Intro to Linear programming

Optimization:
Intro to Linear programming

Optimization: Mathematical Program

- Optimization problem in which the objective and constraints are given as mathematical functions and functional relationships.

Minimize $f(x_1, x_2, \ldots, x_n)$

Subject to:

$$g_1(x_1, x_2, \ldots, x_n) \;=, \geq, \leq \; b_1$$

$$g_2(x_1, x_2, \ldots, x_n) \;=, \geq, \leq \; b_2$$

…

$$g_m(x_1, x_2, \ldots, x_n) \;=, \geq, \leq \; b_m$$

Linear Programming (LP)

- **Linear** – all the functions are linear

Ex: $f(x_1, x_2, \ldots, x_n) = c_1 x_1 + c_2 x_2 + \ldots c_n x_n$

- **Programming** – does not refer to computer programming but rather "planning" - planning of activities to obtain an optimal result i.e., it reaches the specified goal best (according to the mathematical model) among all feasible alternatives.

Components of a Linear Programming Model

- A linear programming model consists of:
  - A set of decision variables
  - A (linear) objective function
  - A set of (linear) constraints

Nature Connection: Recreational Sites

Nature Connection is planning two new public recreational sites: **a forested wilderness area** and a **sightseeing and hiking park**. They own 80 hectares of forested wilderness area and 20 hectares suitable for the sightseeing and hiking park but they don't have enough resources to make the entire areas available to the public. They have a **budget of $120K** per year. They estimate a yearly management and maintenance cost of $1K per hectare for the forested wilderness area, and $4K per hectare for the sightseeing and hiking park. The expected average number of visiting hours a day per hectare are: 10 for the forest and 20 for the sightseeing and hiking park.

Question: How many hectares should Nature Connection allocate to the public sightseeing and hiking park and to the public forested wilderness area, in order to maximize the amount of recreation, (in average number of visiting hours a day for the total area to be open to the public, for both sites**) given their budget constraint?**

Steps in setting up a LP

1. Determine and label the *decision variables*.
2. Determine the objective and use the decision variables to write an expression for the *objective function*.
3. Determine the constraints - *feasible region*.
    1. Determine the *explicit constraints* and write a functional expression for each of them.
    2. Determine the *implicit constraints (e.g., nonnegativity constraints)*.

Formulation of the problem as a Linear Program

1 Decision Variables

x1 – # hectares to allocate to the public forested wilderness area

X2 – # hectares to allocate the public sightseeing and hiking park

2 Objective Function

Max 10 x1 + 20 x2

3 Constraints

$$
\begin{aligned}
x1 \quad\quad\quad & \le 80 \\
x2 & \le 20 \\
x1 + 4\,x2 & \le 120
\end{aligned}
$$

