Mine Automation and Data Analytics

SWAYAM NPTEL COURSE ON

MINE AUTOMATION AND DATA ANALYTICS

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Module 8: Inferential Statistics

Lecture 20 A: Continuous Random Variable Part I

CONCEPTS COVERED

1. Continuous random variable
2. Probability Distribution Function
3. Uniform distribution
4. Exponential distribution
5. Normal distribution

Discrete and Continuous Random Variables

Definition:

A discrete random variable is characterized by its ability to assume, at most, a countable number of possible values.

Consequently, any random variable capable of adopting either a finite number or a countably infinite number of distinct values qualifies as a discrete random variable.

It's worth noting that there are also random variables whose set of potential values is uncountably infinite.

Definition:

Continuous random variables pertain to scenarios where outcomes of random events are numerical, yet cannot be enumerated and are infinitely divisible.

Discrete Random Variable

- A discrete random variable is characterized by having possible values that are distinct points along the real number line.
- Discrete random variables are often associated with counting scenarios

Continuous Random Variable

- A continuous random variable is defined by its possible values spanning an interval along the real number line.
- Continuous random variables typically involve measurement scenarios.

Probability density function (pdf)

- A Probability Density Function (pdf) is a mathematical function that describes the likelihood of a continuous random variable falling within a particular range of values.
- In other words, it provides a way to represent the probability distribution of a continuous random variable. It is denoted by f(x)

Probability density function (pdf)

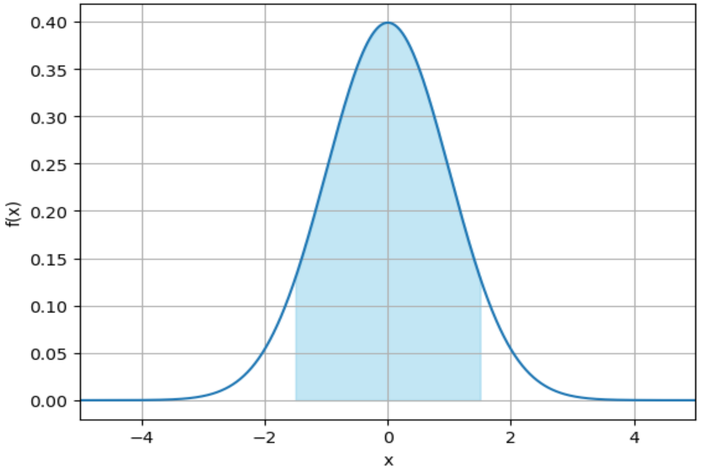
- **The integral of the Probability Density Function curve signifies the likelihood of the random variable existing within a specific interval.**
- **In mathematical terms, for a continuous random variable X with Probability Density Function (f(x)),**

The probability of X lying within the range [a, b] is calculated by integrating the Probability Density Function over that interval.

$$P(X \in [a,b]) = P(a \leq X \leq b) \text{ is area under curve between a and b}$$

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

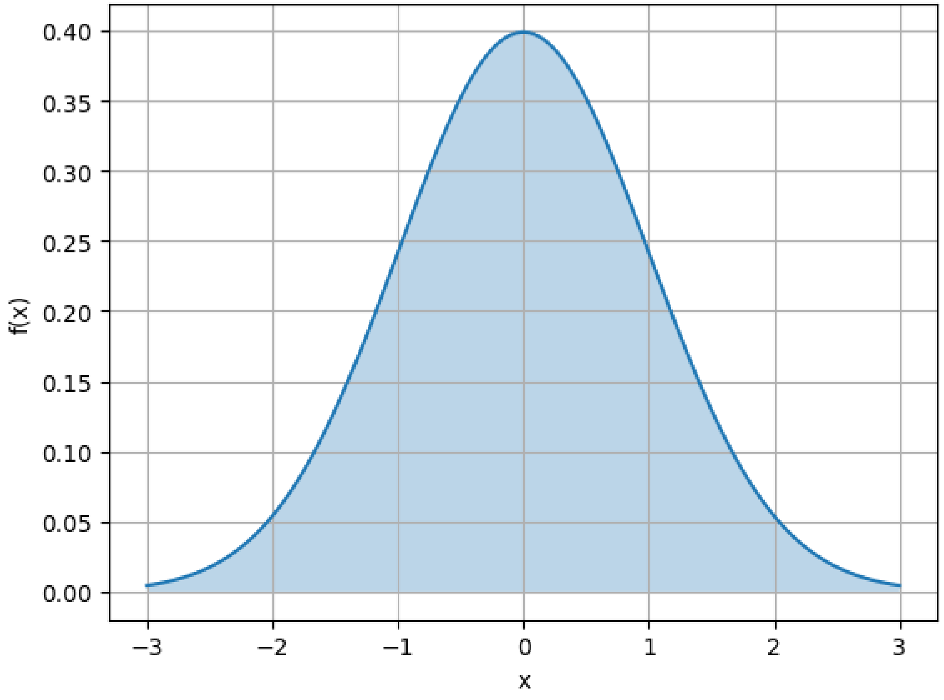
$P(-1.5 \leq X \leq 1.5):$



Properties of pdf

- **The total area under the curve equals 1, ensuring the probabilities encompass all potential outcomes.**

$$\int f(x)dx = 1$$



Properties of pdf

- Therefore, the area beneath the probability distribution curve of a continuous random variable between any two points falls within the range of 0 to 1.

$$0 \leq P(X = x_i) \leq 1$$

- The area beneath the probability density function graph between points a and b remains constant, irrespective of whether the endpoints a and b are inclusive or exclusive.

$$P(a \leq X \leq b) = P(a < X < b) = \int_a^b f(x)\,dx$$

Cumulative Distribution Function

- **For a continuous random variable X**

$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)\,dx$$

- **Given that a probability of continuous random variable (X) taking any single value is zero , we consequently have**

$$P(X \leq a) = P(X < a) = \int_{-\infty}^{a} f(x)\,dx$$

Expectation and Variance of any continuous random variable

Expectation E(X) = $\int x\, f(x)\, dx$

Variance Var(X) = $\int \left(x - E(X)\right)^2 f(x) dx$

Expectation and Variance of any discrete random variable

Expectation E(X) = $\sum_{i=1}^{\infty} x_i P(x_i)$

Variance $Var\ (X) = E(X - \mu)^2$

Continuous Random Variable Distributions

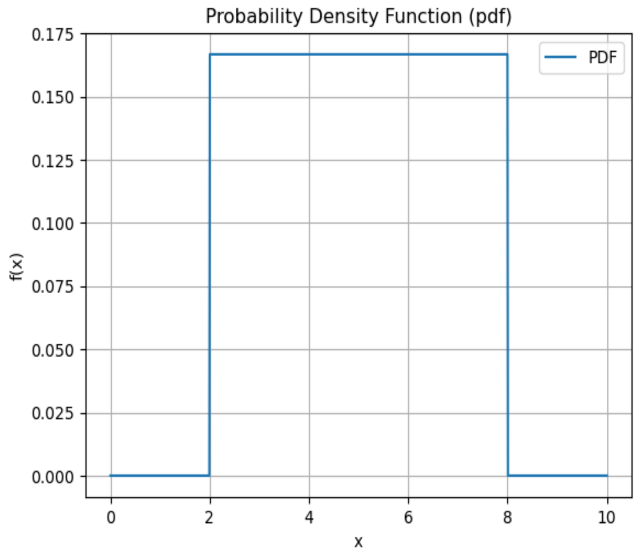
- Uniform Distribution
- Standard Uniform Distribution
- Exponential Distribution
- Normal Distribution
- Standard Normal Distribution
- T distribution
- Chi Squared Distribution

Continuous random variable
Uniform Distribution

A uniform distribution is assigned to a continuous random variable, denoted as X ~ U(a, b).

