9-7. Show, for the vacuum state, the expectation value of $\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}}$ is $(\hbar/2kc)\delta_{\mathbf{k},\mathbf{q}}$ and that of $\bar{a}_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}}$ is $(\hbar/2kc)\delta_{-\mathbf{k},\mathbf{q}}$.

We will use the following integrals.

$$\int_{-\infty}^{\infty} \exp(-ax^2+b)\,dx = \sqrt{\frac{\pi}{a}}\exp(b) \tag{1}$$

$$\int_{-\infty}^{\infty} x\exp(-ax^2+b)\,dx = 0 \tag{2}$$

$$\int_{-\infty}^{\infty} x^2\exp(-ax^2+b)\,dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}\exp(b) \tag{3}$$

For simplicity of notation, let

$$A = \bar{a}^c_{1,\mathbf{k}} \qquad B = \bar{a}^s_{1,\mathbf{k}} \qquad C = \bar{a}^c_{1,\mathbf{q}} \qquad D = \bar{a}^s_{1,\mathbf{q}}$$

From problem 9-6

$$\begin{aligned} \bar{a}_{1,\mathbf{k}} &= \frac{1}{\sqrt{2}}(A - iB) \\ \bar{a}_{1,\mathbf{q}} &= \frac{1}{\sqrt{2}}(C - iD) \end{aligned} \tag{4}$$

Adapted from equation (8.84)

$$\langle\Phi_0|f|\Phi_0\rangle = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \Phi_0^* f \Phi_0\,dA\,dB\,dC\,dD$$

Adapted from problem 9-6 with q as a mode number (not electric charge).

$$\Phi_0 = \exp\left(-\frac{kc}{4\hbar}A^2 - \frac{kc}{4\hbar}B^2 - \frac{qc}{4\hbar}C^2 - \frac{qc}{4\hbar}D^2\right)$$

It follows that

$$\Phi_0^*\Phi_0 = \exp\left(-\frac{kc}{2\hbar}A^2 - \frac{kc}{2\hbar}B^2 - \frac{qc}{2\hbar}C^2 - \frac{qc}{2\hbar}D^2\right)$$

Compute the normalization constant K.

$$K = \langle\Phi_0|1|\Phi_0\rangle = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \Phi_0^*\Phi_0\,dA\,dB\,dC\,dD$$

Apply integral (1) for each factor in the measure (see problem 8-5).

$$K = \left(\frac{2\pi\hbar}{kc}\right)^{1/2}\left(\frac{2\pi\hbar}{kc}\right)^{1/2}\left(\frac{2\pi\hbar}{qc}\right)^{1/2}\left(\frac{2\pi\hbar}{qc}\right)^{1/2}$$

Compute the expectation value for $\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{k}}$. From (4) we have

$$\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{k}} = \frac{A^2+B^2}{2}$$

Hence

$$\langle\Phi_0|\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{k}}|\Phi_0\rangle = \frac{1}{K}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Phi_0^*\Phi_0\frac{A^2+B^2}{2}\,dA\,dB\,dC\,dD$$

Rewrite as

$$\langle\Phi_0|\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{k}}|\Phi_0\rangle = \frac{1}{2K}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Phi_0^*\Phi_0A^2\,dA\,dB\,dC\,dD$$
$$+\frac{1}{2K}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Phi_0^*\Phi_0B^2\,dA\,dB\,dC\,dD$$

By integrals (1) and (3) we have

$$\langle\Phi_0|\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{k}}|\Phi_0\rangle = \frac{1}{K}\frac{\hbar}{kc}\left(\frac{2\pi\hbar}{kc}\right)^{1/2}\left(\frac{2\pi\hbar}{kc}\right)^{1/2}\left(\frac{2\pi\hbar}{qc}\right)^{1/2}\left(\frac{2\pi\hbar}{qc}\right)^{1/2}$$

Hence

$$\langle\Phi_0|\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{k}}|\Phi_0\rangle = \frac{\hbar}{kc} \tag{5}$$

Compute the expectation value for $\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}}$. From (4) we have

$$\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}} = \frac{AC+BD-iAD+iBC}{2}$$

Hence

$$\langle\Phi_0|\bar{a}^*_{1,\mathbf{k}}\bar{a}_{1,\mathbf{q}}|\Phi_0\rangle =$$
$$\frac{1}{K}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Phi_0^*\Phi_0\frac{AC+BD-iAD+iBC}{2}\,dA\,dB\,dC\,dD$$

By integral (2) all terms are zero, hence

$$\langle \Phi_0 | \bar{a}^*_{1,\mathbf{k}} \bar{a}_{1,\mathbf{q}} | \Phi_0 \rangle = 0 \tag{6}$$