Sensitivity Analysis of Objective Function Coefficients

- The range of values over which an objective function coefficient may vary without causing any change in the values of the decision variables in the optimal solution is called <u>range of optimality.</u>
- Managers should focus on those objective coefficients that have a narrow range of optimality and coefficients near the endpoints of the range.

Sensitivity Analysis of Objective Function Coefficients

- The optimal solution will remain unchanged as long as:
  - An objective function coefficient lies within its range of optimality
  - There are no changes in any other input parameters.
- The value of the objective function will change if
  - the coefficient is multiplied by a non-zero number.

Sensitivity Analysis of Objective Function Coefficients

- The optimal solution will not change as long as the following expression is satisfied

$$\text{slope of one of the binding constraints} \leq -\frac{c_1}{c_2} \leq \text{slope of one of the binding constraints}$$

Sensitivity Analysis of Objective Function Coefficients

- Simultaneous Changes:
  - The range of optimality for objective function coefficient is applicable for changes made to one coefficient at a time.
  - If two or more objective function coefficients are changed simultaneously, further analysis is needed to determine whether the optimal solution will change.

The 100% Rule for Objective Function Coefficients

- The 100% rule states that simultaneous changes in objective function coefficients will not change the optimal solution as long as the sum of the percentages of the change divided by the corresponding maximum allowable change in the range of optimality for each coefficient does not exceed 100%.

The 100% Rule for Objective Function Coefficients

- If the sum of the percentage changes does not exceed 100%, the optimal solution will not change.
- The 100 percent rule does not, however, say that the optimal solution will change if the sum of the percentage changes exceeds 100%. It is possible that the optimal solution will not change even though the sum of the percentage changes exceeds 100%.

Reduced Cost

- The amount by which an objective function coefficient would have to improve (increase for a maximization problem and decrease for a minimization problem) before it would b it would be possible for the corresponding variable to assume a positive value in the optimal solution is called "***Reduced cost***."

Sensitivity Analysis of The Right hand side of the constraints

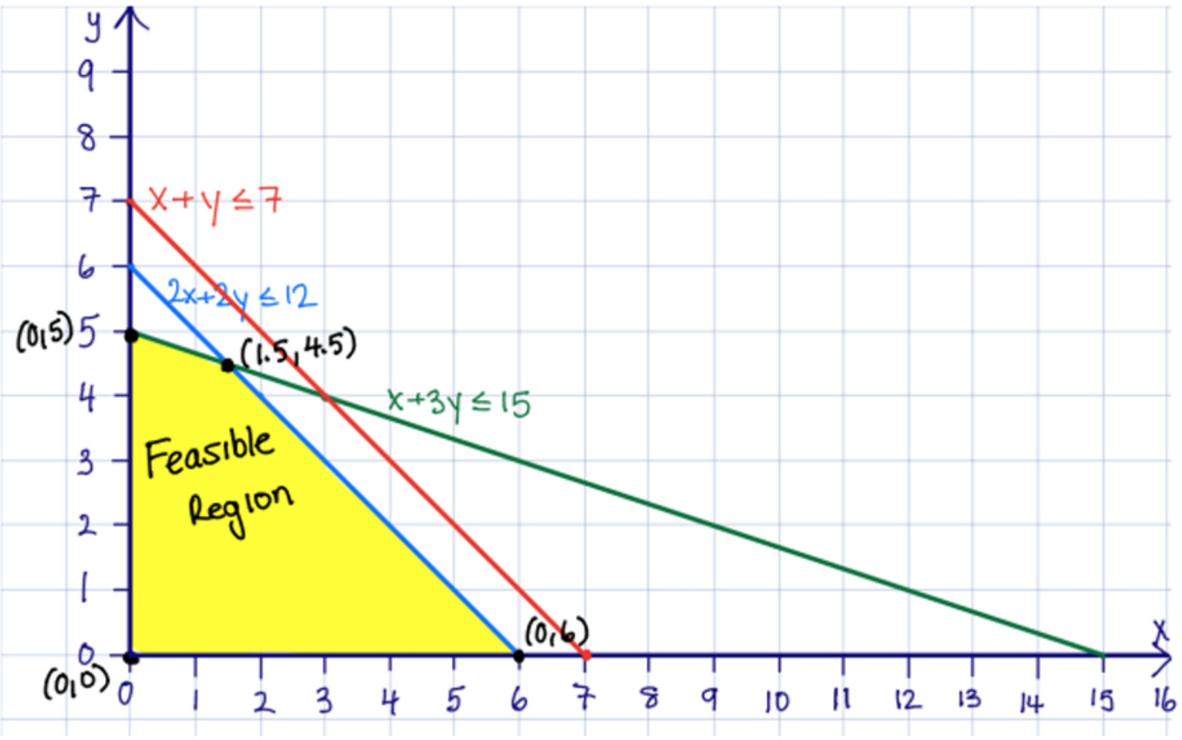
- In sensitivity analysis of right-hand sides of constraints we are interested in the following questions:
  - Keeping all other factors the same, how much would the optimal value of the objective function (for example, the profit) change if the right-hand side of a constraint changed by one unit?
  - For how many additional or fewer units will this per unit change be valid?