The Probability Density Function for a uniform distribution is expressed as : $f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & otherwise \end{cases}$

We know that the total area under the probability density curve of any continuous distribution equals 1



$$\int_a^b f(x)dx = 1$$

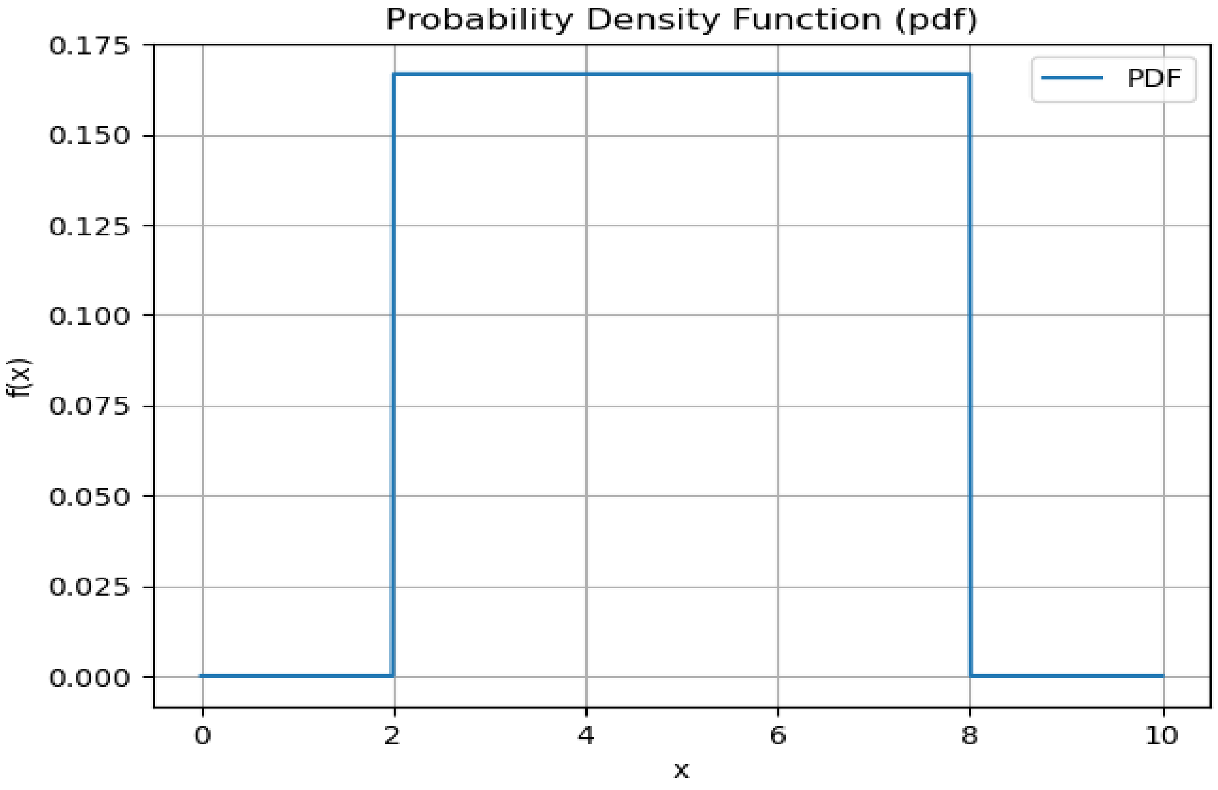
$$f(x) \int_a^b dx = 1$$

$$f(x)\ (b - a) = 1$$

$$f(x) = \frac{1}{b-a}$$

Continuous random variable
Uniform Distribution

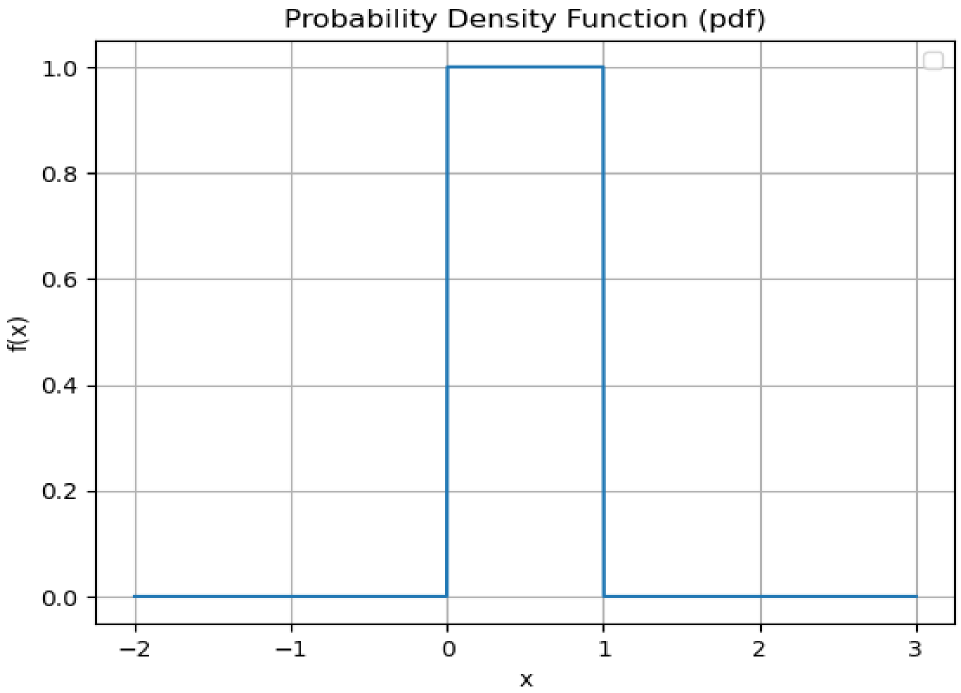
Example: Uniform Distribution U(2,8)



Standard Uniform Distribution

A continuous random variable follows the standard uniform distribution with a minimum value of 0 and a maximum value of 1, represented as X ~ U(0, 1).

Probability Density Function of uniform distribution is given by : $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & otherwise \end{cases}$



$$\int_0^1 f(x)dx = 1$$

$$f(x) \int_0^1 dx = 1$$

$$f(x)\ (1 - 0) = 1$$

$$f(x) = 1$$

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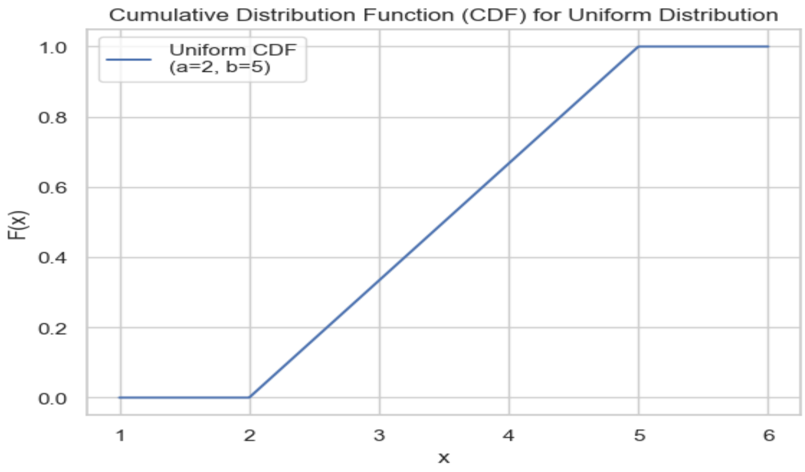
Cumulative Distribution Function
Uniform Distribution

The cumulative distribution function for a uniform distribution, denoted by F(x), is given by:

$$\boldsymbol{F(x)} = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x < b \\ 1 & x \ge b \end{cases}$$

