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B1.1 Logic

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Contents

Part I

Propositional Calculus

1 Syntax of Propositional Calculus

Definition 1.1 (Alphabet of propositional calculus)

The alphabet of propositional calculus, denoted by $\mathcal{L}_{\text{prop}}$, consists of the following symbols:

- propositional variables: $p_0, p_1, \ldots$
- negation: $\neg$
- binary connectives: $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$
- punctuation marks: (and)

Once we have the alphabet, we introduce the notion of strings, then formulas (which are the 'grammatical' strings).

Definition 1.2 (String)

A <u>*string*</u> (of $\mathcal{L}_{\text{prop}}$) is any finite sequence of symbols from $\mathcal{L}_{\text{prop}}$.[1] The <u>*length*</u> of a string is the number of symbols it has.

[1] There must be no gaps between the symbols.

Definition 1.3 (Formula)

A <u>*formula*</u> (of $\mathcal{L}_{\text{prop}}$) is defined recursively by the following rules:

1. Every propositional variable is a formula
2. If the string ϕ is a formula, then so is $\neg\phi$
3. If the strings ϕ, χ are both formulas, then so are the following:[2]

$$(\phi \wedge \chi) \qquad (\phi \vee \chi) \qquad (\phi \rightarrow \chi) \qquad (\phi \leftrightarrow \chi)$$

4. Nothing else is a formula

We denote the set of all formulas of $\mathcal{L}_{\text{prop}}$ by $\text{Form}(\mathcal{L}_{\text{prop}})$.

[2] Note the parentheses!

When we want to prove a result about a formula, it often is useful to use induction, as demonstrated by the following lemma.

Lemma 1.1

If ϕ is a formula, then exactly one of the following statements is true:

- ϕ is a propositional variable
- the first symbol of ϕ is $\neg$
- the first symbol of ϕ is (

Theorem 1.2 (Unique readability theorem)

A formula can be constructed in only one way. In other words, if ϕ is a formula, then, exactly one of the following holds:

(i) ϕ is p_i for some i

(ii) ϕ is $\neg\psi$ for some unique formula ψ

(iii) ϕ is $(\psi * \chi)$ for some unique formulas ψ, χ and a unique binary connective $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

2 Valuations

Valuation concerns the determination of truth values of formulas.

Definition 2.1 (Valuation)

A *valuation* v is a function $v : \{p_0, p_1, \ldots\} \rightarrow \{T, F\}$.

However, a valuation alone does not provide much power, as we can only determine the truth value of variables and not of complex formulas. To achieve the latter, we need to extend a valuation to all formulas.

2.1 Truth Tables

Given a valuation v, we can extend it uniquely to a function $\tilde{v} : \mathrm{Form}(\mathcal{L}_{\mathrm{prop}}) \rightarrow \{T, F\}$.

ψ	$\neg\psi$
T	F
F	T

ψ	χ	$\psi \wedge \chi$	$\psi \vee \chi$	$\psi \rightarrow \chi$	$\psi \leftrightarrow \chi$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

2.2 Logical Consequence and Logical Validity

Definition 2.2 (Satisfiability)

Let ϕ be a formula and v be a valuation. If $\widetilde{v}(\phi) = T$, we say v *satisfies* ϕ. If ϕ is satisfied by some valuation, then it is *satisfiable*.

Definition 2.3 (Tautology)

Let ϕ be a formula. If ϕ is satisfied by every valuation, then we say ϕ is a *tautology* (or *logically valid*), denoted as $\models \phi$.

Definition 2.4 (Logical consequence)

Let ψ, ϕ be formulas. Then we say ϕ is a *logical consequence* of ψ, denoted $\psi \models \phi$, if for every valuation v,

$$\text{if } \widetilde{v}(\psi) = T \text{ then } \widetilde{v}(\phi) = T$$

Let Γ be a (possibly infinite) set of formulas and ϕ be a formula. Then we say ϕ is a *logical consequence* of Γ, denoted $\Gamma \models \phi$, if for every valuation v,

$$\text{if } \widetilde{v}(\psi) = T \text{ for all } \psi \in \Gamma, \text{ then } \widetilde{v}(\phi) = T$$

Lemma 2.1

Let ψ, ϕ be formulas. Then $\psi \models \phi$ if and only if $\models (\psi \to \phi)$.