Sensitivity Analysis of The Right hand side of the constraints

- Any change to the right hand side of a binding constraint will change the optimal solution.
- Any change to the right-hand side of a non-binding constraint that is less than its slack or surplus, will cause no change in the optimal solution.

Maximize $f = 2x + 9y$ given the following constraints

$$x + y \leq 7$$
$$2x + 2y \leq 12$$
$$x + 3y \leq 15$$



Corners: (0,0), (0,5), (6,0), (1.5, 4.5)

Step 3: Find the Maximum and Optimal Point

- Substitute each corner point into the objective function. The highest value is the maximum value. Do this for every corner value.
    - (0,0) = 2(0) + 9(0) = 0
    - (0,5) = 2(0) + 9(5) = 45
    - (6,0) = 2(6) + 9(0) = 12
    - (1.5,4.5) = 2(1.5) + 9(4.5) = 43.5
- Maximum Value = 45, Optimal Point = (0,5)

Identify Binding and Non-Binding Constraints

Step One: Substitute the optimal point into all the constraints

Optimal Point is (0,5)

$x + y \leq 7 \rightarrow 5 \leq 7$ (Non-Binding Constraint)

$2x + 2y \leq 12 \rightarrow 10 \leq 12$ (Non-Binding Constraint)

$x + 3y \leq 15 \rightarrow 15 \leq 15$ (Binding Constraint)

Step Two:

- If your answer is equal to the Right Hand Side (RHS) of the inequality, then the constraint is **BINDING**.
- If your answer is not equal to the RHS of the inequality, then the constraint is **NON-BINDING**.
- If your answer is greater than the RHS of the inequality, you've made an error with the beginning steps, go back and correct the error.

Find Redundant Constraints

- Any lines that are NOT on the boundary of the feasible region as shown in the graph above, are REDUNDANT CONSTRAINTS
- $x + y \leq 7$ IS A REDUNDANT CONSTRAINT

Find the Shadow Price

- What will happen to the objective function if the right side of the constraints increase by 1?
- You must find the shadow prices for BINDING CONSTRAINTS.
- NON-BINDING CONSTRAINTS always have a shadow price of 0.

Step One: Remember the binding and non-binding constraints that were found earlier in the problem. You can state that the shadow price of a non-binding constraint is 0 (there is nothing to solve).

$x + y \ \leq 7 \rightarrow 5 \leq 7$ (Non-Binding Constraint) Shadow Price = 0

$2x + 2y \leq 12 \rightarrow 10 \leq 12$ (Non-Binding Constraint) Shadow Price = 0

$x + 3y \leq 15 \rightarrow 15 \leq 15$ (Binding Constraint)

Step Two: For the only binding constraint, increase the RHS by 1. (Note: If there were two binding constraints, you would increase the RHS of the first one, leave the other one unchanged, then solve for X and Y as in Step 3. Then for the second binding constraint, increase the RHS of it by 1, leave the first binding constraint unchanged and proceed by solving for X and Y as in Step 3).

$$x + 3y \leq 15$$
$$x + 3y = 16$$
$$y \leq \frac{16}{3}$$
$$x = 0$$

Step Three: Solve for x and y of the binding constraint. Using either substitution or elimination.

$$x = 0, y = \frac{16}{3}$$

Step Four: Substitute the values of x and y into the objective function and solve. We'll call this F_N or F (New).

$$F = 2x + 9y$$
$$F_N = 2(0) + 9(\frac{16}{3})$$
$$F_N = 48$$

Step Five: Subtract the New value from the Optimal value.

$$SP_3 = F_N - F_{Optimal}$$
$$48 - 45 = 3$$

Therefore, the shadow price of $x + 3y \leq 15$ is 3.

Since for this problem there is only one binding constraint, you would just do this process for that one. If there were more than one binding constraints,

Find the Amount of Change in Objective Function if RHS of Constant Increases by 8%.

Step One: Multiply the RHS of the binding constraint by 0.08.

$$x + 3y \leq 15 \times 0.08$$
$$k = 1.2$$

Step Two: Multiply the shadow price of the binding constraint by your answer from step one.

The Triangle known as delta Δ represents change in.

$$\boxed{\Delta Objective\ Function = SP \times k}$$

$$\Delta Objective\ Function = 3 \times 1.2$$
$$\Delta Objective\ Function = 3.6$$

Shadow Price

- Assuming there are no other changes to the input parameters, the change to the objective function value per unit increase to a right hand side of a constraint is called the ***Shadow Price***

Range of Feasibility

- Assuming there are no other changes to the input parameters, the *range of feasibility* is
  - The range of values for a right hand side of a constraint, in which the shadow prices for the constraints remain unchanged.
  - In the range of feasibility the objective function value changes as follows:
    Change in objective value =
    [Shadow price][Change in the right hand side value]

Dual Price

- The concept of a ***dual price*** is closely related to the concept of a ***shadow price***.
- The dual price associated with a constraint is the *improvement* in the value of the optimal solution per unit increase in the right hand side of the constraint.
- In general, the dual price and the shadow price are the same for all the maximization linear programs.

