SWAYAM NPTEL COURSE ON

MINE AUTOMATION AND DATA ANALYTICS

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Module 8: Inferential Statistics

Lecture 19A: Discrete Random Variable Part I

CONCEPTS COVERED

- Definition of Random Variable with an example
- Types of Random Variables: Discrete and Random Variable
- Probability Mass Function, graph and examples.
- Cumulative Distribution Function, graph and examples.
- Definition of Expectation and its Properties.
- Variance: Definition

Random Variable

- In probability experiments, our focus often lies not in every detail of the experiment's outcome, but rather in the numerical value of certain quantities derived from the result.
- For instance, when rolling a dice twice, we may only be concerned with the sum of the outcomes rather than the specific values on each individual dice. This means we might only care about knowing that the sum is seven, without being interested in whether the actual outcome was (1,6), (2,5), (3,4), (4,3), (5,2), or (6,1).
- These quantities of interest, or more formally, these real-valued functions defined on the sample space, are referred to as random variables.
- Since the outcome of the experiment determines the value of a random variable, we can assign probabilities to the possible values of the random experiment.

Random Variable

Rolling a dice: Sample Space

- The sample space for this experiment, denoted as S, consists of all possible outcomes when a dice is rolled twice.
- {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),

  (2,1),(2,2),(2,3),(2,4),(2,5),(2,6),

  (3,1),(3,2),(3,3),(3,4),(3,5),(3,6),

  (4,1),(4,2),(4,3),(4,4),(4,5),(4,6),

  (5,1),(5,2),(5,3),(5,4),(5,5),(5,6),

  (6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}
- Given the probabilities associated with the question, we need to determine how many outcomes will yield a sum of 3.
- It's important to note that the experiment and sample space used to address this question remain the same.

Random Variable

- Let X represent the sum of outcomes from the two rolls.
- Therefore, X can take on values ranging from 2 to 12.

P(X=2)=P[1,1]=1/36,

P(X=3)=P[(1,2),(2,1)]=2/36

P(X=4)=P[(1,3),(2,2),(3,1)]=3/36

P(X=5)=P[(1,4),(2,3),(3,2),(4,1)]=4/36

P(X=6)=P[(1,5),(2,4),(3,3),(4,2),(5,1)]=5/36

P(X=7)=P[(1,6),(2,5),(3,4),(4,3),(5,2)(6,1)]=6/36

P(X=8)=P[(2,6),(3,5),(4,4),(5,3),(6,2)]=5/36

P(X=9)=P[(3,6),(4,5),(5,4),(6,3)]=4/36

P(X=10)=P[(4,6),(5,5),(6,4)]=3/36

P(X=11)=P[(5,6),(6,5)]=2/36,

P(X=12)=P[6,6]=1/36

Discrete and Continuous Random Variables

Definition:

A discrete random variable is characterized by its ability to assume, at most, a countable number of possible values.

Consequently, any random variable capable of adopting either a finite number or a countably infinite number of distinct values qualifies as a discrete random variable.

It's worth noting that there are also random variables whose set of potential values is uncountably infinite.

Definition:

Continuous random variables pertain to scenarios where outcomes of random events are numerical, yet cannot be enumerated and are infinitely divisible.

Discrete Random Variable

- A discrete random variable is characterized by having possible values that are distinct points along the real number line.
- Discrete random variables are often associated with counting scenarios

Continuous Random Variable

- A continuous random variable is defined by its possible values spanning an interval along the real number line.
- Continuous random variables typically involve measurement scenarios.

Discrete and Continuous Random Variable Examples

Examples of discrete random variables include:

- The number of people in a house
- The number of languages a person can speak
- The number of times a student takes a particular test before qualifying
- The number of collisions at an intersection
- The number of spelling mistakes in a document

Examples of Continuous random variables include:

- Temperature of a patient.
- Height of an athlete
- Speed of a vehicle.
- Time taken by a person to come home from the office.

Probability Mass Function (p.m.f)

- A random variable characterized by its ability to assume, at most, a countable number of potential values is termed a discrete random variable.

