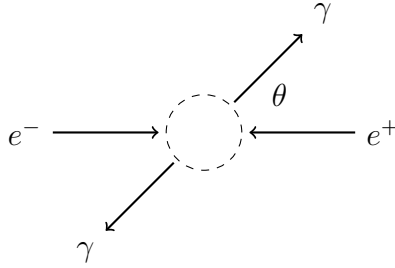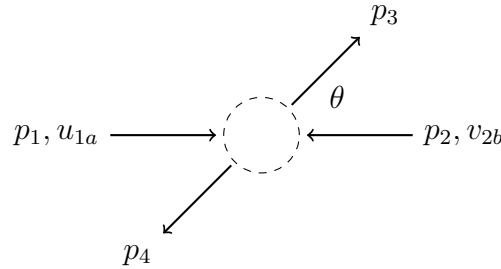# Annihilation

Electron-positron annihilation emits two photons.



Here is the same diagram with momentum and spinor labels.



In a typical collider experiment the momentum vectors are

$$
p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \quad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} \quad p_3 = \begin{pmatrix} E \\ E\sin\theta\cos\phi \\ E\sin\theta\sin\phi \\ E\cos\theta \end{pmatrix} \quad p_4 = \begin{pmatrix} E \\ -E\sin\theta\cos\phi \\ -E\sin\theta\sin\phi \\ -E\cos\theta \end{pmatrix}
$$
$$
\text{inbound electron} \qquad\quad \text{inbound positron} \qquad\qquad \text{outbound photon} \qquad\qquad\quad \text{outbound photon}
$$

Symbol $p$ is incident momentum. Symbol $E$ is total energy $E = \sqrt{p^2 + m^2}$ where $m$ is electron mass. Polar angle $\theta$ is the observed scattering angle. Azimuth angle $\phi$ cancels out in scattering calculations.

The spinors are

$$
u_{11} = \begin{pmatrix} E + m \\ 0 \\ p \\ 0 \end{pmatrix} \qquad\qquad v_{21} = \begin{pmatrix} -p \\ 0 \\ E + m \\ 0 \end{pmatrix}
$$
$$
\text{inbound electron} \qquad\qquad\qquad \text{inbound positron}
$$
$$
\text{spin up} \qquad\qquad\qquad\qquad \text{spin up}
$$

$$
u_{12} = \begin{pmatrix} 0 \\ E + m \\ 0 \\ -p \end{pmatrix} \qquad\qquad v_{22} = \begin{pmatrix} 0 \\ p \\ 0 \\ E + m \end{pmatrix}
$$
$$
\text{inbound electron} \qquad\qquad\qquad \text{inbound positron}
$$
$$
\text{spin down} \qquad\qquad\qquad\qquad \text{spin down}
$$

Spinor subscripts have 1 for spin up and 2 for spin down. The spinors are not individually normalized. Instead, a combined spinor normalization constant $N = (E + m)^2$ will be used.

This is the probability density for spin state $ab$. The formula is derived from Feynman diagrams for annihilation.

$$|\mathcal{M}_{ab}|^2 = \frac{e^4}{N} \left| -\frac{\bar{v}_{2b}\gamma^\mu(\slashed{q}_1 + m)\gamma^\nu u_{1a}}{t - m^2} - \frac{\bar{v}_{2b}\gamma^\nu(\slashed{q}_2 + m)\gamma^\mu u_{1a}}{u - m^2} \right|^2$$

Symbol $e$ is electron charge and

$$q_1 = p_1 - p_3$$
$$q_2 = p_1 - p_4$$

Symbols $t$ and $u$ are Mandelstam variables

$$t = q_1^2 = (p_1 - p_3)^2$$
$$u = q_2^2 = (p_1 - p_4)^2$$

Let

$$a_1 = \bar{v}_{2b}\gamma^\mu(\slashed{q}_1 + m)\gamma^\nu u_{1a}, \quad a_2 = \bar{v}_{2b}\gamma^\nu(\slashed{q}_2 + m)\gamma^\mu u_{1a}$$

Then

$$\begin{aligned}
|\mathcal{M}_{ab}|^2 &= \frac{e^4}{N} \left| -\frac{a_1}{t - m^2} - \frac{a_2}{u - m^2} \right|^2 \\
&= \frac{e^4}{N} \left( -\frac{a_1}{t - m^2} - \frac{a_2}{u - m^2} \right) \left( -\frac{a_1}{t - m^2} - \frac{a_2}{u - m^2} \right)^* \\
&= \frac{e^4}{N} \left( \frac{a_1 a_1^*}{(t - m^2)^2} + \frac{a_1 a_2^*}{(t - m^2)(u - m^2)} + \frac{a_1^* a_2}{(t - m^2)(u - m^2)} + \frac{a_2 a_2^*}{(u - m^2)^2} \right)
\end{aligned}$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is computed by summing $|\mathcal{M}_{ab}|^2$ over all spin and polarization states and then dividing by the number of inbound states. There are four inbound states. The sum over polarization states is already accomplished by contraction of $aa^*$ over $\mu$ and $\nu$.

