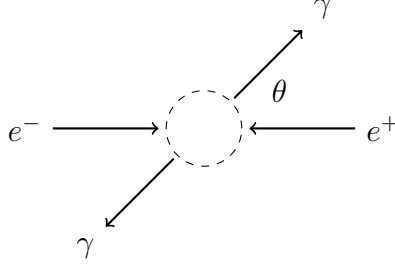# Annihilation

Annihilation is the interaction $e^- + e^+ \rightarrow \gamma + \gamma$.

In the center-of-mass frame we have the following momentum vectors where $E = \sqrt{p^2 + m^2}$.

$$
p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \quad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} \quad p_3 = \begin{pmatrix} E \\ E \sin\theta \cos\phi \\ E \sin\theta \sin\phi \\ E \cos\theta \end{pmatrix} \quad p_4 = \begin{pmatrix} E \\ -E \sin\theta \cos\phi \\ -E \sin\theta \sin\phi \\ -E \cos\theta \end{pmatrix}
$$

| inbound electron | inbound positron | outbound photon | outbound photon |

Spinors for the inbound electron.

$$
u_{11} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix} \qquad u_{12} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix}
$$

inbound electron spin up   inbound electron spin down

Spinors for the inbound positron.

$$
v_{21} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} -p \\ 0 \\ E+m \\ 0 \end{pmatrix} \qquad v_{22} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ p \\ 0 \\ E+m \end{pmatrix}
$$

inbound positron spin up   inbound positron spin down

Let a be the spin state of the inbound electron and let b be the spin state of the inbound positron such that subscript $ba \in \{11, 12, 21, 22\}$. The probability amplitude $\mathcal{M}_{ba}$ for spin state ba is

$$
\mathcal{M}_{ba} = \mathcal{M}_{1ba} + \mathcal{M}_{2ba}
$$

where

$$
\mathcal{M}_{1ba} = \frac{\bar{v}_{2b}(-ie\gamma^\mu)(\slashed{q}_1 + m)(-ie\gamma^\nu)u_{1a}}{t - m^2}, \quad \mathcal{M}_{2ba} = \frac{\bar{v}_{2b}(-ie\gamma^\nu)(\slashed{q}_2 + m)(-ie\gamma^\mu)u_{1a}}{u - m^2}
$$

Symbol e is elementary charge and

$$\slashed{q}_1 = (p_1 - p_3)^\alpha g_{\alpha\beta}\gamma^\beta$$

$$\slashed{q}_2 = (p_1 - p_4)^\alpha g_{\alpha\beta}\gamma^\beta$$

$$t = (p_1 - p_3)^2$$

$$u = (p_1 - p_4)^2$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is the average probability density for all four spin states.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}|\mathcal{M}_{ba}|^2$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\left(\frac{\mathcal{M}_{1ba}\mathcal{M}_{1ba}^*}{(t-m^2)^2} + \frac{\mathcal{M}_{1ba}\mathcal{M}_{2ba}^* + \mathcal{M}_{2ba}\mathcal{M}_{1ba}^*}{(t-m^2)(u-m^2)} + \frac{\mathcal{M}_{2ba}\mathcal{M}_{2ba}^*}{(u-m^2)^2}\right)$$

To understand how $\mathcal{M}_{1ba}\mathcal{M}_{1ba}^*$ is calculated, write $\mathcal{M}_{1ba}$ in component form.

$$(\mathcal{M}_{1ba})^{\mu\nu} = \frac{(\bar{v}_{2b})_\alpha(-ie\gamma^{\mu\alpha}{}_\beta)(\slashed{q}_1 + m)^\beta{}_\rho(-ie\gamma^{\nu\rho}{}_\sigma)(u_{1a})^\sigma}{t-m^2}$$

Metric tensor $g_{\mu\nu}$ is required to sum over indices μ and ν.

$$\mathcal{M}_{1ba}\mathcal{M}_{1ba}^* = (\mathcal{M}_{1ba})^{\mu\nu}(\mathcal{M}_{1ba}^*)_{\mu\nu} = (\mathcal{M}_{1ba})^{\mu\nu}g_{\mu\alpha}(\mathcal{M}_{1ba}^*)^{\alpha\beta}g_{\beta\nu}$$

