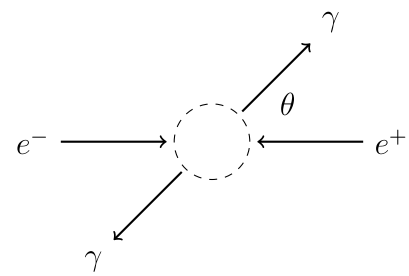
Annihilation

Annihilation is the interaction $e^- + e^+ \to \gamma + \gamma$.



In the center-of-mass frame we have the following momentum vectors where $E = \sqrt{p^2 + m^2}$.

$$p_1 = \underset{\substack{\text{inbound}\\\text{electron}}}{\begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}} \qquad p_2 = \underset{\substack{\text{inbound}\\\text{positron}}}{\begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix}} \qquad p_3 = \underset{\substack{\text{outbound}\\\text{photon}}}{\begin{pmatrix} E \\ E\sin\theta\cos\phi \\ E\sin\theta\sin\phi \\ E\cos\theta \end{pmatrix}} \qquad p_4 = \underset{\substack{\text{outbound}\\\text{photon}}}{\begin{pmatrix} E \\ -E\sin\theta\cos\phi \\ -E\sin\theta\sin\phi \\ -E\cos\theta \end{pmatrix}}$$

Spinors for the inbound electron.

$$u_{11} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound electron}\\\text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix}} \qquad u_{12} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound electron}\\\text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix}}$$

Spinors for the inbound positron.

$$v_{21} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound positron}\\\text{spin up}}}{\begin{pmatrix} -p \\ 0 \\ E+m \\ 0 \end{pmatrix}} \qquad v_{22} = \frac{1}{\sqrt{E+m}} \underset{\substack{\text{inbound positron}\\\text{spin down}}}{\begin{pmatrix} 0 \\ p \\ 0 \\ E+m \end{pmatrix}}$$

The probability amplitude $\mathcal{M}_{ab}$ for spin state ab is

$$\mathcal{M}_{ab} = \mathcal{M}_{1ab} + \mathcal{M}_{2ab}$$

where

$$\mathcal{M}_{1ab} = \frac{\bar{v}_{2b}(-ie\gamma^\mu)(\not{q}_1 + m)(-ie\gamma^\nu)u_{1a}}{t - m^2}, \quad \mathcal{M}_{2ab} = \frac{\bar{v}_{2b}(-ie\gamma^\nu)(\not{q}_2 + m)(-ie\gamma^\mu)u_{1a}}{u - m^2}$$

Matrices $\not{q}_1$ and $\not{q}_2$ represent momentum transfer.

$$\not{q}_1 = (p_1 - p_3)^\alpha g_{\alpha\beta}\gamma^\beta$$
$$\not{q}_2 = (p_1 - p_4)^\alpha g_{\alpha\beta}\gamma^\beta$$

Scalars t and u are Mandelstam variables.

$$
\begin{aligned}
t &= (p_1 - p_3)^2 \\
u &= (p_1 - p_4)^2
\end{aligned}
$$

In component form (note that indices μ and ν are interchanged for $\mathcal{M}_{2ab}$)

$$
\begin{aligned}
(\mathcal{M}_{1ab})^{\mu\nu} &= \frac{(\bar{v}_{2b})_\alpha(-ie\gamma^{\mu\alpha}{}_\beta)(\not{q}_1 + m)^\beta{}_\rho(-ie\gamma^{\nu\rho}{}_\sigma)(u_{1a})^\sigma}{t - m^2} \\
(\mathcal{M}_{2ab})^{\nu\mu} &= \frac{(\bar{v}_{2b})_\alpha(-ie\gamma^{\nu\alpha}{}_\beta)(\not{q}_2 + m)^\beta{}_\rho(-ie\gamma^{\mu\rho}{}_\sigma)(u_{1a})^\sigma}{u - m^2}
\end{aligned}
$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is the average of spin states.

$$
\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}|\mathcal{M}_{ab}|^2
$$

Hence

$$
\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\left(\mathcal{M}_{1ab}\mathcal{M}^*_{1ab} + \mathcal{M}_{1ab}\mathcal{M}^*_{2ab} + \mathcal{M}_{2ab}\mathcal{M}^*_{1ab} + \mathcal{M}_{2ab}\mathcal{M}^*_{2ab}\right)
$$

Metric tensor $g_{\mu\nu}$ lowers indices.