Proof. ($\Rightarrow$) Assume $\psi \models \phi$. Let v be any valuation. Whenever $\widetilde{v}(\psi) = T$, we have $\widetilde{v}(\phi) = T$ by definition. Hence, $\widetilde{v}((\psi \to \phi)) = T$ by truth table[3]. On the other hand, if $\widetilde{v}(\psi) = F$ then $\widetilde{v}((\psi \to \phi)) = T$ by TT $\to$. Therefore, $(\psi \to \phi)$ is satisfied by every valuation, so $\models (\psi \to \phi)$.
($\Leftarrow$) Assume $\models (\psi \to \phi)$. Let v be any valuation such that $\widetilde{v}(\psi) = T$. We also have $\widetilde{v}((\psi \to \phi)) = T$, so $\widetilde{v}(\phi) = T$ by TT $\to$. Therefore, $\psi \models \phi$. $\square$

[3] We often abbreviate 'the truth table of the connective $*$' by 'TT $*$'

Lemma 2.2

Let Γ be a set of formulas and ψ, ϕ be formulas. Then $\Gamma \cup \{\psi\} \models \phi$ if and only if $\Gamma \models (\psi \to \phi)$.

3 Logical Equivalence and Adequacy

Definition 3.1 (Logical equivalence)

Two formulas ϕ and ψ are *logically equivalent*, denoted as $\phi \models\!\!\!\!\!\dashv \psi$, if $\phi \models \psi$ and $\psi \models \phi$.[4]

[4] Logical equivalence is what permits us to drop parentheses in some cases, for example chaining $\wedge$ or $\vee$.

Lemma 3.1

Let ϕ, ψ be formulas, then $(\phi \vee \psi) \models\!\!\!\!\!\dashv \neg(\neg\phi \wedge \neg\psi)$.

Proof. Let v be a valuation. Then,

$$\begin{array}{lll} & \tilde{v}(\neg(\neg\phi \wedge \neg\psi)) = F & \\ \text{iff} & \tilde{v}((\neg\phi \wedge \neg\psi)) = T & \text{by TT } \neg \\ \text{iff} & \tilde{v}(\neg\phi) = \tilde{v}(\neg\psi) = T & \text{by TT } \wedge \\ \text{iff} & \tilde{v}(\phi) = \tilde{v}(\psi) = F & \text{by TT } \neg \\ \text{iff} & \tilde{v}((\phi \vee \psi)) = F & \text{by TT } \vee \end{array}$$

The result then follows. □

Below are some more logical equivalences:

Proposition 3.2

- $\neg \bigvee_{i=1}^{n} \phi_i \mathrel{\models\!\!\!\Dashv} \bigwedge_{i=1}^{n} \neg\phi_i$ and $\neg \bigwedge_{i=1}^{n} \phi_i \mathrel{\models\!\!\!\Dashv} \bigvee_{i=1}^{n} \neg\phi_i$ (De Morgan's)
- $(\phi \to \psi) \mathrel{\models\!\!\!\Dashv} (\neg\phi \vee \psi)$
- $(\phi \vee \psi) \mathrel{\models\!\!\!\Dashv} ((\phi \to \psi) \to \psi)$

3.1 Adequacy

Definition 3.2 (Truth function)

The set of partial valuations V_n contains all functions $v : \{p_0, \ldots, p_{n-1}\} \to \{T, F\}$. Then, an *n-ary truth function* is a function $J : V_n \to \{T, F\}$.[5]

[5] Intuitively, a truth function 'evaluates' a certain valuation of variables, similar to how a row in a truth table evaluates the input truth values.

Note that by definition, V_n contains 2^n functions, and thus the number of all n-ary truth functions is 2^{2^n}.

We write $\text{Form}_n(\mathcal{L}_{\text{prop}}) \subset \text{Form}(\mathcal{L}_{\text{prop}})$ for the set of formulas of $\mathcal{L}_{\text{prop}}$ which only contain variables from $\{p_0, \ldots, p_{n-1}\}$. Let $\phi \in \text{Form}_n(\mathcal{L}_{\text{prop}})$, then it uniquely determines an n-ary truth function J_ϕ by

$$\begin{aligned} J_\phi : V_n &\to \{T, F\} \\ v &\mapsto \tilde{v}(\phi) \end{aligned}$$

Intuitively, the evaluation of ϕ on a valuation is essentially a truth function. Hence, J_ϕ is given by the truth table for ϕ.

