Mott problem

Consider the emission of an α particle in a cloud chamber. The quantum mechanical model of the particle is a spherical wave emanating from the origin. A spherical wave should ionize atoms throughout the cloud chamber. However, only straight tracks are observed. Neville Mott used the Schrodinger equation to explain why straight tracks are observed.

Let $\mathbf{R}$ be the position of the α particle, let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the positions of the free electrons, and let $\mathbf{a}_1$ and $\mathbf{a}_2$ be the positions of the first two atoms ionized by the α particle. The Hamiltonian for the system is

$$\hat{H} = \hat{K}_\alpha + \hat{K}_1 + \hat{K}_2 + V_1 + V_2 + U_1 + U_2$$

where

$\hat{K}_\alpha = -\dfrac{\hbar^2}{2M}\nabla_\alpha^2$	kinetic energy of α particle
$\hat{K}_1 = -\dfrac{\hbar^2}{2m}\nabla_1^2$	kinetic energy of 1st electron
$\hat{K}_2 = -\dfrac{\hbar^2}{2m}\nabla_2^2$	kinetic energy of 2nd electron
$V_1 = -\dfrac{e^2}{\lvert\mathbf{r}_1 - \mathbf{a}_1\rvert}$	potential energy of 1st electron
$V_2 = -\dfrac{e^2}{\lvert\mathbf{r}_2 - \mathbf{a}_2\rvert}$	potential energy of 2nd electron
$U_1 = -\dfrac{2e^2}{\lvert\mathbf{R} - \mathbf{r}_1\rvert}$	potential energy of α and 1st electron
$U_2 = -\dfrac{2e^2}{\lvert\mathbf{R} - \mathbf{r}_2\rvert}$	potential energy of α and 2nd electron

Let ψ_1 and ψ_2 be atomic wavefunctions such that

$$\hat{H}\psi_1 = E_1\psi_1, \quad \hat{H}\psi_2 = E_2\psi_2$$

We want to find a wavefunction $F(\mathbf{R}, \mathbf{r}_1, \mathbf{r}_2)$ such that

$$\hat{H}F = EF$$

Let

$$F = F_0 + F_1 + F_2 + \cdots$$

and let

$$\hat{H}_0 = \hat{K}_\alpha + E_1 + E_2$$

Start by finding an F_0 such that

$$\hat{H}_0 F_0 = EF_0$$

The solution is

$$F_0(\mathbf{R}, \mathbf{r}_1, \mathbf{r}_2) = f_0(\mathbf{R})\psi_1(\mathbf{r}_1 - \mathbf{a}_1)\psi_2(\mathbf{r}_2 - \mathbf{a}_2) \tag{1}$$

where

$$f_0(\mathbf{R}) = \frac{1}{|\mathbf{R}|}\exp\left(\frac{ik|\mathbf{R}|}{\hbar}\right), \quad k = \sqrt{2M(E - E_1 - E_2)}$$

It follows that for the full Hamiltonian

$$\hat{H}F_0 = EF_0 + (U_1 + U_2)F_0$$

To cancel $(U_1 + U_2)F_0$ from the full Hamiltonian, find an F_1 such that

$$\hat{H}_0 F_1 = EF_1 - (U_1 + U_2)F_0$$

Rewrite as

$$\left(\hat{H}_0 - E\right) F_1 = -(U_1 + U_2)F_0$$

Expand F_1 and F_0.

$$\left(\hat{H}_0 - E\right) f_1(\mathbf{R})\psi_1(\mathbf{r}_1 - \mathbf{a}_1)\psi_2(\mathbf{r}_2 - \mathbf{a}_2) = -(U_1 + U_2)f_0(\mathbf{R})\psi_1(\mathbf{r}_1 - \mathbf{a}_1)\psi_2(\mathbf{r}_2 - \mathbf{a}_2)$$

To solve for $f_1(\mathbf{R})$ multiply both sides by

$$\psi_1^*(\mathbf{r}_1 - \mathbf{a}_1)\psi_2^*(\mathbf{r}_2 - \mathbf{a}_2)$$

and integrate over $\mathbf{r}_1$ and $\mathbf{r}_2$ to obtain

$$\left(\hat{H}_0 - E\right) f_1(\mathbf{R}) = V(\mathbf{R})f_0(\mathbf{R}) \tag{2}$$

where

$$V(\mathbf{R}) = 2e^2 \int \frac{|\psi_1(\mathbf{r}_1 - \mathbf{a}_1)|^2\,|\psi_2(\mathbf{r}_2 - \mathbf{a}_2)|^2}{|\mathbf{R} - \mathbf{r}_1|}\,d\mathbf{r}_1\,d\mathbf{r}_2 + 2e^2 \int \frac{|\psi_1(\mathbf{r}_1 - \mathbf{a}_1)|^2\,|\psi_2(\mathbf{r}_2 - \mathbf{a}_2)|^2}{|\mathbf{R} - \mathbf{r}_2|}\,d\mathbf{r}_1\,d\mathbf{r}_2$$

