

6-13. Assume that $V(\mathbf{r}, t)$ is independent of time. Substitute the free particle kernel K_0 into equation (6.61) and integrate over t_c to show that

$$\begin{aligned} \psi(b) = & \exp\left(-\frac{iE_a t_b}{\hbar}\right) \exp\left(\frac{i\mathbf{p}_a \cdot \mathbf{x}_b}{\hbar}\right) \\ & - \exp\left(-\frac{iE_a t_b}{\hbar}\right) \frac{m}{2\pi\hbar^2} \int \frac{1}{R_{bc}} \exp\left(\frac{ipR_{bc}}{\hbar}\right) V(\mathbf{x}_c) \exp\left(\frac{i\mathbf{p}_a \cdot \mathbf{x}_c}{\hbar}\right) d^3\mathbf{x}_c \end{aligned} \quad (6.62)$$

where R_{bc} is the distance from the variable point of integration \mathbf{x}_c to the final point \mathbf{x}_b and p is the magnitude of the momentum of the electron.

This is equation (6.61).

$$\begin{aligned} \psi(b) = & \exp\left(-\frac{iE_a t_b}{\hbar}\right) \exp\left(\frac{i\mathbf{p}_a \cdot \mathbf{x}_b}{\hbar}\right) \\ & - \frac{i}{\hbar} \int_0^{t_b} \int K_0(b, c) V(c) \exp\left(-\frac{iE_a t_c}{\hbar}\right) \exp\left(\frac{i\mathbf{p}_a \cdot \mathbf{x}_c}{\hbar}\right) d^3\mathbf{x}_c dt_c \end{aligned} \quad (6.61)$$

This is the integral over t_c from (6.61) with $V(c)$ independent of time.

$$I = \int_0^{t_b} K_0(b, c) \exp\left(-\frac{iE_a t_c}{\hbar}\right) dt_c$$

Substitute $E_a = p^2/2m$.

$$I = \int_0^{t_b} K_0(b, c) \exp\left(-\frac{ip^2 t_c}{2m\hbar}\right) dt_c$$

Substitute K_0 from problem 4-12.

$$I = \int_0^{t_b} \left(\frac{m}{2\pi i\hbar(t_b - t_c)}\right)^{3/2} \exp\left(\frac{imR_{bc}^2}{2\hbar(t_b - t_c)}\right) \exp\left(-\frac{ip^2 t_c}{2m\hbar}\right) dt_c$$

Let

$$\begin{aligned} \lambda &= \frac{1}{\hbar} \\ f &= \left(\frac{m}{2\pi i\hbar(t_b - t_c)}\right)^{3/2} \\ g &= \frac{mR_{bc}^2}{2(t_b - t_c)} - \frac{p^2 t_c}{2m} \end{aligned}$$

Then

$$I = \int_0^{t_b} f \exp(i\lambda g) dt_c$$

The phase of the exponential is stationary (i.e., $g' = 0$) for

$$t_c = t_b - \frac{mR_{bc}}{p}$$

By the method of stationary phase

$$I \approx \pm \left(\frac{2\pi i}{\lambda g''} \right)^{1/2} f \exp(i\lambda g) \Big|_{t_c}$$

Hence

$$I \approx -\frac{im}{2\pi\hbar R_{bc}} \exp\left(\frac{ipR_{bc}}{\hbar} - \frac{ip^2 t_b}{2m\hbar}\right)$$

The integral can also be written as

$$-\frac{im}{2\pi\hbar R_{bc}} \exp\left(\frac{ipR_{bc}}{\hbar}\right) \exp\left(-\frac{iE_a t_b}{\hbar}\right) \quad (1)$$

Substitute (1) into (6.61) to obtain (6.62).

Note that the method of stationary phase requires $0 < t_c < t_b$ so the above solution is valid for physical values that satisfy

$$0 < \frac{mR_{bc}}{p} < t_b$$

Hence (6.62) is not true in general but is *probably* true if t_b is large.

Just for the fun of it, check physical dimensions.

$$\frac{mR_{bc}}{p} \propto \frac{\text{mass} \times \text{length}}{\text{mass} \times \text{length}/\text{time}} = \text{time}$$