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B1.2 Set Theory

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1 First Axioms of Set Theory

The language of set theory is the language of first-order predicate calculus (FOPC) equipped with an extra non-logical symbol “ \in ”, which denotes the binary membership relation.

As this is not a course in logic, we will be informal in some formulas, dropping parentheses occasionally, using informal symbols (e.g. \neq)

One of set theory’s objectives is to reduce all of mathematics to set theory, treating all mathematical objects, theorems, and the sort all as objects of set theory. Therefore, in formulating the axioms, we will often assume something is a set, but that is essentially saying it could be anything — a number, a function, a set, etc.

For each axiom, we will formulate it both in natural language and in \mathcal{L} .

Axiom ZF1 (The axiom of extensionality)

Two sets are equal if and only if they have the same elements.

$$\forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y))$$

Axiom ZF2 (Null set axiom)

There exists an empty set containing no elements.

$$\exists x \forall y \neg (y \in x)$$

Theorem 1.1

There is a unique set containing no elements.¹

Proof. By ZF2, there exists an empty set x . Suppose y is another empty set. Then x and y have the same elements (i.e. no elements), so $x = y$ by ZF1. \square

Now that we have the empty set, we need to expand our ‘universe’ of sets to include the non-empty ones.

Axiom ZF3 (Unordered pairs)

Let x, y be two sets, then there exists a set whose elements are exactly x and y .

$$\forall x \forall y \exists z ((x \in z) \wedge (y \in z) \wedge \forall w (w \in z \rightarrow (w = x \vee w = y)))$$

As with ZF2, such an unordered pair z is unique given x, y , and we can introduce the notation $\{x, y\}$ for better readability.² Note also we can form unordered pairs with the same element to get $\{x, x\} = \{x\}$.

¹ Informally, we denote this set by \emptyset , and we can construct an equivalent restatement of any statement containing \emptyset .

² Again, we can still reduce any formula

Definition 1.1 (Ordered pair)

Let x, y be sets. The set $\langle x, y \rangle := \{\{x\}, \{x, y\}\}$ is the *ordered pair* of x and y . We call x the *first coordinate* and y the *second coordinate*.

Theorem 1.2

Let x, y, u, v be sets and suppose $\langle x, y \rangle = \langle u, v \rangle$, then $x = u$ and $y = v$.

Proof. Suppose $\{\{x\}, \{x, y\}\} = \{\{u\}, \{u, v\}\}$.

First suppose $x = y$. Then $\{x, y\} = \{x\}$ and thus $\{\{x\}, \{x, y\}\} = \{\{x\}\}$. By ZF1, each element of $\langle u, v \rangle$ is an element of $\langle x, y \rangle = \{\{x\}\}$, so $\{u\} = \{x\}$. Applying ZF1 again, we get $u = x$. But we also have $\{u, v\} = \{x\} = \{u\}$, so $\{u, v\} = \{u\}$ and thus $v = u$ by ZF1. Therefore $x = y = u = v$. If instead we suppose $u = v$, then we still arrive at the same conclusion by a symmetrical argument.

Therefore we can assume $x \neq y$ and $u \neq v$. Again, by ZF1, $\{x\}$ is an element of $\{\{u\}, \{u, v\}\}$. There are two cases: 1. if $\{x\} = \{u\}$, then $x = u$ by ZF1; 2. if $\{x\} = \{u, v\}$, then $x = u = v$ by ZF1, a contradiction \times .

Similarly, $\{x, y\}$ is an element of $\{\{u\}, \{u, v\}\}$. There are two cases: 1. if $\{x, y\} = \{u\}$, then $u = x = y$, a contradiction \times ; 2. if $\{x, y\} = \{u, v\}$, then y is an element of $\{u, v\}$ by ZF1, but if $y = u$ then $x = u = y$, a contradiction, so $y = v$.

Thus we conclude $x = u$ and $y = v$. \square

Axiom ZF4 (Unions)

Let x be a set, then there exists a set consisting of exactly all the elements of the elements of x .³

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (z \in w \wedge w \in x))$$

³ This 'union' is denoted by $\bigcup x$. Note that this axiom, in contrast to defining $x \cup y$, permits infinite unions.

