(37.1) (a) Verify eqn 37.4.

(b) Show that the rotation matrix

$$D(\theta^3) = e^{-\frac{i}{2}\sigma^3\theta^3} = \begin{pmatrix} e^{-\frac{i\theta^3}{2}} & 0 \\ 0 & e^{\frac{i\theta^3}{2}} \end{pmatrix} \tag{37.22}$$

(a) This is equation (37.4).

$$D(\theta^1) = \exp\left(-\frac{i}{2}\sigma^1\theta^1\right) = I\cos\frac{\theta^1}{2} - i\sigma^1\sin\frac{\theta^1}{2} \tag{37.4}$$

To verify (37.4), start with equation (37.2).

$$D(\boldsymbol{\theta}) = \exp\left(-\frac{i}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\theta}\right) \tag{37.2}$$

Expand the argument of the exponential.

$$D(\boldsymbol{\theta}) = \exp\left(-\frac{i}{2}\left(\sigma^1\theta^1 + \sigma^2\theta^2 + \sigma^3\theta^3\right)\right)$$

Then for $\theta^2 = \theta^3 = 0$ we have

$$D(\theta^1) = \exp\left(-\frac{i}{2}\sigma^1\theta^1\right)$$

Convert to rectangular coordinates.

$$D(\theta^1) = \cos\left(\frac{1}{2}\sigma^1\theta^1\right) - i\sin\left(\frac{1}{2}\sigma^1\theta^1\right) \tag{1}$$

Noting that

$$\left(\sigma^1\right)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

we have

$$\left(\sigma^1\right)^{2n} = I^n = I$$

and

$$\left(\sigma^1\right)^{2n+1} = I^{2n}\sigma^1 = \sigma^1$$

Hence by considering the Taylor expansion of sine and cosine in (1) we have

$$D(\theta^1) = I\cos\frac{\theta^1}{2} - i\sigma^1\sin\frac{\theta^1}{2}$$

(b) Note that

$$\left(\sigma^3\right)^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Hence by the same argument as in part (a) we have

$$D(\theta^3) = I\cos\frac{\theta^3}{2} - i\sigma^3\sin\frac{\theta^3}{2}$$

Rewrite in matrix form.

$$D(\theta^3) = \begin{pmatrix} \cos\frac{\theta^3}{2} - i\sin\frac{\theta^3}{2} & 0 \\ 0 & \cos\frac{\theta^3}{2} + i\sin\frac{\theta^3}{2} & 0 \end{pmatrix} = \begin{pmatrix} e^{-\frac{i\theta^3}{2}} & 0 \\ 0 & e^{\frac{i\theta^3}{2}} \end{pmatrix}$$