LINEAR PROGRAMMING: THE GRAPHICAL METHOD

- Linear Programming Problem
- Properties of LPs
- LP Solutions
- Graphical Solution
- Introduction to Sensitivity Analysis

Linear Programming (LP) Problem

- A mathematical programming problem is one that seeks to maximize or minimize an objective function subject to constraints.
- If both the objective function and the constraints are linear, the problem is referred to as a linear programming problem.
- Linear functions are functions in which each variable appears in a separate term raised to the first power and is multiplied by a constant (which could be 0).
- Linear constraints are linear functions that are restricted to be "less than or equal to", "equal to", or "greater than or equal to" a constant.

Linear Programming (LP) M

- There are five common types of decisions in which LP may play a role
  - Product mix
  - Production plan
  - Ingredient mix
  - Transportation
  - Assignment

Steps in Developing a Linear Programming (LP) Model

1) Formulation

2) Solution

3) Interpretation and Sensitivity Analysis

Steps in Formulating LP Problems

1. Define the objective. (min or max)
2. Define the decision variables. (positive, binary)
3. Write the mathematical function for the objective.
4. Write a 1- or 2-word description of each constraint.
5. Write the right-hand side (RHS) of each constraint.
6. Write $\leq$, =, or $\geq$ for each constraint.
7. Write the decision variables on LHS of each constraint.
8. Write the coefficient for each decision variable in each constraint.

Properties of LP Models

1) Seek to minimize or maximize
2) Include "constraints" or limitations
3) There must be alternatives available
4) All equations are linear

LP Problems in: Product Mix

- Objective

  To select the mix of products or services that results in maximum profits for the planning period

- Decision Variables

  How much to produce and market of each product or service for the planning period

- Constraints

  Maximum amount of each product or service demanded; Minimum amount of product or service policy will allow; Maximum amount of resources available

Example LP Model Formulation: The Product Mix Problem

Decision: How much to make of $\geq$ 2 products?

Objective: Maximize profit

Constraints: Limited resources

Example: Pine Furniture Co.

Two products: Chairs and Tables

Decision: How many of each to make this month?

Objective: Maximize profit

Pine Furniture Data

	Tables (per table)	Chairs (per chair)	Hours Available
Profit Contribution	$7	$5	
Carpentry	3 hrs	4 hrs	2400
Painting	2 hrs	1 hr	1000

Other Limitations:

- Make no more than 450 chairs
- Make at least 100 tables

Constraints:

- Have 2400 hours of carpentry time available

$$3\,T + 4\,C \leq 2400 \quad \text{(hours)}$$

- Have 1000 hours of painting time available

$$2\,T + 1\,C \leq 1000 \quad \text{(hours)}$$

More Constraints:

- Make no more than 450 chairs

  $C \leq 450$ (num. chairs)

- Make at least 100 tables

  $T \geq 100$ (num. tables)

Nonnegativity:

Cannot make a negative number of chairs or tables

$$T \geq 0$$

$$C \geq 0$$

Model Summary

Max $7T + 5C$ (profit)

Subject to the constraints:

$$
\begin{array}{rll}
3T + 4C & \le 2400 & \text{(carpentry hrs)} \\
2T + 1C & \le 1000 & \text{(painting hrs)} \\
C & \le 450 & \text{(max \# chairs)} \\
T & \ge 100 & \text{(min \# tables)} \\
T, C & \ge 0 & \text{(nonnegativity)}
\end{array}
$$

Graphical Solution

- Graphing an LP model helps provide insight into LP models and their solutions.

- While this can only be done in two dimensions, the same properties apply to all LP models and solutions.