Axiom ZF5 (Comprehension scheme⁴)

Let x be a set and $\phi(z, w_1, \dots, w_k)$ be a formula in \mathcal{L} , with free variables among z, w_1, \dots, w_k . Then, for any sets w_1, \dots, w_k , there is a set y consisting of the elements z of x for which $\phi(z, w_1, \dots, w_k)$ holds.

$$\forall x \forall w_1 \dots \forall w_k \exists y \forall z (z \in y \leftrightarrow (z \in x \wedge \phi(z, w_1, \dots, w_k)))$$

⁴ also known as the subset axiom scheme or the separation scheme

The above is an axiom scheme, meaning for each formula ϕ , there is an instantiation of an axiom. Note that this scheme provides a way to define sets of the form $\{z \in x : \phi(z, w_1, \dots, w_k)\}$, as compared to the much less restricted version $\{z : \phi(z, w_1, \dots, w_k)\}$ (naive comprehension). A problem which arises from the latter is that it allows z to quantify over a domain that includes y (the set being defined). This gives rise to the (in)famous Russell's paradox. Therefore, with ZF5, we only allow com-

prehension of subsets of a set.

Theorem 1.3

Let x, y be sets. Then their intersection $x \cap y$ and their difference $x \setminus y$ are both sets.

Proof. Considering $\phi(z, w) = z \in w$ and $\phi(z, w) = \neg(z \in w)$ respectively, the result follows by comprehension. \square

Theorem 1.4 (Intersections)

Let x be a non-empty set. Then the intersection $\bigcap x$ of all elements of x is a set.⁵

Proof. By ZF4, we are able to construct $\bigcup x$. The set of interest here is a subset of $\bigcup x$. Hence, we can construct it by comprehension:

$$\bigcap x = \left\{ z \in \bigcup x : \forall y (y \in x \rightarrow z \in y) \right\} \quad \square$$

⁵ We avoid the case where $x = \emptyset$. As we will see in the proof below, it will lead to an invalidity.

Proposition 1.5

Let x be a set. There is no set whose elements are all those sets which are not elements of x .⁶

Proof. \square

⁶ i.e. there is no ‘absolute complement’ of a set

Theorem 1.6

There is no set of all sets.

Proof. ASFOC x is the set of all sets, then by comprehension, we can define $y = \{z \in x : z \notin z\}$. Then, as $y \in x$ we have $y \notin y \leftrightarrow y \in x$, which is the Russell’s paradox, a contradiction \times . \square

1.1 Classes

Classes are an **informal** notion (i.e. not part of the formal language) which can be useful when we want to refer to a collection of sets (e.g. the collection of all sets). Classes are defined by naive comprehension.

Definition 1.2

Let $\phi(x)$ be a formula in \mathcal{L} where x is any set (it is allowed for $\phi(x)$ to contain other free variables). Then a class \mathbf{X} can be defined as

$$\mathbf{X} = \{x : \phi(x)\}$$

2 The Power Set Axiom

Axiom ZF6 (Power Set Axiom)

Let x be a set, then there exists a set whose elements are precisely the subsets of x .⁷

⁷ This (unique) set is the power set of x , denoted $\mathcal{P}(x)$.

Theorem 2.1

The power set of x is uniquely determined by x .

Proposition 2.2 (Cartesian products)

Let X, Y be sets, then there exists a set whose elements are precisely the ordered pairs $\langle x, y \rangle$ for all $x \in X$ and $y \in Y$.⁸

Proof. Let $x \in X$ and $y \in Y$. Then $\{x\}, \{x, y\}$ are both subsets of $X \cup Y$, so $\{x\}, \{x, y\} \in \mathcal{P}(X \cup Y)$. Now let $Z = \mathcal{P}(\mathcal{P}(X \cup Y))$. Then $\{\{x\}, \{x, y\}\}$ is a subset of $\mathcal{P}(X \cup Y)$, so $\langle x, y \rangle = \{\{x\}, \{x, y\}\} \in Z$.

