Stabilized SVRG: Simple Variance Reduction for Nonconvex Optimization

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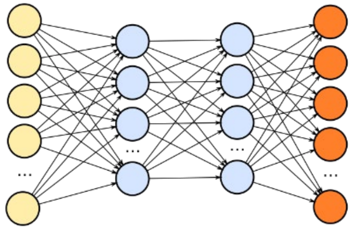
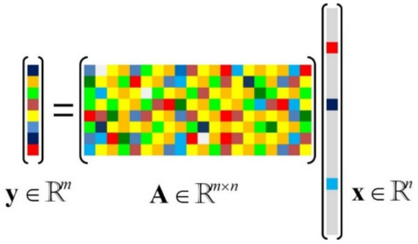
Non-convex Optimization

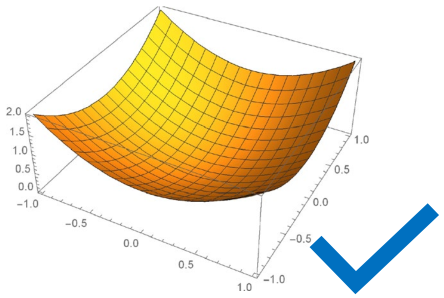


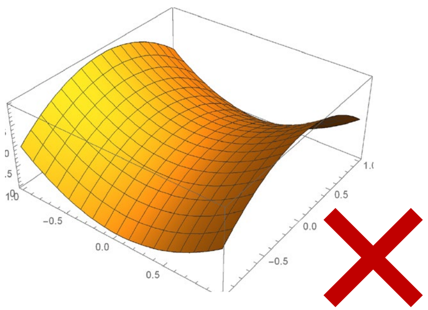


- In theory, finding global minima is NP-Hard.
- In practice, just run (stochastic) gradient descent.

  All local minima are global minima, all saddle points are strict. (e.g. matrix completion [GLM16], dictionary learning [SQW17], certain objectives of neural networks [GLM17].)

Goal: find second-order stationary points (0 gradient and psd Hessian).





Empirical Risk Minimization

- Empirical risk minimization:

  min empirical risk = $\frac{1}{n}\sum_{i=1}^{n}$ (risk over sample i)

- Problem:

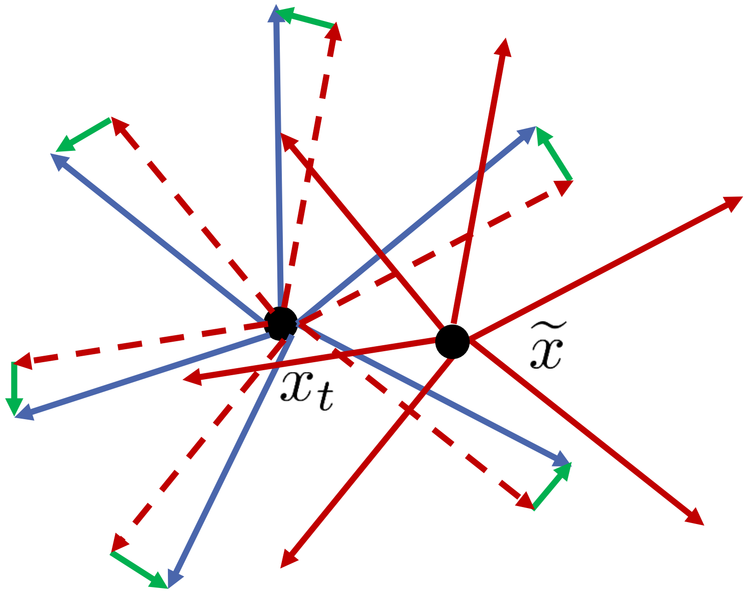
$$\min_{x\in\mathbb{R}^d}\left\{f(x) := \frac{1}{n}\sum_{i=1}^{n} f_i(x)\right\}$$

- Both $f_i(\cdot)$ and $f(\cdot)$ can be non-convex.
- $f_i(x)$: risk over one sample
- $f(x)$: empirical risk

SVRG (Stochastic Variance Reduced Gradient)

- SGD: $x_{t+1} = x_t - \eta \nabla f_i(x_t),\ i \sim [n]$
  Converges to an ε-first-order stationary point ($\|\nabla f(x)\| \leq \epsilon$) in $O(\frac{\sigma^2}{\epsilon^4})$
- SVRG [JZ13]: in each epoch, compute the full gradient of the first point (snapshot point) and use it to reduce variance in the following iterates
- SVRG: $O(\frac{n^{2/3}}{\epsilon^2} + n)$ [AH16] [RHSPS16] [LL18]

$$x_{t+1} = x_t - \eta(\underbrace{\nabla f_i(x_t) - \nabla f_i(\widetilde{x})}_{\text{small}} + \underbrace{\nabla f(\widetilde{x})}_{\approx \nabla f(x_t)})$$



Our Results

Theorem. We design an algorithm (Stabilized SVRG) that can find an ε-second order stationary point using

$$\widetilde{O}\left(\frac{n^{2/3}}{\epsilon^2} + \frac{n}{\epsilon^{1.5}}\right)$$

stochastic gradients.

$$\|\nabla f(x)\| \leq \epsilon \text{ and } \lambda_{\min}\left(\nabla^2 f(x)\right) \geq -\sqrt{\epsilon}$$

1. The first simple variant of SVRG with similar guarantee.
2. Stabilization technique might be applicable to other algorithms.