Dual Price

- In minimization linear programs, the shadow price is the negative of the corresponding dual price.
- The negative dual price tells us that the objective function will not improve if the value of the right hand side is increased by one unit.

Example: (TOYCO Model)

TOYCO assembles three types of toys-trains, trucks, and cars-using three operations. The daily limits on the available times for the three operations are 430,460, and 420 minutes, respectively, and the revenues per unit of toy train, truck, and car are \$3, \$2, and \$5, respectively. The assembly times per train at the three operations are 1, 3, and 1 minutes, respectively. The corresponding times per train and per car are (2,0,4) and (1,2,0) minutes (a zero time indicates that the operation is not used).

Let $x1$=daily number of units assembled of trains

$x2$= daily number of units assembled of trucks

$x3$= daily number of units assembled of cars

the associated LP model is given as:

$$\text{maximize } z=3x1+2x2+5x3$$

subject to

$$x1+2x2+x3 \leq 430 \text{ (operation 1)}$$

$$3x1+ \quad +2x3 \leq 460 \text{ (operation 2)}$$

$$x1+4x2 \qquad \leq 420 \text{ (operation 3)}$$

$$x1, x2, x3 \geq 0$$

Post-Optimal Analysis

- Changes affecting **feasibility**
  LP Model: right-hand side change or **a new constraint**
  How to recover optimal if the perturbation causes the change in basic optimal solution?

- Changes affecting **optimality**
  LP Model: objective coefficient or **new variable**
  How to find new optimal?

LP formulation:

$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &\le 430 \text{ (Operation 1)} \\ 3x_1 + 2x_3 &\le 460 \text{ (Operation 2)} \\ x_1 + 4x_2 &\le 420 \text{ (Operation 3)} \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

Using x_4, x_5, and x_6 as the slack variables for the constraints of operations 1, 2, and 3, respectively, the optimum tableau is

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

Current optimal:

$z=1350$ $x1=0$, $x2=100$, $x3=230$

Algebraic Sensitivity Analysis (First Case)

Determination of dual prices and feasibility ranges

- Suppose that D_1, D_2, and D_3 are the (positive or negative) changes made in the allotted daily manufacturing time of operations 1,2,and 3
- **The Toyko model can be changed to**

$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3$$

$$\begin{aligned}
x_1 + 2x_2 + x_3 &\le 430 + D_1 \quad \text{(Operation 1)}\\
3x_1 + 2x_3 &\le 460 + D_2 \quad \text{(Operation 2)}\\
x_1 + 4x_2 &\le 420 + D_3 \quad \text{(Operation 3)}\\
x_1, x_2, x_3 &\ge 0
\end{aligned}$$

Algebraic Sensitivity Analysis (First Case)

Determination of dual prices and feasibility ranges

- To express the optimum simplex tableau of the modified problem in terms of the changes, D_1, D_2, and D_3,
  - we first rewrite the starting tableau using the new right-hand sides, $430+D_1$, $460+D_2$, and $420+D_3$

							Solution			
Basic	x_1	x_2	x_3	x_4	x_5	x_6	*RHS*	D_1	D_2	D_3
z	−3	−2	−5	0	0	0	0	0	0	0
x_4	1	2	1	1	0	0	430	1	0	0
x_5	3	0	2	0	1	0	460	0	1	0
x_6	1	4	0	0	0	1	420	0	0	1

  - The two shaded areas are identical.

Determination of dual prices and feasibility ranges

- Hence, if we repeat the same simplex iterations as in the original model,
  - the columns in the two highlighted area will also be __________ in the optimal tableau

							Solution			
Basic	x_1	x_2	x_3	x_4	x_5	x_6	RHS	D_1	D_2	D_3
z	4	0	0	1	2	0	1350	1	2	0
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100	$\frac{1}{2}$	$-\frac{1}{4}$	0
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230	0	$\frac{1}{2}$	0
x_6	2	0	0	-2	1	1	20	-2	1	1

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution RHS	D_1	D_2	D_3
z	4	0	0	1	2	0	1350	1	2	0
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100	$\frac{1}{2}$	$-\frac{1}{4}$	0
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230	0	$\frac{1}{2}$	0
x_6	2	0	0	-2	1	1	20	-2	1	1

- The new optimum tableau provides the following optimal solution:

$z = 1350 + 1D_1 + 2D_2$

$x_2 = 100 + \frac{1}{2}D_1 + -\frac{1}{4}D_2$

$x_3 = 230 + \frac{1}{2}D_2$

$x_6 = 20 - 2D_1 + D_2 + D_3$

- Optimal solution:

$$z = 1350 + D_1 + 2D_2$$
$$x_2 = 100 + \tfrac{1}{2} D_1 - \tfrac{1}{4} D_2$$
$$x_3 = 230 + \tfrac{1}{2} D_2$$
$$x_6 = 20 - 2D_1 + D_2 + D_3$$

- Dual prices: The value of the objective function can be written as

$$z = 1350 + 1\,D_1 + 2\,D_2 + 0\,D_3$$

The equation shows that

1. A unit change in operation 1 capacity changes z ______
2. A unit change in operation 2 capacity changes z ______
3. A unit change in operation 3 capacity changes z ______

- This means that, by definition, the corresponding _____________ are 1, 2, and 0 ($/min) for operations 1, 2, and 3, respectively.

