MND 500

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Table 1.1. Standard and Canonical Forms

	MINIMIZATION PROBLEM	MAXIMIZATION PROBLEM
STANDARD FORM	Minimize $\sum_{j=1}^{n} c_j x_j$ subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1,\ldots, m$ $x_j \geq 0, \quad j = 1,\ldots, n.$	Maximize $\sum_{j=1}^{n} c_j x_j$ subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1,\ldots, m$ $x_j \geq 0, \quad j = 1,\ldots, n.$
CANONICAL FORM	Minimize $\sum_{j=1}^{n} c_j x_j$ subject to $\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = 1,\ldots, m$ $x_j \geq 0, \quad j = 1,\ldots, n.$	Maximize $\sum_{j=1}^{n} c_j x_j$ subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1,\ldots, m$ $x_j \geq 0, \quad j = 1,\ldots, n.$

Canonical is converted to Standard by adding the slack or surplus variables

Canonical-Primal (CP) $\min\langle \vec{c}, \vec{x} \rangle$ $s.t. \begin{cases} A\vec{x} \geq \vec{b} \\ \vec{x} \geq \vec{0} \end{cases}$	Canonical-Dual (CD) $\max\langle \vec{b}, \vec{y} \rangle$ $s.t. \begin{cases} A^T\vec{y} \leq \vec{c} \\ \vec{y} \geq \vec{0} \end{cases}$
Standard-Primal (SP) $\min\langle \vec{c}, \vec{x} \rangle$ $s.t. \begin{cases} A\vec{x} = \vec{b} \\ \vec{x} \geq \vec{0} \end{cases}$	Standard-Dual (SD) $\max\langle \vec{b}, \vec{y} \rangle$ $s.t. \begin{cases} A^T\vec{y} \leq \vec{c} \\ \text{(no sign constraints on } \vec{y}\text{)} \end{cases}$

Introduction to Duality

- For every linear programming problem there is a corresponding linear programming problem called the **dual**.
- If the **original problem is a maximization problem** then the **dual problem is minimization problem** and if the original problem is a minimization problem then the dual problem is maximization problem.
- In either case the **final table of the dual problem will contain both the solution to the dual problem and the solution to the original problem**.
- The solution of the dual problem is readily obtained from the original problem solution if the simplex method is used.
- The formulation of the dual problem also sometimes referred as the concept of duality is helpful for the understanding of the linear programming. The variable of the dual problem is known as the dual variables or shadow price of the various resources.

Dual Problem Formulation

If the original problem is in the standard form then the dual problem can be formulated using the following rules:

- The number of constraints in the original problem is equal to the number of dual variables. The number of constraints in the dual problem is equal to the number of variables in the original problem.
- The original problem profit coefficients appear on the right hand side of the dual problem constraints.
- If the original problem is a maximization problem then the dual problem is a minimization problem. Similarly, if the original problem is a minimization problem then the dual problem is a maximization problem.
- The original problem has less than or equal to ($\leq$) type of constraints while the dual problem has greater than or equal to ($\geq$) type constraints.
- The coefficients of the constraints of the original problem which appear from left to right are placed from top to bottom in the constraints of the dual problem and vice versa.

Rules for decision variables in Dual

Assume that the primal LP is a maximization problem and the dual LP is a minimization problem. Then we have the following rules for obtaining the dual:

Primal (maximization)	Dual (minimization)
Equality constraint	Unrestricted-sign variable
Inequality constraint ($\leq$)	Nonnegative variable
Unrestricted-sign variable	Equality constraint
Nonnegative variable	Inequality constraint ($\geq$)

Example 25 Write the dual of the problem

$$\text{Min } Z = 3x_1 + x_2$$

$$\begin{aligned} \text{S.T.} \quad & 2x_1 + 3x_2 \geq 2 \\ & x_1 + x_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution

The given L.P.P. is in the standard primal form. In matrix notation it is written as

$$\text{Min } Z_P = (3, 1)\,[x_1, x_2] = CX$$

$$\text{S.T.} \qquad \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$AX \geq b$$

