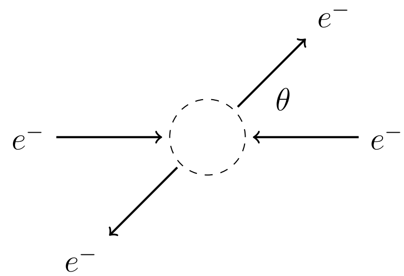
Moller scattering

Moller scattering is the interaction $e^- + e^- \to e^- + e^-$.



Define the following momentum vectors and spinors. Symbol p is incident momentum. Symbol E is total energy $E = \sqrt{p^2 + m^2}$ where m is electron mass. Polar angle θ is the observed scattering angle. Azimuth angle ϕ cancels out in scattering calculations.

$$p_1 = \underset{\text{inbound } e^-}{\begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}} \qquad u_{11} = \underset{\substack{\text{inbound } e^- \\ \text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix}} \qquad u_{12} = \underset{\substack{\text{inbound } e^- \\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix}}$$

$$p_2 = \underset{\text{inbound } e^-}{\begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix}} \qquad u_{21} = \underset{\substack{\text{inbound } e^- \\ \text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ -p \\ 0 \end{pmatrix}} \qquad u_{22} = \underset{\substack{\text{inbound } e^- \\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ p \end{pmatrix}}$$

$$p_3 = \underset{\text{outbound } e^-}{\begin{pmatrix} E \\ p\sin\theta\cos\phi \\ p\sin\theta\sin\phi \\ p\cos\theta \end{pmatrix}} \qquad u_{31} = \underset{\substack{\text{outbound } e^- \\ \text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ p_{3z} \\ p_{3x} + ip_{3y} \end{pmatrix}} \qquad u_{32} = \underset{\substack{\text{outbound } e^- \\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ p_{3x} - ip_{3y} \\ -p_{3z} \end{pmatrix}}$$

$$p_4 = \underset{\text{outbound } e^-}{\begin{pmatrix} E \\ -p\sin\theta\cos\phi \\ -p\sin\theta\sin\phi \\ -p\cos\theta \end{pmatrix}} \qquad u_{41} = \underset{\substack{\text{outbound } e^- \\ \text{spin up}}}{\begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix}} \qquad u_{42} = \underset{\substack{\text{outbound } e^- \\ \text{spin down}}}{\begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}}$$

The spinors are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^4$ will be used.

This is the probability density for spin state *abcd*. The formula is derived from Feynman diagrams for Moller scattering.

$$|\mathcal{M}_{abcd}|^2 = \frac{e^4}{N}\Bigg|\underbrace{\frac{1}{t}(\bar{u}_{3c}\gamma^\mu u_{1a})(\bar{u}_{4d}\gamma_\mu u_{2b})}_{\text{no electron interchange}} - \underbrace{\frac{1}{u}(\bar{u}_{4d}\gamma^\nu u_{1a})(\bar{u}_{3c}\gamma_\nu u_{2b})}_{\text{electron interchange}}\Bigg|^2$$

Symbol e is electron charge and

$$\begin{aligned} t &= (p_1 - p_3)^2 = (p_1 - p_3)^\mu g_{\mu\nu}(p_1 - p_3)^\nu \\ u &= (p_1 - p_4)^2 = (p_1 - p_4)^\mu g_{\mu\nu}(p_1 - p_4)^\nu \end{aligned}$$

Let

$$a_1 = (\bar{u}_{3c}\gamma^\mu u_{1a})(\bar{u}_{4d}\gamma_\mu u_{2b}), \quad a_2 = (\bar{u}_{4d}\gamma^\nu u_{1a})(\bar{u}_{3c}\gamma_\nu u_{2b})$$

Then

$$\begin{aligned} |\mathcal{M}_{abcd}|^2 &= \frac{e^4}{N}\left|\frac{a_1}{t} - \frac{a_2}{u}\right|^2 \\ &= \frac{e^4}{N}\left(\frac{a_1}{t} - \frac{a_2}{u}\right)\left(\frac{a_1}{t} - \frac{a_2}{u}\right)^* \\ &= \frac{e^4}{N}\left(\frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{tu} - \frac{a_1^* a_2}{tu} + \frac{a_2 a_2^*}{u^2}\right) \end{aligned}$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is computed by summing $|\mathcal{M}_{abcd}|^2$ over all spin states and dividing by the number of inbound states. There are four inbound states.

$$\begin{aligned} \langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}|\mathcal{M}_{abcd}|^2 \\ &= \frac{e^4}{4N}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}\left(\frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{tu} - \frac{a_1^* a_2}{tu} + \frac{a_2 a_2^*}{u^2}\right) \end{aligned}$$

The Casimir trick uses matrix arithmetic to compute sums.