Definition 3.3 (Adequacy)

A language $\mathcal{L}_{\text{prop}}$ is *adequate* if for every $n \geqslant 1$ and every truth function $J : V_n \to \{T, F\}$, there is some $\phi \in \text{Form}_n(\mathcal{L}_{\text{prop}})$ such that $J_\phi = J$.[6]

[6] As a counterexample, if a language only contains $\wedge$, then no formulas can 'replicate' the case where $v(p_0) = F$, but $J(v) = T$.

Theorem 3.3

The language $\mathcal{L}_{\text{prop}}$ is adequate. Moreover, the subset of $\mathcal{L}_{\text{prop}}$ which only uses the connectives $\neg, \wedge, \vee$ is already adequate (i.e. $\to$ and $\leftrightarrow$ does not add to the expressive power of $\mathcal{L}_{\text{prop}}$).

Proof. We will prove the stronger statement, as the weaker one follows trivially. The idea of the proof is to construct a formula to explicitly describe each 'true' row of the truth table as a possible case, and connect the cases with disjunction.
Let $n \in \mathbb{N}$ and $J : V_n \to \{T, F\}$ be any n-ary truth function.
If $J(v) = F$ for all $v \in V_n$ (a contradiction), then take $\phi := (p_0 \wedge \neg p_0)$. Then for each $n \in V_n$ we have $J_\phi(v) = \widetilde{v}(p_0 \wedge \neg p_0) = F = J(v)$.
Otherwise, define $U := \{v \in V_n : J(v) = T\}$ which is nonempty. For each $v \in U$ and $i < n$, define ψ_i^v as follows:

$$\psi_i^v := \begin{cases} p_i & \text{if } v(p_i) = T \\ \neg p_i & \text{if } v(p_i) = F \end{cases}$$

Finally, let $\psi^v := \bigwedge_{i=0}^{n-1} \psi_i^v$ and $\phi := \bigvee_{v \in U} \psi_v$.
Note, for any valuation $w \in V_n$, the following equivalence holds

$$\begin{array}{lll} & \widetilde{w}(\psi^v) = T & \\ \text{iff} & \widetilde{w}(\psi_i^v) = T \text{ for all } i < n & \text{by TT } \wedge \\ \text{iff} & w = v & \text{by definition of } \psi_i^v \end{array}$$

Therefore,

$$\begin{array}{lll} & \widetilde{w}(\phi) = J_\phi(w) = T & \\ \text{iff} & \widetilde{w}(\psi^v) = T \text{ for some } v \in U & \text{by TT } \vee \\ \text{iff} & w = v \text{ for some } v \in U & \text{by the above} \\ \text{iff} & w \in U & \text{as } v \in U \\ \text{iff} & J(w) = T & \text{by def. of } U \end{array}$$

We have now shown that $J_\phi(w) = J(w)$ for any $w \in V_n$, so $J_\phi = J$. □

In this proof, we come across a useful concept.

Definition 3.4 (Disjunctive normal form)

A *conjunctive clause* is a conjunction of only atoms (i.e. p_i's and $\neg p_i$'s).
A formula is in *disjunctive normal form* (DNF) if it is the disjunction of conjunctive clauses.

Corollary 3.4

Every formula in $\mathcal{L}_{\text{prop}}$ is logically equivalent to one in DNF.

Definition 3.5 (Connective adequacy)

Let S be a set of (truth-functional[7]) connectives. We write $\mathcal{L}_{\text{prop}}[S]$ for the language with connectives S (all else is equal). We say S is *adequate* (or *truth-functionally complete*) if $\mathcal{L}_{\text{prop}}[S]$ is adequate.

[7] each is given by some truth table

We have shown $\{\neg, \wedge, \vee\}$ is adequate in Theorem 3.3. Hence by De

Morgan's, both $\{\neg, \wedge\}$ and $\{\neg, \vee\}$ are adequate. But $\vee$ can be expressed with $\to$ (Proposition 3.2), so $\{\neg, \to\}$ is also adequate.

4 Deductive System for Propositional Calculus

In this section, we aim to prove any logical consequences.

