Fine structure 1

Consider the fine structure formula for hydrogen.

$$E = -\mu c^2 \left[1 - \frac{1}{\sqrt{1 + \left(\frac{\alpha}{n - j - \frac{1}{2} + \sqrt{\left(j + \frac{1}{2}\right)^2 - \alpha^2}} \right)^2}} \right]$$

Show that to order α^4

$$E = -\frac{\mu c^2 \alpha^2}{2n^2} \left(1 + \frac{\alpha^2}{n\left(j + \frac{1}{2}\right)} - \frac{3\alpha^2}{4n^2} \right)$$

Let

$$f = \frac{1}{\sqrt{1 + \left(\frac{\alpha}{A + \sqrt{B^2 - \alpha^2}} \right)^2}}$$

Expand f as a Taylor series.

$$f = 1 - \frac{\alpha^2}{2(A+B)^2} - \frac{\alpha^4}{2B(A+B)^3} + \frac{3\alpha^4}{8(A+B)^4} + \mathcal{O}(\alpha^6)$$

Substitute $A = n - j - 1/2$ and $B = j + 1/2$ to obtain $A + B = n$ and

$$f = 1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^3\left(j + \frac{1}{2}\right)} + \frac{3\alpha^4}{8n^4} \tag{1}$$

Hence

$$E = -\mu c^2 (1 - f) = -\mu c^2 \left(\frac{\alpha^2}{2n^2} + \frac{\alpha^4}{2n^3\left(j + \frac{1}{2}\right)} - \frac{3\alpha^4}{8n^4} \right)$$

Factor out $\alpha^2/(2n^2)$.

$$E = -\frac{\mu c^2 \alpha^2}{2n^2} \left(1 + \frac{\alpha^2}{n\left(j + \frac{1}{2}\right)} - \frac{3\alpha^2}{4n^2} \right) \tag{2}$$

Note that

$$-\frac{\mu c^2 \alpha^2}{2} = -13.6\,\mathrm{eV}$$