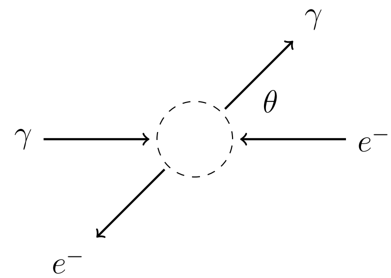
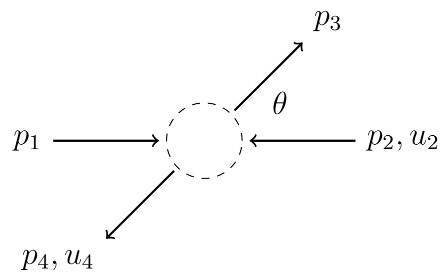
Compton scattering occurs when a high energy photon such as a gamma ray interacts with an electron. In typical Compton scattering experiments the incident electron is at rest with zero velocity. However, it is easier to develop a theory using the center of mass frame in which the photon and the electron have equal and opposite momentum. The following diagram shows the photon and electron scattering through angle θ in the center of mass frame.



Here is the same diagram with momentum and spinor labels.



Here are the momentum vectors for center of mass coordinates.

$$p_1 = \begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix} \qquad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -\omega \end{pmatrix} \qquad p_3 = \begin{pmatrix} \omega \\ \omega\sin\theta\cos\phi \\ \omega\sin\theta\sin\phi \\ \omega\cos\theta \end{pmatrix} \qquad p_4 = \begin{pmatrix} E \\ -\omega\sin\theta\cos\phi \\ -\omega\sin\theta\sin\phi \\ -\omega\cos\theta \end{pmatrix}$$

inbound photon | inbound electron | outbound photon | outbound electron

Symbol ω is the photon energy and E is total energy $E = \sqrt{\omega^2 + m^2}$ where m is electron mass. Polar angle θ is the observed scattering angle of the photon. Azimuth angle ϕ cancels out in scattering calculations.

The spinors are

$$u_{21} = \begin{pmatrix} E+m \\ 0 \\ -\omega \\ 0 \end{pmatrix} \qquad u_{22} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ \omega \end{pmatrix} \qquad u_{41} = \begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix} \qquad u_{42} = \begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}$$

inbound electron, spin up | inbound electron, spin down | outbound electron, spin up | outbound electron, spin down

The spinors shown above are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^2$ will be used.

The following formula computes a probability density $|\mathcal{M}_{ab}|^2$ for Compton scattering where a is the spin state of the inbound electron and b is the spin state of the outbound electron.

$$|\mathcal{M}_{ab}|^2 = \frac{e^4}{N} \left| -\frac{\bar{u}_{4b}\gamma^\mu(\not{q}_1 + m)\gamma^\nu u_{2a}}{s - m^2} - \frac{\bar{u}_{4b}\gamma^\nu(\not{q}_2 + m)\gamma^\mu u_{2a}}{u - m^2} \right|^2$$

Symbol e is electron charge. Symbols s and u are Mandelstam variables $s = (p_1 + p_2)^2$ and $u = (p_1 - p_4)^2$. Symbol $q_1 = p_1 + p_2$ and $q_2 = p_2 - p_3$.

Let

$$a_1 = \bar{u}_{4b}\gamma^\mu(\not{q}_1 + m)\gamma^\nu u_{2a} \qquad a_2 = \bar{u}_{4b}\gamma^\nu(\not{q}_2 + m)\gamma^\mu u_{2a}$$

Then

$$\begin{aligned}
|\mathcal{M}_{ab}|^2 &= \frac{e^4}{N}\left|-\frac{a_1}{s-m^2} - \frac{a_2}{u-m^2}\right|^2 \\
&= \frac{e^4}{N}\left(-\frac{a_1}{s-m^2} - \frac{a_2}{u-m^2}\right)\left(-\frac{a_1}{s-m^2} - \frac{a_2}{u-m^2}\right)^* \\
&= \frac{e^4}{N}\left(\frac{a_1 a_1^*}{(s-m^2)^2} + \frac{a_1 a_2^*}{(s-m^2)(u-m^2)} + \frac{a_1^* a_2}{(s-m^2)(u-m^2)} + \frac{a_2 a_2^*}{(u-m^2)^2}\right)
\end{aligned}$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is computed by summing $|\mathcal{M}_{ab}|^2$ over all spin and polarization states and then dividing by the number of inbound states. There are four inbound states. The sum over polarizations is already accomplished by contraction of aa^* over μ and ν.

