HT 2020

Elements of Deductive Logic

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1 Determinants

Geometrically, we can view the determinant of an $n \times n$ square matrix by the amount its corresponding linear map scales volumes in n-dimensional space. As an example, the absolute value of the determinant of a 3×3 matrix is the volume of the parallelepiped formed by the transformed unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

We will then try to define the determinant in an algebraic way. For such a map $D : \mathcal{M}_{n\times n}(\mathbb{R}) \to \mathbb{R}$, we want it have the following properties (deduced from our intuitive geometric understanding):

1. Alternating[1]: $\mathbf{b} = \mathbf{c} \implies D([\dots,\mathbf{b},\mathbf{c},\dots]) = 0$
2. Homogeneous: $D([\dots,\lambda\mathbf{b},\mathbf{c},\dots]) = \lambda D([\dots,\mathbf{b},\mathbf{c},\dots])$
3. Linear: $D([\dots,\mathbf{b}+\mathbf{c},\dots]) = D([\dots,\mathbf{b},\dots]) + D([\dots,\mathbf{c},\dots])$
4. Preserves identity: $D(I_n) = 1$

[1] the reason being, if two unit vectors get transformed onto the same vector by a linear map, then the image of that linear map is of a lower dimension, hence its volume becomes 0 in the original dimension

Definition 1.1

A map $D : \mathcal{M}_{n\times n}(\mathbb{R}) \to \mathbb{R}$ is *determinantal* if it satisfies the above 4 properties.

From these basic properties, we can deduce the following proposition about determinantal maps:

Proposition 1.1

Let $D : \mathcal{M}_{n\times n}(\mathbb{R}) \to \mathbb{R}$ be a determinantal map, then

(i) $D([\dots,\mathbf{b},\mathbf{c},\dots]) = -D([\dots,\mathbf{c},\mathbf{b},\dots])$

(ii) $\mathbf{b} = \mathbf{c} \implies D([\dots,\mathbf{b},\dots,\mathbf{c},\dots]) = 0$

(iii) $D([\dots,\mathbf{b},\dots,\mathbf{c},\dots]) = -D([\dots,\mathbf{c},\dots,\mathbf{b},\dots])$

Proof. We will prove one-by-one.

(i) We have

$$\begin{aligned}
0 &= D([\dots,\mathbf{b}+\mathbf{c},\mathbf{b}+\mathbf{c},\dots]) \\
&= D([\dots,\mathbf{b}+\mathbf{c},\mathbf{b},\dots]) + D([\dots,\mathbf{b}+\mathbf{c},\mathbf{c},\dots]) \\
&= D([\dots,\mathbf{b},\mathbf{b}\dots]) + D([\dots,\mathbf{b},\mathbf{c},\dots]) \\
&\qquad + D([\dots,\mathbf{c},\mathbf{b}\dots]) + D([\dots,\mathbf{c},\mathbf{c}\dots])
\end{aligned}$$

Then, by the alternating property, we get

$$0 = D([\dots,\mathbf{b},\mathbf{c},\dots]) + D([\dots,\mathbf{c},\mathbf{b}\dots])$$

and the result follows.

(ii) We can repeatedly apply (i) above to move the columns $\mathbf{b}$ and $\mathbf{c}$ adjacent to each other. Then,

$$D([\ldots,\mathbf{b},\ldots,\mathbf{c},\ldots]) = \pm D([\ldots,\mathbf{b},\mathbf{c},\ldots])$$

Since $\mathbf{b} = \mathbf{c}$, we have, by alternating property,

$$D([\ldots,\mathbf{b},\ldots,\mathbf{c},\ldots]) = 0$$

(iii) This proof is analogous to that of (i), but using (ii). □

Given these properties, we can start finding examples of determinantal maps. For $n = 1$, we have that

$$D([a]) = aD([1]) = a$$

For $n = 2$, we have that

$$\begin{aligned} D\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) &= D\left(a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \end{bmatrix}, b\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= ab \cdot D\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) + ad \cdot D\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &\quad + cb \cdot D\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) + cd \cdot D\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right) \\ &= ad - bc \end{aligned}$$

We see that in these two cases, we can explicitly write down the uniquely-defined determinant. We will show that this existence and uniqueness generalizes to all n.

Theorem 1.2

A determinantal map exists for each $n \in \mathbb{Z}^+$.

