Green's function

In this section we will find the Green's function $G(\mathbf{x})$ such that

$$(\nabla^2 + k^2)G(\mathbf{x}) = \delta^3(\mathbf{x}) \tag{1}$$

Let $g(\mathbf{y})$ be the Fourier transform of $G(\mathbf{x})$ such that

$$G(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \exp(i\mathbf{x} \cdot \mathbf{y}) g(\mathbf{y}) \, d\mathbf{y} \tag{2}$$

Substitute (2) into (1) to obtain

$$(\nabla^2 + k^2) \left[\frac{1}{(2\pi)^{\frac{3}{2}}} \int \exp(i\mathbf{x} \cdot \mathbf{y}) g(\mathbf{y}) \, d\mathbf{y} \right] = \delta(\mathbf{x})$$

By linearity of differentiation the $(\nabla^2 + k^2)$ can be moved inside the integral.

$$\frac{1}{(2\pi)^{\frac{3}{2}}} \int (\nabla^2 + k^2) \exp(i\mathbf{x} \cdot \mathbf{y}) g(\mathbf{y}) \, d\mathbf{y} = \delta(\mathbf{x})$$

Noting that

$$\nabla^2 \exp(i\mathbf{x} \cdot \mathbf{y}) = -y^2 \exp(i\mathbf{x} \cdot \mathbf{y})$$

and

$$\delta(\mathbf{x}) = \frac{1}{(2\pi)^3} \int \exp(i\mathbf{x} \cdot \mathbf{y}) \, d\mathbf{y}$$

we have

$$\frac{1}{(2\pi)^{\frac{3}{2}}} \int (-y^2 + k^2) \exp(i\mathbf{x} \cdot \mathbf{y}) g(\mathbf{y}) \, d\mathbf{y} = \frac{1}{(2\pi)^3} \int \exp(i\mathbf{x} \cdot \mathbf{y}) \, d\mathbf{y}$$

Hence

$$g(\mathbf{y}) = \frac{1}{(2\pi)^{\frac{3}{2}} (k^2 - y^2)}$$

Substitute for $g(\mathbf{y})$ in (2) to obtain

$$G(\mathbf{x}) = \frac{1}{(2\pi)^3} \int \frac{\exp(i\mathbf{x} \cdot \mathbf{y})}{k^2 - y^2} \, d\mathbf{y}$$

Change to polar coordinates where $x = |\mathbf{x}|$, $y = |\mathbf{y}|$, and θ and ϕ are the angular distance from $\mathbf{x}$ to $\mathbf{y}$.

$$G(\mathbf{x}) = \frac{1}{(2\pi)^3} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{\exp(ixy \cos\theta)}{k^2 - y^2} y^2 \sin\theta \, dy \, d\theta \, d\phi$$

For the integrals over θ and ϕ we have

$$\int_0^\pi \int_0^{2\pi} \exp(ixy \cos\theta) \sin\theta \, d\theta \, d\phi = \frac{4\pi \sin(xy)}{xy}$$

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Hence
$$G(\mathbf{x}) = \frac{1}{2\pi^2 x} \int_0^\infty \frac{y\sin(xy)}{k^2 - y^2}\, dy$$

Noting that $y\sin(xy)$ is an even function of y we can change the integral limits as follows.

$$G(\mathbf{x}) = \frac{1}{4\pi^2 x} \int_{-\infty}^\infty \frac{y\sin(xy)}{k^2 - y^2}\, dy$$

Negate the denominator.

$$G(\mathbf{x}) = \frac{1}{4\pi^2 x} \int_{-\infty}^\infty -\frac{y\sin(xy)}{y^2 - k^2}\, dy$$

Change the sine function to exponential form and factor the denominator.

$$G(\mathbf{x}) = \frac{i}{8\pi^2 x} \left[\int_{-\infty}^\infty \frac{y\exp(ixy)}{(y-k)(y+k)}\, dy - \int_{-\infty}^\infty \frac{y\exp(-ixy)}{(y-k)(y+k)}\, dy\right]$$