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}|\mathcal{M}_{ab}|^2 \\
&= \frac{e^4}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\frac{1}{N}\left(\frac{a_1 a_1^*}{(s-m^2)^2} + \frac{a_1 a_2^*}{(s-m^2)(u-m^2)} + \frac{a_1^* a_2}{(s-m^2)(u-m^2)} + \frac{a_2 a_2^*}{(u-m^2)^2}\right)
\end{aligned}$$

Use the Casimir trick to replace sums over spins with matrix products.

$$\begin{aligned}
f_{11} &= \frac{1}{N}\sum_{a=1}^{2}\sum_{b=1}^{2} a_1 a_1^* = \operatorname{Tr}\Big((\not{p}_2 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\Big) \\
f_{12} &= \frac{1}{N}\sum_{a=1}^{2}\sum_{b=1}^{2} a_1 a_2^* = \operatorname{Tr}\Big((\not{p}_2 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\Big) \\
f_{22} &= \frac{1}{N}\sum_{a=1}^{2}\sum_{b=1}^{2} a_2 a_2^* = \operatorname{Tr}\Big((\not{p}_2 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\nu(\not{q}_2 + m)\gamma_\mu\Big)
\end{aligned}$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left(\frac{f_{11}}{(s-m^2)^2} + \frac{f_{12}}{(s-m^2)(u-m^2)} + \frac{f_{12}^*}{(s-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2}\right) \tag{1}$$

Run "compton-scattering-1.txt" to verify the Casimir trick for Compton scattering.

The following momentum formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$\begin{aligned}
f_{11} = &-16(p_1\cdot p_1)(p_2\cdot p_4) + 32(p_1\cdot p_2)(p_1\cdot p_4) + 32(p_1\cdot p_4)(p_2\cdot p_2) + 16(p_2\cdot p_2)(p_2\cdot p_4) \\
&+ 64m^2(p_1\cdot p_1) + 64m^2(p_1\cdot p_2) - 64m^2(p_1\cdot p_4) - 48m^2(p_2\cdot p_4) + 64m^4 \\
f_{12} = &-32(p_1\cdot p_2)(p_2\cdot p_4) + 32(p_1\cdot p_3)(p_2\cdot p_4) - 32(p_2\cdot p_2)(p_2\cdot p_4) + 32(p_2\cdot p_3)(p_2\cdot p_4) \\
&+ 32m^2(p_1\cdot p_2) - 16m^2(p_1\cdot p_3) + 16m^2(p_1\cdot p_4) \\
&+ 48m^2(p_2\cdot p_2) - 32m^2(p_2\cdot p_3) + 48m^2(p_2\cdot p_4) - 16m^2(p_3\cdot p_4) - 32m^4 \\
f_{22} = &\ 16(p_2\cdot p_2)(p_2\cdot p_4) - 32(p_2\cdot p_2)(p_3\cdot p_4) + 32(p_2\cdot p_3)(p_3\cdot p_4) - 16(p_2\cdot p_4)(p_3\cdot p_3) \\
&- 64m^2(p_2\cdot p_3) - 48m^2(p_2\cdot p_4) + 64m^2(p_3\cdot p_3) + 64m^2(p_3\cdot p_4) + 64m^4
\end{aligned}$$

In terms of Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, and $u = (p_1 - p_4)^2$ the formulas are

$$\begin{aligned}
f_{11} &= -8su + 24sm^2 + 8um^2 + 8m^4 \\
f_{12} &= 8sm^2 + 8um^2 + 16m^4 \\
f_{22} &= -8su + 8sm^2 + 24um^2 + 8m^4
\end{aligned} \tag{2}$$

For high energy experiments where $E \gg m$ the approximation $m = 0$ can be used resulting in the following simplified probability density.

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{1 + \cos\theta}{2} + \frac{2}{1 + \cos\theta} \right)$$

Run "compton-scattering-2.txt" to verify momentum formulas for Compton scattering.

