

Consider the harmonic oscillator wave function

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi \hbar} \right)^{\frac{1}{4}} H_n \left(\sqrt{\frac{m\omega}{\hbar}} x \right) \exp \left(-\frac{m\omega x^2}{2\hbar} \right)$$

and its propagator

$$K(x_b, t_b, x_a, t_a) = \left(\frac{m\omega}{2\pi i \hbar \sin(\omega T)} \right)^{\frac{1}{2}} \exp \left[\frac{i m \omega}{2 \hbar \sin(\omega T)} (x_a^2 \cos(\omega T) - 2x_a x_b + x_b^2 \cos(\omega T)) \right]$$

where $T = t_b - t_a$.

We should have

$$\psi_n(x_b) = \int_{-\infty}^{\infty} K(x_b, t_b, x_a, t_a) \psi_n(x_a) dx_a$$

Try for $n = 1$.

$$\psi_1(x_a) = \sqrt{2} \left(\frac{m^2 \omega^3}{\pi \hbar^3} \right)^{\frac{1}{4}} x_a \exp \left(-\frac{m\omega x_a^2}{2\hbar} \right)$$

The path integral is

$$I = \int_{-\infty}^{\infty} K(x_b, t_b, x_a, t_a) \psi_1(x_a) dx_a = \frac{1}{2} \left(-\frac{m^9 \omega^9}{\pi \hbar^9} \right)^{\frac{1}{4}} \left(\frac{m\omega}{2\hbar} \sin(\omega T) - \frac{i m \omega}{2\hbar} \cos(\omega T) \right)^{-\frac{3}{2}} \\ \times x_b \exp \left(-\frac{m^2 \omega^2 x_b^2}{4 \hbar^2 \left(\frac{m\omega}{2\hbar} \sin^2(\omega T) - \frac{i m \omega}{2\hbar} \sin(\omega T) \cos(\omega T) \right)} + \frac{i m \omega x_b^2 \cos(\omega T)}{2 \hbar \sin(\omega T)} \right)$$

Let $T = \pi/(2\omega)$ to cancel time dependency. The result is

$$I = \psi_1(x_b) \exp \left(\frac{i\pi}{4} \right)$$

The phase shift is inconsequential and can be discarded.

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