- Let X be a discrete random variable, and suppose that it has n possible values, which we will label $x_1, x_2, \ldots x_n$. For a discrete random variable X, we can define the probability mass function $p(x)$ of X by

$$P(x_i) = P(X = x_i)$$

- Represent in Tabular Form

X	x_1	x_2	x_3	…	…	x_n
$P(X = x_i)$	$P(x_1)$	$P(x_2)$	$P(x_3)$	…	…	$P(x_n)$

Key Properties of p.m.f

- The probability mass function $p(x)$ is positive for, at most, a countable number of x values.
- if X must assume one of the values $x_1, x_2, \dots x_n$.

➢ $p(x_i) >= 0, \; i = 1, 2, 3 \dots n$

➢ $p(x_i) = 0$, for all other x values

- Since X must take one of the values x_i, we have

$$\sum_{i=1}^{\infty} p(x_i) = 1$$

X	x_1	x_2	x_3	…	…	x_n
$P(X = x_i)$	$P(x_1)$	$P(x_2)$	$P(x_3)$	…	…	$P(x_n)$

Example 1 of p.m.f

X is a random variable that assumes three values: 0, 1, and 2, with the corresponding probabilities as follows:

$$P(X=0): 1/3$$

$$P(X=1): 1/3$$

$$P(X=2): 1/3$$

1. Each probability is greater than or equal to 0, i.e., non-zero.
2. Sum of probabilities = 1/3 + 1/3 + 1/3 = 1

- Two key properties are satisfied.
- It is p.m.f.

Example 2 of p.m.f

- Flipping a coin three times.
- Sample Space $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
- Let X denote the random variable representing the count of heads in the tosses.
- What is the Probability Mass Function?

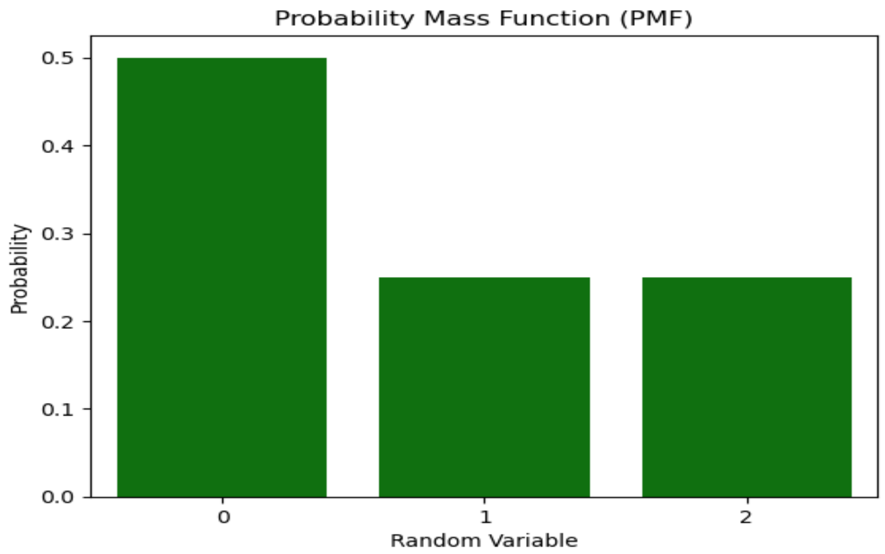
Solution:

X	0	1	2	3
$P(X=x_i)$	$1/8$	$3/8$	$3/8$	$1/8$

- Key properties are satisfied.
- Hence it is Probability Mass Function (p.m.f)

