Atomic transitions 5

From the previous section

$$R_{a\to b} = \frac{\pi e^2 \rho(\omega_0)}{\varepsilon_0 \hbar^2} \left| \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle \right|^2$$

Recalling that

$$\boldsymbol{\epsilon} \cdot \mathbf{r} = \epsilon_x x + \epsilon_y y + \epsilon_z z$$

we have

$$\left| \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle \right|^2 = \left| \epsilon_x \langle \psi_b | x | \psi_a \rangle + \epsilon_y \langle \psi_b | y | \psi_a \rangle + \epsilon_z \langle \psi_b | z | \psi_a \rangle \right|^2$$

Average over $\boldsymbol{\epsilon}$ to eliminate it. Let

$$\begin{aligned} e_x &= \sin\theta \cos\phi \\ e_y &= \sin\theta \sin\phi \\ e_z &= \cos\theta \end{aligned}$$

Then

$$\begin{aligned} \left| \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle \right|^2 &= \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \left| \epsilon_x \langle \psi_b | x | \psi_a \rangle + \epsilon_y \langle \psi_b | y | \psi_a \rangle + \epsilon_z \langle \psi_b | z | \psi_a \rangle \right|^2 \sin\theta \, d\theta \, d\phi \\ &= \tfrac{1}{3} \left| \langle \psi_b | x | \psi_a \rangle \right|^2 + \tfrac{1}{3} \left| \langle \psi_b | y | \psi_a \rangle \right|^2 + \tfrac{1}{3} \left| \langle \psi_b | z | \psi_a \rangle \right|^2 \qquad (1) \end{aligned}$$

Hence

$$R_{a\to b} = \frac{\pi e^2 \rho(\omega_0)}{3\varepsilon_0 \hbar^2} \left[\left| \langle \psi_b | x | \psi_a \rangle \right|^2 + \left| \langle \psi_b | y | \psi_a \rangle \right|^2 + \left| \langle \psi_b | z | \psi_a \rangle \right|^2 \right]$$