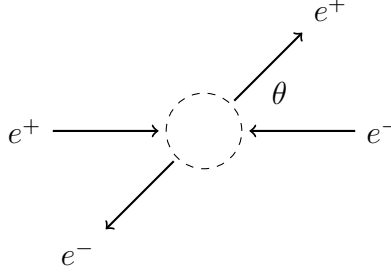


Bhabha scattering

Bhabha scattering is the interaction $e^- + e^+ \rightarrow e^- + e^+$.



The following momentum vectors are for the center-of-mass frame with $E = \sqrt{p^2 + m^2}$.

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \quad \text{inbound } e^+ \quad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} \quad \text{inbound } e^- \quad p_3 = \begin{pmatrix} E \\ p \sin \theta \cos \phi \\ p \sin \theta \sin \phi \\ p \cos \theta \end{pmatrix} \quad \text{outbound } e^+ \quad p_4 = \begin{pmatrix} E \\ -p \sin \theta \cos \phi \\ -p \sin \theta \sin \phi \\ -p \cos \theta \end{pmatrix} \quad \text{outbound } e^-$$

Spinors for the inbound positron.

$$v_{11} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} p \\ 0 \\ E+m \\ 0 \end{pmatrix} \quad \text{inbound } e^+ \text{ spin up} \quad v_{12} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ -p \\ 0 \\ E+m \end{pmatrix} \quad \text{inbound } e^+ \text{ spin down}$$

Spinors for the inbound electron.

$$u_{21} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ -p \\ 0 \end{pmatrix} \quad \text{inbound } e^- \text{ spin up} \quad u_{22} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ 0 \\ p \end{pmatrix} \quad \text{inbound } e^- \text{ spin down}$$

Spinors for the outbound positron.

$$v_{31} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} p_{3z} \\ p_{3x} + ip_{3y} \\ E+m \\ 0 \end{pmatrix} \quad \text{outbound } e^+ \text{ spin up} \quad v_{32} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} p_{3x} - ip_{3y} \\ -p_{3z} \\ 0 \\ E+m \end{pmatrix} \quad \text{outbound } e^+ \text{ spin down}$$

Spinors for the outbound electron.

$$u_{41} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix} \quad \text{outbound } e^- \text{ spin up} \quad u_{42} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix} \quad \text{outbound } e^- \text{ spin down}$$

The scattering amplitude \mathcal{M}_{abcd} for spin state $abcd$ is

$$\mathcal{M}_{abcd} = \mathcal{M}_{1abcd} + \mathcal{M}_{2abcd}$$

where

$$\mathcal{M}_{1abcd} = -\frac{e^2}{t} (\bar{v}_{1a} \gamma^\nu v_{3c}) \underset{\text{scattering}}{(\bar{u}_{4d} \gamma_\nu u_{2b})}, \quad \mathcal{M}_{2abcd} = \frac{e^2}{s} (\bar{v}_{1a} \gamma^\mu u_{2b}) \underset{\text{annihilation}}{(\bar{u}_{4d} \gamma_\mu v_{3c})}$$

Symbol e is elementary charge and

$$s = (p_1 + p_2)^2$$

$$t = (p_1 - p_3)^2$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is the average over spin states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{abcd} |\mathcal{M}_{abcd}|^2$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{abcd} (\mathcal{M}_{1abcd} \mathcal{M}_{1abcd}^* + \mathcal{M}_{1abcd} \mathcal{M}_{2abcd}^* + \mathcal{M}_{2abcd} \mathcal{M}_{1abcd}^* + \mathcal{M}_{2abcd} \mathcal{M}_{2abcd}^*)$$

The Casimir trick uses matrix arithmetic to sum over spin states.

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left(\frac{f_{11}}{t^2} + \frac{2f_{12}}{st} + \frac{f_{22}}{s^2} \right)$$

where

$$f_{11} = \text{Tr} \left[(\not{p}_1 - m) \gamma^\mu (\not{p}_3 - m) \gamma^\nu \right] \text{Tr} \left[(\not{p}_4 + m) \gamma_\mu (\not{p}_2 + m) \gamma_\nu \right]$$

$$f_{12} = -\text{Tr} \left[(\not{p}_1 - m) \gamma^\mu (\not{p}_2 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\mu (\not{p}_3 - m) \gamma_\nu \right]$$

