Cold emission

Consider the following potential energy function where Q is a positive constant.

$$V(x) = \begin{cases} 0, & x < 0 \\ V_0 - Qx, & x \geq 0 \end{cases}$$

Suppose a particle of mass m and energy $E < V_0$ is traveling from left to right along the x axis. The particle is in a potential energy barrier for

$$E \leq V_0 - Qx$$

Solving for x we have the particle in the barrier for

$$x \leq \frac{V_0 - E}{Q}$$

Hence the particle has a Schrodinger equation for each the following regions.

$$\begin{aligned}
&-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_1 = E\psi_1, && x < 0 \\
&-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_2 + (V_0 - Qx)\psi_2 = E\psi_2, && 0 \leq x \leq \frac{V_0 - E}{Q} \\
&-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_3 = E\psi_3, && x > \frac{V_0 - E}{Q}
\end{aligned}$$

Let ψ_1 and ψ_3 have the most general free-particle solutions.

$$\begin{aligned}
\psi_1(x) &= A \exp\left(i\sqrt{\frac{2mE}{\hbar^2}}x\right) + B \exp\left(-i\sqrt{\frac{2mE}{\hbar^2}}x\right) \\
\psi_3(x) &= F \exp\left(i\sqrt{\frac{2mE}{\hbar^2}}x\right) + G \exp\left(-i\sqrt{\frac{2mE}{\hbar^2}}x\right)
\end{aligned}$$

Let $W = V_0 - E$ and use the WKB approximation to solve for ψ_2.

$$\begin{aligned}
\psi_2(x) &= C \exp\left(-\frac{1}{\hbar}\int \sqrt{2m(W - Qx)}\, dx\right) + D \exp\left(\frac{1}{\hbar}\int \sqrt{2m(W - Qx)}\, dx\right) \\
&= C \exp\left(\frac{(2m(W - Qx))^{\frac{3}{2}}}{3Qm\hbar}\right) + D \exp\left(-\frac{(2m(W - Qx))^{\frac{3}{2}}}{3Qm\hbar}\right)
\end{aligned}$$

To simplify the formulas let

$$k = \frac{\sqrt{2mE}}{\hbar}, \quad \beta = \frac{(2m(W - Qx))^{\frac{3}{2}}}{3Qm\hbar}$$

and write

$$\psi_1(x) = A\exp(ikx) + B\exp(-ikx)$$
$$\psi_2(x) = C\exp(\beta x) + D\exp(-\beta x)$$
$$\psi_3(x) = F\exp(ikx) + G\exp(-ikx)$$

Exponentials of $-i$ represent particles moving from right to left. The B exponential represents a particle reflected from the boundary at $x = 0$. There is no reflection for $x > (V_0 - E)/Q$ hence $G = 0$.

Let us now solve for the coefficients using boundary conditions. Let $L = (V_0 - E)/Q$. Four boundary conditions are needed to ensure continuity at $x = 0$ and $x = L$.

$$\psi_1(0) = \psi_2(0)$$
$$\psi_1'(0) = \psi_2'(0)$$
$$\psi_2(L) = \psi_3(L)$$
$$\psi_2'(L) = \psi_3'(L)$$

From the boundary condition $\psi_2(L) = \psi_3(L)$ we have

$$C\exp(\beta L) + D\exp(-\beta L) = F\exp(ikL) \tag{1}$$

From the boundary condition $\psi_2'(L) = \psi_3'(L)$ we have

$$\beta C\exp(\beta L) - \beta D\exp(-\beta L) = ikF\exp(ikL) \tag{2}$$

Add β times (1) to (2) to obtain

$$2\beta C\exp(\beta L) = (\beta + ik)F\exp(ikL)$$

Hence

$$C = \frac{(\beta + ik)F\exp(ikL - \beta L)}{2\beta} \tag{3}$$

Add minus β times (1) to (2) to obtain

$$-2\beta D\exp(-\beta L) = (-\beta + ik)F\exp(ikL)$$

Hence

$$D = \frac{(\beta - ik)F\exp(ikL + \beta L)}{2\beta} \tag{4}$$

From the boundary condition $\psi_1(0) = \psi_2(0)$ we have

$$A + B = C + D \tag{5}$$

From the boundary condition $\psi_1'(0) = \psi_2'(0)$ we have

$$ik(A - B) = \beta(C - D) \tag{6}$$

Add ik times (5) to (6) to obtain

$$2ikA = \beta(C - D) + ik(C + D)$$

Hence

$$A = \frac{\beta(C - D)}{2ik} + \frac{C + D}{2}$$