1 Introduction

We consider two-dimensional correlated Brownian motion with absorbing boundaries:

$$X(t) = x_0 + \mu_x t + \sigma_x W_x(t) \qquad a_x < X(t) < b_x \tag{1}$$
$$Y(t) = y_0 + \mu_y t + \sigma_y W_y(t) \qquad a_y < Y(t) < b_y \tag{2}$$

where W_i are standard Brownian motions with $\mathrm{Cov}(W_1(t), W_2(t)) = \rho t$ for $0 < t' \leq t$. In particular, we find the joint transition density function for $(X(t), Y(t))$ under the boundary conditions:

$$p\left(X(t) = x, Y(t) = y | a_x < X(t') < b_x, a_y < Y(t') < b_y, 0 < t' \leq t, X(0) = x_0, Y(0) = y_0\right). \tag{3}$$

This density, which we shorten to $q(x,y,t)$ from now on, is the solution to the Fokker-Planck equation [Oksendal, 2013]:

$$\frac{\partial}{\partial t} q(x,y,t') = -\mu_x \frac{\partial}{\partial x} q(x,y,t') - \mu_y \frac{\partial}{\partial y} q(x,y,t') + \frac{1}{2}\sigma_x^2 \frac{\partial^2}{\partial x^2} q(x,y,t') + \rho\sigma_x\sigma_y \frac{\partial^2}{\partial x \partial y} q(x,y,t') + \frac{1}{2}\sigma_y^2 \frac{\partial^2}{\partial y^2} q(x,y,t'), \tag{4}$$
$$q(a_x,y,t') = q(b_x,y,t') = q(x,a_y,t') = q(x,b_y,t') = 0, \tag{5}$$
$$0 < t' \leq t.$$

Differentiating $q(x,y,t)$ with respect to the boundaries produces the transition density of a particle beginning and ending at the points $(X_1(0), X_2(0))$ and $(X_1(t), X_2(t))$ respectively, while attaining the minima a_x/a_y and maxima b_x/b_y in each coordinate direction:

$$\frac{\partial^4}{\partial a_x \partial b_x \partial a_y \partial b_y} q(x,y,t) =$$

$$p\left(X(t) = x, Y(t) = y \,\middle|\, \min_{t'} X(t') = a_x, \max_{t'} X(t') = b_x, \min_{t'} Y(t') = a_y, \max_{t'} Y(t') = b_y, 0 < t' \leq t, X(0) = x_0, Y(0) = y_0\right). \tag{6}$$

The transition density for the considered system has been used in computing first passage times [Kou et al., 2016, Sacerdote et al., 2016], with application to structural models in credit risk and default correlations [Haworth et al., 2008, Ching et al., 2014]. He et al. [1998] use variants of the differentiated solutions with respect to some of the boundaries to price financial derivative instruments whose payoff depends on observed maxima/minima.

Closed-form solutions to (4) - (5) are available for some parameter regimes. When $\rho = 0$, the transition density of the process can be obtained with a Fourier expansion. When $a_1 = -\infty$ and $b_1 = \infty$, the method of images can be used to enforce the remaining boundaries. For either $a_1, a_2 = -\infty$ or $b_1, b_2 = \infty$, the Fokker-Plank equation is a Sturm-Liouville problem in radial coordinates. Both of these techniques are used by He et al. [1998]. However, to the best of our knowledge, there is no closed-form solution to the general problem in (4) - (5).

It is still possible to approach the general problem by proposing a Fourier expansions. However, a drawback of this out-of-the-box solution is that the system matrix for the corresponding eigenvalue problem is large and dense. An alternative is to use a finite difference scheme. However, discretization of the initial condition introduces a numerical bias in the estimation procedure.

In this paper, we propose a solution to the general problem (4) - (5) which is obtained by combining a small-time analytic solution with a finite-element method. We apply our computational method for finding (3) and (6) to estimate the equation parameters in settings where the model assumption is appropriate. application is the maximal likelihood estimation.