So P contains all the desired ordered pairs. Next we use comprehension to pick out exactly those ordered pairs. Construct the set

$$P = \{z \in Z : \exists x \exists y (x \in X \wedge y \in Y \wedge z = \langle x, y \rangle)\}$$

By definition, all elements of P are ordered pairs of the form $\langle x, y \rangle$ where $x \in X$ and $y \in Y$. Vice versa, all the ordered pairs $\langle x, y \rangle$ are in Z and satisfies the comprehension condition, and so are also in P . \square

⁸ In fact, as we know, this set is also uniquely determined by X and Y . It is the Cartesian product of X and Y , denoted $X \times Y$.

2.1 Relations and Functions

Definition 2.1 (Relation)

Let X, Y be sets. A relation between X and Y is any subset R of $X \times Y$. If $\langle x, y \rangle \in R$, then we write xRy .

Definition 2.2 (Function)

A relation $f \subseteq X \times Y$ is a function (or map) from X to Y if, for all $x \in X$, there exists a unique $y \in Y$ such that $\langle x, y \rangle \in f$. Symbolically,

$$(f \subseteq X \times Y) \wedge \forall x \left(x \in X \rightarrow \exists y \left(y \in Y \wedge \langle x, y \rangle \in f \wedge \forall z (\langle x, z \rangle \in f \rightarrow z = y) \right) \right)$$

If $\langle x, y \rangle \in f$, then we write $y = f(x)$.

The set of all functions from X to Y is denoted Y^X or ${}^X Y$.⁹

Note that since $f \subseteq X \times Y \subseteq \mathcal{P}(\mathcal{P}(X \cup Y))$, we can ‘recover’ parts of $X \cup Y$ from f by $\bigcup \bigcup f$. Namely, $\bigcup \bigcup f = X \cup f[X] \subseteq X \cup Y$.

In fact, $\bigcup \mathcal{P}(X) = X$ always holds, but not vice versa.

⁹ Intuitively, for finite sets X, Y , a function has to choose from $|Y|$ elements for each of the $|X|$ elements, so the set has $|Y|^{|X|}$ elements, and hence the notation.

Definition 2.3 (Domain)

The domain of a function f is a set defined using comprehension by

$$\text{Dom}(f) = \left\{ x \in \bigcup \bigcup f : \exists y (y \in \bigcup \bigcup f \wedge \langle x, y \rangle \in f) \right\}$$

Definition 2.4 (Range)

The range of a function f is a set defined using comprehension by

$$\text{Range}(f) = \left\{ y \in \bigcup \bigcup f : \exists x (x \in \bigcup \bigcup f \wedge \langle x, y \rangle \in f) \right\}$$

Specifically, note that \emptyset is itself a function on \emptyset (definition holds vacuously), which is known as the empty function. In fact, \emptyset is also both injective and surjective as a function, hence $\emptyset^\emptyset = \{\emptyset\}$.

2.2 Equivalence Relations and Order Relations

We now introduce two specific relations.

Definition 2.5 (Equivalence relation)

A relation E on a set X is an equivalence relation if it is

- (i) reflexive: for all $x \in X$, $\langle x, x \rangle \in E$
- (ii) transitive: for all $x, y, z \in X$, if $\langle x, y \rangle, \langle y, z \rangle \in E$, then $\langle x, z \rangle \in E$
- (iii) symmetric: for all $x, y \in X$, if $\langle x, y \rangle \in E$, then $\langle y, x \rangle \in E$

Definition 2.6 (Weak partial order)

A relation R on a set X is a weak partial order if it is

- (i) reflexive: for all $x \in X$, $\langle x, x \rangle \in R$
- (ii) transitive: for all $x, y, z \in X$, if $\langle x, y \rangle, \langle y, z \rangle \in R$, then $\langle x, z \rangle \in R$
- (iii) anti-symmetric¹⁰: for all $x, y \in X$, if $\langle x, y \rangle, \langle y, x \rangle \in R$, then $x = y$

¹⁰ another way to formulate this is: if $\langle x, y \rangle \in R$ and $x \neq y$, then $\langle y, x \rangle \notin R$

Definition 2.7 (Weak total order)

A weak partial order R on a set X is a weak total order if any two elements are comparable, i.e.

