2

Linear Programming

2.1 INTRODUCTION

Linear programming was developed in 1947 by George B. Dantzig, Marshal Wood and their associates. It deals with the optimization (maximization or minimization) of a function of variables, known as objective functions. It is a set of linear equalities/inequalities known as constraint. Basically, linear programming is a mathematical technique, which involves the allocations of limited resources in an optimal manner on the basis of a given criterion of optimality. Linear programming is an optimization method applicable for the solution of problems in which the objective function and the constraints appear as linear functions of decision variables.

2.2 BASIC DEFINITIONS

1. Decision Variables

These are the variables, whose quantitative values are to be found from the solution of the model so as to maximize or minimize the objective function. The decision variables are usually denoted by $x_1, x_2, x_3, \ldots x_n$. It may be controllable or uncontrollable.

Controllable variables are those, whose values are under control of the decision makers. Uncontrollable variables are those, whose values are not under control.

2. Objective Function

It is the determinants of quantity either to be maximized or to be minimized. An objective function must include all the possibilities with profit or cost coefficient per unit of output. It is denoted by Z. The objective function can be stated as

$$\text{Max } Z \text{ or min } Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

3. Constraints (Inequalities)

These are the restrictions imposed on decision variables. It may be in terms of availability of raw materials, machine hours, man-hours, etc.

$$a_{i1}\, x_1 + a_{i2}\, x_2 + a_{i3}\, x_3 + \ldots + a_{in}\, x_n\ (\leq, =, \geq)\ b_i$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$a_{m1}\, x_1 + a_{m2}\, x_2 + a_{m3}\, x_3 + \ldots + a_{mn}\, x_n\ (\leq, =, \geq)\ b_m$$

and $$x_1, x_2, x_3 \ldots x_n \geq 0 \qquad (3)$$

Equation (1) is known as objective function.

Equation (2) represents the role of constants.

Equation (3) is non-negative restrictions.

Also $a'_{ij}s$ $b'_j s$ and $c'_j s$ are constants and $x'_j s$ are decision variables.

The above L.P.P. can be expressed in the form of matrix as follows:

Opt. $Z = CX$,

Subject to

$$AX\ (\leq, =, \geq)\ B$$

and $X \geq 0$

where
$$C = c_{1,}\, c_{2,}\, c_3 \ldots c_n$$
$$X = x_1, x_2, x_3 \ldots x_n$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m \times n} = [a_{ij}]_{m \times n}$$

Example 1 A manufacturer produces two types of models M_1 & M_2. Each model of type M_1 requires 4 hr of grinding and 2 hr of polishing. Whereas model M_2 requires 2 hr of grinding and 5 hr of polishing. The manufacturer has 2 grinders and 3 polishers. Each grinder works 60 hr a week and each polisher works 50 hr a week. Profit on model M_1 is Rs 4.00 and on model M_2 is Rs 5.00. How should the manufacturer allocate his production capacity to the two types of models, so that he may make the maximum profit in a weak? Formulate it as linear programming problem.

Solution

Decision Variables Let x_1 and x_2 be the number of units produced model M_1 and model M_2. Therefore, x_1 and x_2 be treated as decision variables.

Objective Function Since the profit on both the models is given and we have to maximize the profit. Therefore,

$$\text{Max } Z = 4x_1 + 5x_2 \qquad \ldots(1)$$

Constraints There are two constraints one for grinding and other for polishing. Two grinders are working. Therefore, number of hours available for grinding = $60 \times 2 = 120$ hours

Model M_1 requires 4 hr of grinding and Model M_2 requires 2 hours of grinding. Hence, the grinding constraint is given by

$$4x_1 + 2x_2 \leq 120 \qquad \text{...(2)}$$

There are 3 polishers. Total no. of hr available for polishing = $50 \times 3 = 150$ hr.

Model M_1 requires 2 hr of polishing, whereas model M_2 requires 5 hr of polishing. Therefore, we have

$$2x_1 + 5x_2 \leq 150 \qquad \text{...(3)}$$

Non-negative Restriction

$$x_1, x_2, \geq 0 \qquad \text{...(4)}$$

From equations (1), (2), (3), and (4), we have

$$\begin{aligned} \text{Max } Z &= 4x_1 + 5x_2 \\ \text{S.T. } 4x_1 + 2x_2 &\leq 120 \\ 2x_1 + 5x_2 &\leq 150 \\ x_1, x_2, &\geq 0 \end{aligned}$$

Example 2 A paper mill produces two grades of papers X and Y. Because of raw material restrictions it cannot produce more than 500 tonnes of grade X and 400 tonnes of grade Y in a week. There are 175 production hr in a week. It requires 0.2 and 0.4 hr to produce one tonne of product X and Y respectively with corresponding profit of Rs 4.00 and 5.00 per tonne. Formulate the above as L.P.P. to maximize the profit.

Solution

Decision Variables Let x_1 and x_2 be the number of units of two grades of papers X and Y. Therefore, x_1 and x_2 can be treated as decision variables.

Objective Function Since the profit of two grades of papers X and Y are given and we have to maximize the profit.

$$\therefore \qquad \text{Max } Z = 400\, x_1 + 500\, x_2 \qquad \text{...(1)}$$

Constraints There are two constraints one with respect to raw materials and other with respect to production hours.

$$\left.\begin{aligned} x_1 &\leq 500 \\ x_2 &\leq 400 \\ 0.2\, x_1 + 0.4\, x_2 &\leq 175 \end{aligned}\right\} \qquad \text{...(2)}$$

Example 4 A manufacturer produces three models I, II and III of a certain product. He uses two types of raw materials (A and B) of which 5000 and 8000 units respectively are available. Raw material of type A requires 3, 4 and 6 units of each model. Whereas type B requires 6, 4 and 8 of model I, II and III respectively. The labour time of each unit of model I is twice that of model II and three times of model III. The entire labour force of the factory can produce equivalent of 3000 units of model I. A market survey indicates that the minimum demand of three models is 600, 400 and 350 units respectively. However, the ratios of number of units produced must be equal to 3 : 2 : 5. Assume that the profit per unit of models I, II and III are Rs 80, 50, and 120 respectively. Formulate this problem as linear programming model to determine the number of units of each product which will maximize the profit.

Solution

The above problem can be tabulated as given below:

Raw materials	Requirement per unit model			Quantity of raw material available (units)
	I	II	III	
A	3	4	6	5000
B	6	4	8	8000
Profit/unit (Rs)	80	50	120	
Proportion of labour time	1	$\frac{1}{2}$	$\frac{1}{3}$	Production equivalent of model I = 3000 units

Decision Variables Let x_1, x_2, x_3 be the number of units of models I, II and III respectively. Therefore, it will be treated as decision variables.

Objective Function Since profit per units of models are given and we have to maximize the profit. Therefore,

$$\text{Max } Z = 80x_1 + 50x_2 + 120x_3 \qquad \ldots(1)$$

Constraints As per the statement of problem constraints are given as (as per tabulated value)

$$\left.\begin{aligned} 3x_1 + 4x_2 + 6x_3 &\leq 5000 \\ 6x_1 + 4x_2 + 8x_3 &\leq 8000 \\ x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 &\leq 3000 \\ x_1 &\leq 600 \\ x_2 &\leq 400 \\ x_1 &\leq 350 \end{aligned}\right\} \qquad \ldots(2)$$

Non-negative Restrictions

$$x_1, x_2, x_3 \geq 0 \qquad \ldots(3)$$

From equations (1), (2) and (3) finally, we have

$$\text{Max } Z = 80x_1 + 50x_2 + 120x_3$$

S·T·

$$x_1 \le 600$$
$$x_2 \le 400$$
$$x_3 \le 350$$
$$3x_1 + 4x_2 + 6x_3 \le 5000$$
$$6x_1 + 4x_2 + 8x_3 \le 8000$$
$$x_1 + \frac{1}{2} x_2 + \frac{1}{3} x_3 \le 3000$$
$$x_1, x_2, x_3 \ge 0$$

Example 5 A research laboratory has two melts *A* and *B* of copper (Cu), Nickel (Ni) and Zinc (Zn) alloy to make up a new alloy. The composition of metals are as follows.

Melt	Composition (Parts)		
	Cu	Ni	Zn
A	3	2	1
B	2	2	1

To make up a new alloy at least 15 kg of copper, 10 kg of nickel, and 6 kg of zinc are needed. Melt *A* cost Rs 45 per kg and melt *B* cost Rs 50 per kg. Formulate the L.P.P. for the quantities of each melt to be used to minimized cost.

Solution

The above data can be tabulated as follows.

Composition	Melt		Requirement of elements (Rs)
	A	*B*	
Cu	3	2	15
Ni	2	2	10
Zn	1	1	6
Cost per kg of melt (Rs)	45	50	

Decision Variables Let x_1 and x_2 be the quantity of melt *A* and *B* respectively. Therefore, x_1 and x_2 can be treated as decision variables.

Objective Function Since cost per kg melt of product *A* and *B* are given and we have to minimize the cost. Therefore,

$$\text{Min } Z = 45x_1 + 50x_2 \qquad \ldots(1)$$

Constraints As per the statement of problem, we have

$$\frac{3}{6}x_1 + \frac{2}{5} x_2 \ge 15$$

$$\left.\begin{array}{ll} \text{or} & 5x_1 + 4x_2 \geq 150 \\ & \frac{2}{6}x_1 + \frac{2}{5}x_2 \geq 10 \\ \text{or} & 5x_1 + 6x_2 \geq 150 \\ & \frac{1}{6}x_1 + \frac{1}{5}x_2 \geq 6 \\ \text{or} & 5x_1 + 6x_2 \geq 180 \end{array}\right\} \qquad \ldots(2)$$

Non-negative Restrictions

$$x_1, x_2, >, 0 \qquad \ldots(3)$$

From equation (1), (2) and (3), we have

$$\text{Min } Z = 45x_1 + 50x_2$$

$$\begin{aligned} \text{S.T.} \quad & 5x_1 + 4x_2 \geq 150 \\ & 5x_1 + 6x_2 \geq 150 \\ & 5x_1 + 6x_2 \geq 180 \\ & x_1 + x_2 \geq 0 \end{aligned}$$

Example 6 The objective of a diet problem is to ascertain the quantities of a certain foods that should be eaten to meet certain nutritional requirement at a minimum cost. The consideration is limited to milk, beef and eggs and to vitamines A, B and C. The number of milligrams of each of these vitamines contained within a unit of each food is given below.

Vitamin	Gallon of milk	Pound of beef	Dozen of eggs	Minimum daily requirement
A	1	1	10	1 mg
B	100	10	10	50 mg
C	10	100	10	10 mg
Cost	Rs 1.00	Rs 1.10	Rs 0.50	—

What is the L.P.P. for this problem?

Solution

Decision Variables Let the daily diet consist of x_1 gallons of milk, x_2 pounds of beef and x_3 dozens of eggs. Therefore, x_1, x_2 and x_3 can be treated as decision variables.

Objective Function Since cost per day of milk, beef and eggs are given and we have to minimize the total cost, therefore, we have

$$\text{Min } Z = 1.00x_1 + 1.10x_2 + 0.50x_3 \qquad \ldots(1)$$

Constraints As per the statement of problem, we have

$$\left.\begin{array}{rcl} x_1+x_2+10x_3 & \geq & 1 \\ 100x_1+10x_2+10x_3 & \geq & 50 \\ 10x_1+100x_2+10x_3 & \geq & 10 \end{array}\right\} \quad \ldots(2)$$

Non-negative Restrictions

$$x_1, x_2, x_3 \geq 0 \quad \ldots(3)$$

From equation (1), (2) and (3), we have

$$\text{Min } Z = x_1 + 1.10x_2 + 0.50x_3$$

$$\begin{array}{ll} \text{S.T.} & x_1 + x_2 + 10x_3 \geq 1 \\ & 100x_1 + 10x_2 + 10x_3 \geq 50 \\ & 10x_1 + 100x_2 + 10x_3 \geq 10 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Example 7 A firm can produce three types of cloth A, B and C. Three kinds of wool is required for it, say red, green and blue wools. One unit length of type A cloth needs 2 yards of red wool, 5 yards of blue wools, one unit length of type B cloth needs 3 yards of red wool, 4 yards of green wool, and 2 yards of blue wool, and one unit length of type C cloth needs 6 yards of green and 5 yards of blue wools. The firm has only a stock of 10 yards of red wool, 12 yards of green wool, and 17 yards of blue wool. It is assumed that the income obtained from one unit length of type A, B and C are Rs 4.00, 5.00 and 6.00 respectively. Determine how the firm should use the available material, so as to maximize the income from the finished cloths.

Solution

The above problem can be tabulated as:

Kinds of wool	Types of cloth			Stock of wool (yards)
	A	B	C	
Red	2	3	0	10
Green	0	4	6	12
Blue	5	2	5	17
Income from one unit of clothes (Rs)	4.00	5.00	6.00	

Decision Variables Let the firm produce x_1, x_2, x_3 yards of three types of cloth A, B and C respectively. Therefore, x_1, x_2 and x_3 can be treated as decision variables.

Objective Function Since the profit per unit length of type A, B and C are given and we have to maximize the profit, therefore, we have

$$\text{Max } Z = 4x_1 + 5x_2 + 6x_3 \quad \ldots(1)$$

Constraints As per the statement of given problem, we have

$$\left.\begin{aligned} 2x_1 + 3x_2 + 0x_3 &\le 10 \\ 0x_1 + 4x_2 + 6x_3 &\le 12 \\ 5x_1 + 2x_2 + 5x_3 &\le 17 \end{aligned}\right\} \quad \ldots(2)$$

Non-negative Restrictions

$$x_1, x_2, x_3 \ge 0 \quad \ldots(3)$$

From equations (1), (2) and (3), we have

$$\text{Max } Z = 4x_1 + 5x_2 + 6x_3$$

$$\begin{aligned} \text{S.T.} \quad 2x_1 + 3x_2 + 0x_3 &\le 10 \\ 0x_1 + 4x_2 + 6x_3 &\le 12 \\ 5x_1 + 2x_2 + 5x_3 &\le 17 \end{aligned}$$

and $$x_1, x_2, x_3 \ge 0$$

Example 8 An oil refinery uses blending process to produce gasoline in a typical manufacturing process. Crude A and B are mixed to produce gasoline G_1 and G_2. The input and output of the process are as follows:

Process	Input (tonnes)		Output (tonnes)	
	Crude A	Crude B	G_1	G_2
1	1	2	6	8
2	6	8	5	7

Availability of crude A is only 200 tonnes and B is 500 tonnes. Market demand of G_1 is 150 tonnes and G_2 is 200 tonnes. Profit on process 1 and process 2 is Rs 300 and 500 per tonne. What is the optimal mixture of two blending processes so that refinery can maximize its profit?

