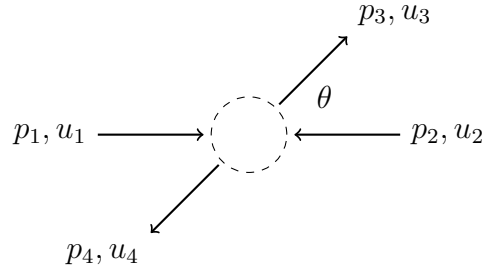Moller scattering is the result of interactions between electrons. The following diagram shows the geometry of a typical collider experiment that generates Moller scattering data.

Here is the same diagram with momentum and spinor labels.

In center of mass coordinates the momentum vectors are

$$
p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \qquad
p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} \qquad
p_3 = \begin{pmatrix} E \\ p \sin\theta \cos\phi \\ p \sin\theta \sin\phi \\ p \cos\theta \end{pmatrix} \qquad
p_4 = \begin{pmatrix} E \\ -p \sin\theta \cos\phi \\ -p \sin\theta \sin\phi \\ -p \cos\theta \end{pmatrix}
$$
$$
\text{inbound electron} \qquad \text{inbound electron} \qquad \text{outbound electron} \qquad \text{outbound electron}
$$

Symbol p is incident momentum, E is total energy $E = \sqrt{p^2 + m^2}$, and m is electron mass. Polar angle θ is the observed scattering angle. Azimuth angle ϕ cancels out in scattering calculations.

The spinors are

$$
u_{11} = \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix} \qquad
u_{12} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix} \qquad
u_{21} = \begin{pmatrix} E+m \\ 0 \\ -p \\ 0 \end{pmatrix} \qquad
u_{22} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ p \end{pmatrix}
$$
$$
\text{inbound electron, spin up} \quad \text{inbound electron, spin down} \quad \text{inbound electron, spin up} \quad \text{inbound electron, spin down}
$$

$$
u_{31} = \begin{pmatrix} E+m \\ 0 \\ p_{3z} \\ p_{3x}+ip_{3y} \end{pmatrix} \qquad
u_{32} = \begin{pmatrix} 0 \\ E+m \\ p_{3x}-ip_{3y} \\ -p_{3z} \end{pmatrix} \qquad
u_{41} = \begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x}+ip_{4y} \end{pmatrix} \qquad
u_{42} = \begin{pmatrix} 0 \\ E+m \\ p_{4x}-ip_{4y} \\ -p_{4z} \end{pmatrix}
$$
$$
\text{outbound electron, spin up} \quad \text{outbound electron, spin down} \quad \text{outbound electron, spin up} \quad \text{outbound electron, spin down}
$$

The spinors shown above are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^4$ will be used.

The following formula computes a probability density $|\mathcal{M}_{abcd}|^2$ for Moller scattering where the subscripts $abcd$ are the spin states of the electrons.

$$|\mathcal{M}_{abcd}|^2 = \frac{e^4}{N} \left| \frac{1}{t}(\bar{u}_{3c}\gamma^\mu u_{1a})(\bar{u}_{4d}\gamma_\mu u_{2b}) - \frac{1}{u}(\bar{u}_{4d}\gamma^\nu u_{1a})(\bar{u}_{3c}\gamma_\nu u_{2b}) \right|^2$$

Symbol e is electron charge. Symbols t and u are Mandelstam variables

$$t = (p_1 - p_3)^2$$
$$u = (p_1 - p_4)^2$$

Let

$$a_1 = (\bar{u}_{3c}\gamma^\mu u_{1a})(\bar{u}_{4d}\gamma_\mu u_{2b}) \qquad a_2 = (\bar{u}_{4d}\gamma^\nu u_{1a})(\bar{u}_{3c}\gamma_\nu u_{2b})$$

Then

$$\begin{aligned}
|\mathcal{M}_{abcd}|^2 &= \frac{e^4}{N} \left| \frac{a_1}{t} - \frac{a_2}{u} \right|^2 \\
&= \frac{e^4}{N} \left(\frac{a_1}{t} - \frac{a_2}{u} \right) \left(\frac{a_1}{t} - \frac{a_2}{u} \right)^* \\
&= \frac{e^4}{N} \left(\frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{tu} - \frac{a_1^* a_2}{tu} + \frac{a_2 a_2^*}{u^2} \right)
\end{aligned}$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is computed by summing $|\mathcal{M}_{abcd}|^2$ over all spin states and dividing by the number of inbound states. There are four inbound states.

