Rotating wave approximation

Let $\Psi(\mathbf{r},t)$ be the following wave function for a two state system.

$$\Psi(\mathbf{r},t) = \psi_a(\mathbf{r})c_a(t)\exp(-\tfrac{i}{\hbar}E_a t) + \psi_b(\mathbf{r})c_b(t)\exp(-\tfrac{i}{\hbar}E_b t)$$

Let $\hat{H}(\mathbf{r},t)$ be the Hamiltonian

$$\hat{H}(\mathbf{r},t) = \hat{H}_0(\mathbf{r}) + \hat{H}_1(\mathbf{r},t)$$

where

$$\hat{H}_0\psi_a = E_a\psi_a, \quad \hat{H}_0\psi_b = E_b\psi_b, \quad \hat{H}_0\Psi = (E_a + E_b)\Psi$$

It was shown that if Ψ is a solution to the Schrödinger equation then

$$\frac{d}{dt}c_a(t) = -\frac{i}{\hbar}\langle\psi_a|\hat{H}_1|\psi_b\rangle\exp(-i\omega_0 t)c_b(t), \quad \frac{d}{dt}c_b(t) = -\frac{i}{\hbar}\langle\psi_b|\hat{H}_1|\psi_a\rangle\exp(i\omega_0 t)c_a(t) \tag{1}$$

where

$$\omega_0 = \frac{E_b - E_a}{\hbar}$$

Let $\hat{H}_1(\mathbf{r},t)$ be the perturbation

$$\hat{H}_1(\mathbf{r},t) = \hat{V}(\mathbf{r})\cos(\omega t)$$

Then

$$\langle\psi_a|\hat{H}_1|\psi_b\rangle = \langle\psi_a|\hat{V}|\psi_b\rangle\left[\tfrac{1}{2}\exp(i\omega t) + \tfrac{1}{2}\exp(-i\omega t)\right]$$

The rotating wave approximation discards the second term and asserts

$$\langle\psi_a|\hat{H}_1|\psi_b\rangle = \tfrac{1}{2}\langle\psi_a|\hat{V}|\psi_b\rangle\exp(i\omega t) \tag{2}$$

Substitute equation (2) into (1) to obtain

$$\frac{d}{dt}c_a(t) = -\frac{i}{2\hbar}\langle\psi_a|\hat{V}|\psi_b\rangle\exp\big(i(\omega - \omega_0)t\big)c_b(t) \tag{3}$$

and

$$\frac{d}{dt}c_b(t) = -\frac{i}{2\hbar}\langle\psi_b|\hat{V}|\psi_a\rangle\exp\big(i(\omega_0 - \omega)t\big)c_a(t) \tag{4}$$

Use Laplace transforms to solve for $c_b(t)$ with initial conditions $c_a(0) = 1$ and $c_b(0) = 0$.

$$c_b(t) = -\frac{i}{2\hbar\omega_r}\langle\psi_b|\hat{V}|\psi_a\rangle\sin(\omega_r t)\exp\left(\tfrac{i}{2}(\omega_0 - \omega)t\right)$$

Symbol ω_r is the Rabi flopping frequency

$$\omega_r = \frac{1}{2}\sqrt{(\omega_0 - \omega)^2 + \left|\langle\psi_a|\hat{V}|\psi_b\rangle\right|^2/\hbar^2}$$

Use the latter part of equation (1) and the solution for $c_b(t)$ to solve for $c_a(t)$.

$$c_a(t) = \left[\cos(\omega_r t) + i\left(\frac{\omega_0 - \omega}{2\omega_r}\right)\sin(\omega_r t)\right]\exp\left(-\tfrac{i}{2}(\omega_0 - \omega)t\right)$$

Rewrite ω_r as

$$\omega_r = \frac{1}{2\hbar}\sqrt{\hbar^2(\omega_0 - \omega)^2 + \left|\langle\psi_a|\hat{V}|\psi_b\rangle\right|^2}$$

and note that for

$$\hbar^2(\omega_0 - \omega)^2 \gg \left|\langle\psi_a|\hat{V}|\psi_b\rangle\right|^2$$

we have

$$\omega_r \approx \tfrac{1}{2}|\omega_0 - \omega|$$

Using this approximation, the transition probability $P_{a\to b}(t) = |c_b(t)|^2$ becomes identical to the first order perturbation result.