

Feynman and Hibbs problem 4-1

Show that for a single particle moving in three dimensions in a potential energy $V(\mathbf{x}, t)$ the Schrodinger equation is

$$\frac{\partial \psi(\mathbf{x}, t)}{\partial t} = -\frac{i}{\hbar} \left(-\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{x}, t) + V(\mathbf{x}, t) \psi(\mathbf{x}, t) \right)$$

This is the Lagrangian.

$$L(\dot{\mathbf{x}}, \mathbf{x}) = \frac{m}{2} \dot{\mathbf{x}}^2 - V(\mathbf{x}, t) \quad (1)$$

Extend equation (4.3) from one dimension to three dimensions.

$$\psi(\mathbf{x}, t + \epsilon) = \frac{1}{A} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left(\frac{i}{\hbar} \epsilon L \left(\frac{\mathbf{x} - \mathbf{y}}{\epsilon}, \frac{\mathbf{x} + \mathbf{y}}{2} \right) \right) \psi(\mathbf{y}, t) dy_1 dy_2 dy_3$$

From (1) we have

$$L \left(\frac{\mathbf{x} - \mathbf{y}}{\epsilon}, \frac{\mathbf{x} + \mathbf{y}}{2} \right) = \frac{m}{2\epsilon^2} (\mathbf{x} - \mathbf{y})^2 - V \left(\frac{\mathbf{x} + \mathbf{y}}{2}, t \right)$$

Hence

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) = \frac{1}{A} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left(\frac{i}{\hbar} \left(\frac{m}{2\epsilon} (\mathbf{x} - \mathbf{y})^2 - \epsilon V \left(\frac{\mathbf{x} + \mathbf{y}}{2}, t \right) \right) \right) \\ \times \psi(\mathbf{y}, t) dy_1 dy_2 dy_3 \end{aligned}$$

Let

$$\mathbf{y} = \mathbf{x} + \boldsymbol{\eta}$$

Then

$$(\mathbf{x} - \mathbf{y})^2 = \boldsymbol{\eta}^2$$

and

$$\frac{\mathbf{x} + \mathbf{y}}{2} = \mathbf{x} + \frac{1}{2} \boldsymbol{\eta}$$

Hence

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) = \frac{1}{A} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left(\frac{i}{\hbar} \left(\frac{m}{2\epsilon} \boldsymbol{\eta}^2 - \epsilon V \left(\mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \right) \\ \times \psi(\mathbf{x} + \boldsymbol{\eta}, t) d\eta_1 d\eta_2 d\eta_3 \end{aligned}$$

Factor the exponential.

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) = \frac{1}{A} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left(\frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 \right) \exp \left(-\frac{i\epsilon}{\hbar} V \left(\mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \\ \times \psi(\mathbf{x} + \boldsymbol{\eta}, t) d\eta_1 d\eta_2 d\eta_3 \quad (2) \end{aligned}$$

Now we are going to use an approximation for the second exponential. From the identity $\exp(i\theta) = \cos(\theta) + i \sin(\theta)$ we have

$$\begin{aligned} \exp \left(-\frac{i\epsilon}{\hbar} V \left(\mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) = \\ \cos \left(-\frac{\epsilon}{\hbar} V \left(\mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) + i \sin \left(-\frac{\epsilon}{\hbar} V \left(\mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \end{aligned}$$

For very small ϵ we have the approximation

$$\exp \left(-\frac{i\epsilon}{\hbar} V \left(\mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \approx 1 - \frac{i\epsilon}{\hbar} V \left(\mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right)$$

The authors write that the $\frac{1}{2} \boldsymbol{\eta}$ term can be dropped “because the error is of higher order than ϵ .” Hence

$$\exp \left(-\frac{i\epsilon}{\hbar} V \left(\mathbf{x} + \frac{1}{2} \boldsymbol{\eta}, t \right) \right) \approx 1 - \frac{i\epsilon}{\hbar} V \left(\mathbf{x}, t \right) \quad (3)$$

Substituting (3) into (2) yields

$$\begin{aligned} \psi(\mathbf{x}, t + \epsilon) = \frac{1}{A} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left(\frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2 \right) \left(1 - \frac{i\epsilon}{\hbar} V \left(\mathbf{x}, t \right) \right) \\ \times \psi(\mathbf{x} + \boldsymbol{\eta}, t) d\eta_1 d\eta_2 d\eta_3 \quad (4) \end{aligned}$$

Next we will use the following Taylor series approximations.

$$\begin{aligned}\psi(\mathbf{x}, t + \epsilon) &\approx \psi(\mathbf{x}, t) + \epsilon \frac{\partial \psi}{\partial t} \\ \psi(\mathbf{x} + \boldsymbol{\eta}, t) &\approx \psi(\mathbf{x}, t) + \boldsymbol{\eta} \cdot \nabla \psi + \frac{1}{2} \boldsymbol{\eta} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \psi)\end{aligned}\tag{5}$$

Note: In component notation

$$\boldsymbol{\eta} \cdot \nabla \psi = \eta_1 \frac{\partial \psi}{\partial x_1} + \eta_2 \frac{\partial \psi}{\partial x_2} + \eta_3 \frac{\partial \psi}{\partial x_3}$$

and

$$\begin{aligned}\boldsymbol{\eta} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \psi) &= \eta_1^2 \frac{\partial^2 \psi}{\partial x_1^2} + \eta_2^2 \frac{\partial^2 \psi}{\partial x_2^2} + \eta_3^2 \frac{\partial^2 \psi}{\partial x_3^2} \\ &\quad + 2\eta_1 \eta_2 \frac{\partial^2 \psi}{\partial x_1 \partial x_2} + 2\eta_1 \eta_3 \frac{\partial^2 \psi}{\partial x_1 \partial x_3} + 2\eta_2 \eta_3 \frac{\partial^2 \psi}{\partial x_2 \partial x_3}\end{aligned}$$