$$\forall x \forall y \left((x \in X \wedge y \in X) \rightarrow (\langle x, y \rangle \in R \vee \langle y, x \rangle \in R) \right)$$

Definition 2.8 (Strict partial order)

A relation S on a set X is a strict partial order if it is

- (i) irreflexive: for all $x \in X$, $\langle x, x \rangle \notin S$
- (ii) transitive: for all $x, y, z \in X$, if $\langle x, y \rangle, \langle y, z \rangle \in S$, then $\langle x, z \rangle \in S$
- (iii) *strictly anti-symmetric¹¹: for all $x, y \in X$, at most one of $\langle x, y \rangle \in S$ and $\langle y, x \rangle \in S$ holds

¹¹ in fact, this property already follows from (i) and (ii) and therefore is only added for intuitive purposes

Definition 2.9 (Strict total order)

A strict partial order S on a set X is a strict total order if any two distinct elements are comparable, i.e.

$$\forall x \forall y \left((x \in X \wedge y \in X \wedge x \neq y) \rightarrow (\langle x, y \rangle \in S \vee \langle y, x \rangle \in S) \right)$$

Notation-wise, we often use $<, >$ for strict total orders, \prec, \succ for strict partial orders, \leq, \geq for weak total orders, and \preceq, \succeq for weak partial orders; we often use \sim, \approx for equivalence relations.

3 The Axiom of Infinity and The Natural Numbers

We first give one of the many (set-theoretical) constructions of the natural numbers. We informally define the following symbols:¹²

$$0 = \emptyset, \quad 1 = 0^+ = \{0\} = \{\emptyset\}, \quad 2 = 1^+ = \{0, 1\} = \{\emptyset, \{\emptyset\}\}, \quad \dots$$

¹² This construction has 2 advantages: the set representing n has n elements, and $m < n$ iff $m \in n$.

Given a natural number¹³ n , we can further construct

$$n + 1 = n^+ = \{0, \dots, n - 1, n\} = n \cup \{n\}$$

¹³ let's pretend we know what this is for now!

Definition 3.1 (Successor)

Let x be a set. Then the successor of x is the set $x \cup \{x\}$, denoted by x^+ .

We see immediately from ZF3 (unordered pairs) and ZF4 (unions) that every set has a successor set.

Definition 3.2 (Inductive set)

A set x is an inductive set (also successor set) if $\emptyset \in x$ and whenever $y \in x$, then $y^+ \in x$ as well (in other words, x is closed under successor).

From the axioms we have so far, each individual natural number does exist as a set, however the infinite collection of all natural numbers does not. This is why we need to introduce the following axiom.

Axiom ZF7 (Axiom of infinity)

There is an inductive set.

$$\exists x(\exists y(y \in x \wedge \forall z \neg(z \in y)) \wedge \forall z(z \in x \rightarrow z^+ \in x))$$

Theorem 3.1 (Natural numbers)

There is a unique inductive set which is a subset of any inductive set.¹⁴¹⁵

Proof. Let $\phi(x)$ denote the formula expressing “ x is an inductive set”. Let X be any inductive set. Now define

$$\mathbb{N} := \{x \in X : \forall y(\phi(y) \rightarrow x \in y)\}$$

We claim \mathbb{N} is the set with the desired property. It is immediate from definition that \mathbb{N} is a subset of every inductive set. To show \mathbb{N} is inductive, suppose $x \in \mathbb{N}$, then $x \in X$ so $x^+ \in X$. Also, by definition, x is an element of any inductive set y , hence, x^+ is also an element of any inductive set. So $x^+ \in \mathbb{N}$. Therefore \mathbb{N} is inductive.

It remains to prove the uniqueness of \mathbb{N} . Suppose N is also an inductive set which is a subset of any inductive set. Then $N \subseteq \mathbb{N}$ and $\mathbb{N} \subseteq N$, so $N = \mathbb{N}$. \square

Theorem 3.2 (Proof by induction)

Let $\phi(x)$ be a property of any natural number x . If $\phi(0)$ holds and, for any $x \in \mathbb{N}$, if $\phi(x)$ holds then $\phi(x^+)$ holds, then $\phi(x)$ holds for every $x \in \mathbb{N}$.

