Spin

Spin state $|s\rangle$ is a normalized vector in $\mathbb{C}^2$.

$$|s\rangle = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}, \quad |c_+|^2 + |c_-|^2 = 1$$

Spin measurement probabilities are the transition probabilities from $|s\rangle$ to an eigenstate.

For spin measurements in the z direction we have

$$\begin{aligned} \Pr\left(S_z = +\tfrac{\hbar}{2}\right) &= |\langle z_+|s\rangle|^2 \\ \Pr\left(S_z = -\tfrac{\hbar}{2}\right) &= |\langle z_-|s\rangle|^2 \end{aligned}$$

Define the z eigenstates as

$$|z_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |z_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

By definition of expectation value we have

$$\langle S_z\rangle = \tfrac{\hbar}{2}\Pr\left(S_z = +\tfrac{\hbar}{2}\right) - \tfrac{\hbar}{2}\Pr\left(S_z = -\tfrac{\hbar}{2}\right)$$

Rewrite as

$$\langle S_z\rangle = \tfrac{\hbar}{2}|\langle z_+|s\rangle|^2 - \tfrac{\hbar}{2}|\langle z_-|s\rangle|^2$$

Rewrite again as

$$\langle S_z\rangle = \tfrac{\hbar}{2}\langle s|z_+\rangle\langle z_+|s\rangle - \tfrac{\hbar}{2}\langle s|z_-\rangle\langle z_-|s\rangle$$

Then by

$$\langle S_z\rangle = \langle s|S_z|s\rangle$$

we have

$$S_z = \tfrac{\hbar}{2}|z_+\rangle\langle z_+| - \tfrac{\hbar}{2}|z_-\rangle\langle z_-| = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

From the commutator

$$S_+S_- - S_-S_+ = 2\hbar S_z$$

we have

$$S_+S_- - S_-S_+ = \hbar^2|z_+\rangle\langle z_+| - \hbar^2|z_-\rangle\langle z_-|$$

Rewrite as

$$S_+S_- - S_-S_+ = \hbar^2|z_+\rangle\langle z_-|z_-\rangle\langle z_+| - \hbar^2|z_-\rangle\langle z_+|z_+\rangle\langle z_-|$$

Hence

$$S_+ = \hbar|z_+\rangle\langle z_-| = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$S_- = \hbar|z_-\rangle\langle z_+| = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then by

$$S_+ = S_x + iS_y$$
$$S_- = S_x - iS_y$$

we obtain

$$S_x = \frac{S_+ + S_-}{2} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$S_y = \frac{S_+ - S_-}{2i} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

By solving for the eigenstates in

$$S_x|x_\pm\rangle = \pm\tfrac{\hbar}{2}|x_\pm\rangle$$
$$S_y|y_\pm\rangle = \pm\tfrac{\hbar}{2}|y_\pm\rangle$$

we obtain

$$|x_+\rangle = \frac{|z_+\rangle + |z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$|x_-\rangle = \frac{|z_+\rangle - |z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and

$$|y_+\rangle = \frac{|z_+\rangle + i|z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$
$$|y_-\rangle = \frac{|z_+\rangle - i|z_-\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

The expected spin direction vector is

$$\langle \mathbf{S} \rangle = \begin{pmatrix} \langle s|S_x|s\rangle \\ \langle s|S_y|s\rangle \\ \langle s|S_z|s\rangle \end{pmatrix}, \quad |\langle \mathbf{S}\rangle| = \frac{\hbar}{2}$$

To convert a direction vector to a spin state, let θ and ϕ be the polar and azimuth angles such that

$$\langle \mathbf{S} \rangle = \frac{\hbar}{2}\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

Then

$$|s\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\exp(i\phi) \end{pmatrix}$$

By the identities

$$\cos^2(\theta/2) = \frac{\cos\theta + 1}{2}, \quad \sin^2(\theta/2) = \frac{1 - \cos\theta}{2}$$

and noting that $0 \le \theta \le \pi$ we have

$$\cos(\theta/2) = \sqrt{\frac{\langle z\rangle + 1}{2}}, \quad \sin(\theta/2)\exp(i\phi) = \sqrt{\frac{1-\langle z\rangle}{2}}\frac{\langle x\rangle + i\langle y\rangle}{\sqrt{\langle x\rangle^2 + \langle y\rangle^2}}$$

where

$$\begin{aligned}
\langle x\rangle &= \tfrac{2}{\hbar}\langle S_x\rangle = \sin\theta\cos\phi \\
\langle y\rangle &= \tfrac{2}{\hbar}\langle S_y\rangle = \sin\theta\sin\phi \\
\langle z\rangle &= \tfrac{2}{\hbar}\langle S_z\rangle = \cos\theta
\end{aligned}$$

