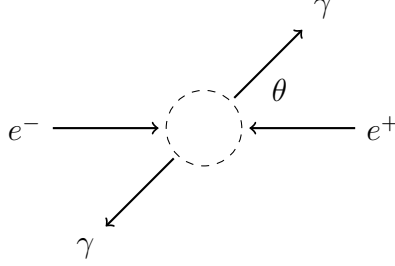


Annihilation

Annihilation is the interaction $e^- + e^+ \rightarrow \gamma + \gamma$.



In the center-of-mass frame we have the following momentum vectors where $E = \sqrt{p^2 + m^2}$.

$$\begin{aligned}
 p_1 &= \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} & p_2 &= \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} & p_3 &= \begin{pmatrix} E \\ E \sin \theta \cos \phi \\ E \sin \theta \sin \phi \\ E \cos \theta \end{pmatrix} & p_4 &= \begin{pmatrix} E \\ -E \sin \theta \cos \phi \\ -E \sin \theta \sin \phi \\ -E \cos \theta \end{pmatrix} \\
 &\text{inbound electron} & &\text{inbound positron} & &\text{outbound photon} & &\text{outbound photon}
 \end{aligned}$$

Spinors for the inbound electron.

$$\begin{aligned}
 u_{11} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix} & u_{12} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix} \\
 &\text{inbound electron spin up} & &\text{inbound electron spin down}
 \end{aligned}$$

Spinors for the inbound positron.

$$\begin{aligned}
 v_{21} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} -p \\ 0 \\ E+m \\ 0 \end{pmatrix} & v_{22} &= \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ p \\ 0 \\ E+m \end{pmatrix} \\
 &\text{inbound positron spin up} & &\text{inbound positron spin down}
 \end{aligned}$$

The scattering amplitude $\mathcal{M}_{ab}{}^{\mu\nu}$ for spin ab and polarization $\mu\nu$ is

$$\mathcal{M}_{ab}{}^{\mu\nu} = \mathcal{M}_{1ab}{}^{\mu\nu} + \mathcal{M}_{2ab}{}^{\nu\mu}$$

where

$$\begin{aligned}
 \mathcal{M}_{1ab}{}^{\mu\nu} &= \frac{\bar{v}_{2b}(-ie\gamma^\mu)(\not{q}_1 + m)(-ie\gamma^\nu)u_{1a}}{t - m^2} \\
 \mathcal{M}_{2ab}{}^{\nu\mu} &= \frac{\bar{v}_{2b}(-ie\gamma^\nu)(\not{q}_2 + m)(-ie\gamma^\mu)u_{1a}}{u - m^2}
 \end{aligned}$$

Matrices \not{q}_1 and \not{q}_2 represent momentum transfer.

$$\begin{aligned}\not{q}_1 &= (p_1 - p_3)^\alpha g_{\alpha\beta} \gamma^\beta \\ \not{q}_2 &= (p_1 - p_4)^\alpha g_{\alpha\beta} \gamma^\beta\end{aligned}$$

Scalars t and u are Mandelstam variables.

$$\begin{aligned}t &= (p_1 - p_3)^2 \\ u &= (p_1 - p_4)^2\end{aligned}$$

In component form (note that indices μ and ν are interchanged for \mathcal{M}_{2ab})

$$\begin{aligned}\mathcal{M}_{1ab}^{\mu\nu} &= \frac{(\bar{v}_{2b})_\alpha (-ie\gamma^{\mu\alpha}_\beta)(\not{q}_1 + m)^\beta_\rho (-ie\gamma^{\nu\rho}_\sigma)(u_{1a})^\sigma}{t - m^2} \\ \mathcal{M}_{2ab}^{\nu\mu} &= \frac{(\bar{v}_{2b})_\alpha (-ie\gamma^{\nu\alpha}_\beta)(\not{q}_2 + m)^\beta_\rho (-ie\gamma^{\mu\rho}_\sigma)(u_{1a})^\sigma}{u - m^2}\end{aligned}$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is the average over spin and polarization states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{a,b} \sum_{\mu,\nu} |\mathcal{M}_{ab}^{\mu\nu}|^2$$

Summing over μ and ν requires $g_{\mu\nu}$ to lower indices.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{a,b} \mathcal{M}_{ab}^{\mu\nu} (g_{\mu\alpha} \mathcal{M}_{ab}^{\alpha\beta} g_{\beta\nu})^*$$

Substitute $\mathcal{M}_{1ab} + \mathcal{M}_{2ab}$ for \mathcal{M}_{ab} . (Note that $P_{12} = P_{21}^*$ hence by the property that probabilities are real we have $P_{12} = P_{21}$.)

