MT 2020

# Analysis I

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## Contents

## 1  Real Numbers

In this course, we assume real numbers' existence and investigate their properties and operations we can perform on them. Below are the arithmetics of real numbers.

1. $\forall a, b \in \mathbb{R} : \exists!(a + b) \in \mathbb{R}$ <span style="float:right">*sum*</span>

2. $\forall a, b \in \mathbb{R} : \exists!(a \cdot b) \in \mathbb{R}$ <span style="float:right">*product*</span>

3. $\forall a \in \mathbb{R} : \exists!(-a) \in \mathbb{R}$ <span style="float:right">*negative* / *additive inverse*</span>

4. $\forall a \in \mathbb{R}, a \neq 0 : \exists!(\frac{1}{a}) \in \mathbb{R}$ <span style="float:right">*reciprocal* / *multiplicative inverse*</span>

5. $\exists 0 \in \mathbb{R}$ <span style="float:right">*zero* / *additive identity*</span>

6. $\exists 1 \in \mathbb{R}$ <span style="float:right">*one* / *multiplicative identity*</span>

Next, we introduce the key arithmetic properties of the real numbers:

**Axiom 1.1**

$\forall a, b, c \in \mathbb{R} :$

(i) $a + b = b + a$ <span style="float:right">($+$ is commutative)</span>

(ii) $a + (b + c) = (a + b) + c$ <span style="float:right">($+$ is associative)</span>

(iii) $a + 0 = a$

(iv) $a + (-a) = 0$

(v) $a \cdot b = b \cdot a$ <span style="float:right">($\cdot$ is commutative)</span>

(vi) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ <span style="float:right">($\cdot$ is associative)</span>

(vii) $a \cdot 1 = a$

(viii) $a \cdot \frac{1}{a} = 1$ given that $a \neq 0$

(ix) $a \cdot (b + c) = a \cdot b + a \cdot c$ <span style="float:right">($\cdot$ distributes over $+$)</span>

(x) $0 \neq 1$

**Definition 1.1**

Let $\mathbb{F}$ be a set with operations $+$ and $\cdot$ satisfying Axiom 1.1. Then we say $\mathbb{F}$ is a *field*.

Sets such as $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{C}$ are fields.[1]

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[1] Note that $\mathbb{Z}$ does not form a field

From the axioms of arithmetics, we can derive the following properties:

**Proposition 1.2**

$\forall a, b, c, x, y \in \mathbb{R}:$

   (i) $a + x = a \implies x = 0$                                   (uniqueness of 0)

   (ii) $a + x = a + y \implies x = y$                          (cancellation for $+$)

   (iii) $-0 = 0$

   (iv) $-(-a) = a$

   (v) $-(a + b) = (-a) + (-b)$

   (vi) $a \cdot x = a,\ a \neq 0 \implies x = 1$                       (uniqueness of 1)

   (vii) $a \cdot x = a \cdot y,\ a \neq 0 \implies x = y$              (cancellation for $\cdot$)

  (viii) $a \neq 0 \implies \dfrac{1}{\frac{1}{a}} = a$

   (ix) $(a + b) \cdot c = a \cdot c + b \cdot c$

   (x) $a \cdot 0 = 0$

   (xi) $a \cdot (-b) = -(a \cdot b)$. In particular, $(-1) \cdot a = -a$

   (xii) $(-1) \cdot (-1) = 1$

  (xiii) $a \cdot b = 0 \implies a = 0 \vee b = 0$. Also, $a \neq 0 \wedge b \neq 0 \implies \frac{1}{a \cdot b} = \frac{1}{a} \cdot \frac{1}{b}$

We can also now define integer exponentiation: take $a \in \mathbb{R} \setminus \{0\}$; we define $a^0 = 1$ and then inductively, for integers $k \geqslant 0$, we define positive powers of $a$ as $a^{k+1} = a^k \cdot a$; lastly, for integers $l \leqslant -1$, we define $a^l = \frac{1}{a^{-l}}$.

## 1.1   Ordering the Real Numbers

The key property of the real numbers which differs them from complex numbers is that they can be ordered. The following are the axioms for the usual ordering on $\mathbb{R}$.

**Axiom 1.3**

There exists a subset $\mathbb{P}$ of $\mathbb{R}$ such that $\forall a, b \in \mathbb{R}$ :

  (i) $a, b \in \mathbb{P} \implies a + b \in \mathbb{P}$                         (+ and ordering)

  (ii) $a, b \in \mathbb{P} \implies a \cdot b \in \mathbb{P}$                           ($\cdot$ and ordering)

  (iii) exactly one of $a \in \mathbb{P}$, $a = 0$, and $-a \in \mathbb{P}$ holds

The elements of $\mathbb{P}$ are the *positive numbers*; the elements of $\mathbb{P} \cup \{0\}$ are the *non-negative numbers*.

    Importantly, we write $a < b$ or $b > a$ exactly when $b - a \in \mathbb{P}$; and we write $a \leqslant b$ or $b \geqslant a$ exactly when $b - a \in \mathbb{P} \cup \{0\}$.

