

# Quantum harmonic oscillator

*Anything quadratic is called harmonic.* —A. Zee

A harmonic oscillator is anything with potential energy proportional to displacement squared.

$$V(x) \propto x^2$$

For a quantum harmonic oscillator

$$V(x) = \frac{m\omega^2 x^2}{2}$$

Hence the hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 x^2}{2}$$

We seek to solve the eigenvalue equation

$$\hat{H}\psi_n = E_n\psi_n$$

The solution is

$$\psi_n(x) = C_n \exp\left(-\frac{m\omega x^2}{2\hbar}\right) H_n\left(x\sqrt{m\omega/\hbar}\right), \quad n = 0, 1, 2, \dots$$

$C_n$  is the normalization constant

$$C_n = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}$$

$H_n$  is the  $n$ th hermite polynomial

$$H_n(y) = (-1)^n \exp(y^2) \frac{d^n}{dy^n} \exp(-y^2)$$

The eigenvalues are

$$E_n = \hbar\omega \left(n + \frac{1}{2}\right)$$

The ladder operators are

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{i\hat{p}}{m\omega}\right)$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{i\hat{p}}{m\omega}\right)$$

Operator  $\hat{a}$  lowers  $\psi_n$ .

$$\hat{a}\psi_n = \sqrt{n}\psi_{n-1}$$

Operator  $\hat{a}^\dagger$  raises  $\psi_n$ .

$$\hat{a}^\dagger\psi_n = \sqrt{n+1}\psi_{n+1}$$

This is how  $\psi_n$  can be obtained from  $\psi_0$ .

$$\psi_n = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} \psi_0$$

## Exercises

1. Verify  $\psi_n$  and  $E_n$ .
2. Verify ladder operators.
3. Let

$$\Psi(x) = \frac{\psi_2(x) + \psi_3(x)}{\sqrt{2}}$$

Verify that

$$\Pr(x \geq 0) = \int_0^\infty \Psi^* \Psi \, dx \approx 0.85$$

4. Let

$$m = 6.64 \times 10^{-27} \text{ kilogram}, \quad V(10^{-6} \text{ meter}) = 1 \text{ electronvolt}$$

Verify that

$$\omega = \sqrt{\frac{2V(x)}{mx^2}} = 6.95 \times 10^9 \text{ second}^{-1}$$

For  $\Psi = (\psi_2 + \psi_3)/\sqrt{2}$  verify that

$$\begin{aligned} \langle x \rangle &= \int_{-\infty}^{\infty} x \Psi^* \Psi \, dx = 1.85 \times 10^{-9} \text{ meter} \\ \langle E \rangle &= \int_{-\infty}^{\infty} \Psi^* \hat{H} \Psi \, dx = 1.37 \times 10^{-5} \text{ electronvolt} = \frac{E_2 + E_3}{2} \end{aligned}$$