$$\begin{aligned}\langle |\mathcal{M}|^2 \rangle &= \frac{1}{4} \sum_{a,b} \left[\underbrace{\mathcal{M}_{1ab}^{\mu\nu} (g_{\mu\alpha} \mathcal{M}_{1ab}^{\alpha\beta} g_{\beta\nu})^*}_{P_{11ab}} + \underbrace{\mathcal{M}_{1ab}^{\mu\nu} (g_{\nu\alpha} \mathcal{M}_{2ab}^{\alpha\beta} g_{\beta\mu})^*}_{P_{12ab}} \right. \\ &\quad \left. + \underbrace{\mathcal{M}_{2ab}^{\nu\mu} (g_{\mu\alpha} \mathcal{M}_{1ab}^{\alpha\beta} g_{\beta\nu})^*}_{P_{21ab}} + \underbrace{\mathcal{M}_{2ab}^{\nu\mu} (g_{\nu\alpha} \mathcal{M}_{2ab}^{\alpha\beta} g_{\beta\mu})^*}_{P_{22ab}} \right]\end{aligned}$$

The Casimir trick uses matrix arithmetic to sum over spin and polarization states:

$$\begin{aligned}\sum_{a,b} P_{11ab} &= \frac{e^4}{(t - m^2)^2} \text{Tr} \left[(\not{p}_1 + m) \gamma^\mu (\not{q}_1 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\nu (\not{q}_1 + m) \gamma_\mu \right] \\ \sum_{a,b} P_{12ab} &= \frac{e^4}{(t - m^2)(u - m^2)} \text{Tr} \left[(\not{p}_1 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\mu (\not{q}_1 + m) \gamma_\nu \right] \\ \sum_{a,b} P_{22ab} &= \frac{e^4}{(u - m^2)^2} \text{Tr} \left[(\not{p}_1 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\nu (\not{q}_2 + m) \gamma_\mu \right]\end{aligned}$$

Let

$$\begin{aligned} f_{11} &= \text{Tr} \left[(\not{p}_1 + m) \gamma^\mu (\not{q}_1 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\nu (\not{q}_1 + m) \gamma_\mu \right] \\ f_{12} &= \text{Tr} \left[(\not{p}_1 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\mu (\not{q}_1 + m) \gamma_\nu \right] \\ f_{22} &= \text{Tr} \left[(\not{p}_1 + m) \gamma^\mu (\not{q}_2 + m) \gamma^\nu (\not{p}_2 - m) \gamma_\nu (\not{q}_2 + m) \gamma_\mu \right] \end{aligned}$$

so that

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left[\frac{f_{11}}{(t - m^2)^2} + \frac{2f_{12}}{(t - m^2)(u - m^2)} + \frac{f_{22}}{(u - m^2)^2} \right]$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$\begin{aligned} f_{11} &= 32(p_1 \cdot p_3)(p_1 \cdot p_4) + 32(p_1 \cdot p_3)m^2 - 32m^4 \\ f_{12} &= 16(p_1 \cdot p_2)m^2 - 16m^4 \\ f_{22} &= 32(p_1 \cdot p_3)(p_1 \cdot p_4) + 32(p_1 \cdot p_4)m^2 - 32m^4 \end{aligned}$$

In Mandelstam variables

$$\begin{aligned} f_{11} &= 8tu - 24tm^2 - 8um^2 - 8m^4 \\ f_{12} &= 8sm^2 - 32m^4 \\ f_{22} &= 8tu - 8tm^2 - 24um^2 - 8m^4 \end{aligned}$$

For $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$\begin{aligned} f_{11} &= 8tu \\ f_{12} &= 0 \\ f_{22} &= 8tu \end{aligned}$$

Hence

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left(\frac{f_{11}}{(t - m^2)^2} + \frac{2f_{12}}{(t - m^2)(u - m^2)} + \frac{f_{22}}{(u - m^2)^2} \right) \\ &= \frac{e^4}{4} \left(\frac{8tu}{t^2} + \frac{8tu}{u^2} \right) \\ &= 2e^4 \left(\frac{u}{t} + \frac{t}{u} \right) \end{aligned}$$

For $m = 0$ the Mandelstam variables are

$$\begin{aligned} t &= -2E^2(1 - \cos \theta) \\ u &= -2E^2(1 + \cos \theta) \end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right)$$

Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{4(4\pi\epsilon_0)^2 s}$$

where

$$s = (p_1 + p_2)^2 = 4E^2$$

For high energy experiments we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\epsilon_0)^2 s} \left(\frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right)$$

Noting that

$$e^2 = 4\pi\epsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s} \left(\frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right)$$

Noting that

$$d\Omega = \sin \theta d\theta d\phi$$

we also have

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{2s} \left(\frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right) \sin \theta d\theta d\phi$$

Let $S(\theta_1, \theta_2)$ be the following integral of $d\sigma$.

$$S(\theta_1, \theta_2) = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{\pi\alpha^2(\hbar c)^2}{s} [I(\theta_2) - I(\theta_1)]$$

where

$$I(\theta) = 2 \cos \theta + 2 \log(1 - \cos \theta) - 2 \log(1 + \cos \theta)$$

The cumulative distribution function is

$$F(\theta) = \frac{S(a, \theta)}{S(a, \pi - a)} = \frac{I(\theta) - I(a)}{I(\pi - a) - I(a)}, \quad a \leq \theta \leq \pi - a$$

Angular support is reduced by an arbitrary angle $a > 0$ because $I(0)$ and $I(\pi)$ are undefined.

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 < \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi - a) - I(a)} \left(\frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right) \sin \theta$$