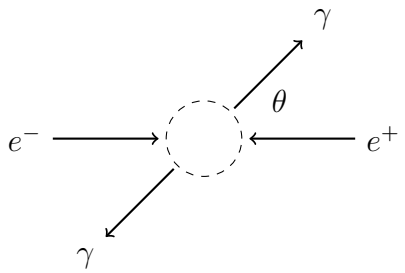
Annihilation

Annihilation is the interaction $e^- + e^+ \to \gamma + \gamma$.



Define the following momentum vectors and spinors. Symbol p is incident momentum. Symbol E is total energy $E = \sqrt{p^2 + m^2}$ where m is electron mass. Polar angle θ is the observed scattering angle. Azimuth angle ϕ cancels out in scattering calculations.

$$p_1 = \underset{\text{inbound electron}}{\begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}} \qquad p_2 = \underset{\text{inbound positron}}{\begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix}} \qquad p_3 = \underset{\text{outbound photon}}{\begin{pmatrix} E \\ E\sin\theta\cos\phi \\ E\sin\theta\sin\phi \\ E\cos\theta \end{pmatrix}} \qquad p_4 = \underset{\text{outbound photon}}{\begin{pmatrix} E \\ -E\sin\theta\cos\phi \\ -E\sin\theta\sin\phi \\ -E\cos\theta \end{pmatrix}}$$

$$u_{11} = \underset{\substack{\text{inbound electron} \\ \text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix}} \qquad v_{21} = \underset{\substack{\text{inbound positron} \\ \text{spin up}}}{\begin{pmatrix} -p \\ 0 \\ E+m \\ 0 \end{pmatrix}}$$

$$u_{12} = \underset{\substack{\text{inbound electron} \\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix}} \qquad v_{22} = \underset{\substack{\text{inbound positron} \\ \text{spin down}}}{\begin{pmatrix} 0 \\ p \\ 0 \\ E+m \end{pmatrix}}$$

The spinors are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^2$ will be used.

This is the probability density for spin state ab. The formula is derived from Feynman diagrams for annihilation.

$$|\mathcal{M}_{ab}|^2 = \frac{e^4}{N}\left|-\frac{\bar{v}_{2b}\gamma^\mu(\not{q}_1 + m)\gamma^\nu u_{1a}}{t - m^2} - \frac{\bar{v}_{2b}\gamma^\nu(\not{q}_2 + m)\gamma^\mu u_{1a}}{u - m^2}\right|^2$$

Symbol e is electron charge and

$$\not{q}_1 = (p_1 - p_3)^\mu g_{\mu\nu}\gamma^\nu$$
$$\not{q}_2 = (p_1 - p_4)^\mu g_{\mu\nu}\gamma^\nu$$

Symbols t and u are Mandelstam variables

$$t = (p_1 - p_3)^2 = (p_1 - p_3)^\mu g_{\mu\nu} (p_1 - p_3)^\nu$$
$$u = (p_1 - p_4)^2 = (p_1 - p_4)^\mu g_{\mu\nu} (p_1 - p_4)^\nu$$

Let

$$a_1 = \bar{v}_{2b} \gamma^\mu (\not{p}_1 + m) \gamma^\nu u_{1a}, \quad a_2 = \bar{v}_{2b} \gamma^\nu (\not{p}_2 + m) \gamma^\mu u_{1a}$$

Then

$$\begin{aligned}
|\mathcal{M}_{ab}|^2 &= \frac{e^4}{N} \left| -\frac{a_1}{t - m^2} - \frac{a_2}{u - m^2} \right|^2 \\
&= \frac{e^4}{N} \left(-\frac{a_1}{t - m^2} - \frac{a_2}{u - m^2} \right) \left(-\frac{a_1}{t - m^2} - \frac{a_2}{u - m^2} \right)^* \\
&= \frac{e^4}{N} \left(\frac{a_1 a_1^*}{(t - m^2)^2} + \frac{a_1 a_2^*}{(t - m^2)(u - m^2)} + \frac{a_1^* a_2}{(t - m^2)(u - m^2)} + \frac{a_2 a_2^*}{(u - m^2)^2} \right)
\end{aligned}$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is computed by summing $|\mathcal{M}_{ab}|^2$ over all spin and polarization states and then dividing by the number of inbound states. There are four inbound states. The sum over polarization states is already accomplished by contraction of aa^* over μ and ν.

