Exercise 6.5. Prove the following theorem:

When any one of Alice's or Bob's spin operators acts on a product state, the result is still a product state.

Show that in a product state, the expectation value of any component of $\vec{\sigma}$ or $\vec{\tau}$ is exactly the same as it would be in the individual single spin states.

Let ψ be the product state

$$\psi = \psi_A \otimes \psi_B$$

where

$$\psi_A = \alpha_u|u\} + \alpha_d|d\}, \qquad \psi_B = \beta_u|u\rangle + \beta_d|d\rangle$$

Let σ be one of Alice's spin operators. Then

$$\sigma\psi = \sigma\psi_A \otimes \psi_B \tag{1}$$

Let τ be one of Bob's spin operators. Then

$$\tau\psi = \psi_A \otimes \tau\psi_B \tag{2}$$

By the fact that (1) and (2) are product states, the theorem is proved.

For the second part of the exercise, let

$$\sigma\psi_A = \alpha'_u|u\} + \alpha'_d|d\}$$

It follows that

$$\sigma\psi = \alpha'_u\beta_u|uu\rangle + \alpha'_u\beta_d|ud\rangle + \alpha'_d\beta_u|du\rangle + \alpha'_d\beta_d|dd\rangle$$

The expectation value for Alice's spin operator is

$$\langle\sigma\rangle = \psi^*\sigma\psi \tag{3}$$

The result is

$$\langle\sigma\rangle = \alpha_u^*\alpha'_u\beta_u^*\beta_u + \alpha_u^*\alpha'_u\beta_d^*\beta_d + \alpha_d^*\alpha'_d\beta_u^*\beta_u + \alpha_d^*\alpha'_d\beta_d^*\beta_d$$

Rewrite as

$$\langle\sigma\rangle = \alpha_u^*\alpha_u'(\beta_u^*\beta_u + \beta_d^*\beta_d) + \alpha_d^*\alpha_d'(\beta_u^*\beta_u + \beta_d^*\beta_d)$$

By normalization we have $\beta_u^*\beta_u + \beta_d^*\beta_d = 1$ hence

$$\langle\sigma\rangle = \alpha_u^*\alpha_u' + \alpha_d^*\alpha_d' = \psi_A^*\sigma\psi_A \tag{4}$$

By (3) and (4) we have

$$\langle\sigma\rangle = \psi^*\sigma\psi = \psi_A^*\sigma\psi_A$$

By a similar argument for Bob's spin operator we have

$$\langle\tau\rangle = \psi^*\tau\psi = \psi_B^*\tau\psi_B$$

Hence the expectation value of a spin operator is the same for a product state and its corresponding single spin state.