Perturbation example

Let

$$\dot{c}_a = -\frac{i}{\hbar} c_b H_{ab} e^{-i\omega t}$$
$$\dot{c}_b = -\frac{i}{\hbar} c_a H_{ba} e^{i\omega t}$$

Find $c_a(t)$ and $c_b(t)$ to second order for $c_a(0) = a$ and $c_b(0) = b$. (See Griffiths and Schroeter problem 11.6.)

First order:

$$\begin{aligned}
\dot{c}_a^{(1)} &= -\frac{i}{\hbar} H_{ab} e^{-i\omega t} c_b \Big|_{c_b = b} &&= -\frac{ib}{\hbar} H_{ab} e^{-i\omega t} \\
\dot{c}_b^{(1)} &= -\frac{i}{\hbar} H_{ba} e^{i\omega t} c_a \Big|_{c_a = a} &&= -\frac{ia}{\hbar} H_{ba} e^{i\omega t}
\end{aligned} \tag{1}$$

Hence

$$\begin{aligned}
c_a^{(1)}(t) &= a + \int_0^t \dot{c}_a^{(1)}(t')\,dt' &&= a - \frac{ib}{\hbar} \int_0^t H_{ab}(t') e^{-i\omega t'}\,dt' \\
c_b^{(1)}(t) &= b + \int_0^t \dot{c}_b^{(1)}(t')\,dt' &&= b - \frac{ia}{\hbar} \int_0^t H_{ba}(t') e^{i\omega t'}\,dt'
\end{aligned} \tag{2}$$

Second order:

$$\begin{aligned}
\dot{c}_a^{(2)} &= -\frac{i}{\hbar} H_{ab} e^{-i\omega t} c_b \Big|_{c_b = c_b^{(1)}} &&= -\frac{ib}{\hbar} H_{ab} e^{-i\omega t} - \frac{a}{\hbar^2} H_{ab} e^{-i\omega t} \int_0^t H_{ba}(t') e^{i\omega t'}\,dt' \\
\dot{c}_b^{(2)} &= -\frac{i}{\hbar} H_{ba} e^{i\omega t} c_a \Big|_{c_a = c_a^{(1)}} &&= -\frac{ia}{\hbar} H_{ba} e^{i\omega t} - \frac{b}{\hbar^2} H_{ba} e^{i\omega t} \int_0^t H_{ab}(t') e^{-i\omega t'}\,dt'
\end{aligned} \tag{3}$$

Hence

$$\begin{aligned}
c_a^{(2)}(t) &= a + \int_0^t \dot{c}_a^{(2)}(t')\,dt' \\
&= a - \frac{ib}{\hbar} \int_0^t H_{ab}(t') e^{-i\omega t'}\,dt' - \frac{a}{\hbar^2} \int_0^t H_{ab}(t') e^{-i\omega t'} \left[\int_0^{t'} H_{ba}(t'') e^{i\omega t''}\,dt'' \right] dt'
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
c_b^{(2)}(t) &= b + \int_0^t \dot{c}_b^{(2)}(t')\,dt' \\
&= b - \frac{ia}{\hbar} \int_0^t H_{ba}(t') e^{i\omega t'}\,dt' - \frac{b}{\hbar^2} \int_0^t H_{ba}(t') e^{i\omega t'} \left[\int_0^{t'} H_{ab}(t'') e^{-i\omega t''}\,dt'' \right] dt'
\end{aligned} \tag{5}$$