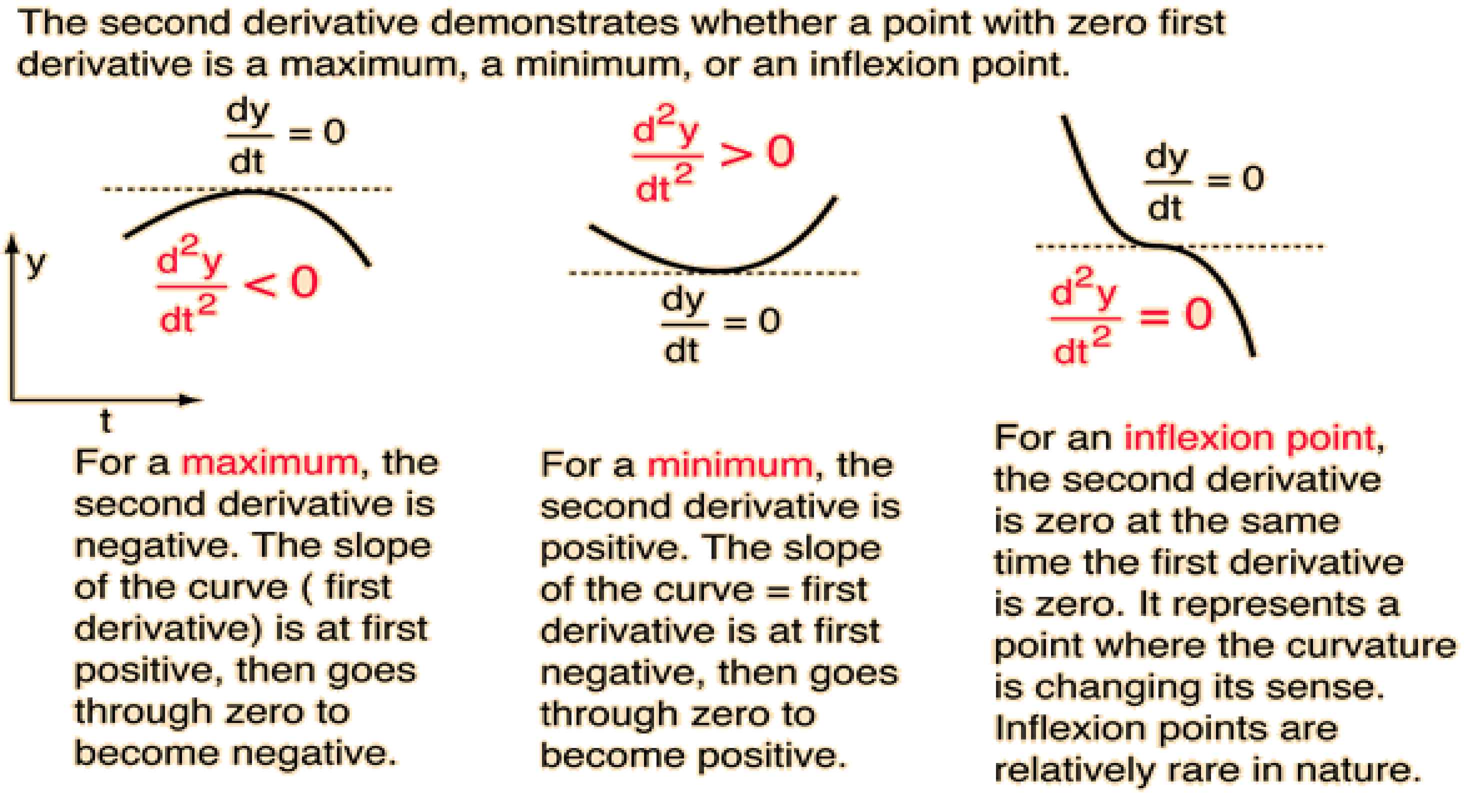
What is Optimization

Dr. Siddhartha Agarwal

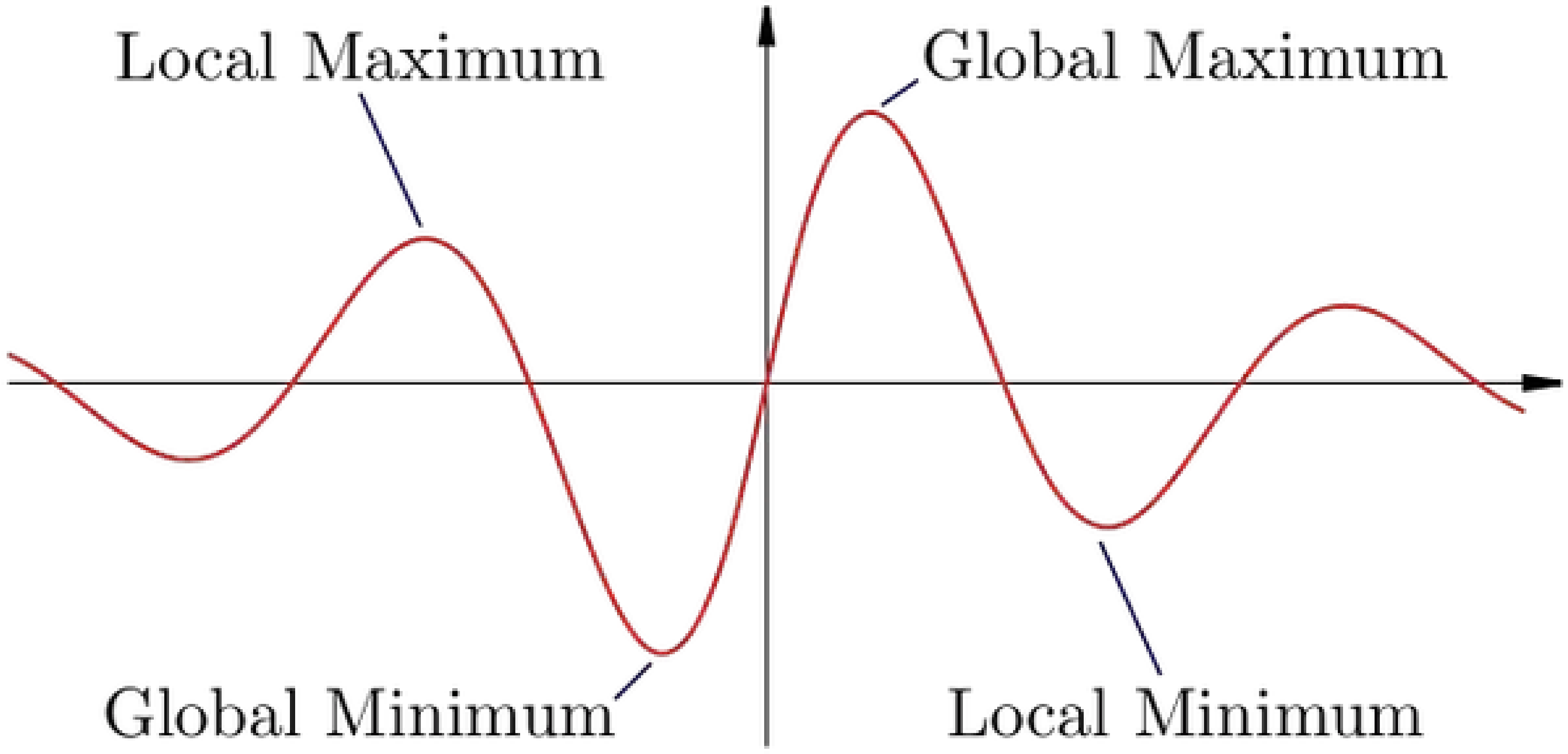
MND 404

Optimization

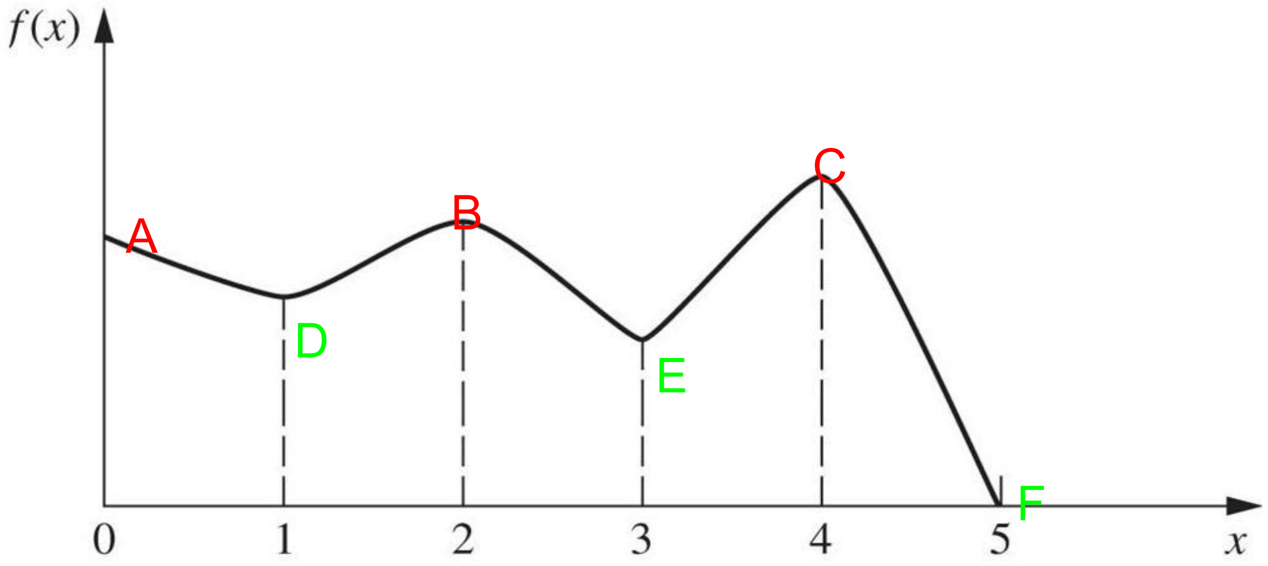
- Simply means finding the maximum or minimum of a function



Local and Global Minima & Maxima



A local maximum need not be a global maximum



This function has three local maxima—$x = 0$, $x = 2$, and $x = 4$—but only one of these—$x = 4$—is a *global maximum.* (Similarly, there are local minima at $x = 1$, 3, and 5, but only $x = 5$ is a *global minimum.*)

Classical Optimization Methods

Unconstrained optimization of a function of a single variable

Consider a function of a single variable $f(x)$. A necessary condition for a particular solution $x = x^*$ to be either a minimum or a maximum is that

$$\frac{df(x)}{dx} = 0 \quad \text{at } x = x^*.$$

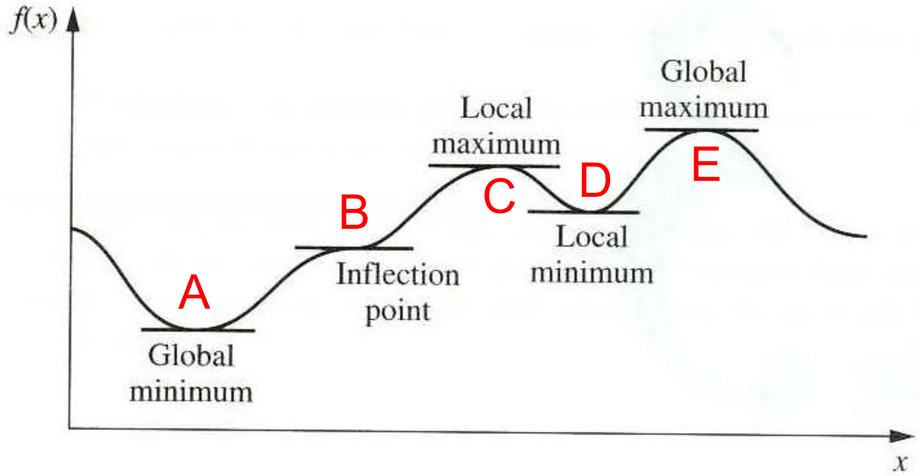