$$
\begin{aligned}
\mathcal{M}_{1ab}\mathcal{M}^*_{1ab} &= (\mathcal{M}_{1ab})^{\mu\nu}(\mathcal{M}^*_{1ab})_{\mu\nu} = (\mathcal{M}_{1ab})^{\mu\nu}\, g_{\mu\alpha}(\mathcal{M}^*_{1ab})^{\alpha\beta}g_{\beta\nu} \\
\mathcal{M}_{1ab}\mathcal{M}^*_{2ab} &= (\mathcal{M}_{1ab})^{\mu\nu}(\mathcal{M}^*_{2ab})_{\nu\mu} = (\mathcal{M}_{1ab})^{\mu\nu}\, g_{\nu\alpha}(\mathcal{M}^*_{2ab})^{\alpha\beta}g_{\beta\mu} \\
\mathcal{M}_{2ab}\mathcal{M}^*_{2ab} &= (\mathcal{M}_{2ab})^{\nu\mu}(\mathcal{M}^*_{2ab})_{\nu\mu} = (\mathcal{M}_{2ab})^{\nu\mu}\, g_{\nu\alpha}(\mathcal{M}^*_{2ab})^{\alpha\beta}g_{\beta\mu}
\end{aligned}
$$

The Casimir trick uses matrix arithmetic to sum over spin states.

$$
\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left(\frac{f_{11}}{(t-m^2)^2} + \frac{2f_{12}}{(t-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2}\right)
$$

where

$$
\begin{aligned}
f_{11} &= \mathrm{Tr}\left((\not{p}_1 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\right) \\
f_{12} &= \mathrm{Tr}\left((\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\right) \\
f_{22} &= \mathrm{Tr}\left((\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_2 + m)\gamma_\mu\right)
\end{aligned}
$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$
\begin{aligned}
f_{11} &= 32(p_1\cdot p_3)(p_1\cdot p_4) - 32m^2(p_1\cdot p_2) + 64m^2(p_1\cdot p_3) + 32m^2(p_1\cdot p_4) - 64m^4 \\
f_{12} &= 16m^2(p_1\cdot p_3) + 16m^2(p_1\cdot p_4) - 32m^4 \\
f_{22} &= 32(p_1\cdot p_3)(p_1\cdot p_4) - 32m^2(p_1\cdot p_2) + 32m^2(p_1\cdot p_3) + 64m^2(p_1\cdot p_4) - 64m^4
\end{aligned}
$$

In Mandelstam variables

$$
\begin{aligned}
f_{11} &= 8tu - 24tm^2 - 8um^2 - 8m^4 \\
f_{12} &= 8sm^2 - 32m^4 \\
f_{22} &= 8tu - 8tm^2 - 24um^2 - 8m^4
\end{aligned}
$$

For $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$
\begin{aligned}
f_{11} &= 8tu \\
f_{12} &= 0 \\
f_{22} &= 8tu
\end{aligned}
$$

Hence

$$
\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left(\frac{f_{11}}{(t-m^2)^2} + \frac{2f_{12}}{(t-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2} \right) \\
&= \frac{e^4}{4} \left(\frac{8tu}{t^2} + \frac{8tu}{u^2} \right) \\
&= 2e^4 \left(\frac{u}{t} + \frac{t}{u} \right)
\end{aligned}
$$

For $m = 0$ the Mandelstam variables are

$$
\begin{aligned}
t &= -2E^2(1 - \cos\theta) \\
u &= -2E^2(1 + \cos\theta)
\end{aligned}
$$

Hence

$$
\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta} \right)
$$

Cross section

The differential cross section is

$$
\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{4(4\pi\varepsilon_0)^2 s}
$$

where

$$
s = (p_1 + p_2)^2 = 4E^2
$$

For high energy experiments we have

$$
\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta} \right)
$$

Hence

$$
\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\varepsilon_0)^2 s} \left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta} \right)
$$

Noting that

$$e^2 = 4\pi\varepsilon_0 \alpha \hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s}\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)$$

Noting that

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

we also have

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{2s}\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)\sin\theta \, d\theta \, d\phi$$

Let $S(\theta_1, \theta_2)$ be the following integral of $d\sigma$.

$$S(\theta_1, \theta_2) = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{\pi\alpha^2(\hbar c)^2}{s}[I(\theta_2) - I(\theta_1)]$$

where

$$I(\theta) = 2\cos\theta + 2\log(1-\cos\theta) - 2\log(1+\cos\theta)$$

The cumulative distribution function is

$$F(\theta) = \frac{S(a,\theta)}{S(a,\pi-a)} = \frac{I(\theta) - I(a)}{I(\pi-a) - I(a)}, \quad a \le \theta \le \pi - a$$

Angular support is reduced by an arbitrary angle $a > 0$ because $I(0)$ and $I(\pi)$ are undefined.

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 < \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi-a) - I(a)}\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)\sin\theta$$