SWAYAM NPTEL COURSE ON

MINE AUTOMATION AND DATA ANALYTICS

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Module 9 : Hypothesis Testing

Lecture 21B : Hypothesis Testing - II

CONCEPTS COVERED

1) Z test: Tests concerning the mean of a normal population
2) Solved Example using Z test: 1
3) Solved Example using Z test: 2
4) Solved Example using Z test: 3
5) Solved Example using Z test : 4

Tests concerning the mean of a normal population
Z test

1. **Case of Known Variance**

Suppose that $X_1, \dots, X_n$ is a sample of size n from a normal distribution having an unknown mean μ and a known variance σ^2 and suppose we are interested in testing the null hypothesis

$$H_0 : \mu = \mu_0$$

against the alternative hypothesis

$$H_1 : \mu \neq \mu_0$$

Since $\overline{X} = \sum_{i=1}^{n} \frac{X_i}{n}$ is a natural point estimator of μ, it seems reasonable to accept H_0 if X is not too far from μ_0. That is, the critical region of the test would be of the form

$$C = \{ X_1, \dots, X_n : |X - \mu_0| > c\}$$

for some suitably chosen value *c*

If we desire that the test has significance level α, then we must determine the critical value c that will make the type I error equal to α. That is, c must be such that

$$P_{\mu_0}\{|\bar{X} - \mu_0| > c\} = \alpha$$

where we write P_{μ_0} to mean that the preceding probability is to be computed under the assumption that $\mu = \mu_0$. However, when $\mu = \mu_0$, X will be normally distributed with mean μ_0 and variance σ^2/n and so Z, defined by

$$Z \equiv \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} = \frac{\sqrt{n}(\bar{X} - \mu_0)}{\sigma}$$ will have a standard normal distribution

$$P\left\{|Z| > \frac{c\sqrt{n}}{\sigma}\right\} = \alpha$$

$$2P\left\{Z > \frac{c\sqrt{n}}{\sigma}\right\} = \alpha$$

where Z is a standard normal random variable. However, we know that

$$P\{Z > z_{\alpha/2}\} = \alpha/2$$

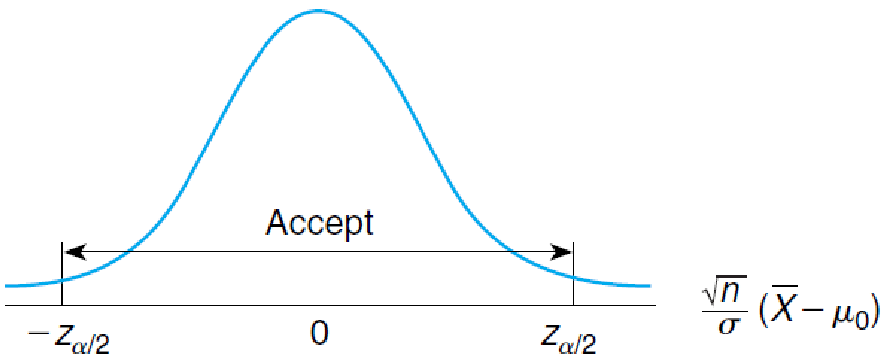
$$\frac{c\sqrt{n}}{\sigma} = z_{\alpha/2}$$

$$c = \frac{z_{\alpha/2}\sigma}{\sqrt{n}}$$

Thus, the significance level α test is to reject H_0 if $|\overline{X} - \mu_0| > z_{\alpha/2}\sigma/\sqrt{n}$ **and accept otherwise; or, equivalently, to**

$$\text{reject} \quad H_0 \quad \text{if} \quad \frac{\sqrt{n}}{\sigma}|\overline{X} - \mu_0| > z_{\alpha/2}$$

$$\text{accept} \quad H_0 \quad \text{if} \quad \frac{\sqrt{n}}{\sigma}|\overline{X} - \mu_0| \leq z_{\alpha/2}$$



we have superimposed the standard normal density function [which is the density of the test statistic $\sqrt{n}(\overline{X} - \mu_0)/\sigma$ **when** H_0 **is true].**

