9-5. The momentum in the field is given by

$$\frac{1}{4\pi c} \int \mathbf{E} \times \mathbf{B} \, d^3\mathbf{r}$$

In the absence of matter (so $\phi_{\mathbf{k}} = 0$), show that this is

$$i \int \mathbf{k} \, (\mathbf{a}_{\mathbf{k}}^* \cdot \dot{\mathbf{a}}_{\mathbf{k}}) \, \frac{d^3\mathbf{k}}{(2\pi)^3}$$

Let

$$D_{\mathbf{k}} = \exp(i\mathbf{k} \cdot \mathbf{r}) \, \frac{d^3\mathbf{k}}{(2\pi)^3}$$

Consider equation (9.14).

$$\mathbf{A} = \sqrt{4\pi}c \int \mathbf{a}_{\mathbf{k}} \, D_{\mathbf{k}} \qquad\qquad (9.14)$$

By equation (9.9) with $\phi = 0$ and (9.14)

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = -\sqrt{4\pi} \int \dot{\mathbf{a}}_{\mathbf{k}} \, D_{\mathbf{k}}$$

By equation (9.7) and (9.14)

$$\mathbf{B} = \nabla \times \mathbf{A} = \sqrt{4\pi}ic \int \mathbf{k} \times \mathbf{a}_{\mathbf{k}} \, D_{\mathbf{k}}$$

Hence

$$\mathbf{E} \times \mathbf{B} = -4\pi ic \int \dot{\mathbf{a}}_{\mathbf{k}} \times (\mathbf{k} \times \mathbf{a}_{\mathbf{k}}) \, D_{\mathbf{k}}$$

By the triple product formula

$$\mathbf{E} \times \mathbf{B} = -4\pi ic \int \left((\dot{\mathbf{a}}_{\mathbf{k}} \cdot \mathbf{a}_{\mathbf{k}})\mathbf{k} - (\dot{\mathbf{a}}_{\mathbf{k}} \cdot \mathbf{k})\mathbf{a}_{\mathbf{k}} \right) D_{\mathbf{k}}$$

By orthogonality of $\mathbf{E}$ and $\mathbf{k}$ (see problem 9-1), $\dot{\mathbf{a}}_{\mathbf{k}} \cdot \mathbf{k} = 0$ hence

$$\mathbf{E} \times \mathbf{B} = -4\pi ic \int (\dot{\mathbf{a}}_{\mathbf{k}} \cdot \mathbf{a}_{\mathbf{k}})\mathbf{k} \, D_{\mathbf{k}}$$

Integrate over all space.

$$\int \mathbf{E} \times \mathbf{B} \, d^3\mathbf{r} = -4\pi ic \int (\dot{\mathbf{a}}_{\mathbf{k}} \cdot \mathbf{a}_{\mathbf{k}})\mathbf{k} \, \frac{d^3\mathbf{k}}{(2\pi)^3}$$

Note: Integrating over all $\mathbf{r}$ is the same as integrating over all $\mathbf{k}$, hence the exponential in $D_{\mathbf{k}}$ is cancelled.

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