x1 >= 0; x2 >= 0 Non-negativity constraints

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1 Decision Variables

x1 – # hectares to allocate to the public forested wilderness area

X2 – # hectares to allocate the public sightseeing and hiking park

2 Objective Function

Max 10 x1 + 20 x2

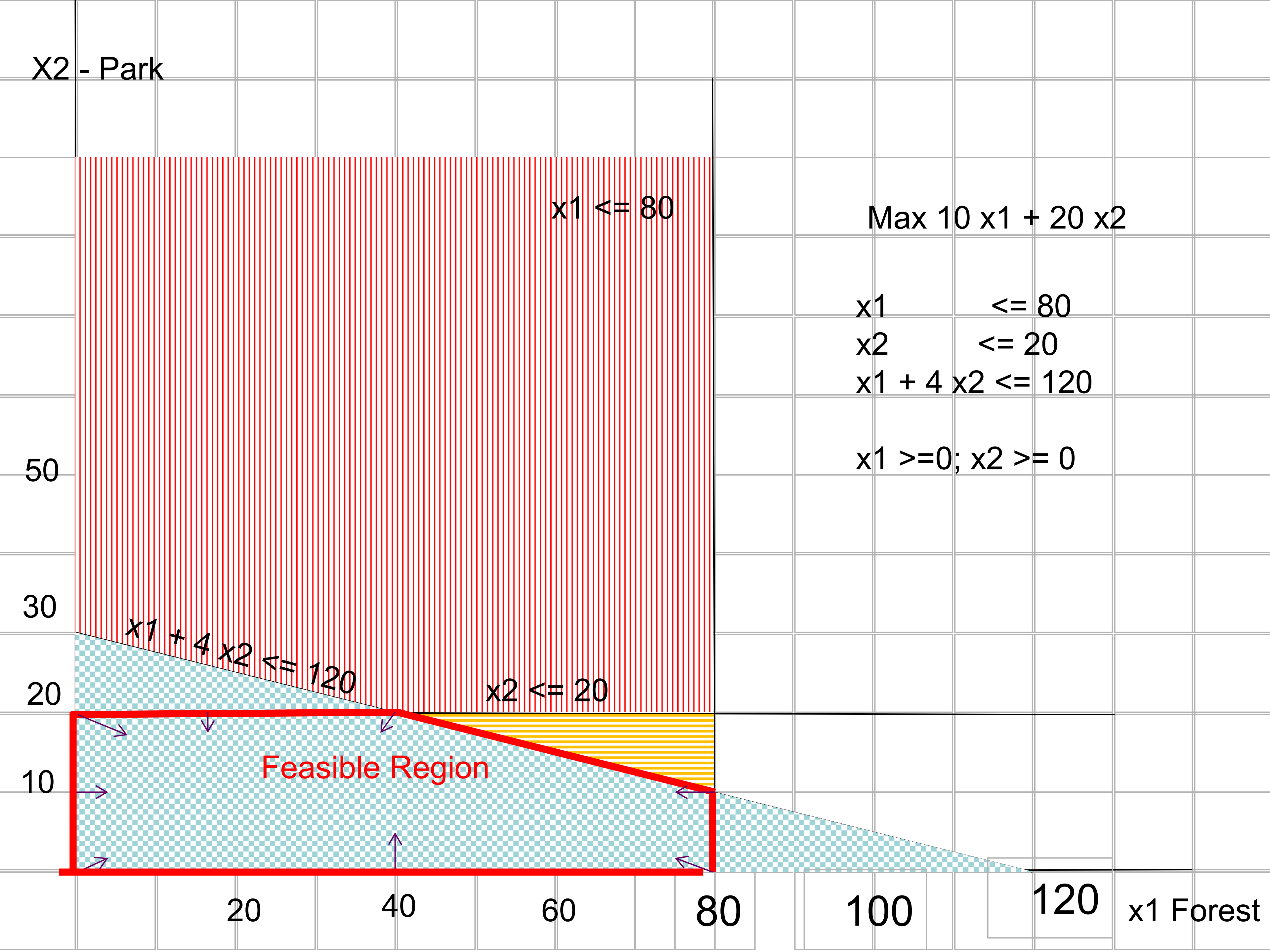
3 Constraints

x1 <= 80 Land for forest

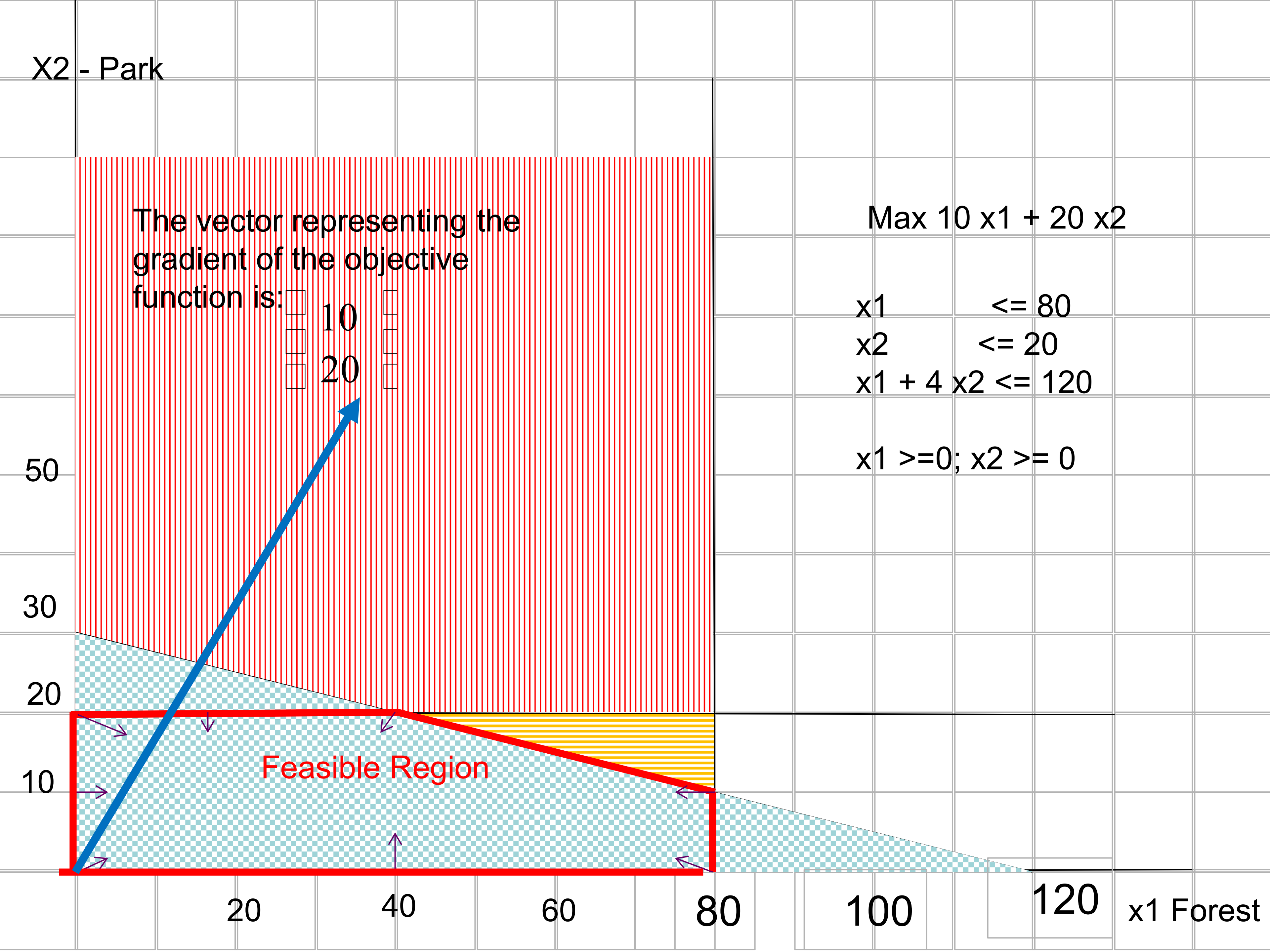
x2 <= 20 Land for Park

x1 + 4 x2 <= 120 Budget

x1 >= 0; x2 >= 0 Non-negativity constraints



X2 - Park
x1 <= 80
Max 10 x1 + 20 x2
x1 <= 80
x2 <= 20
x1 + 4 x2 <= 120
x1 >=0; x2 >= 0
50
30
x1 + 4 x2 <= 120
20
x2 <= 20
Feasible Region
10
20
40
60
80
100
120
x1 Forest


