Rutherford scattering 1

Find the scattering cross section for Coulomb potential $V(r)$.

$$V(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r}$$

Start with the Born approximation.

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |Q|^2, \quad Q = \int \exp\left(\frac{i\mathbf{p}\cdot\mathbf{r}}{\hbar}\right) V(\mathbf{r})\, d\mathbf{r}^3$$

Convert Q to polar coordinates.

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) V(r,\theta,\phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Substitute the Coulomb potential for $V(r,\theta,\phi)$ and note r^2 becomes r.

$$Q = -\frac{Ze^2}{4\pi\varepsilon_0} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) r \sin\theta\, dr\, d\theta\, d\phi$$

Integrate over ϕ (multiplies Q by 2π).

$$Q = -\frac{Ze^2}{2\varepsilon_0} \int_0^{\pi} \int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) r \sin\theta\, dr\, d\theta$$

Transform the integral over θ to an integral over y where $y = \cos\theta$ and $dy = -\sin\theta\, d\theta$. The minus sign in dy is canceled by interchanging integration limits $\cos 0 = 1$ and $\cos\pi = -1$.

$$Q = -\frac{Ze^2}{2\varepsilon_0} \int_{-1}^{1} \int_0^{\infty} \exp\left(\frac{ipry}{\hbar}\right) r\, dr\, dy$$

Solve the integral over y (note r in the integrand cancels).

$$Q = -\frac{Ze^2}{2\varepsilon_0} \int_0^{\infty} \frac{\hbar}{ip} \left[\exp\left(\frac{ipr}{\hbar}\right) - \exp\left(-\frac{ipr}{\hbar}\right)\right] dr$$

Solve the integral over r.

$$Q = -\frac{Ze^2}{2\varepsilon_0}\frac{\hbar}{ip} \left[\frac{\hbar}{ip}\exp\left(\frac{ipr}{\hbar}\right) + \frac{\hbar}{ip}\exp\left(-\frac{ipr}{\hbar}\right)\right]_0^{\infty}$$

The first exponential is a problem so go back and multiply the integrand by $\exp(-\epsilon r)$.

$$Q = -\frac{Ze^2}{2\varepsilon_0} \int_0^{\infty} \frac{\hbar}{ip} \left[\exp\left(\frac{ipr}{\hbar} - \epsilon r\right) - \exp\left(-\frac{ipr}{\hbar} - \epsilon r\right)\right] dr$$

Solve the integral.

$$Q = -\frac{Ze^2}{2\varepsilon_0}\frac{\hbar}{ip} \left[\frac{1}{ip/\hbar - \epsilon}\exp\left(\frac{ipr}{\hbar} - \epsilon r\right) + \frac{1}{ip/\hbar + \epsilon}\exp\left(-\frac{ipr}{\hbar} - \epsilon r\right)\right]_0^{\infty}$$

Evaluate the limits.

$$Q = -\frac{Ze^2}{2\varepsilon_0}\frac{\hbar}{ip}\left(-\frac{1}{ip/\hbar - \epsilon} - \frac{1}{ip/\hbar + \epsilon}\right) = -\frac{Ze^2}{2\varepsilon_0}\frac{2}{(p/\hbar)^2 + \epsilon^2} \tag{1}$$

Set $\epsilon = 0$ to obtain

$$Q = -\frac{Ze^2\hbar^2}{\varepsilon_0 p^2}$$

Calculate the cross section.

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |Q|^2 = \frac{m^2Z^2e^4}{4\pi^2\varepsilon_0^2p^4} \tag{2}$$

Substitute $16\pi^2\varepsilon_0^2\alpha^2\hbar^2c^2$ for e^4.

$$\frac{d\sigma}{d\Omega} = \frac{4m^2Z^2\alpha^2\hbar^2c^2}{p^4}$$

Symbol p is momentum transfer $|\mathbf{p}_i| - |\mathbf{p}_f|$ such that

$$p^2 = 4mE(1 - \cos\theta)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2\hbar^2c^2}{4E^2(1-\cos\theta)^2} \tag{3}$$

Noting that

$$(1-\cos\theta)^2 = 4\sin^4(\theta/2)$$

we have the alternative form of (3)

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2\hbar^2c^2}{16E^2\sin^4(\theta/2)}$$