# Numerical Optimization Methods in Imaging

**Exercice** .

Let $\mathcal{H}$ and $\mathcal{G}$ be real Hilbert spaces (e.g., $\mathcal{H} = \mathbb{R}^N$ and $\mathcal{G} = \mathbb{R}^M$). Let $f \in \Gamma_0(\mathcal{H})$, let $g \in \Gamma_0(\mathcal{G})$, and let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ (e.g., $L \in \mathbb{R}^{M\times N}$). Let $\widetilde{x} \in \mathcal{H}$ and $\gamma \in \,]0, +\infty[$. We denote by $L^*$ the adjoint of $L$ (e.g., $L^* = L^\top$ in the finite dimensional case).

1. Consider the operator defined as

$$(\forall y \in \mathcal{G}) \quad B(y) = -L\text{prox}_f(\widetilde{x} - L^* y).$$

   Show that $B$ is $\beta_B$-cocoercive where $\beta_B$ will be specified.

2. In the following, we assume that $\gamma \in ]0, 2/\|L\|^2[$. Deduce from the previous question that the operator defined as

$$(\forall y \in \mathcal{G}) \quad T(y) = \text{prox}_{\gamma g^*}(y + \gamma L\text{prox}_f(\widetilde{x} - L^* y))$$

   is $\alpha_T$-averaged where $\alpha_T$ will be specified.

3. Assume that $\text{Fix}T \neq \varnothing$. Consider the iterative algorithm :

$$(\forall n \in \mathbb{N}) \quad y_{n+1} = y_n + \lambda_n (T(y_n) - y_n).$$

   where $y_0 \in \mathcal{G}$. Give a sufficient condition on $(\lambda_n)_{n\in\mathbb{N}}$ for the weak convergence of $(y_n)_{n\in\mathbb{N}}$. What can be said about the limit point ?

4. Show that the fixed points of $T$ are the minimizers of a function to be determined.

5. Let $y \in \text{Fix}T$. Show that $x = \text{prox}_f(\widetilde{x} - L^* y)$ is the solution to a minimization problem.