Free particle action

This is the Lagrangian for a free particle.

$$L = \frac{m\dot{x}^2}{2}$$

Show that

$$S = \int_0^T L\,dt = \frac{m(x_b - x_a)^2}{2T}, \quad T = t_b - t_a$$

The first step is to derive $x(t)$ and $\dot{x}(t)$ from L and the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

From L we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}, \quad \frac{\partial L}{\partial x} = 0$$

and by Euler-Lagrange

$$m\ddot{x} = 0$$

Acceleration is zero hence velocity is constant and equals distance divided by time.

$$\dot{x} = \frac{x_b - x_a}{T}$$

The action is

$$\begin{aligned} S &= \int_0^T L\,dt \\ &= \frac{m}{2}\int_0^T \dot{x}^2\,dt \\ &= \frac{m}{2}\int_0^T \frac{(x_b - x_a)^2}{T^2}\,dt \\ &= \frac{m(x_b - x_a)^2}{2T^2}[t]_0^T \\ &= \frac{m(x_b - x_a)^2}{2T} \end{aligned} \tag{1}$$