(1.6) Show that if $Z_0[J]$ is given by

$$Z_0[J] = \exp\left(-\frac{1}{2}\int d^4x\, d^4y\, J(x)\Delta(x-y)J(y)\right) \tag{1.49}$$

where $\Delta(x) = \Delta(-x)$ then

$$\frac{\delta Z_0[J]}{\delta J(z_1)} = -\left[\int d^4y\, \Delta(z_1 - y)J(y)\right] Z_0[J] \tag{1.50}$$

Let

$$I = \int d^4x\, d^4y\, \big(J(x) + \epsilon\delta(x-z_1)\big)\Delta(x-y)\big(J(y) + \epsilon\delta(y-z_1)\big)$$

Then

$$\frac{\delta Z_0[J]}{\delta J(z_1)} = \lim_{\epsilon\to 0}\frac{\exp(-I/2) - Z_0[J]}{\epsilon} \tag{1}$$

Expand the integrand of I to obtain

$$I = I_1 + \epsilon I_2 + \epsilon I_3 + \epsilon^2 I_4$$

where

$$\begin{aligned}
I_1 &= \int d^4x\, d^4y\, J(x)\Delta(x-y)J(y)\\
I_2 &= \int d^4x\, d^4y\, \delta(x-z_1)\Delta(x-y)J(y) = \int d^4y\, \Delta(z_1-y)J(y)\\
I_3 &= \int d^4x\, d^4y\, J(x)\Delta(x-y)\delta(y-z_1) = \int d^4x\, \Delta(x-z_1)J(x)\\
I_4 &= \int d^4x\, d^4y\, \delta(x-z_1)\Delta(x-y)\delta(y-z_1) = \Delta(0)
\end{aligned}$$

Noting that $\Delta(x-y) = \Delta(y-x)$ we have

$$I_2 + I_3 = 2I_2$$

The product $\epsilon^2 I_4$ vanishes in the limit hence

$$I = I_1 + 2\epsilon I_2 \tag{2}$$

Substitute (2) into (1) to obtain

$$\frac{\delta Z_0[J]}{\delta J(z_1)} = \lim_{\epsilon\to 0} \frac{\exp(-I_1/2 - \epsilon I_2) - Z_0[J]}{\epsilon}$$

Let $X = I_1/2$ and $\epsilon = h/I_2$. Then

$$\begin{aligned}\frac{\delta Z_0[J]}{\delta J(z_1)} &= I_2 \lim_{h\to 0} \frac{\exp(-X-h) - \exp(-X)}{h} \\ &= I_2 \frac{d}{dX} \exp(-X) \\ &= -I_2 \exp(-X)\end{aligned}$$

Hence

$$\frac{\delta Z_0[J]}{\delta J(z_1)} = -\left(\int d^4y\, \Delta(z_1 - y) J(y)\right) Z_0[J]$$