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{1}{4} \sum_{a=1}^{2} \sum_{b=1}^{2} \sum_{c=1}^{2} \sum_{d=1}^{2} |\mathcal{M}_{abcd}|^2 \\
&= \frac{e^4}{4N} \sum_{a=1}^{2} \sum_{b=1}^{2} \sum_{c=1}^{2} \sum_{d=1}^{2} \left(\frac{a_1 a_1^*}{t^2} - \frac{a_1 a_2^*}{tu} - \frac{a_1^* a_2}{tu} + \frac{a_2 a_2^*}{u^2} \right)
\end{aligned}$$

Use the Casimir trick to replace sums over spins with matrix products.

$$f_{11} = \frac{1}{N} \sum_{abcd} a_1 a_1^* = \text{Tr}\left((\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu \right) \text{Tr}\left((\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu \right)$$

$$f_{12} = \frac{1}{N} \sum_{abcd} a_1 a_2^* = \text{Tr}\left((\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu \right)$$

$$f_{22} = \frac{1}{N} \sum_{abcd} a_2 a_2^* = \text{Tr}\left((\not{p}_4 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu \right) \text{Tr}\left((\not{p}_3 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu \right)$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left(\frac{f_{11}}{t^2} - \frac{f_{12}}{tu} - \frac{f_{12}^*}{tu} + \frac{f_{22}}{u^2} \right)$$

Run "moller-scattering-1.txt" to verify the Casimir trick.

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$\begin{aligned}
f_{11} &= 32(p_1 \cdot p_2)^2 + 32(p_1 \cdot p_4)^2 - 64m^2(p_1 \cdot p_3) + 64m^4 \\
f_{12} &= -32(p_1 \cdot p_2)^2 + 32m^2(p_1 \cdot p_2) + 32m^2(p_1 \cdot p_3) + 32m^2(p_1 \cdot p_4) - 32m^4 \\
f_{22} &= 32(p_1 \cdot p_2)^2 + 32(p_1 \cdot p_3)^2 - 64m^2(p_1 \cdot p_4) + 64m^4
\end{aligned}$$

In Mandelstam variables

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$
$$u = (p_1 - p_4)^2$$

the formulas are

$$f_{11} = 8s^2 + 8u^2 - 64sm^2 - 64um^2 + 192m^4$$
$$f_{12} = -8s^2 + 64sm^2 - 96m^4$$
$$f_{22} = 8s^2 + 8t^2 - 64sm^2 - 64tm^2 + 192m^4$$

Run "moller-scattering-2.txt" to verify.

High energy approximation

When $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$f_{11} = 8s^2 + 8u^2$$
$$f_{12} = -8s^2$$
$$f_{22} = 8s^2 + 8t^2$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left(\frac{f_{11}}{t^2} - \frac{f_{12}}{tu} - \frac{f_{12}^*}{tu} + \frac{f_{22}}{u^2} \right)$$
$$= \frac{e^4}{4} \left(\frac{8s^2 + 8u^2}{t^2} - \frac{-8s^2}{tu} - \frac{-8s^2}{tu} + \frac{8s^2 + 8t^2}{u^2} \right)$$
$$= 2e^4 \left(\frac{s^2 + u^2}{t^2} + \frac{2s^2}{tu} + \frac{s^2 + t^2}{u^2} \right)$$

Combine terms so $\langle |\mathcal{M}|^2 \rangle$ has a common denominator.

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{u^2(s^2 + u^2) + 2s^2 tu + t^2(s^2 + t^2)}{t^2 u^2} \right)$$

For $m = 0$ the Mandelstam variables are

$$s = 4E^2$$
$$t = 2E^2(\cos\theta - 1)$$
$$u = -2E^2(\cos\theta + 1)$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{32E^8 \cos^4\theta + 192E^8 \cos^2\theta + 288E^8}{16E^8 (\cos\theta - 1)^2 (\cos\theta + 1)^2} \right)$$
$$= 4e^4 \frac{(\cos^2\theta + 3)^2}{(\cos\theta - 1)^2 (\cos\theta + 1)^2}$$
$$= 4e^4 \left(\frac{\cos^2\theta + 3}{\cos^2\theta - 1} \right)^2$$

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Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle |\mathcal{M}|^2 \rangle}{64\pi^2 s} = \frac{e^4}{16\pi^2 s} \left(\frac{\cos^2\theta + 3}{\cos^2\theta - 1} \right)^2, \quad s \gg m$$

Substituting $e^4 = 16\pi^2\alpha^2$ yields

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s} \left(\frac{\cos^2\theta + 3}{\cos^2\theta - 1} \right)^2$$

We can integrate $d\sigma$ to obtain a cumulative distribution function. Recall that

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

Hence

$$d\sigma = \frac{\alpha^2}{s} \left(\frac{\cos^2\theta + 3}{\cos^2\theta - 1} \right)^2 \sin\theta \, d\theta \, d\phi$$

Let $I(\theta)$ be the following integral of $d\sigma$.

$$I(\theta) = \frac{s}{2\pi\alpha^2} \int_0^{2\pi} \int d\sigma$$
$$= \int \left(\frac{\cos^2\theta + 3}{\cos^2\theta - 1} \right)^2 \sin\theta \, d\theta, \quad a \le \theta \le \pi - a$$

Angular support is limited to an arbitrary $a > 0$ because $I(0)$ and $I(\pi)$ are undefined. Assume that $I(\theta) - I(a)$ is computable given θ by either symbolic or numerical integration.

