilon}{2A\hbar} V\left(\mathbf{x}, t\right) \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \boldsymbol{\eta} \cdot \nabla(\boldsymbol{\eta} \cdot \nabla\psi)\, d\eta_1\, d\eta_2\, d\eta_3 \quad (12)$$

To solve the above integrals, we will use the following formulas provided by the authors.

$$\int_{-\infty}^{\infty} \exp\left(\frac{imx^2}{2\hbar\epsilon}\right)\,dx = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{1/2} \tag{13}$$

$$\int_{-\infty}^{\infty} \exp\left(\frac{imx^2}{2\hbar\epsilon}\right) x\,dx = 0 \tag{14}$$

$$\int_{-\infty}^{\infty} \exp\left(\frac{imx^2}{2\hbar\epsilon}\right) x^2\,dx = \left(-\frac{2\pi i\hbar^3\epsilon^3}{m^3}\right)^{1/2} \tag{15}$$

Rewrite the integral in (7) and (10) in component notation.

$$\begin{aligned}
&\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\,d\eta_1\,d\eta_2\,d\eta_3 \\
&\qquad = \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right)\,d\eta_1\,d\eta_2\,d\eta_3
\end{aligned}$$

Then by equation (13)

$$\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right)\,d\eta_1\,d\eta_2\,d\eta_3 = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \tag{16}$$

Rewrite the integral in (8) and (11) in component notation.

$$\begin{aligned}
&\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \boldsymbol{\eta}\cdot\nabla\psi\,d\eta_1\,d\eta_2\,d\eta_3 \\
&\quad = \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right) \eta_1 \frac{\partial\psi}{\partial x_1}\,d\eta_1\,d\eta_2\,d\eta_3 \\
&\qquad + \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right) \eta_2 \frac{\partial\psi}{\partial x_2}\,d\eta_1\,d\eta_2\,d\eta_3 \\
&\qquad + \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right) \eta_3 \frac{\partial\psi}{\partial x_3}\,d\eta_1\,d\eta_2\,d\eta_3
\end{aligned}$$

Then by equation (14)

$$\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \boldsymbol{\eta}\cdot\nabla\psi\,d\eta_1\,d\eta_2\,d\eta_3 = 0 \tag{17}$$

Rewrite the integral in (9) and (12) in component notation.

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi)\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\quad = \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right)\exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right)\exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right)\eta_1^2\frac{\partial^2\psi}{\partial x_1^2}\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad + \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right)\exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right)\exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right)\eta_2^2\frac{\partial^2\psi}{\partial x_2^2}\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad + \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right)\exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right)\exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right)\eta_3^2\frac{\partial^2\psi}{\partial x_3^2}\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad + \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right)\exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right)\exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right)2\eta_1\eta_2\frac{\partial^2\psi}{\partial x_1\partial x_2}\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad + \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right)\exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right)\exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right)2\eta_1\eta_3\frac{\partial^2\psi}{\partial x_1\partial x_3}\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad + \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right)\exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right)\exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right)2\eta_2\eta_3\frac{\partial^2\psi}{\partial x_2\partial x_3}\, d\eta_1\, d\eta_2\, d\eta_3
\end{aligned}
$$

By equations (13) and (15)

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right)\exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right)\exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right)\eta_1^2\frac{\partial^2\psi}{\partial x_1^2}\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\quad = \frac{i\hbar\epsilon}{m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\frac{\partial^2\psi}{\partial x_1^2} \\
&\int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right)\exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right)\exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right)\eta_2^2\frac{\partial^2\psi}{\partial x_2^2}\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\quad = \frac{i\hbar\epsilon}{m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\frac{\partial^2\psi}{\partial x_2^2} \\
&\int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right)\exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right)\exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right)\eta_3^2\frac{\partial^2\psi}{\partial x_3^2}\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\quad = \frac{i\hbar\epsilon}{m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\frac{\partial^2\psi}{\partial x_3^2}
\end{aligned}
$$

By equation (14)

$$\int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right) 2\eta_1\eta_2 \frac{\partial^2\psi}{\partial x_1 \partial x_2}\, d\eta_1\, d\eta_2\, d\eta_3 = 0$$
$$\int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right) 2\eta_1\eta_3 \frac{\partial^2\psi}{\partial x_1 \partial x_3}\, d\eta_1\, d\eta_2\, d\eta_3 = 0$$
$$\int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right) 2\eta_2\eta_3 \frac{\partial^2\psi}{\partial x_2 \partial x_3}\, d\eta_1\, d\eta_2\, d\eta_3 = 0$$

Hence

$$\begin{aligned}
&\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\psi)\, d\eta_1\, d\eta_2\, d\eta_3 \\
&\qquad = \frac{i\hbar\epsilon}{m}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\psi(\mathbf{x},t) \qquad (18)
\end{aligned}$$

Substitute the solved integrals into (6) to obtain

$$\begin{aligned}
&\psi(\mathbf{x},t) + \epsilon\frac{\partial\psi}{\partial t} \\
&\quad = \frac{1}{A}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\psi(\mathbf{x},t) && \text{from (7) and (16)} \\
&\qquad + \frac{i\hbar\epsilon}{2Am}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\nabla^2\psi(\mathbf{x},t) && \text{from (9) and (18)} \\
&\qquad - \frac{i\epsilon}{A\hbar}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}V(\mathbf{x},t)\psi(\mathbf{x},t) && \text{from (10) and (16)} \\
&\qquad + \frac{\epsilon^2}{2Am}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}V(\mathbf{x},t)\nabla^2\psi(\mathbf{x},t) && \text{from (12) and (18)}
\end{aligned}$$

In the limit as $\epsilon \to 0$ we have

$$\psi(\mathbf{x},t) = \frac{1}{A}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}\psi(\mathbf{x},t)$$

Hence

$$A = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}$$

Cancel A and drop the order ϵ^2 term.

$$\psi(\mathbf{x},t)+\epsilon\frac{\partial\psi}{\partial t}=\psi(\mathbf{x},t)+\frac{i\hbar\epsilon}{2m}\nabla^2\psi(\mathbf{x},t)-\frac{i\epsilon}{\hbar}V(\mathbf{x},t)\psi(\mathbf{x},t)$$

The $\psi(\mathbf{x},t)$ terms cancel.

$$\epsilon\frac{\partial\psi}{\partial t}=\frac{i\hbar\epsilon}{2m}\nabla^2\psi(\mathbf{x},t)-\frac{i\epsilon}{\hbar}V(\mathbf{x},t)\psi(\mathbf{x},t)$$

Divide through by ϵ.

$$\frac{\partial\psi}{\partial t}=\frac{i\hbar}{2m}\nabla^2\psi(\mathbf{x},t)-\frac{i}{\hbar}V(\mathbf{x},t)\psi(\mathbf{x},t)$$