X2 - Park
The vector representing the gradient of the objective function is: 10 20
Max 10 x1 + 20 x2
x1 <= 80
x2 <= 20
x1 + 4 x2 <= 120
x1 >=0; x2 >= 0
50
30
20
10
Feasible Region
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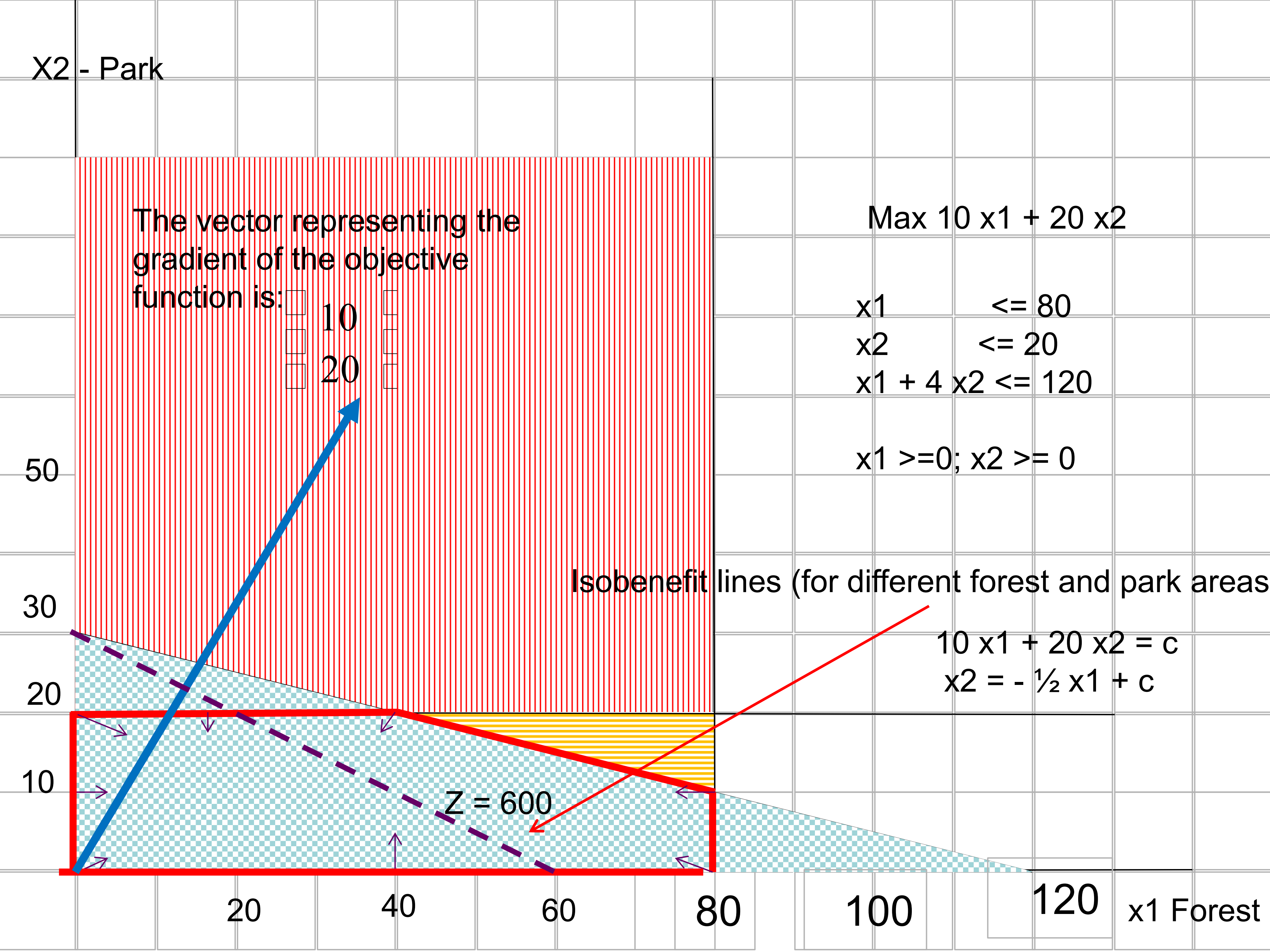
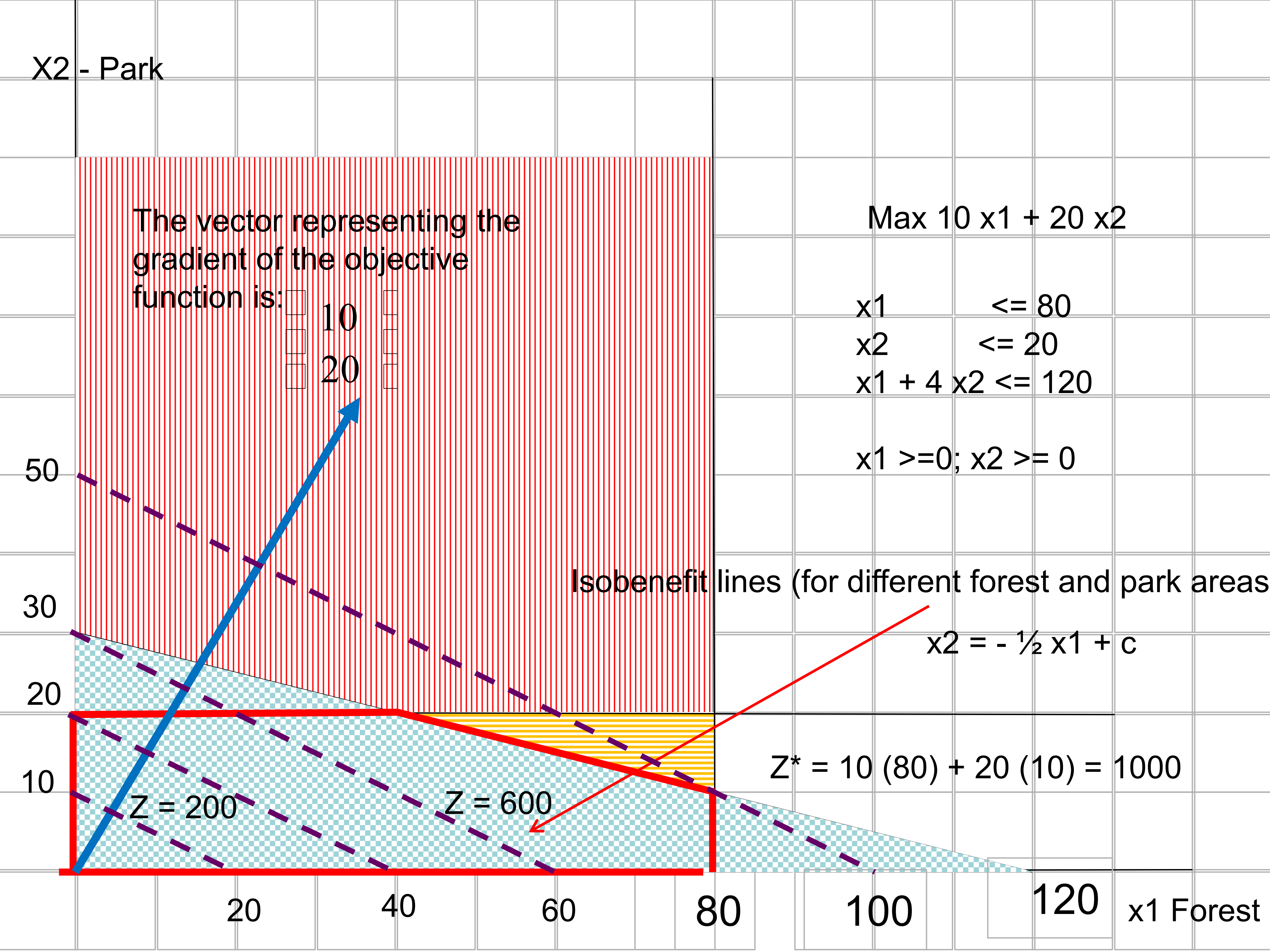
x1 Forest



