

# Harmonic oscillator propagator

Let  $H$  be the harmonic oscillator Hamiltonian.

$$H = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2}$$

Let  $K(b, a)$  be the propagator for the harmonic oscillator.

$$K(b, a) = \langle x_b | \exp\left(-\frac{iHt}{\hbar}\right) | x_a \rangle \quad (1)$$

This is van Kortryk's identity.

$$\exp\left(-\frac{iHt}{\hbar}\right) = \exp\left(-\frac{im\omega}{2\hbar} \hat{x}^2 \tan \frac{\omega t}{2}\right) \exp\left(-\frac{i}{2m\omega\hbar} \hat{p}^2 \sin(\omega t)\right) \exp\left(-\frac{im\omega}{2\hbar} \hat{x}^2 \tan \frac{\omega t}{2}\right) \quad (2)$$

Substitute (2) into (1) to obtain

$$K(b, a) = \langle x_b | \exp\left(-\frac{im\omega}{2\hbar} \hat{x}^2 \tan \frac{\omega t}{2}\right) \exp\left(-\frac{i}{2m\omega\hbar} \hat{p}^2 \sin(\omega t)\right) \exp\left(-\frac{im\omega}{2\hbar} \hat{x}^2 \tan \frac{\omega t}{2}\right) | x_a \rangle$$

Replace operator  $\hat{x}$  with its eigenvalues.

$$K(b, a) = \exp\left(-\frac{im\omega}{2\hbar} x_b^2 \tan \frac{\omega t}{2}\right) \langle x_b | \exp\left(-\frac{i}{2m\omega\hbar} \hat{p}^2 \sin(\omega t)\right) | x_a \rangle \exp\left(-\frac{im\omega}{2\hbar} x_a^2 \tan \frac{\omega t}{2}\right) \quad (3)$$

Let

$$K_0(b, a) = \langle x_b | \exp\left(-\frac{i}{2m\omega\hbar} \hat{p}^2 \sin(\omega t)\right) | x_a \rangle$$

Substitute  $T$  for  $\sin(\omega t)/\omega$ .

$$K_0(b, a) = \langle x_b | \exp\left(-\frac{iT}{2m\hbar} \hat{p}^2\right) | x_a \rangle$$

$K_0$  is now a free particle propagator hence

$$K_0(b, a) = \left(\frac{m}{2\pi i \hbar T}\right)^{1/2} \exp\left(\frac{im}{2\hbar T} (x_b - x_a)^2\right)$$

Substitute  $\sin(\omega t)/\omega$  for  $T$ .

$$K_0(b, a) = \left(\frac{m\omega}{2\pi i \hbar \sin(\omega t)}\right)^{1/2} \exp\left(\frac{im\omega}{2\hbar \sin(\omega t)} (x_b - x_a)^2\right) \quad (4)$$

Substitute (4) into (3) and combine exponentials.

$$K(b, a) = \left(\frac{m\omega}{2\pi i \hbar \sin(\omega t)}\right)^{1/2} \exp\left(-\frac{im\omega}{2\hbar} (x_b^2 + x_a^2) \tan \frac{\omega t}{2} + \frac{im\omega}{2\hbar \sin(\omega t)} (x_b - x_a)^2\right)$$

By the identity

$$\cot \theta = -\tan \frac{\theta}{2} + \frac{1}{\sin \theta}$$

we have

$$K(b, a) = \left( \frac{m\omega}{2\pi i \hbar \sin(\omega t)} \right)^{1/2} \exp \left( \frac{im\omega}{2\hbar} (x_b^2 + x_a^2) \cot(\omega t) - \frac{im\omega}{\hbar \sin(\omega t)} x_b x_a \right)$$