1 Introduction

We consider two-dimensional correlated Brownian motion with absorbing boundaries:

$$X(t) = x_0 + \mu_x t + \sigma_x W_x(t) \qquad a_x < X(t) < b_x \tag{1}$$
$$Y(t) = y_0 + \mu_y t + \sigma_y W_y(t) \qquad a_y < Y(t) < b_y \tag{2}$$

where W_i are standard Brownian motions with $\text{Cov}(W_1(t), W_2(t)) = \rho t$ for $0 < t' \leq t$. In particular, we find the joint transition density function for $(X(t), Y(t))$ under the boundary conditions:

$$\Pr\left(X(t) \in dx, Y(t) \in dy, a_x < X(t') < b_x, a_y < Y(t') < b_y | 0 < t' \leq t, X(0) = x_0, Y(0) = y_0, \theta\right), \tag{3}$$

with $\theta := (\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$. This function, which we shorten to $q(x,y,t)$ from now on, is the solution to the Fokker-Planck equation [Oksendal, 2013]:

$$\frac{\partial}{\partial t} q(x,y,t') = -\mu_x \frac{\partial}{\partial x} q(x,y,t') - \mu_y \frac{\partial}{\partial y} q(x,y,t') + \frac{1}{2}\sigma_x^2 \frac{\partial^2}{\partial x^2} q(x,y,t') + \rho\sigma_x\sigma_y \frac{\partial^2}{\partial x \partial y} q(x,y,t') + \frac{1}{2}\sigma_y^2 \frac{\partial^2}{\partial y^2} q(x,y,t'), \tag{4}$$

$$q(a_x,y,t') = q(b_x,y,t') = q(x,a_y,t') = q(x,b_y,t') = 0, \qquad 0 < t' \leq t. \tag{5}$$

Differentiating $q(x,y,t)$ with respect to the boundaries produces the transition density of a particle beginning and ending at the points $(X_1(0), X_2(0))$ and $(X_1(t), X_2(t))$, respectively, while attaining the minima a_x/a_y and maxima b_x/b_y in each coordinate direction:

$$\frac{\partial^4}{\partial a_x \partial b_x \partial a_y \partial b_y} q(x,y,t) =$$

$$\Pr\left(X(t) \in dx, Y(t) \in dy, \min_{t'} X(t') = a_x, \max_{t'} X(t') = b_x, \min_{t'} Y(t') = a_y, \max_{t'} Y(t') = b_y \,\middle|\, 0 < t' \leq t, X(0) = x_0, Y(0) = y_0, \theta\right). \tag{6}$$

The transition density (3) with less than 4 boundaries has been used in computing first passage times [Kou et al., 2016, Sacerdote et al., 2016], with application to structural models in credit risk and default correlations [Haworth et al., 2008, Ching et al., 2014]. He et al. [1998] use variants of (6) with respect to some of the boundaries to price financial derivative instruments whose payoff depends on **some** of the observed maxima/minima.

Closed-form solutions to (4) - (5) are available for some parameter regimes. When $\rho = 0$, the transition density of the process is the solution to a well-understood Sturm-Liouville problem where the eigenfunctions of the differential operator are sine functions. When $a_1 = -\infty$ and $b_1 = \infty$, the method of images can be used to enforce the remaining boundaries. For either $a_1, a_2 = -\infty$ or $b_1, b_2 = \infty$, eigenfunction of the Fokker-Plank equation can be found in radial coordinates. Both of these techniques are used and detailed by He et al. [1998]. However, to the best of our knowledge, there is no closed-form solution to the general problem in (4) - (5). This also limits the available ways to compute (6), with the most straightforward approach being finite difference with respect to the boundary conditions. This, however, requires one to solve at least 16 eigenvalue problems to evaluate the density function for a single observation, motivating the need for an efficient numerical method to solve (4) - (5).

At some point you need to explain why you say this (i.e., you need to explicitly say that because a_1, a_2, b_1, b_2 appear in the boundaries there is no closed-form solution to the derivate even if you have an explicit representation for the solution. The right place might not be here though.