- Optimal solution:

$$z = 1350 + D_1 + 2D_2$$
$$x_2 = 100 + \tfrac{1}{2} D_1 - \tfrac{1}{4} D_2$$
$$x_3 = 230 + \tfrac{1}{2} D_2$$
$$x_6 = 20 - 2D_1 + D_2 + D_3$$

- Dual prices: The value of the objective function can be written as

$$z = 1350 + 1\,D_1 + 2\,D_2 + 0\,D_3$$

The equation shows that

1. A unit change in operation 1 capacity changes z by \$1
2. A unit change in operation 2 capacity changes z by \$2
3. A unit change in operation 3 capacity changes z by \$0

- This means that, by definition, the corresponding dual prices are 1, 2, and 0 (\$/min) for operations 1, 2, and 3, respectively.

- Feasibility range: The current solution remains feasible if all the basic variables remain nonnegative-that is,

- Simultaneous changes D_1, D_2, and D_3 that satisfy these inequalities will keep the solution ___________.
- The new optimum solution can be found by substituting out the values of D_1, D_2, and D_3

- Feasibility range: The current solution remains feasible if all the basic variables remain nonnegative-that is,

$$x_2 = 100 + \frac{1}{2}D_1 - \frac{1}{4}D_2 \geq 0$$

$$x_3 = 230 + \frac{1}{2}D_2 \geq 0$$

$$x_6 = 20 - 2D_1 + D_2 + D_3 \geq 0$$

- Simultaneous changes D_1, D_2, and D_3 that satisfy these inequalities will keep the solution feasible.
- The new optimum solution can be found by substituting out the values of D_1, D_2, and D_3

- The given conditions can produce the individual feasibility ranges associated with changing the resources one at a time
- **Example:** a change in operation 1 time only means that $D_2 = D_3 = 0$.
- The simultaneous conditions thus reduce

$$x_2 = 100 + \tfrac{1}{2}D_1 - \tfrac{1}{4}D_2 \geq 0$$
$$x_3 = 230 + \tfrac{1}{2}D_2 \geq 0$$
$$x_6 = 20 - 2D_1 + D_2 + D_3 \geq 0$$

- This means that the dual price for operation 1 is valid in the feasibility range ______________

- The given conditions can produce the individual feasibility ranges associated with changing the resources one at a time
- **Example:** a change in operation 1 time only means that $D_2 = D_3 = 0$.
- The simultaneous conditions thus reduce

$$x_2 = 100 + \tfrac{1}{2}D_1 - \tfrac{1}{4}D_2 \geq 0$$
$$x_3 = 230 + \tfrac{1}{2}D_2 \geq 0$$
$$x_6 = 20 - 2D_1 + D_2 + D_3 \geq 0$$

$x_2 = 100 + \frac{1}{2}D_1 \geq 0 \Rightarrow D_1 \geq -200$

$x_3 = 230 \geq 0 \Rightarrow$

$x_6 = 20 - 2D_1 \geq 0 \Rightarrow D_1 \leq 10$

$-200 \leq D_1 \leq 10$

- This means that the dual price for operation 1 is valid in the feasibility range $-200 \leq D_1 \leq 10$

Error: read 440 in current slide as 460

- We can show in a similar manner that the feasibility ranges for operations 2 and 3 are $-20 \leq D_2 \leq 400$ and $-20 \leq D_3 \leq \infty$
- We can now summarize the dual prices and their feasibility ranges for the TOYCO model as follows

			Resource amount (minutes)		
Resource	Dual price ($)	Feasibility range	*Minimum*	*Current*	*Maximum*
Operation 1	1	$-200 \leq D_1 \leq 10$	230	430	440
Operation 2	2	$-20 \leq D_2 \leq 400$	440	440	860
Operation 3	0	$-20 \leq D_3 < \infty$	400	420	∞

Optimality Ranges

- The development is based on the definition of reduced cost
- In the TOYCO model, let d_1, d_2, and d_3 represent the change in unit revenues for toy trucks, trains, and cars, respectively.
  - The objective function then becomes
- We first consider the general situation in which all the objective coefficients are ______________________.
- With the simultaneous changes, the z-row in the starting tableau appears as:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	$-3 - d_1$	$-2 - d_2$	$-5 - d_3$	0	0	0	0

Optimality Ranges

- The development is based on the definition of reduced cost
- In the TOYCO model, let d_1, d_2, and d_3 represent the change in unit revenues for toy trucks, trains, and cars, respectively.
  - The objective function then becomes

$$\boxed{\text{Max } z = (3+d_1)x_1 + (2+d_2)x_2 + (5+d_3)x_3}$$

- We first consider the general situation in which all the objective coefficients are changed simultaneously.
- With the simultaneous changes, the z-row in the starting tableau appears as:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	$-3-d_1$	$-2-d_2$	$-5-d_3$	0	0	0	0

Optimality Ranges

- When we generate the simplex tableaus the optimal iteration will appear as follows

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	$4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1$	0	0	$1 + \frac{1}{2}d_2$	$2 - \frac{1}{4}d_2 + \frac{1}{2}d_3$	0	$1350 + 100d_2 + 230d_3$
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	$-\frac{1}{4}$	0	0	-2	1	1	20

- The new optimal tableau is the same as in the original optimal tableau, except for the ________________
- This means that changes in the objective-function coefficients can affect the __________________________.