Carpentry

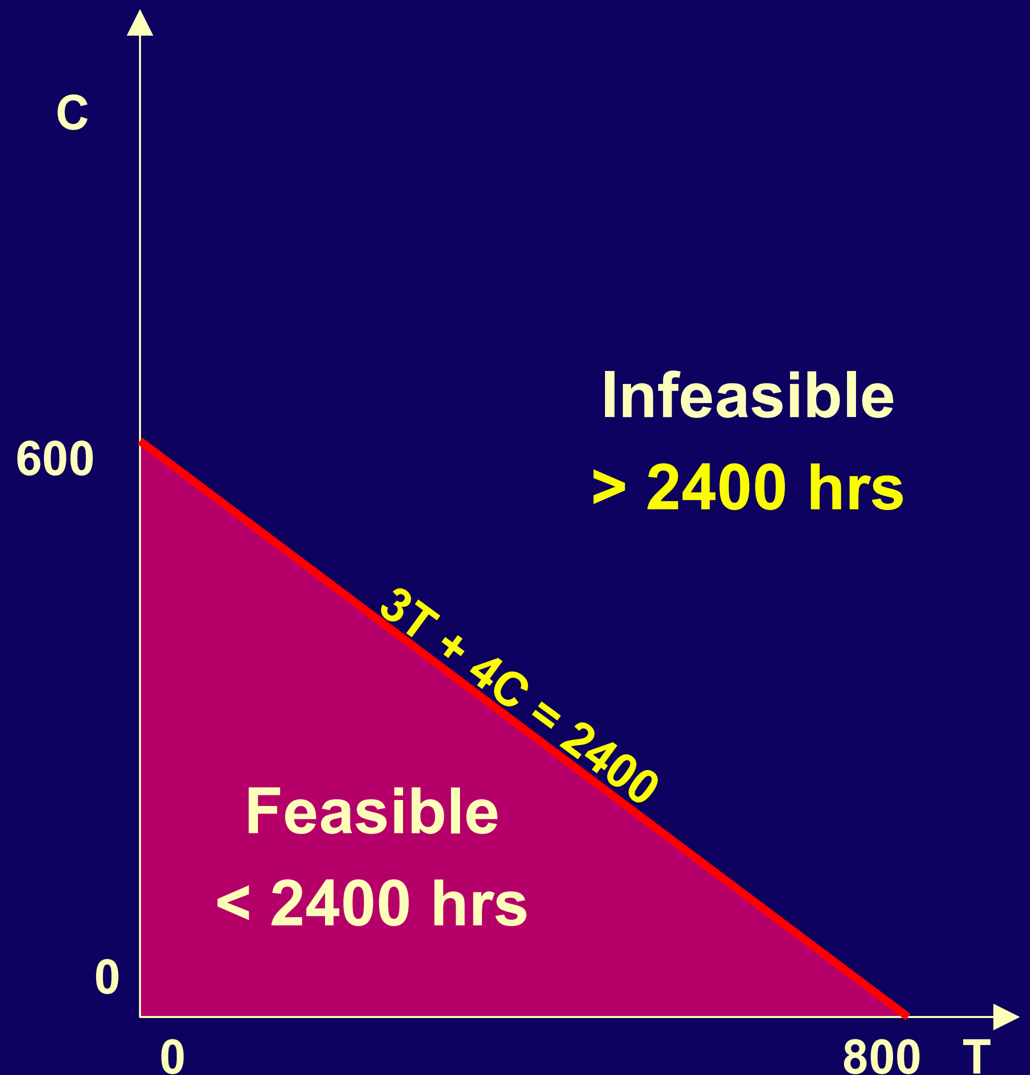
Constraint Line

$3T + 4C = 2400$

Intercepts

$(T = 0, C = 600)$

$(T = 800, C = 0)$



Painting

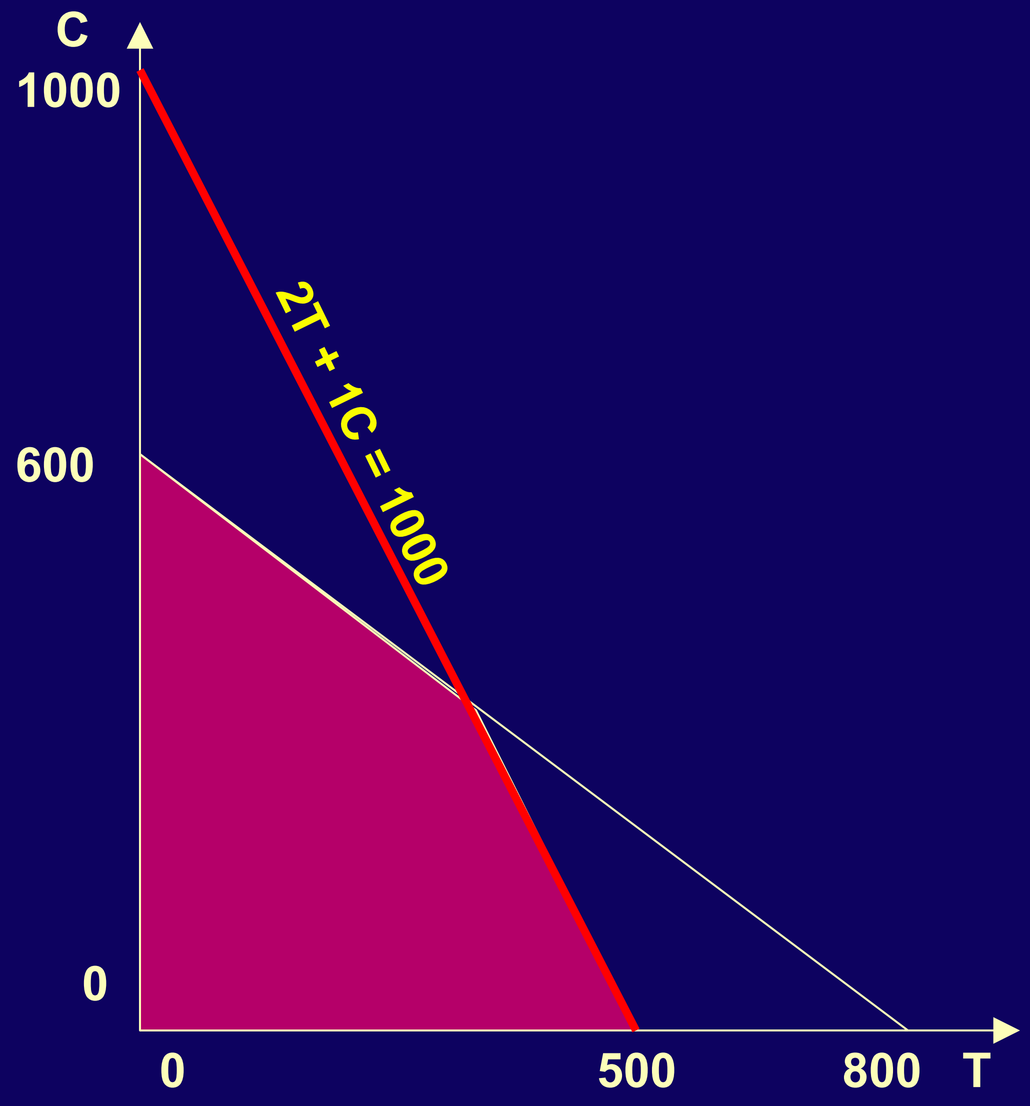
Constraint Line

2T + 1C = 1000

Intercepts

(T = 0, C = 1000)

(T = 500, C = 0)