Lab frame

Compton scattering experiments are typically done in the "lab" frame where the electron is at rest. The following Lorentz boost Λ transforms momentum vectors from the center of mass frame to the lab frame.

$$\Lambda = \begin{pmatrix} E/m & 0 & 0 & \omega/m \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega/m & 0 & 0 & E/m \end{pmatrix}, \qquad \Lambda p_2 = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Mandelstam variables are invariant under a boost.

$$\begin{aligned}
s &= (p_1 + p_2)^2 = (\Lambda p_1 + \Lambda p_2)^2 \\
t &= (p_1 - p_3)^2 = (\Lambda p_1 - \Lambda p_3)^2 \\
u &= (p_1 - p_4)^2 = (\Lambda p_1 - \Lambda p_4)^2
\end{aligned}$$

In the lab frame, let ω_L be the angular frequency of the incident photon and let ω_L' be the angular frequency of the scattered photon.

$$\begin{aligned}
\omega_L &= \Lambda p_1 \cdot (1, 0, 0, 0) = \frac{\omega^2}{m} + \frac{\omega E}{m} \\
\omega_L' &= \Lambda p_3 \cdot (1, 0, 0, 0) = \frac{\omega^2 \cos\theta}{m} + \frac{\omega E}{m}
\end{aligned}$$

It follows that

$$\begin{aligned}
s &= (p_1 + p_2)^2 = 2m\omega_L + m^2 \\
t &= (p_1 - p_3)^2 = 2m(\omega_L' - \omega_L) \\
u &= (p_1 - p_4)^2 = -2m\omega_L' + m^2
\end{aligned}$$

Compute $\langle |\mathcal{M}|^2 \rangle$ using equations (1) and (2) and the above s, t, and u that involve ω_L and ω_L'.

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{\omega_L}{\omega_L'} + \frac{\omega_L'}{\omega_L} + \left(\frac{m}{\omega_L} - \frac{m}{\omega_L'} + 1 \right)^2 - 1 \right)$$

From the Compton formula

$$\frac{1}{\omega_L'} - \frac{1}{\omega_L} = \frac{1 - \cos\theta_L}{m}$$

we have

$$\cos\theta_L = \frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1$$

Hence

$$\langle|\mathcal{M}|^2\rangle = 2e^4\left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \cos^2\theta_L - 1\right)$$

Run “compton-scattering-3.txt” to verify lab frame formulas for Compton scattering.

Cross section

Now that we have derived $\langle|\mathcal{M}|^2\rangle$ we can investigate the angular distribution of scattered photons. For simplicity let us drop the L subscript from lab variables. From now on the symbols ω, ω', and θ will be lab frame variables.

The differential cross section for Compton scattering is

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{\hbar^2}{64\pi^2 m^2 c^4}\left(\frac{\omega'}{\omega}\right)^2\langle|\mathcal{M}|^2\rangle \\ &= \frac{e^4\hbar^2}{32\pi^2 m^2 c^4}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + \cos^2\theta - 1\right)\end{aligned}$$

From $e^4 = 16\pi^2\alpha^2$ we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2\hbar^2}{2m^2c^4}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + \cos^2\theta - 1\right)$$

The scattered photon frequency ω' is computed from the Compton equation.

$$\omega' = \frac{m\omega}{m + \omega(1-\cos\theta)}$$

We can integrate $d\sigma$ to obtain a cumulative distribution function. Recall that

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

Hence

$$d\sigma = \frac{\alpha^2\hbar^2}{2m^2c^4}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + \cos^2\theta - 1\right)\sin\theta\, d\theta\, d\phi$$

Let $I(\theta)$ be the following integral of $d\sigma$.

$$\begin{aligned}I(\theta) &= \left(\frac{2m^2c^4}{\alpha^2\hbar^2}\right)\frac{1}{2\pi}\int_0^{2\pi}\int d\sigma \\ &= \int\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + \cos^2\theta - 1\right)\sin\theta\, d\theta, \quad 0 \le \theta \le \pi\end{aligned}$$

Assume that $I(\theta) - I(0)$ is computable given θ by either symbolic or numerical integration.

Let C be the normalization constant

$$C = I(\pi) - I(0)$$

Then the cumulative distribution function $F(\theta)$ is

$$F(\theta) = \frac{I(\theta) - I(0)}{C}, \quad 0 \leq \theta \leq \pi$$

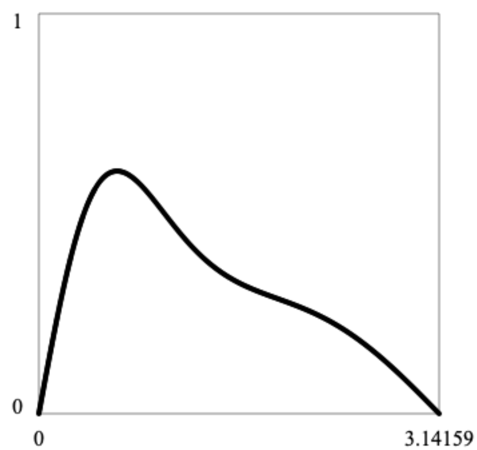
The probability of observing scattering events in the interval θ_1 to θ_2 can now be computed.

$$P(\theta_1 \leq \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density function $f(\theta)$ is the derivative of $F(\theta)$.

