HT 2020

Elements of Deductive Logic

Jiaming (George) Yu
jiaming.yu@jesus.ox.ac.uk

September 24, 2023

Contents

1 Mathematical Induction

There are numerous formulations of mathematical induction.

Axiom 1.1 (Weak Principle of Induction, WPI)

If (i) some P is true for the first member of a sequence S (ordered by the natural numbers), and (ii) if P is true for any n^{th} member of S, then P is true for the $(n+1)^{\text{th}}$ member of S; then for every $x \in S$, P is true for x.

We call condition (i) the *base case* and condition (ii) the *induction case*. Notation-wise, we can write $P(x)$ for the statement 'P is true for x'. This way, we can rewrite WPI symbolically as:

$$P(s_0) \wedge \forall n(P(s_n) \models P(s_{n+1})) \models (\forall x \in S)P(x)$$

Axiom 1.2 (Strong Principle of Induction, SPI)

If, for every n, if P is true for all of s_m whenever $m < n$, then $P(s_n)$; then for every $x \in S$, $P(x)$.[1]

$$\forall n\big(\forall m(m < n \models P(s_m)) \models P(s_n)\big) \models (\forall x \in S)P(x)$$

[1] The base case is implied when $n = 0$

Axiom 1.3 (The Least Number Principle, LNP)

For any non-empty subset M of $\mathbb{N}$, M has a least member.

We now claim that all of WPI, SPI, and LNP are logically equivalent.

Theorem 1.4

The SPI follows from the WPI.

Proof. Let P be a proposition defined on $\mathbb{N}$.[2] Assume the WPI and that $\forall n\big(\forall m(m < n \models P(m)) \models P(n)\big)$. Define $Q(n)$ as the proposition $\forall m(m < n \models P(m))$. $Q(0)$ is vacuously true. Now assume $Q(n)$; combining $Q(n)$ and the assumption, we have that $P(n)$, hence by definition $Q(n+1)$ is true. Therefore, applying the WPI, we have that $(\forall n \in \mathbb{N})Q(n)$, and therefore $(\forall n \in \mathbb{N})P(n)$.[3] $\square$

[2] We can always transform those ordered by $\mathbb{N}$ to ones defined on $\mathbb{N}$

[3] Some extra reasoning could be used here but I will leave as is

Theorem 1.5

The WPI follows from the LNP.

Proof. Let P be a proposition defined on $\mathbb{N}$. Assume $P(0)$ and $P(n) \models P(n+1)$. Also let $M \subseteq \mathbb{N}$ be the set containing those m where $\neg P(m)$.[4] Assume M has a least member m (with $m > 0$), then by definition, $\neg P(m)$ and $P(m-1)$. But by assumption, $P(m-1)$ entails $P(m)$. Hence by contradiction, M has no least member. Hence by the contrapositive of LNP, it follows that M is empty. So we conclude $(\forall n \in \mathbb{N})P(n)$. $\square$

[4] Observe that we are assuming we can construct a set with all and only those elements satisfying a property, it turns out that this isn't a trivial result, but it is outside the scope of this course.

Theorem 1.6

The LNP follows from the SPI.

Proof. Let $M \subseteq \mathbb{N}$ be a set without a least member. Let $P(n)$ be the proposition $n \notin M$. Now assume, for a given n, all the m's strictly less than n satisfy $P(m)$. Since n is the least number larger than all the possible m's, this entails that $n \notin M$, because otherwise it would mean that M does have a least member — n; therefore $P(n)$. Applying SPI, we conclude that $(\forall n \in \mathbb{N})P(n)$, and thus M is an empty set. This proves the contrapositive of the LNP and hence the LNP. $\square$

These principles of induction can be very helpful when proving results about formulas. To do this, we usually want to perform induction on the complexity of formulas.

Definition 1.1

Let ϕ be an -formula. Denote the set of all connectives and sentence letters in ϕ by $\mathrm{Conn}(\phi)$ and $\mathrm{SenLett}(\phi)$ respectively. The *complexity* of ϕ, denoted by $\mathrm{Comp}(\phi)$, is the total number of connectives in ϕ, that is, $\mathrm{Comp}(\phi) = |\mathrm{Conn}(\phi)|$.

2 Metatheory of

Definition 2.1

A <u>*literal*</u> is any sentence letter or negated sentence letter.

