

## Atomic transitions 3

From the previous section

$$c_b(t) = \frac{ieE_0}{m\hbar\omega} \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle \frac{\sin(\frac{1}{2}(\omega_0 - \omega)t)}{\omega_0 - \omega} \exp(\frac{i}{2}(\omega_0 - \omega)t)$$

The transition probability is

$$\text{Pr}_{a \rightarrow b}(t) = |c_b(t)|^2 = \frac{e^2 E_0^2}{m^2 \hbar^2 \omega^2} |\langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle|^2 \frac{\sin^2(\frac{1}{2}(\omega_0 - \omega)t)}{(\omega_0 - \omega)^2}$$

We are now going to pivot from  $E_0$  to a full spectrum electric field. Let  $u$  be energy density such that

$$E_0^2 = \frac{2u}{\epsilon_0}$$

By substitution

$$\text{Pr}_{a \rightarrow b}(t) = \frac{2u}{\epsilon_0} \frac{e^2}{m^2 \hbar^2 \omega^2} |\langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle|^2 \frac{\sin^2(\frac{1}{2}(\omega_0 - \omega)t)}{(\omega_0 - \omega)^2}$$

For incoherent radiation

$$u = \int_{-\infty}^{\infty} \rho(\omega) d\omega$$

where  $\rho(\omega)$  is energy per volume per hertz. By substitution

$$\text{Pr}_{a \rightarrow b}(t) = \frac{2}{\epsilon_0} \frac{e^2}{m^2 \hbar^2} |\langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle|^2 \int_{-\infty}^{\infty} \frac{\rho(\omega)}{\omega^2} \frac{\sin^2(\frac{1}{2}(\omega_0 - \omega)t)}{(\omega_0 - \omega)^2} d\omega$$

The integrand is sharply peaked at  $\omega = \omega_0$  so we use the approximation

$$\int_{-\infty}^{\infty} \frac{\rho(\omega)}{\omega^2} \times \dots d\omega \approx \frac{\rho(\omega_0)}{\omega_0^2} \int_{-\infty}^{\infty} \dots d\omega$$

Using change of variable

$$y = \frac{1}{2}(\omega - \omega_0)t, \quad dy = \frac{t}{2} d\omega, \quad \omega - \omega_0 = \frac{2y}{t}$$

we have

$$\int_{-\infty}^{\infty} \frac{\sin^2(\frac{1}{2}(\omega_0 - \omega)t)}{(\omega_0 - \omega)^2} d\omega = \frac{t}{2} \int_{-\infty}^{\infty} \frac{\sin^2(-y)}{y^2} dy = \frac{\pi}{2} t$$

Hence

$$\text{Pr}_{a \rightarrow b}(t) = \frac{\pi e^2 \rho(\omega_0)}{\epsilon_0 m^2 \hbar^2 \omega_0^2} |\langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle|^2 t$$

The transition rate is

$$R_{a \rightarrow b} = \frac{d}{dt} \text{Pr}_{a \rightarrow b}(t) = \frac{\pi e^2 \rho(\omega_0)}{\epsilon_0 m^2 \hbar^2 \omega_0^2} |\langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle|^2$$

Verify dimensions.

$$R_{a \rightarrow b} \propto \frac{\frac{e^2}{\text{C}^2} \frac{\rho(\omega_0)}{\text{J m}^{-3} \text{s}}}{\frac{\epsilon_0}{\text{C}^2 \text{J}^{-1} \text{m}^{-1}} \frac{m^2}{\text{kg}^2} \frac{\hbar^2}{\text{J}^2 \text{s}^2} \frac{\omega_0^2}{\text{s}^{-2}}} \times \left| \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle \right|^2 = \text{s}^{-1}$$