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{C}\left(\frac{\omega'}{\omega}\right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + \cos^2\theta - 1\right)\sin\theta$$

Run "compton-scattering-4.txt" to plot $f(\theta)$. The following plot is for $\omega = 500\,\text{keV} = 1.2 \times 10^{20}\,\text{Hz}$.



Probability distribution for 45° bins ($\omega = 500\,\text{keV} = 1.2 \times 10^{20}\,\text{Hz}$).

θ_1	θ_2	$P(\theta_1 \leq \theta \leq \theta_2)$
0°	45°	0.35
45°	90°	0.34
90°	135°	0.22
135°	180°	0.09

Thomson scattering

When ω is much smaller than the electron mass m we have

$$\frac{m}{m + \omega(1 - \cos\theta)} \approx 1$$

Hence for $\omega \ll m$ the differential cross section is approximately

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2}(1 + \cos^2\theta)$$

which is the formula for Thomson scattering.

LEP data

The following Compton scattering data is from the paper "Compton Scattering of Quasi-Real Virtual Photons at LEP" (`arxiv.org/abs/hep-ex/0504012`).

x	y
-0.74	13380
-0.60	7720
-0.47	6360
-0.34	4600
-0.20	4310
-0.07	3700
0.06	3640
0.20	3340
0.33	3500
0.46	3010
0.60	3310
0.73	3330

The data are for the center of mass frame and have the following relationship with the differential cross section formula.

$$x = \cos\theta \qquad y = \frac{d\sigma}{d\cos\theta}$$

This is the differential cross section formula for Compton scattering in the center of mass frame with high energy approximation $m = 0$.

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{s}\left(\frac{1+\cos\theta}{2} + \frac{2}{1+\cos\theta}\right)$$

To compute predicted values $\hat{y}$ from the above formula, use $s = 40$ to approximate the QED values in the paper. Multiply the result by $(\hbar c)^2$ to convert to SI and multiply by 10^{40} to convert square meters to picobarns.

$$\hat{y} = \frac{\pi\alpha^2}{s}\left(\frac{1+x}{2} + \frac{2}{1+x}\right) \times (\hbar c)^2 \times 10^{40}$$

The following table includes the predicted cross section $\hat{y}$.

x	y	$\hat{y}$
-0.74	13380	12739
-0.60	7720	8468
-0.47	6360	6577
-0.34	4600	5472
-0.20	4310	4723
-0.07	3700	4259
0.06	3640	3936
0.20	3340	3691
0.33	3500	3532
0.46	3010	3420
0.60	3310	3338
0.73	3330	3291

The coefficient of determination R^2 measures how well predicted values fit the real data.

$$R^2 = 1 - \frac{\sum (y - \hat{y})^2}{\sum (y - \bar{y})^2} = 0.97$$

The result indicates that the model $d\sigma$ explains 97% of the variance in the data.

Notes on Eigenmath scripts

Write a_1 and a_2 in full component form.

$$a_1^{\mu\nu} = \bar{u}_{4\alpha}\gamma^{\mu\alpha}{}_\beta(\not{q}_1 + m)^\beta{}_\rho\gamma^{\nu\rho}{}_\sigma u_2^\sigma \qquad a_2^{\nu\mu} = \bar{u}_{4\alpha}\gamma^{\nu\alpha}{}_\beta(\not{q}_2 + m)^\beta{}_\rho\gamma^{\mu\rho}{}_\sigma u_2^\sigma$$

Transpose γ tensors to form inner products over α and ρ.

$$a_1^{\mu\nu} = \bar{u}_{4\alpha}\gamma^{\alpha\mu}{}_\beta(\not{q}_1 + m)^\beta{}_\rho\gamma^{\rho\nu}{}_\sigma u_2^\sigma \qquad a_2^{\nu\mu} = \bar{u}_{4\alpha}\gamma^{\alpha\nu}{}_\beta(\not{q}_2 + m)^\beta{}_\rho\gamma^{\rho\mu}{}_\sigma u_2^\sigma$$

Convert transposed γ to Eigenmath code.

