

8-5. A transition element which employs the same wave function as both the initial and final states is called an expectation value. Thus the expectation value of  $F$  for the ground state  $\Phi_0$  of equation (8.83) is

$$\langle \Phi_0 | F | \Phi_0 \rangle = \int \cdots \int \Phi_0^* F \Phi_0 dQ_1 dQ_2 \cdots dQ_{N-1} \quad (8.84)$$

(The integral over complex variables is defined as equal to the corresponding integral over real normal coordinates  $Q_\alpha^c$  and  $Q_\alpha^s$ .) Show that the following expectation values are correct (for  $\alpha \neq \beta$ ).

$$\begin{aligned} \langle \Phi_0^* | Q_\alpha | \Phi_0 \rangle &= \langle \Phi_0^* | Q_\alpha^* | \Phi_0 \rangle = 0 \\ \langle \Phi_0^* | Q_\alpha^2 | \Phi_0 \rangle &= \langle \Phi_0^* | Q_\alpha^{*2} | \Phi_0 \rangle = 0 \\ \langle \Phi_0^* | Q_\alpha^* Q_\alpha | \Phi_0 \rangle &= \frac{\hbar}{2\omega_\alpha} \langle \Phi_0^* | 1 | \Phi_0 \rangle \\ \langle \Phi_0^* | Q_\alpha^* Q_\beta | \Phi_0 \rangle &= 0, \quad \alpha \neq \beta \end{aligned} \quad (8.85)$$

From problem 8-4,

$$|\Phi_0|^2 = \Phi_0^* \Phi_0 = \exp \left( -\frac{1}{\hbar} \sum_{\alpha=1}^{N-1} \omega_\alpha ((Q_\alpha^c)^2 + (Q_\alpha^s)^2) \right)$$

We will use the following integrals.

$$\int_{-\infty}^{\infty} \exp(-ax^2 + b) dx = \sqrt{\frac{\pi}{a}} \exp(b) \quad (1)$$

$$\int_{-\infty}^{\infty} x \exp(-ax^2 + b) dx = 0 \quad (2)$$

$$\int_{-\infty}^{\infty} x^2 \exp(-ax^2 + b) dx = \frac{1}{2a} \sqrt{\frac{\pi}{a}} \exp(b) \quad (3)$$

Here are some specific examples with  $a = \omega_1/\hbar$ .

$$\int_{-\infty}^{\infty} |\Phi_0|^2 dQ_1^c = \left( \frac{\pi\hbar}{\omega_1} \right)^{1/2} \exp \left( \frac{\omega_1}{\hbar} (Q_1^c)^2 \right) |\Phi_0|^2 \quad (4)$$

$$\int_{-\infty}^{\infty} Q_1^c |\Phi_0|^2 dQ_1^c = 0 \quad (5)$$

$$\int_{-\infty}^{\infty} (Q_1^c)^2 |\Phi_0|^2 dQ_1^c = \frac{\hbar}{2\omega_1} \left( \frac{\pi\hbar}{\omega_1} \right)^{1/2} \exp \left( \frac{\omega_1}{\hbar} (Q_1^c)^2 \right) |\Phi_0|^2 \quad (6)$$

Multiplying  $|\Phi_0|^2$  by an exponential cancels a factor, i.e., in (1) and (4),

$$\exp(b) = \exp\left(\frac{\omega_1}{\hbar}(Q_1^c)^2\right) |\Phi_0|^2$$

Compute the expectation value for  $Q_1$ . Let

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{Q_1^c - iQ_1^s}{\sqrt{2}} \right) |\Phi_0|^2 dQ_1^c dQ_1^s$$

By integral (2),  $I = 0$  hence by equation (8.84)

$$\langle \Phi_0^* | Q_1 | \Phi_0 \rangle = 0$$

Compute the expectation value for  $Q_1^*$ . Let

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{Q_1^c + iQ_1^s}{\sqrt{2}} \right) |\Phi_0|^2 dQ_1^c dQ_1^s$$

As above,  $I = 0$  hence

$$\langle \Phi_0^* | Q_1^* | \Phi_0 \rangle = 0$$

Compute the expectation value for  $Q_1^2$ . Let

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{Q_1^c - iQ_1^s}{\sqrt{2}} \right)^2 |\Phi_0|^2 dQ_1^c dQ_1^s$$

