2-3. Find S_{cl} for a particle under a constant force f, that is,

$$L = \frac{m}{2}\dot{x}^2 + fx$$

We will need equation (2.7) to determine the classical motion $x(t)$.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \tag{2.7}$$

For the Lagrangian L given in problem 2-3 we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} \qquad \frac{\partial L}{\partial x} = f$$

By equation (2.7) we have for the classical acceleration $\ddot{x}(t)$

$$\ddot{x}(t) = \frac{f}{m}$$

Hence $x(t)$ must have the following quadratic form.

$$x(t) = \frac{f}{2m}t^2 + Bt + C \tag{1}$$

We have the following boundary conditions.

$$x(t_a) = x_a \qquad x(t_b) = x_b$$

Subtract boundary conditions to solve for B.

$$x(t_b) - x(t_a) = x_b - x_a = \frac{f}{2m}\left(t_b^2 - t_a^2\right) + B(t_b - t_a)$$

Solve for B.

$$B = \frac{x_b - x_a}{t_b - t_a} - \frac{f\ (t_b^2 - t_a^2)}{2m(t_b - t_a)} \tag{2}$$

Solve for C.

$$\begin{aligned} C &= x_a - \frac{f}{2m}t_a^2 - Bt_a \\ &= \frac{f\ (t_a t_b^2 - t_a^2 t_b)}{2m(t_b - t_a)} + \frac{t_b x_a - t_a x_b}{t_b - t_a} \end{aligned} \tag{3}$$

Substitute (2) and (3) into (1) to obtain

$$x(t) = \frac{ft^2}{2m} + \left(\frac{x_b - x_a}{t_b - t_a} - \frac{f\ (t_b^2 - t_a^2)}{2m(t_b - t_a)} \right) t + \frac{f\ (t_a t_b^2 - t_a^2 t_b)}{2m(t_b - t_a)} + \frac{t_b x_a - t_a x_b}{t_b - t_a} \tag{4}$$

Equation (4) is too complicated to continue so translate the time coordinate as follows.

$$t_a = 0 \qquad t_b = T$$

We now have

$$x(t) = \frac{ft^2}{2m} + \left(\frac{x_b - x_a}{T} - \frac{fT}{2m} \right) t + x_a \tag{5}$$

Differentiate $x(t)$ to obtain velocity $\dot{x}(t)$.

$$\dot{x}(t) = \frac{d}{dt} x(t) = \frac{ft}{m} - \frac{fT}{2m} + \frac{x_b - x_a}{T} \tag{6}$$

Substitute (5) and (6) into L.

$$\begin{aligned} L &= \frac{m}{2}\dot{x}^2 + fx \\ &= \frac{f^2 t^2}{m} + \left(\frac{2f\ (x_b - x_a)}{T} - \frac{f^2 T}{m} \right) t + \frac{f^2 T^2}{8m} + \frac{f\ (3x_a - x_b)}{2} + \frac{m(x_b - x_a)^2}{2T^2} \end{aligned} \tag{7}$$

Integrate (7) to obtain S_{cl}.

$$S_{cl} = \int_0^T L\, dt = \frac{m(x_b - x_a)^2}{2T} + \frac{fT(x_b + x_a)}{2} - \frac{f^2 T^3}{24m} \tag{8}$$

where $T = t_b - t_a$.