8-3. Show that Q^c_α, Q^s_α are normal coordinates corresponding to standing wave normal modes $\cos(2\pi\alpha j/N)$ and $\sin(2\pi\alpha j/N)$, in the sense that (for N odd)

$$q_j(t) = \sqrt{\frac{2}{N}}\left(\frac{1}{2}Q^c_0(t) + \sum_{\alpha=1}^{(N-1)/2}\left(Q^c_\alpha(t)\cos\frac{2\pi\alpha j}{N} + Q^s_\alpha(t)\sin\frac{2\pi\alpha j}{N}\right)\right) \tag{8.82}$$

Consider the following equations.

$$Q_\alpha(t) = \frac{1}{\sqrt{N}}\sum_{k=1}^{N} q_k(t)\left(\cos\frac{2\pi\alpha k}{N} - i\sin\frac{2\pi\alpha k}{N}\right) \tag{8.77}$$

$$Q^c_\alpha = \frac{1}{\sqrt{2}}(Q_\alpha + Q^*_\alpha) \tag{8.79}$$

$$Q^s_\alpha = \frac{i}{\sqrt{2}}(Q_\alpha - Q^*_\alpha) \tag{8.80}$$

Substitute (8.77) into (8.82).

$$q_j = \frac{1}{\sqrt{2N}}Q^c_0 + \frac{1}{N}\sum_{\alpha=1}^{(N-1)/2}\sum_{k=1}^{N} q_k(T_1 + T_2 + T_3 + T_4) \tag{1}$$

where

$$\begin{aligned}
T_1 &= \cos\frac{2\pi\alpha k}{N}\cos\frac{2\pi\alpha j}{N} - i\sin\frac{2\pi\alpha k}{N}\cos\frac{2\pi\alpha j}{N}\\
T_2 &= \cos\frac{2\pi\alpha k}{N}\cos\frac{2\pi\alpha j}{N} + i\sin\frac{2\pi\alpha k}{N}\cos\frac{2\pi\alpha j}{N}\\
T_3 &= i\cos\frac{2\pi\alpha k}{N}\sin\frac{2\pi\alpha j}{N} + \sin\frac{2\pi\alpha k}{N}\sin\frac{2\pi\alpha j}{N}\\
T_4 &= -i\cos\frac{2\pi\alpha k}{N}\sin\frac{2\pi\alpha j}{N} + \sin\frac{2\pi\alpha k}{N}\sin\frac{2\pi\alpha j}{N}
\end{aligned}$$

It follows that

$$T_1 + T_2 + T_3 + T_4 = 2\cos\frac{2\pi\alpha k}{N}\cos\frac{2\pi\alpha j}{N} + 2\sin\frac{2\pi\alpha k}{N}\sin\frac{2\pi\alpha j}{N}$$

By trigonometric identities

$$T_1 + T_2 + T_3 + T_4 = 2\cos\left(\frac{2\pi\alpha}{N}(j-k)\right) \tag{2}$$

Substitute (2) into (1) to obtain

$$q_j = \frac{1}{\sqrt{2N}} Q_0^c + \frac{2}{N} \sum_{\alpha=1}^{(N-1)/2} \sum_{k=1}^{N} q_k \cos\left(\frac{2\pi\alpha}{N}(j-k)\right) \tag{3}$$

By equations (8.77) and (8.79) with $\alpha = 0$ we have

$$Q_0^c = \sqrt{\frac{2}{N}} \sum_{k=1}^{N} q_k \tag{4}$$

Substitute (4) into (3).

$$q_j = \frac{1}{N} \sum_{k=1}^{N} q_k + \frac{2}{N} \sum_{\alpha=1}^{(N-1)/2} \sum_{k=1}^{N} q_k \cos\left(\frac{2\pi\alpha}{N}(j-k)\right)$$

Rewrite as

$$q_j = \sum_{k=1}^{N} q_k \left(\frac{1}{N} + \frac{2}{N} \sum_{\alpha=1}^{(N-1)/2} \cos\left(\frac{2\pi\alpha}{N}(j-k)\right) \right) \tag{5}$$

For the sum over α in (5) we have

$$\sum_{\alpha=1}^{(N-1)/2} \cos\left(\frac{2\pi\alpha}{N}(j-k)\right) = \begin{cases} (N-1)/2 & j = k \\ -1/2 & j \ne k \quad \text{(see proof below)} \end{cases}$$

Hence for $j = k$

$$\frac{1}{N} + \frac{2}{N} \sum_{\alpha=1}^{(N-1)/2} \cos\left(\frac{2\pi\alpha}{N}(j-k)\right) = 1$$

and for $j \ne k$

$$\frac{1}{N} + \frac{2}{N} \sum_{\alpha=1}^{(N-1)/2} \cos\left(\frac{2\pi\alpha}{N}(j-k)\right) = 0$$

It follows that (5) reduces to the following tautology.

$$q_j = \sum_{k=1}^{N} q_k \delta(j-k) = q_j$$

Hence (8.82) is proven to be correct.

We will now show that for N odd and $j \neq k$,

$$\sum_{\alpha=1}^{(N-1)/2} \cos\left(\frac{2\pi\alpha}{N}(j-k)\right) = -\frac{1}{2}$$

Consider the following geometric series formula.

$$\sum_{k=0}^{N-1} z^k = \frac{1-z^N}{1-z}, \qquad |z| < 1 \tag{6}$$

Let

$$z = \exp\left(\frac{2\pi i n}{N}\right)$$

where n is an integer such that $2n < N$ so that $|z| < 1$. Then

$$z^N = \exp(2\pi i n) = 1$$

Hence by (6)

$$\sum_{k=0}^{N-1} z^k = 0 \tag{7}$$

Noting that

$$z^k = \exp\left(\frac{2\pi i n k}{N}\right) = \cos\left(\frac{2\pi n k}{N}\right) + i\sin\left(\frac{2\pi n k}{N}\right)$$

we have by (7)

$$\sum_{k=0}^{N-1} \cos\left(\frac{2\pi n k}{N}\right) + i\sum_{k=0}^{N-1} \sin\left(\frac{2\pi n k}{N}\right) = 0$$

Hence

$$\sum_{k=0}^{N-1} \cos\left(\frac{2\pi nk}{N}\right) = 0 \tag{8}$$

We need to get rid of the restriction $2n < N$. Consider

$$\cos\left(\frac{2\pi(N-n)k}{N}\right) \tag{9}$$

By trigonometric identities (9) is equivalent to

$$\cos\left(\frac{2\pi nk}{N}\right)\cos(2\pi k) + \sin\left(\frac{2\pi nk}{N}\right)\sin(2\pi k)$$

By $\cos(2\pi k) = 1$ and $\sin(2\pi k) = 0$ we have

$$\cos\left(\frac{2\pi(N-n)k}{N}\right) = \cos\left(\frac{2\pi nk}{N}\right) \tag{10}$$

Hence (8) is valid for $0 < n < N$.

Returning to (8) and starting the summation at $k = 1$, we have

$$\sum_{k=1}^{N-1} \cos\left(\frac{2\pi nk}{N}\right) = -1$$

By (10) we have for N odd

$$\sum_{k=1}^{(N-1)/2} \cos\left(\frac{2\pi nk}{N}\right) = -\frac{1}{2}$$

Finally, replace k with α and n with $j-k$. (The sign of $j-k$ doesn't matter.)

$$\sum_{\alpha=1}^{(N-1)/2} \cos\left(\frac{2\pi\alpha}{N}(j-k)\right) = -\frac{1}{2}$$