    With these axioms of ordering, we can deduce the following properties:

**Proposition 1.4**

$\forall a, b, c \in \mathbb{R}$ :

  (i) $a \leqslant a$                                         (reflexivity)

  (ii) $a \leqslant b \wedge b \leqslant a \implies a = b$               (antisymmetry)

  (iii) $a \leqslant b \wedge b \leqslant c \implies a \leqslant c$               (transitivity)

  (iv) exactly one of $a < b$, $a = b$, and $a > b$ holds     (trichotomy)

    Combining inequalities and arithmetics, we have the following:

**Proposition 1.5**

$\forall a, b, c \in \mathbb{R}$ :

  (i) $0 < 1$

  (ii) $a < b \iff -b < -a$. In particular, $a > 0 \iff -a < 0$

  (iii) $a < b \implies a + c < b + c$

  (iv) $a < b \wedge 0 < c \implies ac < bc$

  (v) $a^2 \geqslant 0$ with equality if and only if $a = 0$

  (vi) $a > 0 \iff \frac{1}{a} > 0$

  (vii) $a, b > 0 \wedge a < b \implies \frac{1}{b} < \frac{1}{a}$

**Theorem 1.6** (Bernoulli's Inequality)

Let $x \in \mathbb{R}$ be such that $x > -1$ and $n$ be a positive integer. Then $(1+x)^n \geqslant 1 + nx$.

*Proof.* Fix $x > -1$. We will perform induction on $n$.
When $n = 1$, $1 + x \geqslant 1 + x$ trivially.
Suppose the result holds for some $n \geqslant 1$, so $(1+x)^n \geqslant 1 + nx$. Then,

$$
\begin{aligned}
(1+x)^{n+1} &= (1+x)^n(1+x) \\
&\geqslant (1+nx)(1+x) \\
&= 1 + (n+1)x + nx^2 \\
&\geqslant 1 + (n+1)x
\end{aligned}
$$

So the result is proved by induction. $\qquad\square$

## 1.2 The Modulus of a Real Number

**Definition 1.2**

Let $a \in \mathbb{R}$. The *modulus*, or *absolute value*, $|a|$ of $a$ is defined to be

$$
|a| = \begin{cases} a, & a > 0 \\ 0, & a = 0 \\ -a, & a < 0 \end{cases}
$$

Note that the trichotomy properties makes this the modulus well-defined.
Again, we look at the properties of the modulus of real numbers:

**Proposition 1.7**

$\forall a, b, c \in \mathbb{R}$ :

(i) $|-a| = |a|$

(ii) $|a| \geqslant 0$

(iii) $|a|^2 = a^2$

(iv) $|ab| = |a||b|$

(v) $-|a| \leqslant a \leqslant |a|$

(vi) $c \geqslant 0 \implies (|a| \leqslant c \iff -c \leqslant a \leqslant c)$ and similarly for strict inequalities

With these properties in mind, we can arrive at an important theorem:

**Theorem 1.8** (Triangle Inequality)

Let $a, b \in \mathbb{R}$. Then $|a + b| \leqslant |a| + |b|$.

*Proof.* Let $a, b \in \mathbb{R}$, we have that $-|a| \leqslant a \leqslant |a|$ and $-|b| \leqslant b \leqslant |b|$. Since addition preserves inequalities[2], we have

$$-(|a| + |b|) \leqslant a + b \leqslant |a| + |b|$$

Since $|a| + |b| \geqslant 0$, by property (vi), we have $|a + b| \leqslant |a| + |b|$. $\qquad\square$

[2] a result that is not proved here

**Theorem 1.9** (Reverse Triangle Inequality)

Let $a, b \in \mathbb{R}$. Then $|a - b| \geqslant \big||a| - |b|\big|$.

*Proof.* By Theorem 1.8, we have

$$|a| = |a - b + b| \leqslant |a - b| + |b|$$

So $|a - b| \geqslant |a| - |b|$. However, the above argument also works if we swap $a$ and $b$, resulting in $|a - b| \geqslant |b| - |a|$. And we know that $\big||a| - |b|\big|$ is one of $|a| - |b|$ or $|b| - |a|$, so we can conclude that $|a - b| \geqslant \big||a| - |b|\big|$. $\quad\square$

## 1.3   Completeness

Informally, we view rational numbers as having 'gaps' between them while real numbers don't. We will try to formalize this idea.

**Upper and Lower Bounds**

**Definition 1.3**

Let $S \subseteq \mathbb{R}$. Take $b \in \mathbb{R}$. We say that

- $b$ is an *upper bound* of $S$ if $\forall s \in S : s \leqslant b$

- $b$ is a *lower bound* of $S$ if $\forall s \in S : s \geqslant b$

- $S$ is *bounded above* if $S$ has an upper bound

- $S$ is *bounded below* if $S$ has a lower bound

- $S$ is *bounded* if $S$ is bounded above or below

While a set might have an infinite number of bounds, there are certain bounds that are more interesting than others.

**Definition 1.4**

Let $S \subseteq \mathbb{R}$. We say that $\alpha \in \mathbb{R}$ is the _supremum_ of $S$, written sup $S$, if $\alpha$ is an upper bound of $S$ and $\alpha$ is the least upper bound of $S$. That is, concretely, if

(i) $\forall s \in S : s \leqslant \alpha$

(ii) $(\forall s \in S : s \leqslant b) \implies \alpha \leqslant b$

Importantly, remark that if a set $S$ has a supremum, then it is unique.

**Axiom 1.10** (Completeness Axiom of Real Numbers)

Let $S$ be a non-empty subset of $\mathbb{R}$ that is bounded above. Then $S$ has a supremum.

Note that the non-empty condition is crucial as every real number is an upper bound to the empty set, hence no supremum.

Similarly, we have the analogous definition for lower bounds:

**Definition 1.5**

Let $S \subseteq \mathbb{R}$. We say that $\alpha \in \mathbb{R}$ is the _infimum_ of $S$, written inf $S$, if $\alpha$ is a lower bound of $S$ and $\alpha$ is the greatest lower bound of $S$. That is, concretely, if

(i) $\forall s \in S : s \geqslant \alpha$

(ii) $(\forall s \in S : s \geqslant b) \implies \alpha \geqslant b$

We have the following properties of suprema and infima of real numbers:

**Proposition 1.11**

Let $S, T \subseteq \mathbb{R}$ be non-empty,

(i) if $S \subseteq T$ and $T$ is bounded above, then $S$ is bounded above, and $\sup S \leqslant \sup T$

(ii) if $T$ is bounded below and we let $S = \{-t : t \in T\}$, then $S$ is bounded above; furthermore, inf $T$ exists, and $\inf T = -\sup S$.