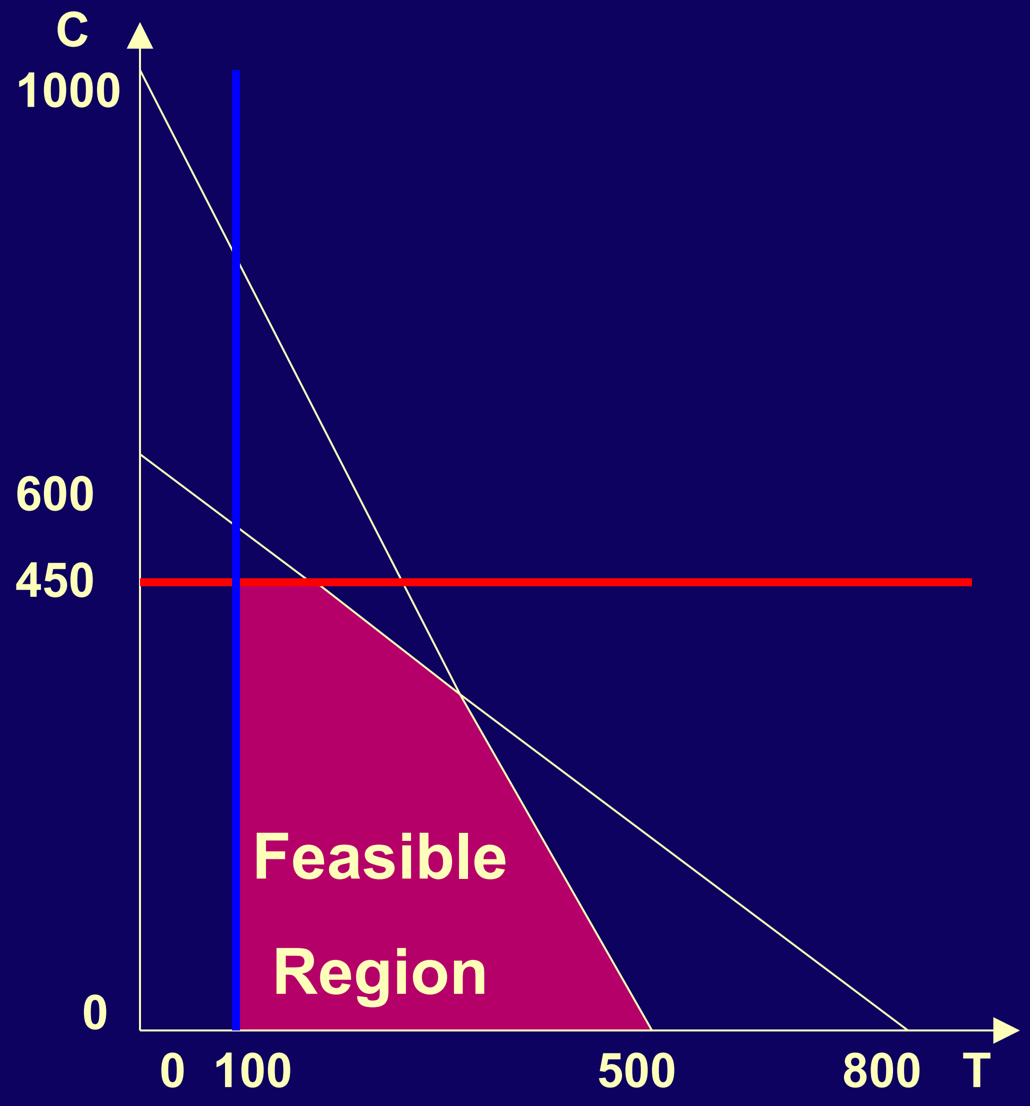


Max Chair Line

C = 450

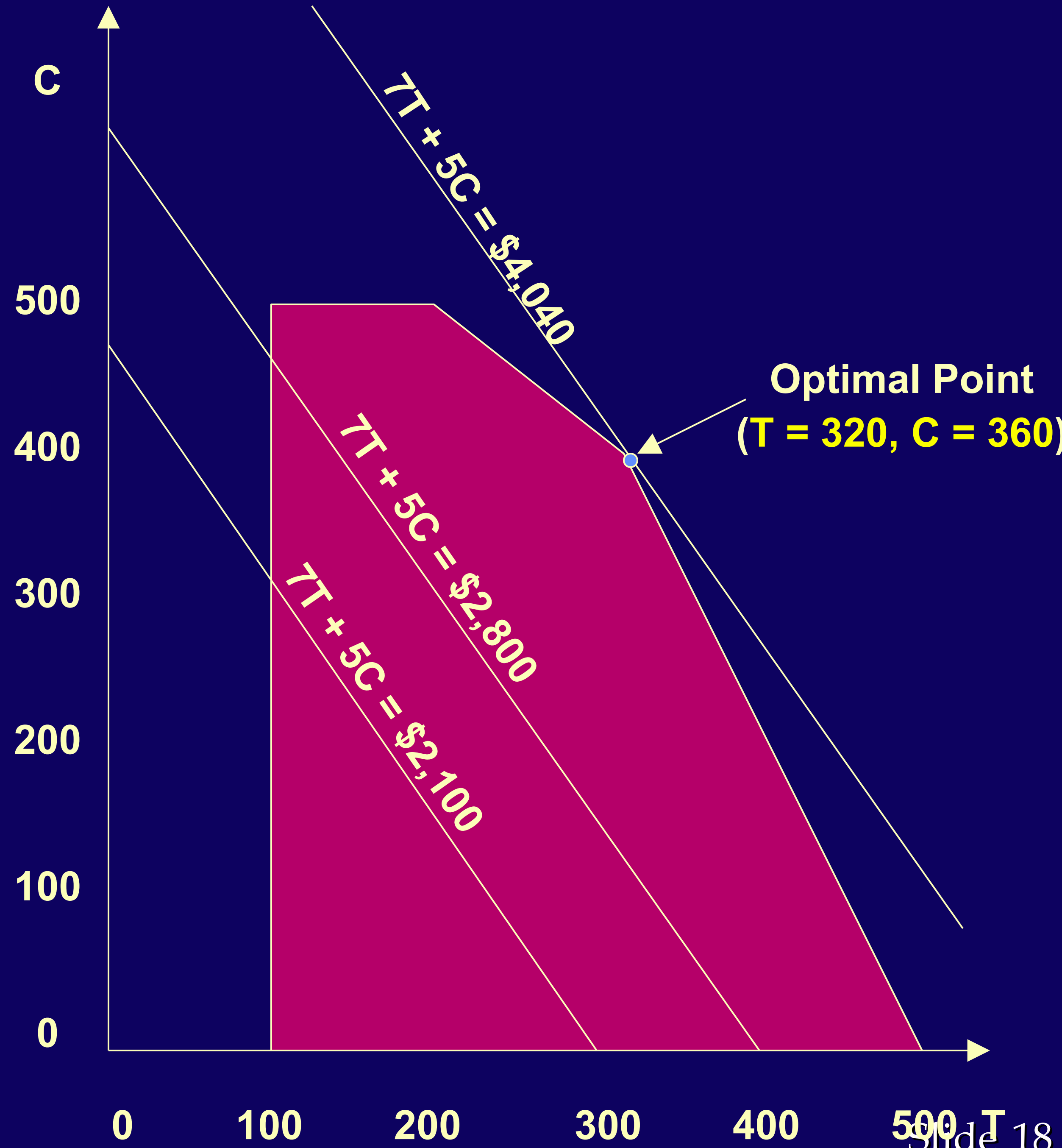
Min Table Line

T = 100



Objective
Function Line

7T + 5C = Profit



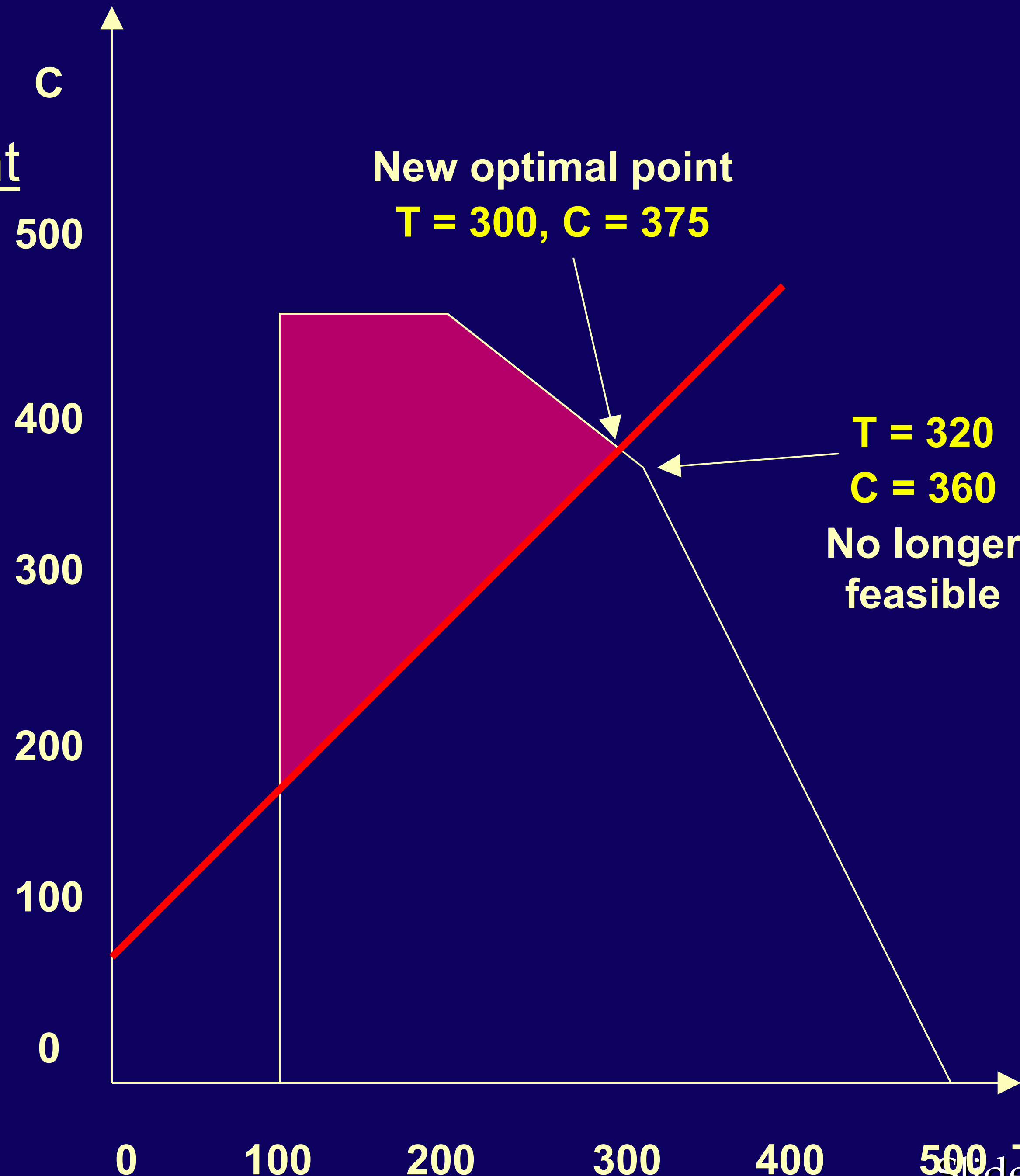
<u>Additional Constraint</u>

Need at least 75 more chairs than tables

$C \geq T + 75$

Or

$C - T \geq 75$



LP Characteristics

- **Feasible Region**: The set of points that satisfies all constraints
- **Corner Point Property**: An optimal solution must lie at one or more corner points
- **Optimal Solution**: The corner point with the best objective function value is optimal

Special Situation in LP

1. **Redundant Constraints** - do not affect the feasible region

Example: $x \leq 10$

$x \leq 12$

The second constraint is redundant because it is *less* restrictive.

Special Situation in LP

2. **Infeasibility** – when no feasible solution exists (there is no feasible region)

Example: $x \leq 10$

$x \geq 15$

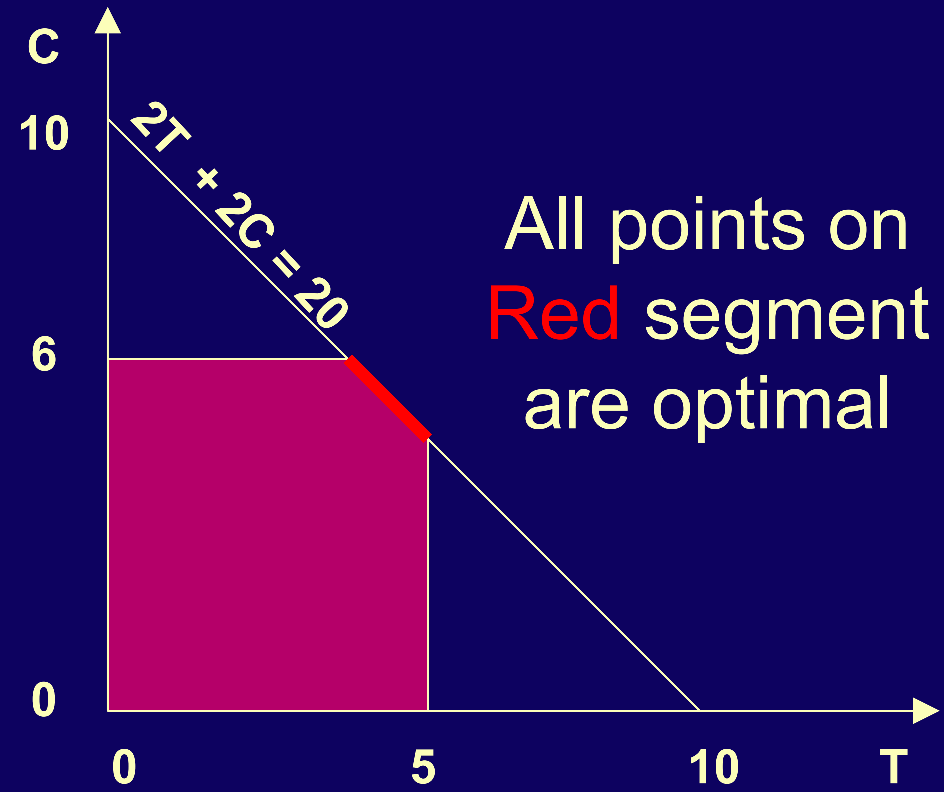
Special Situation in LP

3. **Alternate Optimal Solutions** – when there is more than one optimal solution

Max $2T + 2C$

