Pure Mathematics Cheat Sheet

Prelims 2021

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February 9, 2022

Contents

Part I

Linear Algebra

1 Matrices

(a) A square matrix A is *skew-symmetric* or *antisymmetric* if $A^T = -A$.

(b) A square matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ is *orthogonal* (i.e. $A^T = A^{-1}$) if and only if its columns or rows are mutually orthogonal vectors. A is also *orthogonal* if and only if $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{\text{col}}$

2 Vector Spaces and Subspaces

(a) If for some non-empty set V, we have $\mathbf{0}_V \in V$, V is closed under addition, and V is closed under scalar multiplication. Then we can say V (along with the addition and scalar multiplication maps) is a vector space given natural definitions of vector addition and scalar multiplication.

(b) Let V be a vector field over $\mathbb{F}$. A subspace U of V is a non-empty (usually, show $\mathbf{0}_V \in U$) subset of V that is closed under addition and scalar multiplication. We denote $U \leqslant V$.

The *zero subspace* or *trivial subspace* $\{\mathbf{0}_V\}$ and V are always subspaces of V.

Note subspaces are transitive.

(c) **Subspace Test**

For a vector field V over $\mathbb{F}$ and $U \subseteq V$, $U \leqslant V$ if and only if $\mathbf{0}_V \in U$ and $\lambda u_1 + u_2 \in U$ for all $u_1, u_2 \in U$ and $\lambda \in \mathbb{F}$.

(d) For $U, W \leqslant V$, both $U + W$ and $U \cap W$ are subspaces of V. Specifically, $U + W$ is the smallest subspace of V containing both U and W; $U \cap W$ is the largest subspace of V contained in both U and W.

(e) The only subspaces of $\mathbb{R}^n$ are $\mathbb{R}^k$ hyperplanes which intersect the origin, where $k = 0, 1, \ldots, n$.

3 Bases

(a) The *span* of $u_1, u_2, \ldots, u_n \in V$ is $u_1, u_2, \ldots, u_n \leqslant V$ where

$$u_1, u_2, \ldots, u_n = \{\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n : \alpha_1, \ldots, \alpha_n \in \mathbb{F}\}$$

This is the smallest subspace of V containing $u_1, \ldots, u_n$.

More generally, the *span* of $S \subseteq V$ is

$$S = \{\alpha_1 v_1 + \cdots + \alpha_m v_m : m \geqslant 0,\ v_1, \ldots, v_m \in S,\ \alpha_1, \ldots, \alpha_n \in \mathbb{F}\}$$

While S may potentially be infinite, all linear combinations must be finite.

(b) For some $S \subseteq V$, we say S *spans* V or S is a *spanning set* for V, if $S = V$.

(c) For a matrix M, we write (M) for the *row space* of M (i.e. the span of the rows of M) and (M) for the *column space* of M.

(d) We say $v_1, \ldots, v_n \in V$ are *linearly independent* if the only solution to

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = \mathbf{0}_V$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ is $\alpha_1 = \cdots = \alpha_n = \mathbf{0}_V$.

We say $S \subseteq V$ is linearly independent if every finite subset of S is linearly independent.

(e) For linearly independent $v_1, \ldots, v_n \in V$, then

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = \beta_1 v_1 + \cdots + \beta_n v_n$$

if and only if $\alpha_i = \beta_i$ for $i = 1, \ldots, n$.

(f) For linearly independent $v_1, \ldots, v_n \in V$, if for some $v_{n+1} \in V$ there is $v_{n+1} \notin v_1, \ldots, v_n$, then $v_1, \ldots, v_n, v_{n+1}$ are linearly independent.

(g) A *basis* of V is a linearly independent, spanning set of V.

If V has a finite basis, we say V is *finite-dimensional*.

(h) $S \subseteq V$ is a basis of V if and only if every $v \in V$ can be uniquely expressed as a linear combination of elements of S. The scalars in such an expression are the *coordinates* of v w.r.t. the basis S.

Part II
Groups and Group Actions

4 Groups

(a) a

Part III
Analysis

5 Sequences

(a) **Scenic Viewpoints Theorem**

Any real sequence (a_n) has a monotone subsequence.

(b) **Bolzano–Weierstrass Theorem**

Any bounded real sequence (a_n) has a convergent subsequence.

(c) A sequence (a_n) is *Cauchy* if

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall m, n \geqslant N : |a_n - a_m| < \varepsilon$$

(d) A sequence (a_n) converges if and only if it is Cauchy.