Solution

Decision Variables Let x_1 and x_2 be the number of tonnes to be produced by process 1 and process 2. Therefore, x_1 and x_2 can be treated as decision variables.

Objective Functions Since the profit on process 1 and process 2 is given and we have to maximize the profit. Therefore, we have

$$\text{Max } Z = 300x_1 + 500x_2 \quad \ldots(1)$$

Constraints As per the statement of problem, we have

$$\left.\begin{aligned} 5x_1 + 6x_2 &\le 200 \\ 2x_1 + 8x_2 &\le 500 \\ 6x_1 + 5x_2 &\le 150 \\ 8x_1 + 7x_2 &\le 200 \end{aligned}\right\} \quad \ldots(2)$$

Non-negative Restrictions

$$x_1 \text{ and } x_2 \ge 0 \quad \ldots(3)$$

From equation (1), (2) and (3), we have

$$\text{Max } Z = 300x_1 + 500x_2$$

$$\begin{aligned} \text{S.T. } 5x_1 + 6x_2 &\leq 200 \\ 2x_1 + 8x_2 &\leq 500 \\ 6x_1 + 5x_2 &\leq 150 \\ 8x_1 + 7x_2 &\leq 200 \\ x_1 \text{ and } x_2 &\geq 0 \end{aligned}$$

EXERCISE

1. A company produces two types of leather belts A and B. A is of superior quality and B is of inferior quality. The respective profits are Rs 10 and Rs 5.00 per belt. The supply of raw material is sufficient for making 850 belts per day. For belt A special type of buckle is required and 500 are available per day. There are 700 buckles available for belt B per day. Belt A needs twice as much as time as that required for belt B and the company can produce 500 belts if all of them were of the type A. Formulate L.P.P. for above problem.

Ans. $\text{Max } Z = 10x_1 + 5x_2$

$$\begin{aligned} \text{S.T. } x_1 + x_2 &\leq 850 \\ x_1 &\leq 500 \\ x_2 &\leq 700 \\ 2x_1 + x_2 &\leq 1000 \\ x_1, x_2 &\geq 0 \end{aligned}$$

2. A company produces two types of caps. Each cap of the first type requires as much labour time as the second type. If all caps are of second type only; the company can produce a total of 500 caps a day. The market limits daily sales of the first and second type to 150 and 250 caps. Assume that the profit per cap are Rs 10 for type B. Formulate the problem as a linear programing model in order to determine the number of caps to be produced of each type as to maximize the profit.

Ans. $\text{Max } Z = 10x_1 + 5x_2$

$$\begin{aligned} \text{S.T. } 2x_1 + x_2 &\leq 500 \\ x_1 &\leq 150 \\ x_2 &\leq 250 \\ x_1, x_2 &\geq 0 \end{aligned}$$

3. An oil refinery can blend three grades of crude oil to produce quality P and quality Q petrol. Two blending processes are available. For each production run the older process uses 5 units of crude A, 7 units of crude B and 2 units of crude C to produce 9 units of P and 7 units of Q. The newer processes uses 3 units of crude A. 9 units of crude B, and 4 units of crude C to produce 5 units of P and 9 units of Q petrol. Because of prior contract commitments the refinery must produce at least 500 units of P and 300 units of Q for the next month. It has available 1500 units of crude A, 1900 units of crude B and 1000 units of crude C. For each unit of P, the refinery receives Rs 60.00, while for each unit of Q it receives Rs 90.00. Find out the linear programming formulation of the problem to maximize the revenue.

$$
\begin{aligned}
\textbf{Ans.}\quad & \text{Max } Z = 1170x_1 + 1110x_2 \\
\text{S.T.}\quad & 5x_1 + 3x_2 \le 150 \\
& 7x_1 + 9x_2 \le 1900 \\
& 2x_1 + 4x_2 \le 1000 \\
& 9x_1 + 5x_2 \le 500 \\
& 7x_1 + 9x_2 \le 300 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

4. Orient Paper Mill produces two grades of papers X and Y. Because of raw material restrictions not more than 400 tonnes of grade X and 300 tonnes of grade Y can be produced in a week. There are 160 production hours in a week. It requires 0.2 and 0.4 hours to produce one tonne of products X and Y respectively with corresponding profit of Rs 35.00 and Rs 60.00 per tonne. Formulate a linear programming model to optimize the product mixture for maximum profit.

$$
\begin{aligned}
\textbf{Ans.}\quad & \text{Max } Z = 35X + 60Y \\
\text{S.T.}\quad & 0.2X + 0.4Y \le 160 \\
& X \le 400 \\
& Y \le 300 \\
& X, Y, \ge 0
\end{aligned}
$$

5. Garima Enterprises manufactures three types of dolls. The boy requires $\frac{1}{2}$ metre of red cloth, $1\frac{1}{2}$ metre of green and $1\frac{1}{2}$ metre of black cloth and 5 kg of fibre. The girl requires $\frac{1}{2}$ metre of red cloth, 2 metre of green cloth and 1 metre of black and 6 kg of fibre. The dog requires of $\frac{1}{2}$ metre of red, 1 metre of green. $\frac{1}{4}$ metre of black, and 2 kg of fibre. The profit on the three are 3.00, 5.00 and 2.00 respectively. The firm has 1000 metres of red, 1500 metres of green, 2000 metre of black and 6000 kg of fibre. Set up a linear programming for maximum profit to find the number of dolls of each type to be manufactured.

$$
\begin{aligned}
\textbf{Ans.}\quad & \text{Max } Z = 3x_1 + 5x_2 + 2x_3 \\
\text{S.T.}\quad & 0.5x_1 + 0.5x_2 + 0.5x_3 \le 1000 \\
& 1.5x_1 + 2x_2 + x_3 \le 1500 \\
& 0.5x_1 + x_2 + 0.25x_3 \le 2000 \\
& 5x_1 + 6x_2 + 2x_3 \le 6000 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}
$$

6. A resourceful home decorator manufactures two types of lamps say, P and Q. Both lamps go through two technicians first a cutter, second a finisher. Lamp A requires 2 hr of the cutters time, and 1 hour of finisher time. Lamp B requires 1 hours of cutters, 2 hours of finisher time. The cutter has 104 hours and finisher has 76 hours of available time each month. Profit on one lamps is Rs 6.00 and on one B Lamp is Rs 11.00. Assuming that he can sell all that he produces, how many of each types of lamp should be manufacturer to obtain the best return.

Ans. Max $Z = 6x_1 + 11x_2$

S.T. $2x_1 + x_2 \leq 104$

$x_1 + 2x_2 \leq 76$

$x_1, x_2 \geq 0$

7. A firm manufactures two types of products A and B and sells them at a profit of Rs 2.00 on type A and Rs 3.00 on type B. Each product is processed on two machines M_1 and M_2. Type A requires one minute of processing time on M_1 and 2 minutes on M_2, type B requires one minute on M_1 and one minute on M_2. The machine M_1 is available for not more than 6 hr 40 minutes, while machine M_2 is available for 10 hours during any working day. Formulate the problem as an L·P·P. and find how many products of each type should the firm produce each day in order to get maximum profit.

Ans. Max $Z = 2x_1 + 3x_2$

S.T. $x_1 + x_2 \leq 400$

$2x_1 + x_2 \leq 600$

$x_1, x_2 \geq 0$

8. A cold drink plant has two bottling machines A and B. It produces and sells 8-ounce and 16-ounce bottles. The following data is available

Machine	8-ounce	16-ounce
A	100/minute	40/minute
B	60/minute	75/minute

The machines can be run 8 hr per day 5 days per week. Weekly production of the drinks cannot exceed 3,00,000 ounces and the market can absorb 25,000 eight-ounce bottles and 7000 sixteen-ounce bottles per week. Profit on these bottles as 35 paise and 25 paise per bottle respectively. The planner wishes to maximize his profit subject to all the production and marketing restrictions. Formulate it as an L·P·P.

Ans. Max $Z = 0.35x_1 + 0.25x_2$

S.T. $8x_1 + 16x_2 \leq 300000$

$2x_1 + 5x_2 \leq 480{,}000$

$5x_1 + 4x_2 \leq 720{,}000$

$x_1 \leq 25000$

$x_2 \leq 7000$

$x_1, x_2 \geq 0$

9. A company manufactures two products A and B. These products are processed in the same machine. It takes 10 minutes to process one unit of product A and 2 minutes for each unit of product B and the machine operates for a maximum of 35 hr in a week. Product A requires 1.0 kg and B 0.5 kg of raw material per unit the supply of which is 600 kg per week. Product A costs Rs 5.00 per unit and sold at Rs 10. Product B costs Rs 6.00 per unit and can be sold in the market at a unit price of Rs 8.00. Determine the number of units per week to maximize the profit.

Ans. Max $Z = 5x_1 + 2x_2$

$$\text{S.T.} \quad \begin{aligned} 10x_1 + 2x_2 &\leq 2100 \\ x_1 + 0.5x_2 &\leq 600 \\ x_2 &\leq 800 \\ x_1, x_2 &\geq 0 \end{aligned}$$

10. An electric appliance company produces two products: refrigerators and ranges. Production takes place in two separate departments I and II. Refrigerators are produced in department 1 and ranges in department II. There products are sold on weekly basis. The weekly production cannot exceed 25 refrigerators, and 35 ranges. The company regularly employs a total of 60 workers in two departments. A refrigerator requires 2 man weeks labour while a range requires 1 man week labour. A refrigerator contributes a profit of Rs 60.00 and a range contributes a profit of Rs 40.00. How many units of refrigerators and ranges should the company produce to realize the maximum profit. Formulate the above as an L.P.P.

Ans. Max $Z = 60x_1 + 40x_2$

$$\text{S.T.} \quad \begin{aligned} x_1 &\leq 25 \\ x_2 &\leq 35 \\ 2x_1 + x_2 &\leq 60 \\ x_1, x_2 &\geq 0 \end{aligned}$$

2.5 GRAPHICAL METHOD OF SOLVING LINEAR PROGRAMMING PROBLEMS

Graphical method is applicable to find the simple linear programming problem with two decision variables. Various steps for solving the problems are given below:

1. Consider each inequality constraint as equation.
2. Plot each equation on the graph such that each will geometrically respect a straight line.
3. Identify the feasible region. If the inequality constraint corresponding to that line is $\leq$, then the region below the line in the first quadrant is to be shaded. For the inequality constraint $\geq$, then the region above the line in the first quadrant is shaded. The points lying in common region will satisfy all the constraints simultaneously. This common region is called feasible region.
4. Locate the corner points of the feasible region.
5. Draw the straight line to represent the objective function.
6. Test the objective function at each corner point of the feasible region and choose the point, where objective function obtains optimal value.

Example 9 Solve the following L.P.P. by graphical method

$$\text{Min } Z = 20x_1 + 10x_2$$

$$\text{Subject to } x_1 + 2x_2 \leq 40$$

$$3x_1 + x_2 \geq 30$$

$$4x_1 + 3x_2 \geq 60$$

$$x_1, x_2 \geq 0$$

Solution

Convert all the inequalities of the constraints into equations, we have

$$x_1 + 2x_2 = 40$$

$$3x_1 + x_2 = 30$$

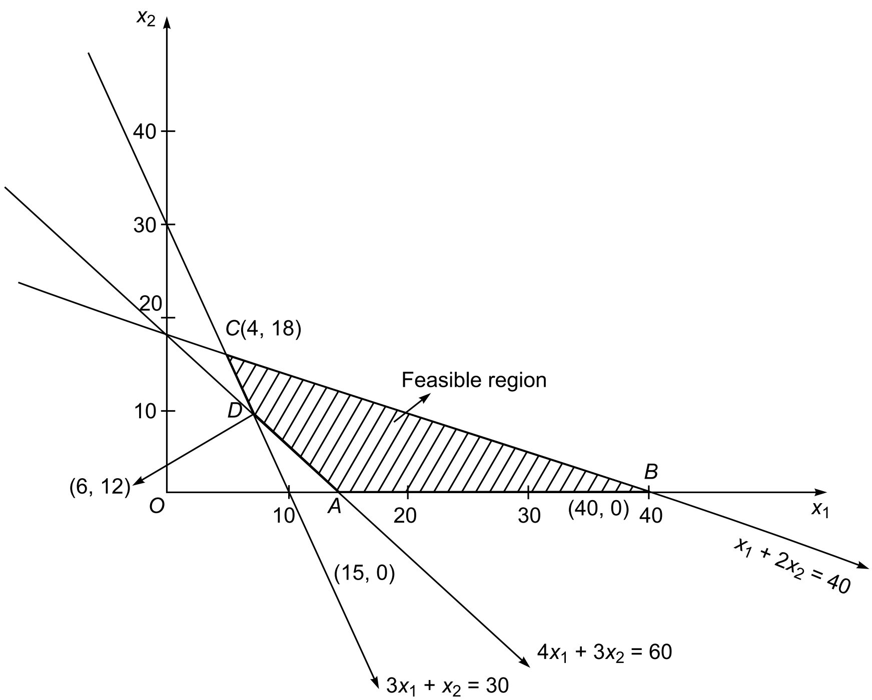
$$4x_1 + 3x_2 = 60$$

$x_1 + 2x_2 = 40$ passes through (0, 20) (40, 0)

$3x_1 + x_2 = 30$ passes through (0, 30) (10, 0)

$4x_1 + 3x_2 = 60$ passes through (0, 20) (15, 0)

Plot above equations on graph, we have



Here feasible region is *ABCD*.

The coordinates of *ABCD* are *A*(15, 0) *B*(40, 0), *C*(4, 18), *D*(6, 12)

Now

Corner Points	Coordinate	Value of Z
A	(15, 0)	300
B	(40, 0)	800
C	(4, 18)	260
D	(6, 12)	240

Therefore, minimum value of Z occurs at $D(6, 12)$. Hence, optimal solution is $x_1 = 6, x_2 = 12$.