Graph of Probability Mass function

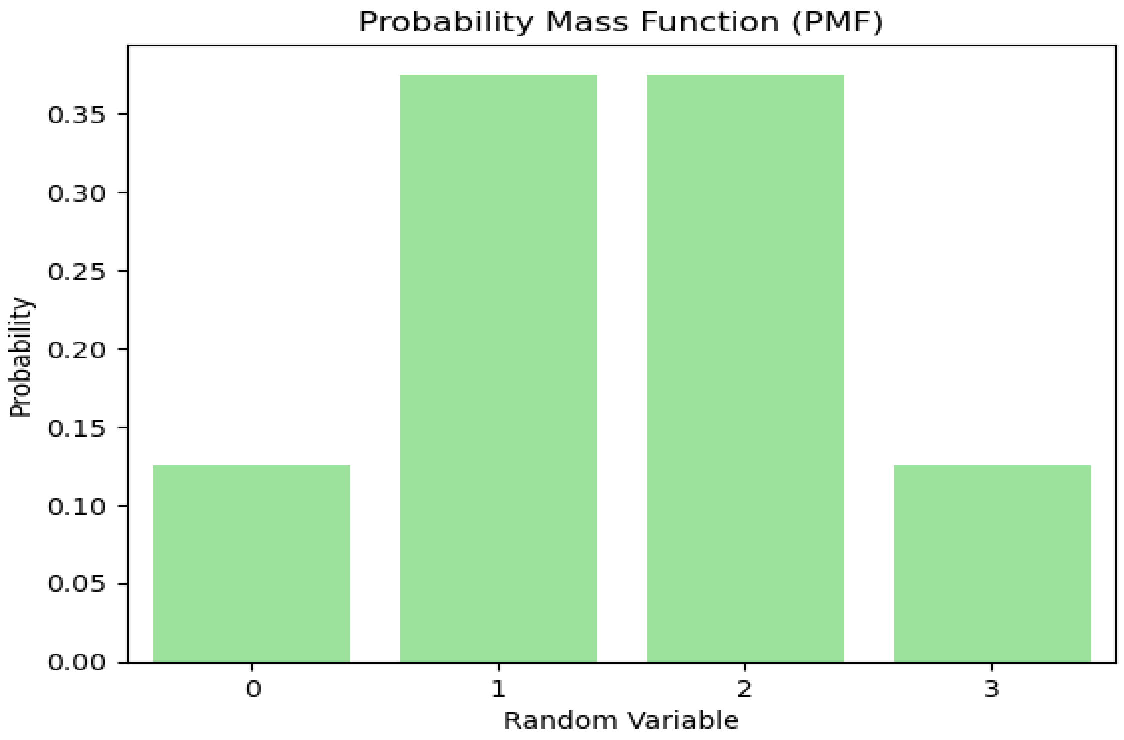
- **Illustrating the probability mass function graphically can be beneficial, often done by plotting P(X=x_i) on the y-axis against x_i on the x-axis.**



X	0	1	2
P(X = x_i)	1/2	1/4	1/4

Tossing a coin thrice
X = the number of heads

X	0	1	2	3
$P(X=x_i)$	1/8	3/8	3/8	1/8



Cumulative distribution function

- The cumulative distribution function (CDF), denoted as F, can be represented as follows:

$$F(a) = P(X \leq a)$$

- If X is a discrete random variable whose possible values are x_1, x_2, x_3 ,…, where $x_1 < x_2 < x_3$ …, then a step function will be the distribution function F of X.

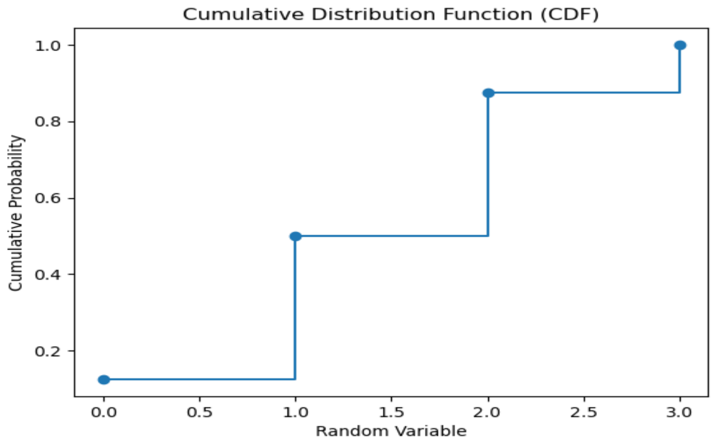
Cumulative distribution function
Step Function

