Eigenvalues of angular momentum

We will derive eigenvalues for L^2 and L_z from the following commutation relations.

$$
\begin{aligned}
[L_x, L_y] &= i\hbar L_z \\
[L_y, L_z] &= i\hbar L_x \\
[L_z, L_x] &= i\hbar L_y
\end{aligned}
$$

$$
[L^2, L_z] = 0
$$

Start by defining the following ladder operators.

$$
\begin{aligned}
L_+ &= L_x + iL_y \\
L_- &= L_x - iL_y
\end{aligned}
$$

We have the following commutation relations for ladder operators.

$$
\begin{aligned}
[L_z, L_+] &= [L_z, L_x] + i[L_z, L_y] \\
&= i\hbar L_y + i(-i\hbar L_x) \\
&= \hbar L_+
\end{aligned}
$$

$$
\begin{aligned}
[L_z, L_-] &= [L_z, L_x] - i[L_z, L_y] \\
&= i\hbar L_y - i(-i\hbar L_x) \\
&= -\hbar L_-
\end{aligned}
$$

We also have

$$
\begin{aligned}
L_- L_+ &= (L_x - iL_y)(L_x + iL_y) \\
&= L_x^2 + L_y^2 + i[L_x, L_y] \\
&= L^2 - L_z^2 - \hbar L_z
\end{aligned}
$$

$$
\begin{aligned}
L_+ L_- &= (L_x + iL_y)(L_x - iL_y) \\
&= L_x^2 + L_y^2 - i[L_x, L_y] \\
&= L^2 - L_z^2 + \hbar L_z
\end{aligned}
$$

Operators L^2 and L_z commute hence they share eigenfunctions ψ.

Let λ be an eigenvalue of L^2 and let μ be an eigenvalue of L_z such that

$$
L^2 \psi = \lambda \psi
$$

and

$$
L_z \psi = \mu \psi
$$

We will now show that

$$
\lambda \geq \mu^2
$$

By definition of L^2 we have

$$L^2\psi = \left({L_x}^2 + {L_y}^2 + {L_z}^2\right)\psi$$

Substitute λ for L^2 and μ for L_z to obtain

$$\lambda\psi = \left({L_x}^2 + {L_y}^2 + \mu^2\right)\psi$$

Rewrite as

$$\left({L_x}^2 + {L_y}^2\right)\psi = (\lambda - \mu^2)\psi$$

The eigenvalues of squared Hermitian operators are nonnegative hence $\lambda - \mu^2 \geq 0$. Hence

$$\lambda \geq \mu^2$$

The property $\lambda \geq \mu^2$ means that μ has an upper limit. Let μ_m be the maximum μ and ψ_m its eigenfunction such that

$$L_z\psi_m = \mu_m\psi_m$$

Apply L_+ to both sides.

$$L_+L_z\psi_m = \mu_m L_+\psi_m$$

Expand the left hand side.

$$(L_zL_+ - L_zL_+ + L_+L_z)\psi_m = \mu_m L_+\psi_m$$

Substitute $\hbar L_+$ for $[L_z, L_+]$.

$$L_zL_+\psi_m - \hbar L_+\psi_m = \mu_m L_+\psi_m$$

Hence

$$L_zL_+\psi_m = (\mu_m + \hbar)L_+\psi_m$$

Because μ_m is the maximum eigenvalue and $\mu_m + \hbar > \mu_m$ we must have

$$L_+\psi_m = 0$$

Consequently

$$L_-L_+\psi_m = 0$$

Recalling that

$$L_-L_+ = L^2 - L_z^2 - \hbar L_z$$

we have

$$(L^2 - L_z^2 - \hbar L_z)\psi_m = (\lambda - \mu_m^2 - \hbar\mu_m)\psi_m = 0$$

Hence

$$\lambda = \mu_m^2 + \hbar\mu_m \tag{1}$$

Let μ_k be the minimum eigenvalue of L_z and ψ_k its eigenfunction such that

$$L_z\psi_k = \mu_k\psi_k$$

Apply L_- to both sides.

$$L_- L_z \psi_k = \mu_k L_- \psi_k$$

Expand the left hand side.

$$(L_z L_- - L_z L_- + L_- L_z)\psi_k = \mu_m L_- \psi_k$$

Substitute $-\hbar L_-$ for $[L_z, L_-]$.

$$L_z L_- \psi_k + \hbar L_- \psi_k = \mu_k L_- \psi_k$$

Hence

$$L_z L_- \psi_k = (\mu_k - \hbar) L_- \psi_k$$

Because μ_k is the minimum eigenvalue and $\mu_k - \hbar < \mu_k$ we must have

$$L_- \psi_k = 0$$

Consequently

$$L_+ L_- \psi_k = 0$$

Recalling that

$$L_+ L_- = L^2 - L_z^2 + \hbar L_z$$

we have

$$(L^2 - L_z^2 + \hbar L_z)\psi_k = (\lambda - \mu_k^2 + \hbar \mu_k)\psi_k = 0$$

Hence

$$\lambda = \mu_k^2 - \hbar \mu_k \tag{2}$$

By equivalence of (1) and (2) we have

$$\mu_m^2 + \hbar \mu_m - \mu_k^2 + \hbar \mu_k = 0 \tag{3}$$

By ladder operators there is an integer n such that

$$\mu_m = \mu_k + n\hbar$$

Substitute $\mu_k + n\hbar$ for μ_m in (3) to obtain

$$\mu_k^2 + 2\mu_k n\hbar + n^2\hbar^2 + \hbar\mu_k + n\hbar^2 - \mu_k^2 + \hbar\mu_k = 0$$

Cancel μ_k^2 and rewrite the remaining terms as

$$2\mu_k(n+1)\hbar + n(n+1)\hbar^2 = 0$$

Divide through by $(n+1)\hbar$ to obtain

$$2\mu_k + n\hbar = 0$$

Hence

$$\mu_k = -\frac{n\hbar}{2}$$

and

$$\mu_m = \mu_k + n\hbar = \frac{n\hbar}{2}$$

Define quantum number l as

$$l = \frac{n}{2} = 0,\ \tfrac{1}{2},\ 1,\ \tfrac{3}{2},\ 2,\ \ldots$$

Then

$$\mu_m = l\hbar$$

By equation (1) we have

$$\lambda = (l\hbar)^2 + l\hbar^2 = l(l+1)\hbar^2$$

Hence $l(l+1)\hbar^2$ are eigenvalues for L^2.

$$L^2\psi = \lambda\psi = l(l+1)\hbar^2\psi$$

Define quantum number m as

$$m\hbar = \mu$$

Hence $m\hbar$ are eigenvalues for L_z.

$$L_z\psi = \mu\psi = m\hbar\psi$$

For a given l, operator L_z has eigenvalues

$$\mu = \mu_k, \ldots, \mu_m = -l\hbar, (-l+1)\hbar, \ldots, (l-1)\hbar, l\hbar$$

Hence

$$m = -l, -l+1 \ldots, l-1, l$$