(13.4) The Lagrangian for electromagnetism in vacuo is $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$. Show that this can be rewritten as

$$\mathcal{L} = -\frac{1}{2} \left(\partial_{\mu} A_{\nu} \partial^{\mu} A^{\nu} - \partial_{\mu} A_{\nu} \partial^{\nu} A^{\mu} \right) \tag{13.42}$$

and hence show that using the transverse projection operator, it may be expressed as

$$\mathcal{L} = \frac{1}{2} A^{\mu} P_{\mu\nu}^T \partial^2 A^{\nu} \tag{13.43}$$

This shows that \mathcal{L} only includes the transverse components of the field, squaring with the idea of electromagnetic waves only representing vibrations transverse to the direction of propagation.

Consider equation (5.28).

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \tag{5.28}$$

It follows that

$$F^{\mu\nu}F_{\mu\nu} = (\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu})(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})$$

Expand the right-hand side.

$$F^{\mu\nu}F_{\mu\nu} = \partial^{\mu}A^{\nu}\partial_{\mu}A_{\nu} - \partial^{\mu}A^{\nu}\partial_{\nu}A_{\mu} - \partial^{\nu}A^{\mu}\partial_{\mu}A_{\nu} + \partial^{\nu}A^{\mu}\partial_{\nu}A_{\mu}$$

Make sums explicit.

$$F^{\mu\nu}F_{\mu\nu} = \underbrace{\sum_{\mu,\nu} \partial^{\mu}A^{\nu}\partial_{\mu}A_{\nu}}_{\Sigma_{1}} - \underbrace{\sum_{\mu,\nu} \partial^{\mu}A^{\nu}\partial_{\nu}A_{\mu}}_{\Sigma_{2}} - \underbrace{\sum_{\mu,\nu} \partial^{\nu}A^{\mu}\partial_{\mu}A_{\nu}}_{\Sigma_{3}} + \underbrace{\sum_{\mu,\nu} \partial^{\nu}A^{\mu}\partial_{\nu}A_{\mu}}_{\Sigma_{4}}$$

Noting that $\Sigma_1 = \Sigma_4$ and $\Sigma_2 = \Sigma_3$ we have

$$F^{\mu\nu}F_{\mu\nu} = 2\sum_{\mu,\nu}\partial^{\mu}A^{\nu}\partial_{\mu}A_{\nu} - 2\sum_{\mu,\nu}\partial^{\mu}A^{\nu}\partial_{\nu}A_{\mu}$$

Hence

$$F^{\mu\nu}F_{\mu\nu} = 2\left(\partial^{\mu}A^{\nu}\partial_{\mu}A_{\nu} - \partial^{\mu}A^{\nu}\partial_{\nu}A_{\mu}\right)$$

and

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{2}\left(\partial^{\mu}A^{\nu}\partial_{\mu}A_{\nu} - \partial^{\mu}A^{\nu}\partial_{\nu}A_{\mu}\right)$$