- **We know for any continuous random variable X :** $\boldsymbol{F(a) = P(X \le a) = \int_{-\infty}^{a} f(x)dx}$



$$\boldsymbol{F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)dx}$$

$$\boldsymbol{= \int_{-\infty}^{a} f(x)dx + \int_{a}^{x} f(x)dx}$$

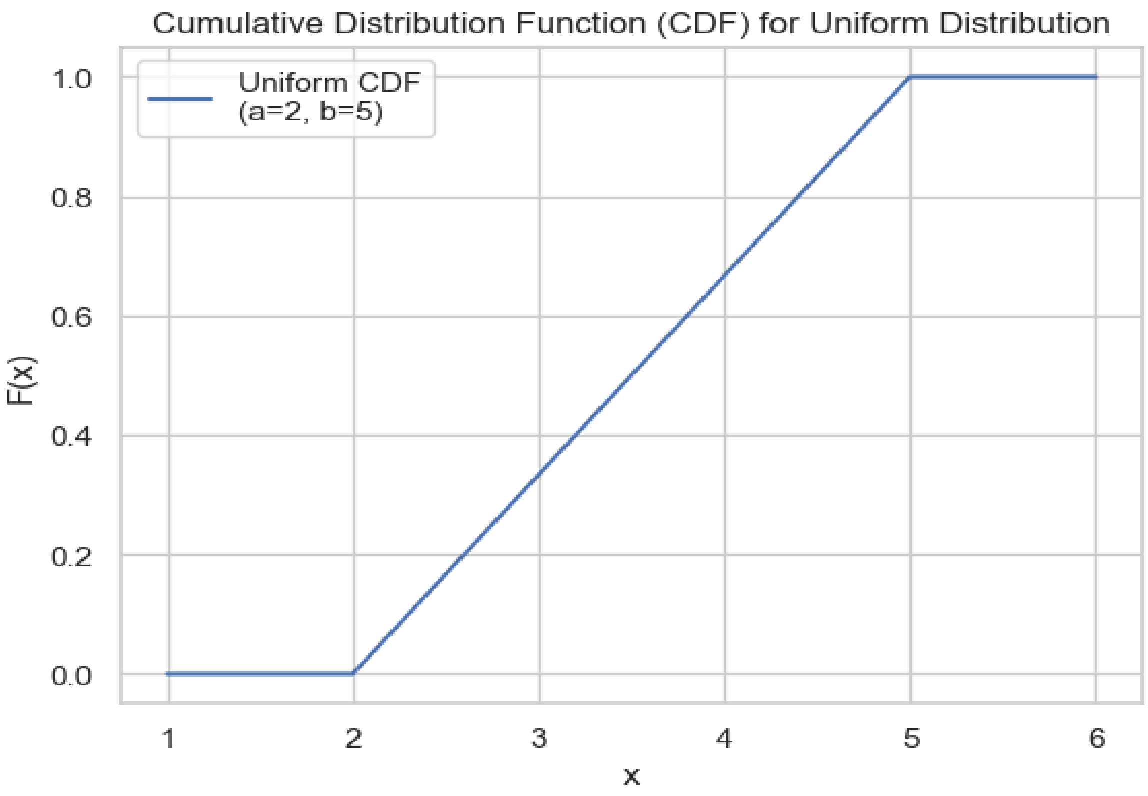
$$\boldsymbol{= 0 + \int_{a}^{x} f(x)dx}$$

$$\boldsymbol{= \int_{a}^{x} \frac{1}{b-a} dx}$$

$$\boldsymbol{= \frac{x-a}{b-a}}$$

Cumulative Distribution Function
Uniform Distribution

Example : cumulative distribution function for uniform distribution U (2,5)

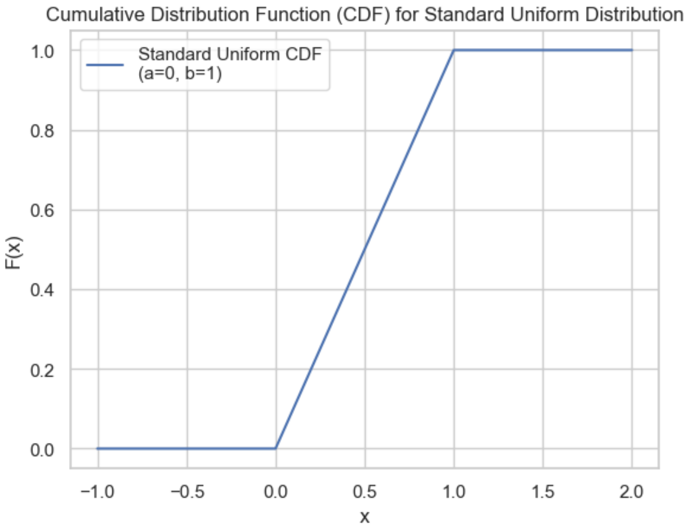


Cumulative Distribution Function
Standard Uniform Distribution

The cumulative distribution function for the standard uniform distribution is denoted by F(x).

$$\boldsymbol{F(x)} = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

- **We know for any continuous random variable X :** $\boldsymbol{F(a) = P(X \le}$ **a) =** $\boldsymbol{\int_{-\infty}^{a} f(x)dx}$



$\boldsymbol{F(x) = P(X \le}$ **x) =** $\boldsymbol{\int_{-\infty}^{x} f(x)dx}$

= $\boldsymbol{\int_{-\infty}^{0} f(x)dx}$ **+** $\boldsymbol{\int_{0}^{x} f(x)dx}$

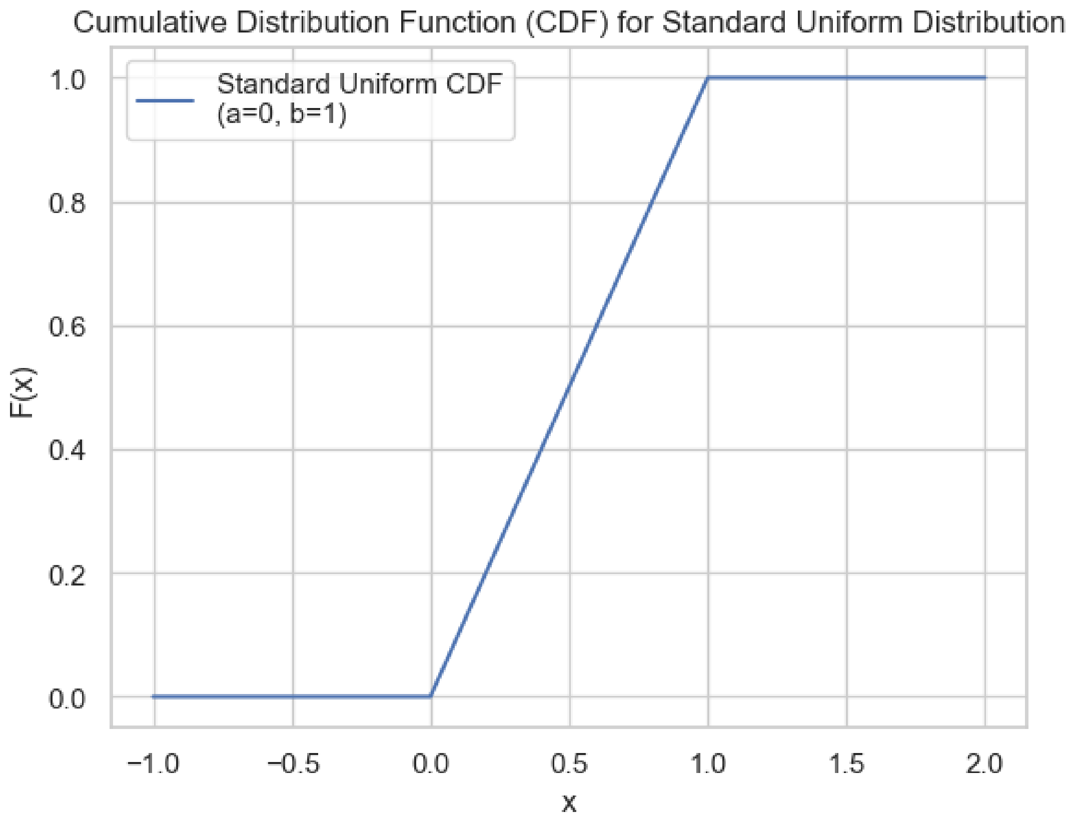
= 0 + $\boldsymbol{\int_{0}^{x} f(x)dx}$