$$\begin{aligned} f_{11} &= \frac{1}{N}\sum_{abcd} a_1 a_1^* = \operatorname{Tr}\Big((\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu\Big)\operatorname{Tr}\Big((\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\Big) \\ f_{12} &= \frac{1}{N}\sum_{abcd} a_1 a_2^* = \operatorname{Tr}\Big((\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\Big) \\ f_{22} &= \frac{1}{N}\sum_{abcd} a_2 a_2^* = \operatorname{Tr}\Big((\not{p}_4 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu\Big)\operatorname{Tr}\Big((\not{p}_3 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\Big) \end{aligned}$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left(\frac{f_{11}}{t^2} - \frac{f_{12}}{tu} - \frac{f_{12}^*}{tu} + \frac{f_{22}}{u^2}\right)$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$\begin{aligned} f_{11} &= 32(p_1 \cdot p_2)^2 + 32(p_1 \cdot p_4)^2 - 64m^2(p_1 \cdot p_3) + 64m^4 \\ f_{12} &= -32(p_1 \cdot p_2)^2 + 32m^2(p_1 \cdot p_2) + 32m^2(p_1 \cdot p_3) + 32m^2(p_1 \cdot p_4) - 32m^4 \\ f_{22} &= 32(p_1 \cdot p_2)^2 + 32(p_1 \cdot p_3)^2 - 64m^2(p_1 \cdot p_4) + 64m^4 \end{aligned}$$

For Mandelstam variables

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = 4E^2 \\
t &= (p_1 - p_3)^2 \\
u &= (p_1 - p_4)^2
\end{aligned}
$$

the formulas are

$$
\begin{aligned}
f_{11} &= 8s^2 + 8u^2 - 64sm^2 - 64um^2 + 192m^4 \\
f_{12} &= -8s^2 + 64sm^2 - 96m^4 \\
f_{22} &= 8s^2 + 8t^2 - 64sm^2 - 64tm^2 + 192m^4
\end{aligned}
$$

For high energy experiments $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$
\begin{aligned}
f_{11} &= 8s^2 + 8u^2 \\
f_{12} &= -8s^2 \\
f_{22} &= 8s^2 + 8t^2
\end{aligned}
$$

Hence

$$
\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4}\left(\frac{f_{11}}{t^2} - \frac{f_{12}}{tu} - \frac{f_{12}^*}{tu} + \frac{f_{22}}{u^2}\right) \\
&= \frac{e^4}{4}\left(\frac{8s^2 + 8u^2}{t^2} - \frac{-8s^2}{tu} - \frac{-8s^2}{tu} + \frac{8s^2 + 8t^2}{u^2}\right) \\
&= 2e^4\left(\frac{s^2 + u^2}{t^2} + \frac{2s^2}{tu} + \frac{s^2 + t^2}{u^2}\right)
\end{aligned}
$$

Combine terms so $\langle |\mathcal{M}|^2 \rangle$ has a common denominator.

$$
\langle |\mathcal{M}|^2 \rangle = 2e^4\left(\frac{u^2(s^2 + u^2) + 2s^2tu + t^2(s^2 + t^2)}{t^2u^2}\right)
$$

For $m = 0$ the Mandelstam variables are

$$
\begin{aligned}
s &= 4E^2 \\
t &= 2E^2(\cos\theta - 1) \\
u &= -2E^2(\cos\theta + 1)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= 2e^4\left(\frac{32E^8\cos^4\theta + 192E^8\cos^2\theta + 288E^8}{16E^8(\cos\theta - 1)^2(\cos\theta + 1)^2}\right) \\
&= 4e^4\frac{\left(\cos^2\theta + 3\right)^2}{(\cos\theta - 1)^2(\cos\theta + 1)^2} \\
&= 4e^4\frac{(\cos^2\theta + 3)^2}{\sin^4\theta}
\end{aligned}
$$

The following equivalent formula can also be used.

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= 2e^4\left(\frac{s^2+u^2}{t^2}+\frac{2s^2}{tu}+\frac{s^2+t^2}{u^2}\right)\\ &= 2e^4\Bigg(\underbrace{\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)}}_{\text{no electron interchange}}+\underbrace{\frac{2}{\sin^2(\theta/2)\cos^2(\theta/2)}}_{\text{interaction term}}+\underbrace{\frac{1+\sin^4(\theta/2)}{\cos^4(\theta/2)}}_{\text{electron interchange}}\Bigg)\end{aligned}$$

Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega}=\frac{\langle|\mathcal{M}|^2\rangle}{4(4\pi\varepsilon_0)^2 s}$$

where

$$s=(p_1+p_2)^2=4E^2$$

For high energy experiments we have

$$\langle|\mathcal{M}|^2\rangle = 4e^4\frac{(\cos^2\theta+3)^2}{\sin^4\theta}$$

Hence for high energy experiments

$$\frac{d\sigma}{d\Omega}=\frac{e^4}{(4\pi\varepsilon_0)^2 s}\frac{(\cos^2\theta+3)^2}{\sin^4\theta}$$

Noting that

$$e^2=4\pi\varepsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2(\hbar c)^2}{s}\frac{(\cos^2\theta+3)^2}{\sin^4\theta}$$

Noting that

$$d\Omega=\sin\theta\,d\theta\,d\phi$$

we also have

$$d\sigma=\frac{\alpha^2(\hbar c)^2}{s}\frac{(\cos^2\theta+3)^2}{\sin^4\theta}\sin\theta\,d\theta\,d\phi$$