If $\frac{d^2f(x)}{dx^2} > 0$ at $x = x^*$ then x^* must be at least a **local minimum**

(If $f(x)$ is strictly convex in the vicinity of x*, $\frac{df(x)}{dx} = 0$ at $x = x^*$ is the necessary and sufficient condition for the solution $x = x^*$ to be a local minimum)

Global minimum: point A

Global maximum: point E

If f (x) is strictly convex everywhere, there is only one local minimum, which is the global minimum



Example:

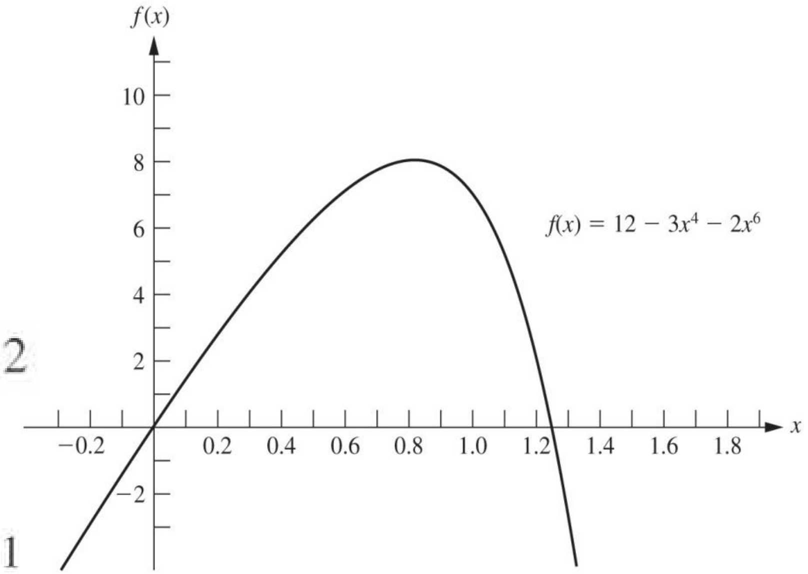
Suppose that the function to be maximized is

$$f(x) = 12x - 3x^4 - 2x^6,$$

Its first two derivatives are

$$\frac{df(x)}{dx} = 12(1 - x^3 - x^5) \begin{cases} = \mathbf{12 > 0} \text{ at } x = \mathbf{0} \\ = -\mathbf{468 < 0} \text{ at } x = 2 \end{cases}$$

$$\frac{d^2f(x)}{dx^2} = -12(3x^2 + 5x^4)$$



Initial $\underline{x} = \mathbf{0}$ **and** $\overline{x} = 2$, with their midpoint $x' = 1$

