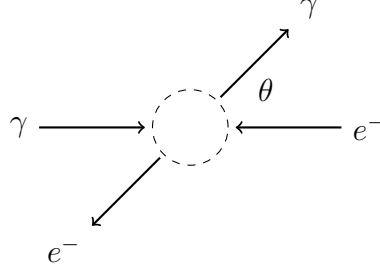


# Klein-Nishina formula

The Klein-Nishina formula is the lab frame cross section for  $\gamma + e^- \rightarrow \gamma + e^-$ .



It is easy to derive the Klein-Nishina formula from Dirac's equation by starting out in the center-of-mass frame and then boosting to the lab frame. The following center-of-mass momentum vectors have  $E = \sqrt{\omega^2 + m^2}$ .

$$\begin{aligned}
 p_1 = \begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix} & \quad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -\omega \end{pmatrix} & \quad p_3 = \begin{pmatrix} \omega \\ \omega \sin \theta \cos \phi \\ \omega \sin \theta \sin \phi \\ \omega \cos \theta \end{pmatrix} & \quad p_4 = \begin{pmatrix} E \\ -\omega \sin \theta \cos \phi \\ -\omega \sin \theta \sin \phi \\ -\omega \cos \theta \end{pmatrix} \\
 \gamma \rightarrow & \quad \leftarrow e^- & \quad \nearrow \gamma & \quad \nwarrow e^-
 \end{aligned}$$

Spinors for  $p_2$ .

$$\begin{aligned}
 u_{21} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ -\omega \\ 0 \end{pmatrix} & u_{22} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ 0 \\ \omega \end{pmatrix} \\
 & \text{spin up} & & \text{spin down}
 \end{aligned}$$

Spinors for  $p_4$ .

$$\begin{aligned}
 u_{41} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix} & u_{42} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix} \\
 & \text{spin up} & & \text{spin down}
 \end{aligned}$$

The scattering amplitude  $\mathcal{M}_{ab}{}^{\mu\nu}$  for spin  $ab$  and polarization  $\mu\nu$  is

$$\mathcal{M}_{ab}{}^{\mu\nu} = \mathcal{M}_{1ab}{}^{\mu\nu} + \mathcal{M}_{2ab}{}^{\nu\mu}$$

where

$$\begin{aligned}
 \mathcal{M}_{1ab}{}^{\mu\nu} &= \frac{\bar{u}_{4b}(-ie\gamma^\mu)(\not{p}_1 + m)(-ie\gamma^\nu)u_{2a}}{s - m^2} \\
 \mathcal{M}_{2ab}{}^{\nu\mu} &= \frac{\bar{u}_{4b}(-ie\gamma^\nu)(\not{p}_2 + m)(-ie\gamma^\mu)u_{2a}}{u - m^2}
 \end{aligned}$$

Matrices  $\not{q}_1$  and  $\not{q}_2$  represent momentum transfer.

$$\begin{aligned}\not{q}_1 &= (p_1 + p_2)^\alpha g_{\alpha\beta} \gamma^\beta \\ \not{q}_2 &= (p_4 - p_1)^\alpha g_{\alpha\beta} \gamma^\beta\end{aligned}$$

Scalars  $s$  and  $u$  are Mandelstam variables.

$$\begin{aligned}s &= (p_1 + p_2)^2 \\ u &= (p_1 - p_4)^2\end{aligned}$$

In component form

$$\begin{aligned}\mathcal{M}_{1ab}{}^{\mu\nu} &= \frac{(\bar{u}_{4b})_\alpha (-ie\gamma^{\mu\alpha}{}_\beta)(\not{q}_1 + m)^\beta{}_\rho (-ie\gamma^{\nu\rho}{}_\sigma)(u_{2a})^\sigma}{s - m^2} \\ \mathcal{M}_{2ab}{}^{\nu\mu} &= \frac{(\bar{u}_{4b})_\alpha (-ie\gamma^{\nu\alpha}{}_\beta)(\not{q}_2 + m)^\beta{}_\rho (-ie\gamma^{\mu\rho}{}_\sigma)(u_{2a})^\sigma}{u - m^2}\end{aligned}$$

Expected probability density  $\langle |\mathcal{M}|^2 \rangle$  is the sum of squared amplitudes divided by the number of inbound states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{ab} \sum_{\mu\nu} |\mathcal{M}_{ab}{}^{\mu\nu}|^2$$

Summing over  $\mu\nu$  requires  $g_{\mu\nu}$  to lower indices.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{ab} \mathcal{M}_{ab}{}^{\mu\nu} (g_{\mu\alpha} \mathcal{M}_{ab}{}^{\alpha\beta} g_{\beta\nu})^*$$

Expand the summand and label the terms. By positivity  $\boxed{2} = \boxed{3}$ .

$$\begin{aligned}\langle |\mathcal{M}|^2 \rangle &= \frac{1}{4} \sum_{ab} \left[ \underbrace{\mathcal{M}_{1ab}{}^{\mu\nu} (g_{\mu\alpha} \mathcal{M}_{1ab}{}^{\alpha\beta} g_{\beta\nu})^*}_{\boxed{1}} + \underbrace{\mathcal{M}_{1ab}{}^{\mu\nu} (g_{\nu\alpha} \mathcal{M}_{2ab}{}^{\alpha\beta} g_{\beta\mu})^*}_{\boxed{2}} \right. \\ &\quad \left. + \underbrace{\mathcal{M}_{2ab}{}^{\nu\mu} (g_{\mu\alpha} \mathcal{M}_{1ab}{}^{\alpha\beta} g_{\beta\nu})^*}_{\boxed{3}} + \underbrace{\mathcal{M}_{2ab}{}^{\nu\mu} (g_{\nu\alpha} \mathcal{M}_{2ab}{}^{\alpha\beta} g_{\beta\mu})^*}_{\boxed{4}} \right]\end{aligned}$$

