Spin state

Fermion spin state $|s\rangle$ is a unit vector in $\mathbb{C}^2$.

$$|s\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad c_1^* c_1 + c_2^* c_2 = 1$$

Here is spin state $|s\rangle$ as a linear combination of basis states "up" and "down."

$$|s\rangle = c_1|u\rangle + c_2|d\rangle, \quad |u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These are the spin operators.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Expectation of spin operators is a projection of $|s\rangle$ onto Euclidean space.

$$\langle x\rangle = \langle s|\sigma_x|s\rangle, \quad \langle y\rangle = \langle s|\sigma_y|s\rangle, \quad \langle z\rangle = \langle s|\sigma_z|s\rangle$$

Let $\mathbf{u}$ be the following spin direction vector determined by $|s\rangle$.

$$\mathbf{u} = \begin{pmatrix} \langle x\rangle \\ \langle y\rangle \\ \langle z\rangle \end{pmatrix} = \langle s|\boldsymbol{\sigma}|s\rangle, \quad \boldsymbol{\sigma} = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}$$

Let θ and ϕ be polar and azimuth angles such that

$$\mathbf{u} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

Then

$$|s\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\exp(i\phi) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

where

$$c_1 = \sqrt{\frac{\langle z\rangle + 1}{2}}, \quad c_2 = \sqrt{\frac{1 - \langle z\rangle}{2}}\,\frac{\langle x\rangle + i\langle y\rangle}{\sqrt{\langle x\rangle^2 + \langle y\rangle^2}}$$

Example. Let

$$|s\rangle = \begin{pmatrix} \frac{1}{3} - \frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$$

It follows that

$$\mathbf{u} = \begin{pmatrix} \langle x\rangle \\ \langle y\rangle \\ \langle z\rangle \end{pmatrix} = \begin{pmatrix} \frac{4}{9} \\ \frac{8}{9} \\ \frac{1}{9} \end{pmatrix}$$

and

$$c_1 = \sqrt{\frac{\langle z\rangle + 1}{2}} = \frac{\sqrt{5}}{3}, \quad c_2 = \sqrt{\frac{1 - \langle z\rangle}{2}} \frac{\langle x\rangle + i\langle y\rangle}{\sqrt{\langle x\rangle^2 + \langle y\rangle^2}} = \frac{2 + 4i}{3\sqrt{5}}$$

These values differ from the original $|s\rangle$ but they do represent the same state.

Note on Eigenmath code. In component notation $\boldsymbol{\sigma} = \sigma^{\alpha\beta}{}_{\gamma}$ hence

$$u^{\alpha} = s^{*}_{\beta}\sigma^{\alpha\beta}{}_{\gamma}s^{\gamma}$$

A transpose swaps α and β so that summed-over indices are adjacent.

$$u^{\alpha} = s^{*}_{\beta}\sigma^{\beta\alpha}{}_{\gamma}s^{\gamma}$$

Hence the Eigenmath code is

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sigma = (sigmax,sigmay,sigmaz)
u = dot(conj(s),transpose(sigma),s)
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