

Spherical harmonics

Verify

$$r^2 \nabla^2 Y_{lm}(\theta, \phi) = -l(l+1)Y_{lm}(\theta, \phi)$$

where $Y_{lm}(\theta, \phi)$ are spherical harmonic functions

$$Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_{lm}(\cos \theta) \exp(im\phi)$$

See arxiv.org/abs/1805.12125 for the following form of $P_{lm}(\cos \theta)$.

$$P_{lm}(\cos \theta) = \begin{cases} \left(\frac{\sin \theta}{2} \right)^m \sum_{k=0}^{l-m} (-1)^k \frac{(l+m+k)!}{(l-m-k)!(m+k)!k!} \left(\frac{1-\cos \theta}{2} \right)^k, & m \geq 0 \\ (-1)^m \frac{(l+m)!}{(l-m)!} P_{l|m|}(\cos \theta), & m < 0 \end{cases}$$

Function $Y_{lm}(\theta, \phi)$ is independent of r hence

$$r^2 \nabla^2 Y_{lm}(\theta, \phi) = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \frac{\partial}{\partial \theta} Y_{lm}(\theta, \phi) \right] + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} Y_{lm}(\theta, \phi)$$