Definition 2.2

Let $^+$ be an extension of with two extra atomic sentences $\top$ and $\bot$ where $|\top|_{\mathcal{A}} = 1$ and $|\bot|_{\mathcal{A}} = 0$ for all structures $\mathcal{A}$. Importantly, $\top$ and $\bot$ are both atoms but not sentence letters.[5]

[5] In , atoms and sentence letters can be used interchangeably, but it is no longer the case in $^+$.

Lemma 2.1 (Relevance Lemma)

Let ϕ be a sentence and $\mathcal{A}, \mathcal{B}$ structures. If for all $\alpha \in \mathrm{SenLett}(\phi)$ there is $|\alpha|_{\mathcal{A}} = |\alpha|_{\mathcal{B}}$, then $|\phi|_{\mathcal{A}} = |\phi|_{\mathcal{B}}$.

Proof. We prove by induction on the complexity of ϕ.
When $\mathrm{Comp}(\phi) = 0$, ϕ is either a sentence letter α or one of $\top$ and $\bot$ (in the case of $^+$). In either case, it follows immediately that $|\phi|_{\mathcal{A}} = |\phi|_{\mathcal{B}}$.
For the inductive step, let $\mathrm{Comp}(\phi) = n$ and assume that the desired result holds for all sentences with complexity strictly less than n. Next, we consider all possible forms of ϕ in terms of ψ and χ (both with complexity strictly less than n):

Case 1. $\phi = \neg\psi$. Then $|\phi|_{\mathcal{A}} = 1 - |\psi|_{\mathcal{A}} = 1 - |\psi|_{\mathcal{B}} = |\phi|_{\mathcal{B}}$.

Case 2. $\phi = \psi \wedge \chi$.

Case 3. $\phi = \psi \vee \chi$.

Case 4. $\phi = \psi \models \chi$.

Case 5. $\phi = \psi \leftrightarrow \chi$.

For cases 2–5, we know that the value of each compound formula is determined exactly by the value of ψ and χ; but we also know by induction hypothesis that $|\psi|_{\mathcal{A}} = |\psi|_{\mathcal{B}}$ and $|\chi|_{\mathcal{A}} = |\chi|_{\mathcal{B}}$, hence $|\phi|_{\mathcal{A}} = |\phi|_{\mathcal{B}}$. $\square$

The above proof serves as an example of a typical proof by induction, and in particularly, by induction on the complexity of sentences. Next, we move on to defining substitutions — a rather complicated one in order to capture our intuitions.

Definition 2.3

Definition 2.4

An sentence ϕ is in *disjunctive normal form* (DNF) if there exist natural numbers $n, m_1, \ldots, m_n$ such that

$$\phi = \bigvee_{i=1}^{n} \left[\bigwedge_{i=1}^{m_i} s_{i,j} \right]$$

where all $s_{i,j}$ are literals.[6]

[6] Conjunctive normal forms (CNF) are defined in a parallel way

Theorem 2.2

Every truth function can be expressed by an sentence in DNF.

2.1 Expressive Adequacy

Definition 2.5

A set of connectives is *expressively adequate* if for any truth function f, there exists a sentence containing only those connectives which expresses f.

2.2 Duality

Duality is a semantic concept from the theory of truth functions.

Definition 2.6

A *truth function* is an n-ary function from the set of n-tuples of T's and F's to the set $\{T, F\}$.

Definition 2.7

For a connective c, we say it *expresses* a truth function f if for any sentences ϕ and ψ and any structure $\mathcal{A}$,

$$f_c\left(|\phi_1|_{\mathcal{A}}, \ldots, |\phi_n|_{\mathcal{A}}\right) = |c(\phi_1, \ldots, \phi_n)|_{\mathcal{A}}$$

3 The Syntax of

Definition 3.1

The *alphabet* of the language consists of the following types of characters:

1. Sentence letters: $P, Q, R, P_1, Q_1, R_1, P_2, Q_2, R_2, \ldots$
2. Logical connectives: $\neg, \wedge, \vee, \models, \leftrightarrow$
3. Parentheses: $(,)$

Definition 3.2

A *string* in a language $\mathcal{L}$ is any finite, ordered sequence of characters from the alphabet of $\mathcal{L}$.

Definition 3.3

The *sentences* of are defined in the following manner:

1. All sentence letters are sentences of .
2. If ϕ and ψ are sentences of , then so are:
   - $\neg\phi$
   - $\phi \wedge \psi$
   - $\phi \vee \psi$
   - $\phi \models \psi$
   - $\phi \leftrightarrow \psi$
3. Nothing else is a sentence of .