The following Casimir trick uses matrix arithmetic to sum over spin and polarization states.

$$\begin{aligned}\sum_{ab} \boxed{1} &= \frac{e^4}{(s - m^2)^2} \text{Tr} \left[ (\not{p}_2 + m) \gamma^\mu (\not{q}_1 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\nu (\not{q}_1 + m) \gamma_\mu \right] \\ \sum_{ab} \boxed{2} &= \frac{e^4}{(s - m^2)(u - m^2)} \text{Tr} \left[ (\not{p}_2 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\mu (\not{q}_1 + m) \gamma_\nu \right] \\ \sum_{ab} \boxed{4} &= \frac{e^4}{(u - m^2)^2} \text{Tr} \left[ (\not{p}_2 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\nu (\not{q}_2 + m) \gamma_\mu \right]\end{aligned}$$

Let

$$\begin{aligned} f_{11} &= \text{Tr} \left[ (\not{p}_2 + m) \gamma^\mu (\not{q}_1 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\nu (\not{q}_1 + m) \gamma_\mu \right] \\ f_{12} &= \text{Tr} \left[ (\not{p}_2 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\mu (\not{q}_1 + m) \gamma_\nu \right] \\ f_{22} &= \text{Tr} \left[ (\not{p}_2 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\nu (\not{q}_2 + m) \gamma_\mu \right] \end{aligned}$$

so that

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left[ \frac{f_{11}}{(s - m^2)^2} + \frac{2f_{12}}{(s - m^2)(u - m^2)} + \frac{f_{22}}{(u - m^2)^2} \right]$$

The following formulas are equivalent to the Casimir trick. (Recall that  $a \cdot b = a^\mu g_{\mu\nu} b^\nu$ .)

$$\begin{aligned} f_{11} &= 32(p_1 \cdot p_2)(p_1 \cdot p_4) + 32(p_1 \cdot p_2)m^2 + 32m^4 \\ f_{12} &= 16(p_1 \cdot p_2)m^2 - 16(p_1 \cdot p_4)m^2 + 32m^4 \\ f_{22} &= 32(p_1 \cdot p_2)(p_1 \cdot p_4) - 32(p_1 \cdot p_4)m^2 + 32m^4 \end{aligned} \tag{2}$$

In Mandelstam variables

$$\begin{aligned} f_{11} &= -8su + 24sm^2 + 8um^2 + 8m^4 \\ f_{12} &= 8sm^2 + 8um^2 + 16m^4 \\ f_{22} &= -8su + 8sm^2 + 24um^2 + 8m^4 \end{aligned} \tag{3}$$

Scattering experiments are typically done in the lab frame. Define Lorentz boost  $\Lambda$  for transforming momentum vectors to the lab frame.

$$\Lambda = \begin{pmatrix} E/m & 0 & 0 & \omega/m \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega/m & 0 & 0 & E/m \end{pmatrix}$$

The electron is at rest in the lab frame.

$$\Lambda p_2 = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Mandelstam variables are invariant under a boost.

$$\begin{aligned} s &= (p_1 + p_2)^2 = (\Lambda p_1 + \Lambda p_2)^2 \\ t &= (p_1 - p_3)^2 = (\Lambda p_1 - \Lambda p_3)^2 \\ u &= (p_1 - p_4)^2 = (\Lambda p_1 - \Lambda p_4)^2 \end{aligned} \tag{4}$$

In the lab frame, let  $\omega_L$  be the angular frequency of the incident photon and let  $\omega'_L$  be the angular frequency of the scattered photon.

$$\omega_L = \Lambda p_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2}{m} + \frac{\omega E}{m}$$

$$\omega'_L = \Lambda p_3 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\omega^2 \cos \theta}{m} + \frac{\omega E}{m}$$

It can be shown that

$$\begin{aligned} s &= m^2 + 2m\omega_L \\ t &= 2m(\omega'_L - \omega_L) \\ u &= m^2 - 2m\omega'_L \end{aligned} \tag{5}$$

Then by (1), (3), and (5) we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left[ \frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \left( \frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1 \right)^2 - 1 \right] \tag{6}$$

Lab scattering angle  $\theta_L$  is given by the Compton equation

$$\cos \theta_L = \frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1$$

Hence

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= 2e^4 \left( \frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \cos^2 \theta_L - 1 \right) \\ &= 2e^4 \left( \frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} - \sin^2 \theta_L \right) \end{aligned}$$

Now that we have derived  $\langle |\mathcal{M}|^2 \rangle$  we can investigate the angular distribution of scattered photons. For simplicity let us drop the  $L$  subscript from lab variables. From now on the symbols  $\omega$ ,  $\omega'$ , and  $\theta$  will be lab frame variables.

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4(4\pi\epsilon_0)^2 s} \left( \frac{\omega'}{\omega} \right)^2 \langle |\mathcal{M}|^2 \rangle$$

where

$$s = m^2 + 2m\omega = (mc^2)^2 + 2(mc^2)(\hbar\omega)$$

and  $\omega'$  is given by the Compton equation

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos \theta)}$$

For the lab frame we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

Hence in the lab frame

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\epsilon_0)^2 s} \left( \frac{\omega'}{\omega} \right)^2 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

Substituting

$$e^2 = 4\pi\epsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s} \left( \frac{\omega'}{\omega} \right)^2 \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right)$$

which is the Klein-Nishina formula.