Determining Critical Values ($Z\alpha/2$)

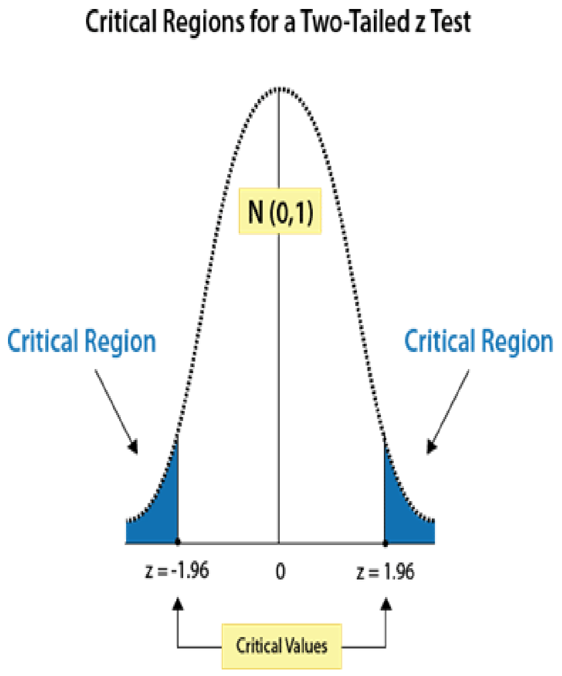
What is the critical value ($Z\alpha/2$) for a 95% confidence level (for α = 0.05) , assuming a two-tailed test?

A 95% confidence level means that a total of 5% of the area under the curve is considered the critical region.

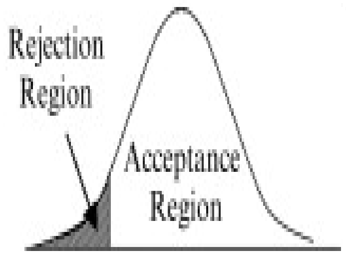
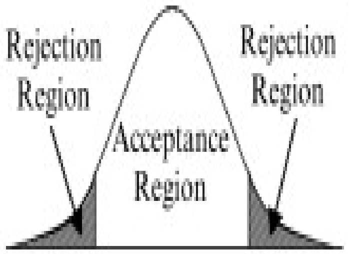
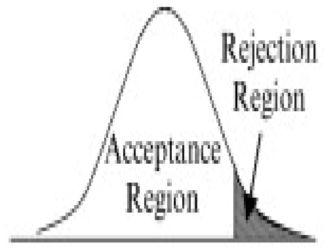
Since this is a two-tailed test, ½ (5%) = 2.5% of the values would be in the left tail, and the other 2.5% would be in the right tail.

Looking up the Z-score associated with 0.025 on a reference table, we find 1.96. Therefore, +1.96 is the critical value of the right tail, and -1.96 is the critical value of the left tail.

The critical value for a 95% confidence level is $Z\alpha/2$ = +-1.96



Rejection regions for different tailed Z test

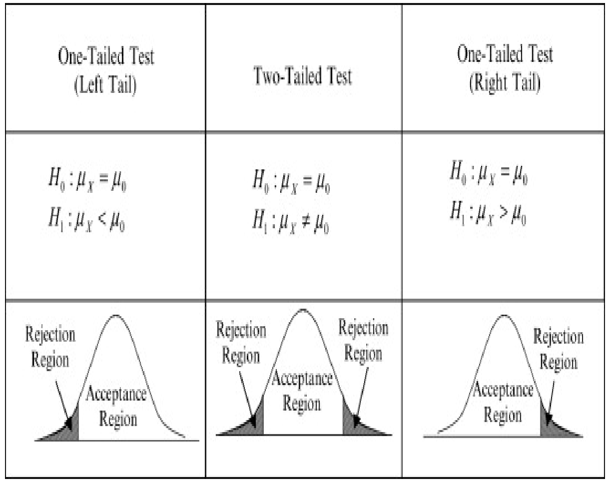
One-Tailed Test (Left Tail)	Two-Tailed Test	One-Tailed Test (Right Tail)
$H_0: \mu_X = \mu_0$ $H_1: \mu_X < \mu_0$	$H_0: \mu_X = \mu_0$ $H_1: \mu_X \neq \mu_0$	$H_0: \mu_X = \mu_0$ $H_1: \mu_X > \mu_0$
		