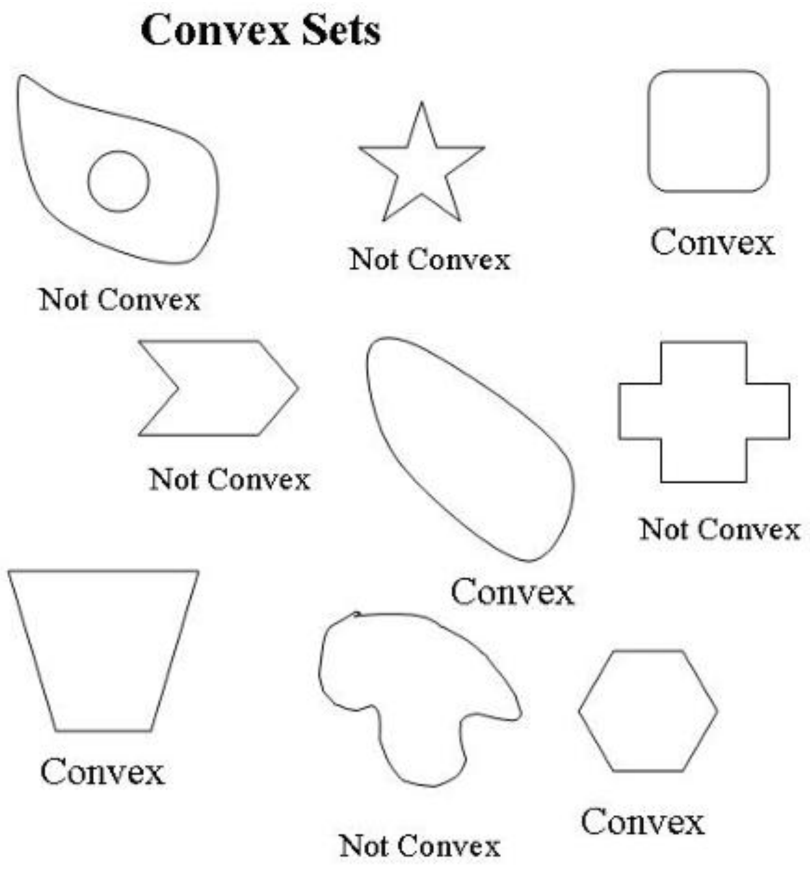
$\epsilon = 0.01$

$x^* \approx 0.836$

Application of the bisection method to the example

Iteration	$\frac{df(x)}{dx}$	$\underline{x}$	$\overline{x}$	New x'	$f(x')$
0		0	2	1	7.0000
1	−12	0	1	0.5	5.7812
2	+10.12	0.5	1	0.75	7.6948
3	+4.09	0.75	1	0.875	7.8439
4	−2.19	0.75	0.875	0.8125	7.8672
5	+1.31	0.8125	0.875	0.84375	7.8829
6	−0.34	0.8125	0.84375	0.828125	7.8815
7	+0.51	0.828125	0.84375	0.8359375	7.8839
Stop					

Why are convex sets important to study optimization ?



Definition of a convex set. A set S in R^n is said to be convex if for each x_1, $x_2 \in S$, the line segment

$\lambda x_1 + (1-\lambda)x_2$ for $\lambda \in (0,1)$ belongs to S.

This says that all points on a line connecting two points in the set are in the set. Or lie within or on the set

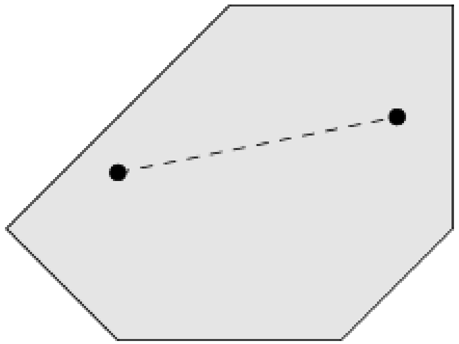
Convexity

Given two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we write $[\mathbf{x}, \mathbf{y}]$ for the line segment whose endpoints are $\mathbf{x}$ and $\mathbf{y}$. (This generalizes the notation $[a, b]$ for the closed interval in $\mathbb{R}$ with endpoints a and b.) The line segment $[\mathbf{x}, \mathbf{y}]$ has a convenient parametrization:

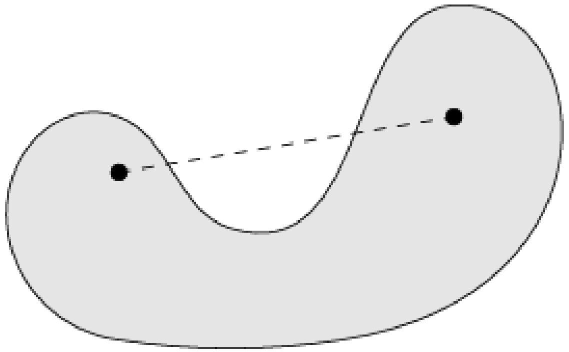
$$[\mathbf{x}, \mathbf{y}] = \{t\mathbf{x} + (1-t)\mathbf{y} : 0 \leq t \leq 1\}.$$

A set $S \subseteq \mathbb{R}^n$ is *convex* if, whenever, $\mathbf{x}, \mathbf{y} \in S$, we have $[\mathbf{x}, \mathbf{y}] \subseteq S$.

In the examples below, the set on the right is not convex: the endpoints of the dashed segment are in S, but some points in the interior are not. The set on the left is convex, though to check this, we would have to verify the definition for all possible segments.