$$f_{22} = \text{Tr} \left[(\not{p}_1 - m) \gamma^\mu (\not{p}_2 + m) \gamma^\nu \right] \text{Tr} \left[(\not{p}_4 + m) \gamma_\mu (\not{p}_3 - m) \gamma_\nu \right]$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$f_{11} = 32(p_1 \cdot p_2)^2 + 32(p_1 \cdot p_4)^2 - 64(p_1 \cdot p_2)m^2 + 64(p_1 \cdot p_4)m^2$$

$$f_{12} = 32(p_1 \cdot p_4)^2 + 64(p_1 \cdot p_4)m^2$$

$$f_{22} = 32(p_1 \cdot p_3)^2 + 32(p_1 \cdot p_4)^2 + 64(p_1 \cdot p_3)m^2 + 64(p_1 \cdot p_4)m^2$$

In Mandelstam variables

$$f_{11} = 8u^2 + 8s^2 - 64um^2 - 64sm^2 + 192m^4$$

$$f_{12} = 8u^2 - 64um^2 + 96m^4$$

$$f_{22} = 8u^2 + 8t^2 - 64um^2 - 64tm^2 + 192m^4$$

For $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$\begin{aligned}f_{11} &= 8u^2 + 8s^2 \\f_{12} &= 8u^2 \\f_{22} &= 8u^2 + 8t^2\end{aligned}$$

For $m = 0$ the Mandelstam variables are

$$\begin{aligned}s &= 4E^2 \\t &= -2E^2(1 - \cos \theta) \\u &= -2E^2(1 + \cos \theta)\end{aligned}$$

Hence

$$\begin{aligned}\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left(\frac{f_{11}}{t^2} + \frac{2f_{12}}{st} + \frac{f_{22}}{s^2} \right) \\&= 2e^4 \left(\frac{u^2 + s^2}{t^2} + \frac{2u^2}{st} + \frac{u^2 + t^2}{s^2} \right) \\&= e^4 \left(\underbrace{\frac{2(1 + \cos \theta)^2 + 8}{(1 - \cos \theta)^2}}_{\text{scattering}} - \underbrace{\frac{2(1 + \cos \theta)^2}{1 - \cos \theta}}_{\text{interaction}} + \underbrace{1 + \cos^2 \theta}_{\text{annihilation}} \right)\end{aligned}$$

The expected probability density can be written more compactly as

$$\langle |\mathcal{M}|^2 \rangle = e^4 \left(\frac{\cos^2 \theta + 3}{\cos \theta - 1} \right)^2$$

Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{4(4\pi\epsilon_0)^2 s}$$

where

$$s = (p_1 + p_2)^2 = 4E^2$$

For high energy experiments we have

$$\langle |\mathcal{M}|^2 \rangle = e^4 \left(\frac{\cos^2 \theta + 3}{\cos \theta - 1} \right)^2$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{4(4\pi\epsilon_0)^2 s} \left(\frac{\cos^2 \theta + 3}{\cos \theta - 1} \right)^2$$

Noting that

$$e^2 = 4\pi\epsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{4s} \left(\frac{\cos^2 \theta + 3}{\cos \theta - 1} \right)^2$$

Noting that

$$d\Omega = \sin \theta d\theta d\phi$$

we also have

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{4s} \left(\frac{\cos^2 \theta + 3}{\cos \theta - 1} \right)^2 \sin \theta d\theta d\phi$$

Let $S(\theta_1, \theta_2)$ be the following integral of $d\sigma$.

$$S(\theta_1, \theta_2) = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1, \theta_2) = \frac{\pi\alpha^2(\hbar c)^2}{2s} [I(\theta_2) - I(\theta_1)]$$

where

$$I(\theta) = \frac{16}{\cos \theta - 1} - \frac{\cos^3 \theta}{3} - \cos^2 \theta - 9 \cos \theta - 16 \log(1 - \cos \theta)$$

The cumulative distribution function is

$$F(\theta) = \frac{S(a, \theta)}{S(a, \pi)} = \frac{I(\theta) - I(a)}{I(\pi) - I(a)}, \quad a \leq \theta \leq \pi$$

Angular support is reduced by an arbitrary angle $a > 0$ because $I(0)$ is undefined.

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1 < \theta \leq \theta_2) = F(\theta_2) - F(\theta_1)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi) - I(a)} \left(\frac{\cos^2 \theta + 3}{\cos \theta - 1} \right)^2 \sin \theta$$