The dual of a given problem is

$$\text{Max } Z_D = b'W$$

$$\text{S.T.} \qquad A'W \leq c'$$

$$\therefore \qquad \text{Max } Z_D = [2, 1]\,[W_1, W_2] = 2W_1 + W_2$$

$$\text{S.T.} \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \leq \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Find dual from primal conversion

$$\text{MAX } z = x_1 - x_2 + 3x_3$$

subject to

$$x_1 + x_2 + x_3 \le 10$$

$$2x_1 - x_2 - x_3 \le 2$$

$$2x_1 - 2x_2 - 3x_3 \le 6$$

and $x_1, x_2, x_3 \ge 0$

In primal, There are 3 variables and 3 constraints, so in dual there must be 3 constraints and 3 variables

In primal, The coefficient of objective function $c_1 = 1, c_2 = -1, c_3 = 3$ becomes right hand side constants in dual

In primal, The right hand side constants $b_1 = 10, b_2 = 2, b_3 = 6$ becomes coefficient of objective function in dual

In primal, objective function is maximizing, so in dual objective function must be minimizing

Let $y1, y2, y3$ be the dual variables

Dual is (Solution steps of Dual by **Simplex method**)

MIN $z_y = 10y_1 + 2y_2 + 6y_3$

subject to

$y_1 + 2y_2 + 2y_3 \geq 1$

$y_1 - y_2 - 2y_3 \geq -1$

$y_1 - y_2 - 3y_3 \geq 3$

and $y_1, y_2, y_3 \geq 0;$

Find dual from primal conversion

MIN $z = x_1 - 3x_2 - 2x_3$

subject to

$3x_1 - x_2 + 2x_3 \le 7$

$2x_1 - 4x_2 \ge 12$

$-4x_1 + 3x_2 + 8x_3 = 10$

and $x_1, x_2 \ge 0$ and x_3 unrestricted in sign

Since objective function is minimizing, all $\leq$ constraints (1) can be converted to $\geq$ type by multipling both sides by -1

$$\text{MIN } z_x = x_1 - 3x_2 - 2x_3$$

subject to

$$-3x_1 + x_2 - 2x_3 \geq -7$$

$$2x_1 - 4x_2 \geq 12$$

$$-4x_1 + 3x_2 + 8x_3 = 10$$

and $x_1, x_2 \geq 0$; x_3 unrestricted in sign

In primal, There are 3 variables and 3 constraints, so in dual there must be 3 constraints and 3 variables

In primal, The coefficient of objective function $c_1 = 1, c_2 = -3, c_3 = -2$ becomes right hand side constants in dual

In primal, The right hand side constants $b_1 = -7, b_2 = 12, b_3 = 10$ becomes coefficient of objective function in dual

In primal, objective function is minimizing, so in dual objective function must be maximizing

The x_3 variable in the primal is unrestricted in sign, therefore the 3^{rd} constraint in the dual shall be equality.

Let $y1, y2, y3$ be the dual variables

Since 3^{rd} constraint in the primal is equality, the corresponding dual variable y_3 will be unrestricted in sign.

Dual is (Solution steps of Dual by **Simplex method**)

MAX $z_y = -7y_1 + 12y_2 + 10y_3$

subject to

$-3y_1 + 2y_2 - 4y_3 \le 1$

$y_1 - 4y_2 + 3y_3 \le -3$

$-2y_1 + 8y_3 = -2$

and $y_1, y_2 \ge 0; y_3$ unrestricted in sign

Non-Standard Form Dual

Primal

$$
\begin{aligned}
\max \quad & z = 6x_1 + x_2 + 3x_3 \\
s.t. \quad & 8x_1 + x_2 \leq 10 \\
& x_2 + x_3 = 4 \\
& 6x_1 - 2x_2 + 4x_3 \geq 2 \\
& x_1 \geq 0, x_2 = urs, x_3 \leq 0
\end{aligned}
$$

Dual

$$
\begin{aligned}
\min \quad & \mathrm{w} = 10u_1 + 4u_2 + 2u_3 \\
s.t. \quad & 8u_1 + 6u_3 \geq 6 \\
& u_1 + u_2 - 2u_3 = 1 \\
& u_2 + 4u_3 \leq 3 \\
& \boxed{u_1 \geq 0, u_2 = urs, u_3 \leq 0}
\end{aligned}
$$

$$
\text{Primal LP :} \quad
\begin{array}{rrcrcrcl}
\max z & = & 2x_1 & + & x_2 & & \\
\text{s.t.} & & x_1 & + & x_2 & = & 2 \\
& & 2x_1 & - & x_2 & \geq & 3 \\
& & x_1 & - & x_2 & \leq & 1 \\
& & & & x_1 & \geq & 0, x_2 \text{ urs}
\end{array}
$$

