1 Introduction

The estimation and prediction of price volatility from market data is an important problem in econometrics and finance [Abramov and Klebaner, 2007], as well as practical risk management [Brandt and Santa-Clara, 2006]. The literature on the subject of volatility estimation is vast. Model-based approaches for a single observable asset begin with the ARCH and GARCH models of Engle [1982] and Bollerslev [1986], moving on to stochastic volatility models (see Shephard [2005], for example).

Multivariate equivalents for each of these model classes exist (see Bauwens et al. [2006] and Asai et al. [2006] for reviews of multivariate GARCH and for multivariate stochastic volatility, respectively). However, the majority of work on the subject uses opening and closing prices as data. This approach invariably disregards information traditionally contained in financial timeseries: the observed high and low price of an asset over the quoted periods. To our current knowledge, only Horst et al. [2012] use the observed maximum and minimum of prices in a likelihood to estimate volatility. They do so, however, in a univariate setting.

Explicit model-based approaches in the multivariate setting which take into account extrema over observational periods are completely lacking in the literature, because deriving an efficient approximation of the corresponding likelihood function has hereto been an open problem. In this paper, we use a result addressing this problem and introduce a *bivariate* stochastic volatility model which takes into account the highest and lowest observed prices of each asset as part of a likelihood-based (Bayesian) estimation procedure.

2 Model

The model we will estimate is a bivaraite, 1-factor stochastic volatility model with leverage:

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} x_{t-\Delta} \\ y_{t-\Delta} \end{pmatrix} + \begin{pmatrix} \mu_x \Delta \\ \mu_y \Delta \end{pmatrix} + \begin{pmatrix} \sqrt{1-\rho_t^2}\sigma_{x,t} & \rho_t \sigma_{x,t} \\ 0 & \sigma_{y,t} \end{pmatrix} \begin{pmatrix} \varepsilon_{x,t} \\ \varepsilon_{y,t} \end{pmatrix}, \tag{1}$$

$$\inf_{[t-\Delta,t]} x_\tau = a_{x,t} \qquad \sup_{[t-\Delta,t]} x_\tau = b_{x,t} \quad \inf_{[t-\Delta,t]} y_\tau = a_{y,t} \quad \sup_{[t-\Delta,t]} y_\tau = b_{y,t}$$

$$\log(\sigma_{x,t+\Delta}) = \alpha_x + \theta_x(\log(\sigma_{x,t}) - \alpha_x) + \tau_x \eta_{x,t}, \tag{2}$$

$$\log(\sigma_{y,t+\Delta}) = \alpha_y + \theta_y(\log(\sigma_{y,t}) - \alpha_y) + \tau_y \eta_{y,t}, \tag{3}$$

$$\mathrm{logit}((\rho_{t+\Delta}+1)/2) = \alpha_\rho + \theta_\rho \left(\mathrm{logit}((\rho_t+1)/2) - \alpha_\rho\right) + \tau_\rho \eta_{\rho,t}. \tag{4}$$

The marginal distribution for all of the innovation terms $\varepsilon_{x,t}, \varepsilon_{y,t}, \eta_{x,t}, \eta_{y,t}, \eta_{\rho,t}$ is the standard Gaussian distribution. The *leverage* terms are defined as $E[\varepsilon_{x,t}\,\eta_{x,t}] = \rho_x$ and $E[\varepsilon_{x,t}\,\eta_{x,t}] = \rho_y$. It should be noted here that we are explicitly allowing the correlation of the process to change over time in a mean-reverting fashion. Finally, we explicitly write down the realized extrema over the periods $[t-\Delta,t]$ to be included as data into the likelihood for the dynamical model. We estimate all parameters and dynamical factors in a fully Bayes framework via the augmented particle filter of **?** which we will describe below.

2.1 Likelihood for the observables

Each period $[t-\Delta,t]$ has six associated observables: opening coordinate $(x_{t-\Delta}, y_{t-\Delta})$, closing coordinate (x_t, y_t), and the observed extrema in each nominal direction $(a_{x,t}, b_{x,t}), (a_{y,t}, b_{y,t})$. Given the evolution model in (1), disregarding the information contained in the extrema yields the usual bivariate Gaussian density in terms of the volatility parameters and the state of the process at time $t-\Delta$:

$$p(x_t, y_t | x_{t-\Delta}, y_{t-\Delta}, \mu_x, \mu_y, \sigma_{x,t}, \sigma_{y,t}, \rho_t) =$$

$$\frac{1}{2\pi\,\Delta\,\sigma_{x,t}\sigma_{y,t}\sqrt{1-\rho_t^2}} \exp\left\{ -\frac{1}{2\Delta(1-\rho_t^2)} \left(\frac{(x_t - x_{t-\Delta})^2}{\sigma_{x,t}^2} - 2\rho_t \frac{(x_t - x_{t-\Delta})(y_t - y_{t-\Delta})}{\sigma_{x,t}\sigma_{y,t}} + \frac{(y_t - y_{t-\Delta})^2}{\sigma_{y,t}^2} \right) \right\}.$$

Incorporating the extreme values over $[t-\Delta, t]$ is accomplished by considering the Fokker-Planck Equation for the forward, continuous-time evolution of the probability density function of (x_t, y_t) and including $(a_{x,t}, b_{x,t}), (a_{y,t}, b_{y,t})$ as boundary conditions where the density is zero. The previous work describes the method by which the full likelihood function is found. However, for the purposes of the particle filter used to estimate the model in this work, we improve upon the computational method by performing a set of normalizing transformations, allowing for more flexibility in the parameters for the basis functions in the Galerkin approximation, and extrapolating over certain low-probability regions of the parameter space.

2.2 Improved likelihood computational method for low probability data

3 Estimation Methodology

Given the highly non-linear hierarchical model (1) - (4) and the non-Gaussian observational likelihood, we use a particle filter to estimate the collection of time-dependent parameters which we abbreviate to

$$\sigma_t := (\sigma_{x,t}, \sigma_{y,t}, \rho_t),$$

as well as all of the time-constant parameters governing the evolution of the process

$$\phi := (\alpha_x, \alpha_y, \alpha_\rho, \theta_x, \theta_y, \theta_\rho, \tau_x, \tau_y, \tau_\rho, \rho_x, \rho_y).$$

Particle filters use a discrete mixture to represent the posterior distribution $p(\sigma_t, \phi | \mathcal{D}_t)$, where $\mathcal{D}_t$ represents all of the observable information up to time t:

$$\mathcal{D}_t = (x_0, y_0, a_{x,\Delta}, b_{x,\Delta}, a_{y,\Delta}, b_{y,\Delta}, x_\Delta, y_\Delta, \ldots, x_{t-\Delta}, y_{t-\Delta}, a_{x,t}, b_{x,t}, a_{y,t}, b_{y,t}, x_t, y_t)$$

4 Calibration Study

5 Application

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