Algebraic Sensitivity Analysis (Second Case)

- You really do not need to carry out the simplex row operation to compute the new reduced costs.
  - An examination of the new z-row shows that the coefficients of are taken directly from the constraint coefficients of the optimum tableau.
- A convenient way for computing the new reduced cost is to add
  - a new top row and
  - a new leftmost column to the optimum tableau,

as shown by the shaded areas in the following illustration.

		d_1	d_2	d_3	0	0	0	
	Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
1	z	4	0	0	1	2	0	1350
d_2	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
d_3	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
0	x_6	2	0	0	-2	1	1	20

Algebraic Sensitivity Analysis (Second Case)

		d_1	d_2	d_3	0	0	0	
	Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
1	z	4	0	0	1	2	0	1350
d_2	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
d_3	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
0	x_6	2	0	0	-2	1	1	20

- For the left-most column, the top element is 1 in the z-row followed by change d_i for basic variable x_i
- Keep in mind that d_i =0 for slack variable x_i
- To compute the new reduced cost for any variable:
  - Multiply the elements of its column by the corresponding elements in the leftmost column,
  - Add them up, and subtract the top-row element from the sum.

		d_1	d_2	d_3	0	0	0	
	Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
1	z	4	0	0	1	2	0	1350
d_2	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
d_3	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
0	x_6	2	0	0	-2	1	1	20

- Reduced cost for x_1

- The current solution remains optimal so long as the new reduced costs (z-equation coefficients) remain nonnegative (maximization case).

- We thus have the following simultaneous optimality conditions corresponding to nonbasic x_1, x_4, and x_5 :

Algebraic Sensitivity Analysis (Second Case)

		d_1	d_2	d_3	0	0	0	
	Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
1	z	4	0	0	1	2	0	1350
d_2	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
d_3	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
0	x_6	2	0	0	-2	1	1	20

- Reduced cost for x_1

$$[4*1 + -(\tfrac{1}{4})*d_2 + \tfrac{3}{2}*d_3 + 2*0] - d_1$$

$$= 4 - \tfrac{1}{4}d_2 + \tfrac{3}{2}d_3 - d_1$$

- The current solution remains optimal so long as the new reduced costs (z-equation coefficients) remain nonnegative (maximization case).
- We thus have the following simultaneous optimality conditions corresponding to nonbasic x_1, x_4, and x_5 :

$4 -$

Algebraic Sensitivity Analysis (Second Case)

- Remember that the reduced cost for a basic variable is always ______, as the modified optimal tableau shows.
- **Example:** suppose that the objective function of TOYCO is changed from $z = 3x_1 + 2x_2 + 5x_3$ to $z = 2x_1 + x_2 + 6x_3$.
- Then ______________, ______________, and ______________
- Substitution in the given conditions yields:

$$\boxed{\begin{aligned} 4 - \tfrac{1}{4}d_2 + \tfrac{3}{2}d_3 - d_1 &\geq 0 \\ 1 + \tfrac{1}{2}d_2 &\geq 0 \\ 2 - \tfrac{1}{4}d_2 + \tfrac{1}{2}d_3 &\geq 0 \end{aligned}}$$

Algebraic Sensitivity Analysis (Second Case)

- Remember that the reduced cost for a basic variable is always zero, as the modified optimal tableau shows.
- **Example:** suppose that the objective function of TOYCO is changed from $z = 3x_1 + 2x_2 + 5x_3$ to $z = 2x_1 + x_2 + 6x_3$.
- Then $d_1 = 2 - 1 = \$1$, $d_2 = 1 - 2 = -\$1$, and $d_3 = 6 - 5 = \$1$
- Substitution in the given conditions yields:

$$\boxed{\begin{aligned} 4 - \tfrac{1}{4}d_2 + \tfrac{3}{2}d_3 - d_1 &\geq 0 \\ 1 + \tfrac{1}{2}d_2 &\geq 0 \\ 2 - \tfrac{1}{4}d_2 + \tfrac{1}{2}d_3 &\geq 0 \end{aligned}}$$

$4 - \frac{1}{4}(-1) + \frac{3}{2}(1) - 1(-1) = 6.75 \geq 0$ (satisfied)

$1 + \frac{1}{2}(-1) = 0.5 \geq 0$ (satisfied)

$2 - \frac{1}{4}(-1) + \frac{1}{2}(1) = 2.75 \geq 0$ (satisfied)

HW DUE Next Week

2. Use the solution and the sensitivity analysis to answer the following questions.

   (a) What is the optimal solution? What is the value of the solution?

   (b) Which constraints are binding constraints? Give the slack or surplus for each of the others

   (c) Why is the reduced cost for the number of trucks not equal to 0?

   (d) How low would the profit per car have to go to change the optimal profit mix? How high?

   (e) If the company decided that it must make some trucks, what effect would that have on the daily profit (increase? decrease? how much per truck?)? How can you tell?