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{1}{4} \sum_{a=1}^{2} \sum_{b=1}^{2} |\mathcal{M}_{ab}|^2 \\
&= \frac{e^4}{4N} \sum_{a=1}^{2} \sum_{b=1}^{2} \left( \frac{a_1 a_1^*}{(t - m^2)^2} + \frac{a_1 a_2^*}{(t - m^2)(u - m^2)} + \frac{a_1^* a_2}{(t - m^2)(u - m^2)} + \frac{a_2 a_2^*}{(u - m^2)^2} \right)
\end{aligned}$$

The Casimir trick uses matrix arithmetic to compute sums.

$$f_{11} = \frac{1}{N}\sum_{a=1}^{2}\sum_{b=1}^{2} a_1 a_1^* = \mathrm{Tr}\left((\not{p}_1 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\right)$$

$$f_{12} = \frac{1}{N}\sum_{a=1}^{2}\sum_{b=1}^{2} a_1 a_2^* = \mathrm{Tr}\left((\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\right)$$

$$f_{22} = \frac{1}{N}\sum_{a=1}^{2}\sum_{b=1}^{2} a_2 a_2^* = \mathrm{Tr}\left((\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_2 + m)\gamma_\mu\right)$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left(\frac{f_{11}}{(t-m^2)^2} + \frac{f_{12}}{(t-m^2)(u-m^2)} + \frac{f_{12}^*}{(t-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2}\right)$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$f_{11} = 32(p_1 \cdot p_3)(p_1 \cdot p_4) - 32m^2(p_1 \cdot p_2) + 64m^2(p_1 \cdot p_3) + 32m^2(p_1 \cdot p_4) - 64m^4$$
$$f_{12} = 16m^2(p_1 \cdot p_3) + 16m^2(p_1 \cdot p_4) - 32m^4$$
$$f_{22} = 32(p_1 \cdot p_3)(p_1 \cdot p_4) - 32m^2(p_1 \cdot p_2) + 32m^2(p_1 \cdot p_3) + 64m^2(p_1 \cdot p_4) - 64m^4$$

In Mandelstam variables

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$
$$u = (p_1 - p_4)^2$$

the formulas are

$$f_{11} = 8tu - 24tm^2 - 8um^2 - 8m^4$$
$$f_{12} = 8sm^2 - 32m^4$$
$$f_{22} = 8tu - 8tm^2 - 24um^2 - 8m^4$$

## High energy approximation

For high energy experiments $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$f_{11} = 8tu$$
$$f_{12} = 0$$
$$f_{22} = 8tu$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left(\frac{8tu}{t^2} + \frac{8tu}{u^2}\right)$$
$$= 2e^4\left(\frac{u}{t} + \frac{t}{u}\right)$$

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For $m = 0$ the Mandelstam variables are

$$s = 4E^2$$
$$t = -2E^2(1 - \cos\theta)$$
$$u = -2E^2(1 + \cos\theta)$$

Hence

$$\langle|\mathcal{M}|^2\rangle = 2e^4 \left( \frac{1 + \cos\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{1 + \cos\theta} \right)$$

## Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{4(4\pi\varepsilon_0)^2 s}$$

For the high energy approximation $m = 0$ we have

$$\langle|\mathcal{M}|^2\rangle = 2e^4 \left( \frac{1 + \cos\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{1 + \cos\theta} \right) \quad \text{and} \quad s = 4E^2$$

Substitute for $\langle|\mathcal{M}|^2\rangle$.