Because $|\psi|^2$ is a normalized probability density function we have

$$V(\mathbf{R}) = 2e^2 \int \frac{|\psi_1(\mathbf{r}_1 - \mathbf{a}_1)|^2}{|\mathbf{R} - \mathbf{r}_1|}\,d\mathbf{r}_1 + 2e^2 \int \frac{|\psi_2(\mathbf{r}_2 - \mathbf{a}_2)|^2}{|\mathbf{R} - \mathbf{r}_2|}\,d\mathbf{r}_2$$

Per Mott the solution to (2) is

$$f_1(\mathbf{R}) = \frac{M}{2\pi\hbar^2}\int \frac{V(\mathbf{r})f_0(\mathbf{r})}{|\mathbf{R} - \mathbf{r}|}\exp\left(\frac{ik|\mathbf{R} - \mathbf{r}|}{\hbar}\right)\,d\mathbf{r}, \quad k = \sqrt{2M(E - E_1 - E_2)}$$

Substitute for $f_0(\mathbf{r})$ to obtain

$$f_1(\mathbf{R}) = \frac{M}{2\pi\hbar^2}\int \frac{V(\mathbf{r})}{|\mathbf{R} - \mathbf{r}||\mathbf{r}|}\exp\left(\frac{ik|\mathbf{R} - \mathbf{r}|}{\hbar} + \frac{ik|\mathbf{r}|}{\hbar}\right)\,d\mathbf{r}$$

Change of variable $\mathbf{r} \to \mathbf{y} + \mathbf{a}_1$.

$$f_1(\mathbf{R}) = \frac{M}{2\pi\hbar^2} \int \frac{V(\mathbf{y}+\mathbf{a}_1)}{|\mathbf{R}-\mathbf{y}-\mathbf{a}_1||\mathbf{y}+\mathbf{a}_1|} \exp\left(\frac{ik|\mathbf{R}-\mathbf{y}-\mathbf{a}_1|}{\hbar} + \frac{ik|\mathbf{y}+\mathbf{a}_1|}{\hbar}\right) d\mathbf{y}$$

Per Mott (see also Figari and Teta)

$$f_1(\mathbf{R}) \approx \frac{\exp(ik|\mathbf{R}-\mathbf{a}_1|)}{|\mathbf{R}-\mathbf{a}_1|} \frac{M}{2\pi\hbar^2} \int \frac{V(\mathbf{y}+\mathbf{a}_1)}{|\mathbf{y}+\mathbf{a}_1|} \exp\left(-\frac{ik\mathbf{u}\cdot\mathbf{y}}{\hbar} + \frac{ik|\mathbf{y}|}{\hbar}\right) d\mathbf{y} \qquad (3)$$

where

$$\mathbf{u} = \frac{\mathbf{R}-\mathbf{a}_1}{|\mathbf{R}-\mathbf{a}_1|}$$

By the method of stationary phase the integral vanishes except for

$$\frac{d}{d\mathbf{y}}\left(-\mathbf{u}\cdot\mathbf{y} + |\mathbf{y}+\mathbf{a}_1|\right) = -\mathbf{u} + \frac{\mathbf{y}+\mathbf{a}_1}{|\mathbf{y}+\mathbf{a}_1|} = 0$$

Note that $V(\mathbf{y}+\mathbf{a}_1)$ is small except for $\mathbf{y} \approx 0$ and $\mathbf{y} \approx \mathbf{a}_2 - \mathbf{a}_1$ so we only need to consider $\mathbf{y}$ near those values. For stationarity at both $\mathbf{y} = 0$ and $\mathbf{y} = \mathbf{a}_2 - \mathbf{a}_1$ we have

$$\mathbf{u} = \frac{\mathbf{a}_1}{|\mathbf{a}_1|} = \frac{\mathbf{a}_2}{|\mathbf{a}_2|}$$

Hence $\mathbf{a}_1$ and $\mathbf{a}_2$ form a line through the origin.