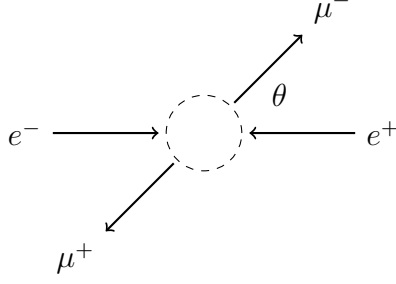
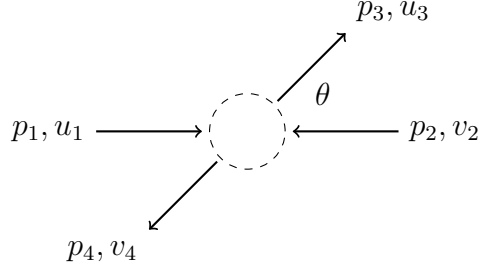


A high energy electron and positron collision can create two muons.



Here is the same diagram with momentum and spinor labels.



In a typical collider experiment the momentum vectors are

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \quad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} \quad p_3 = \begin{pmatrix} E \\ \rho \sin \theta \cos \phi \\ \rho \sin \theta \sin \phi \\ \rho \cos \theta \end{pmatrix} \quad p_4 = \begin{pmatrix} E \\ -\rho \sin \theta \cos \phi \\ -\rho \sin \theta \sin \phi \\ -\rho \cos \theta \end{pmatrix}$$

where E is beam energy, $p = \sqrt{E^2 - m^2}$, $\rho = \sqrt{E^2 - M^2}$, m is electron mass 0.51 MeV, and M is muon mass 106 MeV. The spinors are

$$\begin{aligned} u_{11} &= \begin{pmatrix} E + m \\ 0 \\ p \\ 0 \end{pmatrix} & v_{21} &= \begin{pmatrix} -p \\ 0 \\ E + m \\ 0 \end{pmatrix} & u_{31} &= \begin{pmatrix} E + M \\ 0 \\ p_3^z \\ p_3^x + ip_3^y \end{pmatrix} & v_{41} &= \begin{pmatrix} p_4^z \\ p_4^x + ip_4^y \\ E + M \\ 0 \end{pmatrix} \\ u_{12} &= \begin{pmatrix} 0 \\ E + m \\ 0 \\ -p \end{pmatrix} & v_{22} &= \begin{pmatrix} 0 \\ p \\ 0 \\ E + m \end{pmatrix} & u_{32} &= \begin{pmatrix} 0 \\ E + M \\ p_3^x - ip_3^y \\ -p_3^z \end{pmatrix} & v_{42} &= \begin{pmatrix} p_4^x - ip_4^y \\ -p_4^z \\ 0 \\ E + M \end{pmatrix} \end{aligned}$$

The last digit in a spinor subscript is 1 for spin up and 2 for spin down. Note that the spinors are not individually normalized. Instead, a combined spinor normalization constant $N = (E + m)^2(E + M)^2$ will be used where needed.

This is the probability density for spin state $abcd$. Symbol $s = (p_1 + p_2)^2 = 4E^2$. Symbol e is electron charge.

$$|\mathcal{M}_{abcd}|^2 = \frac{e^4}{N s^2} |(\bar{u}_{3c} \gamma_\mu v_{4d})(\bar{v}_{2b} \gamma^\mu u_{1a})|^2$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is computed by summing $|\mathcal{M}_{abcd}|^2$ over all spin states and dividing by the number of inbound states. There are four inbound states.

$$\begin{aligned}\langle |\mathcal{M}|^2 \rangle &= \frac{1}{4} \sum_{a=1}^2 \sum_{b=1}^2 \sum_{c=1}^2 \sum_{d=1}^2 |\mathcal{M}_{abcd}|^2 \\ &= \frac{e^4}{4N_S^2} \sum_{a=1}^2 \sum_{b=1}^2 \sum_{c=1}^2 \sum_{d=1}^2 |(\bar{u}_3 \gamma_\mu v_4)(\bar{v}_2 \gamma^\mu u_1)|^2\end{aligned}$$

Another way to compute $\langle |\mathcal{M}|^2 \rangle$ is to use the Casimir trick.

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4s^2} \text{Tr} \left((\not{p}_3 + M) \gamma^\mu (\not{p}_4 - M) \gamma^\nu \right) \text{Tr} \left((\not{p}_2 - m) \gamma_\mu (\not{p}_1 + m) \gamma_\nu \right)$$

Here is a third way to compute $\langle |\mathcal{M}|^2 \rangle$.