It is still possible to approach the general problem by proposing a biorthogonal expansion in time and space (Risken [1989], sections 6.2), where the eigenfunctions for the differential operator are approximated as sinusoidal series satisfying the boundary conditions. However, a drawback of this out-of-the-box solution is that the system matrix for the corresponding eigenvalue problem is large and dense. An alternative is to use a finite difference scheme to directly solve the evolution problem after suitable transformations. However, both of these methods need a high degree of numerical resolution to produce practically useful

Not only that, but also that for even moderate \rho you need a large number of basis functions to get a good approximation. Given that the matrix is dense, this means that accurate solutions for moderate \rho become very slow, or even unfeasible.

approximations of the transition density. We conjecture that these inefficiencies come from either using a *separable* representation for the differential operator (trigonometric series) or introducing numerical diffusion (finite difference).

In this paper, we propose a solution to the general problem (4) - (5) which is obtained by combining a small-time analytic solution with a finite-element method. Our method directly takes into account the correlation parameter present in the differential operator in order to efficiently represent the analytic small-time solution and propagate it forward in time. We apply our computational method to estimate equation parameters with a maximum likelihood approach in settings where the model assumptions of constant $(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$ and Brownian motion driving stochastic evolution are appropriate.

The remainder of the paper is organized as follows: .....

2 Approximate Numerical Solutions

Before considering any solutions to the full Fokker-Planck equation (4) - (5), we simplify the PDE by proposing an exponential decomposition of the solution and using the fact that parameters $(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$ are constant:

$$q(x,y,t) = \exp(\alpha x + \beta y + \gamma t) p(x,y,t).$$

Include the scaling in the definition of p(x,y,t) (and write p(x,y,t) = ... instead of q(x,y,t) = ...

Can you give the explicit transformation here and just say that if you define p(x,y) in this way the form of the differential equation reduces to (7)? Then you can say that all the methods you describe focus on the normalized problem.

Also, I do not understand notationwise why you need to introduce \xi, \eta and \tau. You should be able to just use x, y and t.

We can find α, β and γ, as well as a suitable scaling transformation, such that $p(\xi,\eta,\tau)$ satisfies the diffusion equation:

$$\begin{aligned} \frac{\partial}{\partial \tau} p(\xi,\eta,\tau') &= \frac{1}{2}\sigma_\xi^2 \frac{\partial^2}{\partial \xi^2} p(\xi,\eta,\tau') + \rho\sigma_\xi\sigma_\eta \frac{\partial^2}{\partial \xi \partial \eta} p(\xi,\eta,\tau') + \frac{1}{2}\sigma_\eta^2 \frac{\partial^2}{\partial \eta^2} p(\xi,\eta,\tau'), && (7) \\ &:= \mathcal{L} p(\xi,\eta,\tau), && (8) \\ p(\xi,\eta,\tau) &= 0 && \text{for } (\xi,\eta) \in [0,1]\times[0,1], \\ p(\xi,\eta,0) &= \delta(\xi-\xi_0)\delta(\eta-\eta_0) \end{aligned}$$

on the unit square. The transformations $\tilde{\theta} \to \theta$ as well as $(\xi,\eta) \to (x,y)$ allow us to go from $p(\xi,\eta,\tau)$ to $p(x,y,t)$ without trouble. Note here that under this transformation ρ remains the same as in the original coordinate frame. We will call equation (7) the *normalized* problem and will consider its solution without loss of generality.

2.1 Eigenfunction Expansion

Following Section 6.2 of Risken [1989], we may use the biorthogonal decomposition of the solution as a sum of eigenfunctions and time-dependent coefficients determined by eigenvalues:

$$p(\xi,\eta,\tau) = \phi_\nu(\xi,\eta) e^{-\lambda_\nu \tau}, \tag{9}$$

\mathcal{L}

where the eigenfunctions $\phi_\nu(\xi,\eta)$ satisfy the boundary conditions. Because the differential operator ~~in the normalized problem (7)~~ is self-adjoint [PROVE], the family of eigenfunctions is complete in the Hilbert space L^2 [CITE]. Moreover, the eigenvalues are bounded below by 0, so that the solution behaves as expected (see section 6.3 of Risken [1989]).

Same here, why \xi, \eta and \tau?. You should be able to just use x, y and t.