- **Let X be a discrete random variable with the following probability mass function.**

X	0	1	2	3
$P(X = x_1)$	1/8	3/8	3/8	1/8

- **The cumulative distribution function of X is given by**

- $$F(a) = \begin{cases} 0 & a < 0 \\ \frac{1}{8} & 0 \le a < 1 \\ \frac{1}{2} & 1 \le a < 2 \\ \frac{7}{8} & 2 \le a < 3 \\ 1 & 3 \le a < 1 \end{cases}$$



Note that the step size at any of the values 0, 1, 2, and 3 corresponds to the probability that X assumes that specific value.

Expectation of Random Variable

- Let X be a discrete random variable taking values x_1, x_2,…,x_n The expected value of X denoted by E(X) and referred to as Expectation of X is given by

$$E(X) = \sum_{i=1}^{\infty} x_i P(x_i)$$

- The Expectation of a random variable can be interpreted as the "long-run average" value of the random variable across repeated independent observations.

Expectation Example

- Consider X as a discrete random variable with the following probability mass function.

X	0	1	2	3
$P(X = x_i)$	$1/8$	$3/8$	$3/8$	$1/8$

- Expectation of X can be calculated :
- $$E(X) = \sum_{i=1}^{\infty} x_i P(x_i)$$

= (0 x 1/8) + (1 x 3/8) + (2 x 3/8) + (3 x1/8)

= 12/18

= 2/3

Bernoulli random variable

- A random variable that can assume either the value 1 or 0 is referred to as a Bernoulli random variable.
- Let X represent a Bernoulli random variable that assumes the value 1 with probability p.
- The probability distribution of this random variable is as follows:

X	0	1
P(X = x_i)	1- p	p

- Expected value of a Bernoulli random variable: $$\mathrm{E(X)} = \sum_{i=1}^{\infty} x_i P(x_i)$$

E(X) = 0 x (1 - p) + 1 x p = p

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Discrete Uniform random variable

- Let X denote a random variable that is equally probable to assume any of the values 1, 2, 3,..., n.
- Probability mass function is

X	1	2	…	n
P(X = x_i)	1/n	1/n	…	1/n

$$E(X) = \sum_{i=1}^{\infty} x_i P(x_i)$$

- E(X) = (1 x 1/n) + (2 x 1/n) + ….+ (n x 1/n) = n(n+1)/2n
- E(X) = (n+1)/2

Properties of Expectation

- Let *X* represent a discrete random variable with values x_i and its corresponding probability mass function ($X = x_i$).
- Let h be any real values function; the expected value of g(X) is

$$\mathrm{E}(\mathrm{h}(\mathrm{x})) = \sum_i h(x_i)P(X = x_i)$$

- If a and b are constants,

$$E(aX + b) = aE(X) + b$$

Note: $E(x^2) \neq (E(x))^2$

Expectation of sum of two random variables

- The expected value of the sum of random variables equals the sum of the individual expected values.
- In other words, let X and Y be two random variables. Then,

$$E(X+Y) = E(X) + E(Y)$$

Expectation of sum of many random variables

- The result stating that the expected value of the sum of random variables equals the sum of the expected values holds true not only for two but for any number of random variables.
- Let X_1, X_2 ,…, X_n be k discrete random variables. Then,

$$E(\sum_{i=1}^{k} X_i) = \sum_{i=1}^{k} E(X_i)$$

Need for Variance

- The expected value of a random variable provides a weighted average of its potential values, but it does not provide information about the variation or spread of these values. For example, consider the random variables X, Y, and Z, with their respective values and probabilities as follows:

- X = 0 *with probability 1*
- Y = $\begin{cases} -3 & \text{with probability } 1/3 \\ 3 & \text{with probability } 1/3 \end{cases}$
- Z = $\begin{cases} -500 & \text{with probability } 1/2 \\ 500 & \text{with probability } 1/2 \end{cases}$
- E(X)= E(Y) = E(Z) = 0.
- However, it's evident that the spread of Z is greater than that of Y, and Y's spread is greater than that of X.