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\varepsilon_0)^2 s} \left( \frac{1 + \cos\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{1 + \cos\theta} \right)$$

Noting that

$$e^2 = 4\pi\varepsilon_0 \alpha \hbar c$$

we can also write

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s} \left( \frac{1 + \cos\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{1 + \cos\theta} \right)$$

We can integrate $d\sigma$ to obtain a cumulative distribution function. Let $I(\theta)$ be the following integral of $d\sigma$.

$$I(\theta) = \int \left( \frac{1 + \cos\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{1 + \cos\theta} \right) \sin\theta \, d\theta$$

The result is

$$I(\theta) = 2\cos\theta + 2\log(1 - \cos\theta) - 2\log(1 + \cos\theta)$$

The cumulative distribution function is

$$F(\theta) = \frac{I(\theta) - I(a)}{I(\pi - a) - I(a)}, \quad a \leq \theta \leq \pi - a$$

Angular support is limited to an arbitrary $a > 0$ because $I(0)$ and $I(\pi)$ are undefined.

The probability of observing scattering events in the interval $\theta_1$ to $\theta_2$ is

$$P(\theta_1 \leq \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

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## Data from DESY PETRA experiment

See www.hepdata.net/record/ins191231, Table 2, 14.0 GeV.

| $x$ | $y$ |
|---|---|
| 0.0502 | 0.09983 |
| 0.1505 | 0.10791 |
| 0.2509 | 0.12026 |
| 0.3512 | 0.13002 |
| 0.4516 | 0.17681 |
| 0.5521 | 0.1957 |
| 0.6526 | 0.279 |
| 0.7312 | 0.33204 |

Data $x$ and $y$ have the following relationship with the differential cross section formula.

$$x = \cos\theta, \quad y = \frac{d\sigma}{d\Omega}$$

The cross section formula is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\left(\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}\right) \times (\hbar c)^2$$

To compute predicted values $\hat{y}$, multiply by $10^{37}$ to convert square meters to nanobarns.

$$\hat{y} = \frac{\alpha^2}{2s}\left(\frac{1+x}{1-x} + \frac{1-x}{1+x}\right) \times (\hbar c)^2 \times 10^{37}$$

The following table shows predicted values $\hat{y}$ for $s = (14.0\,\text{GeV})^2$.

| $x$ | $y$ | $\hat{y}$ |
|---|---|---|
| 0.0502 | 0.09983 | 0.106325 |
| 0.1505 | 0.10791 | 0.110694 |
| 0.2509 | 0.12026 | 0.120005 |
| 0.3512 | 0.13002 | 0.135559 |
| 0.4516 | 0.17681 | 0.159996 |
| 0.5521 | 0.1957 | 0.198562 |
| 0.6526 | 0.279 | 0.262745 |
| 0.7312 | 0.33204 | 0.348884 |

The coefficient of determination $R^2$ measures how well predicted values fit the data.

$$R^2 = 1 - \frac{\sum(y-\hat{y})^2}{\sum(y-\bar{y})^2} = 0.98$$

The result indicates that the model $d\sigma$ explains 98% of the variance in the data.

## Notes

Here are some notes on how the Eigenmath scripts work.

To convert $a_1$ and $a_2$ to Eigenmath code, it is instructive to write $a_1$ and $a_2$ in full component form.

$$a_1^{\mu\nu} = \bar{v}_{2\alpha}\gamma^{\mu\alpha}{}_\beta(\slashed{q}_1 + m)^\beta{}_\rho\gamma^{\nu\rho}{}_\sigma u_1^\sigma \quad a_2^{\nu\mu} = \bar{v}_{2\alpha}\gamma^{\nu\alpha}{}_\beta(\slashed{q}_2 + m)^\beta{}_\rho\gamma^{\mu\rho}{}_\sigma u_1^\sigma$$

Transpose the $\gamma$ tensors to form inner products over $\alpha$ and $\rho$.

$$a_1^{\mu\nu} = \bar{v}_{2\alpha}\gamma^{\alpha\mu}{}_\beta(\slashed{q}_1 + m)^\beta{}_\rho\gamma^{\rho\nu}{}_\sigma u_1^\sigma \quad a_2^{\nu\mu} = \bar{v}_{2\alpha}\gamma^{\alpha\nu}{}_\beta(\slashed{q}_2 + m)^\beta{}_\rho\gamma^{\rho\mu}{}_\sigma u_1^\sigma$$

Convert transposed $\gamma$ to Eigenmath code.

$$\gamma^{\alpha\mu}{}_\beta \quad \to \quad \texttt{gammaT = transpose(gamma)}$$