Example 10 Solve the following L.P.P. using graphical methods

$$\text{Max } Z = 6x_1 + 8x_2$$
$$\text{Subject to } 5x_1 + 10x_2 \leq 60$$
$$4x_1 + 4x_2 \leq 40$$
$$x_1, x_2 \geq 0$$

Solution

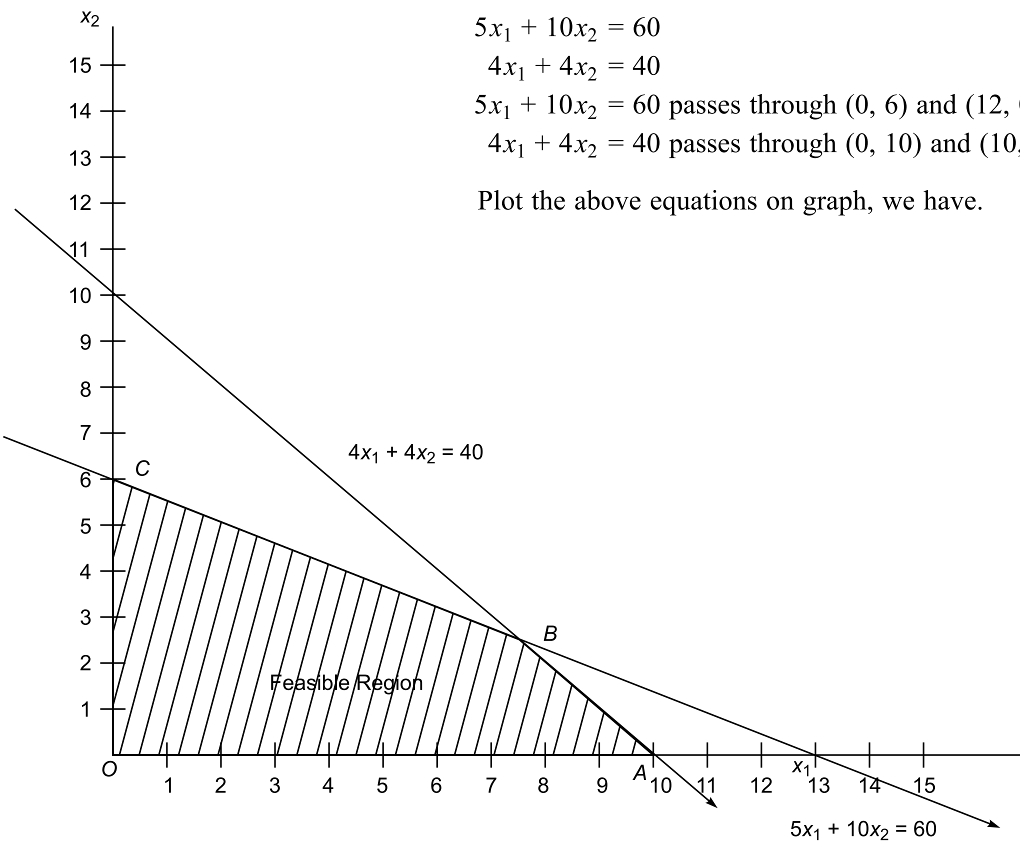
Convert all the equalities of the constraint into equations, we have

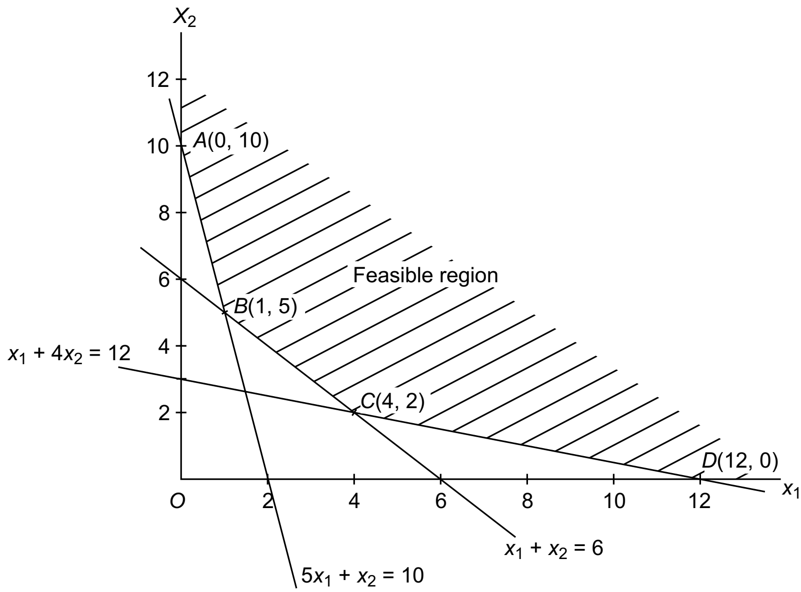
$$5x_1 + 10x_2 = 60$$
$$4x_1 + 4x_2 = 40$$

$5x_1 + 10x_2 = 60$ passes through (0, 6) and (12, 0)

$4x_1 + 4x_2 = 40$ passes through (0, 10) and (10, 0)

Plot the above equations on graph, we have.





Now the coordinates of points *ABCD* are *A*(0, 10), *B*(1, 5), *C*(4, 2), *D*(12, 0)

Corner Points	Coordinate	Value of *Z*
A	(0, 10)	20
B	(1, 5)	13
C	(4, 2)	16
D	(12, 0)	36

Hence, minimum value occurs at point *B*(1, 5). Therefore, optimum solution is given by

$$x_1 = 1,\ x_2 = 5 \text{ and min } Z = 13.$$

Example 12 Solve the following L.P.P. by graphical method

$$\text{Max } Z = 3x_1 + 2x_2$$
$$\text{S.T.} \quad x_1 - x_2 \geq 1$$
$$x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

Solution

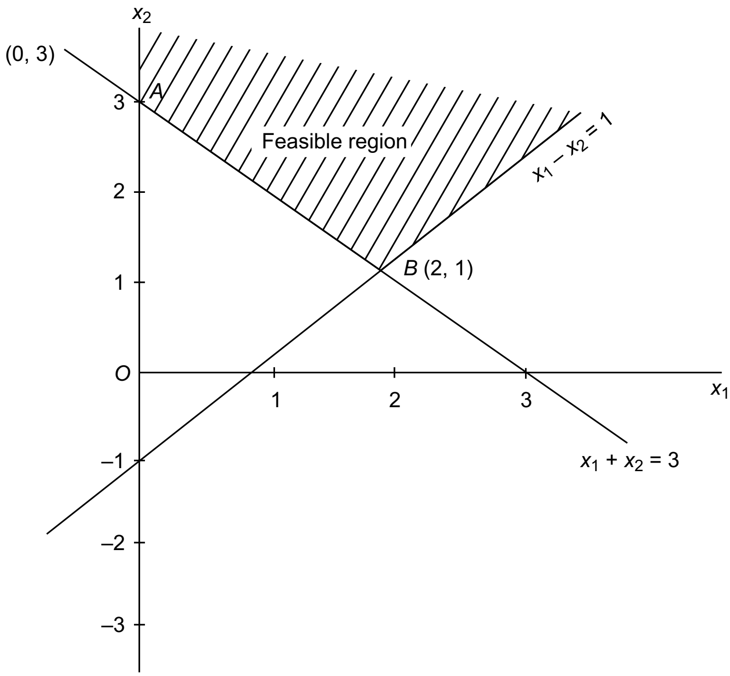
Convert the inequality constraints into equations. We have

$$x_1 - x_2 = 1$$
$$x_1 + x_2 = 3$$

Now

$$x_1 - x_2 = 1 \text{ passes through } (0, -1) \text{ and } (1, 0)$$
$$x_1 + x_2 = 3 \text{ passes through } (0, 3) \text{ and } (3, 0)$$

Plot above equations on graph, we have



Here the solution space is unbounded. The value of objective function at the vertices A and B are $Z(A) = 6$, $Z(B) = 6$. But there exists points in the convex region for which the value of the objective function is more than 8. In fact, the maximum value of Z occurs at infinity. Hence, the problem has an unbounded solution.

Example 13 By graphical method solve the following

$$\text{Max } Z = 3x_1 + 4x_2$$

$$\begin{aligned} \text{S.T. } 5x_1 + 4x_2 &\leq 200 \\ 3x_1 + 5x_2 &\leq 150 \\ 5x_1 + 4x_2 &\geq 100 \\ 8x_1 + 4x_2 &\geq 80 \\ x_1, x_2 &\geq, 0 \end{aligned}$$

Solution

Convert the inequality constraints into equations, we have

$$\begin{aligned} 5x_1 + 4x_2 &= 200 \\ 3x_1 + 4x_2 &= 100 \\ 8x_1 + 4x_2 &= 80 \\ 3x_1 + 5x_2 &= 150 \end{aligned}$$

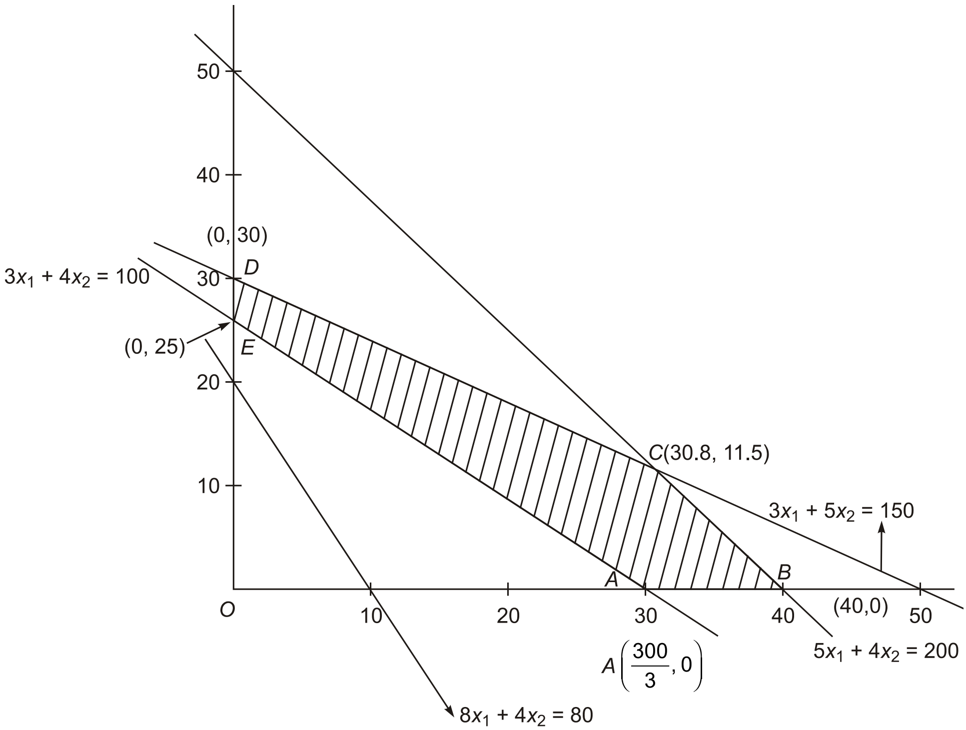
Now $\quad 5x_1 + 4x_2 = 200$ passes through (0, 50) and (40, 0)

$3x_1 + 4x_2 = 100$ passes through (0, 25) and $\left(\frac{100}{3}, 0\right)$

$8x_1 + 4x_2 = 80$ passes through (0, 20) and (10, 0)

$3x_1 + 5x_2 = 150$ passes through (0, 30) and (50, 0)

Plot the above equations on graph, we have



Here feasible region is *ABCDE*. Coordinates are given by $A\left(\frac{100}{3}, 0\right)$, $B(40, 0)$, $C(30.8, 11.5)$, $D(0, 30)$ and $E(0, 25)$.

Corner points	Coordinate	Value of Z
A	$\left(\frac{100}{3}, 0\right)$	100
B	(40, 0)	120
C	(30.8, 11.5)	138.4
D	(0, 30)	120
E	(0, 25)	100

6.
$$\text{Max } Z = 3x_1 - 2x_2$$
$$\text{S.T.} \quad x_1 + x_2 \le 1$$
$$2x_1 + 2x_2 \ge 6$$
$$3x_1 + 2x_2 \ge 48$$
$$x_1, x_2 \ge 0$$

[**Ans.** No feasible solution]

7.
$$\text{Min } Z = 3x_1 - 2x_2$$
$$\text{S.T.} \quad x_1 + x_2 \le 1$$
$$2x_1 + 2x_2 \ge 6$$
$$3x_1 + 2x_2 \ge 48$$
$$x_1, x_2 \ge 0$$

[**Ans.** No feasible solution]

8. A company produces two different products *A* and *B*. The company makes a profit of Rs 40 and Rs 30 per unit on *A* and *B* respectively. The production process has a capacity of 30,000 man hours. It takes 3 hr to produce one unit of a *A* and one hr to produce one unit of *B*. The market survey indicates that the maximum number of units of product *A* that can be sold is 8000 and those of *B* is 12,000 units. Formulate the problem and solve it by graphical method to get maximum profit.

Ans.
$$\text{Max } Z = 40x_1 + 30x_2$$
$$\text{S.T.} \quad 3x_1 + x_2 \le 30{,}000$$
$$x_1 \le 8000$$
$$x_2 \le 12000$$
$$x_1, x_2 \ge 0$$

[**Ans.** $x_1 = 6000$, $x_2 = 1200$, Max $Z = 600000$]

2.6 SIMPLEX METHOD

It is an iterative procedure for solving an L.P.P. in a finite number of steps. This method provides an algorithm which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the successing vertex is less or more as the case may be more than its process vertex. This procedure is repeated and since the number of vertices is finite, the method leads to an optimal vertex in a finite number of steps or indicates the existence of unbounded solutions. It is applicable for any number of decision variables.

2.6.1 Basic Terms Involved in Simplex Method

1. Standard Form of an L.P.P. In standard form of the objective function, namely, maximize or minimize, all the constraints are expressed as equations moreover R.H.S. of each constraint and all variables are non-negative.

2. Slack Variables These variables are added to less than or equal to type constraints to change it into equality.

3. Surplus Variables These variables are substrates from a greater than or equal to type constraint to change it into equality.

4. Basic Solution Given a system of m linear equations with n variables ($m < n$). Any solution which is obtained by solving for m variables keeping the remaining (n–m) variables zero is called a basic solution.

5. Basic Feasible Solution A basic solution, which also satisfies the non-negative constraints, is called basic feasible solution.

6. Non-Degenerate Basic Solution It is the basic feasible solution, which has exactly m positive, i.e., none of basic variables are zero.

7. Degenerate Basic Feasible Solution A B.F.S. is said to be degenerate if one or more basic variables are zero.

8. Feasible Solution Any solution to an L.P.P. which satisfies the non-negative restrictions, is called feasible solution.

9. Optimal Solution A basic feasible solution of an L.P.P. which gives optimum value of the objective function is called optimal solution.

10. Unbounded Solution If the value of the objective function Z can be increased or decreased indefinitely, such solutions are called unbounded solutions.

11. Canonical Form In canonical form, if the objective function is of maximization, all the constraints other than non-negative conditions are ≤ type. If the objective function is of minimization, all the constraints other than non-negative conditions are ≥ type.

2.6.2 Simplex Algorithm

The various steps for the computation of an optimum solution by simplex method are as follows:

1. Check whether the objective function of a given L.P.P. is to be maximized or minimized. If it is to be minimized, then convert into maximization case.
2. Check whether all b_i (i = 1, 2, 3 … n) are positive or not. If any one b_i is negative then make it positive by multiplying –1 in equation of the constraint.
3. Express the problem in standard form by introducing stack/surplus variables to convert the inequality into equation.
4. Find an initial basic feasible solution to the problem and put it in the simplex table.
5. Prepare the initial simplex table.