Definition 4.1 (Proof)

A *proof* of ϕ from a set of premises Γ is a finite sequence of statements $\phi_1, \ldots, \phi_n$ such that $\phi_n = \phi$ and for each ϕ_i, one of the following holds:

- $\phi_i \in \Gamma$; or
- ϕ_i is an axiom; or
- ϕ_i follows from previous statements by some rule of inference

Definition 4.2 (The deductive language $\mathcal{L}_0$)

Define the language $\mathcal{L}_0 := \mathcal{L}_{\text{prop}}[\{\neg, \to\}]$.
The axioms for $\mathcal{L}_0$ are, for any $\alpha, \beta, \gamma \in \text{Form}(\mathcal{L}_0)$:

1. $(\alpha \to (\beta \to \alpha))$
2. $\Big(\big(\alpha \to (\beta \to \gamma)\big) \to \big((\alpha \to \beta) \to (\alpha \to \gamma)\big)\Big)$
3. $((\neg\beta \to \neg\alpha) \to (\alpha \to \beta))$

The single rule of inference for $\mathcal{L}_0$ is *modus ponens*:
MP From α and $(\alpha \to \beta)$ infer β.

4.1 The Deduction Theorem for $\mathcal{L}_0$

Theorem 4.1 (Deduction theorem for $\mathcal{L}_0$)

For any $\Gamma \subseteq \text{Form}(\mathcal{L}_0)$ and $\alpha, \beta \in \text{Form}(\mathcal{L}_0)$, if $\Gamma \cup \{\alpha\} \vdash \beta$, then $\Gamma \vdash (\alpha \to \beta)$.

5 Consistency, Completeness, and Compactness

5.1 Consistency

Definition 5.1 (Consistency)

Let $\Gamma \subseteq \text{Form}(\mathcal{L}_0)$. Γ is *consistent* (specifically, $\mathcal{L}_0$-consistent) if there does not exist a formula α such that $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$. Γ is *inconsistent* otherwise.

Definition 5.2 (Maximal consistency)

Let $\Gamma \subseteq \text{Form}(\mathcal{L}_0)$. Γ is *maximal consistent* if Γ is consistent, and for each $\phi \in \text{Form}(\mathcal{L}_0)$, either $\Gamma \vdash \phi$ or $\Gamma \vdash \neg\phi$. Equivalently, Γ is maximal consistent if for each $\phi \in \text{Form}(\mathcal{L}_0)$, if $\Gamma \cup \{\phi\}$ is consistent, then $\Gamma \vdash \phi$.[8]

[8] Γ is maximal in the sense that any formula which is not inconsistent with Γ is provable from Γ.

5.2 Completeness

Theorem 5.1 (The completeness theorem)

Let $\Gamma \subseteq \mathrm{Form}(\mathcal{L}_0)$ and $\phi \in \mathrm{Form}(\mathcal{L}_0)$. If $\Gamma \models \phi$ then $\Gamma \vdash \phi$.

5.3 Compactness

Theorem 5.2 (The compactness theorem for $\mathcal{L}_0$)

Let $\Gamma \subseteq \mathrm{Form}(\mathcal{L}_0)$. Γ is satisfiable if and only if every finite subset of Γ is also satisfiable.

Part II

Predicate Calculus

6 Substitution

Definition 6.1

For any formula $\phi \in \mathrm{Form}(\mathcal{L}_{\mathrm{prop}})$, variable x_i (not necessarily free in ϕ), and term $t \in \mathrm{Term}(\mathcal{L}_{\mathrm{prop}})$, we say t is *free for x_i in ϕ* if any of the following holds:

(i) ϕ is atomic; or

(ii) $\phi = \neg\psi$, and t is free for x_i in ψ; or

(iii) $\phi = (\psi \to \chi)$, and t is free for x_i in both ψ and χ; or

(iv) $\phi = \forall x_i \psi$; or

(v) $\phi = \forall x_j \psi$, $j \neq i$, and x_j does not occur in t, and t is free for x_i in ψ.

Definition 6.2

For any formula $\phi \in \mathrm{Form}(\mathcal{L}_{\mathrm{prop}})$, variable x_i, and term $t \in \mathrm{Term}(\mathcal{L}_{\mathrm{prop}})$