

Rutherford scattering 1

Use the formula

$$\frac{d\sigma}{d\Omega} = \left(\frac{mQ}{2\pi\hbar} \right)^2, \quad Q = \frac{1}{\hbar} \int \exp \left(\frac{i\mathbf{p} \cdot \mathbf{r}}{\hbar} \right) V(\mathbf{r}) d\mathbf{r}^3$$

to compute the cross section for Rutherford scattering.

For Rutherford scattering $V(\mathbf{r})$ is the Coulomb potential

$$V(\mathbf{r}) = V(r) = -\frac{Ze^2}{r}$$

Convert Q to polar coordinates.

$$Q = \frac{1}{\hbar} \int_0^{2\pi} \int_0^\pi \int_0^\infty \exp \left(\frac{ipr \cos \theta}{\hbar} \right) V(r, \theta, \phi) r^2 \sin \theta dr d\theta d\phi$$

Substitute $V(r)$ for $V(r, \theta, \phi)$.

$$Q = \frac{1}{\hbar} \int_0^{2\pi} \int_0^\pi \int_0^\infty \exp \left(\frac{ipr \cos \theta}{\hbar} \right) V(r) r^2 \sin \theta dr d\theta d\phi$$

Integrate over ϕ .

$$Q = \frac{2\pi}{\hbar} \int_0^\pi \int_0^\infty \exp \left(\frac{ipr \cos \theta}{\hbar} \right) V(r) r^2 \sin \theta dr d\theta$$

Change the complex exponential to rectangular form.

$$Q = \frac{2\pi}{\hbar} \int_0^\pi \int_0^\infty \left[\cos \left(\frac{pr \cos \theta}{\hbar} \right) + i \sin \left(\frac{pr \cos \theta}{\hbar} \right) \right] V(r) r^2 \sin \theta dr d\theta$$

By the integrals

$$\int_0^\pi \cos(a \cos(\theta)) \sin \theta d\theta = \frac{2 \sin a}{a}, \quad \int_0^\pi \sin(a \cos(\theta)) \sin \theta d\theta = 0$$

we have

$$Q = \frac{4\pi}{p} \int_0^\infty \sin \left(\frac{pr}{\hbar} \right) V(r) r dr$$

Substitute the Coulomb potential for $V(r)$.

$$Q = -\frac{4\pi Ze^2}{p} \int_0^\infty \sin \left(\frac{pr}{\hbar} \right) dr$$

To solve the integral, multiply the integrand by $\exp(-\epsilon r)$.

$$Q = -\frac{4\pi Ze^2}{p} \int_0^\infty \sin \left(\frac{pr}{\hbar} \right) \exp(-\epsilon r) dr$$

Convert the integrand to exponential form.

$$Q = -\frac{4\pi Ze^2}{p} \int_0^\infty \frac{i}{2} \left[\exp\left(-\frac{ipr}{\hbar} - \epsilon r\right) - \exp\left(\frac{ipr}{\hbar} - \epsilon r\right) \right] dr$$

Solve the integral.

$$Q = -\frac{4\pi Ze^2}{p} \frac{i}{2} \left(\frac{1}{-ip/\hbar - \epsilon} - \frac{1}{ip/\hbar - \epsilon} \right) \quad (1)$$

Set $\epsilon = 0$.

$$Q = -\frac{4\pi Ze^2}{p} \left(-\frac{\hbar}{p} \right)$$

Hence

$$Q = \frac{4\pi Ze^2 \hbar}{p^2}$$

Compute the differential cross section.

$$\frac{d\sigma}{d\Omega} = \left(\frac{mQ}{2\pi\hbar} \right)^2 = \frac{4m^2 Z^2 e^4}{p^4}$$

Symbol p is momentum transfer $|\mathbf{p}_i| - |\mathbf{p}_f|$ such that

$$p^2 = 2mE(\cos \theta - 1)$$