Then to compute $a_1$ we have

$$a_1 = \bar{v}_{2\alpha}\gamma^{\alpha\mu}{}_\beta(\slashed{q}_1 + m)^\beta{}_\rho\gamma^{\rho\nu}{}_\sigma u_1^\sigma$$

$$\to \quad \texttt{a1 = dot(v2bar[s2],gammaT,qslash1 + m I,gammaT,u1[s1])}$$

where $s_1$ and $s_2$ are spin indices. Similarly for $a_2$ we have

$$a_2 = \bar{v}_{2\alpha}\gamma^{\alpha\mu}{}_\beta(\slashed{q}_2 + m)^\beta{}_\rho\gamma^{\rho\nu}{}_\sigma u_1^\sigma$$

$$\to \quad \texttt{a2 = dot(v2bar[s2],gammaT,qslash2 + m I,gammaT,u1[s1])}$$

In component notation the product $a_1 a_1^*$ is

$$a_1 a_1^* = a_1^{\mu\nu}a_1^{*\mu\nu}$$

To sum over $\mu$ and $\nu$ it is necessary to lower indices with the metric tensor. Also, transpose $a_1^*$ to form an inner product with $\nu$.

$$a_1 a_1^* = a_1^{\mu\nu}a_{1\nu\mu}^*$$

Convert to Eigenmath code. The dot function sums over $\nu$ and the contract function sums over $\mu$.

$$a_1 a_1^* \quad \to \quad \texttt{a11 = contract(dot(a1,gmunu,transpose(conj(a1))),gmunu)}$$

Similarly for $a_2 a_2^*$ we have

$$a_2 a_2^* \quad \to \quad \texttt{a22 = contract(dot(a2,gmunu,transpose(conj(a2))),gmunu)}$$

The product $a_1 a_2^*$ does not require a transpose because $a_2 = a_2^{\nu\mu}$.

$$a_1^{\mu\nu}a_{2\nu\mu}^* \quad \to \quad \texttt{a12 = contract(dot(a1,gmunu,conj(a2)),gmunu)}$$

In component notation, a trace operator becomes a sum over an index, in this case $\alpha$.

$$f_{11} = \text{Tr}\left((\not{p}_1 + m)\gamma^\mu(\not{q}_1 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\nu(\not{q}_1 + m)\gamma_\mu\right)$$

$$= (\not{p}_1 + m)^\alpha{}_\beta\gamma^{\mu\beta}{}_\rho(\not{q}_1 + m)^\rho{}_\sigma\gamma^{\nu\sigma}{}_\tau(\not{p}_2 - m)^\tau{}_\delta\gamma_\nu{}^\delta{}_\eta(\not{q}_1 + m)^\eta{}_\xi\gamma_\mu{}^\xi{}_\alpha$$

As before, transpose $\gamma$ tensors to form inner products.

$$f_{11} = (\not{p}_1 + m)^\alpha{}_\beta\gamma^{\beta\mu}{}_\rho(\not{q}_1 + m)^\rho{}_\sigma\gamma^{\sigma\nu}{}_\tau(\not{p}_2 - m)^\tau{}_\delta\gamma^\delta{}_{\nu\eta}(\not{q}_1 + m)^\eta{}_\xi\gamma^\xi{}_{\mu\alpha}$$

This is the code for transposing $\gamma$.

$$\gamma^{\beta\mu}{}_\beta \quad \rightarrow \quad \texttt{gammaT = transpose(gamma)}$$

$$\gamma^\delta{}_{\nu\eta} \quad \rightarrow \quad \texttt{gammaL = transpose(dot(gmunu,gamma))}$$

To convert $f_{11}$ to Eigenmath code, use an intermediate variable $T$ for the inner product.

$$T^{\alpha\mu\nu}{}_{\nu\mu\alpha} \quad \rightarrow \quad \texttt{T = dot(P1,gammaT,Q1,gammaT,P2,gammaL,Q1,gammaL)}$$

Now sum over the indices of $T$. The innermost contract sums over $\nu$ then the next contract sums over $\mu$. Finally the outermost contract sums over $\alpha$.

$$f_{11} \quad \rightarrow \quad \texttt{f11 = contract(contract(contract(T,3,4),2,3))}$$

Follow suit for $f_{22}$. For $f_{12}$ the order of the rightmost $\mu$ and $\nu$ is reversed.

$$f_{12} = \text{Tr}\left((\not{p}_1 + m)\gamma^\mu(\not{q}_2 + m)\gamma^\nu(\not{p}_2 - m)\gamma_\mu(\not{q}_1 + m)\gamma_\nu\right)$$

The resulting inner product is $T^{\alpha\mu\nu}{}_{\mu\nu\alpha}$ so the contraction is different.

$$f_{12} \quad \rightarrow \quad \texttt{f12 = contract(contract(contract(T,3,5),2,3))}$$

The innermost contract sums over $\nu$ followed by sum over $\mu$ then sum over $\alpha$.