Dirac equation 3

From the previous section

$$\psi_1 = \frac{e^{-i\xi/\hbar}}{\sqrt{E/c+mc}} \underset{\text{fermion spin up}}{\begin{pmatrix} E/c+mc \\ 0 \\ p_z \\ p_x+ip_y \end{pmatrix}} \qquad \psi_2 = \frac{e^{-i\xi/\hbar}}{\sqrt{E/c+mc}} \underset{\text{fermion spin down}}{\begin{pmatrix} 0 \\ E/c+mc \\ p_x-ip_y \\ -p_z \end{pmatrix}}$$

$$\psi_3 = \frac{e^{i\xi/\hbar}}{\sqrt{E/c+mc}} \underset{\text{anti-fermion spin up}}{\begin{pmatrix} p_z \\ p_x+ip_y \\ E/c+mc \\ 0 \end{pmatrix}} \qquad \psi_4 = \frac{e^{i\xi/\hbar}}{\sqrt{E/c+mc}} \underset{\text{anti-fermion spin down}}{\begin{pmatrix} p_x-ip_y \\ -p_z \\ 0 \\ E/c+mc \end{pmatrix}}$$

where

$$\xi = p_\mu x^\mu = Et - p_x x - p_y y - p_z z$$

and

$$E = \sqrt{p_x^2c^2 + p_y^2c^2 + p_z^2c^2 + m^2c^4}$$

Spinors are ψ without the exponentials.

$$u_1 = \frac{1}{\sqrt{E/c+mc}} \underset{\text{fermion spin up}}{\begin{pmatrix} E/c+mc \\ 0 \\ p_z \\ p_x+ip_y \end{pmatrix}} \qquad u_2 = \frac{1}{\sqrt{E/c+mc}} \underset{\text{fermion spin down}}{\begin{pmatrix} 0 \\ E/c+mc \\ p_x-ip_y \\ -p_z \end{pmatrix}}$$

$$v_1 = \frac{1}{\sqrt{E/c+mc}} \underset{\text{anti-fermion spin up}}{\begin{pmatrix} p_z \\ p_x+ip_y \\ E/c+mc \\ 0 \end{pmatrix}} \qquad v_2 = \frac{1}{\sqrt{E/c+mc}} \underset{\text{anti-fermion spin down}}{\begin{pmatrix} p_x-ip_y \\ -p_z \\ 0 \\ E/c+mc \end{pmatrix}}$$

Spinors are solutions to the momentum-space Dirac equations

$$\not{p}u = mcu$$
$$\not{p}v = -mcv$$

where

$$\not{p} = p^\mu g_{\mu\nu}\gamma^\nu$$

and

$$p^\mu = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

Spinors have the following "completeness property."

$$\begin{aligned} u_1\bar{u}_1 + u_2\bar{u}_2 &= \not{p} + mc \\ v_1\bar{v}_1 + v_2\bar{v}_2 &= \not{p} - mc \end{aligned}$$

Adjoints of spinors are formed as

$$\bar{u} = u^\dagger\gamma^0, \quad \bar{v} = v^\dagger\gamma^0$$

hence $u\bar{u}$ and $v\bar{v}$ are outer products that form 4×4 matrices.