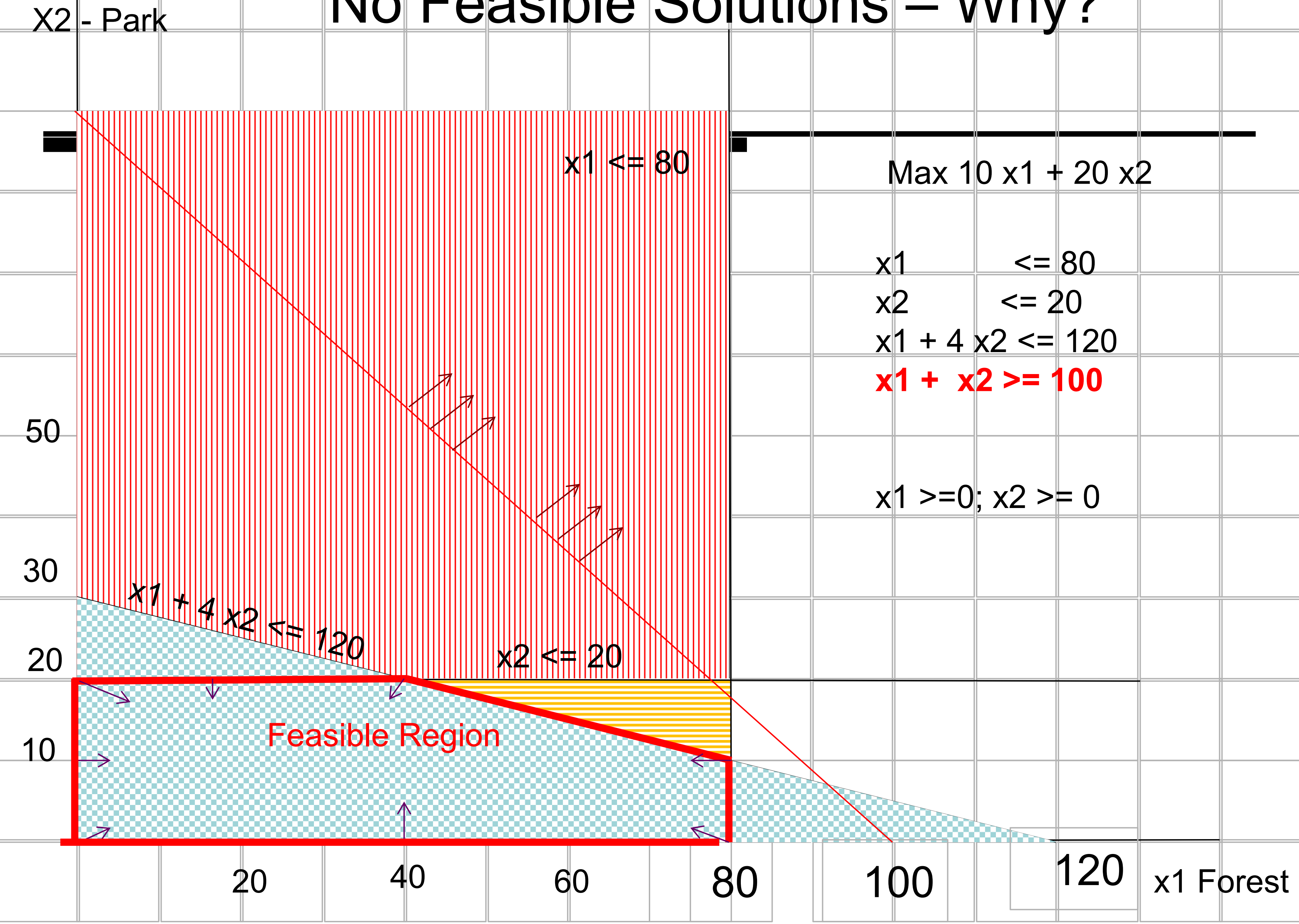
X2 - Park
The vector representing the gradient of the objective function is:
10
20
Max 10 x1 + 20 x2
x1 <= 80
x2 <= 20
x1 + 4 x2 <= 120
x1 >=0; x2 >= 0
Isobenefit lines (for different forest and park areas
10 x1 + 20 x2 = c
x2 = - ½ x1 + c
Z = 600
50
30
20
10
20
40
60
80
100
120
x1 Forest


X2 - Park
The vector representing the gradient of the objective function is:
10
20
Max 10 x1 + 20 x2
x1 <= 80
x2 <= 20
x1 + 4 x2 <= 120
x1 >=0; x2 >= 0
Isobenefit lines (for different forest and park areas
x2 = - ½ x1 + c
Z* = 10 (80) + 20 (10) = 1000
Z = 200
Z = 600
50
30
20
10
20
40
60
80
100
120
x1 Forest

Summary of the Graphical Method

- Draw the constraint boundary line for each constraint. Use the origin (or any point not on the line) to determine which side of the line is permitted by the constraint.
- Find the feasible region by determining where all constraints are satisfied simultaneously.
- Determine the slope of one objective function line (perpendicular to its gradient vector). All other objective function lines will have the same slope.
- Move a straight edge with this slope through the feasible region in the direction of improving values of the objective function (direction of the gradient). Stop at the last instant that the straight edge still passes through a point in the feasible region. This line is the optimal objective function line.
- A feasible point on the optimal objective function line is an optimal solution.

No Feasible Solutions – Why?