Similarly for $\mathcal{M}_{2ba}\mathcal{M}_{2ba}^*$. For $\mathcal{M}_{2ba}$ the index order is ν followed by μ hence

$$\mathcal{M}_{1ba}\mathcal{M}_{2ba}^* = (\mathcal{M}_{1ba})^{\mu\nu}(\mathcal{M}_{2ba}^*)_{\nu\mu} = (\mathcal{M}_{1ba})^{\mu\nu}g_{\nu\beta}(\mathcal{M}_{2ba}^*)^{\beta\alpha}g_{\alpha\mu}$$

The Casimir trick uses matrix arithmetic to sum over spin states.

$$f_{11} = \sum_{a=1}^{2}\sum_{b=1}^{2}\mathcal{M}_{1ba}\mathcal{M}_{1ba}^* = e^4\,\mathrm{Tr}\left((\slashed{p}_1 + m)\gamma^\mu(\slashed{q}_1 + m)\gamma^\nu(\slashed{p}_2 - m)\gamma_\nu(\slashed{q}_1 + m)\gamma_\mu\right)$$

$$f_{12} = \sum_{a=1}^{2}\sum_{b=1}^{2}\mathcal{M}_{1ba}\mathcal{M}_{2ba}^* = e^4\,\mathrm{Tr}\left((\slashed{p}_1 + m)\gamma^\mu(\slashed{q}_2 + m)\gamma^\nu(\slashed{p}_2 - m)\gamma_\mu(\slashed{q}_1 + m)\gamma_\nu\right)$$

$$f_{22} = \sum_{a=1}^{2}\sum_{b=1}^{2}\mathcal{M}_{2ba}\mathcal{M}_{2ba}^* = e^4\,\mathrm{Tr}\left((\slashed{p}_1 + m)\gamma^\mu(\slashed{q}_2 + m)\gamma^\nu(\slashed{p}_2 - m)\gamma_\nu(\slashed{q}_2 + m)\gamma_\mu\right)$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\left(\frac{f_{11}}{(t-m^2)^2} + \frac{2f_{12}}{(t-m^2)(u-m^2)} + \frac{f_{22}}{(u-m^2)^2}\right)$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu}b^\nu$)

$$f_{11} = e^4\left(32(p_1 \cdot p_3)(p_1 \cdot p_4) - 32m^2(p_1 \cdot p_2) + 64m^2(p_1 \cdot p_3) + 32m^2(p_1 \cdot p_4) - 64m^4\right)$$

$$f_{12} = e^4\left(16m^2(p_1 \cdot p_3) + 16m^2(p_1 \cdot p_4) - 32m^4\right)$$

$$f_{22} = e^4\left(32(p_1 \cdot p_3)(p_1 \cdot p_4) - 32m^2(p_1 \cdot p_2) + 32m^2(p_1 \cdot p_3) + 64m^2(p_1 \cdot p_4) - 64m^4\right)$$

In Mandelstam variables

$$f_{11} = e^4 \left(8tu - 24tm^2 - 8um^2 - 8m^4\right)$$
$$f_{12} = e^4 \left(8sm^2 - 32m^4\right)$$
$$f_{22} = e^4 \left(8tu - 8tm^2 - 24um^2 - 8m^4\right)$$

For high energy experiments such that $E \gg m$ let $m = 0$ and obtain

$$f_{11} = e^4\, 8tu$$
$$f_{12} = 0$$
$$f_{22} = e^4\, 8tu$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left(\frac{8tu}{t^2} + \frac{8tu}{u^2}\right)$$
$$= 2e^4 \left(\frac{u}{t} + \frac{t}{u}\right)$$

For $m = 0$ the Mandelstam variables are

$$t = -2E^2(1 - \cos\theta)$$
$$u = -2E^2(1 + \cos\theta)$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{1 + \cos\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{1 + \cos\theta}\right)$$

