1 Introduction

We consider two-dimensional correlated Brownian motion with absorbing boundaries:

$$X(t) = x_0 + \mu_x t + \sigma_x W_x(t) \qquad a_x < X(t) < b_x \tag{1}$$
$$Y(t) = y_0 + \mu_y t + \sigma_y W_y(t) \qquad a_y < Y(t) < b_y \tag{2}$$

where W_i are standard Brownian motions with $\mathrm{Cov}(W_1(t), W_2(t)) = \rho t$ for $0 < t' \leq t$. In particular, we find the joint transition density function for $(X(t), Y(t))$ under the boundary conditions:

$$p\left(X(t) = x, Y(t) = y | a_x < X(t') < b_x, a_y < Y(t') < b_y, 0 < t' \leq t, X(0) = x_0, Y(0) = y_0\right). \tag{3}$$

This density, which we shorten to $p(x, y, t)$ from now on, is the solution to the Fokker-Planck equation [Oksendal, 2013]:

$$\frac{\partial}{\partial t} p(x,y,t') = -\mu_x \frac{\partial}{\partial x} p(x,y,t') - \mu_y \frac{\partial}{\partial y} p(x,y,t') + \frac{1}{2}\sigma_x^2 \frac{\partial^2}{\partial x^2} p(x,y,t') + \rho\sigma_x\sigma_y \frac{\partial^2}{\partial x \partial y} p(x,y,t') + \frac{1}{2}\sigma_y^2 \frac{\partial^2}{\partial y^2} p(x,y,t'), \tag{4}$$
$$p(a_x, y, t') = p(b_x, y, t') = p(x, a_y, t') = p(x, b_y, t') = 0, \tag{5}$$
$$0 < t' \leq t.$$

Differentiating $p(x, y, t)$ with respect to the boundaries produces the transition density of a particle beginning and ending at the points $(X_1(0), X_2(0))$ and $(X_1(t), X_2(t))$ respectively, while attaining the minima a_x/a_y and maxima b_x/b_y in each coordinate direction:

$$\frac{\partial^4}{\partial a_x \partial b_x \partial a_y \partial b_y} p(x,y,t) =$$

$$p\left(X(t) = x, Y(t) = y \,\middle|\, \min_{t'} X(t') = a_x, \max_{t'} X(t') = b_x, \min_{t'} Y(t') = a_y, \max_{t'} Y(t') = b_y, 0 < t' \leq t, X(0) = x_0, Y(0) = y_0\right). \tag{6}$$

The transition density for the considered system has been used in computing first passage times [Kou et al., 2016, Sacerdote et al., 2016], with application to structural models in credit risk and default correlations [Haworth et al., 2008, Ching et al., 2014]. He et al. [1998] use variants of the differentiated solutions with respect to some of the boundaries to price financial derivative instruments whose payoff depends on observed maxima/minima.

Closed-form solutions to (4) - (5) are available for some parameter regimes. When $\rho = 0$, the transition density of the process can be obtained with a Fourier expansion. When $a_1 = -\infty$ and $b_1 = \infty$, the method of images can be used to enforce the remaining boundaries. For either $a_1, a_2 = -\infty$ or $b_1, b_2 = \infty$, the Fokker-Plank equation is a Sturm-Liouville problem in radial coordinates. Both of these techniques are used by He et al. [1998]. However, to the best of our knowledge, there is no closed-form solution to the general problem in (4) - (5).

It is still possible to approach the general problem by proposing a Fourier expansions. However, a drawback of this out-of-the-box solution is that the system matrix for the corresponding eigenvalue problem is large and dense. An alternative is to use a finite difference scheme. However, discretization of the initial condition introduces a numerical bias in the estimation procedure.