= $\boldsymbol{\int_{0}^{x} 1\, dx}$

= $\boldsymbol{x}$

Cumulative Distribution Function
Standard Uniform Distribution

Example : cumulative distribution function for uniform distribution U (0,1)



Expectation

Expectation of continuous random variable

X ~ U(a, b)

$$E(X) = \frac{a+b}{2}$$

We know $$E(X) = \int_a^b xf(x)dx$$

$$= \int_a^b x\frac{1}{b-a}dx = \frac{1}{b-a}\int_a^b xdx$$

$$= \frac{1}{b-a}\frac{b^2-a^2}{2}$$

Expectation of Uniform Continuous random variable = $\frac{b+a}{2}$

Variance

Variance of continuous random variable

X ~ U(a, b)

$$V(X) = \frac{(b-a)^2}{12}$$

$$V(X) = E\left(X^2\right) - (E(X))^2$$

$$E\left(X^2\right) = \int_a^b x^2 f(x)\,dx$$

$$= \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b x^2\,dx$$

$$= \frac{1}{b-a}\frac{b^3-a^3}{3}$$

$$= \frac{b^2+a^2+ab}{3}$$

Variance

Variance of continuous random variable

X ~ U(a, b)

$$V(X) = \frac{(b-a)^2}{12}$$

$$V(X) = E\left(X^2\right) - (E(X))^2$$

$$= \frac{b^2+a^2+ab}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{b^2+a^2+ab}{3} - \frac{a^2+b^2+2ab}{4}$$

$$= \frac{b^2+a^2-2ab}{12}$$

Variance of Uniform Continuous random variable $= \frac{(b-a)^2}{12}$

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Example

X is a uniform random variable over (0,1). Find the probabilities as follows:

1. P(X < 0.5) = 0.5

$P(X \leq x) = P((X \leq x) = P((X < 0.5) = \int_0^{0.5} f(x)dx$

$= \int_0^{0.5} 1.\ dx$ **[We know pdf value for standard uniform distribution is 1]**

= 0.5

1. P(X ≥ 0.8) = 1 – 0.8 = 0.2

$P((X \geq 0.8) = 1 - P((X < 0.8) = 1 - \int_0^{0.8} f(x)dx$

$= 1 - \int_0^{0.8} 1.\ dx$

= 1 - 0.8

= 0.2

Example

X is a uniform random variable over (0,1). Find the probabilities as follows:

1. P(0.2 ≤ X < 0.9) = 0.9 – 0.2 = 0.7

$P(y \leq X \leq$ x) = P (($X \leq$ x) - P (($X \leq$ y) = P (($X \leq$ 0.9) - P (($X \leq$ 0.2)

$$= \int_0^{0.9} f(x)dx - \int_0^{0.2} f(x)dx$$

= 0.9 – 0.2

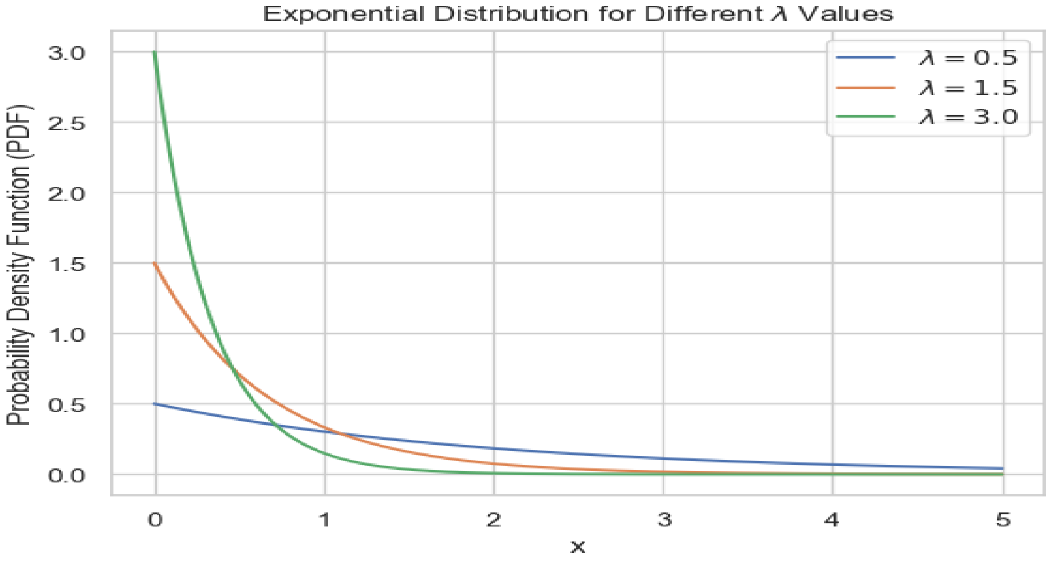
= 0.7

Continuous Random Variable
Exponential Distribution

For some λ > 0, the probability density function (denoted by f(x)) of an exponential continuous random variable is defined as:

$$X \sim Exp(\lambda)$$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & otherwise \end{cases}$$



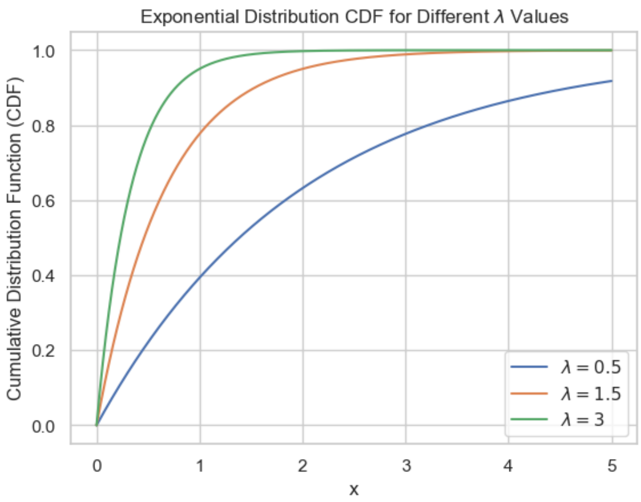
Cumulative Distribution Function
Exponential Distribution

The cumulative distribution function of the exponential distribution is denoted as F(a).