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4s^2} (32(p_1 \cdot p_3)(p_2 \cdot p_4) + 32(p_1 \cdot p_4)(p_2 \cdot p_3) + 32m^2(p_3 \cdot p_4) + 32M^2(p_1 \cdot p_2) + 64m^2M^2)$$

For the momentum vectors given above the result is

$$\langle |\mathcal{M}|^2 \rangle = e^4 \left(1 + \cos^2 \theta + \frac{m^2 + M^2}{E^2} \sin^2 \theta + \frac{m^2 M^2}{E^4} \cos^2 \theta \right)$$

The Stanford Linear Collider had a collision energy of $2E = 91$ GeV. For beam energies such as SLC where $E \gg M$ the above equation can be approximated as

$$\langle |\mathcal{M}|^2 \rangle = e^4 (1 + \cos^2 \theta)$$

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{64\pi^2 s} = \frac{e^4}{256\pi^2 E^2} (1 + \cos^2 \theta)$$

Recall that $e^2 = 4\pi\alpha$ hence

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2} (1 + \cos^2 \theta)$$

The total cross section calculation requires the following definite integral.

$$\int_{\Omega} (1 + \cos^2 \theta) d\Omega = \int_0^{2\pi} \int_0^\pi (1 + \cos^2 \theta) \sin \theta d\theta d\phi = \frac{8}{3} \int_0^{2\pi} d\phi = \frac{16\pi}{3}$$

Hence the total cross section is

$$\sigma = \int_{\Omega} d\sigma = \int_{\Omega} \frac{\alpha^2}{16E^2} (1 + \cos^2 \theta) d\Omega = \frac{\alpha^2}{16E^2} \frac{16\pi}{3} = \frac{\pi\alpha^2}{3E^2}$$

We can integrate the differential cross section to obtain a cumulative distribution function.

Let

$$I(\xi) = 2\pi \int_0^\xi \frac{d\sigma}{d\Omega} \sin \theta d\theta, \quad 0 \leq \xi \leq \pi$$

The result is

$$I(\xi) = 2\pi \left(\frac{\alpha^2}{16E^2} \right) \left(-\frac{1}{3} \cos^3 \xi - \cos \xi + \frac{4}{3} \right)$$

The cumulative distribution function is

$$F(\theta) = \frac{I(\theta)}{I(\pi)}, \quad 0 \leq \theta \leq \pi$$

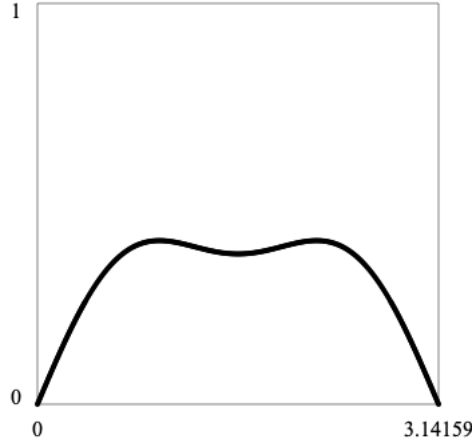
Hence

$$P(\theta_1 \leq \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

The normalized probability density is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{3}{8}(1 + \cos^2 \theta) \sin \theta, \quad 0 \leq \theta \leq \pi$$

Run “muon-production-5.txt” to draw the probability density function.



Run “muon-production-1.txt” to verify that

$$\frac{1}{N} \sum_{a=1}^2 \sum_{b=1}^2 \sum_{c=1}^2 \sum_{d=1}^2 |(\bar{u}_{3c} \gamma_\mu v_{4d})(\bar{v}_{2b} \gamma^\mu u_{1a})|^2 = \text{Tr} \left((\not{p}_3 + M) \gamma^\mu (\not{p}_4 - M) \gamma^\nu \right) \text{Tr} \left((\not{p}_2 - m) \gamma_\mu (\not{p}_1 + m) \gamma_\nu \right)$$

Run “muon-production-2.txt” to verify that

$$\begin{aligned} \frac{1}{64E^4} \text{Tr} \left((\not{p}_3 + M) \gamma^\mu (\not{p}_4 - M) \gamma^\nu \right) \text{Tr} \left((\not{p}_2 - m) \gamma_\mu (\not{p}_1 + m) \gamma_\nu \right) \\ = 1 + \cos^2 \theta + \frac{m^2 + M^2}{E^2} \sin^2 \theta + \frac{m^2 M^2}{E^4} \cos^2 \theta \end{aligned}$$

and to verify that

$$\begin{aligned} \text{Tr} \left((\not{p}_3 + M) \gamma^\mu (\not{p}_4 - M) \gamma^\nu \right) \text{Tr} \left((\not{p}_2 - m) \gamma_\mu (\not{p}_1 + m) \gamma_\nu \right) \\ = 32(p_1 \cdot p_3)(p_2 \cdot p_4) + 32(p_1 \cdot p_4)(p_2 \cdot p_3) + 32m^2(p_3 \cdot p_4) + 32M^2(p_1 \cdot p_2) + 64m^2 M^2 \end{aligned}$$

