

(3.1) For boson operators satisfying

$$[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}^\dagger] = \delta_{\mathbf{p}\mathbf{q}} \quad (3.41)$$

show that

$$\frac{1}{\mathcal{V}} \sum_{\mathbf{p}\mathbf{q}} \exp(i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})) [\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}^\dagger] = \delta^{(3)}(\mathbf{x} - \mathbf{y}) \quad (3.42)$$

where  $\mathcal{V}$  is the volume of space over which the system is defined. Repeat this for fermion commutator operators.

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Substitute (3.41) into the left-hand side of (3.42) and call it  $I$ .

$$I = \frac{1}{\mathcal{V}} \sum_{\mathbf{p}\mathbf{q}} \exp(i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})) \delta_{\mathbf{p}\mathbf{q}}$$

By the sifting property of  $\delta_{\mathbf{p}\mathbf{q}}$  we have

$$I = \frac{1}{\mathcal{V}} \sum_{\mathbf{p}} \exp(i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})) \quad (1)$$

Consider the following identity where  $N$  is the number of bosons.

$$\sum_p \exp(ip(x - y)) = N\delta_{xy} \quad (2)$$

Substitute (2) into (1) to obtain

$$I = \frac{N^3}{\mathcal{V}} \delta_{\mathbf{x}\mathbf{y}}$$

Hence (3.42) is true in the limit  $N \rightarrow \infty$ .

$$\lim_{N \rightarrow \infty} \frac{N^3}{\mathcal{V}} \delta_{\mathbf{x}\mathbf{y}} = \delta^{(3)}(\mathbf{x} - \mathbf{y})$$

Note: The infinity is absorbed by the delta function. Recall that

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Also, see equation (4.6).

$$\frac{1}{\mathcal{V}} \sum_{\mathbf{p}} \exp(i\mathbf{p} \cdot \mathbf{x}) = \delta^{(3)}(\mathbf{x}) \quad (4.6)$$