Problem 1.a

$\tilde{L} = f(x, \mu, \Sigma, \Theta, q) - \zeta(g(\Theta) - c)$ when $g(\Theta) = \sum_{k=1}^{k} \theta_k, c = 1$

$f(x, \mu, \Sigma, \Theta, q) = \sum_{i=1}^{n} \sum_{k=1}^{K} \log(\theta_k \mathcal{N}(x_i|\mu_c, \Sigma_c))$- $\sum_{i=1}^{n} \sum_{k=1}^{K} q(z_i = k)$

Try to solve $\nabla \tilde{L} = 0$

First,

$$\begin{aligned} \frac{\partial \widetilde{L}}{\partial \theta_k} &= \frac{\partial}{\partial \theta_k} \left[f(x, \mu, \Sigma, \theta, q) - \zeta \left(\sum_{k=1}^{k} \theta_k - 1 \right) \right] \\ &= \sum_{i=1}^{n} q\left(z_i = k\right) \frac{1}{\theta_k} - \zeta = 0 \end{aligned} \tag{1}$$

$$\Rightarrow \theta_k = \frac{1}{\zeta} \sum_{i=1}^{n} q\left(z_i = k\right) \tag{2}$$

Second,

$$\frac{\partial \hat{L}}{\partial \zeta} = 0 \tag{3}$$

$$\Rightarrow \sum_{k=1}^{k} \theta_k = 1 \tag{4}$$

Therefore,(2)+(3) is

$$\begin{aligned} \sum_{k=1}^{k} \theta_k &= \sum_{k=1}^{k} \frac{1}{\zeta} \sum_{j=1}^{n} q\left(z_i = k\right) \\ &= \frac{1}{\zeta} \sum_{k=1}^{k} \sum_{i=1}^{n} q\left(z_i = k\right) \\ &= \frac{1}{\zeta} \sum_{i=1}^{n} \sum_{k=1}^{k} q\left(z_i = k\right) = \frac{n}{\zeta} = 1 \end{aligned} \tag{5}$$

In result, since $n = \zeta$ we can substitute ζ to n in equation(2) and get:

$$\theta_k = \frac{1}{n} \sum_{i=1}^{n} q\left(z_i = k\right) \tag{6}$$

which is one of the mixture parameters $p(c)$

Problem 1.b

$\tilde{L} = f(X, \mu, \Sigma, \Theta, q) - \sum_{i=1}^{n} \zeta_i(g_i - c)$ when $g(\theta) = \sum_{k=1}^{k} \theta_k, c = 1$

$f(X, \mu, \Sigma, \Theta, q) = \sum_{i=1}^{n} \sum_{k=1}^{K} \log(\theta_k \mathcal{N}(x_i|\mu_c, \Sigma_c))$ - $\sum_{i=1}^{n} \sum_{k=1}^{K} q(z_i = k)$

$$\begin{aligned} \frac{\partial \tilde{L}}{\partial q(z_i = k)} &= \log(\theta_k N(x_i|\mu_k, \Sigma_k)) - \log q(z_i = k) - q(z_i = k) \frac{1}{q(z_i = k) - \zeta_i} && (7) \\ &= \log(\theta_k \cdot N(x_i|\mu_k, \Sigma_k)) - \log q(z_i = k) - 1 - \zeta_i = 0 && (8) \end{aligned}$$

By equation(8) we can get

$$\begin{aligned} \log(q(z_i = k)) &= \log(\theta_k N(x_i|\mu_k, \Sigma_k) - 1 - \zeta_i \\ &= \log\big(\theta_k N(x_i|\mu_k, \Sigma_k) - \log e - \log e^{\zeta_i} \end{aligned}$$

$$q(z_i = k) = \frac{\theta_k}{e^{(1+\zeta_i)}} N(x_i|\mu_k, \Sigma_k) = 1 \tag{9}$$

$$\frac{\partial \tilde{L}}{\partial \zeta_i} = 0 \tag{10}$$

$$\sum_{k=1}^{K} q(z_i = k) = 1 \tag{11}$$

$$\sum_{k=1}^{k} q(z_i = k) = \sum_{k=1}^{k} \frac{\theta_k}{e^{(1+\zeta_i)}} \cdot N(x_i|\mu_k, \Sigma_k) = 1 \tag{12}$$

$$e^{1+\zeta i} = \sum_{k=1}^{k} \theta_k N(x_i|\mu_k, \Sigma_k) \tag{13}$$

In result, we can get the update of q from equation (9)+(13)

$$q(z_i = k) = \frac{\theta_k \cdot N(x_i|\mu_k, \Sigma_k)}{\sum_{k=1}^{K} \theta_k \cdot N(x_i|\mu_k, \Sigma_k)} \tag{14}$$