Substitute the approximations (5) into (4).

$$\begin{aligned}\psi(\mathbf{x}, t) + \epsilon \frac{\partial \psi}{\partial t} &= \frac{1}{A} \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2\right) \left(1 - \frac{i\epsilon}{\hbar} V(\mathbf{x}, t)\right) \\ &\quad \times \left(\psi(\mathbf{x}, t) + \boldsymbol{\eta} \cdot \nabla \psi + \frac{1}{2} \boldsymbol{\eta} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \psi)\right) d\eta_1 d\eta_2 d\eta_3\end{aligned}$$

Expand the right-hand side.

$$\begin{aligned}&\frac{\psi(\mathbf{x}, t)}{A} \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2\right) d\eta_1 d\eta_2 d\eta_3 \\ &+ \frac{1}{A} \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2\right) \boldsymbol{\eta} \cdot \nabla \psi d\eta_1 d\eta_2 d\eta_3\end{aligned}\tag{6}$$

$$+ \frac{1}{2A} \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2\right) \boldsymbol{\eta} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \psi) d\eta_1 d\eta_2 d\eta_3\tag{7}$$

$$\begin{aligned}&- \frac{i\epsilon}{A\hbar} V(\mathbf{x}, t) \psi(\mathbf{x}, t) \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2\right) d\eta_1 d\eta_2 d\eta_3 \\ &- \frac{i\epsilon}{A\hbar} V(\mathbf{x}, t) \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2\right) \boldsymbol{\eta} \cdot \nabla \psi d\eta_1 d\eta_2 d\eta_3\end{aligned}\tag{8}$$

$$- \frac{i\epsilon}{2A\hbar} V(\mathbf{x}, t) \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon} \boldsymbol{\eta}^2\right) \boldsymbol{\eta} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \psi) d\eta_1 d\eta_2 d\eta_3\tag{9}$$

We will use the following integrals provided by the authors.

$$\int_{-\infty}^{\infty} \exp\left(\frac{imx^2}{2\hbar\epsilon}\right) dx = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{1/2} \quad (10)$$

$$\int_{-\infty}^{\infty} \exp\left(\frac{imx^2}{2\hbar\epsilon}\right) x dx = 0 \quad (11)$$

$$\int_{-\infty}^{\infty} \exp\left(\frac{imx^2}{2\hbar\epsilon}\right) x^2 dx = \left(-\frac{2\pi i\hbar^3\epsilon^3}{m^3}\right)^{1/2} \quad (12)$$

Rewrite the integral in (6) and (8) in component notation.

$$\begin{aligned} & \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \boldsymbol{\eta} \cdot \nabla \psi d\eta_1 d\eta_2 d\eta_3 \\ &= \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right) \eta_1 \frac{\partial \psi}{\partial x_1} d\eta_1 d\eta_2 d\eta_3 \\ & \quad + \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right) \eta_2 \frac{\partial \psi}{\partial x_2} d\eta_1 d\eta_2 d\eta_3 \\ & \quad + \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right) \eta_3 \frac{\partial \psi}{\partial x_3} d\eta_1 d\eta_2 d\eta_3 \end{aligned}$$

Then by (11)

$$\int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \boldsymbol{\eta} \cdot \nabla \psi d\eta_1 d\eta_2 d\eta_3 = 0$$

Rewrite the integral in (7) and (9) in component notation.

$$\begin{aligned}
& \int_{\mathbb{R}^3} \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{\eta}^2\right) \boldsymbol{\eta} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \psi) d\eta_1 d\eta_2 d\eta_3 \\
&= \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right) \eta_1^2 \frac{\partial \psi^2}{\partial x_1^2} d\eta_1 d\eta_2 d\eta_3 \\
&\quad + \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right) \eta_2^2 \frac{\partial \psi^2}{\partial x_2^2} d\eta_1 d\eta_2 d\eta_3 \\
&\quad + \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right) \eta_3^2 \frac{\partial \psi^2}{\partial x_3^2} d\eta_1 d\eta_2 d\eta_3 \\
&\quad + \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right) \eta_1 \eta_2 \frac{\partial \psi^2}{\partial x_1 \partial x_2} d\eta_1 d\eta_2 d\eta_3 \\
&\quad + \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right) \eta_1 \eta_3 \frac{\partial \psi^2}{\partial x_1 \partial x_3} d\eta_1 d\eta_2 d\eta_3 \\
&\quad + \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right) \eta_2 \eta_3 \frac{\partial \psi^2}{\partial x_2 \partial x_3} d\eta_1 d\eta_2 d\eta_3
\end{aligned}$$

By equation (10)

$$\begin{aligned}
& \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right) \eta_1 \eta_2 \frac{\partial \psi^2}{\partial x_1 \partial x_2} d\eta_1 d\eta_2 d\eta_3 = 0 \\
& \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right) \eta_1 \eta_3 \frac{\partial \psi^2}{\partial x_1 \partial x_3} d\eta_1 d\eta_2 d\eta_3 = 0 \\
& \int_{\mathbb{R}^3} \exp\left(\frac{im\eta_1^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_2^2}{2\hbar\epsilon}\right) \exp\left(\frac{im\eta_3^2}{2\hbar\epsilon}\right) \eta_2 \eta_3 \frac{\partial \psi^2}{\partial x_2 \partial x_3} d\eta_1 d\eta_2 d\eta_3 = 0
\end{aligned}$$