Note how the completeness axiom is phrased in terms of the supremum, but part (ii) of the above proposition tells us that infimum also works.

We can define some more familiar concepts:

### Definition 1.6

Let $S \subseteq \mathbb{R}$ be non-empty. Take $M \in \mathbb{R}$. We say that $M$ is the _maximum_ of $S$ if $M \in S$ and $M$ is an upper bound for $S$.

### Definition 1.7

Let $S \subseteq \mathbb{R}$ be non-empty. Take $m \in \mathbb{R}$. We say that $m$ is the _minimum_ of $S$ if $m \in S$ and $m$ is a lower bound for $S$.

Remark that if $S$ is empty or $S$ is not bounded above, then $S$ does not have a maximum. Also, for a non-empty, bounded above set $S \subseteq \mathbb{R}$, $S$ has a maximum if and only if $\sup S \in S$; also, if $S$ has a maximum, then $\max S = \sup S$.

### Proposition 1.12 (Approximation Property)

Let $S \subseteq \mathbb{R}$ be non-empty and bounded above. For any $\varepsilon > 0$, there is some $s_\varepsilon \in S$ such that $\sup S - \varepsilon < s_\varepsilon \leqslant \sup S$.

_Proof._ Let $\varepsilon > 0$. By the definition of supremum, we know that $s \leqslant \sup S$ holds for all $s \in S$. For the first part of the inequality, AFSOC that $\forall s \in S : \sup S - \varepsilon \geqslant s$. So $\sup S - \varepsilon$ is an upper bound by definition, but we also have $\sup S - \varepsilon < \sup S$, which is a contradiction. Hence $\exists s_\varepsilon \in S : \sup S - \varepsilon < s_\varepsilon \leqslant \sup S$. $\qquad\square$

**Existence of Roots**

We can use the ideas of supremum and completeness to prove the existence of roots.

### Theorem 1.13

There exists a unique positive real number $\alpha$ such that $\alpha^2 = 2$.

_Proof._ We will split the proof into two parts.

Existence
Define a set $S = \left\{ s \in \mathbb{R} : s^2 < 2 \right\}$. Note that $1 \in S$ hence $S$ is non-empty. Also, 2 is an upper bound of $S$ since $s > 2 \implies s^2 > 4 \implies s \notin S$. Hence, by completeness, $S$ has a supremum and we can let $\alpha = \sup S$. By definition, $\alpha \geqslant 1$ hence $\alpha$ is positive.
We consider the following cases for $\alpha^2$:

Case 1: $\alpha^2 < 2$. Then we can write $\alpha^2 = 2 - \varepsilon$ for some $\varepsilon > 0$. Consider for some $h \in (0,1)$,

$$(\alpha + h)^2 = \alpha^2 + 2\alpha h + h^2$$
$$\leqslant 2 - \varepsilon + 4h + h$$
$$= 2 - \varepsilon + 5h$$

We want this quantity to be strictly less than 2 (so $\alpha + h \in S$ which creates a contradiction), so let $h = \min\left(\frac{\varepsilon}{10}, \frac{1}{2}\right)$ (this is to ensure $h < 1$). Then $(\alpha + h)^2 \leqslant 2 - \frac{\varepsilon}{2} < 2$ and hence $\alpha + h \in S$. But $h > 0$ so $\alpha + h > \alpha$, which is a contradiction since $\alpha$ is an upper bound.

Case 2: $\alpha^2 > 2$. Then we can write $\alpha^2 = 2 + \varepsilon$ for some $\varepsilon > 0$. Consider for some $h \in (0,1)$,

$$(\alpha - h)^2 = \alpha^2 - 2\alpha h + h^2$$
$$\geqslant 2 + \varepsilon - 4h$$

We want this quantity to be strictly greater than 2, so let $h = \min\left(\frac{\varepsilon}{8}, \frac{1}{2}, \frac{\alpha}{2}\right)$ (this is to ensure $h < 1$ and $\alpha - h > 0$). Also, by approximation, $\exists s \in S : \sup S - h = \alpha - h < s$. Then $2 < 2 + \frac{\varepsilon}{2} \leqslant (\alpha - h)^2 < s^2 < 2$, a contradiction.

Thus, by trichotomy, we conclude that $\alpha^2 = 2$. Hence such an $\alpha$ exists.

Uniqueness
Let $\beta$ be a positive real number with $\beta^2 = 2$. Then $a^2 = 2 = \beta^2$, so

$$\alpha^2 - \beta^2 = 0$$
$$(\alpha + \beta)(\alpha - \beta) = 0$$

Hence $\alpha + \beta = 0$ or $\alpha - \beta = 0$, but $\alpha, \beta > 0$, so $\alpha + \beta > 0$; hence $\alpha - \beta = 0 \implies \alpha = \beta$. $\qquad\square$

With this theorem proved, we have a rather interesting way of proving the non-completeness of the rational numbers.

Theorem 1.14

$\mathbb{Q}$ is not complete (with the ordering inherited from $\mathbb{R}$).

*Proof.* If $\mathbb{Q}$ were complete, then the proof of Theorem 1.13 would work as well in $\mathbb{Q}$. But we know that there is not an element of $\mathbb{Q}$ which squares to 2. So $\mathbb{Q}$ is not complete by contradiction. $\square$

Additionally, we can generalize Theorem 1.13.