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{1}{4} \sum_{a=1}^{2} \sum_{b=1}^{2} |\mathcal{M}_{ab}|^2 \\
&= \frac{e^4}{4N} \sum_{a=1}^{2} \sum_{b=1}^{2} \left(\frac{a_1 a_1^*}{(t - m^2)^2} + \frac{a_1 a_2^*}{(t - m^2)(u - m^2)} + \frac{a_1^* a_2}{(t - m^2)(u - m^2)} + \frac{a_2 a_2^*}{(u - m^2)^2} \right)
\end{aligned}$$

The Casimir trick uses matrix arithmetic to compute sums.

$$f_{11} = \frac{1}{N} \sum_{a=1}^{2} \sum_{b=1}^{2} a_1 a_1^* = \mathrm{Tr} \left((\not{p}_1 + m) \gamma^\mu (\not{p}_1 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\nu (\not{p}_1 + m) \gamma_\mu \right)$$
$$f_{12} = \frac{1}{N} \sum_{a=1}^{2} \sum_{b=1}^{2} a_1 a_2^* = \mathrm{Tr} \left((\not{p}_1 + m) \gamma^\mu (\not{p}_2 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\mu (\not{p}_1 + m) \gamma_\nu \right)$$
$$f_{22} = \frac{1}{N} \sum_{a=1}^{2} \sum_{b=1}^{2} a_2 a_2^* = \mathrm{Tr} \left((\not{p}_1 + m) \gamma^\mu (\not{p}_2 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\nu (\not{p}_2 + m) \gamma_\mu \right)$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left(\frac{f_{11}}{(t - m^2)^2} + \frac{f_{12}}{(t - m^2)(u - m^2)} + \frac{f_{12}^*}{(t - m^2)(u - m^2)} + \frac{f_{22}}{(u - m^2)^2} \right)$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$\begin{aligned}
f_{11} &= 32(p_1 \cdot p_3)(p_1 \cdot p_4) - 32m^2(p_1 \cdot p_2) + 64m^2(p_1 \cdot p_3) + 32m^2(p_1 \cdot p_4) - 64m^4 \\
f_{12} &= 16m^2(p_1 \cdot p_3) + 16m^2(p_1 \cdot p_4) - 32m^4 \\
f_{22} &= 32(p_1 \cdot p_3)(p_1 \cdot p_4) - 32m^2(p_1 \cdot p_2) + 32m^2(p_1 \cdot p_3) + 64m^2(p_1 \cdot p_4) - 64m^4
\end{aligned}$$

For Mandelstam variables

$$\begin{aligned} s &= (p_1 + p_2)^2 = 4E^2 \\ t &= (p_1 - p_3)^2 \\ u &= (p_1 - p_4)^2 \end{aligned}$$

the formulas are

$$\begin{aligned} f_{11} &= 8tu - 24tm^2 - 8um^2 - 8m^4 \\ f_{12} &= 8sm^2 - 32m^4 \\ f_{22} &= 8tu - 8tm^2 - 24um^2 - 8m^4 \end{aligned}$$

High energy approximation

For high energy experiments $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$\begin{aligned} f_{11} &= 8tu \\ f_{12} &= 0 \\ f_{22} &= 8tu \end{aligned}$$

Hence

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left(\frac{8tu}{t^2} + \frac{8tu}{u^2} \right) \\ &= 2e^4 \left(\frac{u}{t} + \frac{t}{u} \right) \end{aligned}$$

For $m = 0$ the Mandelstam variables are

$$\begin{aligned} s &= 4E^2 \\ t &= -2E^2(1 - \cos\theta) \\ u &= -2E^2(1 + \cos\theta) \end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{1 + \cos\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{1 + \cos\theta} \right)$$

Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{4(4\pi\varepsilon_0)^2 s}, \quad s = (p_1 + p_2)^2 = 4E^2$$

For high energy experiments we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{1 + \cos\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{1 + \cos\theta} \right)$$

Substitute for $\langle|\mathcal{M}|^2\rangle$.

