# Numerical Optimization Methods in Imaging

**Exercice** .

Let $\mathcal{H}$ and $\mathcal{G}$ be real Hilbert spaces (e.g., $\mathcal{H} = \mathbb{R}^N$ and $\mathcal{G} = \mathbb{R}^M$). Let $f \in \Gamma_0(\mathcal{H})$, let $g \in \Gamma_0(\mathcal{G})$, and let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ (e.g., $L \in \mathbb{R}^{M \times N}$). Let $\widetilde{x} \in \mathcal{H}$ and $\gamma \in \,]0, +\infty[$. We denote by $L^*$ the adjoint of $L$ (e.g., $L^* = L^\top$ in the finite dimensional case).

1. Consider the operator defined as

$$(\forall y \in \mathcal{G}) \quad B(y) = -L\text{prox}_f(\widetilde{x} - L^* y).$$

Show that $B$ is $\beta_B$-cocoercive where $\beta_B$ will be specified.

2. In the following, we assume that $\gamma \in ]0, 2/\|L\|^2[$. Deduce from the previous question that the operator defined as

$$(\forall y \in \mathcal{G}) \quad T(y) = \text{prox}_{\gamma g^*}(y + \gamma L\text{prox}_f(\widetilde{x} - L^* y))$$

is $\alpha_T$-averaged where $\alpha_T$ will be specified.

3. Assume that $\text{Fix}T \neq \varnothing$. Consider the iterative algorithm :

$$(\forall n \in \mathbb{N}) \quad y_{n+1} = y_n + \lambda_n(T(y_n) - y_n).$$

where $y_0 \in \mathcal{G}$. Give a sufficient condition on $(\lambda_n)_{n\in\mathbb{N}}$ for the weak convergence of $(y_n)_{n\in\mathbb{N}}$. What can be said about the limit point ?

**SOLUTION**

1. We know that $\operatorname{prox}_f$ is firmly nonexpansive, that is
$$(\forall (x,x')\in\mathcal{H}^2)\quad \langle \operatorname{prox}_f(x)-\operatorname{prox}_f(x')\mid x-x'\rangle \geq \|\operatorname{prox}_f(x)-\operatorname{prox}_f(x')\|^2.$$
We deduce that
$$(\forall (x,x')\in\mathcal{H}^2)\quad \langle \operatorname{prox}_f(\widetilde{x}-x)-\operatorname{prox}_f(\widetilde{x}-x')\mid x'-x\rangle$$
$$\geq \|\operatorname{prox}_f(\widetilde{x}-x)-\operatorname{prox}_f(\widetilde{x}-x')\|^2.$$
This shows that $C=-\operatorname{prox}_f(\widetilde{x}-\cdot)$ is firmly nonexpansive, that is $\beta_C$-cocoercive with $\beta_C=1$. From the property given in the lecture, $B=L\circ C\circ L^*$ is thus $\beta_B$-cocoercive with
$$\beta_B=\frac{\beta_C}{\|L\|^2}=\frac{1}{\|L\|^2}.$$

2. We have $T=\operatorname{prox}_{\gamma g^*}\circ(\mathrm{Id}-\gamma B)$. Since $B$ is $\beta_B$-cocoercive, there exists a firmly nonexpansive operator $D\colon\mathcal{G}\to\mathcal{G}$ such that
$$B=\frac{1}{\beta_B}D,$$
which means that there exists a nonexpansive operator $R\colon\mathcal{G}\to\mathcal{G}$ such that
$$B=\frac{1}{2\beta_B}(\mathrm{Id}+R).$$
We have thus
$$\mathrm{Id}-\ \gamma B=\left(1-\frac{\gamma}{2\beta_B}\right)\mathrm{Id}+\frac{\gamma}{2\beta_B}(-R).$$
Since $-R$ is nonexpansive, we deduce that, if $\gamma/(2\beta_B)\in]0,1[$, then $\mathrm{Id}-\ \gamma B$ is $\alpha_B$-averaged with
$$\alpha_B=\frac{\gamma}{2\beta_B}.$$
The condition on $\gamma$ is equivalent to $\gamma\in]0,2/\|L\|^2[$.
Since $\operatorname{prox}_{\gamma g^*}$ is firmly nonexpansive, that is 1/2-averaged, we

deduce from the rule of composition of $\alpha$-averaged operators that $T$ is $\alpha_T$-averaged with

$$\alpha_T = \frac{1}{1 + \left(\frac{1/2}{1-1/2} + \frac{\alpha_B}{1-\alpha_B}\right)^{-1}} = \frac{\frac{1}{2} + \alpha_B - \frac{2}{2}\alpha_B}{1 - \frac{\alpha_B}{2}} = \frac{1}{2 - \alpha_B} \in ]1/2, 1[.$$

3. We recognize a Krasnoselskii-Mann iteration. Therefore $(y_n)_{n\in\mathbb{N}}$ converges weakly to a fixed point of $T$ if $\{\lambda_n\}_{n\in\mathbb{N}} \subset [0, 1/\alpha_T]$ and $\sum_{n=0}^{+\infty} \lambda_n(1 - \alpha_T\lambda_n) = +\infty$. Since $\alpha_T < 1$, the condition is satisfied when, for every $n \in \mathbb{N}$, $\lambda_n = 1$.