Cross section

The differential cross section is
$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{4(4\pi\varepsilon_0)^2 s}$$

where
$$s = (p_1 + p_2)^2 = 4E^2$$

For high energy experiments we have

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{1 + \cos\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{1 + \cos\theta}\right)$$

Hence for high energy experiments

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2(4\pi\varepsilon_0)^2 s} \left(\frac{1 + \cos\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{1 + \cos\theta}\right)$$

Noting that
$$e^2 = 4\pi\varepsilon_0 \alpha \hbar c$$

we have
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{2s}\left(\frac{1+\cos\theta}{1-\cos\theta}+\frac{1-\cos\theta}{1+\cos\theta}\right)$$

Noting that
$$d\Omega = \sin\theta\, d\theta\, d\phi$$

we also have
$$d\sigma = \frac{\alpha^2(\hbar c)^2}{2s}\left(\frac{1+\cos\theta}{1-\cos\theta}+\frac{1-\cos\theta}{1+\cos\theta}\right)\sin\theta\, d\theta\, d\phi$$

Let $S(\theta_1,\theta_2)$ be the following surface integral of $d\sigma$.
$$S(\theta_1,\theta_2) = \int_0^{2\pi}\int_{\theta_1}^{\theta_2} d\sigma$$

The solution is
$$S(\theta_1,\theta_2) = \frac{2\pi\alpha^2(\hbar c)^2}{2s}\big(I(\theta_2)-I(\theta_1)\big)$$

where
$$I(\theta) = 2\cos\theta + 2\log(1-\cos\theta) - 2\log(1+\cos\theta)$$

The cumulative distribution function is
$$F(\theta) = \frac{S(a,\theta)}{S(a,\pi-a)} = \frac{I(\theta)-I(a)}{I(\pi-a)-I(a)}, \quad a \le \theta \le \pi - a$$

Angular support is reduced by an arbitrary angle $a > 0$ because $I(0)$ and $I(\pi)$ are undefined.

The probability of observing scattering events in the interval θ_1 to θ_2 is
$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

Let N be the total number of scattering events from an experiment. Then the number of scattering events in the interval θ_1 to θ_2 is predicted to be
$$NP(\theta_1 \le \theta \le \theta_2)$$

The probability density function is
$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{I(\pi-a)-I(a)}\left(\frac{1+\cos\theta}{1-\cos\theta}+\frac{1-\cos\theta}{1+\cos\theta}\right)\sin\theta$$

Data from DESY PETRA experiment

See www.hepdata.net/record/ins191231, Table 2, 14.0 GeV.

x	y
0.0502	0.09983
0.1505	0.10791
0.2509	0.12026
0.3512	0.13002
0.4516	0.17681
0.5521	0.19570
0.6526	0.27900
0.7312	0.33204

Data x and y have the following relationship with the differential cross section formula.

$$x = \cos\theta, \quad y = \frac{d\sigma}{d\Omega}$$

The cross section formula is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s} \left(\frac{1 + \cos\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{1 + \cos\theta} \right) \times (\hbar c)^2$$

To compute predicted values $\hat{y}$, multiply by 10^{37} to convert square meters to nanobarns.

$$\hat{y} = \frac{\alpha^2}{2s} \left(\frac{1 + x}{1 - x} + \frac{1 - x}{1 + x} \right) \times (\hbar c)^2 \times 10^{37}$$

The following table shows predicted values $\hat{y}$ for $s = (14.0\,\text{GeV})^2$.

x	y	$\hat{y}$
0.0502	0.09983	0.106325
0.1505	0.10791	0.110694
0.2509	0.12026	0.120005
0.3512	0.13002	0.135559
0.4516	0.17681	0.159996
0.5521	0.19570	0.198562
0.6526	0.27900	0.262745
0.7312	0.33204	0.348884

The coefficient of determination R^2 measures how well predicted values fit the data.

$$R^2 = 1 - \frac{\sum (y - \hat{y})^2}{\sum (y - \bar{y})^2} = 0.98$$

The result indicates that the model $d\sigma$ explains 98% of the variance in the data.