CONVEX

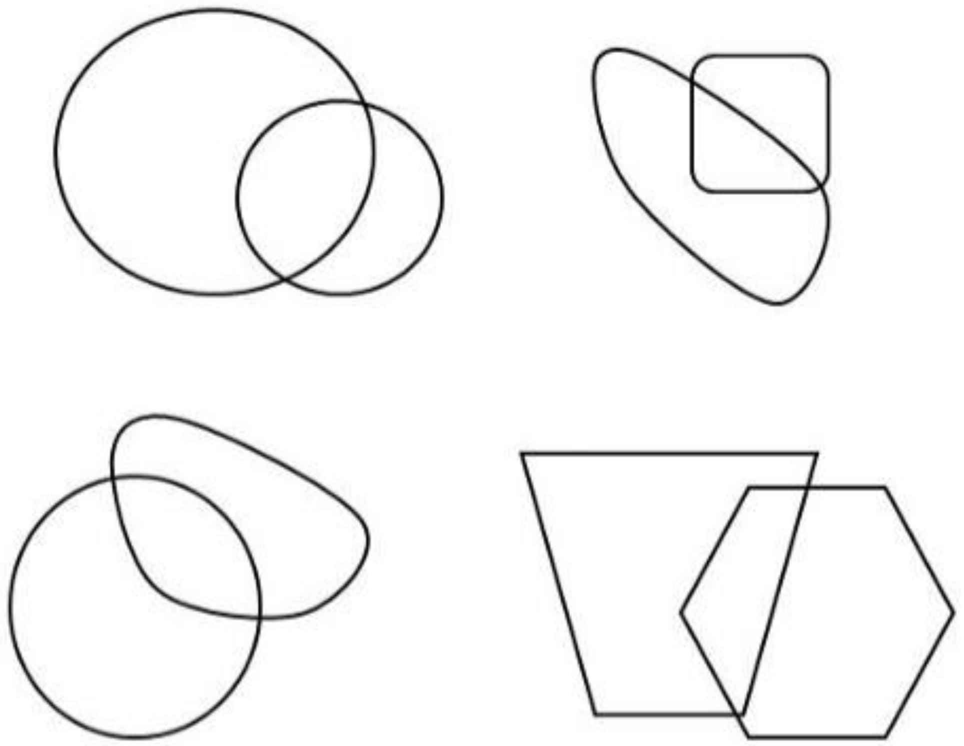


NOT CONVEX

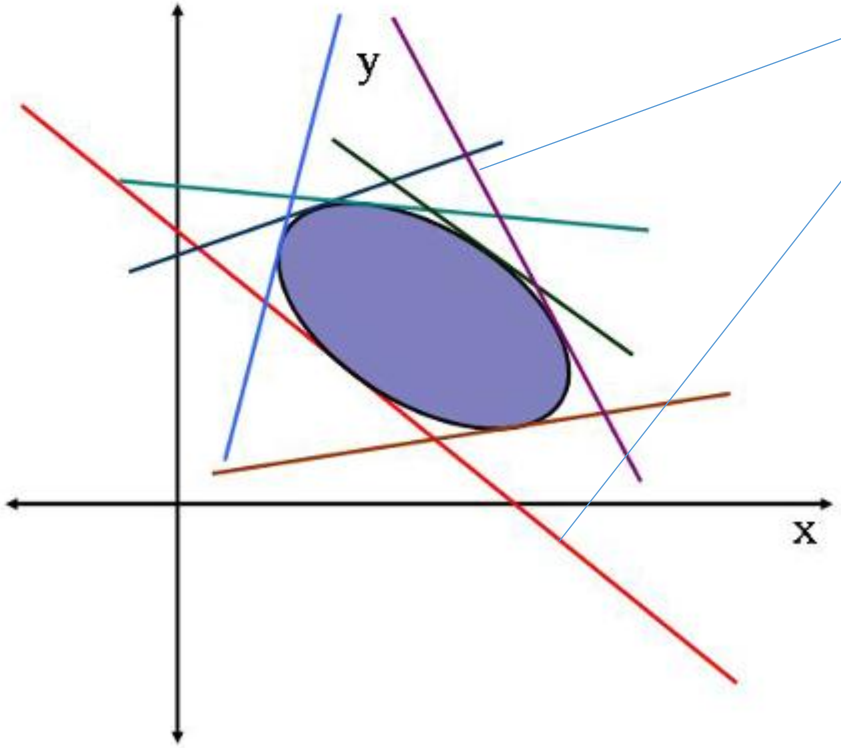
Properties of Convex Sets

- **They are easy to optimize** or easy to find maxima and minima in these functions
- **Intersections of convex sets are convex sets**. The intersection of a finite or infinite number of convex sets is convex. Figure 2 contains some examples of convex set intersections.
- **Addition of Convex Sets** is a Convex Set
- **Half spaces of convex sets are convex sets**
- **A closed, convex set is the intersection of the half spaces that support it. This is illustrated in figure. We can then find a convex set by finding the infinite intersection of half-spaces which support it.**

Intersections of Convex Sets



Minkowski's Theorem



Hyperplanes

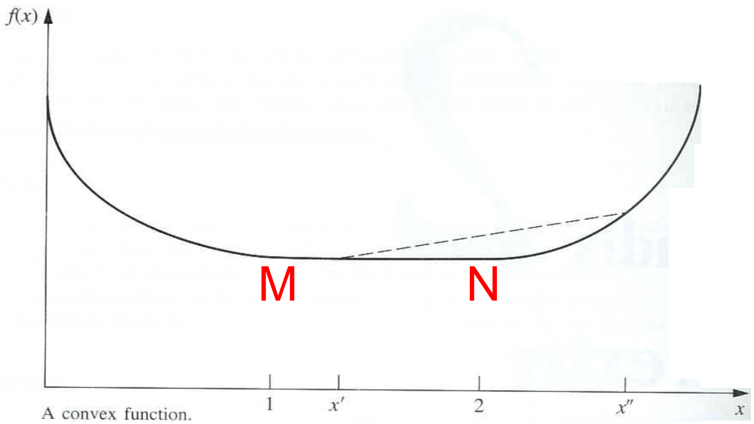
The geometric interpretation indicates that $f(x)$ is convex if it "bend upward".

To be more precise, if $f(x)$ possesses a second derivative everywhere, then $f(x)$ is convex if and only if $d^2f(x)/dx^2 \geq 0$ for all possible values of x

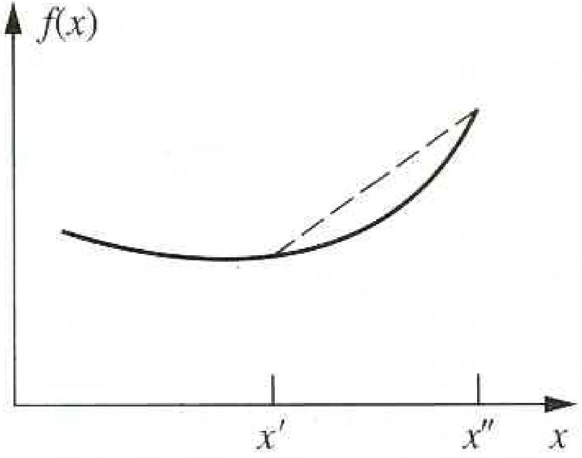
In terms of the second derivative of the function, the convexity test is summarized below.

