

9-6. Show, using sine and cosine modes and real variables, that this expression using complex variables is indeed correct (cf. problem 8-4).

$$\Phi_0 = \exp \left(- \sum_{\mathbf{k}} \frac{kc}{2\hbar} (\bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} + \bar{a}_{2,\mathbf{k}}^* \bar{a}_{2,\mathbf{k}}) \right) \quad (9.43)$$

Let

$$\begin{aligned} \bar{a}_{1,\mathbf{k}}^c &= \frac{1}{\sqrt{2}} (\bar{a}_{1,\mathbf{k}} + \bar{a}_{1,\mathbf{k}}^*) \\ \bar{a}_{1,\mathbf{k}}^s &= \frac{i}{\sqrt{2}} (\bar{a}_{1,\mathbf{k}} - \bar{a}_{1,\mathbf{k}}^*) \end{aligned}$$

$$\begin{aligned} \bar{a}_{2,\mathbf{k}}^c &= \frac{1}{\sqrt{2}} (\bar{a}_{2,\mathbf{k}} + \bar{a}_{2,\mathbf{k}}^*) \\ \bar{a}_{2,\mathbf{k}}^s &= \frac{i}{\sqrt{2}} (\bar{a}_{2,\mathbf{k}} - \bar{a}_{2,\mathbf{k}}^*) \end{aligned}$$

Then

$$\begin{aligned} \bar{a}_{1,\mathbf{k}} &= \frac{1}{\sqrt{2}} (\bar{a}_{1,\mathbf{k}}^c - i\bar{a}_{1,\mathbf{k}}^s) \\ \bar{a}_{2,\mathbf{k}} &= \frac{1}{\sqrt{2}} (\bar{a}_{2,\mathbf{k}}^c - i\bar{a}_{2,\mathbf{k}}^s) \end{aligned}$$

It follows that

$$\begin{aligned} \bar{a}_{1,\mathbf{k}}^* \bar{a}_{1,\mathbf{k}} &= \frac{1}{2} (\bar{a}_{1,\mathbf{k}}^c)^2 + \frac{1}{2} (\bar{a}_{1,\mathbf{k}}^s)^2 \\ \bar{a}_{2,\mathbf{k}}^* \bar{a}_{2,\mathbf{k}} &= \frac{1}{2} (\bar{a}_{2,\mathbf{k}}^c)^2 + \frac{1}{2} (\bar{a}_{2,\mathbf{k}}^s)^2 \end{aligned} \quad (1)$$

Substitute (1) into (9.43).

$$\Phi_0 = \exp \left(- \sum_{\mathbf{k}} \frac{kc}{4\hbar} ((\bar{a}_{1,\mathbf{k}}^c)^2 + (\bar{a}_{1,\mathbf{k}}^s)^2 + (\bar{a}_{2,\mathbf{k}}^c)^2 + (\bar{a}_{2,\mathbf{k}}^s)^2) \right)$$