The following commutator was used to derive S_x and S_y.

$$S_+S_- - S_-S_+ = 2\hbar S_z$$

The commutator is a consequence of the following spin wave equation.

$$\hat{\mathbf{S}}\psi = (\mathbf{r}\times\hat{\mathbf{p}})\psi, \quad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \hat{\mathbf{p}} = -i\hbar\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

Rewrite in component form.

$$\hat{S}_x\psi = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\psi$$

$$\hat{S}_y\psi = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\psi$$

$$\hat{S}_z\psi = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\psi$$

By computer algebra we have

$$\begin{aligned}
(\hat{S}_y\hat{S}_z - \hat{S}_z\hat{S}_y)\psi &= i\hbar\hat{S}_x\psi \\
(\hat{S}_z\hat{S}_x - \hat{S}_x\hat{S}_z)\psi &= i\hbar\hat{S}_y\psi \\
(\hat{S}_x\hat{S}_y - \hat{S}_y\hat{S}_x)\psi &= i\hbar\hat{S}_z\psi
\end{aligned}$$

Let

$$\begin{aligned}
\hat{S}_+ &= \hat{S}_x + i\hat{S}_y \\
\hat{S}_- &= \hat{S}_x - i\hat{S}_y
\end{aligned}$$

By computer algebra

$$(\hat{S}_+\hat{S}_- - \hat{S}_-\hat{S}_+)\psi = 2\hbar\hat{S}_z\psi$$

1. Verify that

$$S_x = \frac{\hbar}{2}(|x_+\rangle\langle x_+| - |x_-\rangle\langle x_-|)$$
$$S_y = \frac{\hbar}{2}(|y_+\rangle\langle y_+| - |y_-\rangle\langle y_-|)$$
$$S_z = \frac{\hbar}{2}(|z_+\rangle\langle z_+| - |z_-\rangle\langle z_-|)$$

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zp = (1,0)
zm = (0,1)

xp = (zp + zm) / sqrt(2)
xm = (zp - zm) / sqrt(2)

yp = (zp + i zm) / sqrt(2)
ym = (zp - i zm) / sqrt(2)

Sx = hbar / 2 ((0,1),(1,0))
Sy = hbar / 2 ((0,-i),(i,0))
Sz = hbar / 2 ((1,0),(0,-1))

check(Sx == hbar / 2 (outer(xp,conj(xp)) - outer(xm,conj(xm))))
check(Sy == hbar / 2 (outer(yp,conj(yp)) - outer(ym,conj(ym))))
check(Sz == hbar / 2 (outer(zp,conj(zp)) - outer(zm,conj(zm))))
```

2. Let $|s\rangle$ be the following spin state.

$$|s\rangle = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$$

Verify that $|s\rangle$ is normalized and that

$$\langle \mathbf{S} \rangle = \langle s|\mathbf{S}|s\rangle = \frac{\hbar}{2} \begin{pmatrix} \frac{4}{9} \\ \frac{8}{9} \\ \frac{1}{9} \end{pmatrix}$$

where

$$\mathbf{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$$

Note: In component form we have

$$\langle s|\mathbf{S}|s\rangle = s^*_\beta S^{\alpha\beta}{}_\gamma s^\gamma$$

Eigenmath requires a transpose so that the β indices are adjacent.

$$\langle s|\mathbf{S}|s\rangle = s^*_\beta S^{\beta\alpha}{}_\gamma s^\gamma$$

```
s = (1/3 - 2/3 i, 2/3)

check(dot(conj(s),s) == 1)

Sx = hbar / 2 ((0,1),(1,0))
Sy = hbar / 2 ((0,-i),(i,0))
Sz = hbar / 2 ((1,0),(0,-1))

S = (Sx,Sy,Sz)

check(dot(conj(s),transpose(S),s) == hbar / 2 (4/9, 8/9, 1/9))
```

3. Let $|s\rangle$ be the following spin state.

$$|s\rangle = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$$

Verify the following measurement probabilities for $|s\rangle$.

$$\Pr\left(S_x = +\tfrac{\hbar}{2}\right) = |\langle x_+|s\rangle|^2 = \tfrac{13}{18}$$
$$\Pr\left(S_x = -\tfrac{\hbar}{2}\right) = |\langle x_-|s\rangle|^2 = \tfrac{5}{18}$$

$$\Pr\left(S_y = +\tfrac{\hbar}{2}\right) = |\langle y_+|s\rangle|^2 = \tfrac{17}{18}$$
$$\Pr\left(S_y = -\tfrac{\hbar}{2}\right) = |\langle y_-|s\rangle|^2 = \tfrac{1}{18}$$

$$\Pr\left(S_z = +\tfrac{\hbar}{2}\right) = |\langle z_+|s\rangle|^2 = \tfrac{5}{9}$$
$$\Pr\left(S_z = -\tfrac{\hbar}{2}\right) = |\langle z_-|s\rangle|^2 = \tfrac{4}{9}$$