## Theorem 1.15

Let $n$ be an integer with $n \geqslant 2$, and let $r$ be a positive real number. Then $r$ has a real $n^{\text{th}}$ root.

### Further Consequences of Completeness

There is also an important property of the natural numbers, which is a consequence of the completeness of real numbers.

## Theorem 1.16 (Archimedean Property of $\mathbb{N}$)

$\mathbb{N}$ is not bounded above.

*Proof.* AFSOC that $\mathbb{N}$ is bounded above. Then $\mathbb{N}$ is non-empty and bounded above, so, by completeness, $\mathbb{N}$ has a supremum. Also, by approximation with $\varepsilon = \frac{1}{2}$, we have that

$$\exists n \in \mathbb{N} : \sup \mathbb{N} - \frac{1}{2} < n \leqslant \sup \mathbb{N}$$

Now $n + 1 \in \mathbb{N}$ and $n + 1 > \sup \mathbb{N}$, a contraction. $\square$

## Corollary 1.17

Let $\varepsilon > 0$. Then there is some $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \varepsilon$.

*Proof.* Otherwise, $\frac{1}{\varepsilon}$ would be an upper bound for $\mathbb{N}$, which contradicts Theorem 1.16. $\square$

We also have the following useful theorem.

## Theorem 1.18

Let $S$ be a non-empty subset of $\mathbb{Z}$.

(i) If $S$ is bounded below, then $S$ has a minimum.

(ii) If $S$ is bounded above, then $S$ has a maximum.

We will provide a proof for the first part of the theorem, as the second part is analogous to the first.

*Proof.* Let $S$ be a non-empty subset of $\mathbb{Z}$ that is bounded below. So $\inf S$ exists by completeness.

By approximation with $\varepsilon = 1$, we have that

$$\exists n \in S : \inf S \leqslant n < \inf S + 1$$

AFSOC that $\inf S < n$. Then we can write $n = \inf S + \delta$ for some $\delta > 0$. Again by approximation with $\varepsilon = \delta$, we have that

$$\exists m \in S : \inf S \leqslant m < \inf S + \delta = n$$

Then $n - m > 0$, but also $n - m \in \mathbb{Z}$ as $\mathbb{Z}$ is closed under subtraction. So $n - m \geqslant 1$. Then $n \geqslant m + 1 \geqslant \inf S + 1$, which is a contraction.

Hence $n = \inf S$ by contraction, so $\inf S \in S$, and $\min S = \inf S$. $\qquad \square$

Furthermore, we can also show the following result to be true.

**Proposition 1.19**

Take $a, b \in \mathbb{R}$ with $a < b$. Then

  (i) $\exists x \in \mathbb{Q} : a < x < b$           (the rationals are dense in the reals)

  (ii) $\exists y \in \mathbb{R} \setminus \mathbb{Q} : a < y < b$     (the irrationals are dense in the reals)

To summarize, up until now, we have shown that $\mathbb{R}$ is a complete ordered field.

## 1.4 Countability

**Definition 1.8**

A set $A$ is *finite* if $A = \varnothing$ or there exists some $n \in \mathbb{N}$ such that there is a bijection $f : A \to \{1, 2, \ldots, n\}$. Otherwise, the set is *infinite*.

**Definition 1.9**

Let $A$ be a set. We say that $A$ is

- *countably infinite* if there is a bijection $f : A \to \mathbb{N}$

- *countable* if there is an injection $f : A \to \mathbb{N}$

- *uncountable* if $A$ is not countable

Note that a set is countable if and only if it is countable infinite or finite.

**Proposition 1.20**

Let $A, B$ be sets. If there is an injection $f : A \to B$ and an injection $g : B \to A$, then there is a bijection $h : A \to B$.

We also have the following proposition:

**Proposition 1.21**

Let $A, B$ be countable sets.

(i) If $A$ and $B$ are disjoint[3], then $A \cup B$ is countable

(ii) $A \times B$ is countable

[3] this condition is actually not necessary, but just makes the proof easier

*Proof.* Since $A, B$ are both countable, there are some injections $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$.

(i) Define a mapping $h : A \cup B \to \mathbb{N}$ given by

$$h(x) = \begin{cases} 2f(x) - 1 & x \in A \\ 2g(x) & x \in B \end{cases}$$

This is well-defined as $A$ and $B$ are disjoint. It can be shown that $h$ is injective. So $A \cup B$ is countable.

(ii) Define a mapping $h : A \times B \to \mathbb{N}$ given by

$$h(a, b) = 2^{f(a)} 3^{g(b)}$$

Since we don't need surjection, $h$ is injective and thus satisfies our need by the uniqueness of prime factorization. $\square$

Using the second part of the proposition, we can prove $\mathbb{Q}^+$ is countable. And further with the first part of the proposition, we can prove $\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$ is also countable.

We can also show that the reals are uncountable by using Cantor's diagonal argument.

## 2   Sequences

### Definition 2.1

A _real sequence_, or _sequence of real numbers_, is a function $\alpha : \mathbb{N} \to \mathbb{R}$. We call $\alpha(n)$, usually written as $a_n$, the $n^{\text{th}}$ term of the sequence. We say that $\alpha$ defines the sequence $(a_n)$ with terms $a_1, a_2, a_3, \ldots$.

Similarly, we can define _complex sequences_ by swapping $\mathbb{R}$ for $\mathbb{C}$.