Since we require $\phi_\nu(\xi,\eta)$ to be zero on the boundaries, we may represent the eigenfunction as a linear combination of sines

$$\phi_\nu(\xi,\eta) = \sum_{l=0}^{L}\sum_{m=0}^{M} c_{l,m,\nu} \sin(2\pi l \xi)\sin(2\pi m \eta) := \Psi(\xi,\eta)^T c_\nu,$$

where we have truncated the infinite series for some suitably large L and M and defined

$$\begin{aligned} \psi_{l,m}(\xi,\eta) &= \sin(2\pi l\xi)\sin(2\pi m\eta), \\ \Psi(\xi,\eta) &= (\psi_{0,0}(\xi,\eta), \ldots, \psi_{L,M}(\xi,\eta))^T, \\ c_\nu &= (c_{0,0,\nu}, \ldots, c_{L,M,\nu})^T. \end{aligned}$$

Isn't this a matrix?

The biorthogonal representation (9) leads to the eigenvalue problem

$$\mathcal{L}\phi_\nu = -\lambda_\nu \phi_\nu, \tag{10}$$

where $\mathcal{L}$ is the differential operator in the normalized Fokker-Planck equation. Applying $\mathcal{L}$ to ϕ_ν produces the linear system

$$\mathcal{L}\phi_\nu = \mathcal{L}(\Psi(\xi,\eta)^T c_\nu) = \mathcal{L}(\Psi(\xi,\eta)^T)c_\nu = (A\Psi(\xi,\eta))^T c_\nu,$$

where A is a constant matrix dependent on $\tilde{\theta}$. In the case where $\rho = 0$, A is diagonal because $\{\psi_{l,m}(\xi,\eta)\}_{l,m}$ are the eigenfunctions to $\mathcal{L}$. When $\rho \neq 0$, A is no longer diagonal and is in fact dense. This caused by the mixing terms

$$\frac{\partial^2}{\partial\xi\partial\eta}\sin(2\pi l\xi)\sin(2\pi m\eta) = (2\pi l)(2\pi m)\cos(2\pi l\xi)\cos(2\pi m\eta)$$

being the products of cosine functions, which have an inefficient sine series representation [CITE]. Substituting the linear representation of $\mathcal{L}\phi_\nu$ into the eigenvalue problem (10), we arrive to the system

$$\Psi(\xi,\eta)^T A^T c_\nu = -\lambda_\nu \Psi(\xi,\eta)^T c_\nu \quad \Leftrightarrow \quad A^T c_\nu = -\lambda_\nu c_\nu$$

whose solution gives the family of orthonormal eigenfunctions. As mentioned already, the efficiency of this approach is dependent on the cost of solving the eigenvalue problem $A^T c_\nu = -\lambda_\nu c_\nu$.

Maybe we want to say something here about how this approximate solution is a function of a_1, a_2, b_1 and b_2 (the c_\nu are a complex function of them whose analytical form is not available), and therefore getting the desity we are interested in requires that the derivative in (6) needs to be numerically approximated.

2.2 Finite Difference

A finite difference method used to solve the problem (7) defines an approximate solution over some grid of points $\{(\xi_l,\eta_m)\}_{l=0,m=0}^{L,M}$ over $[0,1]\times[0,1]$:

$$q(\xi,\eta,\tau) \approx \sum_l \sum_m c_{l,m}(\tau)\delta(\xi-\xi_l)\delta(\eta-\eta_m) = \Delta(\xi,\eta)^T c(t), \tag{11}$$

where $c(\tau) = (c_{0,0}(\tau),\ldots,c_{L,M}(\tau)^T$ and $\Delta(\xi,\eta) = (\delta(\xi-\xi_0)\delta(\eta-\eta_0),\ldots,\delta(\xi-\xi_L)\delta(\eta-\eta_M))$, where we have once again separated the spatial and temporal components of the problem as in the previous section. This is a suitable choice, because the differential operator $\mathcal{L}$ is linear and constant and there is therefore no need to perform approximation in time, i.e. we take derivative with respect to time directly:

The overall flow of the explanation in 2.2 is a bit clunky, but I have no good suggestion on how to improve it.

$$\frac{\partial}{\partial\tau}q(\xi,\eta,\tau) \approx \Delta(\xi,\eta)^T \frac{\partial c(\tau)}{\partial\tau}$$

Isn't this repetitive?

The differential operator $\mathcal{L}$ for the representation in equation (11) is approximated with a finite difference operator $\mathcal{L}_{FD}$ such that approximate derivatives are defined on the grid:

$$\mathcal{L}q(\xi,\eta,\tau) \approx \mathcal{L}_{FD}q(\xi,\eta,\tau) = \sum_l\sum_m f(c_{l,m}(\tau))\delta(\xi-\xi_l)\delta(\eta-\eta_m) := \Delta(\xi,\eta)^T f(c(t))$$

where $f(c_{l,m}(\tau))$ is a function of some neighboring coefficient values at (ξ_l,η_m).

