(1.3) Consider the functional $G[f] = \int g(y, f)\, dy$. Show that

$$\frac{\delta G[f]}{\delta f(x)} = \frac{\partial g(x, f)}{\partial f} \tag{1.41}$$

Now consider the functional $H[f] = \int g(y, f, f')\, dy$ and show that

$$\frac{\delta H[f]}{\delta f(x)} = \frac{\partial g}{\partial f} - \frac{d}{dx}\frac{\partial g}{\partial f'} \tag{1.42}$$

where $f' = \partial f/\partial y$. For the functional $J[f] = \int g(y, f, f', f'')\, dy$ show that

$$\frac{\delta J[f]}{\delta f(x)} = \frac{\partial g}{\partial f} - \frac{d}{dx}\frac{\partial g}{\partial f'} + \frac{d^2}{dx^2}\frac{\partial g}{\partial f''} \tag{1.43}$$

where $f'' = \partial^2 f/\partial y^2$.

Let

$$h = f + \epsilon\delta(y - x) \qquad h' = \frac{\partial h}{\partial y} \qquad h'' = \frac{\partial^2 h}{\partial y^2}$$

Show (1.41).

$$\frac{\delta G[f]}{\delta f(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int g(y, h)\, dy - \int g(y, f)\, dy\right)$$

Taylor expand the first integrand (ϵ^k vanishes in the limit for all $k > 1$).

$$\frac{\delta G[f]}{\delta f(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int g(y, f)\, dy + \int \epsilon\delta(y - x)\frac{\partial g(y, f)}{\partial f}\, dy - \int g(y, f)\, dy\right)$$

Hence

$$\frac{\delta G[f]}{\delta f(x)} = \frac{\partial g(x, f)}{\partial f}$$

Show (1.42).

$$\frac{\delta H[f]}{\delta f(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int g(y, h, h')\, dy - \int g(y, f, f')\, dy\right)$$

Taylor expand the first integrand.

$$g(y, h, h') = g(y, f, f') + \epsilon\delta(y - x)\frac{\partial g(y, f, f')}{\partial f} + \epsilon\frac{\partial\delta(y - x)}{\partial y}\frac{\partial g(y, f, f')}{\partial f'}$$

Hence

$$\frac{\delta H[f]}{\delta f(x)} = \int \delta(y-x)\frac{\partial g(y,f,f')}{\partial f}\,dy + \int \frac{\partial \delta(y-x)}{\partial y}\frac{\partial g(y,f,f')}{\partial f'}\,dy \tag{1}$$

The first integral is solved by (1.41). For the second integral, let

$$u = \frac{\partial g(y,f,f')}{\partial f'} \qquad v = \delta(y-x)$$

Then

$$du = \frac{\partial}{\partial y}\frac{\partial g(y,f,f')}{\partial f'}\,dy \qquad dv = \frac{\partial \delta(y-x)}{\partial y}\,dy$$

Integrate by parts.

$$\begin{aligned}
&\int \frac{\partial \delta(y-x)}{\partial y}\frac{\partial g(y,f,f')}{\partial f'}\,dy \\
&\qquad = uv - \int v\,du \\
&\qquad = \frac{\partial g(y,f,f')}{\partial f'}\delta(y-x) - \int \delta(y-x)\frac{\partial}{\partial y}\frac{\partial g(y,f,f')}{\partial f'}\,dy \\
&\qquad = -\frac{\partial}{\partial x}\frac{\partial g(x,f,f')}{\partial f'} \qquad (2)
\end{aligned}$$

Substitute (1.41) and (2) into (1) to obtain (1.42).

$$\frac{\delta H[f]}{\delta f(x)} = \frac{\partial g(x,f,f')}{\partial f} - \frac{\partial}{\partial x}\frac{\partial g(x,f,f')}{\partial f'}$$

Show (1.43).

$$\frac{\delta J[f]}{\delta f(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int g(y,h,h',h'')\,dy - \int g(y,f,f',f'')\,dy\right)$$

Taylor expand the first integrand.

$$\begin{aligned}
g(y,h,h',h'') &= g(y,f,f',f'') + \epsilon\delta(y-x)\frac{\partial g(y,f,f',f'')}{\partial f} \\
&\quad + \epsilon\frac{\partial \delta(y-x)}{\partial y}\frac{\partial g(y,f,f',f'')}{\partial f'} + \epsilon\frac{\partial^2 \delta(y-x)}{\partial y^2}\frac{\partial g(y,f,f',f'')}{\partial f''}
\end{aligned}$$

Hence

$$\begin{aligned}\frac{\delta J[f]}{\delta f(x)} = \int \delta(y-x)\frac{\partial g(y,f,f',f'')}{\partial f}\,dy \\ + \int \frac{\partial \delta(y-x)}{\partial y}\frac{\partial g(y,f,f',f'')}{\partial f'}\,dy \\ + \int \frac{\partial^2 \delta(y-x)}{\partial y^2}\frac{\partial g(y,f,f',f'')}{\partial f''}\,dy\end{aligned} \tag{3}$$

The first and second integrals are solved by (1.42). For the third integral, let

$$u = \frac{\partial g(y,f,f',f'')}{\partial f''} \qquad v = \frac{\partial \delta(y-x)}{\partial y}$$

Then

$$du = \frac{\partial}{\partial y}\frac{\partial g(y,f,f',f'')}{\partial f''}\,dy \qquad dv = \frac{\partial^2 \delta(y-x)}{\partial y^2}\,dy$$

Integrate by parts.

$$\begin{aligned}&\int \frac{\partial^2 \delta(y-x)}{\partial y^2}\frac{\partial g(y,f,f',f'')}{\partial f''}\,dy \\ &\quad = uv - \int v\,du \\ &\quad = \frac{\partial g(y,f,f',f'')}{\partial f''}\frac{\partial \delta(y-x)}{\partial y} - \int \frac{\partial \delta(y-x)}{\partial y}\frac{\partial}{\partial y}\frac{\partial g(y,f,f',f'')}{\partial f''}\,dy\end{aligned}$$

As before, the uv term vanishes. For the remaining integral, use integration by parts as in (2) to obtain

$$= \frac{\partial^2}{\partial x^2}\frac{\partial g(x,f,f',f'')}{\partial f''} \tag{4}$$

Substitute (1.42) and (4) into (3) to obtain (1.43).

$$\frac{\delta J[f]}{\delta f(x)} = \frac{\partial g(x,f,f',f'')}{\partial f} - \frac{\partial}{\partial x}\frac{\partial g(x,f,f',f'')}{\partial f'} + \frac{\partial^2}{\partial x^2}\frac{\partial g(x,f,f',f'')}{\partial f''}$$