   (f) If the profit per train decreased to \$4, what would be the effect on the optimal production plan? what would be the effect on profit?

   (g) What effect would there be on the profit if the time available for Operation 3 increased to 450 minutes? Why does it make sense that the shadow price for available Operation 3 time is 0? How low would the available time on Operation 3 have to drop to change this shadow price?

   (h) Suppose the company could extend the available time for Operation 1 by 20 minutes per day at a cost of \$.50 (per minute). Use the shadow price to explain why this would be worthwhile for the company. Would this change the optimal production mix?

   (i) If problems with machines reduce the available daily time for Operation 2 to 400 minutes, what affect will this have on profit (be specific — a dollar amount)? Why can't we predict (without re-writing and re-solving the problem) the effect of a decrease to 290 minutes?

   (j) If the contract for toy cars changed to require 90 toy cars per day, how would this affect the optimal solution? The value?

HW Due 7th Feb 2024

- Work as a team, not more than 3 MEMBERS in each group
- Please make a google drive submit all assignments in the drive
- Name of the file contains
  - Name of members
  - MND 500
  - HW 1

Big M Method

A Variant of Simplex Method

Introduction

- A method of solving linear programming problems.
- It is one of the oldest LP techniques.
- **Big M** refers to a large number associated with the artificial variables.
- The Big M method introduces surplus and artificial variables to convert all inequalities into standard form.

Algorithm

- Add **artificial variables** in the model to obtain a feasible solution.
- Added only to the '$\geq$' type or the '=' constraints.
- A value M is assigned to each artificial variable.
- The transformed problem is then solved using simplex eliminating the artificial variables.

Points To Remember

Solve the modified LPP by simplex method, until any one of the three cases may arise:-

- If **no** artificial variable appears in the basis and the optimality conditions are satisfied.
- If **at least one** artificial variable in the basis at zero level and the optimality condition is satisfied .
- If at least one artificial variable appears in the basis at **positive level** and the optimality condition is satisfied, then the original problem has no feasible solution.

Example

- Maximize $Z = x_1 + 5x_2$

  Subject to $4x_1 + 4x_2 \leq 6$
  $x_1 + 3x_2 \geq 2$
  $x_1 , x_2 \geq 0$

Solution : Introducing slack & surplus variables :

$$4x_1 + 4x_2 + S_1 = 6$$
$$x_1 + 3x_2 - S_2 = 2$$

where
S_1 is a slack variable
S_2 is a surplus variable

The surplus variable S_2 represents the extra units.

- Now if we let x_1 & x_2 equal to zero in the initial solution , we will have $S_1=6$, $S_2=-2$, which is not possible because a surplus variable cannot be negative . Therefore , we need **artificial variables**.

  Introducing an artificial variable , say A1.

- <u>Standard Form</u> :

  Maximize $Z = x_1 + 5x_2 + 0s_1 + 0s_2 - M(A1)$

  Subject to

$$4x_1 + 4x_2 + S_1 = 6$$
$$x_1 + 3x_2 - S_2 + A1 = 2$$
$$x_1 , x_2 , S_1 , S_2 , A1 \geq 0$$

		C_j	1	5	0	0	-M	
B	C_B	X_B	X_1	X_2	S_1	S_2	A_1	MinRatio
S_1	0	6	4	4	1	0	0	$^3/_2$
A_1	-M	2	1	3	0	-1	1	$^2/_3$
		Z_j	-M	-3M	0	M	-M	
		$C_j - Z_j$	M+1	3M+5	0	-M	0	

> Entering = X_2, Departing = A_1, Key Element = 3
> R_2(new) = R_2(old) / 3 = R_2(old) × 1/3
> R_1(new) = R_1(old) - 4 R_2(new)

		C_j	1	5	0	0	-M	
B	C_B	X_B	X_1	X_2	S_1	S_2	A_1	MinRatio
S_1	0	$^{10}/_3$	$^8/_3$	0	1	$^4/_3$	$^{-4}/_3$	$^5/_2$
X_2	5	$^2/_3$	$^1/_3$	1	0	$^{-1}/_3$	$^1/_3$	---
		Z_j	5/3	5	0	-5/3	5/3	
		$C_j - Z_j$	-2/3	0	0	5/3	-M-5/3	

> Entering = S_2, Departing = S_1, Key Element = 4/3
> R_1(new) = R_1(old) / 4/3 = R_1(old) × 3/4
> R_2(new) = R_2(old) + 1/3 R_1(new)

		C_j	1	5	0	0	-M	
B	C_B	X_B	X_1	X_2	S_1	S_2	A_1	MinRatio
S_2	0	$^5/_2$	2	0	$^3/_4$	1	-1	
X_2	5	$^3/_2$	1	1	$^1/_4$	0	0	
		Z_j	5	5	5/4	0	0	
		$C_j - Z_j$	-4	0	-5/4	0	-M	

Optimum Solution is arrived at with value of variables as :

$X_1 = 0$

$X_2 = {}^3/_2$

Maximise $Z = 15/2$

Maximize $$z = 20A + 6B + 9C$$

Subject to
$$\begin{aligned} 4A + 3B + C &\leq 24 \\ 2A + 4B + 12C &\geq 30 \\ 2A + 3B + C &= 10 \end{aligned}$$

