From problem 8-4,

$$|\Phi_0|^2 = \Phi_0^*\Phi_0 = \exp\left(-\frac{1}{\hbar}\sum_{\alpha=1}^{N-1}\omega_\alpha\left((Q_\alpha^c)^2 + (Q_\alpha^s)^2\right)\right)$$

We will use the following integrals.

$$\int_{-\infty}^{\infty} \exp(-ax^2 + b)\, dx = \sqrt{\frac{\pi}{a}}\exp(b) \tag{1}$$

$$\int_{-\infty}^{\infty} x\exp(-ax^2 + b)\, dx = 0 \tag{2}$$

$$\int_{-\infty}^{\infty} x^2\exp(-ax^2 + b)\, dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}\exp(b) \tag{3}$$

Here are some specific examples with $a = \omega_1/\hbar$.

$$\int_{-\infty}^{\infty} |\Phi_0|^2\, dQ_1^c = \left(\frac{\pi\hbar}{\omega_1}\right)^{1/2}\exp\left(\frac{\omega_1}{\hbar}(Q_1^c)^2\right)|\Phi_0|^2 \tag{4}$$

$$\int_{-\infty}^{\infty} Q_1^c|\Phi_0|^2\, dQ_1^c = 0 \tag{5}$$

$$\int_{-\infty}^{\infty} (Q_1^c)^2|\Phi_0|^2\, dQ_1^c = \frac{\hbar}{2\omega_1}\left(\frac{\pi\hbar}{\omega_1}\right)^{1/2}\exp\left(\frac{\omega_1}{\hbar}(Q_1^c)^2\right)|\Phi_0|^2 \tag{6}$$

Multiplying $|\Phi_0|^2$ by an exponential cancels a factor, i.e., in (1) and (4),

$$\exp(b) = \exp\left(\frac{\omega_1}{\hbar}(Q_1^c)^2\right)|\Phi_0|^2$$

Compute the expectation value for Q_1. Let

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{Q_1^c - iQ_1^s}{\sqrt{2}}\right)|\Phi_0|^2\, dQ_1^c\, dQ_1^s$$

By integral (2), $I = 0$ hence by equation (8.84)

$$\langle\Phi_0^*|Q_1|\Phi_0\rangle = 0$$

Compute the expectation value for Q_1^*. Let

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{Q_1^c + iQ_1^s}{\sqrt{2}}\right)|\Phi_0|^2\, dQ_1^c\, dQ_1^s$$

As above, $I = 0$ hence

$$\langle\Phi_0^*|Q_1^*|\Phi_0\rangle = 0$$

Compute the expectation value for Q_1^2. Let

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{Q_1^c - iQ_1^s}{\sqrt{2}}\right)^2|\Phi_0|^2\, dQ_1^c\, dQ_1^s$$

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Rewrite as

$$I = -i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_1^c Q_1^s |\Phi_0|^2 \, dQ_1^c \, dQ_1^s$$

$$+ \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (Q_1^c)^2 |\Phi_0|^2 \, dQ_1^c \, dQ_1^s - \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (Q_1^s)^2 |\Phi_0|^2 \, dQ_1^c \, dQ_1^s$$

The first integral vanishes by (2). The remaining integrals cancel hence

$$\langle \Phi_0^* | Q_1^2 | \Phi_0 \rangle = 0$$

Compute the expectation value for Q_1^{*2}. (As above except for a sign change.)

$$I = i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_1^c Q_1^s |\Phi_0|^2 \, dQ_1^c \, dQ_1^s$$

$$+ \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (Q_1^c)^2 |\Phi_0|^2 \, dQ_1^c \, dQ_1^s - \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (Q_1^s)^2 |\Phi_0|^2 \, dQ_1^c \, dQ_1^s$$

By the same arguments as Q_1^2

$$\langle \Phi_0^* | Q_1^{*2} | \Phi_0 \rangle = 0$$

Compute the expectation value for $Q_1^* Q_1$. Let

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{(Q_1^c)^2 + (Q_1^s)^2}{2} \right) |\Phi_0|^2 \, dQ_1^c \, dQ_1^s$$

Rewrite as

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} (Q_1^c)^2 |\Phi_0|^2 \, dQ_1^c \right) dQ_1^s + \frac{1}{2} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} (Q_1^s)^2 |\Phi_0|^2 \, dQ_1^s \right) dQ_1^c$$

By integral (3) with $a = \omega_1/\hbar$,

$$I = \frac{\hbar}{4\omega_1} \left(\frac{\pi \hbar}{\omega_1} \right)^{1/2} \exp \left(\frac{\omega_1}{\hbar} (Q_1^c)^2 \right) \int_{-\infty}^{\infty} |\Phi_0|^2 \, dQ_1^s$$

$$+ \frac{\hbar}{4\omega_1} \left(\frac{\pi \hbar}{\omega_1} \right)^{1/2} \exp \left(\frac{\omega_1}{\hbar} (Q_1^s)^2 \right) \int_{-\infty}^{\infty} |\Phi_0|^2 \, dQ_1^c$$

By integral (1),

$$I = \frac{\hbar}{4\omega_1} \frac{\pi \hbar}{\omega_1} \exp \left(\frac{\omega_1}{\hbar} (Q_1^c)^2 \right) \exp \left(\frac{\omega_1}{\hbar} (Q_1^s)^2 \right) |\Phi_0|^2$$

$$+ \frac{\hbar}{4\omega_1} \frac{\pi \hbar}{\omega_1} \exp \left(\frac{\omega_1}{\hbar} (Q_1^s)^2 \right) \exp \left(\frac{\omega_1}{\hbar} (Q_1^c)^2 \right) |\Phi_0|^2$$

Integrate over the remaining measure as in (8.84).

$$\langle \Phi_0^* | Q_1^* Q_1 | \Phi_0 \rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} I \, dQ_2^c \, dQ_2^s \cdots dQ_{N-1}^c \, dQ_{N-1}^s = \frac{\hbar}{2\omega_1} \frac{\pi \hbar}{\omega_1} \prod_{k=2}^{N-1} \frac{\pi \hbar}{\omega_k} \qquad (7)$$

Note that the exponentials cancel with u.

Compute the normalization constant. By integral (1),

$$\langle \Phi_0^* | 1 | \Phi_0 \rangle = \prod_{k=1}^{N-1} \frac{\pi \hbar}{\omega_k} \qquad (8)$$

Hence by (7) and (8),

$$\langle \Phi_0^* | Q_1^* Q_1 | \Phi_0 \rangle = \frac{\hbar}{2\omega_1} \langle \Phi_0^* | 1 | \Phi_0 \rangle$$

Compute the expectation value for $Q_1^* Q_2$. Let

$$I = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\frac{Q_1^c Q_2^c - Q_1^s Q_2^s - i Q_1^c Q_2^s - i Q_1^s Q_2^c}{2} \right) |\Phi_0|^2 \, dQ_1^c \, dQ_1^s \, dQ_2^c \, dQ_2^s$$

By integral (2) we have $I = 0$, hence by equation (8.84)

$$\langle \Phi_0^* | Q_1^* Q_2 | \Phi_0 \rangle = 0$$

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