```
s = (1/3 - 2/3 i, 2/3)

zp = (1,0)
zm = (0,1)

xp = (zp + zm) / sqrt(2)
xm = (zp - zm) / sqrt(2)

yp = (zp + i zm) / sqrt(2)
ym = (zp - i zm) / sqrt(2)

Pr(a,b) = dot(conj(a),b) dot(conj(b),a)

check(Pr(xp,s) == 13/18)
check(Pr(xm,s) == 5/18)

check(Pr(yp,s) == 17/18)
check(Pr(ym,s) == 1/18)

check(Pr(zp,s) == 5/9)
check(Pr(zm,s) == 4/9)
```

4. Let $|s\rangle$ be the following spin state.

$$|s\rangle = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$$

Verify that the following spin state $|\chi\rangle$ is indistinguishable from $|s\rangle$.

$$|\chi\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\exp(i\phi) \end{pmatrix}$$

where

$$\cos(\theta/2) = \sqrt{\frac{\langle z\rangle + 1}{2}} = \frac{\sqrt{5}}{3}$$

and

$$\sin(\theta/2)\exp(i\phi) = \sqrt{\frac{1 - \langle z\rangle}{2}} \frac{\langle x\rangle + i\langle y\rangle}{\sqrt{\langle x\rangle^2 + \langle y\rangle^2}} = \frac{2 + 4i}{3\sqrt{5}}$$

with

$$\langle x\rangle = \tfrac{2}{\hbar}\langle S_x\rangle$$
$$\langle y\rangle = \tfrac{2}{\hbar}\langle S_y\rangle$$
$$\langle z\rangle = \tfrac{2}{\hbar}\langle S_z\rangle$$

```
s = (1/3 - 2/3 i, 2/3)

Sx = hbar / 2 ((0,1),(1,0))
Sy = hbar / 2 ((0,-i),(i,0))
Sz = hbar / 2 ((1,0),(0,-1))

S = (Sx,Sy,Sz)

x = 2 / hbar dot(conj(s),Sx,s)
y = 2 / hbar dot(conj(s),Sy,s)
z = 2 / hbar dot(conj(s),Sz,s)

cp = sqrt((z + 1) / 2)
cm = sqrt((1 - z) / 2) (x + i y) / sqrt(x^2 + y^2)

check(cp == sqrt(5) / 3)
check(cm == (2 + 4 i) / (3 sqrt(5)))

chi = (cp,cm)

check(dot(conj(s),transpose(S),s) == dot(conj(chi),transpose(S),chi))
```

5. Verify the following commutators for $\mathbf{S}\psi = (\mathbf{r} \times \mathbf{p})\psi$.

$$
\begin{aligned}
[S_x, S_y] &= i\hbar S_z \\
[S_y, S_z] &= i\hbar S_x \\
[S_z, S_x] &= i\hbar S_y
\end{aligned}
$$

$$
\begin{aligned}
[S^2, S_x] &= 0 \\
[S^2, S_y] &= 0 \\
[S^2, S_z] &= 0
\end{aligned}
$$

$$[S_+, S_-] = 2\hbar S_z$$

where

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

and

$$
\begin{aligned}
S_+ &= S_x + iS_y \\
S_- &= S_x - iS_y
\end{aligned}
$$

```
Sx(psi) = -i hbar (y d(psi,z) - z d(psi,y))
Sy(psi) = -i hbar (z d(psi,x) - x d(psi,z))
Sz(psi) = -i hbar (x d(psi,y) - y d(psi,x))

psi = Psi()

check(Sy(Sz(psi)) - Sz(Sy(psi)) == i hbar Sx(psi))
check(Sz(Sx(psi)) - Sx(Sz(psi)) == i hbar Sy(psi))
check(Sx(Sy(psi)) - Sy(Sx(psi)) == i hbar Sz(psi))

S2(psi) = Sx(Sx(psi)) + Sy(Sy(psi)) + Sz(Sz(psi))

check(S2(Sx(psi)) - Sx(S2(psi)) == 0)
check(S2(Sy(psi)) - Sy(S2(psi)) == 0)
check(S2(Sz(psi)) - Sz(S2(psi)) == 0)

Sp(psi) = Sx(psi) + i Sy(psi)
Sm(psi) = Sx(psi) - i Sy(psi)

check(Sp(Sm(psi)) - Sm(Sp(psi)) == 2 hbar Sz(psi))
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