

## Rutherford scattering 2

Find the cross section for Rutherford scattering with a screened Coulomb potential

$$V(r) = -\frac{Ze^2}{4\epsilon_0 r} \exp\left(-\frac{r}{a}\right)$$

Start with

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |Q|^2, \quad Q = \int \exp\left(\frac{i\mathbf{p} \cdot \mathbf{r}}{\hbar}\right) V(\mathbf{r}) d^3\mathbf{r}$$

Convert  $Q$  to polar coordinates.

$$Q = \int_0^{2\pi} \int_0^\pi \int_0^\infty \exp\left(\frac{ipr \cos \theta}{\hbar}\right) V(r, \theta, \phi) r^2 \sin \theta dr d\theta d\phi$$

Substitute the screened Coulomb potential for  $V(r, \theta, \phi)$  and note  $r^2$  becomes  $r$ .

$$Q = -\frac{Ze^2}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^\pi \int_0^\infty \exp\left(\frac{ipr \cos \theta}{\hbar}\right) \exp\left(-\frac{r}{a}\right) r \sin \theta dr d\theta d\phi$$

Integrate over  $\phi$  (multiplies  $Q$  by  $2\pi$ ).

$$Q = -\frac{Ze^2}{2\epsilon_0} \int_0^\pi \int_0^\infty \exp\left(\frac{ipr \cos \theta}{\hbar}\right) \exp\left(-\frac{r}{a}\right) r \sin \theta dr d\theta$$

Transform the integral over  $\theta$  to an integral over  $y$  where  $y = \cos \theta$  and  $dy = -\sin \theta d\theta$ . The minus sign in  $dy$  is canceled by interchanging integration limits  $\cos 0 = 1$  and  $\cos \pi = -1$ .

$$Q = -\frac{Ze^2}{2\epsilon_0} \int_{-1}^1 \int_0^\infty \exp\left(\frac{ipry}{\hbar}\right) \exp\left(-\frac{r}{a}\right) r dr dy$$

Solve the integral over  $y$  (note  $r$  in the integrand cancels).

$$Q = -\frac{Ze^2}{2\epsilon_0} \int_0^\infty \frac{\hbar}{ip} \left[ \exp\left(\frac{ipr}{\hbar}\right) - \exp\left(-\frac{ipr}{\hbar}\right) \right] \exp\left(-\frac{r}{a}\right) dr$$

Solve the integral over  $r$ .

$$Q = -\frac{Ze^2}{2\epsilon_0} \frac{\hbar}{ip} \left[ \frac{1}{ip/\hbar - 1/a} \exp\left(\frac{ipr}{\hbar} - \frac{r}{a}\right) + \frac{1}{ip/\hbar + 1/a} \exp\left(-\frac{ipr}{\hbar} - \frac{r}{a}\right) \right]_0^\infty$$

Evaluate the limits.

$$Q = -\frac{Ze^2}{2\epsilon_0} \frac{\hbar}{ip} \left[ -\frac{1}{ip/\hbar - 1/a} - \frac{1}{ip/\hbar + 1/a} \right] = -\frac{Ze^2}{2\epsilon_0} \frac{2}{(p/\hbar)^2 + (1/a)^2} \quad (1)$$

The cross section is

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |Q|^2 = \frac{m^2 Z^2 e^4}{4\pi^2 \epsilon_0^2 [p^2 + (\hbar/a)^2]^2} \quad (2)$$

Substitute  $16\pi^2\epsilon_0^2\alpha^2\hbar^2c^2$  for  $e^4$ .

$$\frac{d\sigma}{d\Omega} = \frac{4m^2Z^2\alpha^2\hbar^2c^2}{[p^2 + (\hbar/a)^2]^2}$$

Symbol  $p$  is momentum transfer  $|\mathbf{p}_i| - |\mathbf{p}_f|$  such that

$$p^2 = 4mE(1 - \cos \theta)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{4m^2Z^2\alpha^2\hbar^2c^2}{[4mE(1 - \cos \theta) + (\hbar/a)^2]^2}$$

Cancel  $m^2$  in the numerator.

$$\frac{d\sigma}{d\Omega} = \frac{4Z^2\alpha^2\hbar^2c^2}{[4E(1 - \cos \theta) + \frac{1}{m}(\hbar/a)^2]^2} \quad (3)$$

Let  $a \rightarrow \infty$  to obtain the ordinary Rutherford cross section formula

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2\hbar^2c^2}{4E^2(1 - \cos \theta)^2}$$