This is the transition rate A_{nm} for a spontaneous emission process $\psi_n \to \psi_m$.

$$A_{nm} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3}\,\omega_{nm}^3\,|\langle r_{nm}\rangle|^2$$

The angular frequency ω_{nm} is

$$\omega_{nm} = \frac{1}{\hbar}(E_n - E_m)$$

For a hydrogen atom we have

$$E_n = -\frac{\mu}{2n^2}\left(\frac{e^2}{4\pi\varepsilon_0\hbar}\right)^2$$

where μ is reduced electron mass.

The radial density is

$$|\langle r_{nm}\rangle|^2 = |\langle x_{nm}\rangle|^2 + |\langle y_{nm}\rangle|^2 + |\langle z_{nm}\rangle|^2$$

where

$$\begin{aligned}
\langle x_{nm}\rangle &= \int \psi_m^*\,(r\sin\theta\cos\phi)\,\psi_n\,dV \\
\langle y_{nm}\rangle &= \int \psi_m^*\,(r\sin\theta\sin\phi)\,\psi_n\,dV \\
\langle z_{nm}\rangle &= \int \psi_m^*\,(r\cos\theta)\,\psi_n\,dV
\end{aligned}$$

Let us compute A_{21} for a hydrogen atom. For $n = 2$ there are four possible states.

n	ℓ	m
2	1	1
2	1	-1
2	1	0
2	0	0

The following table shows the radial density for every possible transition.

	$\psi_{2,1,1} \to \psi_{1,0,0}$	$\psi_{2,1,-1} \to \psi_{1,0,0}$	$\psi_{2,1,0} \to \psi_{1,0,0}$	$\psi_{2,0,0} \to \psi_{1,0,0}$
$\langle x_{21}\rangle =$	$-\frac{128}{243}\,a_0$	$\frac{128}{243}\,a_0$	0	0
$\langle y_{21}\rangle =$	$-\frac{128}{243}i\,a_0$	$-\frac{128}{243}i\,a_0$	0	0
$\langle z_{21}\rangle =$	0	0	$\frac{128}{243}\sqrt{2}\,a_0$	0
$\lvert\langle r_{21}\rangle\rvert^2 =$	$\frac{32768}{59049}\,a_0^2$	$\frac{32768}{59049}\,a_0^2$	$\frac{32768}{59049}\,a_0^2$	0

Note that the transition rate of $\psi_{2,0,0} \to \psi_{1,0,0}$ is zero. For the allowed transitions, the radial density $|\langle r_{21}\rangle|^2$ is independent of ℓ and m.

Symbol a_0 is the Bohr radius

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{e^2\mu}$$

We have

$$\omega_{21} = \frac{1}{\hbar}(E_2 - E_1) = \frac{3e^4\mu}{128\pi^2\varepsilon_0^2\hbar^3}$$

Hence

$$A_{21} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3} \underbrace{\left(\frac{3e^4\mu}{128\pi^2\varepsilon_0^2\hbar^3}\right)^3}_{\omega_{21}} \underbrace{\frac{32768}{59049}\left(\frac{4\pi\varepsilon_0\hbar^2}{e^2\mu}\right)^2}_{|\langle r_{21}\rangle|^2} = \frac{e^{10}\mu}{26244\pi^5\varepsilon_0^5\hbar^6 c^3} = 6.27 \times 10^8 \,\mathrm{second}^{-1}$$