Example

$$
\begin{array}{llrcrcrcl}
\text{Primal LP :} & \max z & = & 2x_1 & + & x_2 & & & \\
& \text{s.t.} & & x_1 & + & x_2 & = & 2 & \\
& & & 2x_1 & - & x_2 & \geq & 3 & \\
& & & x_1 & - & x_2 & \leq & 1 & \\
& & & & & x_1 & \geq & 0, x_2 \text{ urs} &
\end{array}
$$

$$
\begin{array}{llrcrcrcrcl}
\text{Dual LP :} & \min w & = & 2y_1 & - & 3y_2 & + & y_3 & & \\
& \text{s.t.} & & y_1 & - & 2y_2 & + & y_3 & \geq & 2 \\
& & & y_1 & + & y_2 & - & y_3 & = & 1 \\
& & & & & & y_1 \text{ urs}, y_2, & y_3 & \geq & 0
\end{array}
$$

Duality Theorem

Primal:

$$\begin{aligned} \max z &= \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j &\le b_i \quad (i = 1, \ldots, m) \\ x_1, \ldots, x_n &\ge 0 \end{aligned}$$

Dual:

$$\begin{aligned} \min w &= \sum_{i=1}^{m} b_i y_i \\ \text{s.t.} \quad \sum_{i=1}^{m} a_{ij} y_i &\ge c_j \quad (j = 1, \ldots, n) \\ y_1, \ldots, y_m &\ge 0 \end{aligned}$$

Theorem (Duality theorem)

If the primal problem has an optimal solution x^* then the dual problem also has an optimal solution y^* such that

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*.$$

Weak Duality Theorem

Primal

$$
\begin{aligned}
max \quad z &= \textstyle\sum_{j=1}^{j=n} c_j x_j \\
s.t. \quad & \textstyle\sum_{j=1}^{j=n} a_{i,j} x_j \le b_i, \quad i = 1, \dots, m \\
& x_j \ge 0, \quad j = 1, \dots, n
\end{aligned}
$$

Dual

$$
\begin{aligned}
min \quad \mathrm{w} &= \textstyle\sum_{i=1}^{i=m} u_i b_i \\
s.t. \quad & \textstyle\sum_{i=1}^{i=m} a_{j,i} u_i \ge c_j, \quad j = 1, \dots, n \\
& u_i \ge 0, \qquad i = 1, \dots, m
\end{aligned}
$$

- Let **x** = $[x_1, \dots, x_n]$ be any feasible solution to the primal
- Let **u** = $[u_1, \dots, u_m]$ be any feasible solution to the dual
- (z value for **x**) ≤ (w value for **u**)

Example: Weak Duality Theorem

$$
\begin{array}{llr}
\max & z = 30x_1 + 100x_2 & \\
s.t. & x_1 + \quad x_2 & \le 7 \\
 & 4x_1 + 10x_2 & \le 40 \\
\textbf{Primal} & -x_1 & \le -3 \\
 & x_1, x_2 \ge 0 &
\end{array}
$$

(5, 2) is feasible

z=350

$$
\begin{array}{llr}
\min & w = 7u_1 + 40u_2 - 3u_3 & \\
s.t. & u_1 + \quad 4u_2 - \quad u_3 & \ge 30 \\
 & u_1 + 10u_2 & \ge 100 \\
\textbf{Dual} & u_1, u_2, u_3 \ge 0 &
\end{array}
$$

(0, 10, 0) is feasible

w=400

Strong Duality Lemma

Primal

$$\begin{array}{ll} max & z = \sum_{j=1}^{j=n} c_j x_j \\ s.t. & \sum_{j=1}^{j=n} a_{i,j} x_j \leq b_i, \quad i = 1, \dots, m \\ & x_j \geq 0, \quad j = 1, \dots, n \end{array}$$

Dual

$$\begin{array}{ll} min & \mathrm{w} = \sum_{i=1}^{i=m} u_i b_i \\ s.t. & \sum_{i=1}^{i=m} a_{j,i} u_i \geq c_j, \quad j = 1, \dots, n \\ & u_i \geq 0, \qquad i = 1, \dots, m \end{array}$$