Let $S(\theta_1,\theta_2)$ be the following surface integral of $d\sigma$.

$$S(\theta_1,\theta_2)=\int_0^{2\pi}\int_{\theta_1}^{\theta_2}d\sigma$$

The solution is

$$S(\theta_1,\theta_2)=\frac{2\pi\alpha^2(\hbar c)^2}{s}\big(I(\theta_2)-I(\theta_1)\big)$$

where

$$I(\theta)=-\frac{8\cos\theta}{\sin^2\theta}-\cos\theta$$

The cumulative distribution function is

$$F(\theta) = \frac{S(a,\theta)}{S(a,\pi - a)} = \frac{I(\theta) - I(a)}{I(\pi - a) - I(a)}, \quad a \le \theta \le \pi - a$$

Angular support is reduced by an arbitrary angle $a > 0$ because $I(0)$ and $I(\pi)$ are undefined.

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

Let N be the total number of scattering events from an experiment. Then the number of scattering events in the interval θ_1 to θ_2 is predicted to be

$$NP(\theta_1 \le \theta \le \theta_2)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi - a) - I(a)} \frac{(\cos^2\theta + 3)^2}{\sin^4\theta} \sin\theta$$

Notes

1. A. Zee page 134 has the cross section

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4\pi}\right)^2 \frac{1}{8E^2} f(\theta)$$

where $f(\theta)$ is the probability density function

$$f(\theta) = \frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{2}{\sin^2(\theta/2)\cos^2(\theta/2)} + \frac{1 + \sin^4(\theta/2)}{\cos^4(\theta/2)}$$

The probability density function is equivalent to

$$f(\theta) = \frac{2(\cos^2\theta + 3)^2}{\sin^4\theta}$$

Hence for natural units $\varepsilon_0 = \hbar = c = 1$ and $e^2 = 4\pi\alpha$ the above cross section is equivalent to

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{4E^2} \frac{(\cos^2\theta + 3)^2}{\sin^4\theta}$$

2. In component notation, the trace operators of the Casimir trick become sums over a repeated index, in this case α.

$$\begin{aligned}
f_{11} &= \left((\not p_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not p_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha\right)\left((\not p_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not p_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha\right) \\
f_{12} &= (\not p_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not p_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not p_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not p_2 + m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha \\
f_{22} &= \left((\not p_4 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not p_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha\right)\left((\not p_3 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not p_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha\right)
\end{aligned}$$

To convert the above formulas to Eigenmath code, the γ tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply γ^μ by the metric tensor to lower the index.

$$\gamma^{\beta\mu}{}_{\rho} \quad \rightarrow \quad \texttt{gammaT = transpose(gamma)}$$
$$\gamma^{\beta}{}_{\mu\rho} \quad \rightarrow \quad \texttt{gammaL = transpose(dot(gmunu,gamma))}$$

Define the following 4×4 matrices.

$$(\not{p}_1 + m) \quad \rightarrow \quad \texttt{X1 = pslash1 + m I}$$
$$(\not{p}_2 + m) \quad \rightarrow \quad \texttt{X2 = pslash2 + m I}$$
$$(\not{p}_3 + m) \quad \rightarrow \quad \texttt{X3 = pslash3 + m I}$$
$$(\not{p}_4 + m) \quad \rightarrow \quad \texttt{X4 = pslash4 + m I}$$

Then for f_{11} we have the following Eigenmath code. The contract function sums over α.

$$(\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \quad \rightarrow \quad \texttt{T1 = contract(dot(X3,gammaT,X1,gammaT),1,4)}$$
$$(\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \quad \rightarrow \quad \texttt{T2 = contract(dot(X4,gammaL,X2,gammaL),1,4)}$$

Next, multiply then sum over repeated indices. The dot function sums over ν then the contract function sums over μ. The transpose makes the ν indices adjacent as required by the dot function.

$$f_{11} = \operatorname{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu)\operatorname{Tr}(\cdots\gamma_\mu\cdots\gamma_\nu) \quad \rightarrow \quad \texttt{contract(dot(T1,transpose(T2)))}$$

Follow suit for f_{22}.

$$(\not{p}_4 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \quad \rightarrow \quad \texttt{T1 = contract(dot(X4,gammaT,X1,gammaT),1,4)}$$
$$(\not{p}_3 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \quad \rightarrow \quad \texttt{T2 = contract(dot(X3,gammaL,X2,gammaL),1,4)}$$

Then

$$f_{22} = \operatorname{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu)\operatorname{Tr}(\cdots\gamma_\mu\cdots\gamma_\nu) \quad \rightarrow \quad \texttt{contract(dot(T1,transpose(T2)))}$$

The calculation of f_{12} begins with

$$(\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_2 + m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha$$
$$\rightarrow \quad \texttt{T = contract(dot(X3,gammaT,X1,gammaT,X4,gammaL,X2,gammaL),1,6)}$$

Then sum over repeated indices μ and ν.

$$f_{12} = \operatorname{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu\cdots\gamma_\mu\cdots\gamma_\nu) \quad \rightarrow \quad \texttt{contract(contract(T,1,3))}$$