Note: $r = |\mathbf{r}|$

$$\left(\nabla^2 + k^2\right) G(\mathbf{r}) = \delta^3(\mathbf{r}) \tag{10.52}$$

$$G(\mathbf{r}) = -\frac{e^{ikr}}{4\pi r} \tag{10.65}$$

Recall $\nabla^2 = \nabla \cdot \nabla$ and

$$\nabla \cdot (f\mathbf{A}) = \nabla f \cdot \mathbf{A} + f\nabla\cdot\mathbf{A}$$

Hence

$$\begin{aligned}
\nabla^2 G(\mathbf{r}) &= -\frac{1}{4\pi}\nabla \cdot \nabla\left(\frac{e^{ikr}}{r}\right) \\
&= -\frac{1}{4\pi}\nabla \cdot \left(\frac{1}{r}\nabla e^{ikr} + e^{ikr}\nabla\frac{1}{r}\right) \\
&= -\frac{1}{4\pi}\Bigg(\underbrace{\nabla\frac{1}{r}\cdot\nabla e^{ikr}}_{\text{see (2)}} + \underbrace{\frac{1}{r}\nabla^2 e^{ikr}}_{\text{see (3)}} + \underbrace{\nabla e^{ikr}\cdot\nabla\frac{1}{r}}_{\text{see (2)}} + e^{ikr}\nabla^2\frac{1}{r}\Bigg) \qquad (1)
\end{aligned}$$

In spherical coordinates

$$\nabla\frac{1}{r}\cdot\nabla e^{ikr} = \begin{pmatrix} -1/r^2 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} ike^{ikr} \\ 0 \\ 0 \end{pmatrix} = -\frac{ike^{ikr}}{r^2} \tag{2}$$

and

$$\begin{aligned}
\frac{1}{r}\nabla^2 e^{ikr} &= \frac{1}{r^2}\frac{\partial^2}{\partial r^2}(re^{ikr}) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(e^{ikr} + ikre^{ikr}\right) \\
&= \frac{1}{r^2}\left(2ike^{ikr} - k^2 re^{ikr}\right) \\
&= \frac{2ike^{ikr}}{r^2} - \frac{k^2 e^{ikr}}{r} \qquad (3)
\end{aligned}$$

Substitute into (1) to obtain

$$\begin{aligned}
\nabla^2 G(\mathbf{r}) &= \frac{k^2 e^{ikr}}{4\pi r} - \frac{1}{4\pi}e^{ikr}\nabla^2\frac{1}{r} \\
&= -k^2 G(\mathbf{r}) + \delta^3(\mathbf{r})e^{ikr}
\end{aligned}$$

Noting that $e^{ikr} = 1$ for $r = 0$ we have

$$\nabla^2 G(\mathbf{r}) = -k^2 G(\mathbf{r}) + \delta^3(\mathbf{r}) \tag{4}$$