Z-score values for common confidence levels of a normal distribution

99% Confidence level (i.e alpha = 0.01):
Left-tailed test: Zα = -2.33
Two-tailed test: Zα/2 = +- 2.55 (the critical z-values are +2.55 and -2.55)
Right-tailed test: Zα = +-2.33

95% Confidence level (i.e alpha = 0.05):
Left-tailed test: Zα = -1.65
Two-tailed test: Zα/2 = +-1.96 (the critical z-values are -1.96 and 1.96)
Right-tailed test: Zα = +1.65

90% Confidence level (i.e alpha = 0.1):
Left-tailed test: Zα = -1.2
Two-tailed test: Zα/2 = +-1.65 (the critical z-values are -1.65 and 1.65)
Right-tailed test: Zα = +1.2



If a signal of value μ is sent from location A, then the value received at location B is normally distributed with mean μ and standard deviation 2. The random noise added to the signal is an N(0, 4) random variable. There is reason for the people at location B to suspect that the signal value μ = 8 will be sent today. Test this hypothesis if the same signal value is independently sent five times and the average value received at location B is $\overline{X}$ = 9.5.

$$\frac{\sqrt{n}}{\sigma}|\overline{X} - \mu_0| = \frac{\sqrt{5}}{2}(1.5) = 1.68$$

Since this value is less than $z_{.025}$ = 1.96 , the hypothesis is accepted. In other words, the data are not inconsistent with the null hypothesis in the sense that a sample average as far from the value 8 as observed would be expected , when the true mean is 8, over 5 percent of the time.

Note: however, that if a less stringent significance level were chosen — say α = 0.1 then the null hypothesis would have been rejected. This follows since $z_{.05}$ = 1.645, which is less than 1.68.

Hence, if we had chosen a test that had a 10 percent chance of rejecting H_0 when H_0 was true, then the null hypothesis would have been rejected.

The "correct" level of significance to use in a given situation depends on the individual circumstances involved in that situation.

For instance, if rejecting a null hypothesis H_0 would result in large costs that would thus be lost if H_0 were indeed true, then we might elect to be quite conservative and so choose a significance level of .05 or .01.

Also, if we initially feel strongly that H_0 was correct, then we would require very stringent data evidence to the contrary for us to reject H_0. (That is, we would set a very low significance level in this situation).

Example (1/4) of Z test

Suppose a manufacturer claims that the mean weight of their product is 500 grams. To test this claim, a random sample of 36 products is selected, and their weights are recorded. The sample mean weight is found to be 495 grams, with a sample standard deviation of 10 grams. Assume the weights follow a normal distribution. Using a significance level of 0.05, test the manufacturer's claim.

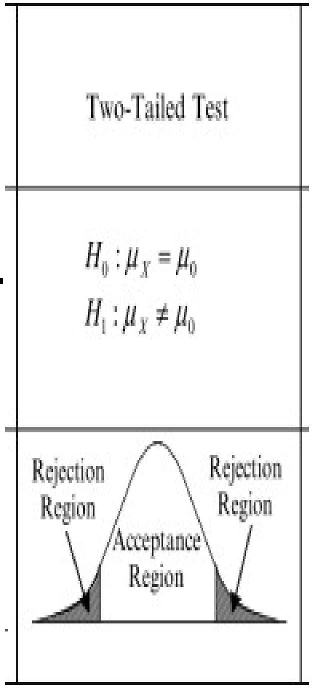
1) State the hypothesis:

Null Hypothesis (H0): The mean weight of the product is 500 grams.