	Initial simplex table						
		C_j	x_1	x_2	…	x_n	0 0 0
C_B	B	X_B	x_1	x_2	…	x_n	S_1, S_2 … S_m
C_{B1}	S_1	b_1	a_{11}	a_{12}	…	a_{1n}	1 0 0
C_{B2}	S_2	b_2	a_{21}	a_{22}	…	a_{2n}	0 1 0

C_B	B	C_j / X_B	3 / x_1	2 / x_2	0 / S_1	0 / S_2	Min $\left(\frac{X_B}{x_1}\right)$
0	S_1	4	1	1	1	0	$\frac{4}{1} = 4$
← 0	S_2	2	$\boxed{1}$	–1	0	1	$\frac{2}{1} = 2$
	Z_j	0	0	0	0	0	
		$Z_j - C_j$	–3 ↑	–2	0	0	

Since all the values of $Z_j - C_j$ is not positive. Therefore, initial basic feasible solution is not optimum. To find optimum solution select the most negative value of $Z_j - C_j$. Here –3 is the most negative value of $Z_j - C_j$. It will enter in the basis and treated as entering variable and corresponding column will known as key column.

Now find leaving variable by taking min $\left(\frac{X_B}{x_1}, x_1 > 0\right)$. Here, minimum value exists in the second row, therefore, it will be treated as key row and S_2 will leave the basis.

Find key element by intersection of key row and key column. Here key element is $\boxed{1}$. Now make all other elements of key column to zero by taking matrix row transformation $R_1 \rightarrow R_1 - R_2$ and prepare the new simplex table.

First simplex table

C_B	B	C_j / X_B	3 / x_1	2 / x_2	0 / S_1	0 / S_2
← 0	S_1	2	0	$\boxed{2}$	1	–1
3	x_2	2	1	–1	0	0
	Z_j	6	3	–3	0	0
		$Z_j - C_j$	0	–5 ↑	0	0

Further, all values of $Z_j - C_j$ is not positive. Therefore solution is not optimal. Here –5 is the most negative number and it will enter in the basic. Corresponding column is treated as key column. Key row is the first row. S_1 will leave the basis. Key element is not unity. Make it unity and then apply $R_2 \rightarrow R_2 + R_1$ to make all after element of key column to zero. Now form the second simplex table.

Second simplex table

C_B	B	C_j / X_B	3 / x_1	2 / x_2	0 / S_1	0 / S_2
2	x_2	1	0	1	1/2	–1/2

Contd.

3	x_1	3	1	0	1/2	1/2
	Z_j	11	3	2	5/2	1/2
		$Z_j - C_j$	0	0	5/2	1/2

Here all the values of $Z_j - C_j$ are positive. Hence, optimum solution will exist and it is given by Max $Z = 11$, $x_1 = 3$, $x_2 = 1$

Example 15 Solve the L.P.P. by simplex method

$$\text{Max } Z = 10x_1 + 6x_2$$
$$\text{S.T.} \quad x_1 + x_2 \le 2$$
$$2x_1 + x_2 \le 4$$
$$3x_1 + 8x_2 \le 12$$
$$x_1, x_2 \ge 0$$

Solution

Given problem is the case of maximization. Also values of b_1 b_2 and b_3 are positive. By introducing the slack variables S_1, S_2 and S_3 convert the problem into standard form.

$$\text{Max } Z = 10x_1 + 6x_2 + 0S_1 + 0S_2 + 0S_3$$
$$\text{S.T.} \quad x_1 + x_2 + S_1 = 2$$
$$2x_1 + x_2 + S_2 = 4$$
$$3x_1 + 8x_2 + S_3 = 12$$
$$x_1, x_2, S_1, S_2, S_3 \ge 0$$

Initial basic feasible solution is given by $x_1 = 0$, $x_2 = 0$, $S_1 = 2$, $S_2 = 4$, $S_3 = 12$

Initial simplex table is given by

Initial simplex table

		C_j	10	6	0	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3
← 0	S_1	2	$\boxed{1}$	1	1	0	0
0	S_2	4	2	1	0	1	0
0	S_3	12	3	8	0	0	1
	Z_j	0	0	0	0	0	0
		$Z_j - C_j$	–10 ↑	–6	0	0	0

Here all the values of $Z_j - C_j$ is not positive.

∴ Optimal solution will not exist. To find optimal solution select the most negative values of $Z_j - C_j$. Here –10 is the most negative number. It will enter in the basis. Corresponding column is treated as key column. Find the key row by taking

Now prepare initial simplex table.

Initial simplex table

C_B	B	C_j X_B	1 x_1	1 x_2	3 x_3	0 S_1	0 S_2
0	S_1	2	3	2	1	1	0
← 0	S_2	2	2	1	$\boxed{2}$	0	1
	Z_j	0	0	0	0	0	0
		$Z_j - C_j$	–1	–1	–3 ↑	0	0

Here all the values of $Z_j - C_j$ are not positive. Hence, the solution is not optimum. To find the optimum solution, select the most negative value of $Z_j - C_j$. Here –3 is the most negative value. It will enter in the basis. Corresponding column is treated as key column. To find the key row, find

$$\min\left(\frac{X_B}{x_3}, x_3 > 0\right) = \min\left(\frac{2}{1}, \frac{2}{2}\right)$$

$$= \min(2, 1) = 1$$

Hence, S_2 will leave the basis.

$\boxed{2}$ is the key element, because it is intersection of key row and key column. Make it unity and then apply matrix row transformation to make other element of key column to zero. Taking $R_1 \to R_2 - R - 1$, we have.

First simplex table.

C_B	B	C_j X_B	1 x_1	1 x_2	3 x_3	0 S_1	0 S_2
0	S_1	1	2	3/2	0	1	–1
3	x_3	1	1	1/2	1	0	1/2
	Z_j	3	1	3/2	3	0	3/2
		$Z_j - C_j$	0	1/2	0	0	3/2

Here, all the values of $Z_j - C_j$ are positive. Hence, optimum solution will exist and it is given by $x_1 = x_2 = 0$, $x_3 = 1$ and

$$\text{Max } Z = 3. \quad \textbf{Ans}$$

Example 17 Use simplex method to solve the L.P.P.

$$\text{Min } Z = x_1 - 3x_2 + 2x_3$$

$$\text{S.T.} \quad 3x_1 - x_2 + 2x_3 \le 7.$$

$$-2x_1 + 4x_2 \le 12$$

$$-4x_1 + 3x_2 + 8x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

Solution

Given problem is the ease of minimization. We shall convert it into maximization.

$$\text{Max } Z = -x_1 + 3x_2 - 2x_3$$
$$\text{S.T.} \quad 3x_1 - x_2 + 2x_3 \leq 7.$$
$$-2x_1 + 4x_2 + 0x_3 \leq 12$$
$$-4x_1 + 3x_2 + 8x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

By introducing the slack variables S_1, S_2 and S_3, we convert inequality into equality, i.e.,

$$\text{Max } Z = -x_1 + 3x_2 - 2x_3 + 0S_1 + 0S_2 + 0S_3$$
$$\text{S.T.} \quad 3x_1 - x_2 + 2x_3 + S_1 = 7$$
$$-2x_1 + 4x_2 + 0x_3 + S_2 = 12$$
$$-4x_1 + 3x_2 + 8x_3 + S_3 = 10$$
$$x_1, x_2, x_3\ S_1, S_2, S_3 \geq 0$$

Initial basic feasible solution is given by $x_1 = x_2 = x_3 = S_1 = S_2 = S_3 = 0$

Now prepare the initial simplex table, we have

Initial simplex table

		C_j	–1	3	–2	0	0	0
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
0	S_1	7	3	–1	2	1	0	0
← 0	S_2	12	–2	$\boxed{4}$	0	0	1	0
0	S_3	10	–4	3	8	0	0	1
	Z_j	0	0	0	0	0	0	0
		$Z_j - C_j$	–1	–3	2 ↑	0	0	0

Here all values of $Z_j - C_j$ are not positive. To find optimum solution select most negative values of $Z_j - C_j$ i.e., –3 is most negative number. Corresponding column will be key column. To find key row, we have

$$\min\left(\frac{X_B}{x_2}\right) = \min\left(\frac{7}{-1}, \frac{12}{4}, \frac{10}{3}\right) = 3$$

$\boxed{4}$ is the key element and corresponding row is key row. Make the key element. Unity and all other elements of key column zero by matrix row transformation. Taking $R_1 \to R_1 + R_2$ and $R_3 \to R_3 - 3R_2$, we have the first simplex table.

$$x_1 + 4x_2 + 0x_3 \le 420$$
$$x_1, x_2, x_3 \ge 0$$

Solution

The given problem is ease of maximization all values of b_i's (i = 1, 2, 3) are positive. By introducing slack variables convert the problem into standard form and inequality into equality; we have

$$\text{Max } Z = 3x_1 + 2x_2 + 5x_3 + 0S_1 + 0S_2 + 0S_3$$
$$\text{S.T.} \quad x_1 + 2x_2 + x_3 + S_1 = 430$$
$$3x_1 + 0x_2 + 2x_3 + S_2 = 460$$
$$x_1 + 4x_2 + 0x_3 + S_3 = 420$$
$$x_1, x_2, x_3, S_1, S_2, S_3 \ge 0$$

Initial basic feasible solution is given by $x_1 = x_2 = x_3 = 0$

$$S_1 = 430, S_2 = 460, S_3 = 420$$

Now prepare initial simplex table.

Initial simplex table

C_B	B	C_j / X_B	3 / x_1	2 / x_2	5 / x_3	0 / S_1	0 / S_2	0 / S_3
0	S_1	430	1	2	1	1	0	0
← 0	S_2	460	3	0	$\boxed{2}$	0	1	0
0	S_3	420	1	4	0	0	0	1
	Z_j	0	0	0	0	0	0	0
		$Z_j - C_j$	–3	–2	–5 ↑	0	0	0

Here all values of $Z_j - C_j$ are not positive. Hence, solution is not optimum. To find optimum solution select the most negative number. Here –5 is the most negative number it will enter in basis. Corresponding column will be treated as key column. To find key row, find min $\left(\frac{X_B}{x_3}\right)$

$$= \min = \left(\frac{430}{1}, \frac{460}{2}, \frac{420}{0}\right)$$
$$= 230$$

∴ The basic variable S_2 will leave the basis.

$\boxed{2}$ is the key element make it unity and other element of key column zero by matrix row transformation. Now we have first simplex table.

First simplex table

		C_j	3	2	5	0	0	0
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
0	S_1	200	$\frac{-1}{2}$	$\boxed{2}$	0	1	1/2	0
5	x_3	230	$\frac{3}{2}$	0	1	0	1/2	0
0	S_3	420	1	4	0	0	0	1
	Z_j	1150	15/2	0	5	0	5/2	0
		$Z_j - C_j$	$\frac{9}{2}$	–2 ↑	0	0	5/2	0

Further, all values of $Z_j - C_j$ are not positive. Hence, repeat the above process.

$\boxed{2}$ is the key element. Make it unity and other elements of key column zero by applying matrix row transformation, we have the second simplex table.

Second simplex table

		C_j	3	2	5	0	0	0
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
2	x_2	100	–1/4	1	0	1/2	–1/4	0
5	x_3	230	3/2	0	1	0	1/2	0
0	s_3	20	2	0	0	–2	1	1
	Z_j	1350	7	2	5	1	2	0
		$Z_j - C_j$	4	0	0	1	2	0

Since all values of $Z_j - C_j \geq 0$. Hence, the solution is optimum. It is given by

$$x_1 = 0,\ x_2 = 100,\ x_3 = 230,\ \text{Max } Z = 1350 \quad \textbf{Ans}$$

EXERCISE

Solve the Following L.P.P. by simplex method.

1. Max $Z = x_1 + 2x_2 + x_3$

S.T. $2x_1 + x_2 - x_3 \geq -2$

$-2x_1 + x_2 - 5x_3 \leq 6$

$4x_1 + x_2 + x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

[**Ans.** $x_1 = 0,\ x_2 = 4,\ x_3 = 2$ Max $Z = 10$]

2.7 ARTIFICIAL VARIABLE TECHNIQUES

If some constraints are of '=' or ≥ type, then they will not contain any basic variables. In such eases, we introduce a new variable called artificial variable. These variables are fictitious and cannot have any physical meaning. Artificial variable is merely a device to get the starting basic feasible solution. To solve the L.P.P. there are two methods:

(i) The Big M method (Method of penalties)
(ii) The Two-phase simplex method.

2.7.1 Big M Method

To solve the L.P.P. by Big M. method, various steps are given below.

(i) Express the problem in standard form by introducing slack variables, surplus variables and artificial variables as required in the problem.
(ii) Add non-negative artificial variables to the left side of each of the equations corresponding to constraint of the type =, and ≥. These variables do not appear in the final solution. This is achieved by assigning very large penalty (–M for maximization) in the objective function.
(iii) Solve the modified L.P.P. by simplex method until any one of three cases may arise:

1. If no artificial variables appears in the basis and optimality conditions of simplex method is satisfied, then initial solution is an optimum basic feasible solution.
2. If at least one artificial variable appears in the optimum basis at zero level and the optimality conditions of simplex method are satisfied, then the current solution is an optimum basic feasible solution.
3. If at least one artificial variable appears in the basis at positive level and optimality condition of simplex method is satisfied, then the problem has no feasible solution.

Example 19 Use Big. M Method to solve the following L.P.P.

$$\text{Max } Z = 3x_1 + 2x_2$$

$$\text{S.T. } 2x_1 + x_2 \le 2$$

$$3x_1 + 4x_2 \ge 12$$

$$x_1, x_2 \ge 0$$

Solution

Express the problem into standard form by introducing slack variables, surplus variable and artificial variable, we have

$$\text{Max } Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 - MA_1$$

$$\text{S.T. } 2x_1 + x_2 + S_1 = 2$$

$$3x_1 + 4x_2 - S_2 + A_1 = 12$$

$$x_1, x_2, S_1, S_2, A_1 \ge 0$$

The initial basic feasible solution is given by

$$x_1 = x_2 = 0,\ S_1 = 2,\ A_1 = 12$$

Now solve the above L.P.P. by general simplex method. Form the initial simplex table.