Variance of a random variable

- Let's denote the expected value of a random variable X by the Greek letter μ.

- If X is a random variable with an expected value μ, then the variance of X, denoted by Var(X) or V(X), is defined by:

$$Var\,(X) = E(X - \mu)^2$$

- In essence, the variance of a random variable X quantifies the squared difference between the random variable and its mean μ on average.

Computational formula for Var(X)

- $Var(X) = E(X - \mu)^2$
- $(X - \mu)^2 = X^2 + \mu^2 - 2\mu X$
- **Using properties of Expectation , We know**

$$
\begin{aligned}
E(X - \mu)^2) &= E(X^2 + \mu^2 - 2\mu X) \\
&= E(X^2) + \mu^2 - 2\mu E(X) \\
&= E(X^2) + \mu^2 - 2\mu^2 \\
&= E(X^2) - \mu^2
\end{aligned}
$$

$$\boxed{Var(X) = E(X^2) - (E(X))^2}$$

Rolling a dice once

- Random Experiment: Roll a dice once
- Sample Space: S = $\{1, 2, 3, 4, 5, 6\}$
- Random variable X is the outcome of the roll.
- The probability distribution is given by

X	1	2	3	4	5	6
X^2	1	4	9	16	25	36
$P(X = x_i)$	1/6	1/6	1/6	1/6	1/6	1/6

$$E(X) = 1\frac{1}{6} + 2.\frac{1}{6} + 3.\frac{1}{6} + 4.\frac{1}{6} + 5.\frac{1}{6} + 6.\frac{1}{6} = 3.5$$

$$E(X^2) = 1.\frac{1}{6} + 4.\frac{1}{6} + 9.\frac{1}{6} + 16.\frac{1}{6} + 25.\frac{1}{6} + 36.\frac{1}{6} = 15.167$$

$$Var(X) = E(X^2) - (E(X))^2 = 15.167 - (3.5)^2 = 2.917$$

Bernoulli random variable

- A random variable that can assume either the value 1 or 0 is referred to as a Bernoulli random variable.
- Let X be a Bernoulli random variable that takes on the value 1 with probability p.
- The probability distribution of the random variable is as follows:

X	0	1
X^2	0	1
$P(X = x_i)$	$1 - p$	p

Expected value of a Bernoulli random variable:

$$E(X) = \sum_{i=1}^{\infty} x_i P(x_i)$$

E(X) = 0 x (1- p) + 1 x p = p

E(X^2) = 0 x (1- p) + 1 x p = p

Var (X) = Variance of Bernoulli's random variable

= E(X^2) – (E(X))^2 = p – p^2 = p(1- p)

Discrete Uniform random variable

- Let X represent a random variable that has an equal likelihood of assuming any of the values 1, 2, 3, ...,n.
- Probability mass function

X	1	2	$\ldots$	n
X^2	1	4	$\ldots$	n^2
$P(X = xi)$	$1/n$	$1/n$	$\ldots$	$1/n$

- E(X) = (1 x 1/n) + (2 x 1/n) + …. + (n x 1/n) = (n+1)/2
- E(X^2) = (1 x 1/n) + (4 x 1/n) + ….+ (n^2 x 1/n) = (n+1)(2n+1)/6
- Var(X) = E(X^2) - $(E(X))^2$ = (n^2-1)/12

REFERENCES

- Introduction to Probability and Statistics for Engineers and Scientists, Sixth Edition, Sheldon M. Ross
- Statistical Methods Combined Edition (Volume I& II), N G Das

CONCLUSION

- Discussed about Random Variable
- Discussed types of Random Variables: Discrete and Random Variable
- Discussed Probability Mass Function, graph, and examples.
- Discussed Cumulative Distribution Function, graph, and examples.
- Discussed Expectation: Definition and its Properties
- Discussed and calculated Variance for different random variables.


THANK YOU