- Let **x** = [x_1, …, x_n] be any feasible solution to the primal
- Let **u** = [u_1, …, u_m] be any feasible solution to the dual
- If (z value for **x**) = (w value for **u**), then **x** is optimal for the primal
  u is optimal for the dual

Example: Strong Duality Lemma

$$\begin{aligned} \max\ \ z &= 30x_1 + 100x_2 \\ s.t.\qquad & x_1 + \quad x_2 \le 7 \\ & 4x_1 + 10x_2 \le 40 \\ & -x_1 \qquad\quad \le -3 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Primal

(3, 2.8) is optimal

z*=370

$$\begin{aligned} \min\ \ \mathrm{w} &= 7u_1 + 40u_2 - 3u_3 \\ s.t.\qquad & u_1 + \ 4u_2 - \ u_3 \ge 30 \\ & u_1 + 10u_2 \qquad\ \ge 100 \\ & u_1, u_2, u_3 \ge 0 \end{aligned}$$

Dual

(0, 10, 10) is optimal

w*=370

Weak and Strong Duality

For any feasible solution x to the primal LP and any feasible solution y to the dual LP we have

$$z = \sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i = w \quad \text{[weak duality]}$$

Therefore, if we find x^* (feasible for the primal LP) and y^* (feasible for the dual LP) such that

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

then x^* is an optimal solution of the primal problem and y^* is an optimal solution of the dual problem!

Relation Between Primal and Dual LP Types

		Dual LP		
		Optimal	**Infeasible**	**Unbounded**
Primal LP	**Optimal**	possible	impossible	impossible
	Infeasible	impossible	possible	possible
	Unbounded	impossible	possible	impossible

Example (An infeasible LP with infeasible dual)

$$
\begin{array}{rrcrcl}
\max z & = & 2x_1 & - & x_2 & \\
\text{s.t.} & & x_1 & - & x_2 & \leq 1 \\
& & -x_1 & + & x_2 & \leq -2 \\
& & & & x_1, x_2 & \geq 0
\end{array}
$$

Canonical Form Primal and Dual

Primal

$$\begin{aligned} max \quad z = \textstyle\sum_{j=1}^{j=n} c_j x_j \\ s.t. \quad \textstyle\sum_{j=1}^{j=n} a_{i,j} x_j \le b_i, \quad i = 1, \dots, m \\ x_j \ge 0, \quad j = 1, \dots, n \end{aligned}$$

Dual

$$\begin{aligned} min \quad \mathrm{w} = \textstyle\sum_{i=1}^{i=m} u_i b_i \\ s.t. \quad \textstyle\sum_{i=1}^{i=m} a_{j,i} u_i \ge c_j, \quad j = 1, \dots, n \\ u_i \ge 0, \quad i = 1, \dots, m \end{aligned}$$

Standard Form Primal and Dual

Primal

$$\begin{aligned}
\max \quad & z = \sum_{j=1}^{j=n} c_j x_j \\
s.t. \quad & \sum_{j=1}^{j=n} a_{i,j} x_j + s_i = b_i, \quad i = 1, \dots, m \\
& x_j \geq 0, \quad j = 1, \dots, n \\
& s_i \geq 0, \quad i = 1, \dots, m
\end{aligned}$$

Dual

$$\begin{aligned}
\min \quad & \mathrm{w} = \sum_{i=1}^{i=m} u_i b_i \\
s.t. \quad & \sum_{i=1}^{i=m} a_{j,i} u_i - e_j = c_j, \quad j = 1, \dots, n \\
& u_i \geq 0, \qquad i = 1, \dots, m \\
& e_j \geq 0, \qquad j = 1, \dots, n
\end{aligned}$$

Complementary Slackness Theorem

- Let **x** = $[x_1, \ldots, x_n]$ be a feasible solution to the primal
- Let **u** = $[u_1, \ldots, u_m]$ be a feasible solution to the dual
- Then, **x** is primal optimal and **u** is dual optimal iff

$$s_i u_i = 0 \;\; (i = 1,2, \ldots m) \text{ and}$$
$$e_j x_j = 0 \;\; (j = 1,2, \ldots n)$$

