2-2. For a harmonic oscillator $L = (m/2)(\dot{x}^2 - \omega^2 x^2)$. With T equal to $t_b - t_a$, show that the classical action is

$$S_{cl} = \frac{m\omega}{2\sin\omega T}\left(\left(x_b^2 + x_a^2\right)\cos\omega T - 2x_b x_a\right) \tag{2.9}$$

We will need equation (2.7) to determine the classical motion $x(t)$.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \tag{2.7}$$

For the Lagrangian L given in problem 2-2 we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \qquad \frac{\partial L}{\partial x} = -m\omega^2 x$$

By equation (2.7) we have for the classical acceleration $\ddot{x}(t)$

$$\ddot{x}(t) = -\omega^2 x \tag{1}$$

The well-known solution to (1) is

$$x(t) = A\sin(\omega t) + B\cos(\omega t)$$

We have the following boundary conditions.

$$x(0) = x_a \qquad x(T) = x_b$$

Solve for B.

$$B = x(0) = x_a$$

For $x(T)$ we have

$$x(T) = A\sin(\omega T) + B\cos(\omega T)$$

Solve for A.

$$A = \frac{x(T) - B\cos(\omega T)}{\sin(\omega T)} = \frac{x_b - x_a\cos(\omega T)}{\sin(\omega T)}$$

Hence the equation of motion is

$$x(t) = \frac{x_b - x_a\cos(\omega T)}{\sin(\omega T)}\sin(\omega t) + x_a\cos(\omega t) \tag{2}$$

Differentiate $x(t)$ to obtain velocity $\dot{x}(t)$.

$$\dot{x}(t) = \frac{d}{dt}x(t) = \omega\left(\frac{x_b - x_a\cos(\omega T)}{\sin(\omega T)}\cos(\omega t) - x_a\sin(\omega t)\right) \tag{3}$$

Using the action integral

$$S = \int_0^T L\,dt$$

we have for the classical action

$$\begin{aligned} S_{cl} &= \frac{m}{2}\int_0^T (\dot{x}^2 - \omega^2 x^2)\,dt \\ &= \frac{m}{2}\left(\int_0^T \dot{x}^2\,dt - \int_0^T \omega^2 x^2\,dt\right) \end{aligned}$$

Let $u = v = \dot{x}$ and note that

$$\dot{u} = \ddot{x}$$

$$\int v\,dt = x$$

Apply integration by parts to the integral of $\dot{x}^2$.

$$\begin{aligned} \int_0^T \dot{x}^2\,dt &= \int_0^T uv\,dt \\ &= \left(u\int v\,dt\right)_0^T - \int_0^T \dot{u}\left(\int v\,dt\right)\,dt \\ &= \dot{x}x\Big|_0^T - \int_0^T \ddot{x}x\,dt \end{aligned}$$

Hence

$$S_{cl} = \frac{m}{2}\left(\dot{x}x\Big|_0^T - \int_0^T \ddot{x}x\,dt - \int_0^T \omega^2 x^2\,dt\right)$$

The two integrals on the right cancel by $\ddot{x} = -\omega^2 x$ from equation (1).

We now have

$$S_{cl} = \frac{m}{2}\dot{x}x\Big|_0^T = \frac{m}{2}\dot{x}(T)x(T) - \frac{m}{2}\dot{x}(0)x(0) \tag{4}$$

By evaluation of $x(t)$ and $\dot{x}(t)$ at $t = T$ and $t = 0$ we have

$$S_{cl} = \frac{m\omega}{2\sin(\omega T)} \left(\left(x_b^2 + x_a^2 \right) \cos(\omega T) - 2x_b x_a \right) \tag{5}$$