Convexity test for a function of a single variable: Consider any function of a single variable $f(x)$ that possesses a second derivative at all possible values of x. Then $f(x)$ is

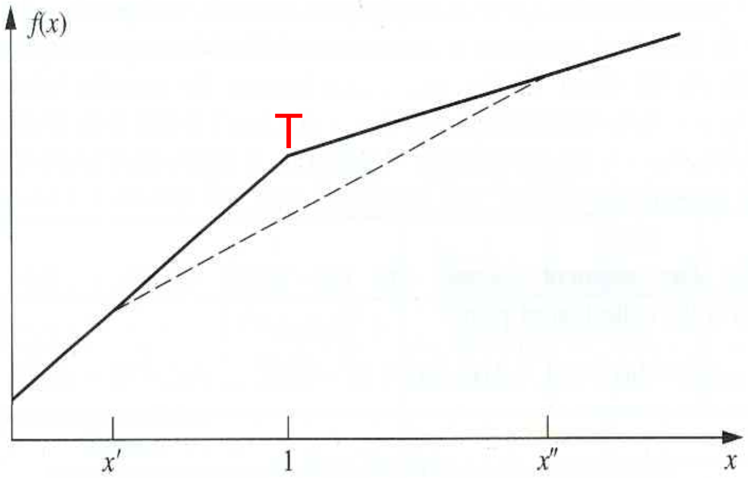
1. *Convex* if and only if $\dfrac{d^2f(x)}{dx^2} \geq 0$ for all possible values of x
2. *Strictly convex* if and only if $\dfrac{d^2f(x)}{dx^2} > 0$ for all possible values of x
3. *Concave* if and only if $\dfrac{d^2f(x)}{dx^2} \leq 0$ for all possible values of x
4. *Strictly concave* if and only if $\dfrac{d^2f(x)}{dx^2} < 0$ for all possible values of x



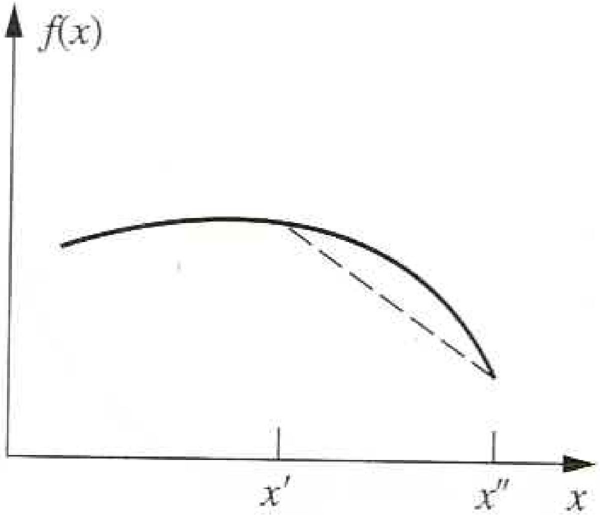
A convex function



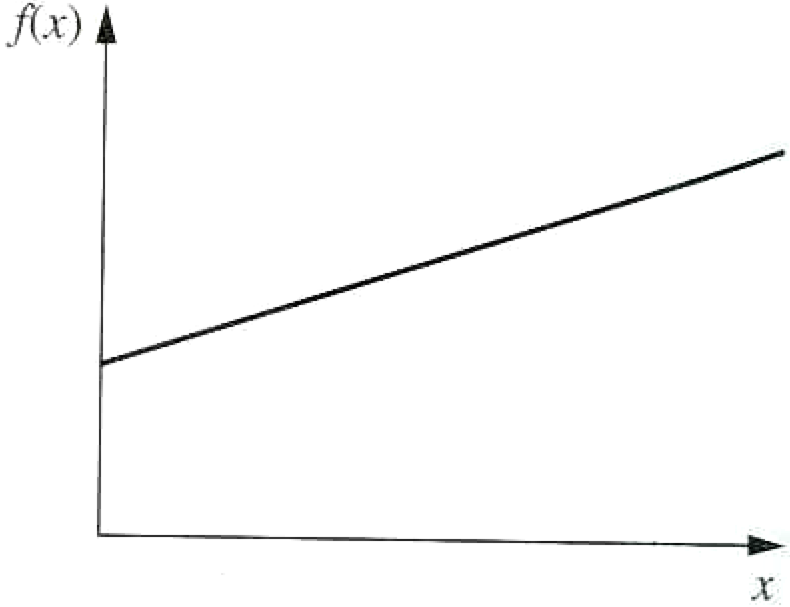
A strictly convex function



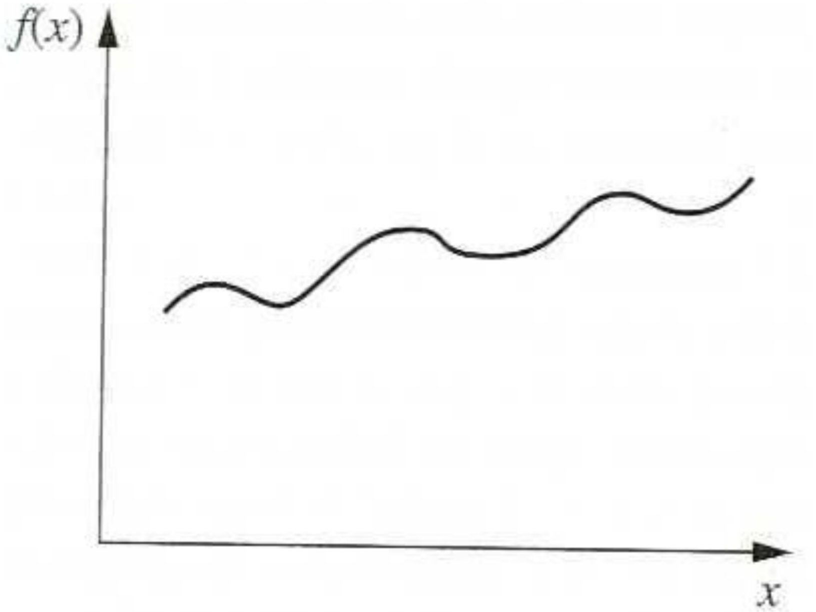
A concave function



A strictly concave function



A function that is both convex and concave



A function that is neither convex nor concave

Just as the second derivative can be used (when it exists everywhere) to check whether a function of a single variable is convex, so second partial derivatives can be used to check functions of several variables, although in a more complicated way. For example, if there are two variables and all partial derivatives exist everywhere, then the convexity test assesses whether *all three quantities* in the first column of Table A2.1 satisfy the inequalities shown in the appropriate column for *all possible values* of (x_1, x_2).