### 2.1   Convergence of Sequences

### Definition 2.2

Let $(a_n)$ be a real sequence, and let $L \in \mathbb{R}$. We say that $(a_n)$ _converges to L as $n \to \infty$_ if

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geqslant N : |a_n - L| < \varepsilon$$

In this case we write $a_n \to L$ as $n \to \infty$ or $\lim\limits_{n \to \infty} a_n = L$, and we say that $L$ is the limit of $(a_n)$.

### Definition 2.3

Let $(a_n)$ be a real sequence. We say that $(a_n)$ _converges_, or is _convergent_, if there is some $L \in \mathbb{R}$ such that $a_n \to L$ as $n \to \infty$. If $(a_n)$ does not converge, then we say it _diverges_, or is _divergent_.

### Definition 2.4

Let $(a_n)$ be a sequence. A _tail_ of $(a_n)$ is a sequence $(b_n)$, where for some natural number $k$ we have $b_n = a_{n+k}$ for $n \geqslant 1$. That is, $(b_n)$ is the sequence obtained by deleting the first $k$ terms of $(a_n)$.

Regarding tails, we have the following lemma:

### Lemma 2.1 (Tails Lemma)

Let $(a_n)$ be a sequence.

(i) If $(a_n)$ converges to a limit $L$, then every tail of $(a_n)$ also converges to the same limit $L$

(ii) If a tail $(b_n) = (a_{n+k})$ of $(a_n)$ converges, then $(a_n)$ converges.

To prove convergence of a sequence, we don't necessarily need to always use the definition of convergence. Below is a useful tool for showing

convergence.

**Proposition 2.2** (Sandwiching, A Special Case)

Let $(a_n)$ and $(b_n)$ be real sequences with $0 \leqslant a_n \leqslant b_n$ for all $n \geqslant 1$. If $b_n \to 0$ as $n \to \infty$, then $a_n \to 0$ as $n \to \infty$.

*Proof.* Let $(a_n)$ and $(b_n)$ be real sequences with $0 \leqslant a_n \leqslant b_n$ for all $n \geqslant 1$. Suppose $b_n \to 0$ as $n \to \infty$, then by definition, if we fix some $\varepsilon > 0$, we know

$$\exists N \in \mathbb{N} : \forall n \geqslant N : |a_n| \leqslant |b_n| < \varepsilon$$

Here, modulus preserves inequality because $a_n, b_n \geqslant 0$.
Hence, by definition, $a_n \to 0$ as $n \to \infty$. $\qquad\square$

**Lemma 2.3**

We have the following convergent sequences:

(i) For some $c \in \mathbb{R}$ with $|c| < 1$. Then $c^n \to 0$ as $n \to \infty$

(ii) Let $a_n = \frac{n}{2^n}$ for $n \geqslant 1$. Then $a_n \to 0$ as $n \to \infty$.

*Proof.*

(i) Write $|c| = \frac{1}{1+y}$ for some $y > 0$. Fix $\varepsilon > 0$. Let $N = \lceil \frac{1}{\varepsilon y} \rceil + 1$. Also, for any $n \geqslant N$, we have, by Bernoulli's inequality (since $y > 0, n \geqslant 1$), that $(1 + y)^n \geqslant 1 + ny$. So

$$|c^n| \leqslant |c|^n = \frac{1}{(1+y)^n} \leqslant \frac{1}{1 + ny} \leqslant \frac{1}{ny} \leqslant \frac{1}{Ny} < \varepsilon$$

So $c^n \to 0$ as $n \to \infty$.

(ii) If $n \geqslant 2$, then, by binomial theorem, $2^n = (1+1)^n \geqslant \binom{n}{2}$. Take $N = \lceil \frac{4}{\varepsilon} + 1 \rceil$, then for any $n \geqslant N$, we have

$$|a_n| = \frac{n}{2^n} \leqslant \frac{n}{\binom{n}{2}} = \frac{2n(n-2)!}{n!} = \frac{2}{n-1} \leqslant \frac{2}{N-1} < \varepsilon$$

So $a_n \to 0$ as $n \to \infty$. $\qquad\square$

We also have the important result.

**Theorem 2.4** (Uniqueness of Limits)

Let $(a_n)$ be a convergent sequence. Then the limit is unique.

*Proof.* Suppose that $a_n \to L_1$ and $a_n \to L_2$ as $n \to \infty$. AFSOC that $L_1 \neq L_2$. Then, we can let $\varepsilon = \frac{1}{2}|L_1 - L_2| > 0$.

Since $a_n \to L_1$ as $n \to \infty$, there is some $N_1$ such that for all $n \geqslant N_1$ there is $|a_n - L_1| < \varepsilon$. Also, since $a_n \to L_2$ as $n \to \infty$, there is also some $N_2$ such that for all $n \geqslant N_2$ there is $|a_n - L_2| < \varepsilon$.

Then, for $n \geqslant \max(N_1, N_2)$, we have $|a_n - L_1| < \varepsilon$ and $|a_n - L_2| < \varepsilon$, so

$$
\begin{aligned}
|L_1 - L_2| &= \big|(L_1 - a_n) + (a_n - L_2)\big| \\
&\leqslant |L_1 - a_n| + |a_n - L_2| \\
&< 2\varepsilon \\
&= |L_1 - L_2|
\end{aligned}
$$

which is a contradiction. So $L_1 = L_2$. $\qquad\square$

## Proposition 2.5

Let $(a_n)$ be a convergent sequence. Then $(|a_n|)$ also converges. Moreover, if $a_n \to L$ as $n \to \infty$, then $|a_n| \to |L|$ as $n \to \infty$.

## Proposition 2.6 (Limits preserve weak inequalities)

Let $(a_n)$ and $(b_n)$ be real sequences with $\forall n \in \mathbb{N} : a_n \leqslant b_n$, and assume that $a_n \to L$ and $b_n \to M$ as $n \to \infty$. Then $L \leqslant M$.

