Show that

$$\int \frac{\psi_1^2 \psi_2^2}{r_{12}}\, dV_1\, dV_2 = \tfrac{5}{8}\alpha$$

where

$$\psi_j = \sqrt{\frac{\alpha^3}{\pi}} \exp(-\alpha r_j)$$

and

$$r_{12} = \sqrt{r_1^2 + r_2^2 - r_1 r_2 \cos\theta_{12}}$$

Symbol θ_{12} is angular separation.

Let $I(r_1)$ be the following integral over V_2.

$$I(r_1) = \int \frac{\psi_2^2}{r_{12}}\, dV_2$$

The measure dV_2 is a volume element in spherical coordinates.

$$dV_2 = r_2^2 \sin\theta_2\, dr_2\, d\theta_2\, d\phi_2$$

Write out the full integral and make $\theta_2 = \theta_{12}$ by independence of the coordinate system.

$$I(r_1) = \frac{\alpha^3}{\pi} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \frac{\exp(-2\alpha r_2)}{\sqrt{r_1^2 + r_2^2 - r_1 r_2 \cos\theta_2}}\, r_2^2 \sin\theta_2\, dr_2\, d\theta_2\, d\phi_2$$

Integrate over ϕ_2.

$$I(r_1) = 2\alpha^3 \int_0^{\pi} \int_0^{\infty} \frac{\exp(-2\alpha r_2)}{\sqrt{r_1^2 + r_2^2 - r_1 r_2 \cos\theta_2}}\, r_2^2 \sin\theta_2\, dr_2\, d\theta_2$$

Expand $1/r_{12}$ in Legendre polynomials. The first integral is over $r_2 < r_1$ and the second is over $r_2 > r_1$.

$$\begin{aligned} I(r_1) = 2\alpha^3 \int_0^{\pi} \int_0^{r_1} \exp(-2\alpha r_2) \left(\sum_{k=0}^{\infty} \frac{r_2^k}{r_1^{k+1}} P_k(\cos\theta_2) \right) r_2^2 \sin\theta_2\, dr_2\, d\theta_2 \\ + 2\alpha^3 \int_0^{\pi} \int_{r_1}^{\infty} \exp(-2\alpha r_2) \left(\sum_{k=0}^{\infty} \frac{r_1^k}{r_2^{k+1}} P_k(\cos\theta_2) \right) r_2^2 \sin\theta_2\, dr_2\, d\theta_2 \end{aligned}$$

It turns out that

$$\int_0^{\pi} P_k(\cos\theta_2) \sin\theta_2\, d\theta_2 = \begin{cases} 2, & k = 0 \\ 0, & k > 0 \end{cases}$$

Hence

$$I(r_1) = \frac{4\alpha^3}{r_1} \int_0^{r_1} \exp(-2\alpha r_2)\, r_2^2\, dr_2 + 4\alpha^3 \int_{r_1}^{\infty} \exp(-2\alpha r_2)\, r_2\, dr_2$$

Solve the integrals.

$$I(r_1) = \frac{4\alpha^3}{r_1}\exp(-2\alpha r_2)\left(-\frac{r_2^2}{2\alpha} - \frac{r_2}{2\alpha^2} - \frac{1}{4\alpha^3}\right)\Bigg|_0^{r_1} + 4\alpha^3\exp(-2\alpha r_2)\left(-\frac{r_2}{2\alpha} - \frac{1}{4\alpha^2}\right)\Bigg|_{r_1}^{\infty}$$

Evaluate per limits.

$$I(r_1) = \frac{1}{r_1} - \frac{1}{r_1}\exp(-2\alpha r_1) - \alpha\exp(-2\alpha r_1)$$

Having obtained $I(r_1)$ we can now evaluate the integral over V_1.

$$I = \frac{\alpha^3}{\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\exp(-2\alpha r_1)I(r_1)\,r_1^2\sin\theta_1\,dr_1\,d\theta_1\,d\phi_1$$

Integrate over θ_1 and ϕ_1.

$$I = 4\alpha^3\int_0^{\infty}\exp(-2\alpha r_1)I(r_1)\,r_1^2\,dr_1$$

Expand the integrand.

$$I = 4\alpha^3\int_0^{\infty}\exp(-2\alpha r_1)\,r_1\,dr_1 - 4\alpha^3\int_0^{\infty}\exp(-4\alpha r_1)\,r_1\,dr_1 - 4\alpha^4\int_0^{\infty}\exp(-4\alpha r_1)\,r_1^2\,dr_1$$

Solve the integrals.

$$I = \exp(-2\alpha r_1)\left(-2\alpha^2 r_1 - \alpha\right)\Bigg|_0^{\infty} - \exp(-4\alpha r_1)(-\alpha^2 r_1 - \tfrac{1}{4}\alpha)\Bigg|_0^{\infty} - \exp(-4\alpha r_1)\left(-\alpha^3 r_1^2 - \tfrac{1}{2}\alpha^2 r_1 - \tfrac{1}{8}\alpha\right)\Bigg|_0^{\infty}$$

The result vanishes for $r_1 = \infty$ hence

$$I = 0 - \left(-\alpha + \tfrac{1}{4}\alpha + \tfrac{1}{8}\alpha\right) = \tfrac{5}{8}\alpha$$