All variables ≥ 0

After adding the appropriate slack variables, the problem is

Maximize $$z = 20A + 6B + 9C$$

Subject to
$$\begin{aligned} 4A + 3B + C + s_1 &= 24 \\ 2A + 4B + 12C - s_2 &= 30 \\ 2A + 3B + C &= 10 \end{aligned}$$

All variables ≥ 0

The variable s1 appears only in the first constraint and has a coefficient of 1, so it can be the basic variable for the first constraint. Although s2 appears only in the second constraint, it has a coefficient of 1; therefore, there is no variable in the second constraint that can act as a basis variable (multiplying the second constraint by 1 makes the right-hand-side coefficient negative). The third constraint also has no unique variable ready to be basic. In this case we create a basic but infeasible solution by adding two new nonnegative variables, A2 A3, to the left-hand side for the second and third constraints. These are called artificial variables. (We use the notation Ai where it is added to the ith constraint.) This gives the new form of the problem, which is in canonical form.

$$
\begin{aligned}
&\text{Maximize} && z = 20A + 6B + 9C \\
&\text{Subject to} && 4A + 3B + C + s_1 = 24 \\
& && 2A + 4B + 12C - s_2 + A_2 = 30 \\
& && 2A + 3B + C + A_3 = 10
\end{aligned}
$$

All variables ≥ 0

The artificial variables are put into the problem for only one purpose: *to act as basic variables to the algorithm started*. If an artificial variable is in the problem, it must be a basic variable. In the current canonical form the basic variables are s_1, A_2, and A_3, and the corresponding basic solution is $A = B = C = s_2 = 0$, $s_1 = 24$, $A_2 = 30$, and $A_3 = 10$. This is a basic solution that occurs at the intersection of the constraint boundaries: $A = 0$, $B = 0$, $C = 0$. However, this solution is infeasible because these lines intersect outside the feasible set.

By modifying the objective function to "encourage" the artificial variables to leave the basis, we use the simplex algorithm to move from an infeasible to a feasible basic solution. The algorithm then proceeds exactly as in the previous section. There are two ways to modify the problem. The first, called the *Big-M method*, is intuitively simple and is demonstrated here.

Analysis of Big M Method

- Problem P : Minimize cx

  Subject to $Ax = b$

  $x \geq 0$

- Problem P(M) : Minimize $cx + M s$

  Subject to $Ax + s = b$

  $x , s \geq 0$

  where,

  "s" is an artificial variable

4.10 – The Big M Method

Letting x1 = number of ounces of orange soda in a bottle of Oranj

x2 = number of ounces of orange juice in a bottle of Oranj

The LP is:

$$\min z = 2x_1 + 3x_2$$

$$\begin{array}{lrll}
\text{st} & 0.5x_1 + 0.25x_2 & \leq 4 & \text{(sugar constraint)} \\
& x_1 + 3x_2 & \geq 20 & \text{(Vitamin C constraint)} \\
& x_1 + x_2 & = 10 & \text{(10 oz in 1 bottle of Oranj)} \\
& x_1, x_2, & > 0 &
\end{array}$$

The LP in standard form is shown on the next slide.

4.10 – The Big M Method

The LP in standard form has z and s_1 which could be used for BVs but row 2 would violate sign restrictions and row 3 no readily apparent basic variable.

Row 1: $z - 2x_1 - 3x_2 = 0$

Row 2: $0.5x_1 + 0.25x_2 + s_1 = 4$

Row 3: $x_1 + 3x_2 - e_2 = 20$

Row 4: $x_1 + x_2 = 10$

In order to use the simplex method, a bfs is needed. To remedy the predicament, **artificial variables** are created. The variables will be labeled according to the row in which they are used as seen below.

Row 1: $z - 2x_1 - 3x_2 = 0$

Row 2: $0.5x_1 + 0.25x_2 + s_1 = 4$

Row 3: $x_1 + 3x_2 - e_2 + a_2 = 20$

Row 4: $x_1 + x_2 + a_3 = 10$

4.10 – The Big M Method

In the optimal solution, all artificial variables must be set equal to zero. To accomplish this, in a min LP, a term Ma_i is added to the objective function for each artificial variable a_i. For a max LP, the term $-Ma_i$ is added to the objective function for each a_i. **M represents some very large number.** The modified Bevco LP in standard form then becomes:

$$
\begin{array}{lllllllll}
\text{Row 1:} & z - 2x_1 - & 3x_2 & & & -Ma_2 & -Ma_3 & = 0 \\
\text{Row 2:} & 0.5x_1 + & 0.25x_2 + s_1 & & & & & = 4 \\
\text{Row 3:} & x_1 + & 3x_2 & -e_2 + & & a_2 & & = 20 \\
\text{Row 4:} & x_1 + & x_2 & & + & & a_3 & = 10
\end{array}
$$

Modifying the objective function this way makes it extremely costly for an artificial variable to be positive. The optimal solution should force $a_2 = a_3 = 0$.

