Atomic transitions 3

From the previous section

$$c_b(t) = \frac{ieE_0}{m\hbar\omega}\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\frac{\sin\left(\frac{1}{2}(\omega_0-\omega)t\right)}{\omega_0-\omega}\exp\left(\tfrac{i}{2}(\omega_0-\omega)t\right)$$

The probability to go from state a to state b is

$$\Pr_{a\to b}(t) = |c_b(t)|^2 = \frac{e^2E_0^2}{m^2\hbar^2\omega^2}\left|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\right|^2\frac{\sin^2\left(\frac{1}{2}(\omega_0-\omega)t\right)}{(\omega_0-\omega)^2}$$

We are now going to pivot from E_0 to a full spectrum electric field. Let u be energy density such that

$$E_0^2 = \frac{2u}{\varepsilon_0}$$

By substitution

$$\Pr_{a\to b}(t) = \frac{2u}{\varepsilon_0}\frac{e^2}{m^2\hbar^2\omega^2}\left|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\right|^2\frac{\sin^2\left(\frac{1}{2}(\omega_0-\omega)t\right)}{(\omega_0-\omega)^2}$$

For incoherent radiation

$$u = \int_{-\infty}^{\infty}\rho(\omega)\,d\omega$$

where $\rho(\omega)$ has dimensions energy per unit volume per hertz. By substitution

$$\Pr_{a\to b}(t) = \frac{2}{\varepsilon_0}\frac{e^2}{m^2\hbar^2}\left|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\right|^2\int_{-\infty}^{\infty}\frac{\rho(\omega)}{\omega^2}\frac{\sin^2\left(\frac{1}{2}(\omega_0-\omega)t\right)}{(\omega_0-\omega)^2}\,d\omega$$

The integrand is sharply peaked at $\omega=\omega_0$ so we use the approximation

$$\int_{-\infty}^{\infty}\frac{\rho(\omega)}{\omega^2}\cdots d\omega \approx \frac{\rho(\omega_0)}{\omega_0^2}\int_{-\infty}^{\infty}\cdots d\omega$$

Using change of variable

$$y = \tfrac{1}{2}(\omega-\omega_0)t,\quad dy = \frac{t}{2}\,d\omega,\quad \omega-\omega_0 = \frac{2y}{t}$$

we have

$$\int_{-\infty}^{\infty}\frac{\sin^2\left(\frac{1}{2}(\omega_0-\omega)t\right)}{(\omega_0-\omega)^2}\,d\omega = \frac{t}{2}\int_{-\infty}^{\infty}\frac{\sin^2(-y)}{y^2}\,dy = \frac{\pi}{2}t$$

Hence

$$\Pr_{a\to b}(t) = \frac{\pi e^2\rho(\omega_0)}{\varepsilon_0 m^2\hbar^2\omega_0^2}\left|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\right|^2 t$$

The transition rate is

$$R_{a\to b} = \frac{d}{dt}\Pr_{a\to b}(t) = \frac{\pi e^2\rho(\omega_0)}{\varepsilon_0 m^2\hbar^2\omega_0^2}\left|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\right|^2$$

Verify dimensions.

$$R_{a \to b} \propto \frac{\overset{e^2}{\underset{C^2}{}} \quad \overset{\rho(\omega_0)}{\underset{J\,m^{-3}\,s}{}}}{\underset{C^2\,J^{-1}\,m^{-1}}{\varepsilon_0} \quad \underset{kg^2}{m^2} \quad \underset{J^2\,s^2}{\hbar^2} \quad \underset{s^{-2}}{\omega_0^2}} \times \underset{kg^2\,m^2\,s^{-2}}{\left| \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r}) | \psi_a \rangle \right|^2} = s^{-1}$$