Lecture 12:

Post-Optimal Analysis

September 25, 2009

Today

- Post-Optimal Analysis:
  Changes affecting optimality

Chapter 4.5.

Post-Optimal Analysis

- Changes affecting **feasibility**
  LP Model: right-hand side change or **a new constraint**
  How to recover optimal if the perturbation causes the change in basic optimal solution?

- Changes affecting **optimality**
  LP Model: objective coefficient or **new variable**
  How to find new optimal?

Changes affecting feasibility

- Increasing/decreasing amount of resources (RHS)
- **Introducing new resources (new constraint in LP)**

Addition of a constraint

TOYCO Revenue Maximization (Primal) Problem

$$
\begin{array}{lll}
\text{maximize} & z = 3x_1 + 2x_2 + 5x_3 & \\
\text{subject to} & x_1 + 2x_2 + x_3 \leq 430 & (machine\ 1) \\
 & 3x_1 \qquad + 2x_3 \leq 460 & (machine\ 2) \\
 & x_1 + 4x_2 \qquad \leq 420 & (machine\ 3) \\
 & x_1, x_2, x_3 \geq 0 &
\end{array}
$$

Current optimal: $z = 1350 \quad x_1 = 0,\ x_2 = 100,\ x_3 = 230$

Suppose TOYCO changes the design of its toys by introducing another operation, with the daily capacity 500 minutes, and operation time 3,3, and 1 for the three products, respectively. How will this operation affect the optimal solution.

Introduce one more constraint:

$$3x_1 + 3x_2 + x_3 \leq 500 \quad (machine\ 4)$$

In the optimal table add a new slack (for the new constraint) and the new row (bring the new slack into the basis):

Basic	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Solution
z	4	0	0	1	2	0	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	0	230
x_6	2	0	0	-2	1	1	0	20
x_7	3	3	1	0	0	0	1	500

The table has to be modified so that x_7 becomes part of the basis

The columns x_2, x_3 and x_6 should have 0 in the x_7-row

After transforming the table to get x_7-row correct, the new table is

Basic	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Solution
z	4	0	0	1	2	0	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	0	230
x_6	2	0	0	-2	1	1	0	20
x_7	$\frac{9}{4}$	0	0	$-\frac{3}{2}$	$\frac{1}{4}$	0	1	-30

The table shows that x_7 is not feasible. Then a dual simplex method can be used to determine a new feasible solution.

We want x_7 to leave (since it is not feasible).

Which variable will enter?

Candidates to enter a nonbasic variables whose entries in x_7-row are negative: the only candidate is x_4.

Modify the table to have x_4 in the basis and x_7 leave.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Solution
z	$\frac{5}{2}$	0	0	0	$\frac{11}{6}$	0	$\frac{2}{3}$	1330
x_2	$\frac{1}{2}$	1	0	0	$-\frac{1}{6}$	0	$\frac{1}{3}$	90
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	0	230
x_6	-1	0	0	0	$\frac{2}{3}$	1	$-\frac{4}{3}$	60
x_4	$-\frac{3}{2}$	0	0	1	$-\frac{1}{6}$	0	$-\frac{2}{3}$	20

This is new optimal solution. The optimal revenue would decrease to 1330.

Changes Affecting Optimality

- Objective cost changes
- Adding new economic activity

Objective Cost Change

TOYCO Revenue Maximization (Primal) Problem

$$
\begin{array}{lll}
\text{maximize} & z = 3x_1 + 2x_2 + 5x_3 & \\
\text{subject to} & x_1 + 2x_2 + x_3 \le 430 & (machine\ 1) \\
 & 3x_1 \qquad + 2x_3 \le 460 & (machine\ 2) \\
 & x_1 + 4x_2 \qquad \le 420 & (machine\ 3) \\
 & x_1, x_2, x_3 \ge 0 & \\
\textit{Current optimal: } z = 1350 & x_1 = 0,\ x_2 = 100,\ x_3 = 230 &
\end{array}
$$

Suppose TOYCO changes the prices for products to 2, 3 and 4, respectively.

The objective function changes to

$$z = 2x_1 + 3x_2 + 4x_3$$

The company wants to find out will this change improve their revenue.

Old Optimal Table

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

The changes in the prices of products (objective) correspond to "perturbed table"

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	?	0	0	?	?	0	?
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

Question: **Why the zeros in the z-row stay unaffected?**

Q: **Why the zeros in the z-row stay unaffected?**

Answer: Because the reduced costs of basic variables stay at 0 at every iteration of the simplex method!!!

So we need to determine 3 unknown reduced costs: reduced cost of nonbasic variables x_1, x_4, and x_5.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	?	0	0	?	?	0	?
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

Question: **What can we do to find the unknown reduced costs?**

Hint: We discuss some relations between simplex table and dual problem....

Relations in Simplex Table

Given a simplex table at any iteration i, we have

- Column Rule

- Reduced Cost Rule (z-row data)
  reduced cost of x_j = l.h.s. of j-th dual constraint − r.h.s. of j-th dual constraint
  The left hand-side of dual constraint is evaluated at the shadow price
  - Shadow Prices
    shadow price = original cost of basis $\times$ inverse
    Here, order in the basis matter.