$\gamma^{\alpha\mu}{}_\beta \quad \rightarrow$ `gammaT = transpose(gamma)`

Then to compute a_1 we have

$a_1 = \bar{u}_{4\alpha}\gamma^{\alpha\mu}{}_\beta(\not{q}_1 + m)^\beta{}_\rho\gamma^{\rho\nu}{}_\sigma u_2^\sigma$

$\rightarrow$ `a1 = dot(u4bar[s4],gammaT,qslash1 + m I,gammaT,u2[s2])`

where s_2 and s_4 are spin indices. Similarly for a_2 we have

$a_2 = \bar{u}_{4\alpha}\gamma^{\alpha\nu}{}_\beta(\not{q}_2 + m)^\beta{}_\rho\gamma^{\rho\mu}{}_\sigma u_2^\sigma$

$\rightarrow$ `a2 = dot(u4bar[s4],gammaT,qslash2 + m I,gammaT,u2[s2])`

In component notation the product $a_1 a_1^*$ is

$$a_1 a_1^* = a_1^{\mu\nu} a_1^{*\mu\nu}$$

To sum over μ and ν it is necessary to lower indices with the metric tensor. Also, transpose a_1^* to form an inner product with ν.

$$a_1 a_1^* = a_1^{\mu\nu} a_{1\nu\mu}^*$$

Convert to Eigenmath code. The dot function sums over ν and the contract function sums over μ.

$a_1 a_1^* \quad \rightarrow$ `a11 = contract(dot(a1,gmunu,transpose(conj(a1)),gmunu))`

Similarly for $a_2 a_2^*$ we have

$a_2 a_2^* \quad \rightarrow$ `a22 = contract(dot(a2,gmunu,transpose(conj(a2)),gmunu))`

The product $a_1 a_2^*$ does not require a transpose because $a_1 a_2^* = a_1^{\mu\nu} a_2^{*\nu\mu}$.

$a_1 a_2^* \quad \rightarrow$ `a12 = contract(dot(a1,gmunu,conj(a2),gmunu))`

In component notation, a trace operator becomes a sum over an index, in this case α.

$$
\begin{aligned}
f_{11} &= \operatorname{Tr}\Big((\not{p}_2+m)\gamma^\mu(\not{q}_1+m)\gamma^\nu(\not{p}_4+m)\gamma_\nu(\not{q}_1+m)\gamma_\mu\Big)\\
&= (\not{p}_2+m)^\alpha{}_\beta\gamma^{\mu\beta}{}_\rho(\not{q}_1+m)^\rho{}_\sigma\gamma^{\nu\sigma}{}_\tau(\not{p}_4+m)^\tau{}_\delta\gamma_\nu{}^\delta{}_\eta(\not{q}_1+m)^\eta{}_\xi\gamma_\mu{}^\xi{}_\alpha
\end{aligned}
$$

As before, transpose γ tensors to form inner products.

$$
f_{11} = (\not{p}_2+m)^\alpha{}_\beta\gamma^{\beta\mu}{}_\rho(\not{q}_1+m)^\rho{}_\sigma\gamma^{\sigma\nu}{}_\tau(\not{p}_4+m)^\tau{}_\delta\gamma^\delta{}_{\nu\eta}(\not{q}_1+m)^\eta{}_\xi\gamma^\xi{}_{\mu\alpha}
$$

To convert to Eigenmath code, use an intermediate variable for the inner product.

$T^{\alpha\mu\nu}{}_{\nu\mu\alpha}$ $\rightarrow$ `T = dot(P2,gammaT,Q1,gammaT,P4,gammaL,Q1,gammaL)`

Now sum over the indices of T. The innermost contract sums over ν then the next contract sums over μ. Finally the outermost contract sums over α.

f_{11} $\rightarrow$ `f11 = contract(contract(contract(T,3,4),2,3))`

Follow suit for f_{22}. For f_{12} the order of the rightmost μ and ν is reversed.

$$
f_{12} = \operatorname{Tr}\Big((\not{p}_2+m)\gamma^\mu(\not{q}_2+m)\gamma^\nu(\not{p}_4+m)\gamma_\mu(\not{q}_1+m)\gamma_\nu\Big)
$$

The resulting inner product is $T^{\alpha\mu\nu}{}_{\mu\nu\alpha}$ so the contraction is different.

f_{12} $\rightarrow$ `f12 = contract(contract(contract(T,3,5),2,3))`

The innermost contract sums over ν followed by sum over μ then sum over α.