Rewrite as

$$\begin{aligned} I &= -i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_1^c Q_1^s |\Phi_0|^2 dQ_1^c dQ_1^s \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (Q_1^c)^2 |\Phi_0|^2 dQ_1^c dQ_1^s - \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (Q_1^s)^2 |\Phi_0|^2 dQ_1^c dQ_1^s \end{aligned}$$

The first integral vanishes by (2). The remaining integrals cancel hence

$$\langle \Phi_0^* | Q_1^2 | \Phi_0 \rangle = 0$$

Compute the expectation value for  $Q_1^{*2}$ . (As above except for a sign change.)

$$\begin{aligned} I &= i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_1^c Q_1^s |\Phi_0|^2 dQ_1^c dQ_1^s \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (Q_1^c)^2 |\Phi_0|^2 dQ_1^c dQ_1^s - \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (Q_1^s)^2 |\Phi_0|^2 dQ_1^c dQ_1^s \end{aligned}$$

By the same arguments as  $Q_1^2$

$$\langle \Phi_0^* | Q_1^{*2} | \Phi_0 \rangle = 0$$

Compute the expectation value for  $Q_1^* Q_1$ . Let

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{(Q_1^c)^2 + (Q_1^s)^2}{2} \right) |\Phi_0|^2 dQ_1^c dQ_1^s$$

Rewrite as

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} (Q_1^c)^2 |\Phi_0|^2 dQ_1^c \right) dQ_1^s + \frac{1}{2} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} (Q_1^s)^2 |\Phi_0|^2 dQ_1^s \right) dQ_1^c$$

By integral (3) with  $a = \omega_1/\hbar$ ,

$$\begin{aligned} I &= \frac{\hbar}{4\omega_1} \left( \frac{\pi\hbar}{\omega_1} \right)^{1/2} \exp \left( \frac{\omega_1}{\hbar} (Q_1^c)^2 \right) \int_{-\infty}^{\infty} |\Phi_0|^2 dQ_1^s \\ &\quad + \frac{\hbar}{4\omega_1} \left( \frac{\pi\hbar}{\omega_1} \right)^{1/2} \exp \left( \frac{\omega_1}{\hbar} (Q_1^s)^2 \right) \int_{-\infty}^{\infty} |\Phi_0|^2 dQ_1^c \end{aligned}$$

By integral (1),

$$\begin{aligned} I &= \frac{\hbar}{4\omega_1} \frac{\pi\hbar}{\omega_1} \exp \left( \frac{\omega_1}{\hbar} (Q_1^c)^2 \right) \exp \left( \frac{\omega_1}{\hbar} (Q_1^s)^2 \right) |\Phi_0|^2 \\ &\quad + \frac{\hbar}{4\omega_1} \frac{\pi\hbar}{\omega_1} \exp \left( \frac{\omega_1}{\hbar} (Q_1^s)^2 \right) \exp \left( \frac{\omega_1}{\hbar} (Q_1^c)^2 \right) |\Phi_0|^2 \end{aligned}$$

Integrate over the remaining measure as in (8.84).

$$\begin{aligned} \langle \Phi_0^* | Q_1^* Q_1 | \Phi_0 \rangle &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} I dQ_2^c dQ_2^s \cdots dQ_{N-1}^c dQ_{N-1}^s \\ &= \frac{\hbar}{2\omega_1} \frac{\pi\hbar}{\omega_1} \prod_{k=2}^{N-1} \frac{\pi\hbar}{\omega_k} \quad (7) \end{aligned}$$

Note that the exponentials cancel with  $u$ .

Compute the normalization constant. By integral (1),

$$\langle \Phi_0^* | 1 | \Phi_0 \rangle = \prod_{k=1}^{N-1} \frac{\pi\hbar}{\omega_k} \quad (8)$$

Hence by (7) and (8),

$$\langle \Phi_0 | Q_1^* Q_1 | \Phi_0 \rangle = \frac{\hbar}{2\omega_1} \langle \Phi_0 | 1 | \Phi_0 \rangle$$

Compute the expectation value for  $Q_1^* Q_2$ . Let

$$I = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left( \frac{Q_1^c Q_2^c - Q_1^s Q_2^s - i Q_1^c Q_2^s - i Q_1^s Q_2^c}{2} \right) |\Phi_0|^2 dQ_1^c dQ_1^s dQ_2^c dQ_2^s$$

By integral (2) we have  $I = 0$ , hence by equation (8.84)

$$\langle \Phi_0 | Q_1^* Q_2 | \Phi_0 \rangle = 0$$