

2-4. Classically, the momentum is defined as

$$p = \frac{\partial L}{\partial \dot{x}} \quad (2.10)$$

Show that the momentum at a final point is

$$\left(\frac{\partial L}{\partial \dot{x}} \right)_{x=x_b} = + \frac{\partial S_{cl}}{\partial x_b} \quad (2.11)$$

while the momentum at an initial point is

$$\left(\frac{\partial L}{\partial \dot{x}} \right)_{x=x_a} = - \frac{\partial S_{cl}}{\partial x_a}$$

Hint: Consider the effect on equation (2.6) of a change in the end points.

$$\delta S = \left(\delta x(t) \frac{\partial L}{\partial \dot{x}} \right)_{t_a}^{t_b} - \int_{t_a}^{t_b} \delta x(t) \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \right) dt \quad (2.6)$$

Consider equation (2.7) for the classical path.

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad (2.7)$$

Substitute (2.7) into (2.6) to obtain

$$\delta S_{cl} = \left(\delta x(t) \frac{\partial L}{\partial \dot{x}} \right)_{t_a}^{t_b} = \delta x_b \left(\frac{\partial L}{\partial \dot{x}} \right)_{x=x_b} - \delta x_a \left(\frac{\partial L}{\partial \dot{x}} \right)_{x=x_a} \quad (1)$$

For endpoints a and b we have $\delta S_{cl} = 0$ by equation (2.3).

$$\delta x_a = \delta x_b = 0 \quad (2.3)$$

However, suppose endpoint b is changed so that $\delta x_b \neq 0$. Then by (1)

$$\delta S_{cl} = \delta x_b \left(\frac{\partial L}{\partial \dot{x}} \right)_{x=x_b}$$

It follows that

$$\frac{\delta S_{cl}}{\delta x_b} = \left(\frac{\partial L}{\partial \dot{x}} \right)_{x=x_b}$$

and in the limit

$$\lim_{\delta \rightarrow 0} \frac{\delta S_{cl}}{\delta x_b} = \frac{\partial S_{cl}}{\partial x_b} = \left(\frac{\partial L}{\partial \dot{x}} \right)_{x=x_b}$$

Likewise, a change of endpoint a yields

$$\lim_{\delta \rightarrow 0} \frac{\delta S_{cl}}{\delta x_a} = \frac{\partial S_{cl}}{\partial x_a} = - \left(\frac{\partial L}{\partial \dot{x}} \right)_{x=x_a}$$