Fermi golden rule step by step

Consider the transition rate $\Gamma_{1\to2}$ where $\psi_1(x)$ is the initial eigenstate and $\psi_2(x)$ is the final eigenstate.

① Let $\Psi(x,t)$ be the following linear combination of eigenstates.

$$\Psi(x,t) = c_1(t)\psi_1(x)\exp(-i\omega_1 t) + c_2(t)\psi_2(x)\exp(-i\omega_2 t)$$

We need to solve for $c_2(t)$ to find the transition rate.

② Let the perturbing Hamiltonian be

$$H'(x,t) = 2V(x)\cos(\omega t + \phi)$$

③ From the time-dependent Schrodinger equation we have

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = H_0(x)\Psi(x,t) + H'(x,t)\Psi(x,t)$$

④ This reduces to

$$i\hbar\frac{\partial c_1(t)}{\partial t}\psi_1(x)\exp(-i\omega_1 t) + i\hbar\frac{\partial c_2(t)}{\partial t}\psi_2(x)\exp(-i\omega_2 t) = H'(x,t)\Psi(x,t)$$

⑤ Take the inner product of $\psi_2^*(x)$ with the above equation to obtain

$$i\hbar\frac{\partial c_2(t)}{\partial t}\exp(-i\omega_2 t) = 2\cos(\omega t + \phi)\Big(c_1(t)M_{21}\exp(-i\omega_1 t) + c_2(t)M_{22}\exp(-i\omega_2 t)\Big)$$

where M_{21} and M_{22} are the matrix elements

$$M_{21} = \int \psi_2^*(x)V(x)\psi_1(x)\,dx$$

$$M_{22} = \int \psi_2^*(x)V(x)\psi_2(x)\,dx$$

⑥ Rewrite as

$$i\hbar\frac{\partial c_2(t)}{\partial t} = 2\cos(\omega t + \phi)\Big(c_1(t)M_{21}\exp\big(i(\omega_2 - \omega_1)t\big) + 2c_2(t)M_{22}\Big)$$

⑦ Let the initial state be $\Psi(x,0) = \psi_1(x)$ hence the initial conditions are

$$c_1(0) = 1, \quad c_2(0) = 0$$

For small t we use the approximations $c_1(t) = 1$ and $c_2(t) = 0$ to obtain

$$i\hbar\frac{\partial c_2(t)}{\partial t} = 2\cos(\omega t + \phi)M_{21}\exp\big(i(\omega_2 - \omega_1)t\big)$$

⑧ Solve for $c_2(t)$ by integrating.

$$c_2(t) = \frac{2M_{21}}{i\hbar} \int_0^t \cos(\omega t' + \phi) \exp\big(i(\omega_2 - \omega_1)t'\big)\, dt'$$

The solution is

$$c_2(t) = -\frac{M_{21}}{\hbar}\left(\frac{\exp\big(i(\omega_2 - \omega_1 - \omega)t\big) - 1}{\omega_2 - \omega_1 - \omega}\right)\exp(-i\phi)$$

$$-\frac{M_{21}}{\hbar}\left(\frac{\exp\big(i(\omega_2 - \omega_1 + \omega)t\big) - 1}{\omega_2 - \omega_1 + \omega}\right)\exp(i\phi)$$

⑨ For ω such that $\omega \approx \omega_2 - \omega_1$ the first term dominates so discard the second term and write

$$c_2(t) = -\frac{M_{21}}{\hbar}\left(\frac{\exp\big(i(\omega_2 - \omega_1 - \omega)t\big) - 1}{\omega_2 - \omega_1 - \omega}\right)\exp(-i\phi)$$

⑩ Rewrite using a sinc function.

$$c_2(t) = -\frac{it}{\hbar}M_{21}\exp\left(i\,\frac{\omega_2 - \omega_1 - \omega}{2}t - i\phi\right)\operatorname{sinc}\left(\frac{\omega_2 - \omega_1 - \omega}{2}t\right)$$

⑪ Hence the transition probability is

$$P(1 \to 2) = |c_2(t)|^2 = \frac{t^2}{\hbar^2}|M_{21}|^2\operatorname{sinc}^2\left(\frac{\omega_2 - \omega_1 - \omega}{2}t\right)$$

⑫ Integrate $P(1 \to 2)$ to obtain the total transition probability $P_{tot}(1 \to 2)$.

$$P_{tot}(1 \to 2) = \frac{t^2}{\hbar^2}|M_{21}|^2 \int_{E-\epsilon}^{E+\epsilon} \operatorname{sinc}^2\left(\frac{E'/\hbar - \omega}{2}t\right)g(E')\,dE'$$

where

$$E = \hbar(\omega_2 - \omega_1)$$

and $g(E')$ is the density of photon states for energy E'.

⑬ Use the approximation $g(E') \approx g(\hbar\omega)$ to obtain

$$P_{tot}(1 \to 2) = \frac{t^2}{\hbar^2}|M_{21}|^2 g(\hbar\omega)\int_{E-\epsilon}^{E+\epsilon}\operatorname{sinc}^2\left(\frac{E'/\hbar - \omega}{2}t\right)dE'$$

⑭ Let

$$y = \frac{E'/\hbar - \omega}{2}t$$

It follows that

$$E' = \frac{2\hbar y}{t} + \hbar\omega$$

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and
$$dE' = \frac{2\hbar}{t}\,dy$$

The integration limits transform as
$$E \pm \epsilon \rightarrow \frac{(E \pm \epsilon)/\hbar - \omega}{2}\,t = \frac{Et}{2\hbar} - \frac{\omega t}{2} \pm \frac{\epsilon t}{2\hbar} \approx \pm\frac{\epsilon t}{2\hbar}$$

Hence the integral is transformed as
$$P_{tot}(1 \rightarrow 2) = \frac{2t}{\hbar}|M_{21}|^2 g(\hbar\omega) \int_{-\epsilon t/2\hbar}^{\epsilon t/2\hbar} \mathrm{sinc}^2\, y\, dy$$

⑮ The sinc squared function is very narrow so use the approximation
$$\int_{-\epsilon t/2\hbar}^{\epsilon t/2\hbar} \mathrm{sinc}^2\, y\, dy \approx \int_{-\infty}^{\infty} \mathrm{sinc}^2\, y\, dy = \pi$$

to obtain
$$P_{tot}(1 \rightarrow 2) = \frac{2\pi t}{\hbar}|M_{21}|^2 g(\hbar\omega)$$

⑯ The transition rate is the derivative of $P_{tot}(1 \rightarrow 2)$.
$$\Gamma_{1\rightarrow 2} = \frac{d}{dt}P_{tot}(1 \rightarrow 2) = \frac{2\pi}{\hbar}|M_{21}|^2 g(\hbar\omega)$$