1 Introduction

We consider two-dimensional correlated Brownian motion with absorbing boundaries:

$$X(t) = x_0 + \mu_x t + \sigma_x W_x(t) \qquad a_x < X(t) < b_x \tag{1}$$
$$Y(t) = y_0 + \mu_y t + \sigma_y W_y(t) \qquad a_y < Y(t) < b_y \tag{2}$$

where W_i are standard Brownian motions with $\mathrm{Cov}(W_1(t), W_2(t)) = \rho t$ for $0 < t' \le t$. In particular, we find the joint transition density function for $(X(t), Y(t))$ under the boundary conditions:

$$p\left(X(t) = x, Y(t) = y | a_x < X(t') < b_x, a_y < Y(t') < b_y, 0 < t' \le t, X(0) = x_0, Y(0) = y_0\right). \tag{3}$$

This density, which we shorten to $p(x,y,t)$ from now on, is the solution to the Fokker-Planck equation [**?**]:

$$\frac{\partial}{\partial t}p(x,y,t') = -\mu_x\frac{\partial}{\partial x}p(x,y,t') - \mu_y\frac{\partial}{\partial y}p(x,y,t') + \frac{1}{2}\sigma_x^2\frac{\partial^2}{\partial x^2}p(x,y,t') + \rho\sigma_x\sigma_y\frac{\partial^2}{\partial x\partial y}p(x,y,t') + \frac{1}{2}\sigma_y^2\frac{\partial^2}{\partial y^2}p(x,y,t'), \tag{4}$$
$$p(a_x,y,t') = p(b_x,y,t') = p(x,a_y,t') = p(x,b_y,t') = 0 \qquad 0 < t' \le t. \tag{5}$$

Differentiating $p(x,y,t)$ with respect to the boundaries produces the transition density of a particle beginning and ending at the points $(X_1(0), X_2(0))$ and $(X_1(t), X_2(t))$ respectively, while attaining the minima a_x/a_y and maxima b_x/b_y in each coordinate direction:

$$\frac{\partial^4}{\partial a_x \partial b_x \partial a_y \partial b_y}p(x,y,t) =$$
$$p\left(X(t) = x, Y(t) = y \,\middle|\, \min_{t'} X(t') = a_x, \max_{t'} X(t') = b_x, \min_{t'} Y(t') = a_y, \max_{t'} Y(t') = b_y, 0 < t' \le t, X(0) = x_0, Y(0) = y_0\right). \tag{6}$$

The transition density for the considered system has been used in computing first passage times [**??**], with application to structural models in credit risk and default correlations [**??**]. [**?**] use variants of the differentiated solutions with respect to some of the boundaries to price financial derivative instruments whose payoff depends on observed maxima/minima.

Closed-form solutions to (4) - (5) are available for some parameter regimes. When $\rho = 0$, the transition density of the process can be obtained with a Fourier expansion. When $a_1 = -\infty$ and $b_1 = \infty$, the method of images can be used to enforce the remaining boundaries. For either $a_1, a_2 = -\infty$ or $b_1, b_2 = \infty$, the Fokker-Plank equation is a Sturm-Liouville problem in radial coordinates. Both of these techniques are used by **?**. However, to the best of our knowledge, there is no closed-form solution to the general problem in (4) - (5).

It is still possible to approach the general problem by proposing a Fourier expansions. However, a drawback of this out-of-the-box solution is that the system matrix for the corresponding eigenvalue problem is large and dense. An alternative is to use a finite difference scheme. However, discretization of the initial condition introduces a numerical bias in the estimation procedure.

In this paper, we propose a solution to the general problem (4) - (5) which is obtained by combining a small-time analytic solution with a finite-element method.

ADD OUT APPLICATION (ESTIMATION)

2 Numerical Method

2.1 Fourier Expansion

The formal Fourier expansion for the problem is

$$f(x,y|t) = \lim_{K\to\infty}\sum_{k=1}^{K} c_k(t) sin(2\pi kx/L).$$

Plugging into the PDE yields an eigenvalue problem which is slow to solve.

2.2 Finite Difference

2.3 Finite Element

References