Alternative Hypothesis (H1): The mean weight of the product is not 500 grams.

H0: μ = 500

H1: μ ≠ 500



2)Determine the significance level (α):

$\alpha = 0.05$

3) Calculate the test statistic (z-score): $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$

The formula for the z-test statistic for the mean is:

Where:

$\overline{x}$ is the sample mean

μ is the population mean under the null hypothesis

σ is the population standard deviation

n is the sample size

Given:

$\overline{x}$ = 495 grams

μ = 500 grams

σ = 10 grams

n = 36

$$z = \frac{495 - 500}{\frac{10}{\sqrt{36}}} = \frac{-5}{\frac{10}{6}} = -3$$

Z-score values for common confidence levels of a normal distribution

99% Confidence level (i.e alpha = 0.01):
Left-tailed test: Zα = -2.33
Two-tailed test: Zα/2 = +- 2.55 (the critical z-values are +2.55 and -2.55)
Right-tailed test: Zα = +-2.33

95% Confidence level (i.e alpha = 0.05):
Left-tailed test: Zα = -1.65
Two-tailed test: Zα/2 = +-1.96 (the critical z-values are -1.96 and 1.96)
Right-tailed test: Zα = +1.65

90% Confidence level (i.e alpha = 0.1):
Left-tailed test: Zα = -1.2
Two-tailed test: Zα/2 = +-1.65 (the critical z-values are -1.65 and 1.65)
Right-tailed test: Zα = +1.2

4) Determine the critical value:

Since it's a two-tailed test, the critical z-values are -1.96 and 1.96 at a significance level of 0.05.

5) Decision rule:

If the absolute value of the z-score is greater than 1.96, we reject the null hypothesis.

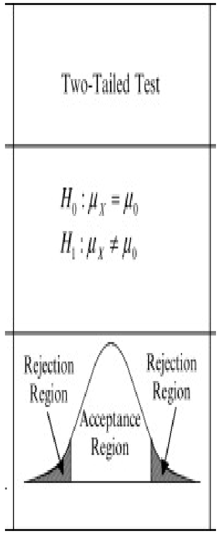
6) Make a decision:

The calculated z-value (-3) falls in the rejection region (less than -1.96), so we reject the null hypothesis.

7) Conclusion:

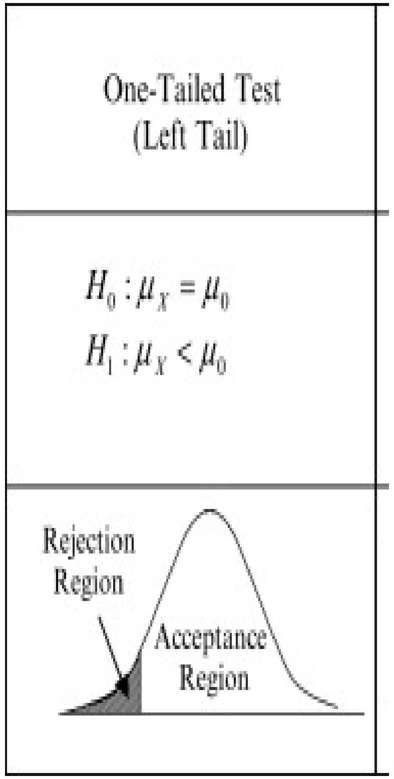
Since we reject the null hypothesis, we have sufficient evidence to conclude that the mean weight of the product is not 500 grams.

Therefore, based on the sample data, there is enough evidence to suggest that the manufacturer's claim is not correct at the 0.05 significance level.