Example

Optimal Primal Solution

$$x_1 = 2, x_2 = 0, x_3 = 8$$
$$s_1 = 24, s_2 = 0, s_3 = 0$$
$$z^* = 280$$

Optimal Dual Solution

$$u_1 = 0, u_2 = 10, u_3 = 10$$
$$e_1 = 0, e_2 = 5, e_3 = 0$$
$$w^* = 280$$

$s_i u_i = 0 \ \ (i = 1,2, \ldots m)$ and
$e_j x_j = 0 \ \ (j = 1,2, \ldots n)$

Use Complementary Slackness to Find the Optimal Sol to the Primal

Primal

$$
\begin{aligned}
max \quad z = 60x_1 + 30x_2 + \ 20x_3 & \\
s.t. \quad 8x_1 + \ 6x_2 + \qquad x_3 + s_1 &= 48 \\
4x_1 + \ 2x_2 + 1.5x_3 + s_2 &= 20 \\
2x_1 + 1.5x_2 + \ 0.5x_3 + s_3 &= 8 \\
x_1, x_2, x_3 \geq 0; s_1, s_2, s_3 &\geq 0
\end{aligned}
$$

Dual Optimal Solution

$$u_1 = 0, u_2 = 10, u_3 = 10$$

$$e_1 = 0, e_2 = 5, e_3 = 0$$

$$w^* = 280$$

Use Complementary Slackness to Find the Optimal Sol to the Primal

Primal

$$
\begin{aligned}
\max \quad z &= 60x_1 + 30x_2 + 20x_3 \\
s.t. \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48 \\
& 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20 \\
& 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8 \\
& x_1, x_2, x_3 \geq 0;\ s_1, s_2, s_3 \geq 0
\end{aligned}
$$

Dual Optimal Solution

$$u_1 = 0, u_2 = 10, u_3 = 10$$

$$e_1 = 0, e_2 = 5, e_3 = 0$$

$$w^* = 280$$

$$s_i u_i = 0 \rightarrow s_1 \geq 0, s_2 = 0, s_3 = 0$$

$$e_j x_j = 0 \rightarrow x_1 \geq 0, x_2 = 0, x_3 \geq 0$$

Primal

$$\begin{aligned} \max \quad & z = 60x_1 + \cancel{30x_2} + 20x_3 \\ s.t. \quad & 8x_1 + \cancel{6x_2} + x_3 + s_1 = 48 \\ & 4x_1 + \cancel{2x_2} + 1.5x_3 + \cancel{s_2} = 20 \\ & 2x_1 + \cancel{1.5x_2} + 0.5x_3 + \cancel{s_3} = 8 \\ & x_1, x_2, x_3 \geq 0; s_1, s_2, s_3 \geq 0 \end{aligned}$$

Dual Optimal Solution

$$u_1 = 0, u_2 = 10, u_3 = 10$$

$$e_1 = 0, e_2 = 5, e_3 = 0$$

$$w^* = 280$$

$$\boxed{\begin{aligned} & s_i u_i = 0 \rightarrow s_1 \geq 0, s_2 = 0, s_3 = 0 \\ & e_j x_j = 0 \rightarrow x_1 \geq 0, x_2 = 0, x_3 \geq 0 \end{aligned}}$$

Primal

$$
\begin{aligned}
max \quad z = 60x_1 + \cancel{30x_2} + 20x_3 \\
s.t. \quad 8x_1 + \cancel{6x_2} + x_3 + s_1 = 48 \\
4x_1 + \cancel{2x_2} + 1.5x_3 + \cancel{s_2} = 20 \\
2x_1 + \cancel{1.5x_2} + 0.5x_3 + \cancel{s_3} = 8 \\
x_1, x_2, x_3 \geq 0; s_1, s_2, s_3 \geq 0
\end{aligned}
$$

Dual Optimal Solution

$$u_1 = 0, u_2 = 10, u_3 = 10$$

$$e_1 = 0, e_2 = 5, e_3 = 0$$

$$w^* = 280$$

$$s_i u_i = 0 \rightarrow s_1 \geq 0, s_2 = 0, s_3 = 0$$

$$e_j x_j = 0 \rightarrow x_1 \geq 0, x_2 = 0, x_3 \geq 0$$

$$s_1 = 24, x_1 = 2, x_3 = 24$$

$$z^* = 280$$

Reverse is also true

- We can take the primal and find the solution to the primal