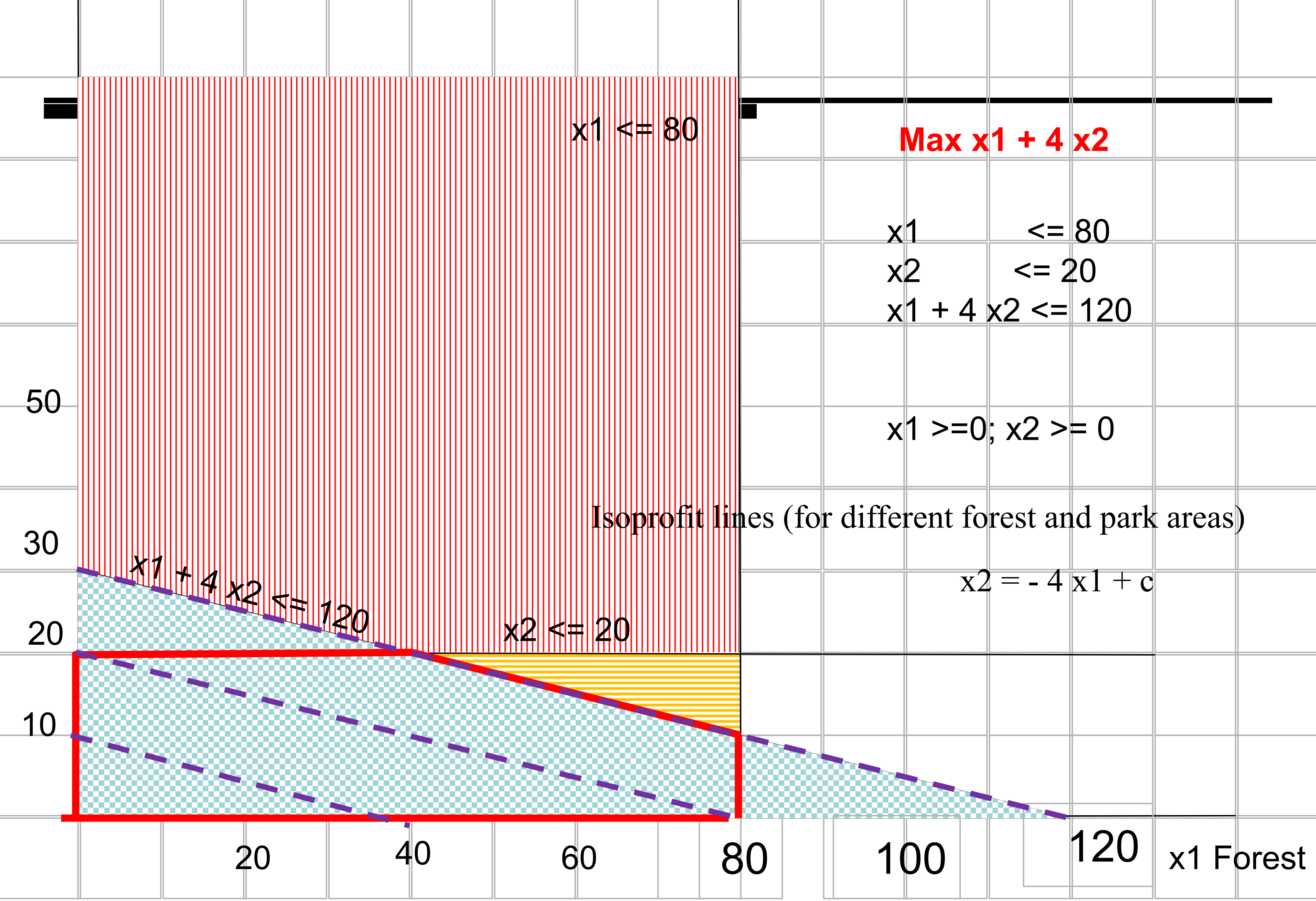


Max 10 x1 + 20 x2

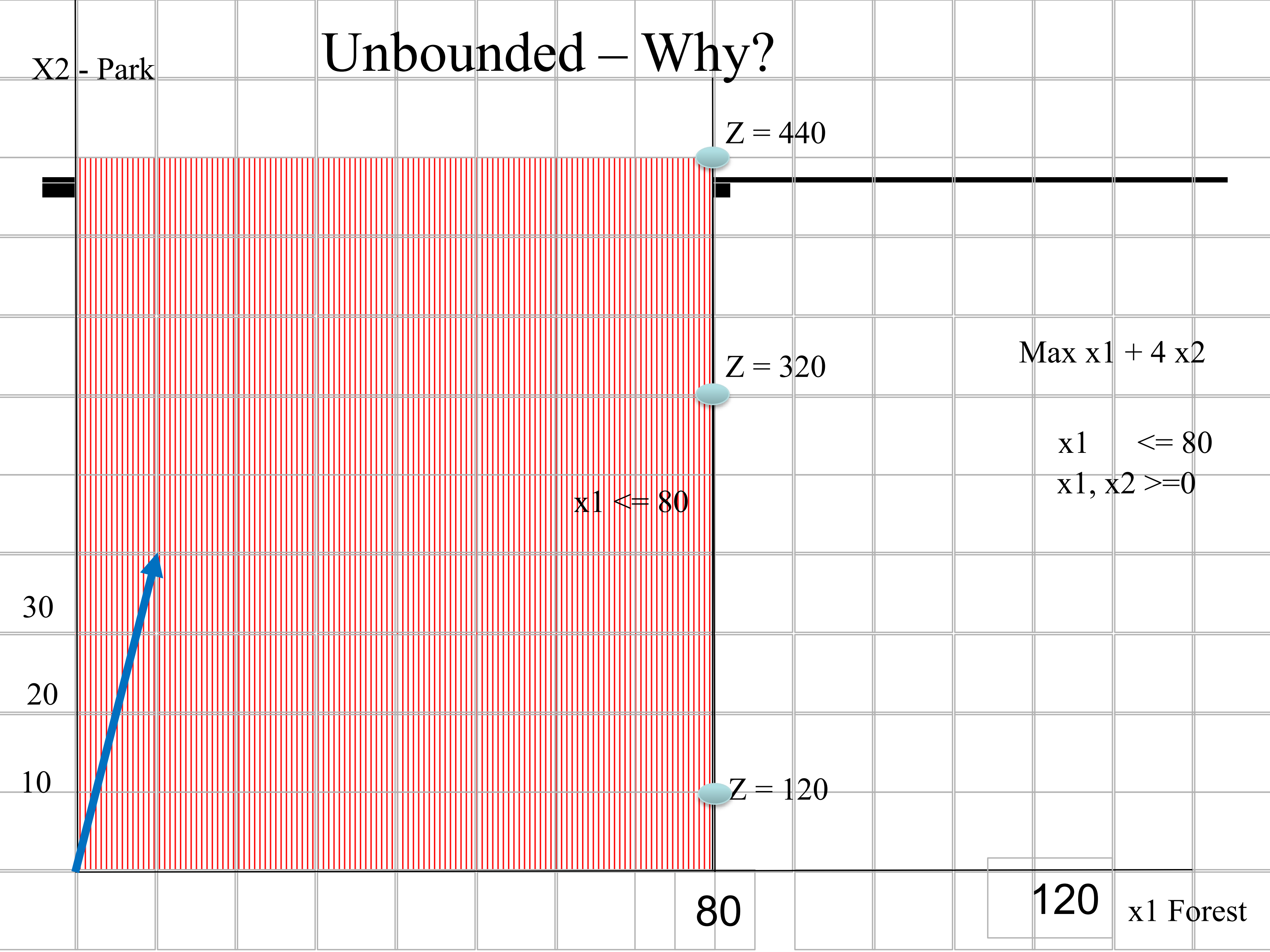
x1 <= 80
x2 <= 20
x1 + 4 x2 <= 120
x1 + x2 >= 100

x1 >=0; x2 >= 0

Multiple Optimal Solutions – Why



Unbounded – Why?



Key Categories of LP problems:

- Resource-Allocation Problems
- Cost-benefit-trade-off problems

Distribution-Network Problems

Mixed problems

Second Example: Keeping the river clean

Cost-benefit-trade-off problems

Choose the <u>mix of levels of various activities</u> to achieve <u>minimum acceptable levels for various benefits</u>

at a minimum cost.

