Groups and Group Actions

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1 Groups

1.1 The Group Axioms

Definition 1.1

Let S be a set, a *binary operation* $*$ on S is a map from $S \times S$ to S, mapping a, b to $a * b$.

There are a few key properties that a binary operation can have:

Definition 1.2

Let $*$ be a binary operation on a set S.

- $*$ is *associative* if $\forall a, b, c \in S : (a * b) * c = a * (b * c)$
- $*$ is *commutative* if $\forall a, b \in S : a * b = b * a$

Definition 1.3

Let $*$ be a binary operation on a set S. We say some $e \in S$ is an *identity element* if $\forall a \in S : a * e = e * a = a$.

Definition 1.4

Let $*$ be a binary operation on a set S with identity e. Let $a \in S$, then we say $b \in S$ is an *inverse* of a is $a * b = b * a = e$. We then write $b = a^{-1}$.

We are now ready to define a group:

Definition 1.5

A *group* $(G, *)$ consists of a set G and a binary operation $*$ on G such that

(i) $*$ is associative

(ii) G has an identity under $*$

(iii) for all $a \in G$, the inverse of a exists

Further, a group $(G, *)$ is said to be *abelian* if $*$ is commutative.