Initial simplex table

		C_j	3	2	0	0	$-M$
C_B	B	X_B	x_1	x_2	S_1	S_2	A_1
$\leftarrow$ 0	S_1	2	2	$\boxed{1}$	1	0	0
$-M$	A_1	12	3	4	0	-1	1
	Z_j	$-12M$	$-3M$	$-4M$	0	M	$-M$
		$Z_j - C_j$	$-3M - 3$	$-4M - 2$ $\uparrow$	0	M	0

Here, all the values of $Z_j - C_j$ are not positive hence, optimality condition of simplex method is not satisfied. To find the optimum solution, select the most negative number of $Z_j - C_j$. Here $-4M - 2$ is the most negative number. Corresponding column is treated as key column. To find key row, find min $\left(\frac{X_B}{x_2}\right) = \min\left(\frac{2}{1}, \frac{12}{4}\right) = (2, 3) = 2.$

$\boxed{1}$ is key element. Therefore, S_1 will leave the basis. Make all elements of key column zero by applying matrix row transformation. i.e., $R_2 \to R_2 - 4R_1$.

Now we have the first simplex table.

First simplex table

		C_j	3	2	0	0	$-M$
C_B	B	X_B	x_1	x_2	S_1	S_2	A_1
2	x_2	2	2	1	1	0	0
$-M$	A_1	4	-5	0	-4	-1	1
	Z_j	$4 - 4M$	$4 + 5M$	2	$2 + 4M$	M	$-M$
		$Z_j - C_j$	$5M + 1$	0	$2 + 4M$	M	0

Here, all values of $Z_j - C_j \geq 0$. Hence, optimality condition of simplex method is satisfied. Also one artificial variable appears in the optimum basis at positive level, therefore the given L.P.P. will have no feasible solution.

Example 20 Solve the following L.P.P., using Big. M. method

$$\begin{aligned} \text{Max } Z &= x_1 + 2x_2 + 3x_3 - x_4 \\ \text{S.T.} \quad x_1 + 2x_2 + 3x_3 &= 15 \\ 2x_1 + x_2 + 5x_3 &= 20 \\ x_1 + 2x_2 + x_3 + x_4 &= 10 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Solution

Here all constraints are in the form of equality. Therefore, we introduce artificial variables A_1, A_2 to convert the problem into standard form.

$$\begin{aligned}
\text{Max } Z &= x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2 - MA_3\\
\text{S.T.} \quad x_1 + 2x_2 + 3x_3 + A_1 &= 15\\
2x_1 + x_2 + 5x_3 + A_2 &= 20\\
x_1 + 2x_2 + x_3 + x_4 &= 10\\
x_1, x_2, x_3, x_4 &\geq 0\\
A_1, A_2, A_3, &\geq 0
\end{aligned}$$

The initial solution is given by

$$x_1 = x_2 = x_3 = 0,\ A_1 = 15,\ A_2 = 20,\ x_4 = 10$$

Now prepare the initial simplex table, which is given by

Initial simplex table

		C_j	1	2	3	–1	$-M$	$-M$
C_B	B	X_B	x_1	x_2	x_3	x_4	A_1	A_2
$\leftarrow -M$	A_1	15	1	2	3	0	1	0
$-M$	A_2	20	2	1	$\boxed{5}$	0	0	1
–1	x_4	10	1	2	1	1	0	0
	Z_j	$-35M - 10$	$-3M - 1$	$-3M - 2$	$-8M - 1$	–1	$-M$	$-M$
		$Z_j - C_j$	$-3M - 2$	$-3M - 4$	$-8M - 4$ $\uparrow$	0	0	0

Here optimality condition of simplex method is not satisfied, i.e., all the values of $Z_j - C_j$ are not positive. To find the optimum value, select the most negative value of $Z_j - C_j$, i.e., $-8M - 4$ will enter in the basis. It is treated as key column. To find key row, find min $\left(\dfrac{X_B}{x_3}\right)$

$$= \min\left(\frac{15}{3}, \frac{20}{5}, \frac{10}{1}\right) = \min(5, 4, 10) = 4$$

Hence, artificial variable A_2 will leave the basis.

$\boxed{5}$ is treated as key element. Make it unity and also other element of key column to zero by taking matrix row transformation.

Now we proceed for the first simplex table.

First simplex table

	B	C_j	1	2	3	–1	$-M$
C_B	A_1	X_B	x_1	x_2	x_3	x_4	A_1
$\leftarrow -M$	x_3	3	–1/2	$\boxed{7/5}$	0	0	1
3	x_4	4	2/5	1/5	1	0	0
–1	Z_j	6	3/5	9/5	0	1	0

Contd.

		$-3M+6$	$\frac{1}{3}M+\frac{3}{5}$	$\frac{-7}{5}M-\frac{6}{5}$	3	–1	$-M$
		$Z_j - C_j$	$\frac{1}{5}M\frac{-2}{5}$	$\frac{-7}{5}M-\frac{16}{5}$ ↑	0	0	0

Further optimality condition is not satisfied. Here, $Z_2 - C_2$ is the most negative number. It will enter in the basis. It is treated as key column. To find key row, find min $\left(\frac{X_B}{x_2}\right)$

$$= \min\left(\frac{3}{7/5}, \frac{4}{1/5}, \frac{6}{9/5}\right)$$

$$= \min\left(\frac{15}{7}, \frac{20}{1} \frac{30}{9}\right) = 15/7$$

$\boxed{7/5}$ is the key element. Make it unity and other elements of key column zero by applying matrix row transformation. We have the second simplex table.

Second simplex table

		C_j	1	2	3	–1
C_B	B	X_B	x_1	x_2	x_3	x_4
2	x_2	15/7	–1/7	1	0	0
3	x_3	25/7	3/7	0	1	0
← –1	x_4	15/7	$\boxed{6/7}$	0	0	1
	Z_j	$\frac{90}{7}$	$\frac{1}{7}$	2	3	–1
		$Z_j - C_j$	$\frac{-6}{7}$ ↑	0	0	0

Here, optimality conditions are not further satisfied $Z_1 - C_1$ is the most negative number. It will enter in the basis. It is treated as key column key row will be the third row $\boxed{6/7}$ is the key element. Make it unity and other elements of key column zero. Now prepare the third simplex table.

Third simplex table

		C_j	1	2	3	–1
C_B	B	X_B	x_1	x_2	x_3	x_4
2	x_2	15/6	0	1	0	1/6
3	x_3	15/6	0	0	1	3/6

Contd.

1	x_4	15/6	1	0	0	7/6
	Z_j	15	1	2	3	3
		$Z_j - C_j$	0	0	0	4

Here, all values of $Z_j - C_j \geq 0$. Hence, optimality. Conditions are satisfied. Hence, optimum six is given by $x_1 = x_2 = x_3 = 15/6$, $x_4 = 0$

and $$\text{Max } Z = 15 \quad \textbf{Ans}$$

Example 21 Solve the following L.P.P. using Big. M. method.

$$\text{Min } Z = x_1 + x_2$$
$$\text{S.T. } 2x_1 + x_2 \geq 4$$
$$x_1 + 7x_2 \geq 7$$
$$x_1, x_2 \geq 0$$

Solution

First convert the problem into maximization form and introducing the surplus variables and artificial variable to convert the problem in standard form.

$$\text{Max } Z = -x_1 - 4x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$$

$$\text{S.T. } 2x_1 + x_2 - S_1 + A_1 = 4$$
$$x_1 + 7x_2 - S_2 + A_2 = 7$$
$$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$$

The initial solution is given by

$$x_1 = x_2 = 0,\ A_1 = 4,\ A_2 = 7.$$

Now prepare the initial simplex table.

Initial simplex table

		C_j	2	4	0	0	$-M$	$-M$
C_B	B	X_B	x_1	x_2	S_1	S_2	A_1	A_2
$-M$	A_1	4	2	1	-1	0	1	0
$\leftarrow -M$	A_2	7	1	$\boxed{7}$	0	-1	0	1
	Z_j	$-11M$	$-3M$	$-8M$	M	M	$-M$	$-M$
		$Z_j - C_j$	$-3M - 2$	$-8M - 4$ $\uparrow$	M	M	0	0

Here, all the values of $Z_j - C_j$ are not positive. Therefore, solution is not optimum. To find optimum solution, select the most negative value of $Z_j - C_j$. Here, $Z_2 - C_2 = -8M - 4$ is the most negative value. It will enter in the basis and corresponding column is treated as key column. Find key row by taking $\min\left(\dfrac{X_B}{x_2}\right) = \min\left(\dfrac{4}{1}, \dfrac{7}{7}\right) = \min(4, 1) = 1.$

∴ Artificial variable A_2 will leave the basis. $\boxed{7}$ is treated as key element.

Make it unity and other element of key column to zero by applying matrix row transformation.

First simplex table

		C_j	2	4	0	0	$-M$
C_B	B	X_B	x_1	x_2	S_1	S_2	A_1
$\leftarrow -M$	A_1	3	$\boxed{13/7}$	0	–1	1/7	1
4	x_2	1	1/7	1	0	–1/7	0
	Z_j	$-3M + 4$	$\frac{-13}{7}M + \frac{4}{7}$	4	M	$\frac{M}{7} - \frac{1}{7}$	M
		$Z_j - C_j$	$\frac{-13}{7}M + \frac{10}{7}$ ↑	0	M	$\frac{M}{7} - \frac{1}{7}$	$2M$

Here, all values of $Z_j - C_j$ are not positive.

Optimality condition is not satisfied.

Now $R_1 \rightarrow \frac{7}{13} R_1$ and then taking

$R_2 \rightarrow R_2 - \frac{1}{7} R_1$, we have second simplex table.

Second simplex table

		C_j	2	4	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2
2	x_1	$\frac{21}{13}$	1	0	$\frac{7}{13}$	$\frac{1}{13}$
4	x_2	$\frac{10}{13}$	0	1	$\frac{1}{13}$	$\frac{1}{13}$
	Z_j	$\frac{82}{13}$	2	4	$\frac{10}{13}$	$\frac{6}{13}$
		$Z_j - C_j$	0	0	$\frac{10}{13}$	$\frac{6}{13}$

Here all the values of $Z_j - C_j$ are positive and no artificial variable appears in optimum basis. Therefore, the required solution is given by

$$x_1 = \frac{21}{13},\ x_2 = \frac{10}{13},\ \text{and min } Z = \frac{82}{13}$$ **Ans.**

7.
$$\text{Min } Z = 12x_1 + 20x_2.$$
$$\begin{aligned} \text{S.T.} \quad 6x_1 + 8x_2 &\geq 100 \\ 7x_1 + 12x_2 &\geq 120 \\ x_1, x_2 &\geq 0 \end{aligned}$$

[**Ans.** Min $Z = 205$, $x_1 = 15$, $x_2 = \frac{5}{4}$]

8.
$$\text{Min } Z = 5x + 3y$$
$$\begin{aligned} \text{S.T.} \quad 2x + 4y &\leq 12 \\ 2x + 2y &= 10 \\ 5x + 2y &\geq 10 \\ x, y &\geq 0 \end{aligned}$$

[**Ans.** $x = 4$, $y = 1$, Min $Z = 23$]

2.7.2 Two-phase Simplex Method

This method is another method to solve a given L.P.P. involving some artificial variable. In this method solution is obtained in two phases.

Phase-I In this phase, we construct an auxiliary L.P.P. leading to a final simplex table. Various steps are given below:

(i) Assign cost – 1 to each artificial variable and cost 0 to all other variables. Also find a new objective function Z^*.

(ii) Solve the auxiliary L.P.P by simplex method until either following three cases arise.

  (i) Max $Z^* < 0$ and at least one artificial variable appears in the optimum basis at positive level. In this case L.P.P. dose not possess any feasible solution.

  (ii) Max $Z^* = 0$ and at least one artificial variable or no artificial variable appears in optimum basis. In both the cases, we go to the phase II.

Phase-II Use solution of phase 1 as the initial value of phase-II. Assign the actual cost to the variables and zero cost of every artificial variable. Delete the artificial variable column from the table. Apply simplex method to the modified simplex table to find the solution.

Example 22 Use two-phase simplex method to solve

$$\text{Max } Z = 5x_1 + 3x_2$$
$$\begin{aligned} \text{S.T.} \quad 2x_1 + x_2 &\leq 1 \\ x_1 + 4x_2 &\geq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution

We convert the given problem in standard form by introducing slack variable, surplus variable and artificial variable. Also assign the cost – 1 to artificial variable and the cost 0 to another variables.

We have

Phase-1

$$\text{Max } Z^* = 0x_1 + 0x_2 + 0S_1 + 0S_2 - 1A_1$$
$$\text{S.T. } 2x_1 + x_2 + S_1 = 1$$
$$x_1 + 4x_2 - S_2 + A_1 = 6$$
$$x_1, x_2, S_1, S_2, A_1 \geq 0$$

The initial basic feasible solution is given by

$$x_1 = x_2 = 0, S_1 = 1, A_1 = 6$$

Now prepare the initial simplex table

		c_j	0	0	0	0	–1
C_B	B	X_B	x_1	x_2	S_1	S_2	A_1
← 0	S_1	1	2	$\boxed{1}$	1	0	0
–1	A_1	6	1	4	0	–1	1
	Z_j	–6	–1	–4	0	1	–1
		$Z_j - C_j$	–1	–4 ↑	0	1	0

Here, all values of $Z_j - C_j$ are not positive so choose the most negative value of $Z_j - C_j$, i.e., $Z_2 - C_2$ is the most negative value. It will enter in the basis and treated as key column. Find key row by taking min $\left\{\dfrac{X_B}{x_2}\right\}$.

$$= \min\left\{\frac{1}{1}, \frac{6}{4}\right\} = 1$$

$\boxed{1}$ is key element. Make other elements of key column zero by applying matrix row transformation. We have the first simplex table.

		C_j	0	0	0	0	–1
C_B	B	X_B	x_1	x_2	S_1	S_2	A_1
0	x_2	1	2	1	1	0	0
–1	A_1	2	–7	0	–4	–1	1
	Z_j	–2	7	0	4	1	–1
		$Z_j - C_j$	7	0	4	1	0

Here all values of $Z_j - C_j \geq 0$. Max $Z^* < 0$ and an artificial variable A_1 appears in the basis at positive level. In this case L.P.P. does not possess any feasible solution.

Example 23 *XYZ* company has two bottling plants. One located at G_1 and the other at J. Each plant produces three drinks A, B and C. The number of bottles produced per day are as follows:

		C_j	0	0	0	0	0	–1	–1	–1
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3	A_1	A_2	A_3
–1	A_1	40	3	3	–1	0	0	1	0	0
← –1	A_2	40	3	$\boxed{1}$	0	–1	0	0	1	0
–1	A_3	44	2	5	0	0	–1	0	0	1
	Z_j	–124	–8	–9	1	1	1	–1	–1	–1
		$Z_j - C_j$	–8	–9 ↑	1	1	1	0	0	0

Here all the values of $Z_j - C_j$ are not positive. Select the most negative number, i.e., $Z_2 - C_2$ is the most negative number. It will enter in the basis and treated as key column. Find the key row by taking min $\left\{\frac{X_B}{x_2}\right\} = \min\left\{\frac{40}{3}, \frac{40}{3}, \frac{44}{2}\right\} = \left\{\frac{40}{3}\right\}$.