Example (2/4) of Z test

Suppose a manufacturer claims that the average lifespan of their light bulbs is at least 1000 hours. You believe that the average lifespan is actually less than that. To test this claim, you collect a sample of 50 light bulbs and find that the average lifespan is 980 hours with a standard deviation of 40 hours. You want to test whether the average lifespan is significantly less than 1000 hours at a 5% significance level.



State the hypothesis:

Null Hypothesis (H0): The average lifespan of the light bulbs is at least 1000 hours.

Alternative Hypothesis (H1): The average lifespan of the light bulbs is less than 1000 hours.

H0: $\mu = 1000$

H1: $\mu < 1000$

Given:

1. Sample mean ($\bar{x}$) = 980 hours
2. Population mean (μ) = 1000 hours
3. Sample standard deviation (σ) = 40 hours
4. Sample size (n) = 50
5. Significance level (α) = 0.05

Calculate the test statistic (z-score):

$$Z = \frac{980 - 1000}{\frac{40}{\sqrt{50}}}$$

$$Z = \frac{-20}{5.6568}$$

$$Z \approx -3.54$$

Z-score values for common confidence levels of a normal distribution

99% Confidence level (i.e alpha = 0.01):
Left-tailed test: Zα = -2.33
Two-tailed test: Zα/2 = +- 2.55 (the critical z-values are +2.55 and -2.55)
Right-tailed test: Zα = +-2.33

95% Confidence level (i.e alpha = 0.05):
Left-tailed test: Zα = -1.65
Two-tailed test: Zα/2 = +-1.96 (the critical z-values are -1.96 and 1.96)
Right-tailed test: Zα = +1.65

90% Confidence level (i.e alpha = 0.1):
Left-tailed test: Zα = -1.2
Two-tailed test: Zα/2 = +-1.65 (the critical z-values are -1.65 and 1.65)
Right-tailed test: Zα = +1.2

Determine the critical value:

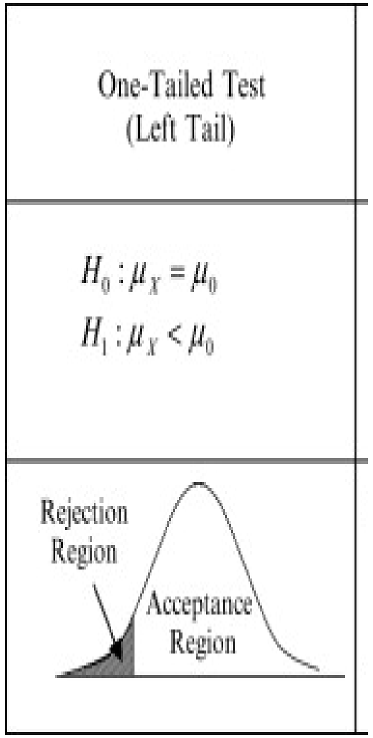
Since this is a left-tailed test and the significance level is 0.05, we find the critical z-value from the standard normal distribution table. At α = 0.05, the critical value is approximately -1.645..

Make a decision:

Since the calculated z-value (-3.54) is less than -1.645 (the critical value for a 5% significance level), we reject the null hypothesis.

Conclusion:

There is enough evidence to suggest that the average lifespan of the light bulbs is significantly less than 1000 hours.



Example (3/4) of Z test

Suppose a company claims that the average response time for their customer service hotline is no more than 3 minutes. You believe that the average response time is actually longer than that. To test this claim, you collect a sample of 40 calls to the hotline and find that the average response time is 3.5 minutes with a standard deviation of 0.8 minutes. You want to test whether the average response time is significantly greater than 3 minutes at a 5% significance level.

State the hypothesis:

Null Hypothesis (H0):

The average response time for the customer service hotline is no more than 3 minutes..

