Consider the following eigenstates of a hypothetical quantum system.[1]

$$
\begin{aligned}
|00\rangle &= (1,0,0,0) && \text{no fermions} \\
|10\rangle &= (0,1,0,0) && \text{one fermion in state } \phi_1 \\
|01\rangle &= (0,0,1,0) && \text{one fermion in state } \phi_2 \\
|11\rangle &= (0,0,0,1) && \text{two fermions, one in state } \phi_1\text{, one in state } \phi_2
\end{aligned}
$$

Let fermion states ϕ_n be modeled by a one dimensional box of length L.

$$
\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)
$$

Creation and annihilation operators are formed from outer products of state vectors. Sign changes make the operators antisymmetric.

$$
\begin{aligned}
\hat{b}_1^\dagger &= |10\rangle\langle 00| - |11\rangle\langle 01| && \text{Create one fermion in state } \phi_1 \\
\hat{b}_1 &= |00\rangle\langle 10| - |01\rangle\langle 11| && \text{Annihilate one fermion in state } \phi_1 \\
\hat{b}_2^\dagger &= |01\rangle\langle 00| + |11\rangle\langle 10| && \text{Create one fermion in state } \phi_2 \\
\hat{b}_2 &= |00\rangle\langle 01| + |10\rangle\langle 11| && \text{Annihilate one fermion in state } \phi_2
\end{aligned}
$$

Given the wavefunction operator

$$
\hat{\psi} = \frac{1}{\sqrt{2}} \sum_{n,m} \phi_n(x)\phi_m(y)\hat{b}_n\hat{b}_m
$$

show that

$$
\hat{\psi}|11\rangle = \frac{1}{\sqrt{2}}\big(\phi_1(x)\phi_2(y) - \phi_1(y)\phi_2(x)\big)|00\rangle
$$

[1] Adapted from problem 16.2.1 of “Quantum Mechanics for Scientists and Engineers.” `https://ee.stanford.edu/~dabm/QMbook.html`