Previous approach: Neon/Neon2 Reduction

- Neon [XRY17] and Neon2 [AL17] can transform an algorithm that finds first-order stationary point to an algorithm with second-order guarantee.

Negative Curvature Search (NC-search)

Given a point x, decide if $\nabla^2 f(x) \succeq -\sqrt{\epsilon} I$ or find a unit vector v such that $v^\top \nabla^2 f(x) v \leq -\frac{\sqrt{\epsilon}}{2}$

- Neon2+SVRG: $\widetilde{O}(\frac{n^{2/3}}{\epsilon^2} + \frac{n}{\epsilon^{1.5}} + \frac{n^{3/4}}{\epsilon^{1.75}})$
- Adding a separate NC-search makes the algorithm complicated, which is not necessary in practice.
- Without NC-Search, our algorithm is simpler!

Stabilized SVRG

- Modifications to original SVRG

At the beginning of each epoch, if the gradient is small

1. add a small perturbation to the current point
2. run SVRG on a shifted function

$$\hat{f}(x) := f(x) - \langle \nabla f(\widetilde{x}), x - \widetilde{x} \rangle$$

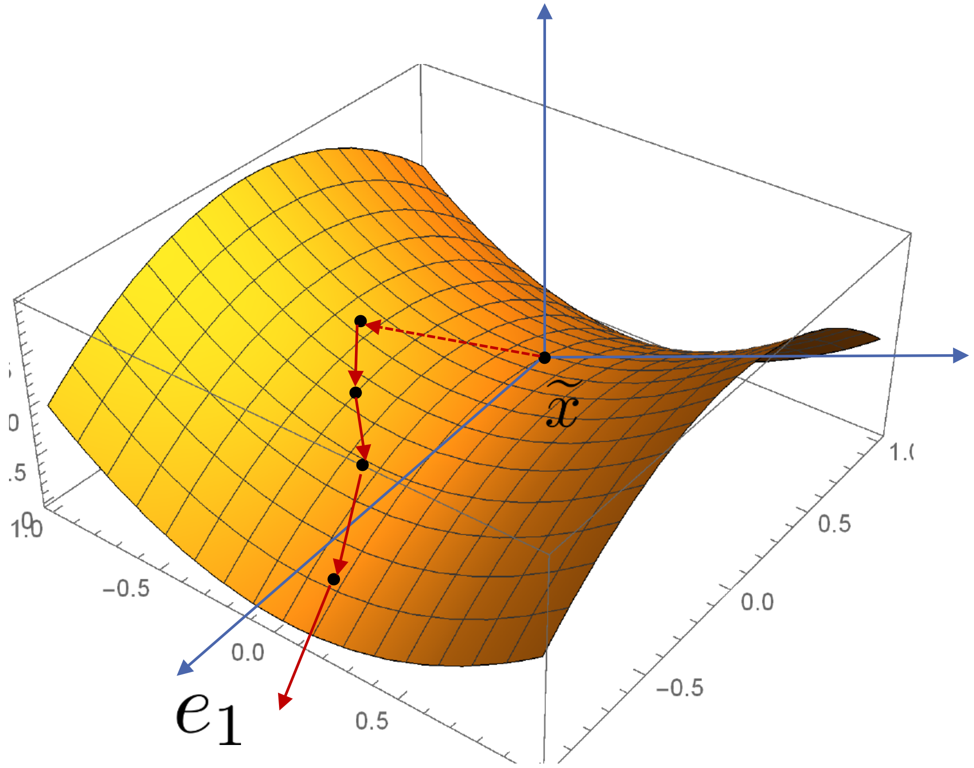
Stabilization

whose gradient at initial point $\widetilde{x}$ is exactly zero.