$\boxed{1}$ is the key element.

Now, make all elements of key column zero by applying matrix row transformation.

First simplex table

		C_j	0	0	0	0	0	–1	–1	–1
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3	A_1	A_2	A_3
← –1	A_1	–80	–6	0	–1	$\boxed{3}$	0	1	–3	0
0	x_2	40	3	$\boxed{1}$	0	–1	0	0	1	0
–1	A_3	–156	–13	0	0	5	–1	0	–5	1
	Z_j	236	19	0	1	–8	1	–1	8	–1
		$Z_j - C_j$	19	0	1	–8 ↑	1	0	9	0

Further, all values of $Z_j - C_j$ are not positive. Select the most negative number, i.e., $Z_4 - C_4$ is most negative number. It will enter in the basis.

Second simplex table

		C_j	0	0	0	0	0	–1	–1	–1
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3	A_1	A_2	A_3
0	S_2	–80/3	–2	0	–1/3	1	0	1/3	–1	0
0	x_2	$\frac{40}{3}$	1	1	$\frac{-1}{3}$	0	0	1/3	–2/3	0
← –1	A_3	$\frac{-68}{3}$	–3	0	5/3	0	–1	$\frac{-5}{3}$	4	1
	Z_j	$\frac{68}{3}$	3	0	$\frac{-5}{3}$	0	1	$\frac{-5}{3}$	4	–1
		$Z_j - C_j$	3	0	$\frac{-5}{3}$	0	2	$\frac{-8}{3}$	5	0

$$R_3 \to \frac{3}{5} R_3, \text{ we have}$$

		C_j	0	0	0	0	0	–1	–1	–1
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3	A_1	A_2	A_3
0	S_2	–80/3	–2	0	$\frac{-1}{3}$	1	0	$\frac{1}{3}$	–1	0
0	x_2	$\frac{40}{3}$	1	1	$\frac{-1}{3}$	0	0	$\frac{1}{3}$	$\frac{-2}{3}$	0
0	A_1	$\frac{-68}{5}$	$\frac{-9}{5}$	0	1	0	$\frac{-3}{5}$	1	$\frac{-12}{5}$	$\frac{3}{5}$
	Z_j	0	0	0	0	0	0	0	0	0
		$Z_j - C_j$	0	0	0	0	0	1	1	1

Here, Max $Z^* = 0$ and at least one artificial variable appears in the optimum basis, therefore, we go to the phase–II.

Phase-II Consider the final simplex table of phase–I. Provide the actual cost to the variables also delete the artificial variable column from the table and then solve by simplex method.

		C_j	600	400	0	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3
0	S_2	–80/3	–2	0	$\frac{-1}{3}$	1	0
400	x_2	40/3	1	1	$\frac{-1}{3}$	0	0
0	A_1	–68/3	$\frac{-9}{5}$	0	1	0	$\frac{-3}{5}$
	Z_j	$\frac{16000}{3}$	800	400	$\frac{400}{3}$	0	0
		$Z_j - C_j$	200	0	$\frac{400}{3}$	0	0

Here, all the values of $Z_j - C_j \geq 0$ and one artificial variable appears in optimum basis.

$$\therefore \qquad \text{Max } Z = \frac{16000}{3}, x_2 = \frac{40}{3}, x_1 = 0 \quad \textbf{Ans.}$$

Example 24 Solve the following L.P.P. by two-phase simplex method.

$$\text{Max } Z = 5x_1 - 4x_2 + 3x_3$$

$$\text{S.T. } 2x_1 + x_2 - 6x_3 = 20$$
$$6x_1 + 5x_2 + 10x_3 \leq 76$$
$$8x_1 - 3x_2 + 6x_3 \leq 50$$
$$x_1, x_2\ x_3 \quad \geq 0$$

Solution

Phase-I By introducing slack variables and artificial variable, convert the problem in to standard form and assign the value –1 to the artificial variable and value 0 to all other variables. Find the modified L.P.P.

$$\text{Max } Z^* = 0x_1 - 0x_2 + 0x_3 + 0S_1 + 0S_2 - 1A_1$$
$$\text{S.T.} \qquad 2x_1 + x_2 - 6x_3 + A_1 = 20$$
$$6x_1 + 5x_2 + 10x_3 + S_1 = 76$$
$$8x_1 - 3x_2 + 6x_3 + S_2 = 50$$
$$x_1, x_2, x_3, S_1, S_2, S_3, A \geq 0$$

Initial basic feasible solution is given by

$$x_1 = x_2 = x_3 = 0,\ A_1 = 20,\ S_1 = 76,\ S_2 = 50.$$

Now prepare the initial simplex table

		C_j	0	0	0	–1	0	0
C_B	B	X_B	x_1	x_2	x_3	A_1	S_2	S_2
–1	A_1	20	2	1	–6	1	0	0
0	S_1	76	6	5	10	0	1	0
← 0	S_2	50	$\boxed{8}$	–3	6	0	0	1
	Z_j	–20	–2	–1	–1	1	0	0
		$Z_j - C_j$	–2 ↑	–1	6	0	0	0

Taking $R_3 \to \frac{R_3}{8}$, then $R_1 \to R_1 - 2R_3$, $R_2 + R_2 - 6R_3$, we have

		C_j	0	0	0	–1	0	0
C_B	B	X_B	x_1	x_2	x_3	A_1	S_1	S_2
–1	A_1	15/2	0	$\boxed{7/4}$	–15/2	1	0	–1/4
0	S_1	77/2	0	$\frac{29}{4}$	11/2	0	1	–3/4
0	x_1	25/4	1	$\frac{-3}{8}$	3/4	0	0	1/8
	Z_j	–15/2	0	$\frac{-7}{4}$	$\frac{15}{2}$	–1	0	1/4
		$Z_j - C_j$	0	$\frac{-7}{4}$ ↑	$\frac{15}{2}$	0	0	$\frac{1}{4}$

$$6x_1 + x_2 + 6x_3 \geq 12$$
$$x_1, x_2, x_3 \geq 0$$

[**Ans.** Max $Z = -15$, $x_1 = \frac{3}{2}$, $x_2 = 3$, $x_3 = 0$]

2. Min $Z = 12x_1 + 18x_2 + 15x_3$

S.T. $4x_1 + 8x_2 + 6x_3 \geq 64$
$$3x_1 + 6x_2 + 12x_3 \geq 96$$
$$x_1, x_2, x_3 \geq 0$$

[**Ans.** $x_1 = 0$, $x_2 = \frac{16}{5}$, $x_3 = \frac{32}{5}$, Min $Z = \frac{768}{5}$]

3. Min $Z = 10x_1 + 6x_2 + 2x_3$

S.T. $-x_1 + x_2 + x_3 \geq 1$
$$3x_1 + x_2 - x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0$$

[**Ans.** $x_1 = \frac{1}{4}$, $x_2 = \frac{5}{4}$, Min $Z = 10$ $x_3 = 0$]

4. Min $Z = -2x_1 - x_2$

S.T. $x_1 + x_2 \geq 2$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

[**Ans.** Min $Z = -8$, $x_1 = 4$, $x_2 = 0$]

5. Max $Z = 2x_1 + x_2 + x_3$

S.T. $4x_1 + 6x_2 + 3x_3 \leq 8$
$$3x_1 - 6x_2 - 4x_3 \leq 1$$
$$2x_1 + 3x_2 - 5x_3 \geq 4$$
$$x_1, x_2, x_3 \geq 0$$

[**Ans.** $x_1 = \frac{9}{7}$, $x_2 = \frac{10}{21}$, $x_3 = 0$, Max $Z = \frac{64}{21}$]

6. Max $Z = 5x_1 + 3x_2$

S.T. $x_1 + x_2 = 5$
$$x_1 + 2x_2 \leq 6$$
$$5x_1 + 2x_2 \geq 10$$
$$x_1, x_2 \geq 0$$

[**Ans.** Min $Z = 23$, $x_1 = 4$, $x_2 = 1$]

7. Max $Z = x_1 + x_2$

S.T. $x_1 + x_2 \geq 2$
$$x_1 + 3x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

[**Ans.** Max $Z = 3$, $x_1 = 3$, $x_2 = 0$]

In matrix notation the primal and dual problem can be written as follows.

Primal Problem Find the column vector X, which

$$\text{Max } Z_P = CX$$
$$\text{S.T. } AX \leq b \text{ and } X \geq 0$$

Dual Problem Find a column vector W, which

$$\begin{aligned} &\text{Min } Z_D = b'W \\ &\text{S.T.} \quad A'W \geq C' \\ &\qquad\quad W \geq 0 \end{aligned}$$

where A', b', c' are the transposes of A, b and c.

Theorem: Dual and dual of a given primal is the primal.

Proof: Consider the L.P.P.

Primal

$$\begin{aligned} \text{Max } Z_P &= c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{S.T. } a_{11}x_1 &+ a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 &+ a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \\ a_{i1}x_1 &+ a_{i2}x_2 + \ldots + a_{in}x_n \leq b_i \\ a_{m1}x_1 &+ a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \end{aligned} \tag{1}$$

and

$$x_1, x_2, x_3 \ldots x_n \geq 0$$

Dual The dual of the above primal (1) is given by

$$\begin{aligned} \text{Min } Z_D &= b_1W_1 + b_2W_2 + \ldots + b_nW_m \\ \text{S.T. } a_{11}W_1 &+ a_{21}W_2 + \ldots + a_{m1}W_m \geq c_1 \\ a_{12}W_1 &+ a_{22}W_2 + \ldots + a_{m2}W_m \geq c_2 \\ &\vdots \\ a_{1n}W_1 &+ a_{2n}W_2 + \ldots + a_{mn}W_m \geq c_m \\ &W_1, W_2 \ldots W_m \geq 0 \end{aligned} \tag{2}$$

Now to write the dual of (2), we first write equation (2) in standard form (1). The dual (2) can be written in standard form as

$$\begin{aligned} &\text{Max } (-Z_D) = -b_1W_1 - b_2W_2 \ldots b_nW_m \\ &\text{S.T.} \qquad -a_{11}W_1 - a_{21}W_2 \ldots a_{m1}\ W_m \leq -c_1 \\ &\qquad\qquad -a_{12}W_1 - a_{22}W_2 \ldots a_{m2}W_m \leq -c_2 \\ &\qquad\qquad -a_{1n}W_1 - a_{2n}W_2 \ldots a_{mn}W_m \leq -c_m \\ &\qquad\qquad W_1, W_2, W_3 \ldots w_m \geq 0 \end{aligned} \tag{3}$$

Dual of Dual Now equation (3) is the form (1). Consider (3) as the primal and its dual is given by

$$\begin{aligned}
\text{Min } Z_D &= -c_1v_1 - c_2v_2 \ldots c_nv_n \\
\text{S.T.} \quad & -a_{11}v_1 - a_{12}v_2 \ldots a_{1n}v_n \geq -b_1 \\
& -a_{21}v_1 - a_{22}v_2 \ldots -a_{2n}v_m \geq -b_2 \\
& \ldots \\
& -a_{m1}v_1 - a_{m2}v_2 \ldots a_{mn}v_n \geq b_m \\
& v_1, v_2 \ldots v_m \geq 0
\end{aligned} \tag{4}$$

The above dual can also be written as

$$\begin{aligned}
\text{Max } Z_D &= c_1v_1 + c_2v_2 + \ldots + c_nv_n \\
\text{S.T.} \quad & a_{11}v_1 + a_{12}v_2 + \ldots + a_{1n}v_n \leq b_1 \\
& a_{21}v_1 + a_{22}v_2 + \ldots + a_{2n}v_n \leq b_2 \\
& \ldots \\
& a_{m1}v_1 + a_{m2}v_2 + \ldots + a_{mn}V_n \leq b_m \\
& v_1, v_2, v_3 \ldots v_n \geq 0
\end{aligned} \tag{5}$$

Equation (5) is identical to (1). Hence, it is proved that dual and dual of a given primal is the primal.

Example 25 Write the dual of the problem

$$\begin{aligned}
\text{Min } Z &= 3x_1 + x_2 \\
\text{S.T.} \quad & 2x_1 + 3x_2 \geq 2 \\
& x_1 + x_2 \geq 1 \\
& x_1, x_2 \geq 0
\end{aligned}$$

Solution

The given L.P.P. is in the standard primal form. In matrix notation it is written as

$$\text{Min } Z_P = (3, 1)\,[x_1, x_2] = CX$$

$$\text{S.T.} \quad \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$AX \geq b$$

The dual of a given problem is

$$\begin{aligned}
\text{Max } Z_D &= b'W \\
\text{S.T.} \quad & A'W \leq c'
\end{aligned}$$

$$\therefore \quad \begin{aligned}
\text{Max } Z_D &= [2, 1]\,[W_1, W_2] \\
&= 2W_1 + W_2
\end{aligned}$$

$$\text{S.T.} \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \leq \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\text{Min } Z = [8, -12, 13]\,[W_1, W_2, W_3]$$

$$\text{S.T.} \begin{bmatrix} 4 & -8 & 5 \\ -1 & -1 & 0 \\ 0 & -3 & -6 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} \geq \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

i.e.,

$$\text{Min } Z_D = 8W_1 - 12W_2 + 13W_3$$

$$\begin{aligned} \text{S.T. } 4W_1 - 8W_2 + 5W_3 &\geq 3 \\ -W_1 - W_2 + 0W_3 &\geq -1 \\ 0W_1 - 3W_2 - 6W_3 &\geq 1 \\ W_1, W_2, W_3 &\geq 0 \end{aligned}$$

Example 27 Find the dual of the following.

$$\text{Min } Z = x_1 + 3x_3$$ **[RTU, B. Tech. Sem. VII 2008]**

$$\begin{aligned} \text{S.T.} \quad 2x_1 + x_3 &\leq 3 \\ x_1 + 2x_2 + 6x_3 &\geq 5 \\ -x_1 + x_3 + 2x_3 &= 2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solution

The given problem is not in canonical form. First we make it in canonical form.

$$\text{Min } Z = x_1 + 3x_3$$

$$\begin{aligned} \text{S.T.} \quad -2x_1 - x_3 &\geq -3 \\ x_1 + 2x_2 + 6x_3 &\geq 5 \\ -x_1 + x_2 + 2x_3 &\geq 2 \\ x_1 - x_2 - 2x_3 &\geq -2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The above problem can be written in matrix form

$$\text{Min } Z = CX$$

$$\text{S.T. } AX \geq b,\ X \geq 0$$

i.e.,

$$\begin{bmatrix} -2 & 0 & -3 \\ 1 & 2 & 6 \\ 1 & -1 & -2 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ -2 \\ 2 \end{bmatrix}$$