Let C be the normalization constant

$$C = I(\pi - a) - I(a)$$

Then the cumulative distribution function $F(\theta)$ is

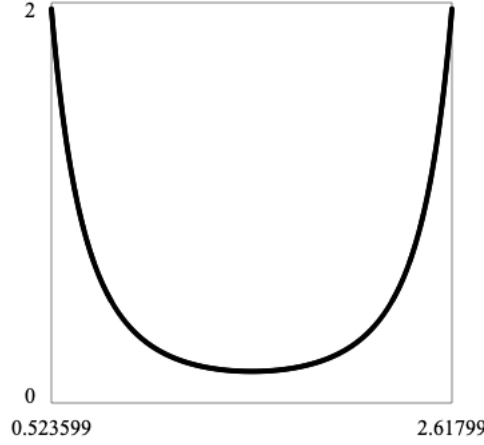$$F(\theta) = \frac{I(\theta) - I(a)}{C}, \quad a \le \theta \le \pi - a$$

The probability of observing scattering events in the interval θ_1 to θ_2 can now be computed.

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

Probability density function $f(\theta)$ is the derivative of $F(\theta)$.

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{C} \left(\frac{\cos^2\theta + 3}{\cos^2\theta - 1} \right)^2 \sin\theta$$

Run "moller-scattering-3.txt" to draw a graph of $f(\theta)$ for $a = \pi/6 = 30°$.

Probability distribution for $30°$ bins ($a = 30°$).

θ_1	θ_2	$P(\theta_1 \leq \theta \leq \theta_2)$
$0°$	$30°$	$-$
$30°$	$60°$	0.40
$60°$	$90°$	0.10
$90°$	$120°$	0.10
$120°$	$150°$	0.40
$150°$	$180°$	$-$

Notes

In component notation, the trace operators of the Casimir trick become sums over a repeated index, in this case α.

$$f_{11} = \left((\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \right) \left((\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \right)$$

$$f_{12} = (\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_2 + m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha$$

$$f_{22} = \left((\not{p}_4 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \right) \left((\not{p}_3 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \right)$$

To convert the above formulas to Eigenmath code, the γ tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply γ^μ by the metric tensor to lower the index.

$$\gamma^{\beta\mu}{}_\rho \quad \to \quad \texttt{gammaT = transpose(gamma)}$$

$$\gamma^\beta{}_{\mu\rho} \quad \to \quad \texttt{gammaL = transpose(dot(gmunu,gamma))}$$

Define the following 4×4 matrices.

$$(\not{p}_1 + m) \quad \to \quad \texttt{X1 = pslash1 + m I}$$

$$(\not{p}_2 + m) \quad \to \quad \texttt{X2 = pslash2 + m I}$$

$$(\not{p}_3 + m) \quad \to \quad \texttt{X3 = pslash3 + m I}$$

$$(\not{p}_4 + m) \quad \to \quad \texttt{X4 = pslash4 + m I}$$

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Then for f_{11} we have the following Eigenmath code. The contract function sums over α.

$$(\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \quad \rightarrow \quad \texttt{T1 = contract(dot(X3,gammaT,X1,gammaT),1,4)}$$

$$(\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \quad \rightarrow \quad \texttt{T2 = contract(dot(X4,gammaL,X2,gammaL),1,4)}$$

Next, multiply then sum over repeated indices. The dot function sums over ν then the contract function sums over μ. The transpose makes the ν indices adjacent as required by the dot function.

$$f_{11} = \text{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu)\,\text{Tr}(\cdots \gamma_\mu \cdots \gamma_\nu) \quad \rightarrow \quad \texttt{contract(dot(T1,transpose(T2)))}$$

Follow suit for f_{22}.

$$(\not{p}_4 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \quad \rightarrow \quad \texttt{T1 = contract(dot(X4,gammaT,X1,gammaT),1,4)}$$

$$(\not{p}_3 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \quad \rightarrow \quad \texttt{T2 = contract(dot(X3,gammaL,X2,gammaL),1,4)}$$

Then

$$f_{22} = \text{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu)\,\text{Tr}(\cdots \gamma_\mu \cdots \gamma_\nu) \quad \rightarrow \quad \texttt{contract(dot(T1,transpose(T2)))}$$

The calculation of f_{12} begins with

$$(\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_2 + m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha$$

$$\rightarrow \quad \texttt{T = contract(dot(X3,gammaT,X1,gammaT,X4,gammaL,X2,gammaL),1,6)}$$

Then sum over repeated indices μ and ν.

$$f_{12} = \text{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu \cdots \gamma_\mu \cdots \gamma_\nu) \quad \rightarrow \quad \texttt{contract(contract(T,1,3))}$$