Spontaneous emission rate

Find the spontaneous emission rate for hydrogen state $2p \to 1s$.

The wave function for hydrogen is

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi)$$

where

$$R_{nl}(r) = \frac{2}{n^2}\sqrt{\frac{(n-l-1)!}{(n+l)!}}\left(\frac{2r}{na_0}\right)^l L_{n-l-1}^{2l+1}\left(\frac{2r}{na_0}\right)\exp\left(-\frac{r}{na_0}\right)a_0^{-3/2}$$

$$L_n^m(x) = (n+m)!\sum_{k=0}^{n}\frac{(-x)^k}{(n-k)!(m+k)!k!}$$

$$Y_{lm}(\theta,\phi) = (-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\exp(im\phi)$$

$$P_l^m(\cos\theta) = \begin{cases} \left(\dfrac{\sin\theta}{2}\right)^m \displaystyle\sum_{k=0}^{l-m}(-1)^k\frac{(l+m+k)!}{(l-m-k)!(m+k)!k!}\left(\frac{1-\cos\theta}{2}\right)^k, & m \ge 0 \\ (-1)^m\dfrac{(l+m)!}{(l-m)!}P_l^{|m|}(\cos\theta), & m < 0 \end{cases}$$

State $2p$ is shorthand for $n=2$ and $l=1$. For $l=1$ there are three ways to choose m hence all of the following processes correspond to the transition $2p \to 1s$. It turns out that all three processes have the same transition rate.

$$\left.\begin{matrix} \psi_{2,1,1} \\ \psi_{2,1,0} \\ \psi_{2,1,-1} \end{matrix}\right\} \to \psi_{100} + \text{photon}$$

The spontaneous emission rate is

$$A_{21} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3}\omega_{21}^3|r_{21}|^2$$

Noting that

$$e^2 = 4\pi\varepsilon_0\hbar c\alpha$$

we can also write

$$A_{21} = \frac{4\alpha}{3c^2}\omega_{21}^3|r_{21}|^2 \tag{1}$$

Verify dimensions:

$$A_{21} \propto (\text{m/s})^{-2} \times \text{s}^{-3} \times \text{m}^2 = \text{s}^{-1}\,(\text{or hertz})$$

For angular frequency ω_{21} we have

$$\omega_{21} = \frac{E_2 - E_1}{\hbar}$$

and for hydrogen

$$E_n = -\frac{\alpha \hbar c}{2n^2 a_0}$$

Hence

$$\omega_{21} = \frac{3\alpha c}{8a_0}$$

For displacement r_{21} we have

$$|r_{21}|^2 = |x_{21}|^2 + |y_{21}|^2 + |z_{21}|^2$$

where

$$x_{21} = \int_0^\infty \int_0^\pi \int_0^{2\pi} x f_{21}\, dV, \quad y_{21} = \int_0^\infty \int_0^\pi \int_0^{2\pi} y f_{21}\, dV, \quad z_{21} = \int_0^\infty \int_0^\pi \int_0^{2\pi} z f_{21}\, dV$$

and

$$f_{21} = \psi_{100}^* \psi_{210}$$

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta$$

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

The integrals work out to be

$$x_{21} = 0, \quad y_{21} = 0, \quad z_{21} = \frac{2^7}{3^5}\sqrt{2}a_0$$

hence

$$|r_{21}|^2 = |z_{21}|^2 = \frac{2^{15}}{3^{10}} a_0^2$$

By equation (1) the spontaneous emission rate is

$$A_{21} = \frac{2^8}{3^8} \frac{\alpha^4 c}{a_0} = 6.26 \times 10^8 \,\mathrm{s}^{-1}$$

Noting that

$$a_0 = \frac{\hbar}{\alpha \mu c}$$

we can also write

$$A_{21} = \frac{2^8}{3^8} \frac{\alpha^5 \mu c^2}{\hbar} = 6.26 \times 10^8 \,\mathrm{s}^{-1}$$

where μ is reduced electron mass

$$\mu = \frac{m_e m_p}{m_e + m_p}$$