HT 2020

Analysis II

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Contents

1 Limits and Continuity

Definition 1.1

A sequence (z_n) of (real or complex) numbers has a *limit* L, if

$$\forall \varepsilon > 0 : \exists N \in \mathbb{R} : \forall n > N : |z_n - L| < \varepsilon$$

This is denoted by $z_n \to L$ as $n \to \infty$ or $\lim_{n\to\infty} z_n = L$. We say a sequence *converges* if it has a limit, and *diverges* otherwise.

Definition 1.2

A sequence (z_n) is a *Cauchy sequence* if

$$\forall \varepsilon > 0 : \exists N \in \mathbb{R} : \forall n, m > N : |z_n - z_m| < \varepsilon$$

2 Differentiability

Definition 2.1

Let f be a function (real or complex) defined on $(a, b) \subseteq \mathbb{R}$ and $x_0 \in (a, b)$. The *derivative* $f'(x_0)$ of f at x_0 is defined as the limit

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

provided the limit exists, in which case we say f is differentiable at x_0. This derivative is also denoted by $\frac{\mathrm{d}f}{\mathrm{d}x}(x_0)$.
We also define the *left derivative* and *right derivative* provided the respective limits exist:

$$f'(x_0^-) = \lim_{x\uparrow x_0} \frac{f(x) - f(x_0)}{x - x_0}, \quad f'(x_0^+) = \lim_{x\downarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Theorem 2.1 (Differentiability Implies Continuity)

Let $f : (a, b) \to \mathbb{R}$ or $\mathbb{C}$. If f is differentiable at $x_0 \in (a, b)$, then f is also continuous at x_0.

Proof. We have

$$\begin{aligned}
\lim_{x\to x_0} \big(f(x) - f(x_0)\big) &= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \\
&= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x\to x_0}(x - x_0) \\
&= 0
\end{aligned}$$

Hence $\lim_{x\to x_0} f(x) = f(x_0)$, therefore f is continuous at x_0. □

A notable result regarding inverse functions is as follows:

Theorem 2.2

Let f be a real-valued, continuous, and bijective function on (a, b). If f is differentiable at $x_0 \in (a, b)$ with $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ with

$$\frac{\mathrm{d}}{\mathrm{d}y} f^{-1}(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{f'(x_0)}$$

2.1 Differentiability of Power Series

Theorem 2.3

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with convergence radius $R > 0$. Its derivative[1] f', also a power series, also has convergence radius R.

[1] differentiating term by term

Theorem 2.4

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with convergence radius $R > 0$. For any $|z| < R$, the derivative $f'(z)$ exists with

$$f'(z) = \frac{\mathrm{d}}{\mathrm{d}z} \sum_{n=0}^{\infty} a_n z^n = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

We can now use power series to study the derivative of several elementary functions.

3 Mean-Value Theorems

We begin with formalizing and proving an intuitive yet important result.

Theorem 3.1 (Fermat's Theorem)

Let $f : E \to \mathbb{R}$ and x_0 a local extremum of f with f differentiable at x_0. Then $f'(x_0) = 0$.

Proof. Without loss of generality, let x_0 be a local minimum. By definition, we can let $\delta > 0$ be such that $\forall x \in (x_0 - \delta, x_0 + \delta) : f(x) \geqslant f(x_0)$.
We now have that $\frac{f(x) - f(x_0)}{x - x_0}$ is negative for any $x \in (x_0 - \delta, x_0)$ and positive for any $x \in (x_0, x_0 + \delta)$. Hence, $f'(x_0^-) \leqslant 0$ and $f'(x_0^+) \geqslant 0$.
But from the differentiability assumption, we know $f'(x_0^-) = f'(x_0^+)$, hence it must be the case that $f'(x_0) = f'(x_0^-) = f'(x_0^+) = 0$. □

Lemma 3.2 (Darboux's IVT)

If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$, then for any $m \in \mathbb{R}$ strictly between $f'(a)$ and $f'(b)$, there exists some $\xi \in (a, b)$ such that $f'(\xi) = m$.

Theorem 3.3 (Rolle's Theorem)

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), and that $f(a) = f(b)$, then there exists some $\xi \in (a, b)$ such that $f'(\xi) = 0$.

Proof. If f is constant on $[a, b]$, then $\forall x \in [a, b] : f'(x) = 0$, which suffices.[2]

Now suppose that f is not constant on $[a, b]$. □

[2] The fact that the derivative of a constant function is always 0 is glossed over here as it is trivial to prove, but still a result worth noting.

Theorem 3.4 (MVT)

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists some $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

Theorem 3.5 (Cauchy's MVT)

If $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b), and $\forall x \in (a, b) : g'(x) \neq 0$, then there exists some $\xi \in (a, b)$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$