For a central difference scheme on a regular $N\times N$ grid aligned with the boundaries (with step size $h = 1/(N-1)$),

$$\frac{\partial^2}{\partial\xi^2}q(\xi_l,\eta_m,\tau) \approx \frac{c_{l+1,m}(\tau) - 2c_{l,m}(\tau) + c_{l-1,m}(\tau)}{h^2} = \frac{1}{h^2}A_{l,m,\xi^2}c(\tau),$$

where A_{l,m,ξ^2} is some all-zero row vector except for three entries of 1 corresponding to the grid points $(\xi_{l-1},\eta_m),(\xi_l,\eta_m),(\xi_{l+1},\eta_m)$. For the mixing term, the approximation is

$$\frac{\partial^2}{\partial\xi\partial\eta}q(\xi_l,\eta_m,\tau) \approx \frac{c_{l+1,m+1}(\tau) - c_{l+1,m-1}(\tau) - c_{l-1,m+1}(\tau) + c_{l-1,m-1}(\tau)}{4h^2} = \frac{1}{4h^2}A_{l,m,\xi\eta}c(\tau).$$

The finite difference approximation of $\mathcal{L}$ can be written as a linear transformation of $c(\tau)$:

$$\mathcal{L}q(\xi,\eta,\tau) \approx \Delta^T(\xi,\eta) \underbrace{\left(\frac{1}{2}\sigma_\xi^2\frac{1}{h^2}A_{\xi^2} + \rho\sigma_\xi\sigma_\eta A_{\xi\eta} + \frac{1}{2}\sigma_\xi^2\frac{1}{h^2}A_{\eta^2}\right)}_{A} c(\tau),$$

where we have composed the row vectors for the different derivative terms as matrices $A_{\xi^2}, A_{\xi\eta}$, and A_{η^2}. The system of differential equations for $c(\tau)$ is therefore completely determined by our choice of step size h, as well as the parameter values $(\sigma_\xi, \sigma_\eta, \rho)$:

$$\begin{aligned} &\frac{\partial c(\tau)}{\partial \tau} = Ac(\tau) \\ \Rightarrow &c(\tau) = \exp(A\tau)\, c(0) \end{aligned} \tag{12}$$

Single line

For non-small τ, we must find the eigenvalue decomposition of A in order to solve the evolution problem. It should be noted here that a regular grid approach with a constant h is appealing, because it allows us to construct once and store the matrices $A_{\xi^2}, A_{\eta^2}, A_{\xi\eta}$, which saves valuable computational resources if we are to solve the finite difference eigenproblem (12) repeatedly for different parameter values $\tilde{\theta}$.

Unlike the system matrix for the trigonometric expansion, the system matrix here is sparse, as was demonstrated for A_{ξ^2} explicitly. The eigenvalue problem is therefore much cheaper to solve. The system matrix can be made even sparser on a regular grid by performing a 45° rotation which removes the mixing term from the problem PDE and preserves the boundaries.

However, the fundamental limitation of using a finite difference method is that differentiation with respect to boundaries is also done using finite differences and it is of fourth order. Given this high order differentiation, h cannot be made too small because roundoff error becomes an issue relatively quickly. [**this needs to be a bit more clear. Why isn't this an issue with a smooth approximation?**] This naturally occurring lower bound on h introduces yet another practical concern: on a regular grid, the delta function initial condition at (ξ_0, η_0) needs to be either rounded to the nearest grid point or represented as a weighted sum of delta functions on the four nearest grid points. Either approach introduces a numerical diffusion into the problem which, for a finite h, can bias the numerical solution. [**for which we have results**]

Yes, this needs an explanation at the end of the previous section.

We will illustrate this issue in Section XXXX

2.3 Finite Element Method

Our contribution in this paper is to propose ....

~~We propose~~ a numerical method which 1) maintains a functional approximation of the differential operator $\mathcal{L}$ while 2) imposing a computational burded comparable to or better than that of the finite difference approach. Further, our method explicitly minimizes the error associated with using a finite, smooth representation of the initial condition $\delta(\xi-\xi_0)\delta(\eta-\eta_0)$.