Subject to:

$$T + C \leq 10$$
$$T \leq 5$$
$$C \leq 6$$
$$T, C \geq 0$$



All points on Red segment are optimal

Special Situation in LP

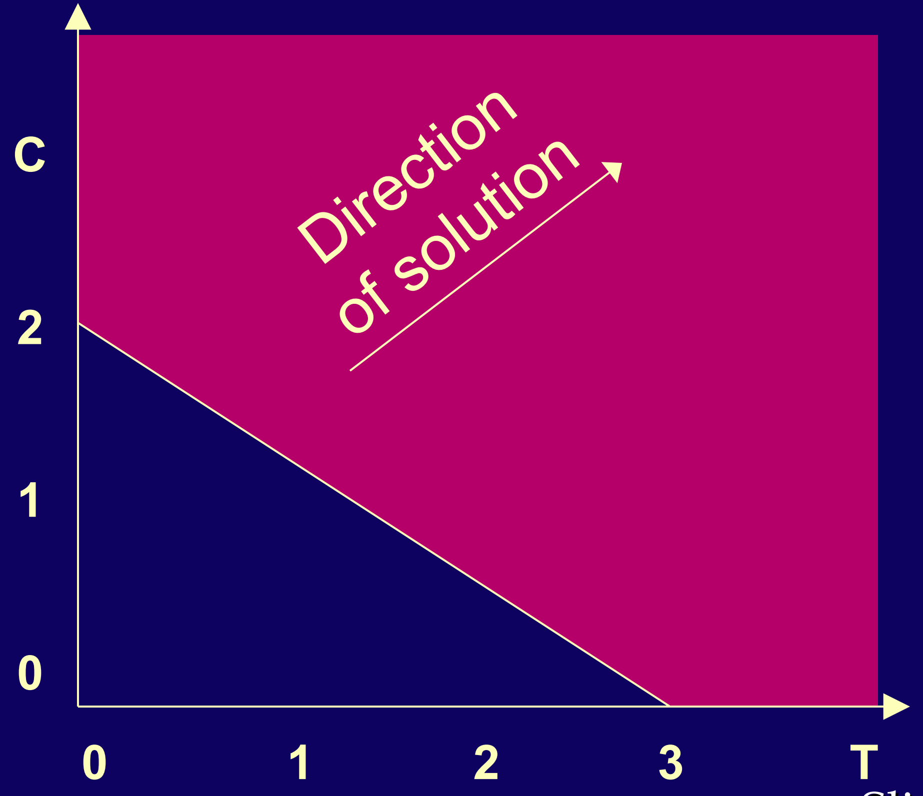
4. **Unbounded Solutions** – when nothing prevents the solution from becoming infinitely large

Max $2T + 2C$

Subject to:

$2T + 3C \geq 6$

$T, C \geq 0$



Building Linear Programming Models

- 1. What are you trying to decide - Identify the decision variable to solve the problem and define appropriate variables that represent them. For instance, in a simple maximization problem, RMC, Inc. interested in producing two products: fuel additive and a solvent base. The decision variables will be X1 = tons of fuel additive to produce, and X2 = tons of solvent base to produce.
- 2. What is the objective to be maximized or minimized? Determine the objective and express it as a linear function. When building a linear programming model, only relevant costs should be included, sunk costs are not included. In our example, the objective function is: z = 40X1 + 30X2; where 40 and 30 are the objective function coefficients.