Proof. Let $X = \{x \in \mathbb{N} : \phi(x)\}$ by comprehension. By assumption, X is an inductive set. Therefore $\mathbb{N} \subseteq X$ so $X = \mathbb{N}$. \square

Definition 3.3 (Usual ordering on \mathbb{N})

Define the relation R_{\leq} on \mathbb{N} by

$$R_{\leq} = \{\langle m, n \rangle \in \mathbb{N} \times \mathbb{N} : m \in n \vee m = n\}$$

We will write $m \leq n$ if $\langle m, n \rangle \in R_{\leq}$.

Proposition 3.3

Let $m, n \in \mathbb{N}$. Then $m \leq n$ if and only if $m \subseteq n$.

Proof. (\Rightarrow) Suppose $m \leq n$. There are two cases:

Case 1. $m \in n$. $n = 0, 1, 2, 3, \dots, m, m+1, \dots, n-1$

Case 2. $m = n$. Then $m \subseteq n$ follows immediately.

(\Leftarrow) Suppose $m \subseteq n$. \square

¹⁴ QUESTION: Which set exactly is this? Can we construct it? Since there are multiple interpretations of the natural numbers, are these in any way related?

Proposition 3.4

The relation R_{\leq} on \mathbb{N} is a weak total order.

Proof. We will first show R_{\leq} is a partial order. \square

3.1 Recursion on ω

Theorem 3.5

Let X be a set, $x_0 \in X$, and $g : X \rightarrow X$ be a function. Then there exists a unique function $f : \omega \rightarrow X$ such that $f(0) = x_0$ and $f(n^+) = g(f(n))$ for all $n \in \omega$.

3.2 Arithmetic on ω

4 Replacement and Foundation Axioms

The above axioms were proposed by Zermelo and they already allow us to construct $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. The following axioms were proposed by others such as Fraenkel. These complete the Zermelo-Fraenkel Set Theory (ZF).

Definition 4.1

A formula $\phi(x, y)$ is a class function if, for all sets x , there exists a unique set y such that $\phi(x, y)$ holds.

Intuitively, a class function is a function-like object (hence the name) from \mathbf{V} to \mathbf{V} .¹⁶ But since \mathbf{V} is not a set, it is therefore not a function.

¹⁶ \mathbf{V} is the class of all sets

Axiom ZF8 (Replacement scheme)

Let $\phi(x, y)$ be a class function. If X is a set, then the collection Y of sets y such that $\phi(x, y)$ holds for some $x \in X$ is a set.¹⁷

¹⁷ Essentially, the image of a set under a class function is also a set.

Axiom ZF9 (Foundation)

Every non-empty set has an \in -minimal element.

$$\forall x (x \neq \emptyset \rightarrow \exists y (y \in x \wedge x \cap y = \emptyset))$$

ZF9 provides a method to restrict certain strange sets such as the universe of sets \mathbf{V} .

Theorem 4.1

Let x be a set, then $x \notin x$.

Proof. AFSOC there is a set x with $x \in x$. Then let $X = \{x\}$. X is non-empty, so by ZF9, there exists $y \in X$ such that $X \cap y = \emptyset$. The only possible choice for y is x , but $x \in X$ and $x \in x$, so $X \cap x = \{x\} \neq \emptyset$, a contradiction. \square

5 Cardinality

Numbers are used in two ways: 1. to assign sizes to sets, and 2. to order elements of a set. These ideas generalize to cardinal and ordinal numbers in the context of infinite sets.

Definition 5.1 (Comparing cardinalities)

Let X, Y be sets. We say X and Y have the same cardinality (or equinumerous or equipotent), denoted $X \sim Y$ or $|X| = |Y|$ or $\#X = \#Y$, if there is a bijection between them.

We say X has cardinality less than or equal to Y , denoted $X \preceq Y$ or $|X| \leq |Y|$ or $\#X \leq \#Y$, if there is an injection from X to Y .

We say X has cardinality less than Y , denoted $X \prec Y$ or $|X| < |Y|$ or $\#X < \#Y$, if $X \preceq Y$ and $X \not\sim Y$.

Lemma 5.1 (Tarski's fixed point theorem)

Suppose X is a set and $H : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$