

Let

$$f(x, y, z, t) = p_x x + p_y y + p_z z - Et$$

where

$$E = \sqrt{p_x^2 + p_y^2 + p_z^2 + m^2}$$

The following solutions to the Dirac equation are used for quantum electrodynamics.

$$\begin{aligned} \psi_1 &= \begin{pmatrix} E + m \\ 0 \\ p_z \\ p_x + ip_y \end{pmatrix} \exp(if(x, y, z, t)) & \psi_2 &= \begin{pmatrix} 0 \\ E + m \\ p_x - ip_y \\ -p_z \end{pmatrix} \exp(if(x, y, z, t)) \\ &\text{fermion spin up} & & \text{fermion spin down} \\ \psi_7 &= \begin{pmatrix} p_z \\ p_x + ip_y \\ E + m \\ 0 \end{pmatrix} \exp(-if(x, y, z, t)) & \psi_8 &= \begin{pmatrix} p_x - ip_y \\ -p_z \\ 0 \\ E + m \end{pmatrix} \exp(-if(x, y, z, t)) \\ &\text{anti-fermion spin up} & & \text{anti-fermion spin down} \end{aligned}$$

A spinor is the vector part of each solution. The following eight spinors are used for scattering calculations. The u spinors are fermions from ψ_1 and ψ_2 . The v spinors are anti-fermions from ψ_7 and ψ_8 . The last digit of the u or v subscript is 1 for spin up and 2 for spin down.

$$\begin{aligned} u_{11} &= \begin{pmatrix} E_1 + m_1 \\ 0 \\ p_{1z} \\ p_{1x} + ip_{1y} \end{pmatrix} & v_{21} &= \begin{pmatrix} p_{2z} \\ p_{2x} + ip_{2y} \\ E_2 + m_2 \\ 0 \end{pmatrix} & u_{31} &= \begin{pmatrix} E_3 + m_3 \\ 0 \\ p_{3z} \\ p_{3x} + ip_{3y} \end{pmatrix} & v_{41} &= \begin{pmatrix} p_{4z} \\ p_{4x} + ip_{4y} \\ E_4 + m_4 \\ 0 \end{pmatrix} \\ u_{12} &= \begin{pmatrix} 0 \\ E_1 + m_1 \\ p_{1x} - ip_{1y} \\ -p_{1z} \end{pmatrix} & v_{22} &= \begin{pmatrix} p_{2x} - ip_{2y} \\ -p_{2z} \\ 0 \\ E_2 + m_2 \end{pmatrix} & u_{32} &= \begin{pmatrix} 0 \\ E_3 + m_3 \\ p_{3x} - ip_{3y} \\ -p_{3z} \end{pmatrix} & v_{42} &= \begin{pmatrix} p_{4x} - ip_{4y} \\ -p_{4z} \\ 0 \\ E_4 + m_4 \end{pmatrix} \end{aligned}$$

These are the associated momentum vectors.

$$p_1 = \begin{pmatrix} E_1 \\ p_{1x} \\ p_{1y} \\ p_{1z} \end{pmatrix} \quad p_2 = \begin{pmatrix} E_2 \\ p_{2x} \\ p_{2y} \\ p_{2z} \end{pmatrix} \quad p_3 = \begin{pmatrix} E_3 \\ p_{3x} \\ p_{3y} \\ p_{3z} \end{pmatrix} \quad p_4 = \begin{pmatrix} E_4 \\ p_{4x} \\ p_{4y} \\ p_{4z} \end{pmatrix}$$

Spinors are solutions to the following momentum-space Dirac equation with $\not{p} = p \cdot (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$.

$$(\not{p} - m)u = 0 \quad (\not{p} + m)v = 0$$

Up and down spinors have the following “completeness property.”

$$u_{11}\bar{u}_{11} + u_{12}\bar{u}_{12} = (E_1 + m_1)(\not{p}_1 + m_1) \quad v_{21}\bar{v}_{21} + v_{22}\bar{v}_{22} = (E_2 + m_2)(\not{p}_2 - m_2)$$

The adjoint of a spinor is $\bar{u} = u^\dagger \gamma^0$. The adjoint is a row vector hence $u\bar{u}$ is an outer product.