Building Linear Programming Models

- 3. What limitations or requirements restrict the values of the decision variables? Identify and write the constraints as linear functions of the decision variables. Constraints generally fall into one of the following categories:
  - **a. Limitations** - The amount of material used in the production process cannot exceed the amount available in inventory. In our example, the limitations are:
  - Material 1 = 20 tons
  - Material 2 = 5 tons
  - Material 3 = 21 tons available.
  - The material used in the production of X1 and X2 are also known.

Building Linear Programming Models

- To produce one ton of fuel additive uses .4 ton of material 1, and .60 ton of material 3. To produce one ton of solvent base it takes .50 ton of material 1, .20 ton of material 2, and .30 ton of material 3. Therefore, we can set the constraints as follows:

  $$.4X1 + .50\,X2 <= 20$$

  $$.20X2 <= 5$$

  $$.6X1 + .3X2 <=21,\ \text{where}$$

  .4, .50, .20, .6, and .3 are called constraint coefficients. The limitations (20, 5, and 21) are called Right Hand Side (RHS).
- **b. Requirements - specifying a minimum levels of performance. For instance, production must be sufficient to satisfy customers' demand.**

LP Solutions

- The maximization or minimization of some quantity is the objective in all linear programming problems.
- A feasible solution satisfies all the problem's constraints.
- Changes to the objective function coefficients do not affect the feasibility of the problem.
- An optimal solution is a feasible solution that results in the largest possible objective function value, *z*, when maximizing or smallest *z* when minimizing.
- In the graphical method, if the objective function line is parallel to a boundary constraint in the direction of optimization, there are alternate optimal solutions, with all points on this line segment being optimal.

LP Solutions

- A graphical solution method can be used to solve a linear program with two variables.
- If a linear program possesses an optimal solution, then an extreme point will be optimal.
- If a constraint can be removed without affecting the shape of the feasible region, the constraint is said to be redundant.
- A nonbinding constraint is one in which there is positive slack or surplus when evaluated at the optimal solution.
- A linear program which is overconstrained so that no point satisfies all the constraints is said to be infeasible.

LP Solutions

- A feasible region may be unbounded and yet there may be optimal solutions. This is common in minimization problems and is possible in maximization problems.
- The feasible region for a two-variable linear programming problem can be nonexistent, a single point, a line, a polygon, or an unbounded area.
- Any linear program falls in one of three categories:
  - is infeasible
  - has a unique optimal solution or alternate optimal solutions
  - has an objective function that can be increased without bound

Example: Graphical Solution

- Solve graphically for the optimal solution:

$$\text{Min} \quad z = 5x_1 + 2x_2$$

$$\begin{aligned} \text{s.t.} \quad 2x_1 + 5x_2 &\geq 10 \\ 4x_1 - x_2 &\geq 12 \\ x_1 + x_2 &\geq 4 \end{aligned}$$

$$x_1, x_2 \geq 0$$

Example: Graphical Solution

- Graph the Constraints

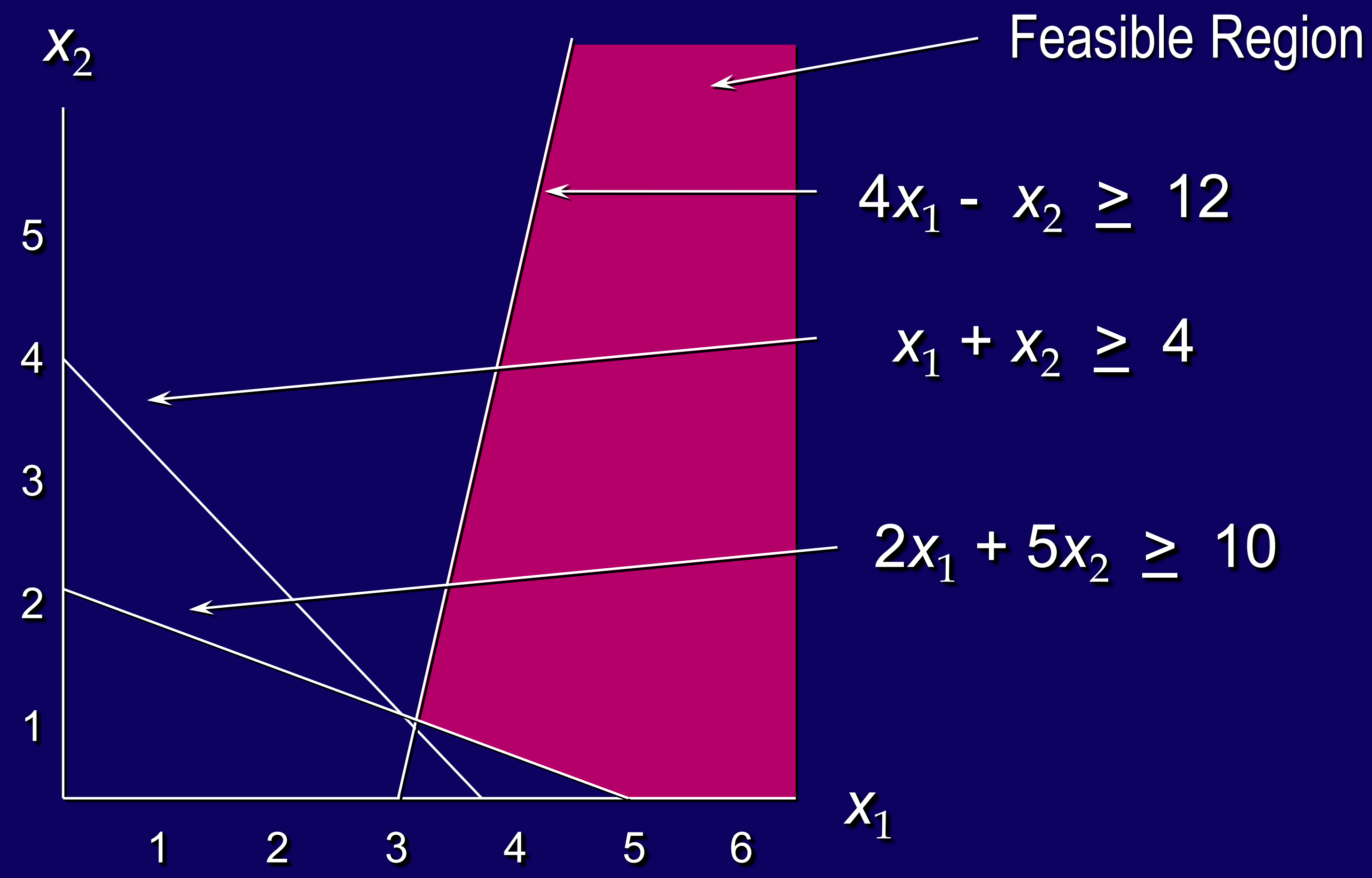
Constraint 1: When $x_1 = 0$, then $x_2 = 2$; when $x_2 = 0$, then $x_1 = 5$. Connect (5,0) and (0,2). The ">" side is above this line.

