Dirac equation 3

From the previous section we have the wavefunctions

$$\psi_1 = \sqrt{E+mc^2}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z c}{E+mc^2} \\ \frac{(p_x+ip_y)c}{E+mc^2} \end{pmatrix} e^{-i\xi/\hbar} \qquad \psi_2 = \sqrt{E+mc^2}\begin{pmatrix} 0 \\ 1 \\ \frac{(p_x-ip_y)c}{E+mc^2} \\ \frac{-p_z c}{E+mc^2} \end{pmatrix} e^{-i\xi/\hbar}$$

wavefunction for fermion spin up　　wavefunction for fermion spin down

$$\psi_3 = \sqrt{E+mc^2}\begin{pmatrix} \frac{p_z c}{E+mc^2} \\ \frac{(p_x+ip_y)c}{E+mc^2} \\ 1 \\ 0 \end{pmatrix} e^{i\xi/\hbar} \qquad \psi_4 = \sqrt{E+mc^2}\begin{pmatrix} \frac{(p_x-ip_y)c}{E+mc^2} \\ \frac{-p_z c}{E+mc^2} \\ 0 \\ 1 \end{pmatrix} e^{i\xi/\hbar}$$

wavefunction for antifermion spin up　　wavefunction for antifermion spin down

where

$$\xi = p_\mu x^\mu = Et - p_x x - p_y y - p_z z$$

and

$$E = \sqrt{p_x^2c^2 + p_y^2c^2 + p_z^2c^2 + m^2c^4}$$

Spinors u and v are equivalent to ψ with the exponentials canceled out.

$$u_1 = \psi_1 e^{i\xi/\hbar} = \sqrt{E+mc^2}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z c}{E+mc^2} \\ \frac{(p_x+ip_y)c}{E+mc^2} \end{pmatrix} \qquad u_2 = \psi_2 e^{i\xi/\hbar} = \sqrt{E+mc^2}\begin{pmatrix} 0 \\ 1 \\ \frac{(p_x-ip_y)c}{E+mc^2} \\ \frac{-p_z c}{E+mc^2} \end{pmatrix}$$

spinor for fermion spin up　　spinor for fermion spin down

$$v_1 = \psi_3 e^{-i\xi/\hbar} = \sqrt{E+mc^2}\begin{pmatrix} \frac{p_z c}{E+mc^2} \\ \frac{(p_x+ip_y)c}{E+mc^2} \\ 1 \\ 0 \end{pmatrix} \qquad v_2 = \psi_4 e^{-i\xi/\hbar} = \sqrt{E+mc^2}\begin{pmatrix} \frac{(p_x-ip_y)c}{E+mc^2} \\ \frac{-p_z c}{E+mc^2} \\ 0 \\ 1 \end{pmatrix}$$

spinor for antifermion spin up　　spinor for antifermion spin down

Spinors are solutions to the momentum-space Dirac equations

$$\not{p}u = mcu$$
$$\not{p}v = -mcv$$

where

$$\not{p} = p^\mu g_{\mu\nu}\gamma^\nu$$

and

$$p^\mu = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

Spinors have the following "completeness property."

$$u_1\bar{u}_1 + u_2\bar{u}_2 = \not{p}c + mc^2$$
$$v_1\bar{v}_1 + v_2\bar{v}_2 = \not{p}c - mc^2$$

Adjoints of spinors are formed as

$$\bar{u} = u^\dagger\gamma^0, \quad \bar{v} = v^\dagger\gamma^0$$

Vector products $uu^\dagger$ and $vv^\dagger$ are outer products that form 4×4 matrices.

Eigenmath script