Table A2.1 **Convexity Test for a Function of Two Variables**

Quantity	Convex	Strictly Convex	Concave	Strictly Concave
$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} \frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} - \left[\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2}\right]^2$	≥ 0	> 0	≥ 0	> 0
$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2}$	≥ 0	> 0	≤ 0	< 0
$\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2}$	≥ 0	> 0	≤ 0	< 0
Values of (x_1, x_2)	All possible values			

To illustrate the convexity test for two variables, consider the function

$$f(x_1, x_2) = (x_1 - x_2)^2 = x_1^2 - 2x_1x_2 + x_2^2.$$

Therefore,

(1) $$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} \frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} - \left[\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \, \partial x_2}\right]^2 = 2(2) - (-2)^2 = 0,$$

(2) $$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} = 2 > 0,$$

(3) $$\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} = 2 > 0.$$

Since ≥ 0 holds for all three conditions, $f(x_1, x_2)$ is convex. However, it is *not* strictly convex because the first condition only gives $= 0$ rather than > 0.

When there are more than two variables, the convexity test is a generalization of the one shown in Table A2.1. For example, in mathematical terminology, $f(x_1, x_2, \ldots, x_n)$ is convex if and only if its $n \times n$ Hessian matrix is positive semidefinite for all possible values of $(x_1, x_2, \ldots, x_n)$.

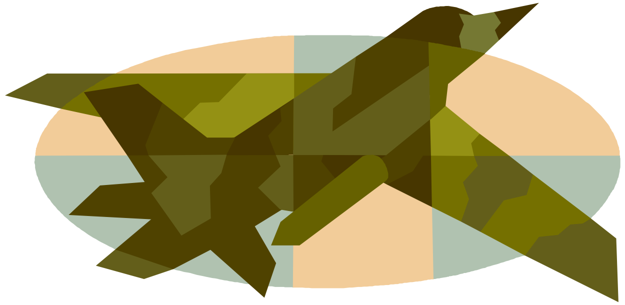
What is Systems Engineering

Dr. Siddhartha Agarwal

MND 404

System Definitions

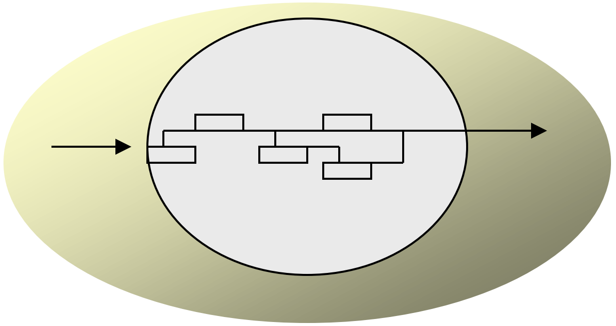
- **System**
  - A combination of elements forming a complex whole
- **System's Function** (Purposeful Action)
  - A system is a set of interrelated components *working together* toward some common objective or purpose
- Systems Engineers deal with *man-made systems*



System Definition

- Systems and Subsystems
  - Systems, segments, subsystems, elements
- Environment
- Inputs/Outputs
- Throughput – what is being transformed within the system!

Defined at System Boundaries



System Classification

- Natural vs. Human–Made Systems
- Physical vs. Conceptual Systems
- Static vs. Dynamic Systems
- Closed vs. Open Systems

Consider Examples – Do they have more than one applicable classification?

Key Takeaway: **Precision of terms** is crucial

Most useful SE tool: Diagramming a sentence

That is: **Precision of usage**, too

Evolution of Systems Engineering

- After World War II, Systems Engineering began to evolve as a branch of engineering as the military services and prime contractors sought tools and techniques that would help them excel at:
  - System performance and mission success
  - Project management: technical performance, delivery schedule, and cost control
- Space Age – more complex systems needed higher reliability
- Computer Age – connected more systems into Systems of Systems and Networked Systems
- **Now we call them Cyber Physical Systems**

Some Definitions of SE

- "System engineering is a robust approach to the design, creation, and operation of systems. In simple terms, the approach consists of identification and quantification of system goals, creation of alternative system design concepts, performance of design trades, selection and implementation of the best design, verification that the design is properly built and integrated, and post-implementation assessment of how well the system meets (or met) the goals."— NASA Systems Engineering Handbook, 1995.
- "An interdisciplinary approach and means to enable the realization of successful systems"— INCOSE handbook, 2004
- More recently the scope of SE has broadened:
  - Design of Enterprises, Infrastructure Networks etc…

Systems Engineering

- An interdisciplinary approach
- A means to enable the realization of successful systems
  - By defining customer <u>needs</u> and required <u>functionality</u> *early* in the development cycle
- Systems engineers are responsible for
  - Design and management of complex systems, guided by systems requirements
- There is a growing need for engineers who are
  - Concerned with the whole system
  - Can take an <u>interdisciplinary</u> *and* <u>top down</u> approach
- Systems engineers need to be
  - Problem *definers*, not just problem solvers
  - Involved with a system *throughout* its entire life cycle, from development through production, deployment, training, support, operation, and disposal

EMERGENCE

- Those collective properties—often called “emergent properties”—are **critical attributes of biological systems, as understanding the individual parts alone is insufficient to understand or predict system behaviour**. Thus, emergent properties necessarily come from the interactions of the parts of the larger system.
- Examples of emergent properties include **cities, the brain, ant colonies and complex chemical systems**
- "The whole is greater than the sum of its parts” – Aristotle.