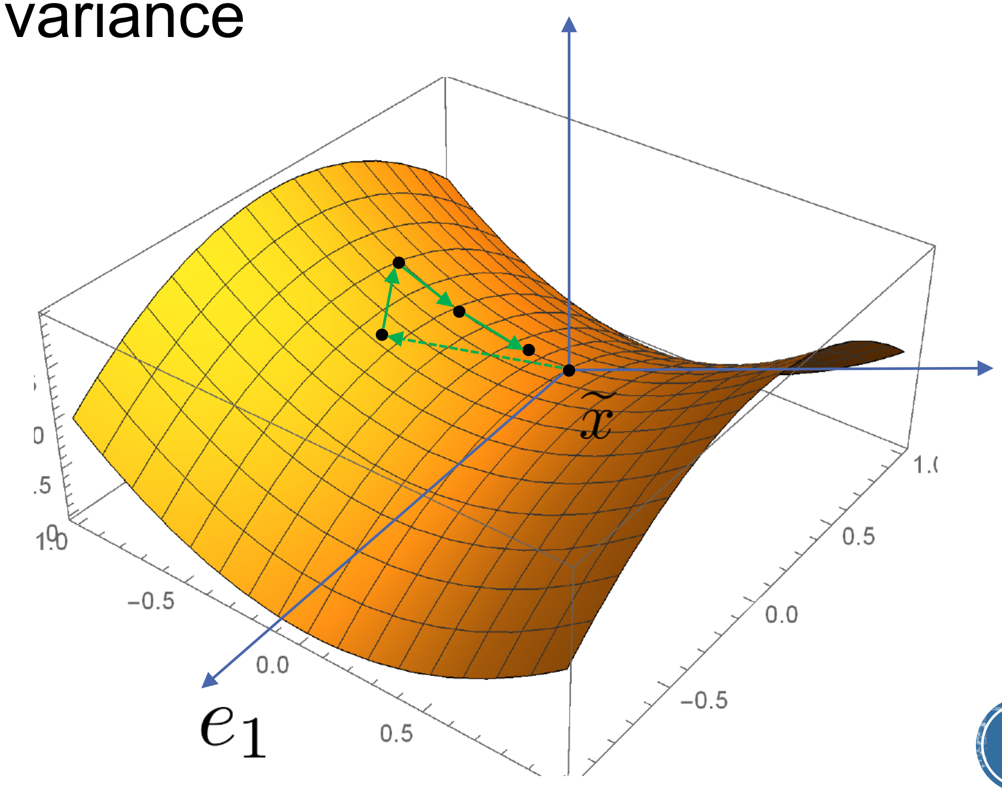
Challenge

Minimum eigenvalue direction of $\nabla^2 f(\tilde{x})$ e_1

- GD: iterates escape along e_1 direction



- SVRG: initial projection along e_1 (only $\frac{\delta}{\sqrt{d}}$) can be canceled by the variance

