

Electromagnetic tensor

This is the standard model for an EM field.

$$\mathbf{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}, \quad A^\mu = \begin{pmatrix} \phi \\ A_x \\ A_y \\ A_z \end{pmatrix}, \quad A_\mu = g_{\mu\nu} A^\nu = \begin{pmatrix} \phi \\ -A_x \\ -A_y \\ -A_z \end{pmatrix}$$

$$g_{\mu\nu} = ((1,0,0,0), (0,-1,0,0), (0,0,-1,0), (0,0,0,-1))$$

$$\begin{aligned} \mathbf{A} &= (A_x(), A_y(), A_z()) \\ \mathbf{A}^\mu &= (\phi(), A_x(), A_y(), A_z()) \\ \mathbf{A}_\mu &= \text{dot}(g_{\mu\nu}, \mathbf{A}^\nu) \end{aligned}$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\begin{aligned} \mathbf{B} &= \text{curl}(\mathbf{A}) \\ \mathbf{E} &= -\text{d}(\phi(), (x, y, z)) - \text{d}(\mathbf{A}, t) \end{aligned}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = A_{\nu,\mu} - A_{\mu,\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$\begin{aligned} \mathbf{X} &= (t, x, y, z) \\ \mathbf{F}_{dd} &= \text{d}(\mathbf{A}_d, \mathbf{X}) \\ \mathbf{F}_{dd} &= \text{transpose}(\mathbf{F}_{dd}) - \mathbf{F}_{dd} \end{aligned}$$

$$\mathbf{T} = ((0, E[1], E[2], E[3]), (-E[1], 0, -B[3], B[2]), (-E[2], B[3], 0, -B[1]), (-E[3], -B[2], B[1], 0))$$

$$\text{check}(\mathbf{F}_{dd} == \mathbf{T})$$

$$F_{\mu\nu} F^{\mu\nu} = 2\mathbf{B}^2 - 2\mathbf{E}^2$$

$$\begin{aligned} \mathbf{F}_{uu} &= \text{dot}(g_{\mu\nu}, \mathbf{F}_{dd}, g^{\mu\nu}) \\ \mathbf{T} &= \text{contract}(\text{dot}(\text{transpose}(\mathbf{F}_{dd}), \mathbf{F}_{uu})) \\ \text{check}(\mathbf{T} == 2 \text{dot}(\mathbf{B}, \mathbf{B}) - 2 \text{dot}(\mathbf{E}, \mathbf{E})) \end{aligned}$$

$$\det(F_{\mu\nu}) = \det(F^{\mu\nu}) = (\mathbf{B} \cdot \mathbf{E})^2$$

$$\begin{aligned} \text{check}(\det(\mathbf{F}_{dd}) == \text{dot}(\mathbf{B}, \mathbf{E})^2) \\ \text{check}(\det(\mathbf{F}_{uu}) == \text{dot}(\mathbf{B}, \mathbf{E})^2) \end{aligned}$$

$$J^\nu = \partial_\mu F^{\mu\nu} = F^{\mu\nu}{}_{,\mu}$$

```
Ju = contract(d(Fuu,X),1,3)
```

$$\partial_\mu J^\mu = J^\mu{}_{,\mu} = 0, \quad \mathbf{J} = \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t}$$

```
check(contract(d(Ju,X)) == 0)
```

```
Jx = Ju[2]
```

```
Jy = Ju[3]
```

```
Jz = Ju[4]
```

```
J = (Jx,Jy,Jz)
```

```
check(J == curl(B) - d(E,t))
```