Second Example: Keeping the River Clean

A pulp mill in Maine makes mechanical and chemical pulp, polluting the river in which it spills its spent waters. This has created several problems, leading to a change in management.

The previous owners felt that it would be too expensive to reduce pollution, so they decided to sell the pulp mill. The mill has been bought back by the employees and local businesses, who now own the mill as a cooperative. The new owners have several objectives:

1 – to keep **at least 300 people employed at the mill** (300 workers a day);

2 – to generate at least $40,000 of revenue a day

They estimate that this will be enough to pay operating expenses and yield a return that will keep the mill competitive in the long run. Within these limits, everything possible should be done to **minimize pollution**.

Both chemical and mechanical pulp require the labor of one worker for 1 day (1 workday, wd) per ton produced;

Mechanical pulp sells at $100 per ton; Chemical pulp sells at $200 per ton;

Pollution is measured by the biological oxygen demand (BOD). One ton of mechanical pulp produces 1 unit of BOD; One ton of chemical pulp produces 1.5 units of BOD.

The maximum capacity of the mill to make mechanical pulp is 300 tons per day; for chemical pulp is 200 tons per day. The two manufacturing processes are independent (I.e., the mechanical pulp line cannot be used to make chemical pulp and vice versa).

- Pollution, employment, and revenues result from the production of both types of pulp. So a natural choice for the variables is:

Decision Variables

- X1 amount of mechanical pulp produced (in tons per day, or t/d) and
- X2 amount of chemical pulp produce (in tons per day, or t/d
- Min Z = 1 X1 + 1.5 X2

(BOD/day)	(BOD/t) (t/d)		(BOD/t) (t/d)	

Subject to:

1 X1	+	1 X2	>= 300 Workers/day
(wd/t) (t/d)		(wd/t)(t/d)	
100 X1	+	200 X2	>= 40,000 revenue/day
($/t) (t/d)	+	($/t)(t/d)	$/d
X1			<= 300 (mechanical pulp)
(t/d)			(t/d)
		X2	<= 200 (mechanical pulp)
		(t/d)	(t/d)

X1 >=0; X2 >= 0

Distribution Network Problems

Distribution-Network Problem

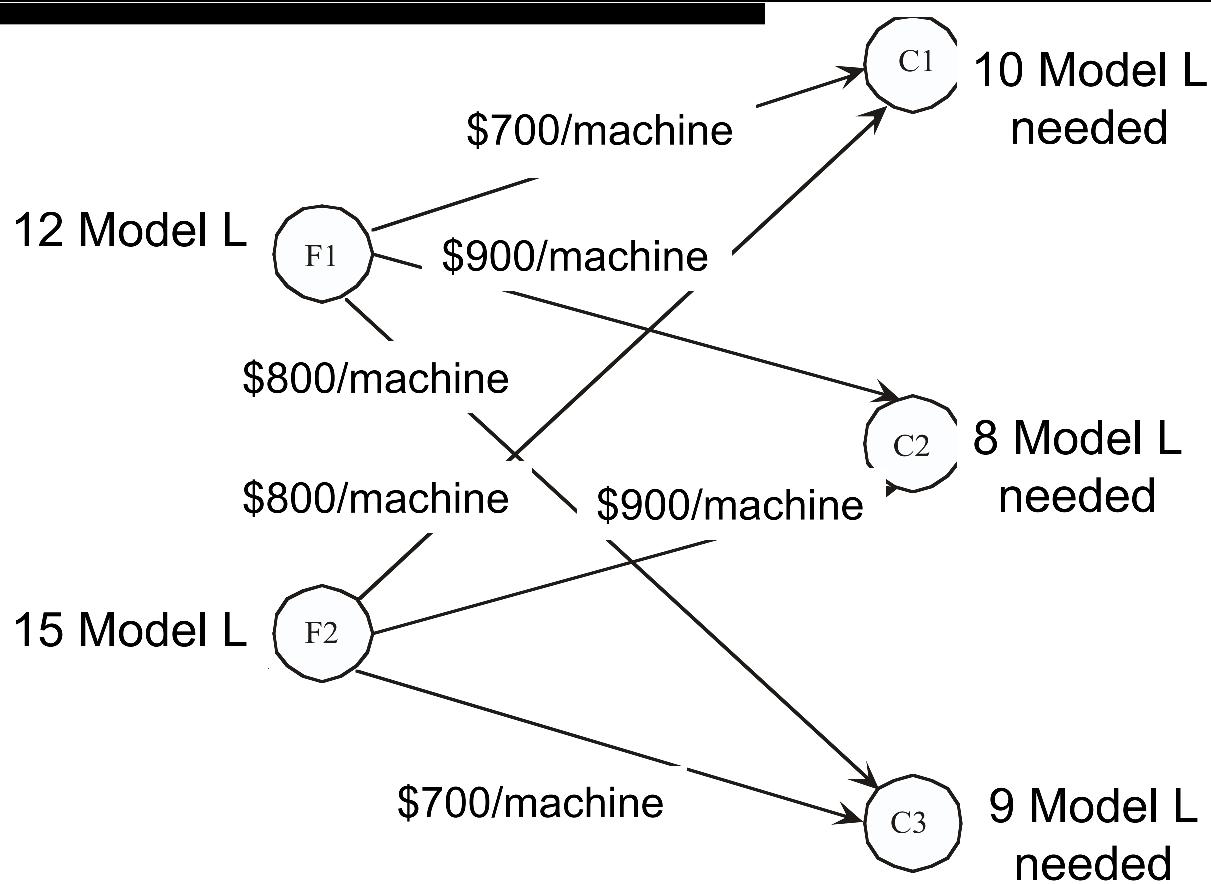
- The International Hospital Share Organization is a non-profit organization that refurbishes a variety of used equipment for hospitals of developing countries at two international factories (F1 and F2). One of its products is a large X-Ray machine.
- Orders have been received from three large communities for the X-Ray machines.

Shipping Cost for Each Machine (Model L)

From \ To	Community 1	Community 2	Community 3	Output
Factory 1	$700	$900	$800	12 X-ray machines
Factory 2	800	900	700	15 X-Ray machines
Order Size	10 X-ray machines	8 X-Ray machines	9 X-Ray machines	

Question: How many X-Ray machines (model L) should be shipped from each factory to each hospital so that shipping costs are minimized?

The Distribution Network



Question: How many machines (model L) should be shipped from each factory to each customer so that sipping costs are minimized?

- Activities – shipping lanes (not the level of production which has already been defined)
  - Level of each activity – number of machines of model L shipped through the corresponding shipping lane.

