

# Harmonic oscillator propagator

Consider the harmonic oscillator wave function

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left( \frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} H_n \left( \sqrt{\frac{m\omega}{\hbar}} x \right) \exp \left( -\frac{m\omega x^2}{2\hbar} \right)$$

and its propagator

$$K(x_b, t_b, x_a, t_a) = \left( \frac{m\omega}{2\pi i \hbar \sin(\omega T)} \right)^{\frac{1}{2}} \exp \left[ \frac{im\omega}{2\hbar \sin(\omega T)} (x_a^2 \cos(\omega T) - 2x_a x_b + x_b^2 \cos(\omega T)) \right]$$

where  $T = t_b - t_a$ .

We should have

$$\psi_n(x_b) = \int_{-\infty}^{\infty} K(x_b, t_b, x_a, t_a) \psi_n(x_a) dx_a$$

Try for  $n = 1$ .

$$\psi_1(x_a) = \sqrt{2} \left( \frac{m^2 \omega^3}{\pi \hbar^3} \right)^{\frac{1}{4}} x_a \exp \left( -\frac{m\omega x_a^2}{2\hbar} \right)$$

By the identity

$$\int_{-\infty}^{\infty} y \exp(-ay^2 + by) dy = \frac{\sqrt{\pi}}{2} \frac{b}{a^{3/2}} \exp \left( \frac{b^2}{4a} \right)$$

the path integral is

$$\begin{aligned} I = \int_{-\infty}^{\infty} K(x_b, t_b, x_a, t_a) \psi_1(x_a) dx_a &= \frac{1}{2} \left( -\frac{m^9 \omega^9}{\pi \hbar^9} \right)^{\frac{1}{4}} \left( \frac{m\omega}{2\hbar} \sin(\omega T) - \frac{im\omega}{2\hbar} \cos(\omega T) \right)^{-\frac{3}{2}} \\ &\times x_b \exp \left( -\frac{m\omega x_b^2}{2\hbar (\sin(\omega T)^2 - i \sin(\omega T) \cos(\omega T))} + \frac{im\omega x_b^2 \cos(\omega T)}{2\hbar \sin(\omega T)} \right) \end{aligned}$$

Let  $T = \pi/(2\omega)$  to cancel time dependency. The result is

$$I = \psi_1(x_b) \exp \left( \frac{i\pi}{4} \right)$$

The phase shift is inconsequential and can be discarded.

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