We know cumulative distribution function for any continuous random variable, $F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)\,dx$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

We Know f(x) = 0 for x less than zero in exponential distribution, So



$$= \int_{0}^{a} \lambda e^{-\lambda x}\,dx$$

$$F(a) = 1 - e^{-\lambda a}$$

Expectation
Exponential Distribution

The Expectation of exponential distribution is given by E(X)

$$E(X^n) = \int_0^\infty x^n \; f(x)dx$$

$$= \int_0^\infty x^n \, \lambda e^{-\lambda x} \, dx$$

$$= \lambda \int_0^\infty x^n \, e^{-\lambda x} \, dx$$

By applying integration by parts formula: $\int udv = uv - \int vdu$

$E(X^n) = \frac{n}{\lambda} E(X^{n-1})$

For n = 1, $E(X^1) = \frac{1}{\lambda} E(X^{1-1}) = \frac{1}{\lambda}$

Expectation of a Exponential Distribution Exp(λ) = $\frac{1}{\lambda}$

Variance
Exponential Distribution

The Variance of exponential distribution is given by V(X)

For n = 1, $\boldsymbol{E(X^1) = \frac{1}{\lambda} E(X^{1-1}) = \frac{1}{\lambda}}$

For n = 2, $\boldsymbol{E(X^2) = \frac{2}{\lambda} E(X^{2-1}) = \frac{2}{\lambda^2}}$

$$\boldsymbol{E(X) = \frac{1}{\lambda}}$$

$$\boldsymbol{V(X) = E(X^2) - (E(X))^2}$$

$$\boldsymbol{= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2}$$

$$\boldsymbol{= \frac{1}{\lambda^2}}$$

The Variance V(x) of an exponential distribution defined by Exp (λ) = $\boldsymbol{\frac{1}{\lambda^2}}$

Normal Distribution

- The normal distribution, alternatively referred to as the Gaussian distribution or bell curve, is a continuous probability distribution exhibiting symmetry around its mean, which coincides with its median and mode.

- Defined by its mean (μ) and standard deviation (σ), the normal distribution takes on a bell-shaped form.

- Considered paramount in statistics for both theoretical understanding and practical application, the normal distribution holds significant importance.

Probability Density Function (pdf)
Normal Distribution

The continuous random variable X follows a normal distribution, denoted as X ~ N (μ, σ^2) , where μ is the mean and σ^2 is the variance.

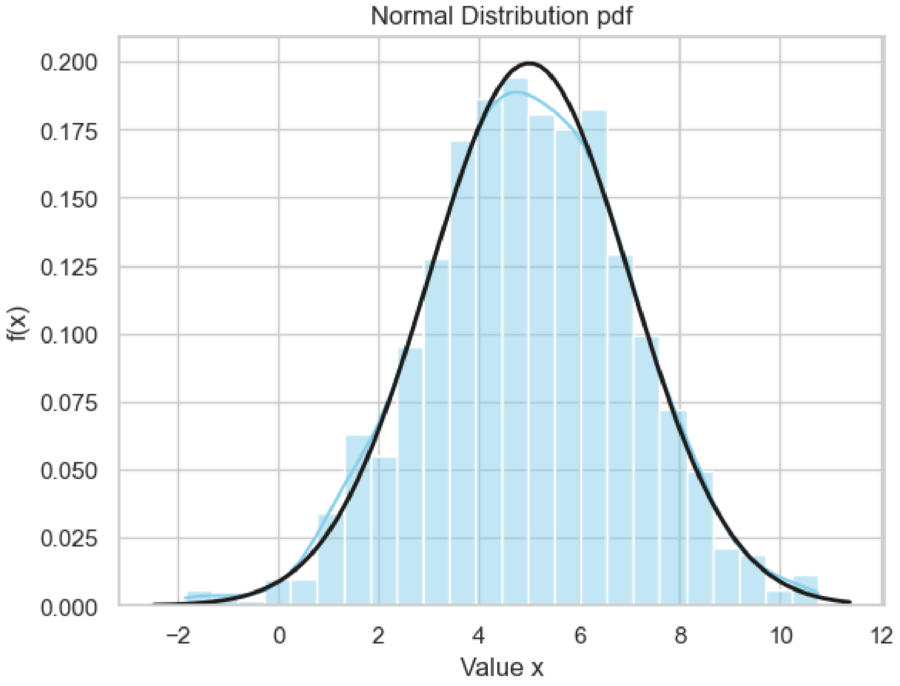
where μ = expected value i.e. mean

σ^2 = variance

Pdf of normal distribution is given by $f(x) = \frac{1}{\sqrt{2\Pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$

Shape of Normal Probability Density Function (pdf)

Pdf of normal distribution is given by $f(x) = \frac{1}{\sqrt{2\Pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$



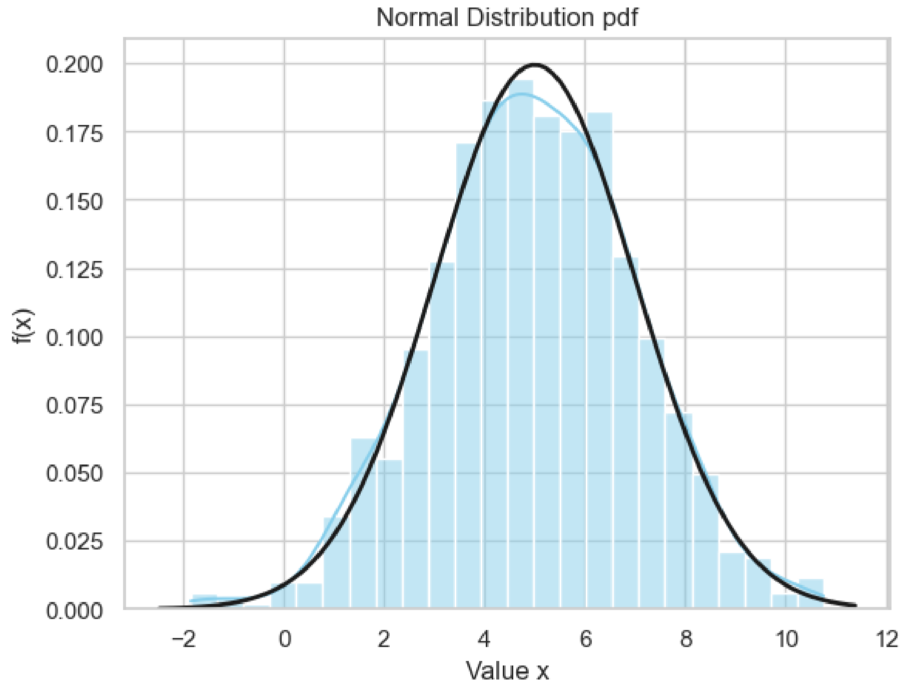
μ = expected value i.e. mean = 5

σ = Standard deviation = 2

Height of the graph

Pdf of normal distribution is given by $f(x) = \frac{1}{\sqrt{2\Pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$

$$f(x) = \frac{1}{\sqrt{2\Pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



$f(x) = 0.199\, for$

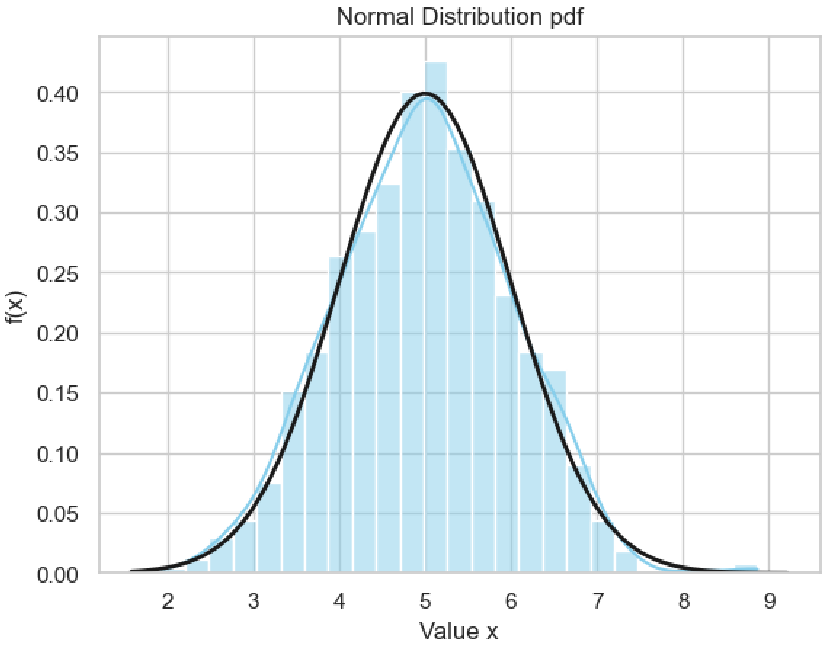
μ = expected value i.e. mean = 5

σ = Standard deviation = 2

Height of the graph

Pdf of normal distribution is given by $f(x) = \frac{1}{\sqrt{2\Pi\sigma}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$

$$f(x) = \frac{1}{\sqrt{2\Pi\sigma}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