Data from SLAC PEP experiment

See www.hepdata.net/record/ins216031, Table 1, 29.0 GeV.

x	y
-0.925	67.08
-0.85	58.67
-0.75	54.66
-0.65	51.72
-0.55	43.70
-0.45	41.12
-0.35	39.71
-0.25	35.34
-0.15	33.35
-0.05	34.69
0.05	34.05
0.15	34.48
0.25	34.66
0.35	35.23
0.45	35.60
0.55	40.13
0.65	42.56
0.75	46.37
0.85	49.28
0.925	55.70

Data x and y have the following relationship with cross section parameters.

$$x = \cos \theta \quad y = (2E)^2 \frac{d\sigma}{d\cos \theta}$$

The differential cross section for muon production is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2} (1 + \cos^2 \theta)$$

Let us compute predicted values \hat{y} from the cross section formula. Start by finding the relationship between $d\Omega$ and $d\cos \theta$. Since $1 + \cos^2 \theta$ has no dependence on ϕ we have

$$\int_{\Omega} (1 + \cos^2 \theta) d\Omega = \int_0^{2\pi} \int_0^{\pi} (1 + \cos^2 \theta) \sin \theta d\theta d\phi = 2\pi \int_0^{\pi} (1 + \cos^2 \theta) \sin \theta d\theta$$

Hence

$$d\Omega = 2\pi \sin \theta d\theta = -2\pi d\cos \theta$$

We want positive cross sections so drop the minus sign and set

$$\frac{d\sigma}{d\cos \theta} = 2\pi \frac{d\sigma}{d\Omega}$$

We can now write

$$\begin{aligned} y &= (2E)^2 \frac{d\sigma}{d\cos \theta} \\ &= (2E)^2 (2\pi) \frac{d\sigma}{d\Omega} \\ &= (2E)^2 (2\pi) \frac{\alpha^2}{16E^2} (1 + \cos^2 \theta) \\ &= \frac{\pi\alpha^2}{2} (1 + \cos^2 \theta) \end{aligned}$$

Multiply by $(\hbar c)^2$ to convert to SI and multiply by 10^{37} to convert square meters to nanobarns.

$$y = \frac{\pi\alpha^2}{2}(1 + \cos^2 \theta) \times (\hbar c)^2 \times 10^{37}$$

Replace $\cos \theta$ with explanatory variable x to obtain \hat{y} .

$$\hat{y} = \frac{\pi\alpha^2}{2}(1 + x^2) \times (\hbar c)^2 \times 10^{37}$$

Here are the predicted values \hat{y} based on the above formula.

x	y	\hat{y}
-0.925	67.08	60.44
-0.85	58.67	56.10
-0.75	54.66	50.89
-0.65	51.72	46.33
-0.55	43.70	42.42
-0.45	41.12	39.17
-0.35	39.71	36.56
-0.25	35.34	34.61
-0.15	33.35	33.30
-0.05	34.69	32.65
0.05	34.05	32.65
0.15	34.48	33.30
0.25	34.66	34.61
0.35	35.23	36.56
0.45	35.60	39.17
0.55	40.13	42.42
0.65	42.56	46.33
0.75	46.37	50.89
0.85	49.28	56.10
0.925	55.70	60.44

The coefficient of determination R^2 measures how well predicted values fit the real data.

$$R^2 = 1 - \frac{\sum(y - \hat{y})^2}{\sum(y - \bar{y})^2} = 0.87$$

The result indicates that the model $d\sigma$ explains 87% of the variance in the data.

Run “muon-production-3.txt” to compute the above results.

Electroweak model

The following differential cross section formula from electroweak theory results in a better fit to the data.¹

$$\frac{d\sigma}{d\Omega} = F(s)(1 + \cos^2 \theta) + G(s) \cos \theta$$

¹F. Mandl and G. Shaw, *Quantum Field Theory Revised Edition*, 316.