Constraint 2: When $x_2 = 0$, then $x_1 = 3$. But setting x_1 to 0 will yield $x_2 = -12$, which is not on the graph. Thus, to get a second point on this line, set x_1 to any number larger than 3 and solve for x_2: when $x_1 = 5$, then $x_2 = 8$. Connect (3,0) and (5,8). The ">" side is to the right.

Constraint 3: When $x_1 = 0$, then $x_2 = 4$; when $x_2 = 0$, then $x_1 = 4$. Connect (4,0) and (0,4). The ">" side is above this line.

Example: Graphical Solution

- Constraints Graphed



Example: Graphical Solution

- Graph the Objective Function

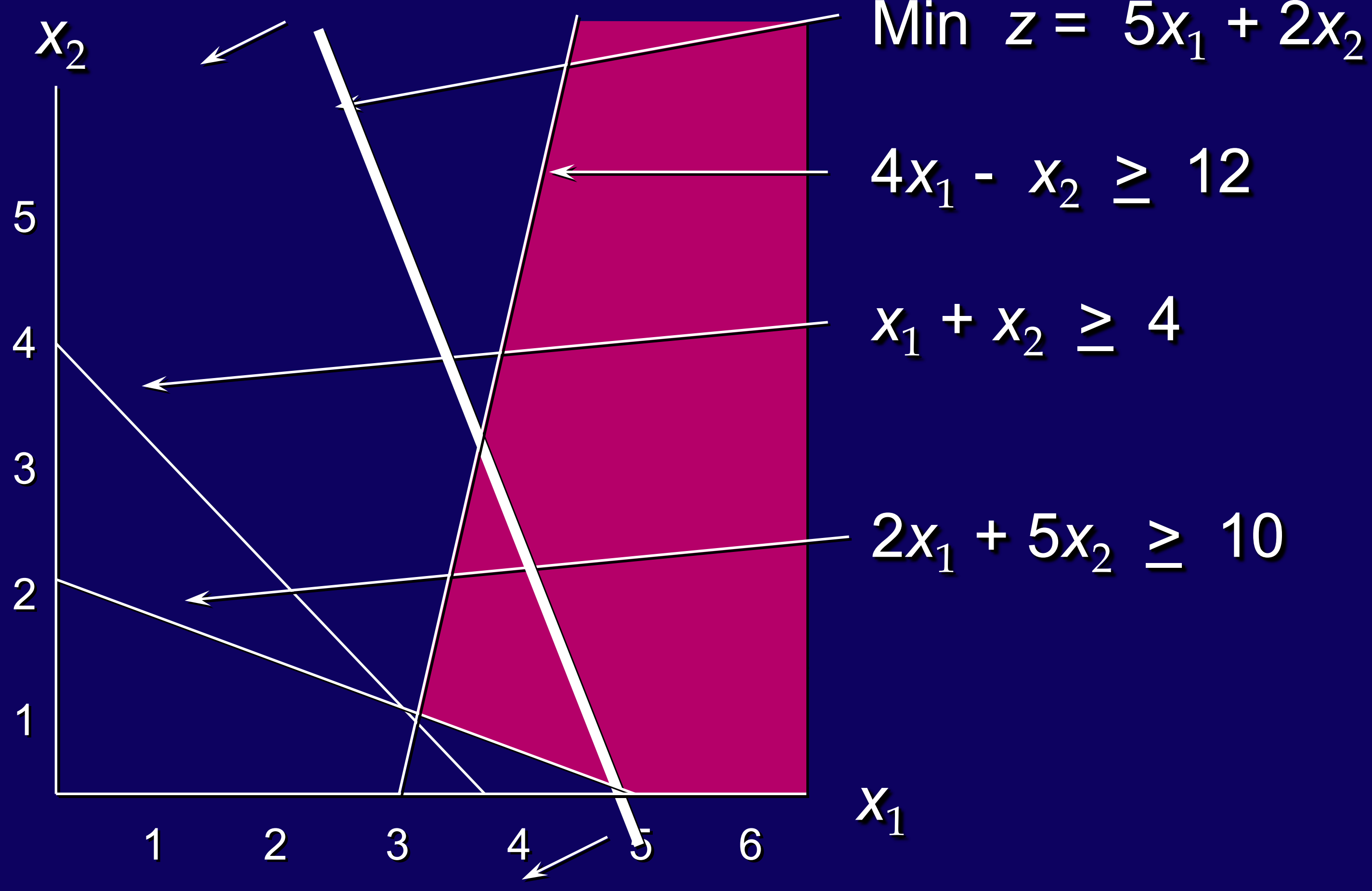
  Set the objective function equal to an arbitrary constant (say 20) and graph it. For $5x_1 + 2x_2 = 20$, when $x_1 = 0$, then $x_2 = 10$; when $x_2 = 0$, then $x_1 = 4$. Connect (4,0) and (0,10).

- Move the Objective Function Line Toward Optimality

  Move it in the direction which lowers its value (down), since we are minimizing, until it touches the last point of the feasible region, determined by the last two constraints. This is called the Iso-Value Line Method.

Example: Graphical Solution

- Objective Function Graphed



Example: Graphical Solution

- Solve for the Extreme Point at the Intersection of the Two Binding Constraints

$$4x_1 - x_2 = 12$$

$$x_1 + x_2 = 4$$

Adding these two equations gives:

