(14.1) Fill in the algebra leading to eqn 14.32.

Consider equation (14.31).

$$\hat{A}^{\mu}(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_p)^{\frac{1}{2}}} \sum_{\lambda=1}^{2} \Big(\epsilon_{\lambda}^{\mu}(p) \hat{a}_{p\lambda} \exp(-ip\cdot x) + \epsilon_{\lambda}^{\mu *}(p) \hat{a}_{p\lambda}^{\dagger} \exp(ip\cdot x) \Big) \tag{14.31}$$

Let

$$\hat{A}^{\mu} = \begin{pmatrix} \hat{A}_0 \\ \hat{A}_1 \\ \hat{A}_2 \\ \hat{A}_3 \end{pmatrix}, \qquad \epsilon_{\lambda}^{\mu}(p) = \begin{pmatrix} \epsilon_{0,p\lambda} \\ \epsilon_{1,p\lambda} \\ \epsilon_{2,p\lambda} \\ \epsilon_{3,p\lambda} \end{pmatrix}$$

Then in vector form we have

$$\begin{pmatrix} \hat{A}_0 \\ \hat{A}_1 \\ \hat{A}_2 \\ \hat{A}_3 \end{pmatrix} = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{\hat{a}}{(2E_p)^{\frac{1}{2}}}$$

where

$$\begin{aligned} \hat{a} = \begin{pmatrix} \epsilon_{0,p1} \\ \epsilon_{1,p1} \\ \epsilon_{2,p1} \\ \epsilon_{3,p1} \end{pmatrix} \hat{a}_{p1} \exp(-ip\cdot x) + \begin{pmatrix} \epsilon_{0,p1}^* \\ \epsilon_{1,p1}^* \\ \epsilon_{2,p1}^* \\ \epsilon_{3,p1}^* \end{pmatrix} \hat{a}_{p1}^{\dagger} \exp(ip\cdot x) \\ + \begin{pmatrix} \epsilon_{0,p2} \\ \epsilon_{1,p2} \\ \epsilon_{2,p2} \\ \epsilon_{3,p2} \end{pmatrix} \hat{a}_{p2} \exp(-ip\cdot x) + \begin{pmatrix} \epsilon_{0,p2}^* \\ \epsilon_{1,p2}^* \\ \epsilon_{2,p2}^* \\ \epsilon_{3,p2}^* \end{pmatrix} \hat{a}_{p2}^{\dagger} \exp(ip\cdot x) \end{aligned}$$

Hence

$$\begin{aligned} \hat{A}_{\nu} = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_p)^{\frac{1}{2}}} \\ \times \Big[(\epsilon_{\nu,p1}\hat{a}_{p1} + \epsilon_{\nu,p2}\hat{a}_{p2}) \exp(-ip\cdot x) + (\epsilon_{\nu,p1}^*\hat{a}_{p1}^{\dagger} + \epsilon_{\nu,p2}^*\hat{a}_{p2}^{\dagger}) \exp(ip\cdot x) \Big] \end{aligned}$$

Recall that

$$p \cdot x = E_p t - p_1 x_1 - p_2 x_2 - p_3 x_3$$

Hence the partial derivative of $\hat{A}_\nu$ is

$$\partial_\mu \hat{A}_\nu = \int \frac{d^3 p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_p)^{\frac{1}{2}}}$$

$$\times \Big[-ip_\mu (\epsilon_{\nu,p1} \hat{a}_{p1} + \epsilon_{\nu,p2} \hat{a}_{p2}) \exp(-ip \cdot x) + ip_\mu (\epsilon^*_{\nu,p1} \hat{a}^\dagger_{p1} + \epsilon^*_{\nu,p2} \hat{a}^\dagger_{p2}) \exp(ip \cdot x) \Big]$$

Consider equation (14.28).

$$\mathcal{H} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \tag{14.28}$$

We need to integrate Hamiltonian density $\mathcal{H}$ to obtain $\hat{H}$, that is,

$$\hat{H} = \int d^3 x \, \mathcal{H} \tag{1}$$

Substitute (14.28) into (1).

$$\hat{H} = \frac{1}{2} \int d^3 x \, \left(E_1^2 + E_2^2 + E_3^2 + B_1^2 + B_2^2 + B_3^2 \right)$$

From equations (5.28) and (5.29) we have

$$\mathbf{E} = \begin{pmatrix} \partial_0 A_1 - \partial_1 A_0 \\ \partial_0 A_2 - \partial_2 A_0 \\ \partial_0 A_3 - \partial_3 A_0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} \partial_3 A_2 - \partial_2 A_3 \\ \partial_1 A_3 - \partial_3 A_1 \\ \partial_2 A_1 - \partial_1 A_2 \end{pmatrix}$$

Start with E_1^2.

$$E_1^2 = (\partial_0 \hat{A}_1)^2 - (\partial_0 \hat{A}_1)(\partial_1 \hat{A}_0) - (\partial_1 \hat{A}_0)(\partial_0 \hat{A}_1) + (\partial_1 \hat{A}_0)^2$$

Let

$$I_1 = \frac{1}{2} \int d^3 x \, (\partial_0 \hat{A}_1)^2$$

Hence

$$I_1 = \frac{1}{2} \int \frac{d^3 x \, d^3 p \, d^3 q}{(2\pi)^3} \frac{1}{(2E_p)^{\frac{1}{2}} (2E_q)^{\frac{1}{2}}}$$

$$\times \sum_{\lambda=1}^{2} \Big(-iE_p \epsilon^1_\lambda(p) \hat{a}_{p\lambda} \exp(-ip \cdot x) + iE_p \epsilon^{1*}_\lambda(p) \hat{a}^\dagger_{p\lambda} \exp(ip \cdot x) \Big)$$

$$\times \sum_{\lambda=1}^{2} \Big(-iE_q \epsilon^1_\lambda(q) \hat{a}_{q\lambda} \exp(-iq \cdot x) + iE_q \epsilon^{1*}_\lambda(q) \hat{a}^\dagger_{q\lambda} \exp(iq \cdot x) \Big)$$

Using the integral

$$\int d^3x \, \exp(ip \cdot x) = (2\pi)^3 \exp(iE_p t)\delta^{(3)}(\mathbf{p})$$

we have

$$\begin{aligned} I_1 = \frac{1}{2} \int & \frac{d^3p \, d^3q}{(2E_p)^{\frac{1}{2}} (2E_q)^{\frac{1}{2}}} \\ & \times \delta^{(3)}(\mathbf{p}) \sum_{\lambda=1}^{2} \Big(-iE_p \epsilon^1_\lambda(p) \hat{a}_{p\lambda} \exp(-iE_p t) + iE_p \epsilon^{1*}_\lambda(p) \hat{a}^\dagger_{p\lambda} \exp(iE_p t) \Big) \\ & \times \delta^{(3)}(\mathbf{q}) \sum_{\lambda=1}^{2} \Big(-iE_q \epsilon^1_\lambda(q) \hat{a}_{q\lambda} \exp(-iE_q t) + iE_q \epsilon^{1*}_\lambda(q) \hat{a}^\dagger_{q\lambda} \exp(iE_q t) \Big) \end{aligned}$$