1 Introduction

We consider two-dimensional correlated Brownian motion with absorbing boundaries:

$$X(t) = x_0 + \mu_x t + \sigma_x W_x(t) \qquad a_x < X(t) < b_x \tag{1}$$
$$Y(t) = y_0 + \mu_y t + \sigma_y W_y(t) \qquad a_y < Y(t) < b_y \tag{2}$$

where W_i are standard Brownian motions with $\text{Cov}(W_1(t), W_2(t)) = \rho t$ for $0 < t' \leq t$. In particular, we find the joint transition density function for $(X(t), Y(t))$ under the boundary restrictions:

$$p\left(X(t) = x, Y(t) = y | a_x < X(t') < b_x, a_y < Y(t') < b_y, 0 < t' \leq t, X(0) = x_0, Y(0) = y_0\right).$$

This density, which we shorten to $p(x, y, t)$ from now on, is the solution to the Fokker-Planck equation [Øksendal, 2003]:

$$\frac{\partial p}{\partial t} = -\mu_x \frac{\partial p}{\partial x} - \mu_y \frac{\partial p}{\partial y} + \frac{1}{2}\sigma_x^2 \frac{\partial^2 p}{\partial x^2} + \rho \sigma_x \sigma_y \frac{\partial^2 p}{\partial x \partial y} + \frac{1}{2}\sigma_y^2 \frac{\partial^2 p}{\partial y^2} \tag{3}$$
$$p(a_x, y \tag{4}$$

where W_i are standard Brownian motions with $\text{Cov}(W_1(t), W_2(t)) = \rho t$.

The solution of the Fokker-Plank equation corresponding to (3) - (4) is the transition density of a particle starting at $(X_1(0), X_2(0))$, terminating at $(X_1(T), X_2(T))$, while never leaving the region Ω defined by the rectangle $(a_1, b_1)x(a_2, b_2)$. Differentiating the solution with respect to the boundaries produces the transition density of a particle beginning and ending at the points $(X_1(0), X_2(0))$ and $(X_1(T), X_2(T))$ respectively, while attaining the minima a_1/a_2 and maxima b_1/b_2 in each coordinate axis.

The transition density for the considered system has been used in computing first passage times [Kou et al., 2016, Sacerdote et al., 2012], with application to structural models in credit risk and default correlations [Haworth et al., 2008, Ching et al., 2014]. [He et al., 1998] use variants of the differentiated solutions with respect to some of the boundaries to price financial derivative instruments whose payoff depends on observed maxima/minima.

Closed-form solutions to (3) - (4) are available for some parameter regimes. When $\rho = 0$, the transition density of the process can be obtained with a Fourier expansion. When $a_1 = -\infty$ and $b_1 = \infty$, the method of images can be used to enforce the remaining boundaries. For either $a_1, a_2 = -\infty$ or $b_1, b_2 = \infty$, the Fokker-Plank equation is a Sturm-Liouville problem in radial coordinates. Both of these techniques are used by He et al. [1998]. However, to the best of our knowledge, there is no closed-form solution to the general problem in (3) - (4).

It is still possible to approach the general problem by proposing a Fourier expansions. However, a drawback of this out-of-the-box solution is that the system matrix for the corresponding eigenvalue problem is large and dense. An alternative is to use a finite difference scheme. However, discretization of the initial condition introduces a numerical bias in the estimation procedure.

In this paper, we propose a solution to the general problem (3) - (4) which is obtained by combining a small-time analytic solution with a finite-element method.

ADD OUT APPLICATION (ESTIMATION)

2 Numerical Method

2.1 Fourier Expansion

The formal Fourier expansion for the problem is

$$f(x, y|t) = \lim_{K \to \infty} \sum_{k=1}^{K} c_k(t) sin(2\pi kx/L).$$

Plugging into the PDE yields an eigenvalue problem which is slow to solve.

2.2 Finite Difference

2.3 Finite Element

References

Wai-Ki Ching, Jia-Wen Gu, and Harry Zheng. On correlated defaults and incomplete information. *arXiv preprint arXiv:1409.1393*, 2014.

Helen Haworth, Christoph Reisinger, and William Shaw. Modelling bonds and credit default swaps using a structural model with contagion. *Quantitative Finance*, 8(7):669–680, 2008.

Hua He, William P Keirstead, and Joachim Rebholz. Double lookbacks. *Mathematical Finance*, 8(3):201–228, 1998.

Steven Kou, Haowen Zhong, et al. First-passage times of two-dimensional brownian motion. *Advances in Applied Probability*, 48(4):1045–1060, 2016.

Bernt Øksendal. Stochastic differential equations. In *Stochastic differential equations*, pages 65–84. Springer, 2003.

Laura Sacerdote, Massimiliano Tamborrino, and Cristina Zucca. First passage times of two-dimensional correlated diffusion processes: analytical and numerical methods. *arXiv preprint arXiv:1212.5287*, 2012.