Proof. We will prove by induction on n. We have shown that the for the base case $n = 1$ there is a determinantal map.
Assume that D_{n-1} is a determinantal map for $(n-1) \times (n-1)$ matrices.
Let A be an $n \times n$ matrix. Let A_{ij} be the $(n-1) \times (n-1)$ matrix formed by removing the i^{th} row and j^{th} column of A.
Let $D_n(A) = a_{11}D_{n-1}(A_{11}) - a_{12}D_{n-1}(A_{12}) + \cdots + (-1)^{n-1}a_{1n}D_{n-1}(A_{1n})$.
We will show that D_n is determinantal.
First note that if $A = I_n$, then $a_{1j} = 0$ for $j \neq 1$. So

$$D_n(I_n) = D_{n-1}(I_{n-1}) + 0 = 1$$

For linearity (homogeneity and additivity), first note importantly that the sum of linear terms is also linear. View D_n as a function of the k^{th} column, then consider the term $a_{1j}D_{n-1}(A_{1j})$: for $j \neq k$, a_{1j} is independent of the k^{th} column and A_{1j} depends linearly on the k^{th} column by induction hypothesis; for $j = k$, A_{1j} is independent of and a_{1j} depends linearly on the k^{th} column. So all such terms depend linearly on the k^{th} column, hence $D_n(A)$ is multilinear. [2]

[2] hello hello

Lastly, to show D_n is alternating, we suppose the j^{th} and $(j+1)^{\text{th}}$ columns of A are the same, so $\mathbf{a}_j = \mathbf{a}_{j+1}$. Then for any $k \neq j, j+1$, there will be two identical columns in A_{1k}, so $D_{n-1}(A_{1k}) = 0$. Now we can simply $D_n(A)$ to

$$D_n(A) = (-1)^{j+1}a_{1j}D_{n-1}(A_{1j}) + (-1)^{j+2}a_{1(j+1)}D_{n-1}\left(A_{1(j+1)}\right)$$

But $\mathbf{a}_j = \mathbf{a}_{j+1}$ implies $a_{1j} = a_{1(j+1)}$ and $A_{1j} = A_{1(j+1)}$, so

$$D_n(A) = (-1)^{j+1}a_{1j}D_{n-1}(A_{1j}) - (-1)^{j+1}a_{1j}D_{n-1}(A_{1j}) = 0$$

So D_n is a determinantal map, and thus a determinantal map exists for each $n \in \mathbb{Z}^+$ by induction. □

Note that we have chosen the first row here to sum over, but we could equally have chosen any other row, and each summation obtained through this method is a *Laplace expansion* of the determinant of the matrix.[3] Importantly, we will also show that no matter which row you choose, you will end up with the same map—in other words, each determinantal map is unique, and there is a unique determinant to each matrix.[4]

[3] hiasdfiou ihaushdfi uahsidf u asdf iauhsdifu hais udfi ausdifu aius dfiausid fuas idf uais difua siudf iaus dfiuahs diufh aisuhd fauis hdfiuah sdf

[4] hello and i can write really long ones like this its pretty cool

We first define the following:

Definition 1.2

A *permutation* on a set $\{1, 2, \ldots, n\}$ is a bijection from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$. We usually denote S_n for the set of all permutations on $\{1, 2, \ldots, n\}$.
A *transposition* is a specific type of permutation which only switches two elements and maps everything else to themself.

Theorem 1.3

There is a unique determinantal map $\det : \mathcal{M}_{n\times n}(\mathbb{R}) \to \mathbb{R}$ for each $n \in \mathbb{Z}^+$.

Proof. □

From this, we can deduce a new formula for the determinant of a matrix. Let A be an $n \times n$ dimensional matrix, and S_n be the set of all permutations of $\{1, \ldots, n\}$, then

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{k=1}^{n} a_{k\sigma(k)} \quad (1.1)$$

Also, we get the following results.

Lemma 1.4

Let $\sigma \in S_n$, then $\operatorname{sign}(\sigma) = \operatorname{sign}(\sigma^{-1})$.

Proof. First, we have that $\sigma \circ \sigma^{-1} = \iota$ is the identity permutation. Also, since σ and σ^{-1} are both products of transpositions, we simply multiply all those transpositions to get $\sigma \circ \sigma^{-1}$, so $\operatorname{sign}(\sigma \circ \sigma^{-1}) = \operatorname{sign}(\sigma) \operatorname{sign}(\sigma^{-1})$. Thus,

$$1 = \operatorname{sign}(\iota) = \operatorname{sign}(\sigma \circ \sigma^{-1}) = \operatorname{sign}(\sigma) \operatorname{sign}(\sigma^{-1})$$

But sign only takes ± 1 as values, so we conclude that $\operatorname{sign}(\sigma) = \operatorname{sign}(\sigma^{-1})$. □

Proposition 1.5

Let A be a square matrix, then $\det(A) = \det\left(A^T\right)$.

Proof. □

From this, we get immediately that the determinant of a matrix is also multilinear in the columns as well as the rows.

There is also one other key property of determinants:

Theorem 1.6 (Multiplicativity)

Let $A, B \in \mathcal{M}_{n \times n}(\mathbb{R})$ be matrices, then $\det(AB) = \det(A) \det(B)$.