

Exercise 5.1. Verify this claim.

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The claim is that any  $2 \times 2$  Hermitian matrix  $\mathbf{L}$  can be written as

$$\mathbf{L} = a\sigma_x + b\sigma_y + c\sigma_z + dI$$

where  $a, b, c$ , and  $d$  are real numbers.

Let  $\mathbf{L}$  be the Hermitian matrix

$$\mathbf{L} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \quad (1)$$

Recall from page 137 that

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It follows that

$$\mathbf{L} = \begin{pmatrix} d + c & a - ib \\ a + ib & d - c \end{pmatrix} \quad (2)$$

Then by equivalence of (1) and (2) we have

$$a = \frac{L_{12} + L_{21}}{2}, \quad b = \frac{i(L_{12} - L_{21})}{2}, \quad c = \frac{L_{11} - L_{22}}{2}, \quad d = \frac{L_{11} + L_{22}}{2}$$

By Hermiticity we have  $\mathbf{L} = \mathbf{L}^\dagger$  hence

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} L_{11}^* & L_{21}^* \\ L_{12}^* & L_{22}^* \end{pmatrix}$$

Therefore  $L_{11}$  and  $L_{22}$  are real, hence  $c$  and  $d$  are real.

Also by Hermiticity,  $L_{21} = L_{12}^*$  hence  $a$  and  $b$  are real.

$$\begin{aligned} a &= \frac{L_{12} + L_{21}}{2} = \frac{L_{12} + L_{12}^*}{2} = \operatorname{Re}(L_{12}) \\ b &= \frac{i(L_{12} - L_{21})}{2} = \frac{i(L_{12} - L_{12}^*)}{2} = -\operatorname{Im}(L_{12}) \end{aligned}$$