Integer Value Optimization

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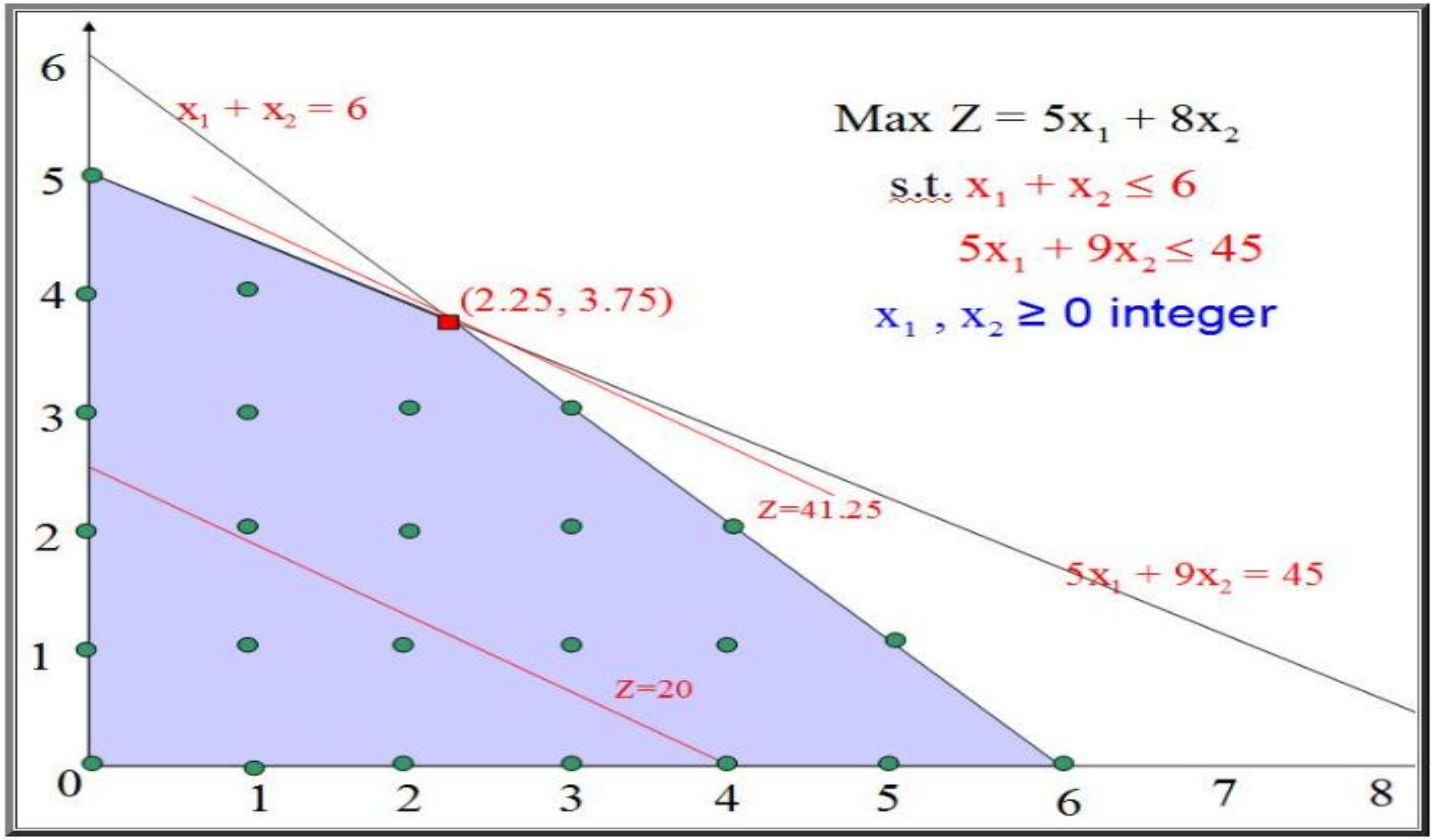
MND 404