2 Approximate Numerical Solutions

Before considering any solutions to (4) - (5), we simplify the PDE in (4) by using the fact that parameters $(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$ are constant and solving for the exponential decomposition

$$p(x,y,t) = \exp(\alpha x + \beta y + \gamma t) q(x,y,t).$$

We can find α, β and γ, as well as a scaling transformation, such that $q(x,y,t)$ satisfies

$$\begin{aligned}
\frac{\partial}{\partial t} q(x,y,t') &= \frac{1}{2}\sigma_x^2 \frac{\partial^2}{\partial x^2} q(x,y,t') + \rho\sigma_x\sigma_y \frac{\partial^2}{\partial x \partial y} q(x,y,t') + \frac{1}{2}\sigma_y^2 \frac{\partial^2}{\partial y^2} q(x,y,t'). \qquad (7)\\
q(x,y,t) &= 0 \qquad \text{for } (x,y) \in \partial\Omega\\
q(x,y,0) &= \delta(x - x_0)\delta(y - y_0)
\end{aligned}$$

on the unit square. We will consider the solution to this PDE without loss of generality.

2.1 Fourier Expansion

The formal Fourier (sinusoidal) expansion for the problem is given by

$$q(x,y,t) = \lim_{K,L\to\infty} \sum_{k=1}^{K} \sum_{l=1}^{L} c_{k,l}(t) \sin\left(2\pi \cdot k \frac{x - a_x}{b_x - a_x}\right) \sin\left(2\pi \cdot l \frac{y - a_y}{b_y - a_y}\right)$$

With $\rho = 0$, the sinusoidal functions are the eigenvectors for the differential operator in (7), and we would proceed by substituting $\hat{q}$ into (7) and deriving a system of ODEs whose solution is the vector $(c_{1,1}(t), \ldots, c_{K,L}(t))$. In this case the system matrix is diagonal so that each $c_{k,l}(t)$ can be written down analytically.

We can proceed in the same manner in the case where $\rho \neq 0$. However, the mixing terms

$$\frac{\partial^2}{\partial x \partial y} \sin\left(2\pi \cdot k \frac{x - a_x}{b_x - a_x}\right) \sin\left(2\pi \cdot l \frac{y - a_y}{b_y - a_y}\right),$$

are cosines and as such have a non-sparse representation in terms of sine series. Because of this, the matrix for the system of ODEs when $\rho \neq 0$ is dense. Moreover, the truncation values for K and L are also large. This makes the computation of the density function prohibitively expensive.

2.2 Finite Difference

The finite difference method we employe uses a regular grid. Because of this, either the initial condition or the boundaries need to be approximated by assigning them to a correct grid point(s). This quantization introduces an aritificial diffusion in the numerical scheme, which biases parameter estimates. Of course, this error can be reduced by using a finer grid. However, the resolution needed is prohibitively high for this problem.

2.3 Finite Element Method

The method we use relies on two parts:

1. a small-time analytic solution $q(x,y,t_\varepsilon)$ for the IC/BC problem,
2. a family of orthonormal basis functions which represent $q(x,y,t_\varepsilon)$ parsimoniously.

By combining 1) and 2), we can efficiently find a weak solution to the PDE (7) via the finite element method [Shaidurov, 2013]. Convergence of our method to the strong solution under the $L^2(\bar{\Omega})$ norm is guaranteed as long as the family we propose is complete in the Banach space of functions induced under $L^2(\bar{\Omega})$ [Salsa, 2016].

The small-time solution is derived by considering the fundamental solution $G(x, y|t, x_0, y_0)$ for the unbounded problem in (7), which is the bivariate Gaussian density with mean and covariance determined by the initial condition and the diffusion parameters [Stakgold and Holst, 2011]. We can then find a small enough t_ε such that $G(x, y|t_\varepsilon, x_0, y_0)$ is numerically zero on three of the four boundaries of $\bar{\Omega}$. The zero-condition on the remaining boundary is enforced by suitably reflecting $G(x, y|t_\varepsilon, x_0, y_0)$ about the boundary. The small-time solution therefore takes on the analytic form

$$q(x, y, t_\varepsilon) = G(x, y|t_\varepsilon, x_0, y_0) - G(x, y|t_\varepsilon, x'_0, y'_0),$$

for some known $(x'_0, y'_0, t_\varepsilon)$.