Importantly, limits do not preserve strict inequalities (as a counterexample, consider the sequence $a_n = \frac{1}{n}$).

## Proposition 2.7 (Sandwiching)

Let $(a_n), (b_n), (c_n)$ be real sequences with $a_n \leqslant b_n \leqslant c_n$ for all $n \geqslant 1$. If $a_n \to L$ and $c_n \to L$, then $b_n \to L$ as $n \to \infty$.

*Proof.* Let $(a_n), (b_n), (c_n)$ be real sequences with $a_n \leqslant b_n \leqslant c_n$ for all $n \geqslant 1$. Suppose $a_n \to L$ and $c_n \to L$ as $n \to \infty$ for some $L \in \mathbb{R}$. We fix some $\varepsilon > 0$.

Then, by definition, we know that there are some $N_1$ and $N_2$ such that

$$
\begin{aligned}
\forall n \geqslant N_1 : |a_n - L| < \varepsilon \\
\forall n \geqslant N_2 : |c_n - L| < \varepsilon
\end{aligned}
$$

Take $N = \max(N_1, N_2)$. Then,

$$
\forall n \geqslant N : L - \varepsilon \leqslant a_n \leqslant b_n \leqslant c_n \leqslant L + \varepsilon
$$

So $|b_n - L| < \varepsilon$. Hence $b_n \to L$ as $n \to \infty$. $\qquad\qquad$ □

Finally, we consider the conditions for convergence of a complex sequence.

### Definition 2.5

Let $(z_n)$ be a complex sequence, and let $L \in \mathbb{C}$. We say that $(z_n)$ _converges_ to $L$ as $n \to \infty$ if

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geqslant N : |z_n - L| < \varepsilon$$

Furthermore, we can use convergence of real sequences to determine that of a complex sequence.

### Theorem 2.8 (Convergence of Complex Sequences)

Let $(z_n)$ be a complex sequence. Write $z_n = x_n + iy_n$ with $x_n, y_n \in \mathbb{R}$, so that $(x_n)$ and $(y_n)$ are real sequences. Then $(z_n)$ converges if and only if both $(x_n)$ and $(y_n)$ converge. Moreover, $\lim\limits_{n\to\infty} z_n = \lim\limits_{n\to\infty} x_n + i \lim\limits_{n\to\infty} y_n$.

## 2.2 Bounded Sequences

### Definition 2.6

Let $(a_n)$ be a sequence. We say that $(a_n)$ is _bounded_ if the set $\{a_n : n \geqslant 1\}$ is bounded; that is, there is some $M$ such that $|a_n| \leqslant M$ for all $n \geqslant 1$. Otherwise we say it is _unbounded_.

### Proposition 2.9 (Convergent sequences are bounded)

Let $(a_n)$ be a convergent sequence. Then $(a_n)$ is bounded.

_Proof._ Assume that $a_n \to L$ as $n \to \infty$. Then, by taking $\varepsilon = 1$, we know $\exists N : \forall n \geqslant N : |a_n - L| < 1$. So, for all $n \geqslant N$,

$$\begin{aligned}
|a_n| &= |a_n - L + L| \\
&\leqslant |a_n - L| + |L| \\
&< 1 + |L|
\end{aligned}$$

This shows that a tail of the sequence is bounded. But since $N$ is finite, we can take the maximum of the first $N - 1$ terms, so let

$$M = \max(|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |L|)$$

This is an upper bound for all terms in the sequence. Hence the sequence

is bounded. □

### Definition 2.7

Let $(a_n)$ be a real sequence. We say that $(a_n)$ _tends to infinity_ as $n \to \infty$ if

$$\forall M \in \mathbb{R} : \exists N \in \mathbb{N} : \forall n \geqslant N : a_n > M$$

Similarly, we say $(a_n)$ _tends to negative infinity_ as $n \to \infty$ if

$$\forall M \in \mathbb{R} : \exists N \in \mathbb{N} : \forall n \geqslant N : a_n < M$$

In each case, we write $a_n \to \infty$ and $a_n \to -\infty$ as $n \to \infty$ respectively.

### Lemma 2.10

We have the following useful results regarding some $\alpha \in \mathbb{R}$:

(i) If $\alpha < 0$, then $n^\alpha \to 0$ as $n \to \infty$.

(ii) If $\alpha > 0$, then $n^\alpha \to \infty$ as $n \to \infty$.

### Lemma 2.11

We also have the following useful results regarding some $c \in \mathbb{R}^+$:

(i) If $c < 1$, then $c^n \to 0$ as $n \to \infty$.

(ii) If $c = 1$, then $c^n \to 1$ as $n \to \infty$.

(iii) If $c > 1$, then $c^n \to \infty$ as $n \to \infty$.

## 2.3 Convergent Subsequences

### Theorem 2.12 (Scenic Viewpoints Theorem)

Let $(a_n)$ be a real sequence. Then $(a_n)$ has a monotone subsequence.

_Proof._ Let $V = \{k \in \mathbb{N} : m > k \implies a_m < a_k\}$ be the set of "peaks" (or "scenic viewpoints") of the sequence (i.e. terms which are greater than all subsequent terms).

Case 1: $V$ is infinite. Then say $V$ has elements $k_1 < k_2 < \cdots$. Then for all $r, s$, we have that $r > s \implies k_r > k_s \implies a_{k_r} < a_{k_s}$, so $(a_{k_r})_r$ is a monotonic decreasing subsequence of $(a_n)$.