In this paper, we propose a solution to the general problem (4) - (5) which is obtained by combining a small-time analytic solution with a finite-element method. Our application is the maximal likelihood estimation

ADD OUT APPLICATION (ESTIMATION)

2 Approximate Numerical Solutions

Before considering any solutions to (4) - (5), we simplify the PDE in (4) by using the fact that parameters $(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$ are constant and solving for the exponential decomposition

$$p(x,y,t) = \exp(\alpha x + \beta y + \gamma t) q(x,y,t).$$

We can find α, β and γ, as well as a scaling transformation, such that $q(x,y,t)$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t} q(x,y,t') &= \frac{1}{2}\sigma_x^2 \frac{\partial^2}{\partial x^2} q(x,y,t') + \rho\sigma_x\sigma_y \frac{\partial^2}{\partial x \partial y} q(x,y,t') + \frac{1}{2}\sigma_y^2 \frac{\partial^2}{\partial y^2} q(x,y,t'). && (7) \\ q(x,y,t) &= 0 && \text{for } (x,y) \in \partial\Omega \\ q(x,y,0) &= \delta(x-x_0)\delta(y-y_0) \end{aligned}$$

on the unit square. We will consider the solution to this PDE without loss of generality.

2.1 Fourier Expansion

The formal Fourier (sinusoidal) expansion for the problem is given by

$$\begin{aligned} q(x,y,t) &= \lim_{K,L\to\infty} \sum_{k=1}^{K} \sum_{l=1}^{L} c_{k,l}(t) \sin\left(2\pi \cdot k \frac{x-a_x}{b_x-a_x}\right) \sin\left(2\pi \cdot l \frac{y-a_y}{b_y-a_y}\right) \\ \hat{q}(x,y,t) &= \sum_{k=1}^{K} \sum_{l=1}^{L} c_{k,l}(t) \sin\left(2\pi \cdot k \frac{x-a_x}{b_x-a_x}\right) \sin\left(2\pi \cdot l \frac{y-a_y}{b_y-a_y}\right) && \text{for some } K, L \end{aligned}$$

With $\rho = 0$, the sinusoidal functions are the eigenvectors for the differential operator in (7), and we would proceed by substituting $\hat{q}$ into (7) and deriving a system of ODEs whose solution is the vector $(c_{1,1}(t), \ldots, c_{K,L}(t))$. In this case the system matrix is diagonal so that each $c_{k,l}(t)$ can be written down analytically.

We can proceed in the same manner in the case where $\rho \neq 0$. However, the mixing terms

$$\frac{\partial^2}{\partial x \partial y} \sin\left(2\pi \cdot k \frac{x-a_x}{b_x-a_x}\right) \sin\left(2\pi \cdot l \frac{y-a_y}{b_y-a_y}\right),$$

are cosines and as such have a non-sparse representation in terms of sine series. Because of this, the matrix for the system of ODEs when $\rho \neq 0$ is dense. Moreover, the truncation values for K and L are also large. Finally, to compute

2.2 Finite Difference

2.3 Finite Element Method

The method we use relies on two pieces:

1. a small-time analytic solution $q(x,y,t_\varepsilon)$ for the IC/BC problem,
2. a family of orthonormal basis functions which represent $q(x,y,t_\varepsilon)$ parsimoniously.

By combining 1) and 2), we can efficiently find a weak solution to the PDE (7) via the finite element method [Shaidurov, 2013]. Convergence of our method to the strong solution under the $L^2(\bar{\Omega})$ norm is guaranteed as long as the family we propose is complete in the Banach space of functions induced under $L^2(\bar{\Omega})$ [Salsa, 2016].

The small-time solution is derived by considering the fundamental solution $G(x,y|t,x_0,y_0)$ for the unbounded problem in (7), which is the bivariate Gaussian density with mean and covariance determined by the initial condition and the diffusion parameters [Stakgold and Holst, 2011]. We can then find a small enough t_ε such that $G(x,y|t_\varepsilon,x_0,y_0)$ is numerically zero on three of the four boundaries of $\bar{\Omega}$. The zero-condition on the remaining boundary is enforced by suitably reflecting $G(x,y|t_\varepsilon,x_0,y_0)$ about the boundary. The small-time solution therefore takes on the analytic form

$$q(x,y,t_\varepsilon) = G(x,y|t_\varepsilon,x_0,y_0) - G(x,y|t_\varepsilon,x_0',y_0'),$$

for some known $(x_0',y_0',t_\varepsilon)$.