4.10 – The Big M Method

Description of the Big M Method

1. Modify the constraints so that the rhs of each constraint is nonnegative. Identify each constraint that is now an = or ≥ constraint.
2. Convert each inequality constraint to standard form (add a slack variable for ≤ constraints, add an excess variable for ≥ constraints).
3. For each ≥ or = constraint, add artificial variables. Add sign restriction $a_i \geq 0$.
4. Let M denote a very large positive number. Add (for each artificial variable) Ma_i to min problem objective functions or $-Ma_i$ to max problem objective functions.
5. Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. Remembering M represents a very large number, solve the transformed problem by the simplex.

4.10 – The Big M Method

If all artificial variables in the optimal solution equal zero, the solution is optimal. If any artificial variables are positive in the optimal solution, the problem is infeasible.

The Bevco example continued:

Initial Tableau

Row	z	x1	x2	s1	e2	a2	a3	rhs
0	1.00	-2.00	-3.00			-M	-M	0.00
1		0.50	0.25	1.00				4.00
2		1.00	3.00		-1.00	1.00		20.00
3		1.00	1.00				1.00	10.00

Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex.

4.10 – The Big M Method

Pivot 1	z	x1	x2	s1	e2	a2	a3	rhs	ratio	ero
0	1.00	2m - 2	4M -3		-M			30M		Row 0 + M(Row 2) + M(Row 3)
1		0.50	0.25	1.00				4.00	16	
2		1.00	3		-1.00	1.00		20.00	6.67	
3		1.00	1.00				1.00	10.00	10	
ero 1	**z**	**x1**	**x2**	**s1**	**e2**	**a2**	**a3**	**rhs**		**ero**
0	1.00	2m - 2	4M -3		-M			30M		
1		0.50	0.25	1.00				4.00		
2		0.33	1		-0.33	0.33		6.67		Row 2 divided by 3
3		1.00	1.00				1.00	10.00		
ero 2	**z**	**x1**	**x2**	**s1**	**e2**	**a2**	**a3**	**rhs**		**ero**
0	1.00	(2M-3)/3			(M-3)/3	(3-4M)/3		(60+10M)/3		Row 0 - (4M-3)*(Row 2)
1		0.50	0.25	1.00				4.00		
2		0.33	1		-0.33	0.33		6.67		
3		1.00	1.00				1.00	10.00		
ero 3	**z**	**x1**	**x2**	**s1**	**e2**	**a2**	**a3**	**rhs**		**ero**
0	1.00	(2M-3)/3			(M-3)/3	(3-4M)/3		(60+10M)/3		
1		0.42		1.00	0.08	-0.08		2.33		Row 1 - 0.25*(Row 2)
2		0.33	1		-0.33	0.33		6.67		
3		1.00	1.00				1.00	10.00		
ero 4	**z**	**x1**	**x2**	**s1**	**e2**	**a2**	**a3**	**rhs**		**ero**
0	1.00	(2M-3)/3			(M-3)/3	(3-4M)/3		(60+10M)/3		
1		0.42		1.00	0.08	-0.08		2.33		
2		0.33	1		-0.33	0.33		6.67		
3		0.67			0.33	-0.33	1.00	3.33		Row 3 - Row 2

4.10 – The Big M Method

Pivot 2	z	x1	x2	s1	e2	a2	a3	rhs	ratio	
0	1.00	(2M-3)/3			(M-3)/3	(3-4M)/3		(60+10M)/3		
1		0.42		1.00	0.08	-0.08		2.33	5.60	
2		0.33	1		-0.33	0.33		6.67	20.00	
3		0.67			0.33	-0.33	1.00	3.33	5.00	

ero 1	z	x1	x2	s1	e2	a2	a3	rhs		ero
0	1.00	(2M-3)/3			(M-3)/3	(3-4M)/3		(60+10M)/3		
1		0.42		1.00	0.08	-0.08		2.33		
2		0.33	1		-0.33	0.33		6.67		
3		1.00			0.50	-0.50	1.50	5.00		(Row 3)*(3/2)

ero 2	z	x1	x2	s1	e2	a2	a3	rhs		ero
0	1.00				-0.50	(1-2M)/2	(3-2M)/2	25.00		Row 0 + (3-2M)*(Row 3)/3
1		0.42		1.00	0.08	-0.08		2.33		
2		0.33	1.00		-0.33	0.33		6.67		
3		1.00			0.50	-0.50	1.50	5.00		

ero 3	z	x1	x2	s1	e2	a2	a3	rhs		ero
0	1.00				-0.50	(1-2M)/2	(3-2M)/2	25.00		
1				1.00	-0.13	0.13	-0.63	0.25		Row 1 - (5/12)*Row 3)
2		0.33	1.00		-0.33	0.33		6.67		
3		1.00			0.50	-0.50	1.50	5.00		

ero 4	z	x1	x2	s1	e2	a2	a3	rhs		ero
0	1.00				-0.50	(1-2M)/2	(3-2M)/2	25.00	Optimal Solution	
1				1.00	-0.13	0.13	-0.63	0.25		
2			1.00		-0.50	0.50	-0.50	5.00		Row 2 -(1/3)*Row 3
3		1.00			0.50	-0.50	1.50	5.00		