Alternative Hypothesis (H1):

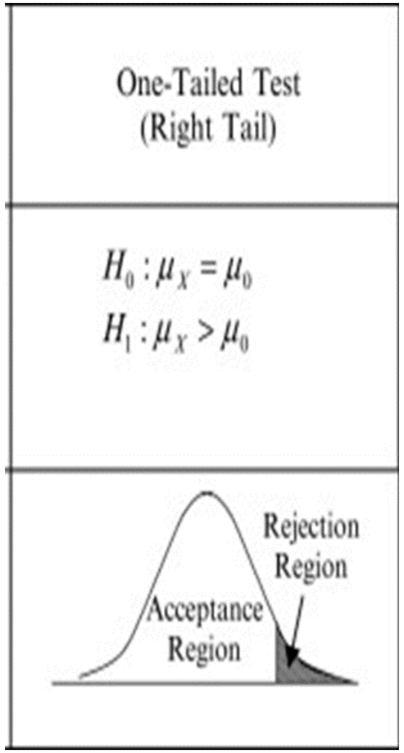
The average response time for the customer service hotline is greater than 3 minutes..

H0: μ = 3

H1: μ > 3

Set Significance Level:

Let's choose a significance level (α) of 0.05.



Calculate the test statistic (z-score):

$$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

Where:

- $\bar{x}$ is the sample mean (3.5 minutes).
- μ is the population mean under the null hypothesis (3 minutes).
- σ is the population standard deviation (0.8 minutes).
- n is the sample size (40).

$$Z = \frac{3.5 - 3}{\frac{0.8}{\sqrt{40}}}$$

$$Z = \frac{0.5}{\frac{0.8}{\sqrt{40}}}$$

$$Z \approx \frac{0.5}{0.1265}$$

$$Z \approx 3.95$$

Z-score values for common confidence levels of a normal distribution

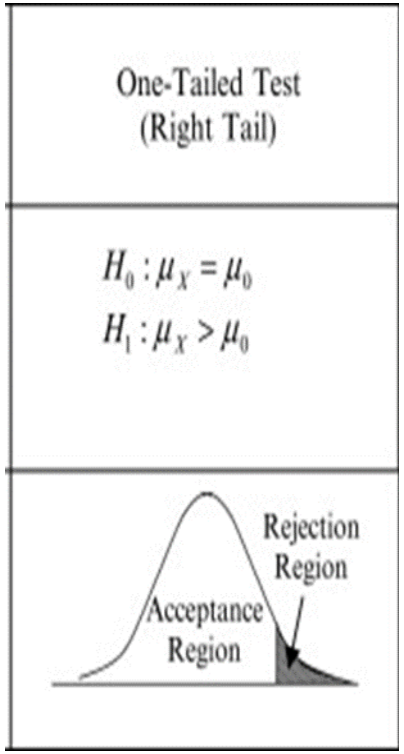
99% Confidence level (i.e alpha = 0.01):
Left-tailed test: $Z\alpha = -2.33$
Two-tailed test: $Z\alpha/2$ = +- 2.55 (the critical z-values are +2.55 and -2.55)
Right-tailed test: $Z\alpha$ = +-2.33

95% Confidence level (i.e alpha = 0.05):
Left-tailed test: $Z\alpha = -1.65$
Two-tailed test: $Z\alpha/2$ = +-1.96 (the critical z-values are -1.96 and 1.96)
Right-tailed test: $Z\alpha = +1.65$

90% Confidence level (i.e alpha = 0.1):
Left-tailed test: $Z\alpha = -1.2$
Two-tailed test: $Z\alpha/2$ = +-1.65 (the critical z-values are -1.65 and 1.65)
Right-tailed test: $Z\alpha = +1.2$

Determine the critical value:

Since this is a right-tailed test and the significance level is 0.05, we find the critical z-value from the standard normal distribution table. At $\alpha = 0.05$, the critical value is approximately 1.645.



Make a decision:

Since the calculated z-value (3.95) is greater than 1.645 (the critical value for a 5% significance level), we reject the null hypothesis.

