Two spins

Spin state $|s\rangle$ for two spins is a unit vector in $\mathbb{C}^4$.

$$|s\rangle = \begin{pmatrix} c_{++} \\ c_{+-} \\ c_{-+} \\ c_{--} \end{pmatrix}, \quad |c_{++}|^2 + |c_{+-}|^2 + |c_{-+}|^2 + |c_{--}|^2 = 1$$

Spin measurement probabilities are the transition probabilities from $|s\rangle$ to an eigenstate.

For spin measurements in the z direction we have

$$\begin{aligned}
\Pr\left(S_{1z} = +\tfrac{\hbar}{2} \text{ and } S_{2z} = +\tfrac{\hbar}{2}\right) &= |\langle z_{++}|s\rangle|^2 = |c_{++}|^2 \\
\Pr\left(S_{1z} = +\tfrac{\hbar}{2} \text{ and } S_{2z} = -\tfrac{\hbar}{2}\right) &= |\langle z_{+-}|s\rangle|^2 = |c_{+-}|^2 \\
\Pr\left(S_{1z} = -\tfrac{\hbar}{2} \text{ and } S_{2z} = +\tfrac{\hbar}{2}\right) &= |\langle z_{-+}|s\rangle|^2 = |c_{-+}|^2 \\
\Pr\left(S_{1z} = -\tfrac{\hbar}{2} \text{ and } S_{2z} = -\tfrac{\hbar}{2}\right) &= |\langle z_{--}|s\rangle|^2 = |c_{--}|^2
\end{aligned}$$

where the eigenstates are

$$z_{++} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad z_{+-} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad z_{-+} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad z_{--} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Spin operators for the first spin ($\otimes$ is kronecker product).

$$S_{1x} = \frac{\hbar}{2}\sigma_x \otimes I = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$S_{1y} = \frac{\hbar}{2}\sigma_y \otimes I = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$S_{1z} = \frac{\hbar}{2}\sigma_z \otimes I = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Spin operators for the second spin.

$$S_{2x} = \frac{\hbar}{2} I \otimes \sigma_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$S_{2y} = \frac{\hbar}{2} I \otimes \sigma_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$S_{2z} = \frac{\hbar}{2} I \otimes \sigma_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Expectation values for the first spin.

$$\begin{aligned}
\langle S_{1x} \rangle &= \langle s | S_{1x} | s \rangle = \frac{\hbar}{2} \left(c_{++} c_{-+}^* + c_{++}^* c_{-+} + c_{+-} c_{--}^* + c_{+-}^* c_{--} \right) \\
\langle S_{1y} \rangle &= \langle s | S_{1y} | s \rangle = \frac{i\hbar}{2} \left(c_{++} c_{-+}^* - c_{++}^* c_{-+} + c_{+-} c_{--}^* - c_{+-}^* c_{--} \right) \\
\langle S_{1z} \rangle &= \langle s | S_{1z} | s \rangle = \frac{\hbar}{2} \left(|c_{++}|^2 + |c_{+-}|^2 - |c_{-+}|^2 - |c_{--}|^2 \right)
\end{aligned}$$

Expectation values for the second spin.

$$\begin{aligned}
\langle S_{2x} \rangle &= \langle s | S_{2x} | s \rangle = \frac{\hbar}{2} \left(c_{++} c_{+-}^* + c_{++}^* c_{+-} + c_{-+} c_{--}^* + c_{-+}^* c_{--} \right) \\
\langle S_{2y} \rangle &= \langle s | S_{2y} | s \rangle = \frac{i\hbar}{2} \left(c_{++} c_{+-}^* - c_{++}^* c_{+-} + c_{-+} c_{--}^* - c_{-+}^* c_{--} \right) \\
\langle S_{2z} \rangle &= \langle s | S_{2z} | s \rangle = \frac{\hbar}{2} \left(|c_{++}|^2 - |c_{+-}|^2 + |c_{-+}|^2 - |c_{--}|^2 \right)
\end{aligned}$$

Expected spin vectors.