  → Best mix of shipping amounts

Example:

Requirement 1: Factory 1 must ship 12 machines
Requirement 2: Factory 2 must ship 15 machines
Requirement 3: Customer 1 must receive 10 machines
Requirement 4: Customer 2 must receive 8 machines
Requirement 5: Customer 3 must receive 9 machines

Algebraic Formulation

Let S_{ij} = Number of machines to ship from i to j (i = F1, F2; j = C1, C2, C3).

$$\text{Minimize Cost} = \$700S_{F1\text{-}C1} + \$900S_{F1\text{-}C2} + \$800S_{F1\text{-}C3} + \$800S_{F2\text{-}C1} + \$900S_{F2\text{-}C2} + \$700S_{F2\text{-}C3}$$

subject to

$$S_{F1\text{-}C1} + S_{F1\text{-}C2} + S_{F1\text{-}C3} = 12$$
$$S_{F2\text{-}C1} + S_{F2\text{-}C2} + S_{F2\text{-}C3} = 15$$
$$S_{F1\text{-}C1} + S_{F2\text{-}C1} = 10$$
$$S_{F1\text{-}C2} + S_{F2\text{-}C2} = 8$$
$$S_{F1\text{-}C3} + S_{F2\text{-}C3} = 9$$

and

$S_{ij} \geq 0$ (i = F1, F2; j = C1, C2, C3).

Algebraic Formulation

Let S_{ij} = Number of machines to ship from i to j (i = F1, F2; j = C1, C2, C3).

Minimize Cost = $\$700S_{F1\text{-}C1} + \$900S_{F1\text{-}C2} + \$800S_{F1\text{-}C3} + \$800S_{F2\text{-}C1} + \$900S_{F2\text{-}C2} + \$700S_{F2\text{-}C3}$

subject to

Factory 1: $S_{F1\text{-}C1} + S_{F1\text{-}C2} + S_{F1\text{-}C3} = 12$

Factory 2: $S_{F2\text{-}C1} + S_{F2\text{-}C2} + S_{F2\text{-}C3} = 15$

Customer 1: $S_{F1\text{-}C1} + S_{F2\text{-}C1} = 10$

Customer 2: $S_{F1\text{-}C2} + S_{F2\text{-}C2} = 8$

Customer 3: $S_{F1\text{-}C3} + S_{F2\text{-}C3} = 9$

and

$S_{ij} \geq 0$ (i = F1, F2; j = C1, C2, C3).

Terminology and Notation

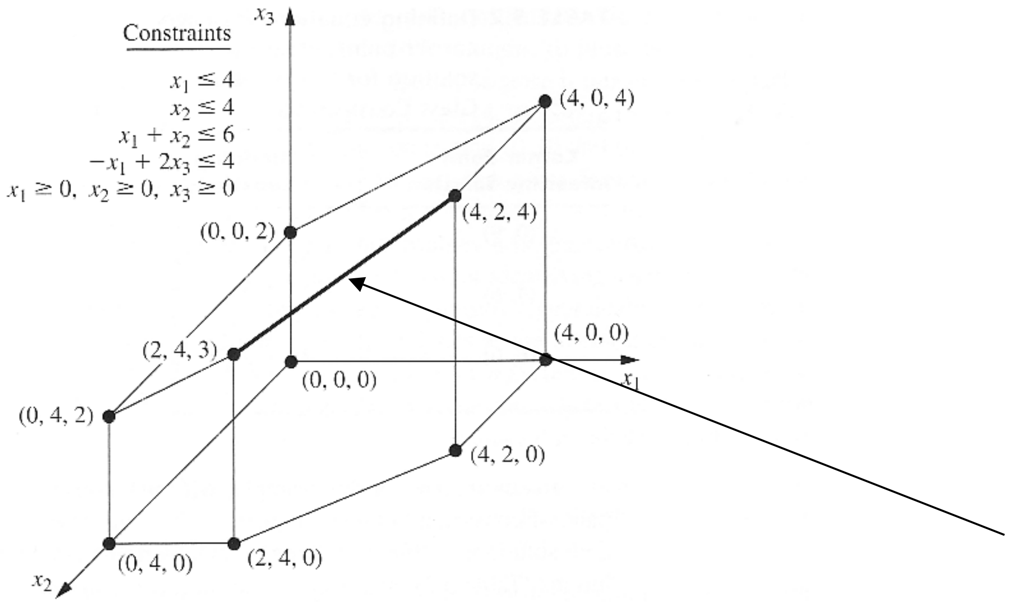
Terminology of solutions in LP model

- Solution – not necessarily the final answer to the problem!!!
- Feasible solution – solution that satisfies all the constraints
- Infeasible solution – solution for which at least one of the constraints is violated
- Feasible region – set of all points that satisfies all the constraints (possible to have a problem without any feasible solutions)
- Binding constraint – the left-hand side and the right-hand side of the constraint are equal, I.e., constraint is satisfied in equality. Otherwise the constraint is nonbinding.
- Optimal solution – feasible solution that has the best value of the objective function.

  Largest value → maximization problems

  Smallest value → minimization problems
- Multiple optimal solutions, no optimal solutions, unbounded Z