As such, our approach consists of two parts

i) a small-time analytic solution $q(x,y,\tau_\varepsilon)$ for the IC/BC problem,

ii) a family of orthonormal basis functions which represent $q(x,y,\tau_\varepsilon)$ parsimoniously.

By combining *i*) and *ii*), we can efficiently find a weak solution to the PDE (7) via the finite element method [Shaidurov, 2013]. Convergence of our method to the strong solution under the $L^2(\bar{\Omega})$ norm is guaranteed as long as the family of basis functions we propose is complete in the Hilbert space under $L^2(\bar{\Omega})$ [Salsa, 2016].

Could be said later once you have described the method.

The small-time solution is derived by considering the fundamental solution $G(\xi,\eta|\tau,\xi_0,\eta_0)$ for the unbounded problem in (7), which is the bivariate Gaussian density with mean and covariance determined by the initial condition and the diffusion parameters [Stakgold and Holst, 2011]. We can then find a small enough t_ε such that $G(\xi,\eta|\tau,\xi_0,\eta_0)$ is numerically zero on three of the four boundaries of $\bar{\Omega}$. The zero-condition on the remaining boundary is enforced by suitably moving the source term for the fundamental solution to some suitable (ξ_0', η_0'). The small-time solution therefore takes on the analytic form

$$q(\xi,\eta,\tau_\varepsilon) = G(\xi,\eta|\tau_\varepsilon,\xi_0,\eta_0) - G(\xi,\eta|\tau_\varepsilon,\xi_0',\eta_0').$$

The construction of the orthonormal basis functions is also motivated by the fundadmental solution for the unbounded problem (7): before performing Gram-Schmidt orthogonalization, the finite family of basis functions $\{\tilde{\psi}_k(x,y|x_k,y_k,\rho,\sigma)\}_k^K$ are of the form

Explicitly write the density of the bivariate normal

$$\tilde{\psi}_k(x,y|x_k,y_k,\rho,\sigma) = N\left((x,y)^T \middle| (x_k,y_k)^T, \begin{pmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{pmatrix}\right) x(1-x)y(1-y).$$

Essentially, the collection $\{\tilde{\psi}_k(x,y|x_k,y_k,\rho,\sigma)\}_k^K$ is composed of fundamental solutions to a heat diffusion problem tuned by σ and ρ, tapered such that their support is on $\bar{\Omega}$, and centered along some grid over $\bar{\Omega}$ and are still smooth.

The advantage of these elements is that they better resolve the fundamental solution for the unbounded problem by taking into account ρ in the covariance of each kernel [**if not prove, demonstrate**]. By performing Gram-Schmidt orthogonalization under the $L^2(\Omega)$ norm, we arrive at a family of orthonormal functions $\{\psi_k(x,y|x_k,y_k,\rho,\sigma)\}_k^K$ which can better resolve small-time solutions having a large correlation coefficient. [This needs a lot more work].

3 Estimation

I would start with something like: "Consider now the problem of estimating the parameters of the Brownian motion from an iid sample Z_1, Z_2, \ldots, Z_n$ where $Z_i = (X_i, Y_i, a_1i, a_2i, b_1i, b_2i)$. Application (why this is important). Maximum likelihood estimation is a natural approach to accomplish this." Since we do not have a simple closed form expression for the likelihood we will use an iterative maximization algorithm, which requires repeated evaluation of the likelihood. For moderate to large sample, this is feasible because our algorithm to evaluate the likelihood is fast.

Our numerical method is specifically designed for computational efficiency for repeatedly evaluating the density function (6). The method is therefore particularly suited for performing maximum likelihood estimation of parameter values with respect to models involving systems of the type (1) - (2). In such a scenario, the maximum likelihood estimator (MLE) for the true parameters θ are those which maximize the (log-)likelihood function conditional on the observed data. MLEs are especially useful in practical settings when they exhibit *consistency*, *i.e.*: the MLEs gets closer to the true parameter θ as more data is collected and included in the likelihood (assuming the model and its parameters remain constant).

In this section, we prove that the MLE based on the **approximate** (weak) finite element solution to the governing Fokker-Planck equation (4) is consistent. We define X as the random variable sampled from the distribution corresponding to the true density (6):

z $X \sim F(\theta)$,

where we have explicitly made the density F dependent on the model parameters θ.

