

## Free particle propagator

A propagator is the amplitude for a particle to go from  $x_a$  at time  $t_a$  to  $x_b$  at time  $t_b$ .

Let  $K(b, a)$  be the propagator

$$K(b, a) = \langle x_b | \exp \left( -\frac{i}{\hbar} H(t_b - t_a) \right) | x_a \rangle$$

where  $H$  is the free particle Hamiltonian

$$H = \frac{\hat{p}^2}{2m}$$

By the identity

$$\int |p\rangle \langle p| dp = 1$$

we can write

$$K(b, a) = \int \langle x_b | \exp \left( -\frac{i}{\hbar} H(t_b - t_a) \right) | p \rangle \langle p | x_a \rangle dp$$

Replace the momentum operator  $\hat{p}$  with its eigenvalue  $p$ .

$$K(b, a) = \int \exp \left( -\frac{i}{2m\hbar} p^2(t_b - t_a) \right) \langle x_b | p \rangle \langle p | x_a \rangle dp$$

Recalling that

$$\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp \left( \frac{ipx}{\hbar} \right)$$

we can write

$$K(b, a) = \frac{1}{2\pi\hbar} \int \exp \left( -\frac{i}{2m\hbar} p^2(t_b - t_a) + \frac{i}{\hbar} p(x_b - x_a) \right) dp$$

By the integral

$$\int_{-\infty}^{\infty} \exp(-ax^2 + bx) dx = \left( \frac{\pi}{a} \right)^{1/2} \exp \left( \frac{b^2}{4a} \right)$$

with

$$a = \frac{i}{2m\hbar}(t_b - t_a) \quad \text{and} \quad b = \frac{i}{\hbar}(x_b - x_a)$$

we have

$$K(b, a) = \frac{1}{2\pi\hbar} \left( \frac{2\pi m\hbar}{i(t_b - t_a)} \right)^{1/2} \exp \left( -\frac{m(x_b - x_a)^2}{2i\hbar(t_b - t_a)} \right)$$

Rewrite as

$$K(b, a) = \left( \frac{m}{2\pi i\hbar(t_b - t_a)} \right)^{1/2} \exp \left( \frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)} \right)$$