Hydrogen selection rules

By computing transition elements, verify the following selection rules.

$$|\Delta l| = 1, \quad |\Delta m| \le 1$$

The transition element $|R_{i\to f}|^2$ for spontaneous emission.

$$|R_{i\to f}|^2 = |\langle\psi_f|\hat{x}|\psi_i\rangle|^2 + |\langle\psi_f|\hat{y}|\psi_i\rangle|^2 + |\langle\psi_f|\hat{z}|\psi_i\rangle|^2$$

Transition amplitudes.

$$\langle\psi_f|\hat{x}|\psi_i\rangle = \int_V \psi_f^*\hat{x}\psi_i\,dV, \quad \langle\psi_f|\hat{y}|\psi_i\rangle = \int_V \psi_f^*\hat{y}\psi_i\,dV, \quad \langle\psi_f|\hat{z}|\psi_i\rangle = \int_V \psi_f^*\hat{z}\psi_i\,dV$$

Operators for dipole approximation.

$$\hat{x} = r\sin\theta\cos\phi, \quad \hat{y} = r\sin\theta\sin\phi, \quad \hat{z} = r\cos\theta$$

Volume measure for spherical coordinates.

$$dV = r^2\sin\theta\,dr\,d\theta\,d\phi$$

Hydrogen wave functions ψ_{nlm} are solutions to the time-independent Schrödinger equation.

$$\hat{H}\psi_{nlm}(r,\theta,\phi) = E_n\psi_{nlm}(r,\theta,\phi)$$

Hydrogen wave functions are formed as

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi)$$

Radial function.

$$R_{nl}(r) = \frac{2}{n^2}\sqrt{\frac{(n-l-1)!}{(n+l)!}}\left(\frac{2r}{na_0}\right)^l L_{n-l-1}^{2l+1}\left(\frac{2r}{na_0}\right)\exp\left(-\frac{r}{na_0}\right)a_0^{-3/2}$$

Associated Laguerre polynomial.

$$L_n^m(x) = (n+m)!\sum_{k=0}^{n}\frac{(-x)^k}{(n-k)!(m+k)!k!}$$

Spherical harmonic.

$$Y_{lm}(\theta,\phi) = (-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\exp(im\phi)$$

Associated Legendre polynomial of $\cos\theta$.

$$P_l^m(\cos\theta) = \begin{cases} \left(\frac{\sin\theta}{2}\right)^m \sum_{k=0}^{l-m}(-1)^k\frac{(l+m+k)!}{(l-m-k)!(m+k)!k!}\left(\frac{1-\cos\theta}{2}\right)^k, & m \ge 0 \\ (-1)^m\frac{(l+m)!}{(l-m)!}P_l^{|m|}(\cos\theta), & m < 0 \end{cases}$$