$$5x_1 = 16 \text{ or } x_1 = 16/5.$$

Substituting this into $x_1 + x_2 = 4$ gives: $x_2 = 4/5$

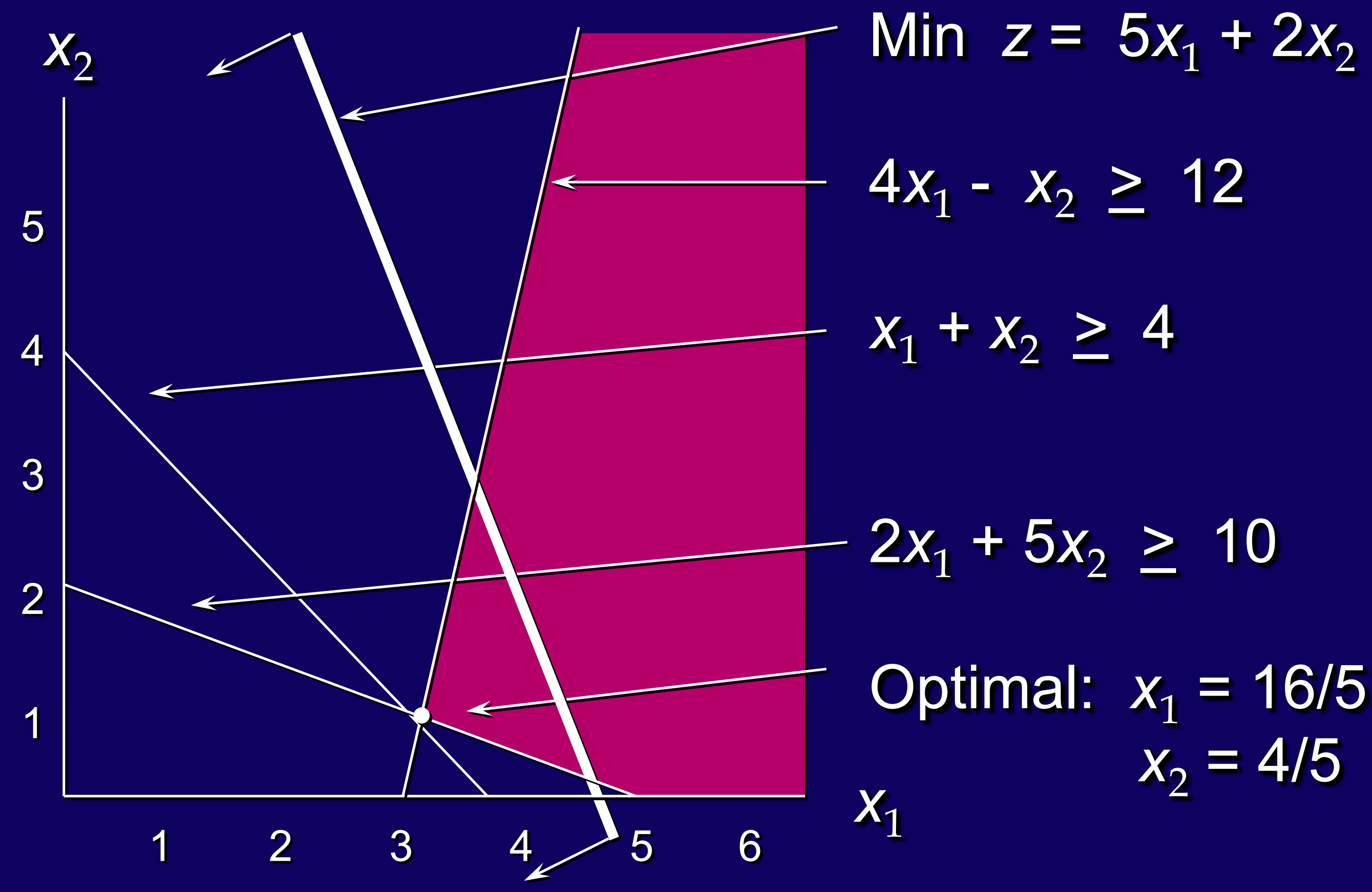
- Solve for the Optimal Value of the Objective Function

Solve for $z = 5x_1 + 2x_2 = 5(16/5) + 2(4/5) = 88/5$.

Thus the optimal solution is

$$\boxed{x_1 = 16/5;\ x_2 = 4/5;\ z = 88/5}$$

Example: Graphical Solution



Sensitivity Analysis

- Sensitivity analysis is used to determine effects on the optimal solution within specified ranges for the objective function coefficients, constraint coefficients, and right hand side values.
- Sensitivity analysis provides answers to certain what-if questions.

Range of Optimality

- A <u>range of optimality</u> of an objective function coefficient is found by determining an interval for the objective function coefficient in which the original optimal solution remains optimal while keeping all other data of the problem constant. The value of the objective function may change in this range.
- Graphically, the limits of a range of optimality are found by changing the slope of the objective function line within the limits of the slopes of the binding constraint lines. (This would also apply to simultaneous changes in the objective coefficients.)
- The slope of an objective function line, Max $c_1x_1 + c_2x_2$, is $-c_1/c_2$, and the slope of a constraint, $a_1x_1 + a_2x_2 = b$, is $-a_1/a_2$.

Example: Sensitivity Analysis

- Solve graphically for the optimal solution:

