1 Introduction

We consider the solution of the Fokker-Planck equation for bivariate Brownian motion with absorbing boundaries and correlated increments. In particular, we find the transition density function for the system

$$X_i(t) = x_i + \mu_i + \sigma_i W_i(t), \qquad i = 1,2, \qquad 0 < t \leq T, \tag{1}$$
$$a_i \leq X_i(t) \leq b_i, \tag{2}$$

where W_i are standard Brownian motions with $\mathrm{Cov}(W_1(t), W_2(t)) = \rho t$.

The solution of the Fokker-Plank equation corresponding to (1) - (2) is the transition density of a particle starting at $(X_1(0), X_2(0))$, terminating at $(X_1(T), X_2(T))$, while never leaving the region Ω defined by the rectangle $(a_1, b_1)x(a_2, b_2)$. Differentiating the solution with respect to the boundaries produces the transition density of a particle beginning and ending at the points $(X_1(0), X_2(0))$ and $(X_1(T), X_2(T))$ respectively, while attaining the minima a_1/a_2 and maxima b_1/b_2 in each coordinate axis.

The transition density for the considered system has been used in computing first passage times [Kou et al., 2016, Sacerdote et al., 2012], with application to structural models in credit risk and default correlations [Haworth et al., 2008, Ching et al., 2014]. [He et al., 1998] use variants of the differentiated solutions with respect to some of the boundaries to price financial derivative instruments whose payoff depends on observed maxima/minima.

Closed-form solutions to (1) - (2) are available for some parameter regimes. When $\rho = 0$, the transition density of the process can be obtained with a Fourier expansion. When $a_1 = -\infty$ and $b_1 = \infty$, the method of images can be used to enforce the remaining boundaries. For either $a_1, a_2 = -\infty$ or $b_1, b_2 = \infty$, the Fokker-Plank equation is a Sturm-Liouville problem in radial coordinates. Both of these techniques are used by He et al. [1998]. However, to the best of our knowledge, there is no closed-form solution to the general problem in (1) - (2).

It is still possible to approach the general problem by proposing a Fourier expansions. However, a drawback of this out-of-the-box solution is that the system matrix for the corresponding eigenvalue problem is large and dense. An alternative is to use a finite difference scheme. However, discretization of the initial condition introduces a numerical bias in the estimation procedure.

In this paper, we propose a solution to the general problem (1) - (2) which is obtained by combining a small-time analytic solution with a finite-element method.

References

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