The construction of the orthonormal basis functions is motivated by the Green's function for the unbounded problem: before performing Gram-Schmidt orthogonalization, the finite family of basis functions are of the form

$$\tilde{\psi}_k(x, y|x_k, y_k, \rho, \sigma) = N\left((x, y)^T \middle| (x_k, y_k)^T, \quad \begin{pmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{pmatrix}\right) x(1-x)y(1-y).$$

The advantage of these basis elements is that they better resolve the fundamental for the unbounded problem by taking into account ρ in the covariance of each kernel. By performing Gram-Schmidt orthogonalization under the $L^2(\Omega)$ norm, we arrive at a family of orthonormal functions which can better resolve small-time solutions having a large correlation coefficient.

3 Estimation

As an application of our computational method, we estimate the parameters in (6). However, before we do so, we prove a lemma to show that maximum likelihood estimates based on the approximate solution are asymptotically efficient.

Lemma 1. *The maximum likelihood estimator is consistent as $n \to \infty$ and $k \to \infty$:*

$$\hat{\theta}_{n,k} \to \theta$$

.

Proof. By the definition of weak convergence, given the weak solution q_k and the classical solution q, for any continuous function f,

$$\langle q_k | f \rangle \to \langle q | f \rangle \text{ as } k \to \infty.$$

Because f can be any function in L^2, we can choose f to be $\exp(ilx)$ for any integer l. This means that the characteristic function of X_k converges pointwise to the characteristic function of X. By Levy's continuity theorem, this means that

$$X_k \xrightarrow{d} X \text{ as } k \to \infty.$$

Next, given Theorem 4.1 in Singler [2008], we know that, for each k, q_k satisfies the criteria A1 - A6 in Casella and Berger [2002] to guarantee that, for data $X_k \sim F_k(\theta)$,

$$\hat{\theta}_{n,k}(X_k) \xrightarrow{p} \theta$$

as $n \to \infty$. Moreover, we are guaranteed asymptotic efficiency. In other words, the MLE estimator for $(\sigma_x, \sigma_y, \rho)$ based on the likelihood function under F_k for data sampled from F_k is asymptotically efficient. Now we need to show that the same holds for data sampled from F as $k \to \infty$.

To do this, we will use Chebyshev's inequality:

$$\Pr_X\left(\left|\hat{\theta}_{n,k}(X)-\theta\right|\geq\varepsilon\right)\leq\frac{\mathrm{E}_X\left[(\hat{\theta}_{n,k}(X)-\theta)^2\right]}{\varepsilon^2}.$$

By the Maximum theorem, $\hat{\theta}_{n,k}(x)$ is a continuous function with respect to x, and further because we have bounded $\hat{\theta}$ from below and above,

$$\mathrm{E}_{X_k}\left[(\hat{\theta}_{n,k}(X_k)-\theta)^2\right]\to\mathrm{E}_X\left[(\hat{\theta}_{n,k}(X)-\theta)^2\right]\text{ as }k\to\infty$$

by the portmanteau lemma. Finally, because $\hat{\theta}_{k,n}$ is asymptotically efficient, we can show that

$$\mathrm{E}_{X_k}\left[(\hat{\theta}_{n,k}(X_k)-\theta)^2\right]\to 0\text{ as }n\to\infty,$$

since the expected value of the estimator tends to θ and its variance goes to 0 when $n\to\infty$. Therefore, given any $\varepsilon>0$ and $\delta>0$, we can find a sufficiently large n and k such that

$$\Pr_X\left(\left|\hat{\theta}_{n,k}(X)-\theta\right|\geq\varepsilon\right)\leq\frac{\mathrm{E}_X\left[(\hat{\theta}_{n,k}(X)-\theta)^2\right]}{\varepsilon^2}<\delta$$

$\square$

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