Math189R SU17
Homework 4
Wednesday, May 31, 2017

Feel free to work with other students, but make sure you write up the homework and code on your own (no copying homework *or* code; no pair programming). Feel free to ask students or instructors for help debugging code or whatever else, though.

The starter files can be found under the Resource tab on course website. The graphs for problem 2 generated by the sample solution could be found in the corresponding zipfile. These graphs only serve as references to your implementation. You should generate your own graphs for submission. Please print out all the graphs generated by your own code and submit them together with the written part, and make sure you upload the code to your Github repository.

1 (Conditioning a Gaussian) Note that from Murphy page 113. "Equation 4.69 is of such importance in this book that we have put a box around it, so you can easily find it." That equation is important. Read through the proof of the result. Suppose we have a distribution over random variables $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$ that is jointly Gaussian with parameters

$$\mu = \begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} \quad \Sigma = \begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{bmatrix},$$

where

$$\mu_1 = \begin{bmatrix}0\\ 0\end{bmatrix}, \ \mu_2 = 5, \ \Sigma_{11} = \begin{bmatrix}6 & 8\\ 8 & 13\end{bmatrix}, \ \Sigma_{21}^T = \Sigma_{12} = \begin{bmatrix}5\\ 11\end{bmatrix}, \ \Sigma_{22} = \begin{bmatrix}14\end{bmatrix}.$$

Compute
(a) The marginal distribution $p(\mathbf{x}_1)$.
(b) The marginal distribution $p(\mathbf{x}_2)$.
(c) The conditional distribution $p(\mathbf{x}_1|\mathbf{x}_2)$
(d) The conditional distribution $p(\mathbf{x}_2|\mathbf{x}_1)$

a) From eqn 4.68, we know

$$p(\vec{x}_1) = \mathcal{N}(x_1 | \vec{\mu}_1, \Sigma_{11}) = \boxed{\mathcal{N}\left(x_1 \middle| \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}6 & 8\\ 8 & 13\end{bmatrix}\right)}$$

also know

$$\mathcal{N}(x|\mu,\Sigma) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right]$$

Plugging things in we get:

$$p(\vec{x}_1) = \frac{1}{(2\pi)^D \begin{vmatrix}6 & 8\\ 8 & 13\end{vmatrix}^{1/2}} \cdot \exp\left[-\frac{1}{2}\left(\vec{x}_1 - \begin{bmatrix}0\\0\end{bmatrix}\right)^T \begin{bmatrix}6 & 8\\ 8 & 13\end{bmatrix}^{-1} \left(\vec{x}_1 - \begin{bmatrix}0\\0\end{bmatrix}\right)\right]$$

$$= \frac{1}{(2\pi)^D (14)^{1/2}} \cdot \exp\left[-\frac{1}{2}\vec{x}_1^T \begin{bmatrix}\frac{13}{14} & -\frac{4}{7}\\ -\frac{4}{7} & \frac{3}{7}\end{bmatrix} \vec{x}_1\right]$$

After checking the answer I realized this was not necessary :(

b) $p(\vec{x}_2) = \mathcal{N}(\vec{x}_2 \mid \vec{\mu}_2, \Sigma_{22})$

$$\boxed{p(\vec{x}_2) = \mathcal{N}(\vec{x}_2 \mid 5, [14])}$$

c) We know that $p(x_1, x_2) = \mathcal{N}(x_1 \mid \mu_{1|2}, \Sigma_{1|2})$ where

$\vec{\mu}_{1|2} = \vec{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\vec{x}_2 - \vec{\mu}_2)$ and $\Sigma_{1|2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$

First, we will find $\vec{\mu}_{1|2}$:

$$\vec{\mu}_{1|2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 11 \end{bmatrix} [14]^{-1} [x_2 - 5]$$

$$= \begin{bmatrix} \frac{5}{14} \\ \frac{11}{14} \end{bmatrix} [x_2 - 5]$$

$$= \begin{bmatrix} \frac{5}{14}(x_2 - 5) \\ \frac{11}{14}(x_2 - 5) \end{bmatrix}$$

Now we will find $\Sigma_{1|2}$

$$\Sigma_{1|2} = \begin{bmatrix} 6 & 8 \\ 8 & 13 \end{bmatrix} - \begin{bmatrix} 5 \\ 11 \end{bmatrix} [14]^{-1} [5 \;\; 11]$$

$$= \begin{bmatrix} 6 & 8 \\ 8 & 13 \end{bmatrix} - \begin{bmatrix} \frac{5}{14} \\ \frac{11}{14} \end{bmatrix} [5 \;\; 11]$$

$$= \begin{bmatrix} 6 & 8 \\ 8 & 13 \end{bmatrix} - \begin{bmatrix} \frac{25}{14} & \frac{55}{14} \\ \frac{55}{14} & \frac{121}{14} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{59}{14} & \frac{57}{14} \\ \frac{57}{14} & \frac{61}{14} \end{bmatrix}$$

So, $\boxed{p(x_1 \mid x_2) = \mathcal{N}\left(\vec{x}_1 \;\middle|\; \begin{bmatrix} \frac{5}{14}(x_2 - 5) \\ \frac{11}{14}(x_2 - 5) \end{bmatrix}, \begin{bmatrix} \frac{59}{14} & \frac{57}{14} \\ \frac{57}{14} & \frac{61}{14} \end{bmatrix}\right)}$

d) $p(x_2 \mid x_1) = \mathcal{N}(x_2 \mid x_{2|1}, \Sigma_{2|1})$ where

$\vec{\mu}_{2|1} = \vec{\mu}_2 + \Sigma_{21}\Sigma_{11}^{-1}(\vec{x}_1 - \vec{\mu}_1)$ & $\Sigma_{2|1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$

Finding $\vec{\mu}_{2|1}$:

$$= 5 + [5 \;\; 11] \begin{bmatrix} 6 & 8 \\ 8 & 13 \end{bmatrix}^{-1} \left[\vec{x}_1 - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]$$

$$= 5 + \left[-\tfrac{23}{14} \;\; \tfrac{13}{7}\right][\vec{x}_1]$$

Finding $\Sigma_{2|1}$

$$\Sigma_{2|1} = [14] - [5 \;\; 11] \begin{bmatrix} 6 & 8 \\ 8 & 13 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$$= [14] - \left[\tfrac{171}{14}\right] = \frac{25}{14}$$

So, $\boxed{p(x_2 \mid x_1) = \mathcal{N}\left(\vec{x}_2 \;\middle|\; 5 + \left[-\tfrac{23}{14} \;\; \tfrac{13}{7}\right][\vec{x}_1], \; \tfrac{25}{14}\right)}$