Rutherford scattering 1

Use the following formula to compute the cross section for Rutherford scattering.

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2} \left(\frac{mQ}{4\pi\hbar^2} \right)^2, \quad Q = \int \exp\left(\frac{i\mathbf{p} \cdot \mathbf{r}}{\hbar} \right) V(\mathbf{r}) \, d\mathbf{r}^3$$

Convert Q to polar coordinates.

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar} \right) V(r, \theta, \phi) \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

For Rutherford scattering $V(\mathbf{r})$ is the Coulomb potential

$$V(\mathbf{r}) = V(r) = -\frac{Ze^2}{r}$$

Substitute the Coulomb potential for $V(r, \theta, \phi)$ and note r^2 becomes r.

$$Q = -Ze^2 \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar} \right) r \sin\theta \, dr \, d\theta \, d\phi$$

Integrate over ϕ.

$$Q = -2\pi Ze^2 \int_0^{\pi} \int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar} \right) r \sin\theta \, dr \, d\theta$$

Change the complex exponential to rectangular form.

$$Q = -2\pi Ze^2 \int_0^{\pi} \int_0^{\infty} \left[\cos\left(\frac{pr\cos\theta}{\hbar} \right) + i\sin\left(\frac{pr\cos\theta}{\hbar} \right) \right] V(r) \, r \sin\theta \, dr \, d\theta$$

By the integrals

$$\int_0^{\pi} \cos\left(a\cos(\theta) \right) \sin\theta \, d\theta = \frac{2\sin a}{a}, \quad \int_0^{\pi} \sin\left(a\cos(\theta) \right) \sin\theta \, d\theta = 0$$

we obtain (note r in the integrand is canceled)

$$Q = -\frac{4\pi\hbar Ze^2}{p} \int_0^{\infty} \sin\left(\frac{pr}{\hbar} \right) dr$$

To solve the integral, multiply the integrand by $\exp(-\epsilon r)$.

$$Q = -\frac{4\pi\hbar Ze^2}{p} \int_0^{\infty} \sin\left(\frac{pr}{\hbar} \right) \exp(-\epsilon r) \, dr$$

Convert the integrand to exponential form.

$$Q = -\frac{4\pi\hbar Ze^2}{p} \int_0^{\infty} \frac{i}{2} \left[\exp\left(-\frac{ipr}{\hbar} - \epsilon r \right) - \exp\left(\frac{ipr}{\hbar} - \epsilon r \right) \right] dr$$

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Solve the integral.

$$Q = -\frac{4\pi\hbar Ze^2}{p}\frac{i}{2}\left(\frac{1}{-ip/\hbar - \epsilon} - \frac{1}{ip/\hbar - \epsilon}\right)$$ (1)

Set $\epsilon = 0$.

$$Q = -\frac{4\pi\hbar Ze^2}{p}\left(-\frac{\hbar}{p}\right)$$

Hence

$$Q = \frac{4\pi\hbar^2 Ze^2}{p^2}$$

Compute the differential cross section.

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}\left(\frac{mQ}{4\pi\hbar^2}\right)^2 = \frac{1}{16\pi^2}\frac{m^2 Z^2 e^4}{p^4}$$ (2)

Substitute $16\pi^2\alpha^2$ for e^4.

$$\frac{d\sigma}{d\Omega} = \frac{m^2 Z^2 \alpha^2}{p^4}$$

Symbol p is momentum transfer $|\mathbf{p}_i| - |\mathbf{p}_f|$ such that

$$p^2 = 2mE(\cos\theta - 1)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4E^2(\cos\theta - 1)^2}$$ (3)