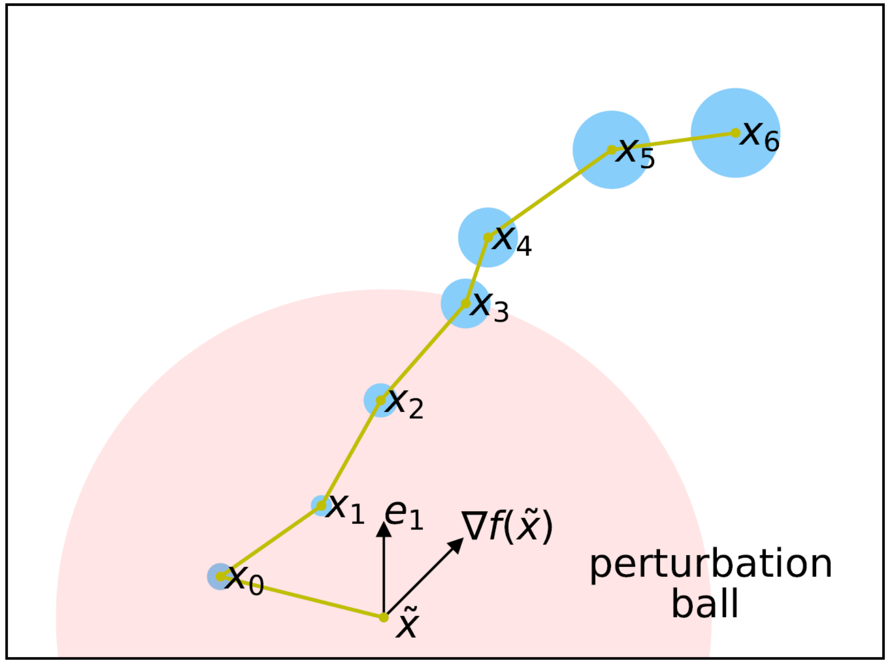


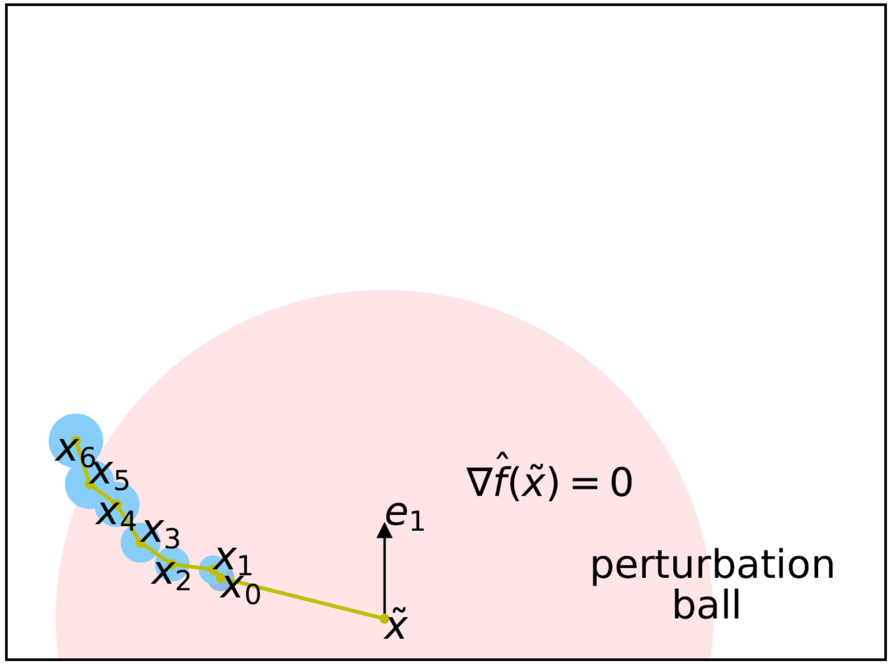
Stabilization

Minimum eigenvalue direction of $\nabla^2 f(\tilde{x})$

e_1

- Variance can be bounded by the distance to the snapshot point.
- Hope the iterates stay close to the initial point for long enough time.





Two Phase Analysis

Minimum eigenvalue direction of $\nabla^2 f(\widetilde{x})$

e_1

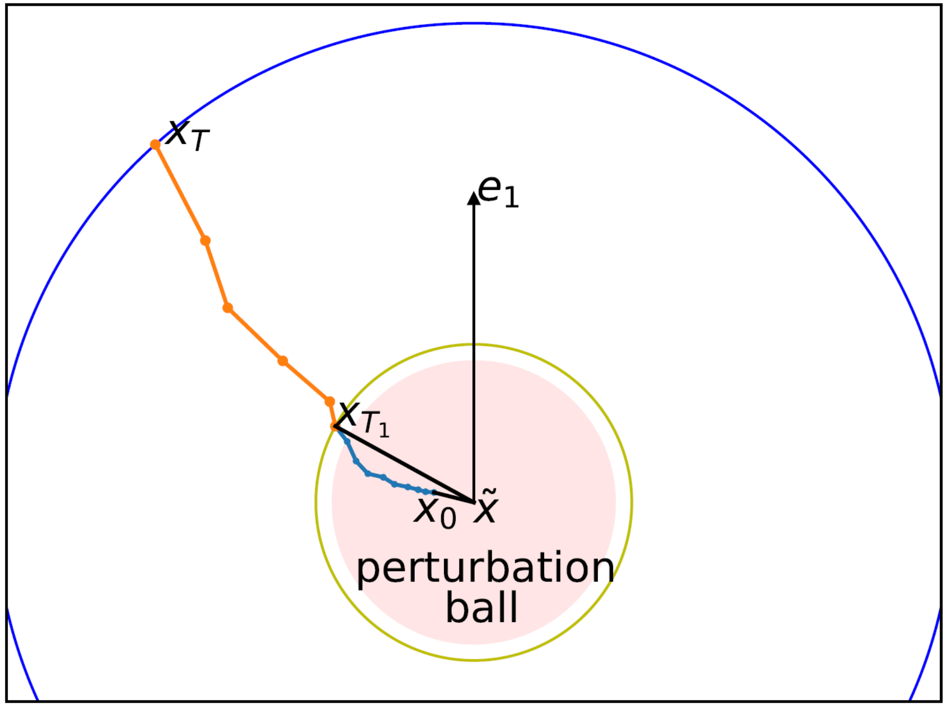
- Phase 1

Bounded in a ball with radius $\widetilde{O}(\delta)$
At the end of Phase 1, the projection along e_1 at least $\delta/2$

Implicit negative curvature search!

- Phase 2

Starting from “a good initial point”, $x_t - \widetilde{x}$ increases exponentially along e_1 direction



Summary

- Main Result:

We give the first simple variant of SVRG which converges to an ε-second-order stationary point within $\widetilde{O}(\frac{n^{2/3}}{\epsilon^2} + \frac{n}{\epsilon^{1.5}})$ time.

- Future work:

1. Formulate the properties that are required for the stabilization idea to work.
2. Give a reduction that produces simpler algorithms with second order guarantees.

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