Quantum angular momentum

For $l = 1$ and $l = 2$ find the eigenfunctions of L_x where

$$L_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right)$$

In other words, find the eigenfunctions $\psi_{l,m}$ such that $L_x\psi_{l,m} = m\hbar\psi_{l,m}$.

Using ladder operators we have

$$L_xY_{l,m} = \tfrac{1}{2}(L_+ + L_-)Y_{l,m} = \frac{\hbar}{2}\left(c^+_{l,m}Y_{l,m+1} + c^-_{l,m}Y_{l,m-1}\right)$$

where

$$\begin{aligned} c^+_{l,m} &= \sqrt{(l-m)(l+m+1)} = \sqrt{l(l+1)-m(m+1)} \\ c^-_{l,m} &= \sqrt{(l+m)(l-m+1)} = \sqrt{l(l+1)-m(m-1)} \end{aligned}$$

Hence for $l = 1$ we have

$$\begin{pmatrix} L_xY_{1,1} \\ L_xY_{1,0} \\ L_xY_{1,-1} \end{pmatrix} = \hbar M_x \begin{pmatrix} Y_{1,1} \\ Y_{1,0} \\ Y_{1,-1} \end{pmatrix}$$

where

$$M_x = \frac{1}{2}\begin{pmatrix} 0 & c^+_{1,0} & 0 \\ c^-_{1,1} & 0 & c^+_{1,-1} \\ 0 & c^-_{1,0} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix}$$

These are the eigenvectors x_m of M_x where m is the eigenvalue.

$$\begin{aligned} x_1 &= \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}^T, & M_xx_1 &= x_1 \\ x_0 &= \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}^T, & M_xx_0 &= 0 \\ x_{-1} &= \begin{pmatrix} -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}^T, & M_xx_{-1} &= -x_{-1} \end{aligned}$$

For eigenfunctions $\psi_{1,m}$ we have

$$\psi_{1,m} = x_m \cdot \begin{pmatrix} Y_{1,1} \\ Y_{1,0} \\ Y_{1,-1} \end{pmatrix}$$

Hence

$$\psi_{1,1} = \frac{1}{2}Y_{1,1} + \frac{1}{\sqrt{2}}Y_{1,0} + \frac{1}{2}Y_{1,-1}, \qquad L_x\psi_{1,1} = \hbar\psi_{1,1}$$

$$\psi_{1,0} = -\frac{1}{\sqrt{2}}Y_{1,1} + \frac{1}{\sqrt{2}}Y_{1,-1}, \qquad L_x\psi_{1,0} = 0$$

$$\psi_{1,-1} = -\frac{1}{2}Y_{1,1} + \frac{1}{\sqrt{2}}Y_{1,0} - \frac{1}{2}Y_{1,-1}, \qquad L_x\psi_{1,-1} = -\hbar\psi_{1,-1}$$

For $l = 2$

$$M_x = \frac{1}{2}\begin{pmatrix} 0 & c^+_{2,1} & 0 & 0 & 0 \\ c^-_{2,2} & 0 & c^+_{2,0} & 0 & 0 \\ 0 & c^-_{2,1} & 0 & c^+_{2,-1} & 0 \\ 0 & 0 & c^-_{2,0} & 0 & c^+_{2,-2} \\ 0 & 0 & 0 & c^-_{2,-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{3/2} & 0 & 0 \\ 0 & \sqrt{3/2} & 0 & \sqrt{3/2} & 0 \\ 0 & 0 & \sqrt{3/2} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

These are the eigenvectors x_m of M_x where m is the eigenvalue.

$$x_2 = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}^T, \qquad M_x x_2 = 2x_2$$

$$x_1 = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}^T, \qquad M_x x_1 = x_1$$

$$x_0 = \begin{pmatrix} \frac{\sqrt{3}}{2\sqrt{2}} & 0 & -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix}^T, \qquad M_x x_0 = 0$$

$$x_{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}^T, \qquad M_x x_{-1} = -x_{-1}$$

$$x_{-2} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} & -\frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2} & -\frac{1}{4} \end{pmatrix}^T, \qquad M_x x_{-2} = -2x_{-2}$$

For eigenfunctions $\psi_{2,m}$ we have

$$\psi_{2,m} = x_m \cdot \begin{pmatrix} Y_{2,2} \\ Y_{2,1} \\ Y_{2,0} \\ Y_{2,-1} \\ Y_{2,-2} \end{pmatrix}$$

Hence

$$\psi_{2,2} = \frac{1}{4}Y_{2,2} + \frac{1}{2}Y_{2,1} + \frac{\sqrt{3}}{2\sqrt{2}}Y_{2,0} + \frac{1}{2}Y_{2,-1} + \frac{1}{4}Y_{2,-2}, \qquad L_x\psi_{2,2} = 2\hbar\psi_{2,2}$$

$$\psi_{2,1} = -\frac{1}{2}Y_{2,2} - \frac{1}{2}Y_{2,1} + \frac{1}{2}Y_{2,-1} + \frac{1}{2}Y_{2,-2}, \qquad L_x\psi_{2,1} = \hbar\psi_{2,1}$$

$$\psi_{2,0} = \frac{\sqrt{3}}{2\sqrt{2}}Y_{2,2} - \frac{1}{2}Y_{2,0} + \frac{\sqrt{3}}{2\sqrt{2}}Y_{2,-2}, \qquad L_x\psi_{2,0} = 0$$

$$\psi_{2,-1} = -\frac{1}{2}Y_{2,2} + \frac{1}{2}Y_{2,1} - \frac{1}{2}Y_{2,-1} + \frac{1}{2}Y_{2,-2}, \qquad L_x\psi_{2,-1} = -\hbar\psi_{2,-1}$$

$$\psi_{2,-2} = -\frac{1}{4}Y_{2,2} + \frac{1}{2}Y_{2,1} - \frac{\sqrt{3}}{2\sqrt{2}}Y_{2,0} + \frac{1}{2}Y_{2,-1} - \frac{1}{4}Y_{2,-2}, \qquad L_x\psi_{2,-2} = -2\hbar\psi_{2,-2}$$