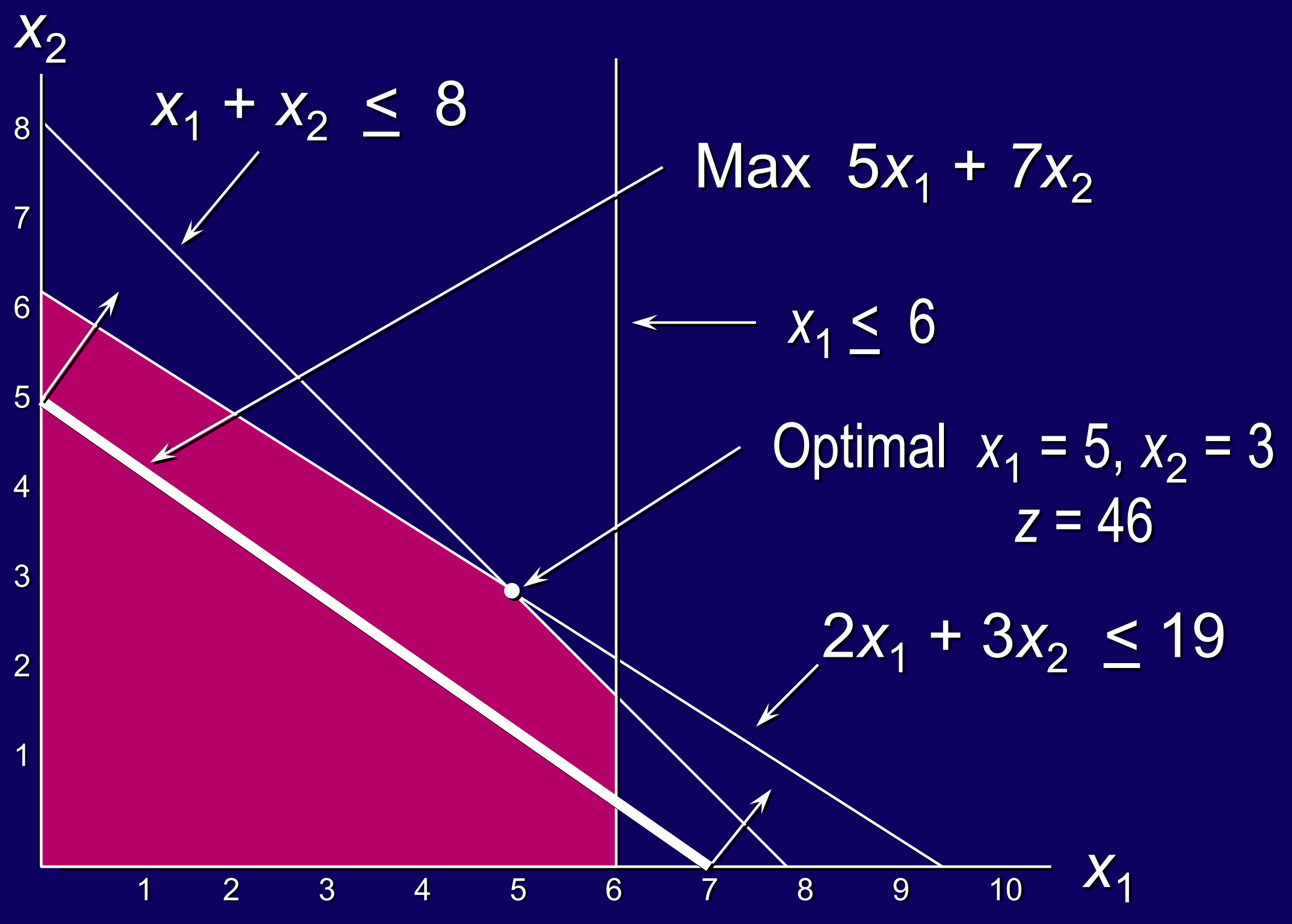
$$\text{Max} \quad z = 5x_1 + 7x_2$$

$$\begin{aligned}
\text{s.t.} \quad x_1 &\le 6 \\
2x_1 + 3x_2 &\le 19 \\
x_1 + x_2 &\le 8
\end{aligned}$$

$$x_1, x_2 \ge 0$$

Example: Sensitivity Analysis

- Graphical Solution



Example: Sensitivity Analysis

- Range of Optimality for c_1

The slope of the objective function line is $-c_1/c_2$. The slope of the first binding constraint, $x_1 + x_2 = 8$, is -1 and the slope of the second binding constraint, $x_1 + 3x_2 = 19$, is -2/3.

Find the range of values for c_1 (with c_2 staying 7) such that the objective function line slope lies between that of the two binding constraints:

$$-1 \leq -c_1/7 \leq -2/3$$

Multiplying through by -7 (and reversing the inequalities):

$$\boxed{14/3 \leq c_1 \leq 7}$$

Example: Sensitivity Analysis

- Range of Optimality for c_2

  Find the range of values for c_2 (with c_1 staying 5) such that the objective function line slope lies between that of the two binding constraints:

$$-1 \leq -5/c_2 \leq -2/3$$

Multiplying by -1: $1 \geq 5/c_2 \geq 2/3$

Inverting, $1 \leq c_2/5 \leq 3/2$

Multiplying by 5: $\boxed{5 \leq c_2 \leq 15/2}$

Example: Sensitivity Analysis

- Shadow Prices

Constraint 1: Since $x_1 \leq 6$ is not a binding constraint, its shadow price is 0.

Constraint 2: Change the RHS value of the second constraint to 20 and resolve for the optimal point determined by the last two constraints:

$$2x_1 + 3x_2 = 20 \quad \text{and} \quad x_1 + x_2 = 8.$$

The solution is $x_1 = 4$, $x_2 = 4$, $z = 48$. Hence, the shadow price = $z_{new} - z_{old} = 48 - 46 = 2$.

Example: Sensitivity Analysis

- Shadow Prices (continued)

<u>Constraint 3</u>: Change the RHS value of the third constraint to 9 and resolve for the optimal point determined by the last two constraints:

$$2x_1 + 3x_2 = 19 \quad \text{and} \quad x_1 + x_2 = 9.$$

The solution is: $x_1 = 8$, $x_2 = 1$, $z = 47$. Hence, the shadow price is $z_{new} - z_{old} = 47 - 46 = 1$.

Example: Infeasible Problem

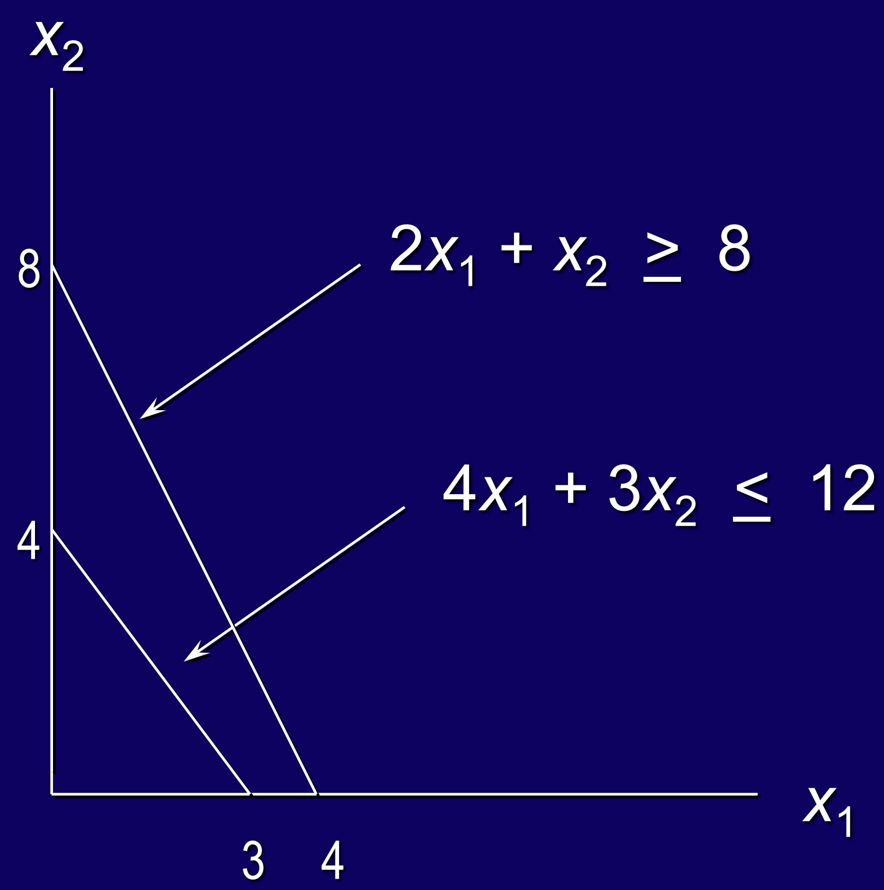
- Solve graphically for the optimal solution:

$$\text{Max} \quad z = 2x_1 + 6x_2$$

$$\begin{aligned} \text{s.t.} \quad & 4x_1 + 3x_2 \leq 12 \\ & 2x_1 + x_2 \geq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Example: Infeasible Problem

- There are no points that satisfy both constraints, hence this problem has no feasible region, and no optimal solution.



Example: Unbounded Problem

- Solve graphically for the optimal solution:

$$\text{Max} \quad z = 3x_1 + 4x_2$$

$$\begin{aligned} \text{s.t.} \qquad x_1 + x_2 &\geq 5 \\ 3x_1 + x_2 &\geq 8 \end{aligned}$$

$$x_1, x_2 \geq 0$$

Example: Unbounded Problem

- The feasible region is unbounded and the objective function line can be moved parallel to itself without bound so that z can be increased infinitely.