Dual of the above primal can be written as

$$\text{Max } Z = b'W$$

$$\text{S.T. } A'W \leq C'$$

$$\begin{bmatrix} -2 & 1 & 1 & -1 \\ -1 & 2 & -1 & 1 \\ 0 & 6 & -2 & 2 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} \le \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

i.e.,

$$\text{Max } Z_D = -2W_1 + 5W_2 - 2W_3 + 2W_4$$

$$\begin{aligned} \text{S.T. } -2W_1 + W_2 + W_3 - W_4 &\le 1 \\ -W_1 + 2W_2 - W_3 + W_4 &\le 0 \\ 0W_1 + 6W_2 - 2W_3 + 2W_4 &\le 3 \\ W_1, W_2, W_3 &\ge 0 \end{aligned}$$ **Ans.**

Example 29 Find the dual of the following L.P.P. and solve it.

$$\text{Max } Z = 4x_1 + 2x_2$$

$$\begin{aligned} \text{S.T. } x_1 + x_2 &\ge 3 \\ x_1 - x_2 &\ge 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Solution

The given problem can be written in matrix notation

$$\text{Max } Z = [4, 2]\,[x_1, x_2] = CX$$

$$\text{S.T. } AX \le b$$

i.e.,

$$\begin{bmatrix} -1 & -1 \\ -1 & +1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

Dual of above primal is given by

$$\text{Min } Z = b'W$$

$$\text{S.T. } A'W \ge C'$$

where A', b' and c' are transposes of A, b and c.

$$\text{Min } Z = [-3, -2]\,[w_1, w_2]$$

$$\text{S.T. } \begin{bmatrix} -1 & -1 \\ -1 & +1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \ge \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\begin{aligned} -W_1 - W_2 &\ge 4 \\ -W_1 + W_2 &\ge 2 \\ W_1, W_2 &\ge 0 \end{aligned}$$

Hence,

$$\text{Min } Z = -3W_1 - 2W_2$$

$$\text{S.T. } -W_1 - W_2 \ge 4$$

$$-W_1 + W_2 \geq 2$$
$$W_1, W_2 \geq 0$$

Now introducing surplus variables and artificial variables convert the problem in standard form.

$$\text{Max } Z^* = 3W_1 + 2W_2 + 0S_1 + 0S_2 - MA_1 - MA_2$$

$$\text{S.T.} \quad -W_1 - W_2 - S_1 + A_1 = 4$$
$$-W_1 + W_2 - S_2 + A_2 = 2$$
$$W_1, W_2, S_1\ S_2, A_1, A_2 \geq 0$$

The initial basic feasible solution is given by

$$W_1 = W_2 = 0, A_1 = 4, A_2 = 2$$

Now prepare the initial simplex table.

		C_j	3	2	0	0	$-M$	$-M$
C_B	B	X_B	W_1	W_2	S_1	S_2	A_2	A_2
$-M$	A_1	4	-1	-1	-1	0	1	0
$\leftarrow -M$	A_2	2	-1	$\boxed{1}$	0	-1	0	1
	Z_j	$-6M$	$2M$	0	M	M	$-M$	$-M$
		$Z_j - C_j$	$2M - 3$	-2 $\uparrow$	M	M	0	0

Here, all values of $Z_j - C_j$ are not positive. Hence, current solution is not optimal. To find optimum solution choose the most negative number, i.e., $Z_2 - C_2$ is most negative. It will enter in basis. Taking $R_1 \to R_1 + R_2$, we have

		C_j	3	2	0	0	$-M$	-1
C_B	B	X_B	W_1	W_2	S_1	S_2	A_1	A_2
$-M$	A_1	6	-2	0	-1	-1	1	–
2	W_2	2	-1	1	0	-1	0	–
	Z_j	$-6M + 4$	$2M - 2$	2	M	$M - 2$	$-M$	–
		$Z_j - C_j$	$2M - 5$	0	M	$M - 2$	0	–

Here, all values of $Z_j - C_j \geq 0$ and one artificial variable appears in the basis at positive level. Thus, the problem has no feasible solution.

Example 30 Solve the following L.P.P. by converting it into its dual.

$$\text{Min } Z = 20x_1 + 10x_2$$

$$\text{S.T.} \quad x_1 + x_2 \geq 10$$
$$3x_1 + 2x_2 \geq 24$$
$$x_1, x_2, \geq 0$$

Solution

The dual of the above L.P.P. can be written as

$$\text{Max } Z_D = 10W_1 + 24W_2$$

$$\begin{aligned} \text{S.T.} \qquad & W_1 + 3W_2 \leq 20 \\ & W_1 + 2W_2 \leq 10 \\ & W_1, W_2 \quad \geq 0 \end{aligned}$$

By using slack variables convert the problem into standard form, we have

$$\text{Max } Z = 10W_1 + 24W_2 + 0S_1 + 0S_2$$

$$\begin{aligned} \text{S.T. } W_1 + 3W_2 + S_1 &= 20 \\ W_1 + 2W_2 + S_2 &= 10 \\ W_1, W_2, S_1, S_2 &\geq 0 \end{aligned}$$

Initial basic feasible solution is given by

$$W_1 = W_2 = 0,\ S_1 = 20,\ S_2 = 10$$

Now prepare initial simplex table, we have

		C_j	10	24	0	0
C_B	B	X_B	W_1	W_2	S_1	S_2
0	S_1	20	1	3	1	0
$\leftarrow 0$	S_2	10	1	$\boxed{2}$	0	1
	Z_j	0	0	0	0	0
		$Z_j - C_j$	–10	–24 $\uparrow$	0	0

Converting the key element $\boxed{2}$ as unity and then taking $R_1 \rightarrow R_1 - 3R_2$, we have

		C_j	10	24	0	0
C_B	B	X_B	W_1	W_2	S_1	S_2
0	S_1	5	–1/2	0	1	–3/2
24	W_2	5	1/2	1	0	1/2
	Z_j	120	12	24	0	12
		$Z_j - C_j$	2	0	0	12

Here, all values of $Z_j - C_j \geq 0$. Hence, optimum solution exists, i.e.,

$$\text{Min } Z = 120,\ x_1 = 0,\ x_2 = 12 \quad \textbf{Ans.}$$

By introducing slack variables, convert the given problem in standard form

$$\text{Max } Z = -3x_1 - x_2 + 0S_1 + 0S_2$$

$$\begin{aligned} \text{S.T.} \quad -x_1 - x_2 + S_1 &= -1 \\ -2x_1 - 3x_2 + S_2 &= -2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Now display all the values in initial simplex table.

		C_j	-3	-1	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2
0	S_1	-1	-1	-1	1	0
$\leftarrow 0$	S_2	-2	-2	$\boxed{-3}$	0	1
	Z_j	0	0	0	0	0
		$Z_j - C_j$	3	1	0	0

Here, all values of $Z_j - C_j \geq 0$ and all $X_{Bi} < 0$. Therefore, the current solution is not an optimum, basic feasible solution.

To find optimum solution, find most negative value of X_{Bi} i.e., -2 is the most negative value of X_{Bi}. Therefore, S_2 will leave the basis. To find the entering variable, find max $\left\{\dfrac{Z_j - C_j}{\text{Second row}}\right\}$

$= \text{Max}\left\{\dfrac{3}{-2}\ \dfrac{1}{-3}\right\} = -\dfrac{1}{3}.$

$\therefore$ x_2 will enter in the basis.

Now prepare the next simplex table.

		C_j	-3	-1	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2
$\leftarrow 0$	S_1	$-1/3$	$-1/3$	0	1	$\boxed{-1/3}$
-1	x_2	$2/3$	$2/3$	1	0	$-1/3$
	Z_j	$\dfrac{-2}{3}$	$\dfrac{-2}{3}$	-1	0	$1/3$
		$Z_j - C_j$	$7/3$	0	0	$1/3$

Since all value of $Z_j - C_j \geq 0$ $X_{Bi} = -1/3 < 0$.

$\therefore$ Current solution is not an optimum solution. Since X_{Bi} is negative therefore, S_1 will leave the basis. Find max $\left\{\dfrac{Z_j - C_j}{\text{First row}}\right\} = \left\{\dfrac{\frac{7}{3}}{-1/3}, \dfrac{\frac{1}{3}}{-1/3}\right\} = -1$

		C_j	–3	–1	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2
0	S_2	1	1	0	–3	1
–1	x_2	1	1	1	–1	0
	Z_j	–1	–1	–1	1	0
		$Z_j - C_j$	2	0	1	0

Since all value of $Z_j - C_j \geq 0$ and $X_{Bi} \geq 0$. Hence, optimum basis feasible solution exists. Also it is given by

$$\text{Max } Z = -1,\ x_1 = 0 \text{ and } x_2 = 1 \textbf{ Ans.}$$

Example 32 Use dual simplex method to solve the following L.P.P.

$$\text{Max } Z = -2x_1 - 3x_2$$

$$\text{S.T.} \quad \begin{aligned} x_1 + x_2 &\geq 2 \\ 2x_1 + x_2 &\leq 10 \\ x_1 + x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution

The above problem can be written in canonical form

$$\text{Max } Z = -2x_1 - 3x_2$$

$$\text{S.T.} \quad \begin{aligned} -x_1 - x_2 &\leq -2 \\ 2x_1 + x_2 &\leq 10 \\ x_1 + x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

By introducing slack variables, convert the problem in standard form

$$\text{Max } Z = -2x_1 - 3x_2 + 0S_1 + 0S_2 + 0S_3$$

$$\text{S.T.} \quad \begin{aligned} -x_1 - x_2 + S_1 &= -2 \\ 2x_1 + x_2 + S_2 &= 10 \\ x_1 + x_2 + S_3 &= 8 \\ x_1, x_2, S_1, S_2, S_3 &\geq 0 \end{aligned}$$

Initial basic feasible solution is given by

$$x_1 = x_2 = 0,\ S_1 = -2,\ S_2 = 10,\ S_3 = 8.$$

Display the values in the initial simplex table.

		C_j	-2	-3	0	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3
$\leftarrow 0$	S_1	-2	$\boxed{-1}$	-1	1	0	0
0	S_2	10	2	1	0	1	0
0	S_3	8	1	1	0	0	1
	Z_j	0	0	0	0	0	0
		$Z_j - C_j$	2 $\uparrow$	3	0	0	0

Here all value of $Z_j - C_j \geq 0$ and $X_{Bi} \leq 0 = -2$. Therefore, current solution is not optimum. S_1 will leave the basis. Now find max $\left\{\frac{2}{-1}, \frac{3}{-1}\right\}$ max $\{-2, -3\} = -2$.

$\therefore$ x_1 is the entering variable and corresponding column is treated as key column.

		C_j	-2	-3	0	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3
-2	x_1	2	1	1	-1	0	0
0	S_2	6	0	-1	2	1	0
0	S_3	6	0	0	1	0	1
	Z_j	-4	-2	-2	1	0	0
		$Z_j - C_j$	0	1	1	0	0

Here, all the values of $Z_j - C_j \geq 0$ and all $X_{Bi} \geq 0$. Hence, optimum solution will exist, i.e.,

$$x_1 = 2, x_2 = 0, \text{Max } Z = -4 \textbf{ Ans.}$$

Example 33 Use dual simplex method to solve the L.P.P.

$$\text{Max } Z = -2x_1 - x_3$$

$$\begin{aligned} \text{S.T.} \quad & x_1 + x_2 - x_3 \geq 5 \\ & x_1 - 2x_2 + 4x_3 \geq 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution

The above problem can be written in canonical form, i.e.,

$$\text{Max } Z = -2x_1 - 0x_2 - x_3$$

$$\begin{aligned} \text{S.T.} \quad & -x_1 - x_2 + x_3 \leq -5 \\ & -x_1 + 2x_2 - 4x_3 \leq -8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

By introducing slack variables, convert the problem in standard form

$$\text{Max } Z = -2x_1 - 0x_2 - x_3 + 0S_1 + 0S_2$$

$$\text{S.T.} \quad -x_1 - x_2 + x_3 + S_1 = -5$$
$$-x_1 + 2x_2 - 4x_3 + S_2 = -8$$
$$x_1, x_2, x_3, S_1, S_2 \geq 0$$

Now display the above values in initial simplex table.

		C_j	–2	0	–1	0	0
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2
0	S_1	–5	–1	–1	1	1	0
← 0	S_2	–8	–1	–2	$\boxed{-4}$	0	1
	Z_j	0	0	0	0	0	0
		$Z_j - C_j$	2	0	1 ↑	0	0

Here all values of $Z_j - C_j \geq 0$ and all $X_{Bi} < 0$. Therefore, solution is not optimum. Here –8 is the most negative value. Hence, S_2 will leave the basis. Now find max $\left\{\frac{2}{-1}, \frac{1}{-4}\right\} = -\frac{1}{4}$

$\therefore$ x_3 will enter in the basis.

		C_j	–2	0	–1	0	0
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2
← 0	S_1	–7	–5/4	$\boxed{-1/2}$	0	1	$\frac{1}{4}$
–1	x_3	2	1/4	–1/2	1	0	–1/4
	Z_j	–2	$-\frac{1}{4}$	$\frac{1}{2}$	–1	0	$\frac{1}{4}$
		$Z_j - C_j$	$\frac{7}{4}$	$\frac{1}{2}$ ↑	0	0	$\frac{1}{4}$

Further all the values of $Z_j - C_j \geq 0$ and $X_{Bi} < 0$. Therefore, solution is not optimum. S_1 will leave the basis. Hence, the next simplex table is given by

		C_j	–2	0	–1	0	0
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2
0	x_2	14	5/2	1	0	–2	–1/2
–1	x_3	9	3/2	0	1	–1	–1/2
	Z_j	–9	–3/2	0	–1	1	1/2
		$Z_j - C_j$	$\frac{1}{2}$	0	0	1	1

$$-4x_1 - x_2 + x_3 \leq -10$$
$$x_1, x_2, x_3 \geq 0$$

[**Ans.** Min $Z = 4W_1 + 15W_2 - 8W_3 + 10W_4 - 10W_5$
S.T. $W_1 + 12W_2 - W_3 + 4W_4 - 4W_5 \geq 20$
$0W_1 + 18W_2 - W_3 + W_4 - W_5 \geq 30$
$-W_1 + 0W_2 - W_3 - W_4 - W_5 \geq 10$
$W_1, W_2, W_3, W_4, W_5 \geq 0$]

5. Max $Z = 2x_1 + 5x_2 + 6x_3$

S.T. $x_1 + 6x_2 - x_3 \leq 3$
$-2x_1 + x_2 + 4x_3 \leq 4$
$x_1 - 5x_2 + 3x_3 \leq 1$
$-3x_1 - 3x_2 + 7x_3 \leq 6$
$x_1, x_2, x_3 \geq 0$

[**Ans.** Min $Z = 3W_1 + 4W_2 + W_3 + 6W_4$
S.T. $W_1 - 2W_2 + W_3 - 3W_4 \geq 2$
$6W_1 + W_2 - 5W_3 - 3W_4 \geq 5$
$-W_1 + 4W_2 + 3W_3 + 7W_4 \geq 6$
$W_1, W_2, W_3, W_4 \geq 0$]

Use duality to solve the L.P.P.