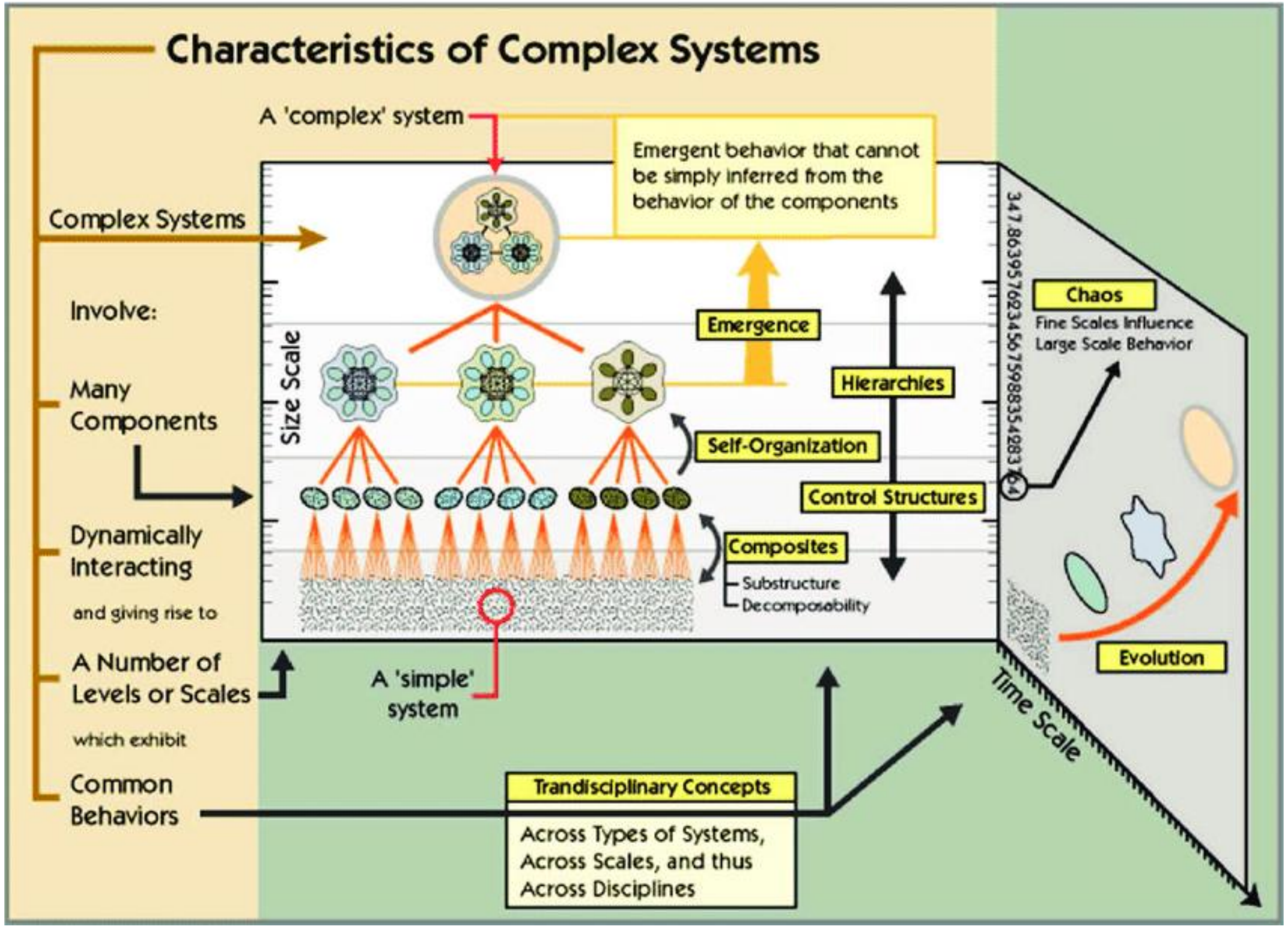
Ant Colonies

- A single ant is a rather limited organism, with little ability to reason or accomplish complex tasks.
- As a whole, however, an ant colony accomplishes astounding tasks, from building hills and dams to finding and moving huge amounts of food.
- In this context, emergent properties are the changes that occur in ant behavior when individual ants work together.
- Alone, an ant behaves erratically and almost at random. But millions of random actions by thousands and thousands of ants can serve to identify necessary tasks and organize other ants to complete them.

The Brain: Human consciousness

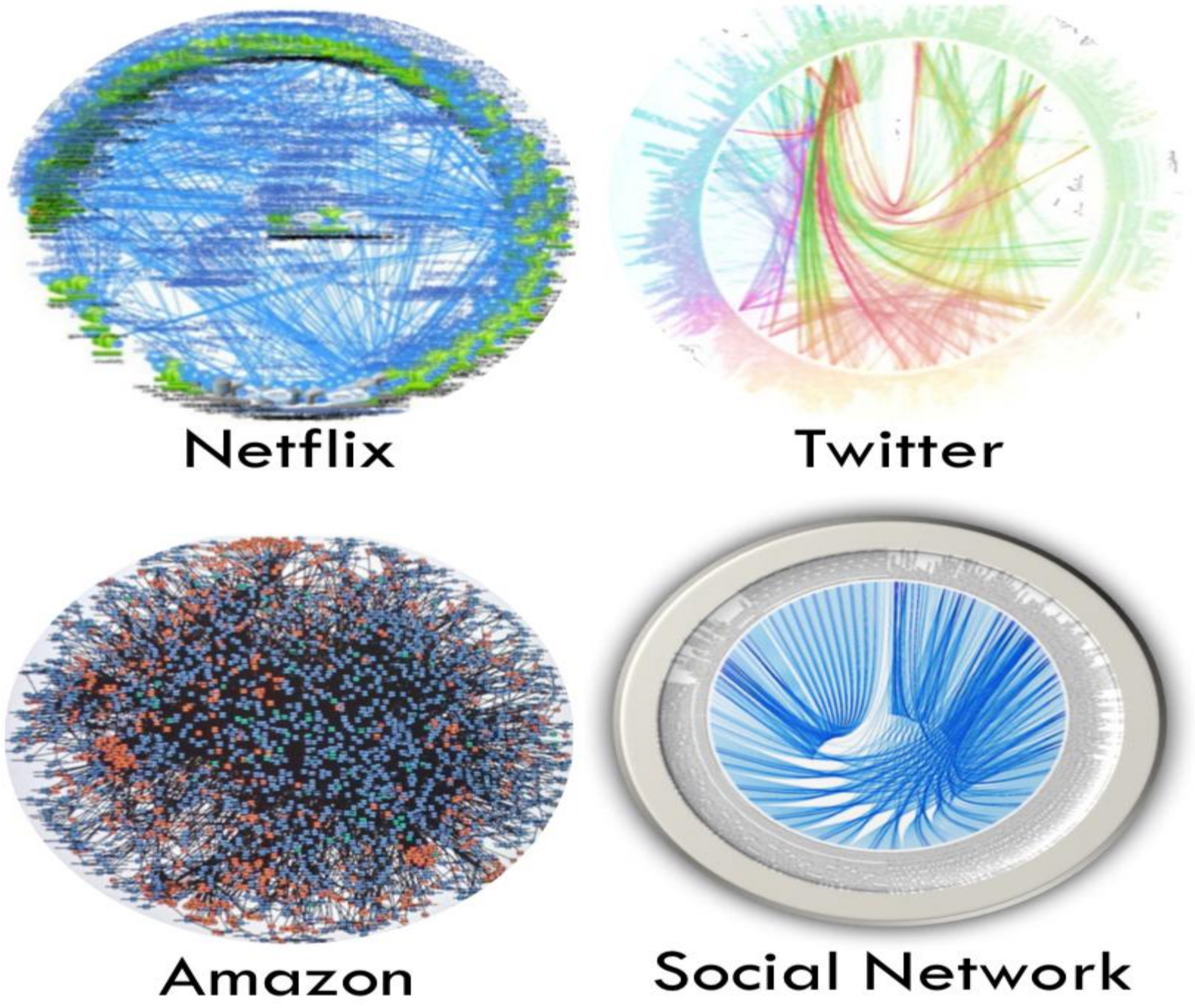
- Human consciousness is often called an emergent property of the human brain.
- Like the ants that make up a colony, no single neuron holds complex information like self-awareness, hope or pride.
- Nonetheless, the sum of all neurons in the nervous system generate complex human emotions like fear and joy, none of which can be attributed to a single neuron.
- Although the human brain is not yet understood enough to identify the mechanism by which emergence functions, most neurobiologists agree that complex interconnections among the parts give rise to qualities that belong only to the whole.