By Cauchy's integral formula we have

$$\int_{-\infty}^\infty \frac{y\exp(ixy)}{y+k}\frac{1}{y-k}\, dy = i\pi \exp(ikx)$$

and

$$\int_{-\infty}^\infty \frac{y\exp(-ixy)}{y-k}\frac{1}{y+k}\, dy = -i\pi \exp(ikx)$$

Hence

$$\boxed{G(\mathbf{x}) = -\frac{\exp(ikx)}{4\pi x}} \tag{3}$$

where
$$x = |\mathbf{x}|$$

Verify that (3) satisfies (1).

We will need the following formula from Griffiths and Schroeter problem 10.8.

$$\nabla^2(1/r) = -4\pi\delta^3(\mathbf{r})$$

Recall that $\nabla^2 = \nabla \cdot \nabla$ and

$$\nabla \cdot (f\mathbf{A}) = \nabla f \cdot \mathbf{A} + f\nabla\cdot\mathbf{A}$$

Substituting $\mathbf{r}$ for $\mathbf{x}$ in (3) we have for the Laplacian of $G(\mathbf{r})$

$$\nabla^2 G(\mathbf{r}) = -\frac{1}{4\pi}\nabla \cdot \nabla \left(\frac{e^{ikr}}{r}\right)$$

$$= -\frac{1}{4\pi}\nabla \cdot \left(\frac{1}{r}\nabla e^{ikr} + e^{ikr}\nabla\frac{1}{r}\right)$$

$$= -\frac{1}{4\pi}\left(\underbrace{\nabla\frac{1}{r}\cdot\nabla e^{ikr}}_{\text{see eq. (5)}} + \underbrace{\frac{1}{r}\nabla^2 e^{ikr}}_{\text{see eq. (6)}} + \underbrace{\nabla e^{ikr}\cdot\nabla\frac{1}{r}}_{\text{see eq. (5)}} + e^{ikr}\nabla^2\frac{1}{r}\right) \tag{4}$$

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In spherical coordinates

$$\nabla \frac{1}{r} \cdot \nabla e^{ikr} = \left(-\frac{1}{r^2}\mathbf{e}_r + 0\mathbf{e}_\theta + 0\mathbf{e}_\phi \right) \cdot \left(ike^{ikr}\mathbf{e}_r + 0\mathbf{e}_\theta + 0\mathbf{e}_\phi \right) = -\frac{ike^{ikr}}{r^2} \tag{5}$$

and

$$\begin{aligned}
\frac{1}{r}\nabla^2 e^{ikr} &= \frac{1}{r^2}\frac{\partial^2}{\partial r^2}(re^{ikr}) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(e^{ikr} + ikre^{ikr} \right) \\
&= \frac{1}{r^2}\left(2ike^{ikr} - k^2 re^{ikr} \right) \\
&= \frac{2ike^{ikr}}{r^2} - \frac{k^2 e^{ikr}}{r}
\end{aligned} \tag{6}$$

Substitute (5) and (6) into (4) to obtain

$$\begin{aligned}
\nabla^2 G(\mathbf{r}) &= \frac{k^2 e^{ikr}}{4\pi r} - \frac{1}{4\pi}e^{ikr}\nabla^2\frac{1}{r} \\
&= -k^2 G(\mathbf{r}) + \delta^3(\mathbf{r})e^{ikr}
\end{aligned}$$

Noting that $e^{ikr} = 1$ at $r = 0$, the e^{ikr} term can be discarded leaving

$$\nabla^2 G(\mathbf{r}) = -k^2 G(\mathbf{r}) + \delta^3(\mathbf{r}) \tag{7}$$

Hence

$$(\nabla^2 + k^2)G(\mathbf{r}) = \delta^3(\mathbf{r})$$

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