Rutherford scattering 1

Use the following formula to compute the cross section for Rutherford scattering.

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}\left(\frac{mQ}{4\pi\hbar}\right)^2, \quad Q = \frac{1}{\hbar}\int \exp\left(\frac{i\mathbf{p}\cdot\mathbf{r}}{\hbar}\right) V(\mathbf{r})\, d\mathbf{r}^3$$

For Rutherford scattering $V(\mathbf{r})$ is the Coulomb potential

$$V(\mathbf{r}) = V(r) = -\frac{Ze^2}{r}$$

Convert Q to polar coordinates.

$$Q = \frac{1}{\hbar}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) V(r,\theta,\phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Substitute $V(r)$ for $V(r,\theta,\phi)$.

$$Q = \frac{1}{\hbar}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) V(r)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Integrate over ϕ.

$$Q = \frac{2\pi}{\hbar}\int_0^{\pi}\int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) V(r)\, r^2 \sin\theta\, dr\, d\theta$$

Change the complex exponential to rectangular form.

$$Q = \frac{2\pi}{\hbar}\int_0^{\pi}\int_0^{\infty} \left[\cos\left(\frac{pr\cos\theta}{\hbar}\right) + i\sin\left(\frac{pr\cos\theta}{\hbar}\right)\right] V(r)\, r^2 \sin\theta\, dr\, d\theta$$

By the integrals

$$\int_0^{\pi} \cos\big(a\cos(\theta)\big)\sin\theta\, d\theta = \frac{2\sin a}{a}, \quad \int_0^{\pi} \sin\big(a\cos(\theta)\big)\sin\theta\, d\theta = 0$$

we obtain (note r^2 in the integrand becomes r)

$$Q = \frac{4\pi}{p}\int_0^{\infty} \sin\left(\frac{pr}{\hbar}\right) V(r)\, r\, dr$$

Substitute the Coulomb potential for $V(r)$ which cancels r in the integrand.

$$Q = -\frac{4\pi Ze^2}{p}\int_0^{\infty} \sin\left(\frac{pr}{\hbar}\right) dr$$

To solve the integral, multiply the integrand by $\exp(-\epsilon r)$.

$$Q = -\frac{4\pi Ze^2}{p}\int_0^{\infty} \sin\left(\frac{pr}{\hbar}\right) \exp(-\epsilon r)\, dr$$

Convert the integrand to exponential form.

$$Q = -\frac{4\pi Ze^2}{p} \int_0^\infty \frac{i}{2} \left[\exp\left(-\frac{ipr}{\hbar} - \epsilon r \right) - \exp\left(\frac{ipr}{\hbar} - \epsilon r \right) \right] dr$$

Solve the integral.

$$Q = -\frac{4\pi Ze^2}{p} \frac{i}{2} \left(\frac{1}{-ip/\hbar - \epsilon} - \frac{1}{ip/\hbar - \epsilon} \right) \tag{1}$$

Set $\epsilon = 0$.

$$Q = -\frac{4\pi Ze^2}{p} \left(-\frac{\hbar}{p} \right)$$

Hence

$$Q = \frac{4\pi Ze^2 \hbar}{p^2}$$

Compute the differential cross section.

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2} \left(\frac{mQ}{4\pi\hbar} \right)^2 = \frac{1}{16\pi^2} \frac{m^2 Z^2 e^4}{p^4} \tag{2}$$

Substitute $16\pi^2\alpha^2$ for e^4.

$$\frac{d\sigma}{d\Omega} = \frac{m^2 Z^2 \alpha^2}{p^4}$$

Symbol p is momentum transfer $|\mathbf{p}_i| - |\mathbf{p}_f|$ such that

$$p^2 = 2mE(\cos\theta - 1)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4E^2(\cos\theta - 1)^2} \tag{3}$$