Case 2: $V$ is finite. Then $\exists N : k > N \implies k \notin V$, so any term beyond $a_N$ is not a "peak". Let $m_1 = N + 1$, then $m_1 \notin V$ so $\exists m_2 > m_1 : a_{m_2} \geqslant a_{m_1}$ by definition of $V$. Similarly, we can inductively construct $m_1 <$

$m_2 < \cdots$ with $a_{m_1} \leqslant a_{m_2} \leqslant \cdots$. So $(a_{m_r})_r$ is a monotonic increasing subsequence of $(a_n)$.

In either case, $(a_n)$ has a monotone subsequence. □

**Theorem 2.13** (Bolzano–Weierstrass Theorem)

Let $(a_n)$ be a bounded real sequence. Then $(a_n)$ has a convergent subsequence.

*Proof.* Let $(a_n)$ be a bounded real sequence. By the Scenic Viewpoints Theorem, $(a_n)$ has a monotone subsequence. Since $(a_n)$ is bounded, this subsequence is also bounded. Hence, by the **??**, this subsequence converges . □

Note that the Bolzano–Weierstrass Theorem potentially could hold for complex sequences as well since it does not involve inequalities. And it turns out that this is true. The result can be obtained by viewing a complex sequence as a sequence of real parts and a sequence of complex parts, and finding a sub-subsequence within a subsequence of the real sequence to guarantee both the real and complex parts converge.

**Corollary 2.14** (L)

t $(z_n)$ be a bounded complex sequence. Then $(z_n)$ has a convergent subsequence.

## 2.4 Cauchy Sequences

To motivate the idea of a Cauchy sequence, note that given a real sequence $(a_n)$ with $a_{n+1} - a_n \to 0$ as $n \to \infty$, it is not necessarily the case that $(a_n)$ converges, as can be demonstrated when $a_n = \sqrt{n}$. Hence we need a stronger condition to assert convergence.

**Definition 2.8** (W)

say that a sequence $(a_n)$ is a *Cauchy sequence* if

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall m, n \geqslant N : |a_n - a_m| < \varepsilon$$

Note that the definition makes sense for both real and complex sequences.

## Proposition 2.15

Let $(a_n)$ be a convergent sequence. Then $(a_n)$ is Cauchy.

*Proof.* Let $(a_n)$ be a sequence which converges to $L$ as $n \to \infty$. Let $\varepsilon > 0$, then $\exists N : n \geqslant N \implies |a_n - L| < \frac{\varepsilon}{2}$. Take any arbitrary $m, n \geqslant N$, then

$$
\begin{aligned}
|a_m - a_n| &= \big|(a_m - L) + (L - a_n)\big| \\
&\leqslant |a_m - L| + |a_n - L| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$
$\square$

## Proposition 2.16

Let $(a_n)$ be a Cauchy sequence. Then $(a_n)$ is bounded.

*Proof.* TODO $\qquad\square$

## Proposition 2.17

Let $(a_n)$ be a Cauchy sequence. Suppose that a subsequence $(a_{n_r})_r$ converges. Then $(a_n)$ converges.

*Proof.* TODO $\qquad\square$

## Theorem 2.18 (Cauchy Convergence Criterion)

Let $(a_n)$ be a sequence. Then $(a_n)$ converges if and only if $(a_n)$ is Cauchy.

*Proof.* ($\Rightarrow$) This is shown in Proposition 2.15.
($\Leftarrow$) Assume that $(a_n)$ is Cauchy. Then $(a_n)$ is bounded by, so by the Bolzano–Weierstrass Theorem, we have that $(a_n)$ has a convergent subsequence. Then, by, $(a_n)$ converges. $\qquad\square$

# 3 Series

## 3.1 Convergence of Series

**Definition 3.1**

Let $(a_k)$ be a sequence. For $n \geqslant 1$, let

$$s_n = a_1 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

This is called a *partial sum* of the series $\sum_{k=1}^{\infty} a_k$.

**Definition 3.2**

Let $(a_k)$ be a sequence. The series $\sum_{k=1}^{n} a_k$ *converges* if the sequence $(s_n)$ of its partial sums converges. If $s_n \to s$ as $n \to \infty$, then we write

$$\sum_{k=1}^{\infty} a_k = s$$

**Proposition 3.1**

Consider the series $\displaystyle\sum_{k=1}^{\infty} a_k$. If it converges, then $a_k \to 0$ as $k \to \infty$.

*Proof.* Let $s_n = \displaystyle\sum_{k=1}^{\infty} a_k$, then $(s_n)$ converges by definition. Say $s_n \to s$ as $n \to \infty$. Then we also have $s_{n-1} \to s$ as $n \to \infty$. So $a_n = s_n - s_{n-1} \to s - s = 0$ as $n \to \infty$. $\qquad\square$

SOME CONTENT IGNORED HERE.

**Definition 3.3**

Let $(a_k)$ be a sequence. We say that $\sum_{k=1}^{\infty} a_k$ *converges absolutely* if $\sum_{k=1}^{\infty} |a_k|$ converges.

Note that the idea of absolute convergence applies for both real and complex series.

CONTENT MISSED HERE.

**Theorem 3.2** (Alternating Series Test)

Let $(u_k)$ be a real sequence, and consider the series $\displaystyle\sum_{k=1}^{\infty}(-1)^{k-1}u_k$. If

- $u_k \geqslant 0$ for $k \geqslant 1$; and

- $(u_k)$ is decreasing; and

- $u_k \to 0$ as $k \to \infty$

then $\displaystyle\sum_{k=1}^{\infty}(-1)^{k-1}u_k$ converges.

## 3.2 Power Series

**Definition 3.4**

A _real power series_ is a series of the form $\displaystyle\sum_{k=0}^{\infty}c_k x^k$, where $\forall k \geqslant 0 : c_k \in \mathbb{R}$ and $x$ is a real variable.