The construction of the orthonormal basis functions is motivated by the Green's function for the unbounded problem: before performing Gram-Schmidt orthogonalization, the finite family of basis functions are of the form

$$\tilde{\psi}_k(x,y|x_k,y_k,\rho,\sigma) = N\left((x,y)^T \middle| (x_k,y_k)^T, \quad \left(\begin{array}{cc} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{array}\right)\right) x(1-x)y(1-y).$$

The advantage of these basis elements is that they better resolve the fundamental for the unbounded problem by taking into account ρ in the covariance of each kernel. By performing Gram-Schmidt orthogonalization under the $L^2(\Omega)$ norm, we arrive at a family of orthonormal functions which can better resolve small-time solutions having a large correlation coefficient.

Lemma 1. *The maximum likelihood estimator is consistent as $n \to \infty$ and $k \to \infty$:*

$$\hat{\theta}_{n,k} \to \theta$$

.

Proof. By the definition of weak convergence, given the weak solution q_k and the classical solution q, for any continuous function f,

$$\langle q_k|f\rangle \to \langle q|f\rangle \text{ as } k \to \infty.$$

Because f can be any function in L^2, we can choose f to be $\exp(ilx)$ for any integer l. This means that the characteristic function of X_k converges pointwise to the characteristic function of X. By Levy's continuity theorem, this means that

$$X_k \xrightarrow{d} X \text{ as } k \to \infty.$$

Next, given Theorem 4.1 in Singler [2008], we know that, for each k, q_k satisfies the criteria in Casella and Berger [2002] to guarantee that, for data $X_k \sim F_k(\theta)$,

$$\hat{\theta}_{n,k}(X_k) \xrightarrow{p} \theta$$

as $n \to \infty$. Moreover, we are guaranteed asymptotic efficiency. In other words, the MLE estimator for (σ_x,σ_y,ρ) based on the likelihood function under F_k for data sampled from F_k is asymptotically efficient. Now we need to show that the same holds for data sampled from F as $k \to \infty$.

To do this, we will use Chebyshev's inequality:

$$\Pr_X\left(\left|\hat{\theta}_{n,k}(X) - \theta\right| \geq \varepsilon\right) \leq \frac{\mathrm{E}_X\left[(\hat{\theta}_{n,k}(X)-\theta)^2\right]}{\varepsilon^2}.$$

By the Maximum theorem, $\hat{\theta}_{n,k}(x)$ is a continuous function with respect to x, and further because we have bounded $\hat{\theta}$ from below and above,

$$\mathrm{E}_{X_k}\left[(\hat{\theta}_{n,k}(X_k)-\theta)^2\right] \to \mathrm{E}_X\left[(\hat{\theta}_{n,k}(X)-\theta)^2\right] \text{ as } k \to \infty$$

by the portmanteau lemma. Finally, because $\hat{\theta}_{k,n}$ is asymptotically efficient, we can show that

$$\mathrm{E}_{X_k}\left[(\hat{\theta}_{n,k}(X_k)-\theta)^2\right] \to 0 \text{ as } n \to \infty,$$

since the expected value of the estimator tends to θ and its variance goes to 0 when $n \to \infty$. Therefore, given any $\varepsilon > 0$ and $\delta > 0$, we can find a sufficiently large n and k such that

$$\Pr_X \left(\left| \hat{\theta}_{n,k}(X) - \theta \right| \geq \varepsilon \right) \leq \frac{\mathrm{E}_X \left[(\hat{\theta}_{n,k}(X) - \theta)^2 \right]}{\varepsilon^2} < \delta$$

□

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