So we need to compute

1. Shadow prices
2. Apply reduced cost rule for nonbasic variables x_j, j=1,4,5 (requires dual problem)

Item 1: **shadow price = new cost of basis $\times$ inverse optimal**

$$(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (3, 4, 0) \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{bmatrix} = \left(\frac{3}{2}, \frac{5}{4}, 0\right)$$

Item 2: TOYCO Dual corresponding to the new data:

$$\begin{array}{ll} \text{maximize} & w = 430y_1 + 460y_2 + 420y_3 \\ \text{subject to} & y_1 + 3y_2 + y_3 \geq \mathbf{2} \quad (x_1) \\ & 2y_1 \qquad + 4y_3 \geq \mathbf{3} \quad (x_2) \\ & y_1 + y_2 \qquad\quad \geq \mathbf{4} \quad (x_3) \\ & y_1 \geq 0 \qquad (x_4) \\ & y_2 \geq 0 \qquad (x_5) \\ & y_3 \geq 0 \qquad (x_6) \end{array}$$

Apply the reduced cost rule to non-basic variables x_1, x_4 and x_5.

reduced cost of nonbasic x_j= coresp. dual constraint lhs − new rhs

Expressions for reduced costs:

$$c_1 = y_1 + 3y_2 + y_3 - 2, \quad c_4 = y_1 - 0, \quad c_5 = y_2 - 0.$$

Evaluate these expressions at the computed shadow prices (item 1) $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (3/2, 5/4, 0)$ and obtain the reduced costs

new reduced cost of x_1 $:= \bar{y}_1 + 3\bar{y}_2 + \bar{y}_3 - 2 = \frac{3}{2} + 3 \cdot \frac{5}{4} + 0 - 2 = \frac{13}{4}$

new reduced cost of x_4 $:= \bar{y}_1 - 0 = \frac{3}{2}$

new reduced cost of x_5 $:= \bar{y}_2 - 0 = \frac{5}{4}$

New Table - with reduced costs updated

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	$\frac{13}{4}$	0	0	$\frac{3}{2}$	$\frac{5}{4}$	0	?
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

To determine z-value in the last column: substitute the current solution in the objective

$$z = 2x_1 + 3x_2 + 4x_3 = 2 \times 0 + 3 \times 100 + 4 \times 230 = 1220.$$

New Table - reflecting the change

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	$-\frac{3}{4}$	0	0	$\frac{3}{2}$	$\frac{5}{4}$	0	1220
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

The revenue of the company would decrease (from 1350 to 1220) - so the considered change in prices is not recommendable.

Adding a new activity

TOYCO Profit Maximization

$$
\begin{array}{lllllllll}
\text{maximize} & z = 3x_1 + 2x_2 + 5x_3 & & & & & & & \\
\text{subject to} & x_1 + 2x_2 + x_3 + x_4 & & & & = 430 & & (machine\ 1) \\
 & 3x_1 \qquad + 2x_3 \qquad + x_5 & & & & = 460 & & (machine\ 2) \\
 & x_1 + 4x_2 \qquad\qquad\qquad + x_6 & & & & = 420 & & (machine\ 3) \\
 & x_1, x_2, x_3 \geq 0 & & & & & & \\
z = 1350 & x_1 = 0,\ x_2 = 100,\ x_3 = 230 & & & & & &
\end{array}
$$

Suppose TOYCO decides to introduce a new product, fire engine having profit \$4, and which assembly requires 1 minute on machines 1 and 2, and 2 minutes on machine 3.

Is this change recommendable to TOYCO?

Assign x_7 for new product. New TOYCO problem is

$$
\begin{array}{ll}
\text{maximize} & z = 3x_1 + 2x_2 + 5x_3 + 4x_7 \\
\text{subject to} & x_1 + 2x_2 + x_3 + x_4 + x_7 = 430 \quad (machine\ 1) \\
 & 3x_1 + 2x_3 + x_5 + x_7 = 460 \quad (machine\ 2) \\
 & x_1 + 4x_2 + x_6 + 2x_7 = 420 \quad (machine\ 3) \\
 & x_1, x_2, x_3 \geq 0
\end{array}
$$

Old Optimal Table

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

Perturbed Table

Basic	x_1	x_2	x_3	x_7	x_4	x_5	x_6	RHS
z	4	0	0	?	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	?	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	?	0	$\frac{1}{2}$	0	230
x_6	2	0	0	?	-2	1	1	20

x_7-column = current inverse $\times$ original x_7-column

$$x_7\text{-column} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Table

Basic	x_1	x_2	x_3	x_7	x_4	x_5	x_6	Solution
z	4	0	0	?	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	230
x_6	2	0	0	1	-2	1	1	20

Reduced cost of x_7

Expression comes from the dual constraint corresponding to x_7

Write down this dual constraint.

$$y_1 + y_2 + 2y_3 \geq 4.$$

The expression for the reduced cost of x_7 is $c_7 = \bar{y}_1 + \bar{y}_2 + 2\bar{y}_3 - 4$, where $\bar{y}_1$, $\bar{y}_2$, and $\bar{y}_3$ are the shadow prices for the current basis.

Shadow prices are

$$[\bar{y}_1, \bar{y}_2, \bar{y}_3] = [2\ 5\ 0] \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{bmatrix} = [1\ 2\ 0]$$

$$x_7\text{-reduced cost} \quad c_7 = \bar{y}_1 + \bar{y}_2 + 2\bar{y}_3 - 4 = 1 + 2 - 4 = -1.$$

New Table

Basic	x_1	x_2	x_3	x_7	x_4	x_5	x_6	Solution
z	4	0	0	-1	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	230
x_6	2	0	0	1	-2	1	1	20

The new table is not optimal and we need to continue simplex method to find it.

Specifically x_7 will enter the basis, and x_6 will leave. The resulting optimal solution will be $x_1 = x_2 = 0$, $x_3 = 125$ and $x_7 = 210$. The optimal value will be \$1465.

Thus, introducing a new product is beneficial for TOYCO.