$$\langle \mathbf{S}_1 \rangle = \langle s | \mathbf{S}_1 | s \rangle = \begin{pmatrix} \langle S_{1x} \rangle \\ \langle S_{1y} \rangle \\ \langle S_{1z} \rangle \end{pmatrix}, \quad \langle \mathbf{S}_2 \rangle = \langle s | \mathbf{S}_2 | s \rangle = \begin{pmatrix} \langle S_{2x} \rangle \\ \langle S_{2y} \rangle \\ \langle S_{2z} \rangle \end{pmatrix}$$

Consider the case of having determined $\langle \mathbf{S}_1 \rangle$ and $\langle \mathbf{S}_2 \rangle$ by experiment. To convert $\langle \mathbf{S}_1 \rangle$ and $\langle \mathbf{S}_2 \rangle$ to a spin state $|s\rangle$, let

$$\begin{aligned}
x_i &= \frac{2}{\hbar} \langle S_{ix} \rangle = \sin\theta_i \cos\phi_i \\
y_i &= \frac{2}{\hbar} \langle S_{iy} \rangle = \sin\theta_i \sin\phi_i \\
z_i &= \frac{2}{\hbar} \langle S_{iz} \rangle = \cos\theta_i
\end{aligned}$$

Then

$$|s_i\rangle = \begin{pmatrix} \cos(\theta_i/2) \\ \sin(\theta_i/2)\exp(i\phi_i) \end{pmatrix}$$

where

$$\cos(\theta_i/2) = \sqrt{\frac{\cos\theta_i + 1}{2}} = \sqrt{\frac{z_i + 1}{2}}$$
$$\sin(\theta_i/2) = \sqrt{\frac{1 - \cos\theta_i}{2}} = \sqrt{\frac{1 - z_i}{2}}$$

and

$$\exp(i\phi_i) = \cos\phi_i + i\sin\phi_i = \frac{x_i + iy_i}{\sqrt{x_i^2 + y_i^2}}$$

Spin state $|s\rangle$ is the kronecker product of $|s_1\rangle$ and $|s_2\rangle$.

$$|s\rangle = |s_1\rangle \otimes |s_2\rangle$$

Spin total angular momentum magnitude squared operator $(\mathbf{S}_1 + \mathbf{S}_2)^2$.

$$(\mathbf{S}_1 + \mathbf{S}_2)^2 = (S_{1x} + S_{2x})^2 + (S_{1y} + S_{2y})^2 + (S_{1z} + S_{2z})^2 = \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Expectation value for $(\mathbf{S}_1 + \mathbf{S}_2)^2$.

$$\langle s|(\mathbf{S}_1 + \mathbf{S}_2)^2|s\rangle = \hbar^2 \left(2|c_{++}|^2 + |c_{+-} + c_{-+}|^2 + 2|c_{--}|^2\right)$$

Exercises

1. Verify spin operators for two spins.

2. Verify expectation values for two spins.

3. Let

$$|s_1\rangle = \begin{pmatrix} \cos(\theta_1/2) \\ \sin(\theta_1/2)\exp(i\phi_1) \end{pmatrix}, \quad |s_2\rangle = \begin{pmatrix} \cos(\theta_2/2) \\ \sin(\theta_2/2)\exp(i\phi_2) \end{pmatrix}$$

and

$$|s\rangle = |s_1\rangle \otimes |s_2\rangle$$

Verify that

$$\langle s|\mathbf{S}_1|s\rangle = \frac{\hbar}{2}\begin{pmatrix} \sin\theta_1\cos\phi_1 \\ \sin\theta_1\sin\phi_1 \\ \cos\theta_1 \end{pmatrix}, \quad \langle s|\mathbf{S}_2|s\rangle = \frac{\hbar}{2}\begin{pmatrix} \sin\theta_2\cos\phi_2 \\ \sin\theta_2\sin\phi_2 \\ \cos\theta_2 \end{pmatrix}$$

4. Verify that for a product state $|s\rangle = |s_1\rangle \otimes |s_2\rangle$ we have

$$\langle S_{1j}S_{2k}\rangle = \langle S_{1j}\rangle\langle S_{2k}\rangle$$

where $j, k \in \{x, y, z\}$.