**Definition 3.5**

A _complex power series_ is a series of the form $\displaystyle\sum_{k=0}^{\infty}c_k z^k$, where $\forall k \geqslant 0 : c_k \in \mathbb{C}$ and $z$ is a complex variable.

Typically we take $x$ or $z$ to be non-zero as a power series always converges when $x = 0$ or $z = 0$.

As an example, we define the following.

**Definition 3.6**

Let the _exponential function_ $\exp : \mathbb{C} \to \mathbb{C}$ be defined by

$$\exp(z) = \sum_{k=0}^{\infty}\frac{z^k}{k!}$$

We also write $e^z$ for $\exp(z)$.

For the domain of this function to be $\mathbb{C}$, we need to make sure that this power series converges for all complex numbers. This is indeed true and we can prove it by using the ratio test.

Similarly, we can now define other familiar functions using power series.

**Definition 3.7**

Let the _sine function_ $\sin : \mathbb{C} \to \mathbb{C}$ be defined by

$$\sin(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

**Definition 3.8**

Let the _cosine function_ $\cos : \mathbb{C} \to \mathbb{C}$ be defined by

$$\cos(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

**Definition 3.9**

Let the _hyperbolic sine function_ $\sinh : \mathbb{C} \to \mathbb{C}$ be defined by

$$\sinh(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$$

**Definition 3.10**

Let the _hyperbolic cosine function_ $\cosh : \mathbb{C} \to \mathbb{C}$ be defined by

$$\cosh(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}$$

Again, by the ratio test, we can show that all these power series converge for all complex numbers.

Intuitively, we think that we can perform algebraic manipulations between convergent series in an expected way. This result can be proven by letting a sequence be the partial sums of a sequence, and applying AoL on that new sequence.

Concretely, if two series $\sum a_k$ and $\sum b_k$ converges to $L$ and $M$ respectively, then the series $\sum (a_k + b_k)$ also converges, and to $L + M$.

Note we can also use this to show the divergence of a series: if we know $\sum a_k$ converges and $\sum b_k$ diverges, then it must be the case that $\sum (a_k + b_k)$ diverges, since otherwise $\sum (a_k + b_k - a_k)$ also converges.

Note also we cannot determine whether the sum of two divergent series diverges or not.

Using this result, we can see that the following is true for some $z \in \mathbb{C}$:

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

$$\cosh z = \frac{1}{2}\left(e^{z} + e^{-z}\right)$$

$$\sinh z = \frac{1}{2}\left(e^{z} - e^{-z}\right)$$

$$e^{iz} = \cos z + i\sin z$$

$$\cos(iz) = \cosh z$$

$$\sin(iz) = \sinh z$$

**Radius of Convergence**

**Definition 3.11**

Let $\sum c_k z^k$ be a power series. We define $R$, its _radius of convergence,_ to be

$$R := \begin{cases} \sup\left\{|z| \in \mathbb{R} : \sum \left|c_k z^k\right| \text{ converges}\right\} & \text{if this exists} \\ \infty & \text{otherwise} \end{cases}$$

**Definition 3.12**

Let $\sum c_k z^k$ be power series with radius of convergence $R$. We call the set $\{z \in \mathbb{C} : |z| < R\}$ the _disc of convergence._

**Proposition 3.3** (Disc of Convergence)

Let $\sum c_k z^k$ be a power series with radius of convergence $R$.

(i) If $R > 0$ and $|z| < R$, then $\sum c_k z^k$ converges absolutely and hence converges

(ii) If $|z| > R$, then $\sum c_k z^k$ diverges

_Proof._ Let $\sum c_k z^k$ be a power series with radius of convergence $R$.

(i) We will consider two cases of $R$.

Case 1: $R \in \mathbb{R}$. Assume $R > 0$ and take $z \in \mathbb{C}$ with $|z| < R$. Then there is some $s$ with $|z| < s < R$. Let $\varepsilon = R - s > 0$. Then by approximation with $\varepsilon$, there is some $\rho$ such that $s = R - \varepsilon < \rho \leqslant R$ and $\sum \left|c_k \rho^k\right|$ converges.

Then $0 \leqslant |z| < \rho$ and $\sum \left| c_k \rho^k \right|$ converges, so by comparison test, $\sum \left| c_k z^k \right|$ converges. Hence this absolute convergence implies convergence of the series $\sum c_k z^k$.

Case 2: $R = \infty$. Similar.

(ii) Take $z \in \mathbb{C}$ with $|z| > R$. AFSOC that $\sum c_k z^k$ converges. Then $c_k z^k \to 0$ as $k \to \infty$, so $(c_k z^k)$ is bounded, so $\exists M$ TODO

$\square$

Note that it is not clear what happens when $|z| = R$—basically, anything can happen.

**Differentiation Theorem**

The differentiation theorem essentially states that within the disc of convergence, we can differentiate term-by-term. More formally,

**Theorem 3.4** (Differentiation Theorem for real power series)

Let $\sum c_k x^k$ be a real power series with radius of convergence $R$. Assume that $0 < R \leqslant \infty$. For $|x| < R$, define $f(x) = \sum_{k=0}^{\infty} c_k x^k$. Then $f(x)$ is well-defined for $|x| < R$. Moreover, if $|x| < R$, then the derivative $f'(x)$ exists, and

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left( \sum_{k=0}^{\infty} c_k x^k \right) = \sum_{k=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left( c_k x^k \right) = \sum_{k=0}^{\infty} k c_k x^{k-1}$$

It is worth noting that this result is not obvious, since it involves exchanging order of two limiting processes (namely, the series and differentiation).