Hydrogen energy four ways

The following formulas for E_n are equivalent.

This is the traditional formula with elementary charge e.

$$E_n = -\frac{\mu}{2n^2}\left(\frac{e^2}{4\pi\varepsilon_0\hbar}\right)^2 \tag{1}$$

This formula eliminates e but still solves the Schrödinger equation easily.

$$E_n = -\frac{\hbar^2}{2n^2\mu a_0^2} \tag{2}$$

This form of E_n is for dividing by $\hbar$ to obtain angular frequency ω.

$$E_n = -\frac{\alpha\hbar c}{2n^2 a_0} \tag{3}$$

This form of E_n is irreducible.

$$E_n = -\frac{\alpha^2\mu c^2}{2n^2} \tag{4}$$

Equation (1) reduces to (4) by substituting

$$e^2 = 4\pi\varepsilon_0\alpha\hbar c$$

Equations (2) and (3) reduce to (4) by substituting

$$a_0 = \frac{\hbar}{\alpha\mu c}$$