Conclusion:

There is enough evidence to suggest that the average response time for the customer service hotline is significantly greater than 3 minutes.

Example (4/4) of Z test

An educational institute claims that the average score of its students on a standardized test is 75. A random sample of 50 students is selected, and their scores are recorded. The sample mean score is found to be 72, with a sample standard deviation of 8. Test the institute's claim at a significance level of 0.01.

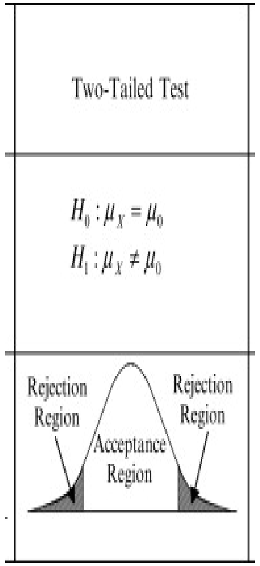
State the hypothesis:

Null Hypothesis (H0): The average score of the institute's students is 75.

Alternative Hypothesis (H1): The average score of the institute's students is not 75.

H0: μ = 75

H1: μ ≠ 75



Given:

1. **Sample mean ($\bar{x}$) = 72**
2. **Population mean (μ) = 75**
3. **Sample standard deviation (σ) = 8**
4. **Sample size (n) = 50**
5. **Significance level (α) = 0.01**

Calculate the test statistic (z-score):

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

$$z = \frac{72 - 75}{\frac{8}{\sqrt{50}}} = \frac{-3}{1.131} \approx -2.65$$

Z-score values for common confidence levels of a normal distribution

99% Confidence level (i.e alpha = 0.01):
Left-tailed test: Zα = -2.33
Two-tailed test: Zα/2 = +- 2.55 (the critical z-values are +2.55 and -2.55)
Right-tailed test: Zα = +-2.33

95% Confidence level (i.e alpha = 0.05):
Left-tailed test: Zα = -1.65
Two-tailed test: Zα/2 = +-1.96 (the critical z-values are -1.96 and 1.96)
Right-tailed test: Zα = +1.65

90% Confidence level (i.e alpha = 0.1):
Left-tailed test: Zα = -1.2
Two-tailed test: Zα/2 = +-1.65 (the critical z-values are -1.65 and 1.65)
Right-tailed test: Zα = +1.2

Determine the critical value:

Since it's a two-tailed test at α = 0.01, the critical z-values are ±2.58 (rounded from z-table).

Decision rule:

If the absolute value of the z-score is greater than 2.58, we reject the null hypothesis.

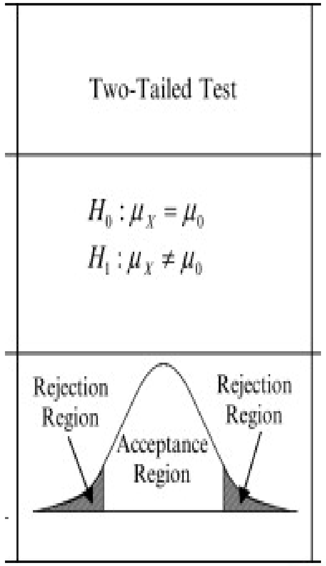
Make a decision:

The absolute value of the calculated z-value (2.65) falls in the rejection region (greater than 2.58), so we reject the null hypothesis.

Conclusion:

Since we reject the null hypothesis, we have sufficient evidence to conclude that the average score of the institute's students is not 75 at a significance level of 0.01.

Therefore, based on the sample data, there is enough evidence to suggest that the institute's claim of the average score being 75 is not supported.



REFERENCES

- Introduction to Probability and Statistics for Engineers and Scientists, Sixth Edition, Sheldon M. Ross
- Statistical Methods Combined Edition (Volume I& II), N G Das

CONCLUSION

- **We have discussed the Z test hypothesis testing in detail, along with four different examples.**


THANK YOU