6 Series

(a) If a series $\sum_{k=1}^{n} a_k$ converges, then $a_k \to 0$ as $k \to \infty$.

(b) **Cauchy Convergence Criterion for Series**

The series $\sum_{k=1}^{\infty} a_k$ converges if and only if

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n > m \geqslant N : \left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon \tag{6.1}$$

(c) Absolute convergence implies convergence.

(d) **Comparison Test**

For real sequences (a_k) and (b_k) with $0 \leqslant a_k \leqslant b_k$ for all $k \geqslant 1$, if $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges too.

(e) **Limit Form of Comparison Test**

For real, positive sequences $(a_k), (b_k)$ such that $\frac{a_k}{b_k} \to L$ as $k \to \infty$, the series $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} a_k$ converges.

(f) **Alternating Series Test**

For a real, non-negative, decreasing (not necessarily strictly) sequence (u_k) with $u_k \to 0$ as $k \to \infty$, the series $\sum_{k=1}^{\infty} (-1)^{k-1} u_k$ converges.

(g) **Ratio Test**

For a real, positive sequence (a_k) with $\frac{a_{k+1}}{a_k} \to L$ as $k \to \infty$, the series $\sum_{k=1}^{\infty} a_k$ converges if $0 \leqslant L < 1$, and diverges if $L > 1$; if $L = 1$, we conclude nothing.

(h) **Integral Test**

Let $f : [1, \infty) \to \mathbb{R}$ be non-negative and decreasing, and that $\int_k^{k+1} f(x)\,\mathrm{d}x$ exists for each $k \geqslant 1$ (can be implied from continuity). Let $s_n = \sum_{k=1}^{n} f(k)$ and $I_n = \int_1^n f(x)\,\mathrm{d}x$, then the series (s_n) converges if and only if (I_n) converges.

Note that if we define $\sigma_n = s_n - I_n$, then (σ_n) converges, say, to σ, and $0 \leqslant \sigma \leqslant f(1)$.

(i) The Euler constant $\gamma \in [0, 1]$ is defined as the limit of (γ_n) where

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{\mathrm{d}x}{x} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$$

7 Continuity

(a) Let $f : E \to \mathbb{R}$ and $p \in E$. We say f is continuous at p if

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in E : \Big(|x - p| < \delta \implies \big| f(x) - f(p) \big| < \varepsilon \Big)$$

In other words, $f(p)$ is well-defined, $\lim_{x \to p} f(x)$ exists and is equal to $f(p)$.

(b) **Intermediate Value Theorem**

Let $f : [a, b] \to \mathbb{R}$ be continuous and c be between $f(a)$ and $f(b)$. Then there exists some $\xi \in [a, b]$ with $f(\xi) = c$.

(c) Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is bounded on $[a, b]$ and attains its bounds (supremum and infimum) on $[a, b]$.

8 Differentiability

(a) Let (f_n) be a sequence of real functions on E and f be a real function on E. We say f_n *converges uniformly* to f on E if

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall x \in E, n \leqslant N : \big| f_n(x) - f(x) \big| < \varepsilon$$

9 Differentiability

(a) Let $f : [a, b] \to \mathbb{R}$ be continuous and 1-1 on (a, b) and $x_0 \in (a, b)$. If f is differentiable at x_0 and $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$\frac{\mathrm{d}}{\mathrm{d}y} f^{-1}(y_0) = \frac{1}{f'(f^{-1}(y_0))} \tag{9.1}$$

(b) Local extrema

(c) **Fermat's Theorem**

Let $f : E \to \mathbb{R}$ and let x_0 be a local extremum of f and f is differentiable at x_0. Then $f'(x_0) = 0$.

(d) **Darboux's Intermediate Value Theorem**

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) and let $A \in \mathbb{R}$ with $f'(a) < A < f'(b)$. Then $\exists \xi \in (a, b) : f'(\xi) = A$.

(e) **Rolle's Theorem**

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$. Then $\exists x_0 \in (a, b) : f'(x_0) = 0$.

(f) **Mean Value Theorem**

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then

$$\exists \xi \in (a, b) : f'(\xi) = \frac{f(b) - f(a)}{b - a} \tag{9.2}$$

Writing $h = b - a$ and $\xi = a + \theta(b - a)$ for some $\theta \in (0, 1)$, we obtain

$$f(a + h) = f(a) + f'(a + \theta h)h \tag{9.3}$$

(g) **Cauchy's Mean Value Theorem**

Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), and $\forall x \in (a, b) : g'(x) \neq 0$. Then

$$\exists \xi \in (a, b) : \frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)} \tag{9.4}$$