(i) Min $Z = 4x_1 + 2x_2 + 3x_3$

S.T. $2x_1 + 4x_3 \geq 5$
$2x_1 + 3x_2 + x_3 \geq 4$
$x_1, x_2, x_3 \geq 0$

[**Ans.** Min $Z_D = \frac{67}{12}$, $W_1 = \frac{7}{2}$, $W_2 = \frac{2}{3}$]

(ii) Max $Z = 3x_1 + 4x_2$

S.T. $x_1 - x_2 \leq 1$
$x_1 + x_2 \geq 4$
$x_1 - 3x_2 \leq 3$
$x_1, x_2 \geq 0$

[No feasible solution exist for dual problem]

(iii) Max $Z = 5x_1 + 12x_2 + 4x_3$

S.T. $x_1 + 2x_2 + x_3 \leq 5$
$2x_1 - x_2 + 3x_3 \leq 2$
$x_1, x_2, x_3 \geq 0$

[Min $Z_D = \frac{141}{5}$, $W_1 = \frac{29}{5}$, $W_2 = \frac{-2}{5}$]

(iv) $\text{Min } Z = 2x_2 + 5x_3$

S.T. $x_1 + x_3 \geq 2$

$2x_1 + x_2 + 6x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

[Max $Z = 27$, $x_2 = 1$, $x_3 = 5$, $x_2 = 0$]

Use dual simplex method to solve the L.P.P.

(i) $\text{Max } Z = -3x_1 - x_2$

S.T. $x_1 + x_2 \geq 1$

$x_1 + 3x_2 \geq 2$

$x_1, x_2, \geq 0$

[Max $Z = -1$, $x_1 = 0$, $x_2 = 1$]

(ii) $\text{Min } Z = 10x_1 + 6x_2 + 2x_3$

S.T. $-x_1 + x_2 + x_3 \geq 1$

$3x_1 + x_2 - x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$

[Min $Z = 10$, $x_1 = \dfrac{1}{4}$, $x_2 = \dfrac{5}{4}$]

(iii) $\text{Min } Z = 30x_1 + 25x_2$

S.T. $2x_1 + 4x_2 \geq 40$

$3x_1 + 2x_2 \geq 50$

$x_1, x_2 \geq 0$

[Min $Z = 512.50$, $x_1 = 15$, $x_2 = \dfrac{5}{2}$]

(iv) $\text{Max } Z = x_1 + x_2$

S.T. $x_1 + x_2 \geq 2$

$x_1 + 3x_2 \leq 3$

$x_1, x_2 \geq 0$

[Max $Z = 3$, $x_1 = 3$, $x_2 = 0$]

(v) $\text{Max } Z = 10x_1 + 20x_2$

S.T. $2x_1 + 4x_2 \geq 16$

$x_1 + 5x_2 \geq 15$

$x_1, x_2 \geq 0$

[Max Z = Unbounded solution]

(vi) $\text{Min } Z = 2x_1 + 2x_2 + 4x_3$

S.T. $2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + x_2 + 7x_3 \leq 3$

$x_1 + 4x_2 + 6x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$

[Min $Z = \dfrac{4}{3}$, $x_1 = 0$, $x_2 = \dfrac{2}{3}$, $x_3 = 0$]

$$4x_1 + 4x_2 + S_2 = 40$$
$$x_1, x_2, S_1, S_2 \geq 0$$

Initial basis solution is given by

$$x_1 = x_2 = 0, S_1 = 60, S_2 = 40$$

Now form a simplex table.

		C_j	6	8	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2
$\leftarrow 0$	S_1	60	5	$\boxed{10}$	1	0
0	S_2	40	4	4	0	1
	Z_j	0	0	0	0	0
		$Z_j - C_j$	-6	-8 $\uparrow$	0	0

Taking $R_2 \to R_2 - 4R_1$, we have

		C_j	6	8	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2
8	x_2	6	$\frac{1}{2}$	1	$\frac{1}{10}$	0
$\leftarrow 0$	S_2	16	$\boxed{2}$	0	$\frac{-2}{5}$	1
	Z_j	48	4	8	4/5	0
		$Z_j - C_j$	-2 $\uparrow$	0	4/5	0

Now we have the next simplex table.

		C_j	6	8	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2
8	x_2	2	0	1	1/5	$-1/4$
6	x_1	8	1	0	$-1/5$	1/2
	Z_j	64	6	8	2/5	1
		$Z_j - C_j$	0	0	2/5	1

Here $Z_j - C_j \geq 0$ optimum solution exist. Max $Z = 64$, $x_1 = 8$, $x_2 = 2$.

(a) The revised right-hand side constants after incorporating the changes in the constraints are obtained by using the formula.

$$\begin{array}{l}\text{Basic variable}\\ \text{in the above}\\ \text{optimum table}\end{array} = \begin{bmatrix}\text{Technological coeff.}\\ \text{columns in the optimal}\\ \text{table w.r.t the basic}\\ \text{variables in intial table}\end{bmatrix}\begin{bmatrix}\text{New R.H.S.}\\ \text{constants}\end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{4} \\ -\frac{1}{5} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 40 \\ 20 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

From the above it is clear that $x_1 = 2, x_2 = 3$. Since these values are non-negative, therefore, revised solution is feasible and optimum. The corresponding optimal solution is 36.

(b) We have

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{4} \\ -\frac{1}{5} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 20 \\ 40 \end{bmatrix} = \begin{bmatrix} -6 \\ 16 \end{bmatrix}$$

$$\Rightarrow x_1 = 16, x_2 = -6.$$

Here $x_2 = -6$ (negative), the solution is infeasible. It can be removed by using dual simplex method, we get

$$x_1 = 4, x_2 = 0, S_1 = 0, S_2 = 24.$$

Hence, $\quad \text{Max } Z = 24$ **Ans.**

(ii) Making Changes in Objective Function Coefficients

The cost coefficient of objective function undergoes changes over a period of time. Under such situation we can obtain the revised optimum solution from the optimum table of original problem. Also it will be interested to know the range of the coefficient of variable in the objective function over which the optimality is unaffected.

Example 35 Solve the following problem

$$\text{Max } Z = 10x_1 + 15x_2 + 20x_3$$

$$\begin{aligned} \text{S.T.} \qquad 2x_1 + 4x_2 + 6x_3 &\le 24 \\ 3x_1 + 9x_2 + 6x_3 &\le 30 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

(a) Find the range of the objective function coefficients c_1 of the variable x_1 such that optimality is unaffected.

(b) Find the range of objective function coefficients c_2 of the variable x_2 such that the optimality is unaffected.

(c) Check whether optimality is affected, if profit coefficients are changed from (10, 15, 20) to (7, 14, 15). If so, find the revized optimum solution.

Solution

By introducing slack variables convert the problem in standard form

$$\text{Max } Z = 10x_1 + 15x_2 + 20x_3 + 0S_1 + 0S_2$$

$$\begin{aligned} \text{S.T. } 2x_1 + 4x_2 + 6x_3 + S_1 &= 24 \\ 3x_1 + 9x_2 + 6x_2 + S_2 &= 30 \\ x_1, x_2, x_3, S_1, S_2 &\geq 0 \end{aligned}$$

The initial basic feasible solution is given by

$$x_1 = x_2 = x_3 = 0, S_1 = 24, S_2 = 30$$

Now form the initial simplex table.

		C_j	10	15	20	0	0
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2
$\leftarrow 0$	S_1	24	2	4	$\boxed{6}$	1	0
0	S_2	30	3	9	6	0	1
	Z_j	0	0	0	0	0	0
		$Z_j - C_j$	–10	–15	–20	0	0
					$\uparrow$		
20	x_3	4	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{1}{6}$	0
0	S_2	6	$\boxed{1}$	5	0	–1	1
	Z_j	80	$\frac{20}{3}$	$\frac{40}{3}$	20	$\frac{20}{6}$	0
		$Z_j - C_j$	$\frac{-10}{3}$	$\frac{-5}{3}$	0	0	0
			$\uparrow$				
20	x_3	2	0	–1	1	$\frac{1}{2}$	$-\frac{1}{3}$
10	x_1	6	1	5	0	–1	1
	Z_j	100	10	30	20	0	10/3
		$Z_j - C_j$	0	15	0	0	10/3

Here all the values of $Z_j - C_j \geq 0$. Hence, optimum solution is given by Max $Z = 100$, $x_1 = 6$, $x_2 = 0$, $x_3 = 2$.

$$Z_2 - C_2 = 14 + [15, 7]\begin{bmatrix} -1 \\ 5 \end{bmatrix} = 6$$

$$Z_3 - C_3 = 15 + [15, 7]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

$$Z_4 - C_4 = 0 + [15, 7]\begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix} = \frac{1}{2}$$

$$Z_5 - C_5 = 0 + [15, 7]\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} = 2$$

Since all $Z_j - C_j \geq 0$ the optimality is unaffected.

(iii) Adding a New Constraint

Sometimes a new constraint may be added to an existing L.P.P. as per changing the realities. Under this situation each of the basic variable in new constraint is substituted with the corresponding expression based on the current optimum table. This will give the modified version of the new constraint in terms of only the current non-basic variables.

If the new constraint is satisfied by the values of the current basic variables the constraint is said to be redundant one. Therefore, optimality of the problem is not affected even after including new constraint into the existing problem.

If the new constraint is not satisfied by the values of the current basic variables the optimality of the problem will be affected. Therefore, modified version of the new constraint is to be augmented to the optimal table of the problem and iterated till the optimality is reached.

Example 36 Solve the problem

$$\text{Max } Z = 6x_1 + 8x_2$$

$$\begin{aligned} \text{S.T.} \quad & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(a) Check whether the addition of constraint $7x_1 + 2x_2 \leq 65$ affects the optimality. If it does, find the new optimum solution.

(b) Check whether the addition of the constraint $6x_1 + 3x_2 \leq 48$ affects the optimality. If it does, find the new solution.

Form simplex method, the optimum simplex table is given by

C_B	B	C_j / X_B	6 / x_1	8 / x_2	0 / S_1	0 / S_2
8	x_2	2	0	1	$\frac{1}{5}$	$-\frac{1}{4}$
6	x_1	8	1	0	$\frac{-1}{5}$	$\frac{1}{2}$
	Z_j	64	6	8	2/5	1
		$Z_j - C_j$	0	0	$\frac{2}{5}$	1

(a) The new constraint is given, i.e.,

$$7x_1 + 2x_2 \leq 65$$

This is satisfied by the values of current basic variables ($x_1 = 8$, $x_2 = 2$). Optimality will not be affected even after including the new constraint into the existing L.P.P.

(b) The new constraint is

$$6x_1 + 3x_2 \leq 48$$

This is not satisfied by the values of the current basic variables ($x_1 = 8$, $x_2 = 2$). So the modified form of the new constraint in terms of only non-basic variables is obtained. The standard form of new constraint after including slack variable S_3 is as follows.

$$6x_1 + 3x_2 + S_3 = 48$$

From the above table, we have

$$x_2 + \frac{1}{5} S_1 - \frac{1}{4} S_2 = 2$$

$$x_1 - \frac{1}{5} S_1 + \frac{1}{2} S_2 = 8$$

From the above equations, we have

$$\frac{3}{5} S_1 - \frac{9}{4} S_2 + S_3 = -6$$

Now we have revised table

C_B	B	C_j / X_B	6 / x_1	8 / x_2	0 / S_1	0 / S_2	0 / S_3
8	x_2	2	0	1	1/5	–1/4	0
6	x_1	8	1	0	–1/5	1/2	0
0	S_3	–6	0	0	3/5	–9/4	1
	Z_j	64	6	8	2/5	1	0
		$Z_j - C_j$	0	0	$\frac{2}{5}$	1	0

Solution

Solve the problem by general simplex method. We have optimal simplex table.

		C_j	6	8	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2
8	x_2	2	0	1	1/5	–1/4
6	x_1	8	1	0	–1/5	1/2
	Z_j	64	6	8	$\frac{2}{5}$	1
		$Z_j - C_j$	0	0	$\frac{2}{5}$	1

Here, all the values of $Z_j - C_j \geq 0$. Hence, optimum solution will exist. $x_1 = 8$, $x_2 = 2$, Max $Z = 64$.

(a) Determination of $Z_3 - C_3$. The relative contribution of the new product P_3 is computed by the following formula.

$$Z_j - C_j = C_j - [C_B]\begin{bmatrix}\text{Technical coefficient}\\ \text{of optimal table w.r.t.}\\ \text{the basic variable}\end{bmatrix} \times \begin{bmatrix}\text{Constraint}\\ \text{coefficients of}\\ \text{new variable}\end{bmatrix}$$

$$= 20 - [8,\ 6]\begin{bmatrix}\frac{1}{5} & -\frac{1}{4}\\ -\frac{1}{5} & \frac{1}{2}\end{bmatrix} \times \begin{bmatrix}6\\ 5\end{bmatrix} = \frac{63}{5}$$

Since the value $Z_3 - C_3$ is greater than zero. The solution is not optimal. It means that the inclusion of new product (new variable) in original problem changes the optimality.

(b) Optimization of the modified problem. The constraint coefficients of the new variable X_3 are determined using the following formula.

$$\begin{bmatrix}\text{Revised constraint}\\ \text{coefficient of the}\\ \text{new variable}\end{bmatrix} = \begin{bmatrix}\text{Technical coefficient}\\ \text{of optimal table w.r.t.}\\ \text{the basic variable}\end{bmatrix} \times \begin{bmatrix}\text{Constraint coefficients}\\ \text{of new variable}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{5} & -\frac{1}{4}\\ -\frac{1}{5} & \frac{1}{2}\end{bmatrix} \times \begin{bmatrix}6\\ 5\end{bmatrix} = \begin{bmatrix}-\frac{1}{20}\\ \frac{13}{10}\end{bmatrix}$$

4. Solve the following L.P.P. by using simplex method.

$$\text{Max } Z = 20x_1 + 80x_2$$

$$\text{S.T.} \quad 4x_1 + 6x_2 \leq 90$$
$$8x_1 + 6x_2 \leq 100$$
$$x_1, x_2 \geq 0$$

If the new constraint $5x_1 + 4x_2 \leq 80$ is added to this L.P.P. Find the solution to the new problem.