where

$$F(s) = \frac{\alpha^2}{4s} \left(1 + \frac{g_V^2}{\sqrt{2}\pi} \left(\frac{m_Z^2}{s - m_Z^2} \right) \left(\frac{sG}{\alpha} \right) + \frac{(g_A^2 + g_V^2)^2}{8\pi^2} \left(\frac{m_Z^2}{s - m_Z^2} \right)^2 \left(\frac{sG}{\alpha} \right)^2 \right)$$

$$G(s) = \frac{\alpha^2}{4s} \left(\frac{\sqrt{2}g_A^2}{\pi} \left(\frac{m_Z^2}{s - m_Z^2} \right) \left(\frac{sG}{\alpha} \right) + \frac{g_A^2 g_V^2}{\pi^2} \left(\frac{m_Z^2}{s - m_Z^2} \right)^2 \left(\frac{sG}{\alpha} \right)^2 \right)$$

and

$$g_A = -0.5$$

$$g_V = -0.0348$$

$$m_Z = 91.17 \text{ GeV}$$

$$G = 1.166 \times 10^{-5} \text{ GeV}^{-2}$$

The corresponding formula for \hat{y} is

$$\hat{y} = 2\pi \left[F(s)(1 + x^2) + G(s)x \right] \times (\hbar c)^2 \times 10^{37}$$

where $\sqrt{s} = 29 \text{ GeV}$ is the center of mass collision energy. Here are the predicted values \hat{y} based on the above formula.

x	y	\hat{y}
-0.925	67.08	65.59
-0.85	58.67	60.84
-0.75	54.66	55.07
-0.65	51.72	49.96
-0.55	43.70	45.49
-0.45	41.12	41.69
-0.35	39.71	38.53
-0.25	35.34	36.02
-0.15	33.35	34.17
-0.05	34.69	32.97
0.05	34.05	32.42
0.15	34.48	32.53
0.25	34.66	33.28
0.35	35.23	34.69
0.45	35.60	36.75
0.55	40.13	39.47
0.65	42.56	42.83
0.75	46.37	46.85
0.85	49.28	51.52
0.925	55.70	55.45

The coefficient of determination R^2 is

$$R^2 = 1 - \frac{\sum (y - \hat{y})^2}{\sum (y - \bar{y})^2} = 0.98$$

The result indicates that electroweak theory explains 98% of the variance in the data.

Run “muon-production-4.txt” to verify.

Notes

Here are a few notes about how the scripts work.

In component notation the traces become sums over the repeated index α .

$$\begin{aligned}\text{Tr} \left((\not{p}_3 + M) \gamma^\mu (\not{p}_4 - M) \gamma^\nu \right) &= (\not{p}_3 + M)^\alpha \gamma^{\mu\beta}_\rho (\not{p}_4 - M)^\rho_\sigma \gamma^{\nu\sigma}_\alpha \\ \text{Tr} \left((\not{p}_2 - m) \gamma_\mu (\not{p}_1 + m) \gamma_\nu \right) &= (\not{p}_2 - m)^\alpha_\beta \gamma^\beta_{\mu\rho} (\not{p}_1 + m)^\rho_\sigma \gamma^\sigma_{\nu\alpha}\end{aligned}$$

To convert the above formulas to Eigenmath code, the γ tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply γ^μ by the metric tensor to lower the index.

$$\begin{aligned}\gamma^{\beta\mu}_\rho &\rightarrow \text{gammaT} = \text{transpose}(\text{gamma}) \\ \gamma^\beta_{\mu\rho} &\rightarrow \text{gammaL} = \text{transpose}(\text{dot}(\text{gmunu}, \text{gamma}))\end{aligned}$$

Define the following 4×4 matrices.

$$\begin{aligned}(\not{p}_1 + m) &\rightarrow \text{X1} = \text{pslash1} + \text{m I} \\ (\not{p}_2 - m) &\rightarrow \text{X2} = \text{pslash2} - \text{m I} \\ (\not{p}_3 + M) &\rightarrow \text{X3} = \text{pslash3} + \text{M I} \\ (\not{p}_4 - M) &\rightarrow \text{X4} = \text{pslash4} - \text{M I}\end{aligned}$$

Then

$$\begin{aligned}(\not{p}_3 + M)^\alpha_\beta \gamma^{\mu\beta}_\rho (\not{p}_4 - M)^\rho_\sigma \gamma^{\nu\sigma}_\alpha &\rightarrow \text{T1} = \text{contract}(\text{dot}(\text{X3}, \text{gammaT}, \text{X4}, \text{gammaT}), 1, 4) \\ (\not{p}_2 - m)^\alpha_\beta \gamma^\beta_{\mu\rho} (\not{p}_1 + m)^\rho_\sigma \gamma^\sigma_{\nu\alpha} &\rightarrow \text{T2} = \text{contract}(\text{dot}(\text{X2}, \text{gammaL}, \text{X1}, \text{gammaL}), 1, 4)\end{aligned}$$

Next, multiply matrices and sum over repeated indices. The dot function sums over ν then the contract function sums over μ . The transpose makes the ν indices adjacent as required by the dot function.

$$\text{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu) \text{Tr}(\cdots \gamma_\mu \cdots \gamma_\nu) \rightarrow \text{contract}(\text{dot}(\text{T1}, \text{transpose}(\text{T2})))$$