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\varepsilon_0)^2 s}\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)$$

Noting that

$$e^2 = 4\pi\varepsilon_0\alpha\hbar c$$

we can also write

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s}\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)$$

We can integrate $d\sigma$ to obtain a cumulative distribution function. Let $I(\theta)$ be the following integral of $d\sigma$. (The $\sin\theta$ is from $d\Omega = \sin\theta\, d\theta\, d\phi$.)

$$I(\theta) = \int\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)\sin\theta\, d\theta$$

The result is

$$I(\theta) = 2\cos\theta + 2\log(1-\cos\theta) - 2\log(1+\cos\theta)$$

The cumulative distribution function is

$$F(\theta) = \frac{I(\theta) - I(a)}{I(\pi - a) - I(a)}, \quad a \le \theta \le \pi - a$$

Angular support is reduced by an arbitrary angle $a > 0$ because $I(0)$ and $I(\pi)$ are undefined.

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

Let N be the number of scattering events from an experiment. Then the number of scattering events in the interval θ_1 to θ_2 is predicted to be

$$N\left(F(\theta_2) - F(\theta_1)\right)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi - a) - I(a)}\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right)\sin\theta$$

Note that if we had carried through the $\alpha^2(\hbar c)^2/2s$ in $I(\theta)$, it would have cancelled out in $F(\theta)$.

Data from DESY PETRA experiment

See www.hepdata.net/record/ins191231, Table 2, 14.0 GeV.

x	y
0.0502	0.09983
0.1505	0.10791
0.2509	0.12026
0.3512	0.13002
0.4516	0.17681
0.5521	0.1957
0.6526	0.279
0.7312	0.33204

Data x and y have the following relationship with the differential cross section formula.

$$x = \cos\theta, \quad y = \frac{d\sigma}{d\Omega}$$

The cross section formula is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right) \times (\hbar c)^2$$

To compute predicted values $\hat{y}$, multiply by 10^{37} to convert square meters to nanobarns.

$$\hat{y} = \frac{\alpha^2}{2s}\left(\frac{1+x}{1-x} + \frac{1-x}{1+x}\right) \times (\hbar c)^2 \times 10^{37}$$

The following table shows predicted values $\hat{y}$ for $s = (14.0\,\mathrm{GeV})^2$.

x	y	$\hat{y}$
0.0502	0.09983	0.106325
0.1505	0.10791	0.110694
0.2509	0.12026	0.120005
0.3512	0.13002	0.135559
0.4516	0.17681	0.159996
0.5521	0.1957	0.198562
0.6526	0.279	0.262745
0.7312	0.33204	0.348884

The coefficient of determination R^2 measures how well predicted values fit the data.

$$R^2 = 1 - \frac{\sum(y-\hat{y})^2}{\sum(y-\bar{y})^2} = 0.98$$

The result indicates that the model $d\sigma$ explains 98% of the variance in the data.

Notes

Here are some notes on how the Eigenmath scripts work.

To convert a_1 and a_2 to Eigenmath code, it is instructive to write a_1 and a_2 in full component form.

$$a_1^{\mu\nu} = \bar{v}_{2\alpha}\gamma^{\mu\alpha}{}_{\beta}(\not{q}_1 + m)^{\beta}{}_{\rho}\gamma^{\nu\rho}{}_{\sigma}u_1^{\sigma} \quad a_2^{\nu\mu} = \bar{v}_{2\alpha}\gamma^{\nu\alpha}{}_{\beta}(\not{q}_2 + m)^{\beta}{}_{\rho}\gamma^{\mu\rho}{}_{\sigma}u_1^{\sigma}$$

Transpose the γ tensors to form inner products over α and ρ.

$$a_1^{\mu\nu} = \bar{v}_{2\alpha}\gamma^{\alpha\mu}{}_{\beta}(\not{q}_1 + m)^{\beta}{}_{\rho}\gamma^{\rho\nu}{}_{\sigma}u_1^{\sigma} \quad a_2^{\nu\mu} = \bar{v}_{2\alpha}\gamma^{\alpha\nu}{}_{\beta}(\not{q}_2 + m)^{\beta}{}_{\rho}\gamma^{\rho\mu}{}_{\sigma}u_1^{\sigma}$$

Convert transposed γ to Eigenmath code.

$$\gamma^{\alpha\mu}{}_{\beta} \quad \rightarrow \quad \texttt{gammaT = transpose(gamma)}$$

