Atomic transitions 4

From the previous section

$$R_{a\to b} = \frac{\pi e^2 \rho(\omega_0)}{\varepsilon_0 m^2 \hbar^2 \omega_0^2} \left| \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} \exp(i\mathbf{k}\cdot\mathbf{r}) | \psi_a \rangle \right|^2$$

Substitute the dipole approximation

$$\exp(i\mathbf{k}\cdot\mathbf{r}) \approx 1$$

to obtain

$$R_{a\to b} = \frac{\pi e^2 \rho(\omega_0)}{\varepsilon_0 m^2 \hbar^2 \omega_0^2} \left| \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle \right|^2$$

By the identity

$$\mathbf{p} = \frac{im}{\hbar}[H_0, \mathbf{r}] \tag{1}$$

we have

$$\begin{aligned}
\langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{p} | \psi_a \rangle &= \frac{im}{\hbar} \langle \psi_b | \boldsymbol{\epsilon} \cdot [H_0, \mathbf{r}] | \psi_a \rangle \\
&= \frac{im}{\hbar} \langle \psi_b | \boldsymbol{\epsilon} \cdot H_0 \mathbf{r} | \psi_a \rangle - \frac{im}{\hbar} \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} H_0 | \psi_a \rangle \\
&= \frac{im}{\hbar} E_b \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle - \frac{im}{\hbar} E_a \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle \\
&= \frac{im}{\hbar} (E_b - E_a) \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle \\
&= im\omega_0 \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle
\end{aligned}$$

Hence

$$R_{a\to b} = \frac{\pi e^2 \rho(\omega_0)}{\varepsilon_0 \hbar^2} \left| \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle \right|^2$$

Verify dimensions.

$$R_{a\to b} \propto \frac{\overset{e^2}{\mathrm{C}^2} \quad \overset{\rho(\omega)}{\mathrm{J\,m^{-3}\,s}}}{\underset{\mathrm{C^2\,J^{-1}\,m^{-1}}}{\epsilon_0} \quad \underset{\mathrm{J^2\,s^2}}{\hbar^2}} \times \underset{\mathrm{m}^2}{\left| \langle \psi_b | \boldsymbol{\epsilon} \cdot \mathbf{r} | \psi_a \rangle \right|^2} = \mathrm{s}^{-1}$$