This is the transition rate $A_{nm}$ for a spontaneous emission process $\psi_n \to \psi_m$.

$$A_{nm} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3}\,\omega_{nm}^3\,|\langle r_{nm}\rangle|^2$$

The transition frequency is

$$\omega_{nm} = \frac{1}{\hbar}(E_n - E_m)$$

For a hydrogen atom we have

$$E_n = -\frac{\mu}{2n^2}\left(\frac{e^2}{4\pi\varepsilon_0\hbar}\right)^2$$

where $\mu$ is reduced electron mass.

The radial density is

$$|\langle r_{nm}\rangle|^2 = |\langle x_{nm}\rangle|^2 + |\langle y_{nm}\rangle|^2 + |\langle z_{nm}\rangle|^2$$

where

$$\langle x_{nm}\rangle = \int \psi_m^*\,(r\sin\theta\cos\phi)\,\psi_n\,dV$$
$$\langle y_{nm}\rangle = \int \psi_m^*\,(r\sin\theta\sin\phi)\,\psi_n\,dV$$
$$\langle z_{nm}\rangle = \int \psi_m^*\,(r\cos\theta)\,\psi_n\,dV$$

Let us compute $A_{21}$ for a hydrogen atom. For $n = 2$ there are four possible states.

| $n$ | $\ell$ | $m$ |
|---|---|---|
| 2 | 1 | 1 |
| 2 | 1 | $-1$ |
| 2 | 1 | 0 |
| 2 | 0 | 0 |

The following table shows the radial density for every possible transition.

| | $\psi_{2,1,1} \to \psi_{1,0,0}$ | $\psi_{2,1,-1} \to \psi_{1,0,0}$ | $\psi_{2,1,0} \to \psi_{1,0,0}$ | $\psi_{2,0,0} \to \psi_{1,0,0}$ |
|---|---|---|---|---|
| $\langle x_{21}\rangle =$ | $-\frac{128}{243}\,a_0$ | $\frac{128}{243}\,a_0$ | 0 | 0 |
| $\langle y_{21}\rangle =$ | $-\frac{128}{243}i\,a_0$ | $-\frac{128}{243}i\,a_0$ | 0 | 0 |
| $\langle z_{21}\rangle =$ | 0 | 0 | $\frac{128}{243}\sqrt{2}\,a_0$ | 0 |
| $\lvert\langle r_{21}\rangle\rvert^2 =$ | $\frac{32768}{59049}\,a_0^2$ | $\frac{32768}{59049}\,a_0^2$ | $\frac{32768}{59049}\,a_0^2$ | 0 |

Note that the transition rate of $\psi_{2,0,0} \to \psi_{1,0,0}$ is zero. For the allowed transitions, the radial density $|\langle r_{21}\rangle|^2$ is independent of $\ell$ and $m$.

Symbol $a_0$ is the Bohr radius

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{e^2 m_e} = 5.29\times 10^{-11}\,\text{meter}$$

For the transition frequency we have

$$\omega_{21} = \frac{1}{\hbar}(E_2 - E_1) = \frac{3e^4\mu}{128\pi^2\varepsilon_0^2\hbar^3} = 1.55 \times 10^{16}\,\text{second}^{-1}$$

Hence

$$A_{21} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3} \times \omega_{21}^3 \times \tfrac{32768}{59049}\,a_0^2 = 6.26 \times 10^8\,\text{second}^{-1}$$

It is interesting to work out $A_{nm}$ symbolically and see how high the powers get.

$$A_{21} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3} \times \underbrace{\left(\frac{3e^4\mu}{128\pi^2\varepsilon_0^2\hbar^3}\right)^3}_{\omega_{21}^3} \times \underbrace{\frac{32768}{59049}\left(\frac{4\pi\varepsilon_0\hbar^2}{e^2 m_e}\right)^2}_{|\langle r_{21}\rangle|^2} = \frac{e^{10}\mu^3}{26244\pi^5\varepsilon_0^5\hbar^6 c^3 m_e^2}$$

Let us analyze the units involved in computing $A_{nm}$. For the coefficient of $A_{nm}$ we have

$$\frac{e^2}{3\pi\varepsilon_0\hbar c^3} \propto \frac{\underbrace{\text{ampere}^2\,\text{second}^2}_{e^2}}{\underbrace{\left(\frac{\text{ampere}^2\,\text{second}^4}{\text{kilogram meter}^3}\right)}_{\varepsilon_0}\underbrace{\left(\frac{\text{kilogram meter}^2}{\text{second}}\right)}_{\hbar}\underbrace{\left(\frac{\text{meter}^3}{\text{second}^3}\right)}_{c^3}} = \frac{\text{second}^2}{\text{meter}^2}$$

For the transition frequency we have

$$\omega_{21} = \frac{3e^4\mu}{128\pi^2\varepsilon_0^2\hbar^3} \propto \frac{\underbrace{\left(\text{ampere}^4\,\text{second}^4\right)}_{e^4}\underbrace{\text{kilogram}}_{\mu}}{\underbrace{\left(\frac{\text{ampere}^4\,\text{second}^8}{\text{kilogram}^2\,\text{meter}^6}\right)}_{\varepsilon_0^2}\underbrace{\left(\frac{\text{kilogram}^3\,\text{meter}^6}{\text{second}^3}\right)}_{\hbar^3}} = \text{second}^{-1}$$

For the Bohr radius we have

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{e^2 m_e} \propto \frac{\underbrace{\left(\frac{\text{ampere}^2\,\text{second}^4}{\text{kilogram meter}^3}\right)}_{\varepsilon_0}\underbrace{\left(\frac{\text{kilogram}^2\,\text{meter}^4}{\text{second}^2}\right)}_{\hbar^2}}{\underbrace{\left(\text{ampere}^2\,\text{second}^2\right)}_{e^2}\underbrace{\text{kilogram}}_{m_e}} = \text{meter}$$

Hence

$$A_{nm} \propto \frac{\text{second}^2}{\text{meter}^2} \times \underbrace{\text{second}^{-3}}_{\omega_{nm}^3} \times \underbrace{\text{meter}^2}_{a_0^2} = \text{second}^{-1}$$