Systems

- **Example:**

Linear Program	Corresponding Integer Program
Max $Z = 5x_1 + 8x_2$ s.t. $x_1 + x_2 \le 6$ $5x_1 + 9x_2 \le 45$	Max $Z = 5x_1 + 8x_2$ s.t. $x_1 + x_2 \le 6$ $5x_1 + 9x_2 \le 45$ $x_1, x_2 \ge 0$ integer

The **feasible regions** of the **Linear Program (LP)** and **Integer Program (IP)** are shown in the following figure:



Explanation:

- The **feasible region** defined by the **constraints** of the **Linear Program** is the ***entire* shaded region**

- The ***integer* program** imposes ***more* constraints**

  Therefore:

  - A ***subset*** of the **feasible region** of the **LP** will statify the **constraints** of the **integer program**

  Specifically:

  - The ***"integral" points* inside the feasible (shaded) region** satisfy the **integer value constraints**

  Feasible "region" of the **Integer Program**:

  - ***All*** the **dots** inside the ***shaded* region** form the **feasible region** for the **Integer Program**.

Optimal solution of IP ≤ Optimal solution of LP

- **Relationship between a LP and its derived IP**
  - **Fact 1:**

    - **Optimal solution of *IP* ≤ Optimal solution of LP**

    Reason:

    - We know that:
      - A solution $x_1, x_2, \ldots x_n$ that **satisfies** the constraints of the ***IP* problem** will ***also*** **satisfy** the constraints of the ***LP* problem**

    - **Therefore:**
      - The ***optimal*** **solution** $x_1^{opt}, x_2^{opt}, \ldots x_n^{opt}$ of the ***IP* problem** is a **solution** of the ***LP* problem**

      - In other words:

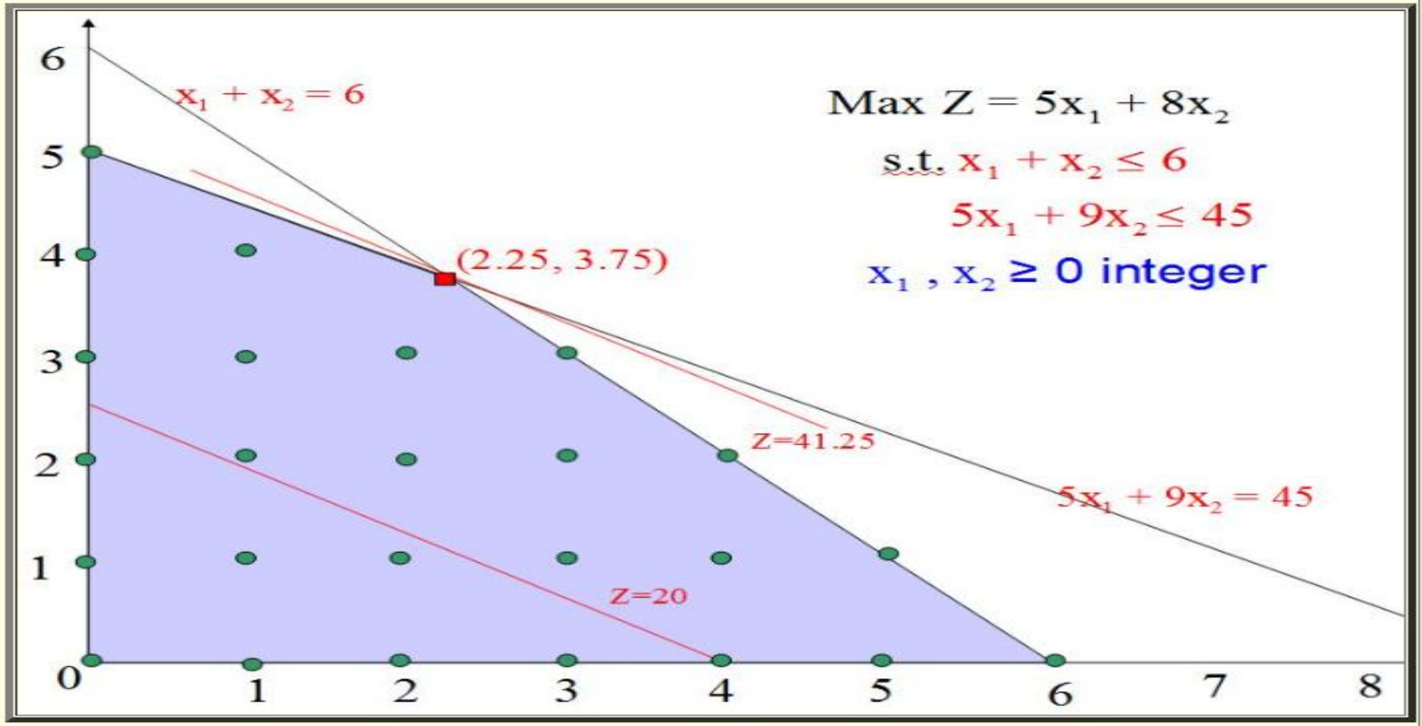
        ```
        OptimalSolution( IP ) ≤ OptimalSolution ( LP )
        ```

