

9-4. In section 2-1 we discussed the mechanisms for obtaining the mechanical equations of motion from the form of the action  $S$  by obtaining the extremum  $S_{cl}$  under the conditions  $\delta S = 0$  for variations of the coordinates,  $\delta \mathbf{x}$ . Show how Maxwell's equations can be derived from the action  $S$  defined in equation (9.23) by requiring  $\delta S$  for first-order variations of  $\mathbf{A}$  and  $\phi$ .

$$S = S_1 + S_2 + S_3 \quad (9.23)$$

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Since  $S_1$  does not depend on  $\mathbf{A}$  or  $\phi$ , we only need  $S_2$  and  $S_3$ .

$$S_2 = - \sum_i e_i \int \left( \phi(\mathbf{x}_i(t), t) - \frac{1}{c} \dot{\mathbf{x}}_i(t) \cdot \mathbf{A}(\mathbf{x}_i(t), t) \right) dt \quad (9.25)$$

$$S_3 = \frac{1}{8\pi} \int \int \left( \left| -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right|^2 - |\nabla \times \mathbf{A}|^2 \right) d^3 \mathbf{r} dt \quad (9.26)$$

Consider equation (2.7), the classical Lagrangian equation of motion.

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \quad (2.7)$$

Extend (2.7) to three dimensions.

$$\frac{d}{dt} \dot{\nabla} L = \nabla L \quad (1)$$

where

$$\dot{\nabla} = \mathbf{i} \frac{\partial}{\partial \dot{x}} + \mathbf{j} \frac{\partial}{\partial \dot{y}} + \mathbf{k} \frac{\partial}{\partial \dot{z}} \quad \nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$$

and

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

From equation (9.25) for a single particle, let

$$L = \phi - \frac{1}{c} (\dot{x} A_x + \dot{y} A_y + \dot{z} A_z)$$

Then

$$\begin{aligned}\frac{d}{dt}\dot{\nabla}L &= -\frac{1}{c}\frac{d}{dt}(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \\ &= -\frac{1}{c}\frac{d}{dt}\mathbf{A}\end{aligned}\tag{2}$$

and

$$\begin{aligned}\nabla L &= \nabla\phi - \frac{1}{c}\left(\dot{x}\frac{\partial A_x}{\partial x}\mathbf{i} + \dot{y}\frac{\partial A_y}{\partial y}\mathbf{j} + \dot{z}\frac{\partial A_z}{\partial z}\mathbf{k}\right) \\ &= \nabla\phi - \frac{1}{c}\nabla(\dot{\mathbf{x}} \cdot \mathbf{A})\end{aligned}\tag{3}$$

Hence by equations (1), (2), and (3)

$$-\frac{1}{c}\frac{d}{dt}\mathbf{A} = \nabla\phi - \frac{1}{c}\nabla(\dot{\mathbf{x}} \cdot \mathbf{A})$$