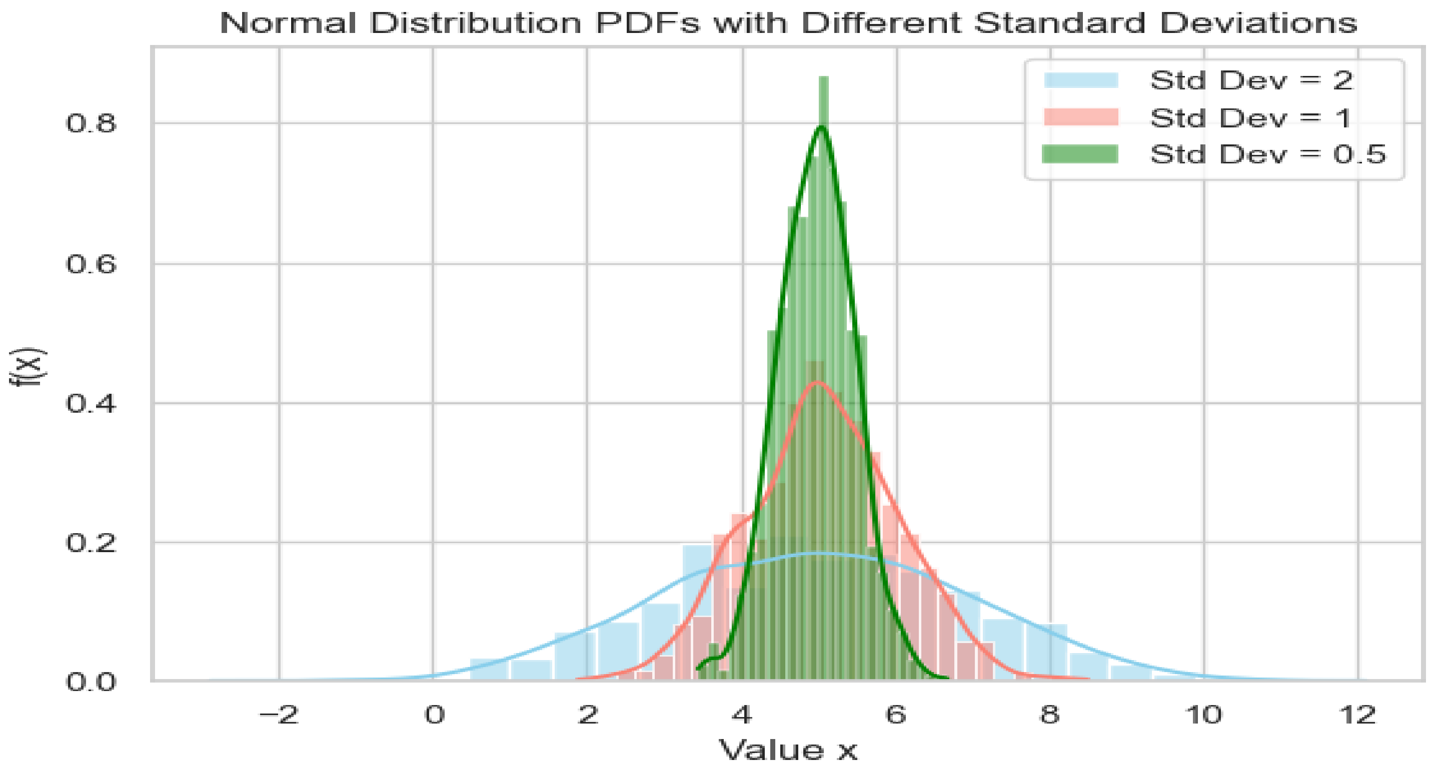
$f(x) = 0.399 \ \ for$

μ = expected value i.e. mean = 5

σ = Standard deviation = 1

Same Mean and Different Standard Deviation

Pdf of normal distribution is given by $f(x) = \frac{1}{\sqrt{2\Pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$



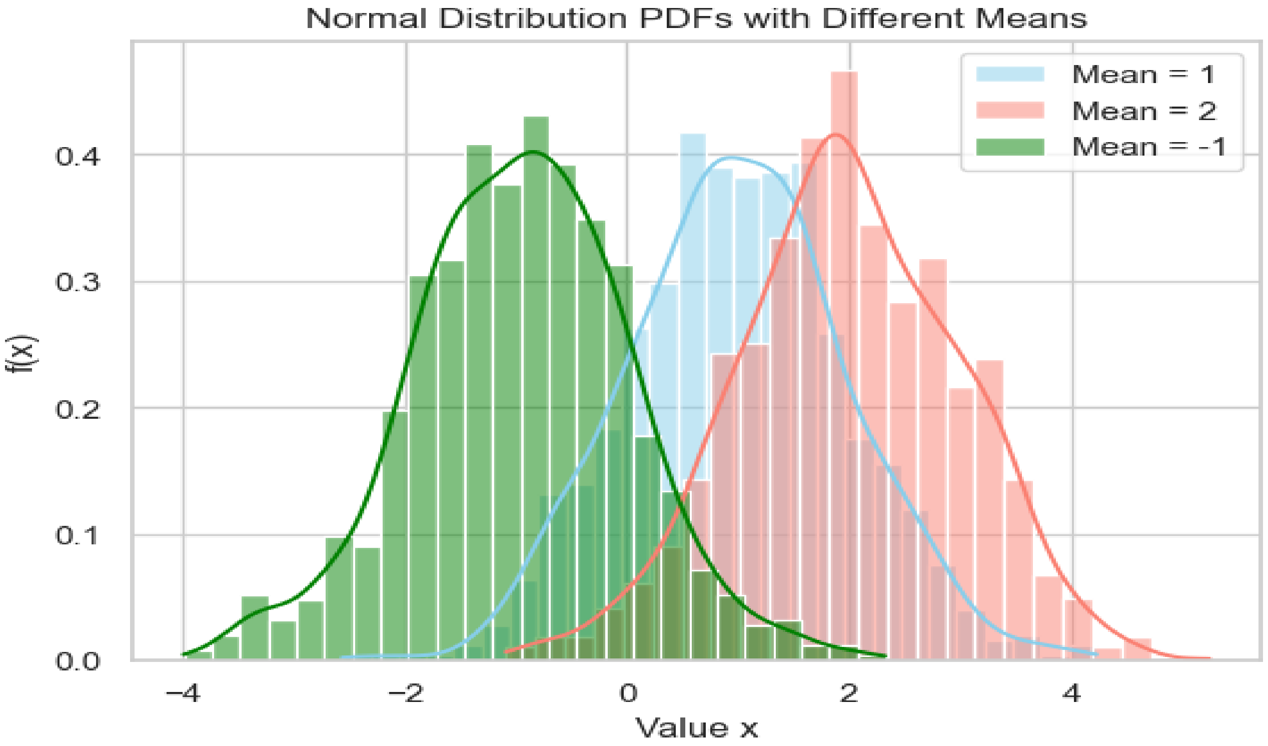
μ = expected value i.e. mean = 5

σ = Standard deviation = 2, 1, 0.5

The larger the standard deviation , the flatter the graph becomes.