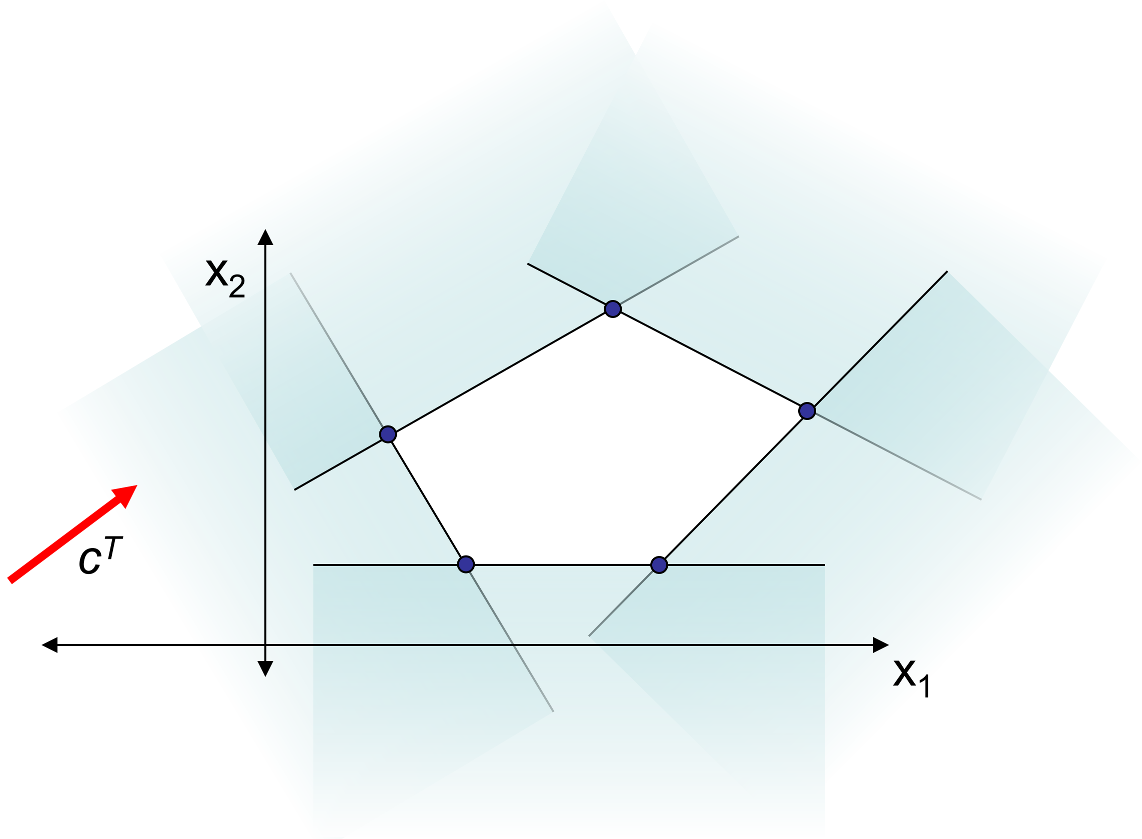
3D feasible region



Solving linear programs

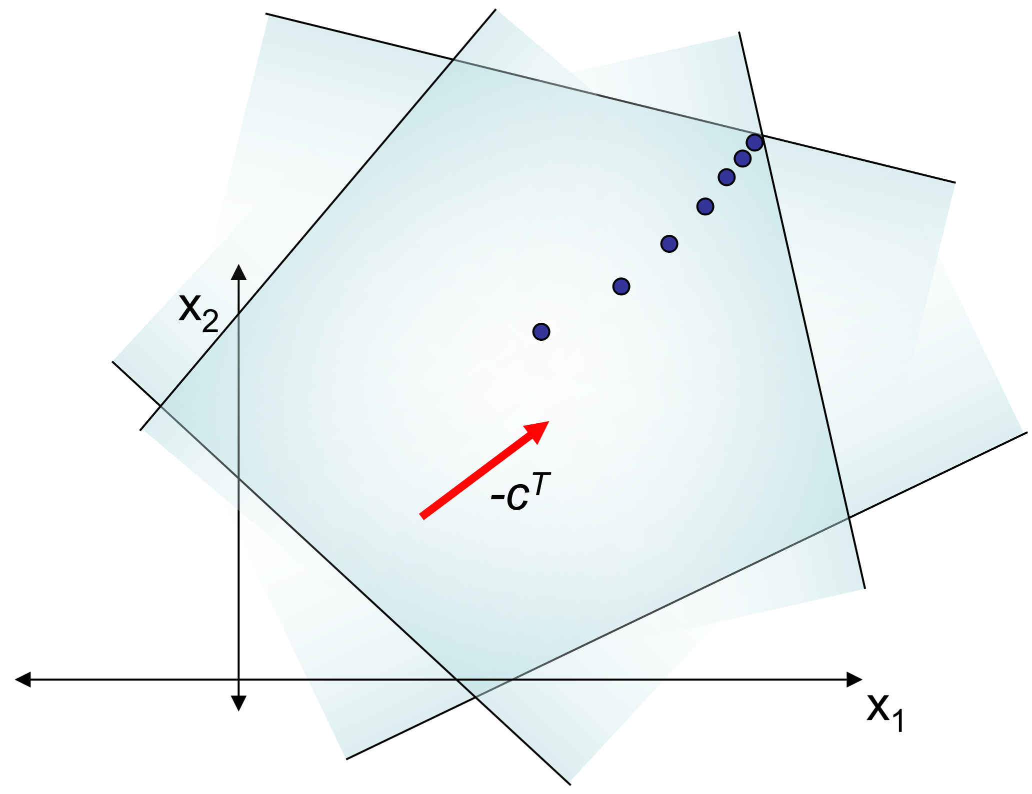
Solution Methods for Linear Programs

- Simplex Method
  - Optimum must be at the intersection of constraints
  - Intersections are easy to find, change inequalities to equalities



Solution Method for Linear Programs

- Interior Point Methods
- Benefits
  - Scales Better than Simplex



Standard form of the LP model

$$\text{minimize} \quad Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

subject to the restrictions

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m,
\end{aligned}$$

$x_i >= 0$, (i =1,2,…,n)

Other variants:

Maximize Z (instead of minimizing Z; but Min Z = - Max -Z)

Some constraints have other signs (=; and >=)

Some variables have unrestricted sign, i.e., they are not subject to the non-negativity constraints

Solving Linear Programs

- To solve LPs, typically need to put them in standard form:

$$\begin{aligned} &\underset{z}{\text{minimize}} \;\; c^T z \\ &\text{subject to} \;\; Az \leq b \end{aligned}$$

  - $z \in \mathbb{R}^n$, $A \in \mathbb{R}^{N_i \times n}$, $b \in \mathbb{R}^{N_i}$

- For absolute loss LP

$$z = \begin{bmatrix} \theta \\ \nu \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} \Phi & -I \\ -\Phi & -I \end{bmatrix}, \quad b = \begin{bmatrix} y \\ -y \end{bmatrix}$$

Is equivalent to

$$\begin{aligned} &\underset{\theta,\nu}{\text{minimize}} \;\; \sum_{i=1}^{m} \nu_i \\ &\text{subject to} \;\; -\nu_i \leq \theta^T \phi(x_i) - y_i \leq \nu_i \end{aligned}$$

Regression with absolute loss

$$\underset{\theta}{\text{minimize}} \sum_{i=1}^{m} |\theta^T \phi(x_i) - y_i| \quad \Longleftrightarrow \quad \begin{array}{ll} \underset{\theta,\nu}{\text{minimize}} & \sum_{i=1}^{m} \nu_i \\ \text{subject to} & -\nu_i \leq \theta^T \phi(x_i) - y_i \leq \nu_i \end{array}$$

- MATLAB code

```
c = [zeros(n,1); ones(m,1)];
A = [Phi -eye(m); -Phi -eye(m)];
b = [y; -y];
z = linprog(c,A,b);
theta = z(1:n)

theta =
    0.0477
   -1.5978
```