Then to compute a_1 we have

$$a_1 = \bar{v}_{2\alpha}\gamma^{\alpha\mu}{}_{\beta}(\not{q}_1 + m)^{\beta}{}_{\rho}\gamma^{\rho\nu}{}_{\sigma}u_1^{\sigma}$$

$$\rightarrow \quad \texttt{a1 = dot(v2bar[s2],gammaT,qslash1 + m I,gammaT,u1[s1])}$$

where s_1 and s_2 are spin indices. Similarly for a_2 we have

$$a_2 = \bar{v}_{2\alpha}\gamma^{\alpha\mu}{}_{\beta}(\not{q}_2 + m)^{\beta}{}_{\rho}\gamma^{\rho\nu}{}_{\sigma}u_1^{\sigma}$$

$$\rightarrow \quad \texttt{a2 = dot(v2bar[s2],gammaT,qslash2 + m I,gammaT,u1[s1])}$$

In component notation the product $a_1 a_1^*$ is

$$a_1 a_1^* = a_1^{\mu\nu} a_1^{*\mu\nu}$$

To sum over μ and ν it is necessary to lower indices with the metric tensor. Also, transpose a_1^* to form an inner product with ν.

$$a_1 a_1^* = a_1^{\mu\nu} a_{1\nu\mu}^*$$

Convert to Eigenmath code. The dot function sums over ν and the contract function sums over μ.

$$a_1 a_1^* \quad \rightarrow \quad \texttt{a11 = contract(dot(a1,gmunu,transpose(conj(a1)),gmunu))}$$

Similarly for $a_2 a_2^*$ we have

$$a_2 a_2^* \quad \rightarrow \quad \texttt{a22 = contract(dot(a2,gmunu,transpose(conj(a2)),gmunu))}$$

The product $a_1 a_2^*$ does not require a transpose because $a_2 = a_2^{\nu\mu}$.

$$a_1^{\mu\nu} a_{2\nu\mu}^* \quad \rightarrow \quad \texttt{a12 = contract(dot(a1,gmunu,conj(a2),gmunu))}$$

In component notation, a trace operator becomes a sum over an index, in this case α.

$$\begin{aligned} f_{11} &= \text{Tr}\left((\not{p}_1 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\right) \\ &= (\not{p}_1 + m)^\alpha{}_\beta\gamma^{\mu\beta}{}_\rho(\not{q}_1 + m)^\rho{}_\sigma\gamma^{\nu\sigma}{}_\tau(\not{p}_2 - m)^\tau{}_\delta\gamma_\nu{}^\delta{}_\eta(\not{q}_1 + m)^\eta{}_\xi\gamma_\mu{}^\xi{}_\alpha \end{aligned}$$

As before, transpose γ tensors to form inner products.

$$f_{11} = (\not{p}_1 + m)^\alpha{}_\beta\gamma^{\beta\mu}{}_\rho(\not{q}_1 + m)^\rho{}_\sigma\gamma^{\sigma\nu}{}_\tau(\not{p}_2 - m)^\tau{}_\delta\gamma^\delta{}_{\nu\eta}(\not{q}_1 + m)^\eta{}_\xi\gamma^\xi{}_{\mu\alpha}$$

This is the code for transposing γ.

$$\begin{aligned} \gamma^{\beta\mu}{}_\beta \quad &\rightarrow \quad \texttt{gammaT = transpose(gamma)} \\ \gamma^\delta{}_{\nu\eta} \quad &\rightarrow \quad \texttt{gammaL = transpose(dot(gmunu,gamma))} \end{aligned}$$

To convert f_{11} to Eigenmath code, use an intermediate variable T for the inner product.

$$T^{\alpha\mu\nu}{}_{\nu\mu\alpha} \quad \rightarrow \quad \texttt{T = dot(P1,gammaT,Q1,gammaT,P2,gammaL,Q1,gammaL)}$$

Now sum over the indices of T. The innermost contract sums over ν then the next contract sums over μ. Finally the outermost contract sums over α.

$$f_{11} \quad \rightarrow \quad \texttt{f11 = contract(contract(contract(T,3,4),2,3))}$$

Follow suit for f_{22}. For f_{12} the order of the rightmost μ and ν is reversed.

$$f_{12} = \text{Tr}\left((\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\right)$$

The resulting inner product is $T^{\alpha\mu\nu}{}_{\mu\nu\alpha}$ so the contraction is different.

$$f_{12} \quad \rightarrow \quad \texttt{f12 = contract(contract(contract(T,3,5),2,3))}$$

The innermost contract sums over ν followed by sum over μ then sum over α.