Same Standard Deviation and Different Mean

Pdf of normal distribution is given by $f(x) = \frac{1}{\sqrt{2\Pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$



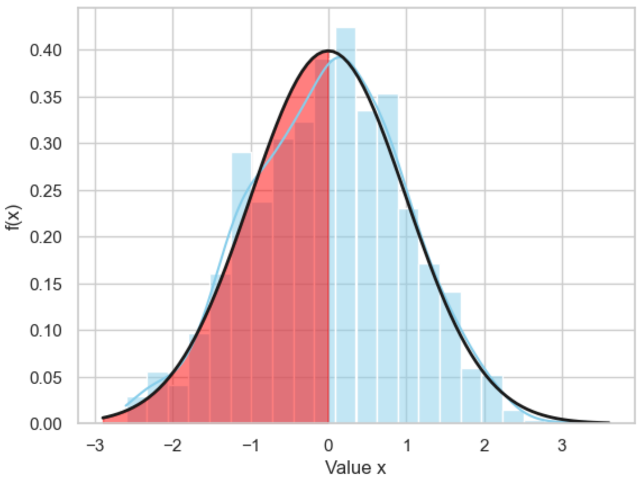
μ = expected value i.e. mean = 1 , 2, -1

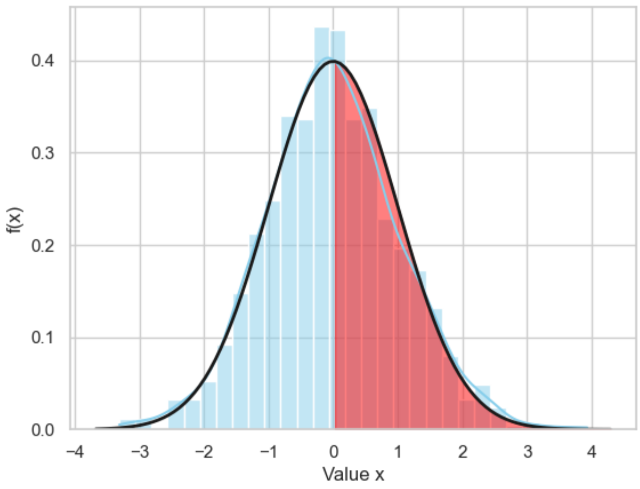
σ = Standard deviation = 1

Symmetric around Mean

- Pdf of normal distribution is given by $f(x) = \frac{1}{\sqrt{2\Pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$
- The probability density function of a normal random variable X exhibits symmetry around its mean.
- This means that X is equally likely to occur on either side of the mean (μ).

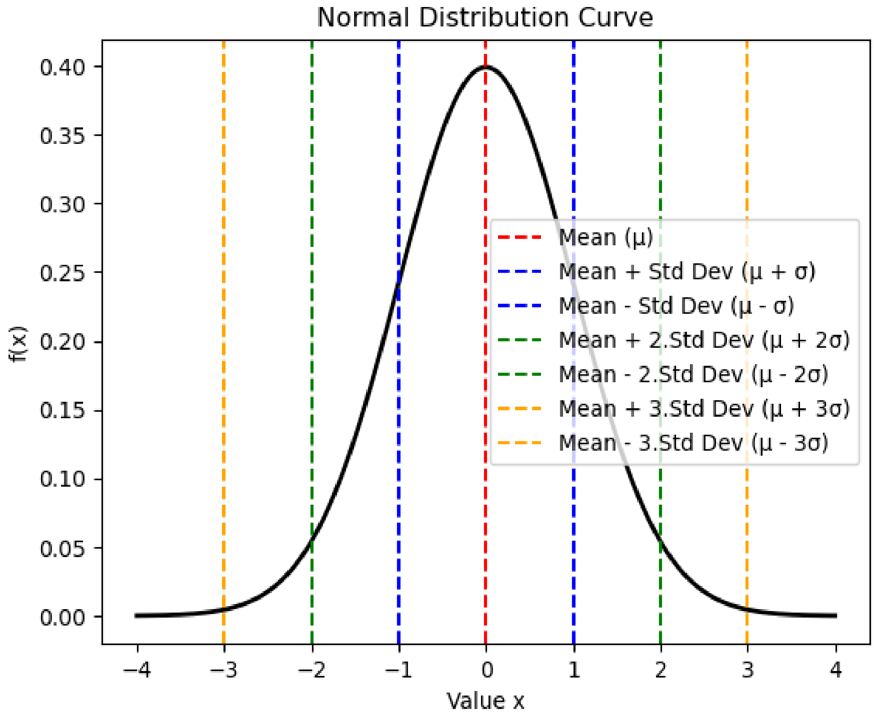
$$P(X \leq \mu) = P(X \geq \mu) = 0.5$$





Normal Distribution

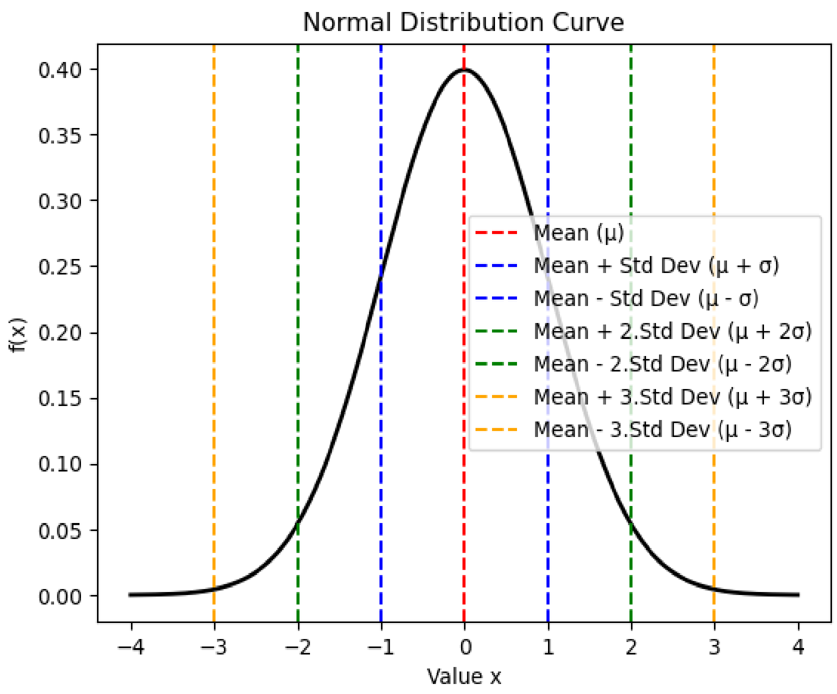
1. **Bell Shaped**
2. **Centered at mean i.e., the expected value**
3. **Close to the horizontal axis outside the range from μ + 3.σ and μ - 3.σ**



Approximation rule for Normal Distribution

For a continuous random variable following a normal distribution with mean μ and standard deviation σ:

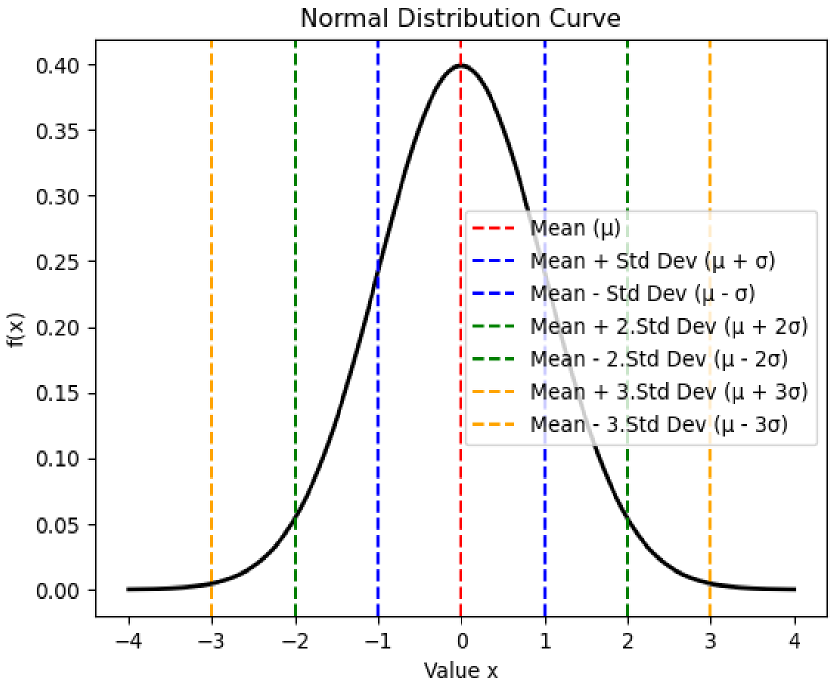
- **The interval between μ + σ and μ - σ encompasses approximately 68% of the distribution.**
- **The interval between μ + 2σ and μ - 2σ encompasses approximately 95% of the distribution.**
- **The interval between μ + 3σ and μ - 3σ encompasses approximately 99% of the distribution.**



Example of Normal Distribution N (μ , σ^2)
X~ N(10,16)

What is the probability that X is between 6 and 14 ?