For a family of basis functions $\{\tilde{\psi}_l(x,y|x_k,y_k,\rho,\sigma)\}_{l=0}^k$, let $q_k(x,y,t|\theta)$ be the weak solution to the governing Fokker-Planck equation (4) - (5). The approximation to the true pdf is then

$$\frac{\partial^4}{\partial a_x \partial b_x \partial a_y \partial b_y} q_k(x,y,t).$$

The distribution corresponding to this density is denoted as $F_k(\theta)$. Let X_k be the random variable corresponding to this distribution:

$$X_k \sim F_k(\theta).$$

We prove Lemma 1, which states that the distribution of random variables sampled according to the approximate solution converge in density to random variables sampled from the true solution:

Lemma 1. *For X_k and X defined above, $X_k \xrightarrow{d} X$ as $k \to \infty$.*

Proof. **do not read below; I'm working on it tonight** The weak solution $q_k(x,y,t)$ has the property that for any $C^\infty(\Omega)$ function $v \in L^2(\Omega)$,

$$\lim_{k\to\infty}(q_k, v) = (q, v),$$

where $(v,u) := \int_{a_x}^{b_x}\int_{a_y}^{b_y} v(x,y,t)u(x,y,t)dxdy$. and q is the strong solution.

Recalling equation (3), the strong solution has the probabilistic interpretation

$$\begin{aligned} q(x,y,t) &= \Pr\left(X(t) \in dx, Y(t) \in dy, a_x < X(t') < b_x, a_y < Y(t') < b_y | 0 < t' \leq t, X(0) = x_0, Y(0) = y_0, \theta\right) \\ &= \Pr\left(X(t) \in dx, Y(t) \in dy, \min_{t'} X(t') \geq a_x, \max_{t'} X(t') \leq b_x, \min_{t'} Y(t') \geq a_y, \max_{t'} Y(t') \leq b_y \,\middle|\, 0 < t' \leq t, X(0) = x_0, Y(0) = y_0, \theta\right) \end{aligned}$$

□

Lemma 2. *The maximum likelihood estimator is consistent as $n \to \infty$ and $k \to \infty$:*

$$\hat{\theta}_{n,k} \to \theta$$

.

Proof. **It's a bit sloppy. Also, we do not need asymptotic efficiency to prove equation (13)** By Lemma 1

$$X_k \xrightarrow{d} X \text{ as } k \to \infty.$$

Next, given Theorem 4.1 in Singler [2008], we know that, for each k, q_k is analytic in both the diffusion parameters and boundary parameters. Hence, the probability density function satisfies the criteria A1 - A6 in Casella and Berger [2002] to guarantee that, for data $X_k \sim F_k(\theta)$,

$$\hat{\theta}_{n,k}(X_k) \xrightarrow{p} \theta \text{ as } n \to \infty.$$

Now we need to show that the same holds for data sampled from F as $k \to \infty$. To do this, we will use Chebyshev's inequality:

$$\Pr_X\left(\left|\hat{\theta}_{n,k}(X) - \theta\right| \geq \varepsilon\right) \leq \frac{\mathrm{E}_X\left[(\hat{\theta}_{n,k}(X) - \theta)^2\right]}{\varepsilon^2}.$$

By the Maximum theorem [REFERENCE], $\hat{\theta}_{n,k}(x)$ is a continuous function with respect to x, and further because we have bounded $\hat{\theta}$ from below and above,

$$\mathrm{E}_{X_k}\left[(\hat{\theta}_{n,k}(X_k) - \theta)^2\right] \to \mathrm{E}_X\left[(\hat{\theta}_{n,k}(X) - \theta)^2\right] \text{ as } k \to \infty$$

by the portmanteau lemma. Finally, we can show that

$$\mathrm{E}_{X_k}\left[(\hat{\theta}_{n,k}(X_k) - \theta)^2\right] \to 0 \text{ as } n \to \infty, \tag{13}$$

since the expected value of the estimator tends to θ and its variance goes to 0 when $n \to \infty$. Therefore, given any $\varepsilon > 0$ and $\delta > 0$, we can find a sufficiently large n and k such that

$$\Pr_X\left(\left|\hat{\theta}_{n,k}(X) - \theta\right| \geq \varepsilon\right) \leq \frac{\mathrm{E}_X\left[(\hat{\theta}_{n,k}(X) - \theta)^2\right]}{\varepsilon^2} < \delta$$

□

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