Optimal Solution of IP may not be equal to the Optimal Solotion of LP

- **Fact 2:**
  - **Optimal solution of LP** may ***not*** **be equal** to the **optimal solution of the derived *IP***

Example:

- **Consider:**



The **optimal solution** of the **Linear Program (LP)** is:

```
x1 = 2.25        (not integer)
x2 = 3.75        (not integer)
```

- This **solution** is ***not feasible*** for the ***IP*** problem

•

•***IF*** the **optimal solution of LP** is ***integral*** (= all values are integers), ***then*** this solution is ***also*** the **optimal solution of the derived *IP***

•

•***IF*** the **Linear Program LP** is ***infeasible*** (i.e., the ***feasible region*** is **empty**), then the **Integer Program IP** is ***also infeasible***

- **Complexity of *Integer* Programming**

    - **Integer Program** is **hard** to solve:

        - The **Integer Program** problem is a **known** ***NP-hard*** **problem**

          Wikipedia link: **click here**

        - ***"NP-hard"*** **problems** (and the related ***"NP-complete"*** **problems**) are problems where:

            - There are **no known** $O(n^k)$ algorithm for solving the problem

              (Algorithm to solve these problems have **exponential order** of **running time** (i.e., $O(2^n)$)

    - **Practice:**

        - **Integer Programs** are solved using **heuristics**

Class Problem

Consider the following optimization model where the decision variables are x_1 and x_2 and where α is a parameter of the problem that is fixed, *i.e.*, it is not a variable:

$$(\mathcal{P}) \quad \begin{array}{ll} \max & x_1 + \alpha x_2 \\ s.t. & x_1 + 2x_2 \leq 5 \\ & x_2 \geq x_1 - 5 \\ & x_2 \geq 0. \end{array}$$

1. Is this optimization model linear or nonlinear, continuous or mixed integer?
2. Represent the feasible region of this problem.
3. For what value of α does $(\mathcal{P})$ have multiple optimal solutions? Explain.
4. For what value α does $(\mathcal{P})$ have a single optimal solution of value 5? Explain.
5. For each one of the following constraints, describe if the optimization model obtained by adding this constraint to $(\mathcal{P})$ is (i) linear or nonlinear, (ii) feasible or infeasible:
   (a) $x_1^2 + x_2^2 \leq 1$.
   (b) $x_1 \geq 3$.
   (c) $x_1 + x_2^2 \geq 4$.
6. If the objective is changed from maximization to minimization, is the optimization model $(\mathcal{P})$ unbounded? Why?