Complex Systems



Systems that are "complex" have distinct properties that arise from these relationships, such as **nonlinearity, emergence, adaptation (response to environmental change), and feedback loops, among others.**

Network Examples of Complex Systems



Complex Systems

- Examples of complex systems include **ant-hills, ants themselves, human economies, climate, nervous systems, cells and living things, including human beings, as well as modern energy or telecommunication infrastructures**

System Elements

- **Components** – operating parts of the system described by input, process, and output
  - Elements of the system perform functions that perform the processes
- **Attributes** – properties of the components, or of the system under consideration
- **Relationships – links between components and attributes**

System Components

The properties and behaviors of each component (or subset) of the system:

- Have an effect on the properties and behaviors of the system as a whole
- Impact depends on the properties and behaviors of at least one other component in the set
- Components can**not** be divided into completely **in**dependent subsets

System Components

- Structural Components
  - Static, but still perform an operation (or function)
- Operating Components
  - Perform processing or control (do or change something)
- Flow Components
  - Material, energy or information being altered

Examples of Systems

- Fire Department
- University
- Interstate Highway System
- Utilities (Gas, Water, Electric, Phone, Cable, Internet)
- Computer
- Cruise ship
- Aircraft
- Corporation
- Swamp…Desert…Tree

Subsystems or… SoS?

Environment?

Inputs?

Outputs?

Throughput?

Examples of Developed System Artifacts

System Engineering

Systems Engineering is a top-down, life-cycle approach to the design, development, and deployment of large-scale systems, processes, or operations to meet the effective needs of users and stakeholders in a cost-effective, high-quality way.

- An organised and systematic way of design
- Considers all the factors involved in the design
- Integrates all the disciplines and specialty groups into a team effort
- Ensures the business and customer needs of all stakeholders and ensures a system that meets the user needs

Engineering

The process of devising a system, component, or process to meet desired needs. It is a decision-making process (often iterative) in which the basic sciences, mathematics, and engineering sciences are applied to convert resources optimally to meet a stated objective.

–Accreditation Board for Engineering and Technology

Then, What is a System?

a. A group of interacting, interrelated, or interdependent elements forming a complex whole.
b. A functionally related group of elements, : The human body regarded as a functional physiological unit.
c. An organism as a whole, especially with regard to its vital processes or functions.
d. A group of physiologically or anatomically complementary organs or parts: *the nervous system; the skeletal system.*
e. A group of interacting mechanical or electrical components.
f. A network of related computer software, hardware, and data transmission devices.

A *system* is commonly defined to be a collection of hardware, software, people, facilities, and procedures organised to accomplish some common objectives.

- Definition of a System
  (NASA Systems Engineering Handbook)
  - ❑ A system is a set of interrelated components which interact with one another in an organized fashion toward a common purpose.
- System components may be quite diverse
  - ❑ Persons and Organizations
  - ❑ Software and Data
  - ❑ Equipment and Hardware
  - ❑ Facilities and Materials
  - ❑ Services and Techniques

Systems Engineering

- Definition of Systems Engineering (NASA SE Handbook)

  - Systems Engineering is a robust approach to the design, creation, and operation of systems.

- Systems Engineering consists of
  - Identification and quantification of system goals
  - Creation of alternative system design concepts
  - Performance of design trades
  - Selection and implementation of the best design (balanced and robust)
  - Verification that the design is actually built and properly integrated in accordance with specifications
  - Assessment of how well the system meets the goals

What is Systems Engineering?

Systems Engineering is a top-down, <u>life-cycle</u> approach to the design, development, and deployment of large-scale systems, processes, or operations to meet the effective needs of users and stakeholders in a cost-effective, high-quality way.

- Systems Engineering typically involves an interdisciplinary approach and means to enable the realization of successful systems.

- It focuses on defining customer needs and required functionality early in the development cycle, documenting requirements, then proceeding with design synthesis and system validation while considering the complete problem.

Building Blocks of Systems Engineering

- Math & Physical Sciences
  - Qualitative modeling
  - Quantitative modeling
  - Physical modeling
  - Theory of Constraints
  - Physical Laws
- Management Sciences
  - Economics
  - Organizational Design
  - Business Decision Analysis
  - Operations Research

Unique to Systems Engineering

Building Blocks of Systems Engineering

- Social Sciences
  - Multi-disciplinary Teamwork
  - Organizational Behavior
  - Leadership

- Body of Knowledge
  - Problem definition
    - System boundaries
    - Objectives hierarchy
    - Concept of operations
    - Originating requirements
  - Concurrent engineering
    - System life cycle phases
    - Integration/Qualification

Unique to Systems Engineering

- Architectures
  - Functional/Logical
  - Physical/Operational
  - Interface
- Trades
  - Concept-level
  - Risk management
  - Key performance parameters

Other Considerations

- Achieving balance between inherent conflicts
  - System functionality and performance
  - Development cost and recurring cost
  - Development schedule
    (Time to Market)
  - Development risk (Probability of Success)
  - Business viability and success

- System Optimization
  - Subsystems often suboptimal to achieve best balance at system level
  - Ultimate system purpose must prevail against conflicting considerations
  - Long-term considerations (e.g., disposal) may drive technical decisions
- Customer Interface
  - Often must act as “honest broker”
  - Carries burden of educating customer on hard choices
  - Must think ahead to the next customer and next application
  - Must “challenge” all requirements

Systems Engineering Heritage

• Water distribution systems in Mesopotamia	4000 BC
• Irrigation systems in Egypt	3300 BC
• Urban systems such as Athens, Greece	400 BC
• Roman highway systems	300 BC
• Water transportation systems like Erie Canal	1800s
• Telephone systems	1877
• Electrical power distribution systems	1880

Modern Origins of the Systems Approach

- British multi-disciplined team formed (1937) to analyze Air Defense System
- Bell Labs supported Nike development (1939-1945)
- SAGE Air defense system defined and managed by MIT (1951-1980)
- ATLAS Intercontinental Ballistic Missile Program managed by systems contractor, Ramo-Wooldridge Corp (1954-1964)

Spread of the Systems Approach[1]

- Early Proponents
  - Research and Development Corporation (RAND)
  - Robert McNamara (Secretary of Defense)
  - Jay Forrester (Modeling Urban Systems at MIT)
- Growth in systems engineering citations (Engineering Index)
  - Nil in 1964
  - One Page in 1966
  - Eight Pages in 1969
- Nine Universities Offered Systems Engineering Programs in 1964

1) Hughes, Thomas P., *Rescuing Prometheus*, Chapter 4, pps. 141-195, Pantheon Books, New York, 1998.