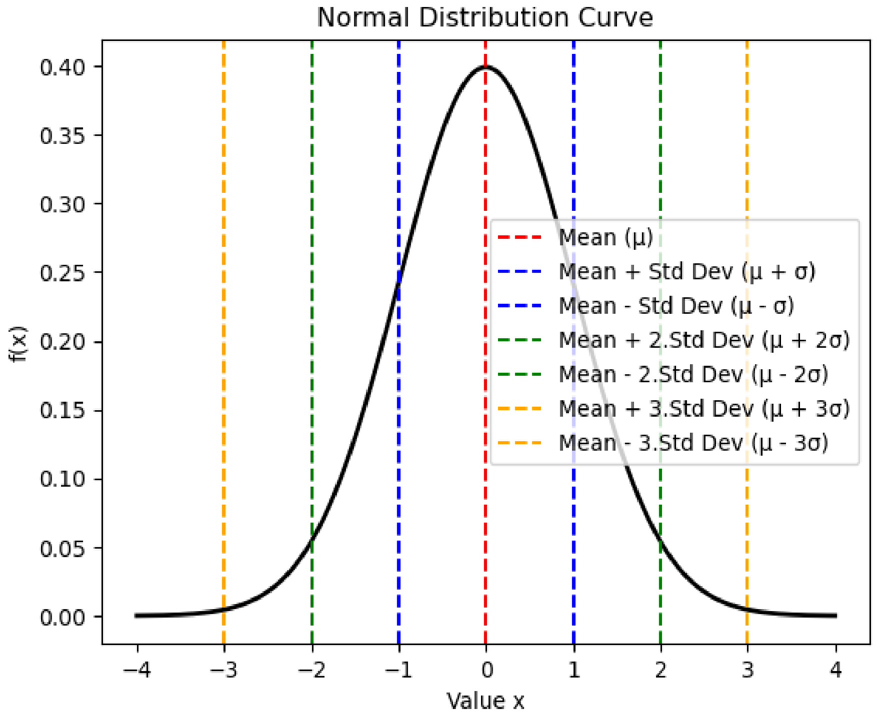
Using the Approximation rule Answer is 0.68



Example of Normal Distribution N (μ , σ^2)
X~ N(10,16)

What is the probability that X is less than 6 ?

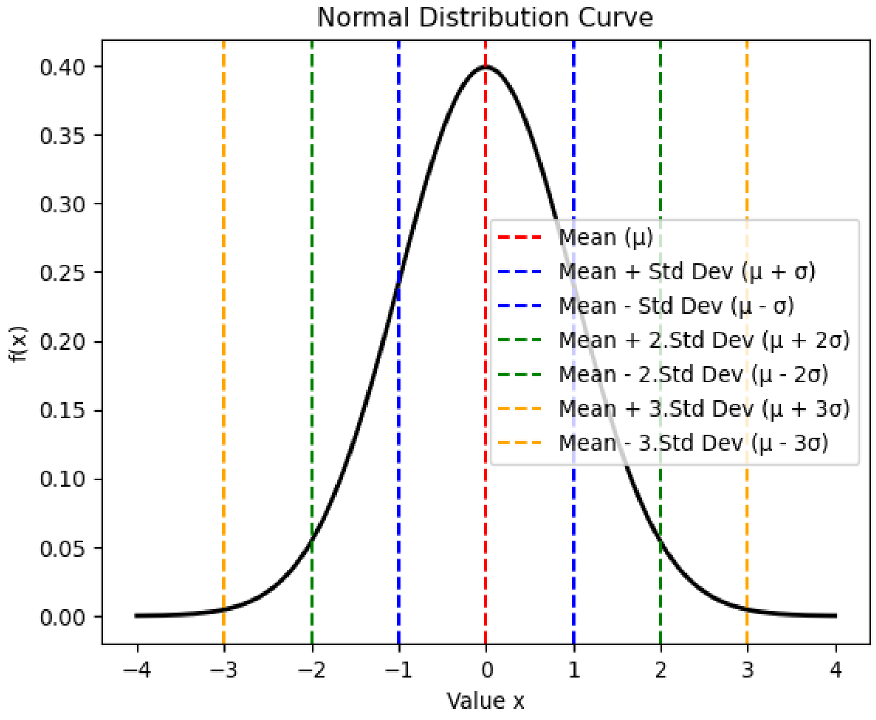
Using the Approximation rule Answer is 0.16



Example of Normal Distribution N (μ , σ^2)
X~ N(10,16)

What is the probability that X is greater than 14 ?

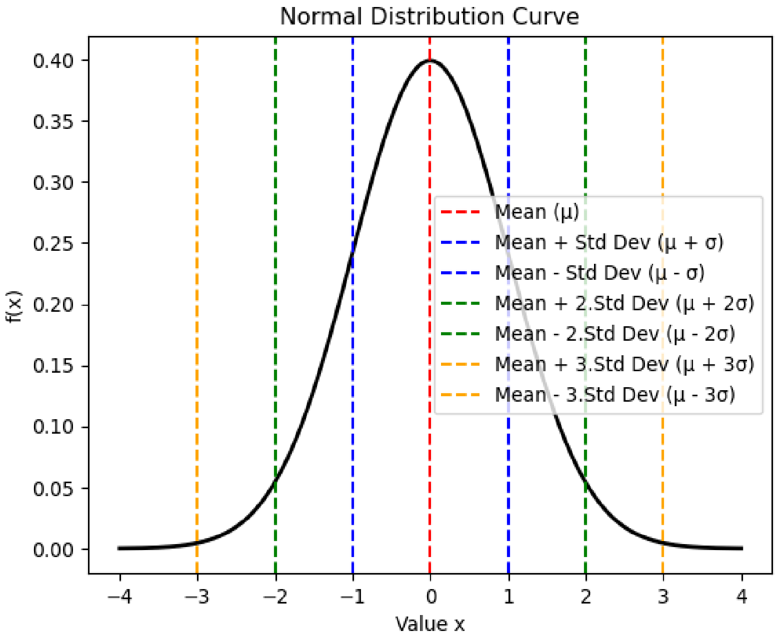
Using the Approximation rule, the Answer is 0.16



Example of Normal Distribution N (μ, σ^2)
X~ N(10,16)

What is the probability that X is greater than 22 ?

Using the Approximation rule, the Answer is 0



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- Statistical Methods Combined Edition (Volume I& II), N G Das

CONCLUSION

- Continuous random variable
- Probability Distribution Function
  Properties of pdf
- Cumulative Distribution Function
- Uniform and Standard Uniform distribution
  Expectation and Variance
- Exponential distribution
  Expectation and Variance
- Normal distribution
  Approximation Rule


THANK YOU

Mine Automation and Data Analytics