Perturbation expansion

This is the propagator for a particle in a potential energy field $V(x,t)$.

$$K(b,a)=\int_{x_a}^{x_b}\exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b}\left(\frac{m\dot{x}^2}{2}-V\big(x(t),t\big)\right)dt\right]\mathscr{D}x(t)$$

Factor the exponential.

$$K(b,a)=\int_{x_a}^{x_b}\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b}\frac{m\dot{x}^2}{2}\,dt\right)\exp\left(-\frac{i}{\hbar}\int_{t_a}^{t_b}V\big(x(t),t\big)\,dt\right)\mathscr{D}x(t)$$

Expand the second exponential as a power series.

$$K(b,a)=\int_{x_a}^{x_b}\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b}\frac{m\dot{x}^2}{2}\,dt\right)\left(1-\frac{i}{\hbar}\int_{t_a}^{t_b}V\big(x(t),t\big)\,dt+\cdots\right)\mathscr{D}x(t)$$

Hence

$$K(b,a)=K_0(b,a)+K_1(b,a)+\cdots$$

where K_0 is a free particle propagator and

$$K_1(b,a)=-\frac{i}{\hbar}\int_{x_a}^{x_b}\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b}\frac{m\dot{x}^2}{2}\,dt\right)\int_{t_a}^{t_b}V\big(x(t_c),t_c\big)\,dt_c\,\mathscr{D}x(t)$$

Interchange the order of the integrals.

$$K_1(b,a)=-\frac{i}{\hbar}\int_{t_a}^{t_b}\int_{x_a}^{x_b}\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b}\frac{m\dot{x}^2}{2}\,dt\right)V\big(x(t_c),t_c\big)\,\mathscr{D}x(t)\,dt_c$$

Factor the exponential with respect to t_c.

$$K_1(b,a)=-\frac{i}{\hbar}\int_{t_a}^{t_b}\int_{x_a}^{x_b}\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_c}\frac{m\dot{x}^2}{2}\,dt\right)\exp\left(\frac{i}{\hbar}\int_{t_c}^{t_b}\frac{m\dot{x}^2}{2}\,dt\right)V\big(x(t_c),t_c\big)\,\mathscr{D}x(t)\,dt_c$$

The two exponentials are free particle propagators.

$$\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_c}\frac{m\dot{x}^2}{2}\,dt\right)=K_0\big(x(t_c),t_c,x_a,t_a\big)\qquad\text{from } x_a,t_a \text{ to } x(t_c),t_c$$

$$\exp\left(\frac{i}{\hbar}\int_{t_c}^{t_b}\frac{m\dot{x}^2}{2}\,dt\right)=K_0\big(x_b,t_b,x(t_c),t_c\big)\qquad\text{from } x(t_c),t_c \text{ to } x_b,t_b$$

Hence

$$K_1(b,a)=-\frac{i}{\hbar}\int_{t_a}^{t_b}\int_{x_a}^{x_b}K_0\big(x(t_c),t_c,x_a,t_a\big)K_0\big(x_b,t_b,x(t_c),t_c\big)V\big(x(t_c),t_c\big)\,\mathscr{D}x(t)\,dt_c$$

The integral is over all possible paths $x(t)$ hence

$$-\infty<x(t_c)<\infty$$

Let $x_c = x(t_c)$ and transform the measure from $\mathscr{D}x(t)$ to dx_c.

$$K_1(b,a) = -\frac{i}{\hbar}\int_{t_a}^{t_b}\int_{-\infty}^{\infty} K_0(x_c,t_c,x_a,t_a)K_0(x_b,t_b,x_c,t_c)V(x_c,t_c)\,dx_c\,dt_c$$

Substitute for K_0.

$$\begin{aligned}K_1(b,a) = -\frac{i}{\hbar}\int_{t_a}^{t_b}\int_{-\infty}^{\infty} &\left(\frac{m}{2\pi i\hbar(t_c-t_a)}\right)^{\frac{1}{2}}\exp\left(\frac{im(x_b-x_a)^2}{2\hbar(t_c-t_a)}\right)\\ &\times\left(\frac{m}{2\pi i\hbar(t_b-t_c)}\right)^{\frac{1}{2}}\exp\left(\frac{im(x_b-x_a)^2}{2\hbar(t_b-t_c)}\right)V(x_c,t_c)\,dx_c\,dt_c\end{aligned}$$