Got it 👍 — you want **examples of Linear Programming Problems (LPPs)** that illustrate the different possible outcomes of the **Simplex Method**:

1. **Unique Solution**

$$\text{Maximize } Z = 3x_1 + 2x_2$$

subject to

$$x_1 + x_2 \leq 4, \quad x_1 \leq 2, \quad x_1, x_2 \geq 0$$

👉 The feasible region is a bounded polygon. The optimum occurs at $(x_1, x_2) = (2, 2)$ with $Z = 10$. **Unique solution.**

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2. **Multiple (Alternative) Solutions**

$$\text{Maximize } Z = 2x_1 + 3x_2$$

subject to

$$x_1 + x_2 \leq 4, \quad 2x_1 + 3x_2 \leq 12, \quad x_1, x_2 \geq 0$$

👉 The objective function coefficients (2,3) are proportional to the second constraint. So the line $2x_1 + 3x_2 = 12$ itself gives the **same optimum value** at multiple points. For example, $(0, 4)$ and $(3, 2)$ are both optimal. **Multiple solutions.**

3. **Infeasible Solution**

$$\text{Maximize } Z = x_1 + x_2$$

subject to

$$x_1 + x_2 \geq 10, \quad x_1 + x_2 \leq 5, \quad x_1, x_2 \geq 0$$

👉 The constraints are contradictory ($x_1 + x_2$ can't be simultaneously ≥10 and ≤5). So **no feasible region**. **Infeasible problem.**

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4. **Unbounded Solution**

$$\text{Maximize } Z = x_1 + x_2$$

subject to

$$x_1 - x_2 \geq 2, \quad x_1, x_2 \geq 0$$

👉 The feasible region is unbounded in the direction of increasing x_1 and x_2. As we increase both, Z can grow indefinitely. **Unbounded solution.**

So, what is the problem with degenerate solutions?

- Degenerate iteration is the process of identifying the same point (solution) using different defining equations (different nonnegativity constraints). If we are lucky enough to obtain defining equations that correctly guides the moving direction, we move to a different point with a nondegenerate pivot.

- If we have a degenerate solution, pivot may continue indefinitely (Example in text p.31, pivoting rule is largest coefficient for entering variable and smallest subscript for leaving variable in case of ties. Then we have the initial dictionary again after 6 pivots.)

- Terminology:
  - Cycling : appearance of the same dictionary (tableau) again in the simplex iterations.

Bland’s rule to prevent cycling

Avoid cycling

- Perturbation method, Lexicographic method
- Smallest-subscript rule (Bland’s rule)

- Smallest-subscript rule (Bland’s rule) :

  Choose the variable having the smallest index among possible candidates (any nonbasic variable with $\bar{c}_j > 0$) as the entering variable.

  Also, if ties occur while choosing the leaving variable, select the smallest indexed variable among the candidates (tied basic variables in the minimum ratio test) as the leaving variable.

  See the proof in the text.

Example

(35) Consider the following simplex tableau:

		x_1	x_2	x_3	x_4	x_5	x_6
	2	-2	-3	1	12	0	0
$x_5 =$	0	-2	-9	1	9	1	0
$x_6 =$	0	$1/3$	1	$-1/3$	-2	0	1

Apply the simplex tableau method with the following rules:

- the non-basic variable with the most negative reduced cost enters; and
- in case of ties for the leaving variable, choose the row that is *higher up* in the tableau as the pivot row (even if the basic variable in that row has a larger subscript or index).

Demonstrate that the simplex method cycles. Now apply the simplex method with Bland's rule (smallest index pivoting rule), and demonstrate that the simplex method does not cycle.

given a final simplex table how can I recognize that this is a multiple solutions, unique solution, infeasible solution, unbounded

- Great question 👍 — if you're only given the **final simplex tableau**, you can still recognize the type of solution by checking certain conditions. Here's the checklist:

Definitions:

- C_j = the original objective function coefficient of variable x_j.
- Z_j = the "contribution" of variable x_j to the objective function based on the current solution (calculated from the basis).

Formula:

$$Z_j = \sum (C_B \cdot a_{ij})$$

where:

- C_B = coefficients in the objective function for the current basic variables,
- a_{ij} = column entries of variable x_j in the tableau.

Then,

$$Z_j - C_j = \text{Net evaluation / opportunity cost of including } x_j$$

Interpretation (for Maximization problems):

- If $Z_j - C_j > 0$ → Including this variable would **decrease** the objective value → not desirable.
- If $Z_j - C_j = 0$ → This variable can **enter the basis without changing the objective value** → possible **alternative optimal solution**.
- If $Z_j - C_j < 0$ → This variable can **improve the objective value** → candidate to enter the basis.

1. Unique Optimal Solution

- **Condition**: All $c_j - z_j$ (a.k.a. relative profits) in the final row are **≤ 0** (for maximization).
- None of the **non-basic variables** have $c_j - z_j = 0$.
  👉 That means there is only one optimal corner point.

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2. Multiple (Alternative) Optimal Solutions

- **Condition**: Optimality is reached ($c_j - z_j \leq 0$), **but at least one non-basic variable has** $c_j - z_j = 0$.
  👉 This indicates that moving along that variable does not change the objective function → another corner point gives the same optimum.

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3. Infeasible Solution

- In **two-phase simplex method**, if after **Phase I** the minimum value of the artificial variables is **> 0**, then the original problem has **no feasible solution**.
  👉 In the final tableau, if any **artificial variable remains in the basis with nonzero value**, the problem is **infeasible**.

4. Unbounded Solution

- **Condition**: In the simplex process, while selecting the **entering variable**, if the **pivot column has all entries ≤ 0** (in the constraint rows), then the ratio test cannot be applied.
  👉 This means the objective function can increase indefinitely → **unbounded solution**.

(We'll assume a **maximization problem**.)

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1. Unique Optimal Solution

Basis	x_1	x_2	s_1	Solution (RHS)
x_1	1	0	0.5	2
x_2	0	1	0.5	2
Zj – Cj	0	0	-0.5	Z = 10

- All $Zj - Cj \leq 0$.
- No non-basic variable has $Zj - Cj = 0$.

  👉 **Unique solution** at $x_1 = 2, x_2 = 2, Z = 10$.

2. Multiple (Alternative) Optimal Solutions

Basis	x_1	x_2	s_1	Solution (RHS)
x_1	1	0	0.5	3
s_1	0	1	0.5	1
Zj – Cj	0	0	0	Z = 12

- Optimality reached.
- But notice: $Zj - Cj = 0$ for **non-basic s_1**.

  👉 Means we can bring s_1 into the basis without changing Z.

  👉 **Multiple solutions.**

3. Infeasible Solution

(Final tableau after **Phase I**)

Basis	x_1	x_2	a_1	Solution (RHS)
x_1	1	0	0	2
a_1	0	1	1	3
Zj – Cj	0	0	0	W = 3

- Artificial variable a_1 is still in the basis with value **3 ≠ 0**.

  👉 Feasible region doesn't exist.

  👉 **Infeasible problem.**

4. Unbounded Solution

(Iteration where problem becomes unbounded)

Basis	x_1	x_2	s_1	Solution (RHS)
x_1	1	-2	0	4
s_1	0	-1	1	2
Zj – Cj	0	-3	-1	Z = 6

- The entering variable is x_2 (since $Zj - Cj = -3 < 0$).
- But look: in the pivot column (x_2), **all entries ≤ 0**.

  👉 Ratio test not possible → objective can increase indefinitely.

  👉 **Unbounded solution.**