1 Introduction

We consider two-dimensional correlated Brownian motion with absorbing boundaries:

$$X(t) = x_0 + \mu_x t + \sigma_x W_x(t) \qquad a_x < X(t) < b_x \tag{1}$$

$$Y(t) = y_0 + \mu_y t + \sigma_y W_y(t) \qquad a_y < Y(t) < b_y \tag{2}$$

where W_i are standard Brownian motions with $\mathrm{Cov}(W_1(t), W_2(t)) = \rho t$ for $0 < t' \leq t$. In particular, we find the joint transition density function for $(X(t), Y(t))$ under the boundary conditions:

$$p\left(X(t) = x, Y(t) = y | a_x < X(t') < b_x, a_y < Y(t') < b_y, 0 < t' \leq t, X(0) = x_0, Y(0) = y_0\right). \tag{3}$$

This density, which we shorten to $p(x, y, t)$ from now on, is the solution to the Fokker-Planck equation [Oksendal, 2013]:

$$\frac{\partial}{\partial t} p(x,y,t') = -\mu_x \frac{\partial}{\partial x} p(x,y,t') - \mu_y \frac{\partial}{\partial y} p(x,y,t') + \frac{1}{2}\sigma_x^2 \frac{\partial^2}{\partial x^2} p(x,y,t') + \rho\sigma_x\sigma_y \frac{\partial^2}{\partial x \partial y} p(x,y,t') + \frac{1}{2}\sigma_y^2 \frac{\partial^2}{\partial y^2} p(x,y,t'), \tag{4}$$

$$p(a_x, y, t') = p(b_x, y, t') = p(x, a_y, t') = p(x, b_y, t') = 0, \qquad 0 < t' \leq t. \tag{5}$$

Differentiating $p(x, y, t)$ with respect to the boundaries produces the transition density of a particle beginning and ending at the points $(X_1(0), X_2(0))$ and $(X_1(t), X_2(t))$ respectively, while attaining the minima a_x/a_y and maxima b_x/b_y in each coordinate direction:

$$\frac{\partial^4}{\partial a_x \partial b_x \partial a_y \partial b_y} p(x,y,t) =$$

$$p\left(X(t) = x, Y(t) = y \,\middle|\, \min_{t'} X(t') = a_x, \max_{t'} X(t') = b_x, \min_{t'} Y(t') = a_y, \max_{t'} Y(t') = b_y, 0 < t' \leq t, X(0) = x_0, Y(0) = y_0\right). \tag{6}$$

The transition density for the considered system has been used in computing first passage times [Kou et al., 2016, Sacerdote et al., 2016], with application to structural models in credit risk and default correlations [Haworth et al., 2008, Ching et al., 2014]. He et al. [1998] use variants of the differentiated solutions with respect to some of the boundaries to price financial derivative instruments whose payoff depends on observed maxima/minima.

Closed-form solutions to (4) - (5) are available for some parameter regimes. When $\rho = 0$, the transition density of the process can be obtained with a Fourier expansion. When $a_1 = -\infty$ and $b_1 = \infty$, the method of images can be used to enforce the remaining boundaries. For either $a_1, a_2 = -\infty$ or $b_1, b_2 = \infty$, the Fokker-Plank equation is a Sturm-Liouville problem in radial coordinates. Both of these techniques are used by He et al. [1998]. However, to the best of our knowledge, there is no closed-form solution to the general problem in (4) - (5).

It is still possible to approach the general problem by proposing a Fourier expansions. However, a drawback of this out-of-the-box solution is that the system matrix for the corresponding eigenvalue problem is large and dense. An alternative is to use a finite difference scheme. However, discretization of the initial condition introduces a numerical bias in the estimation procedure.

In this paper, we propose a solution to the general problem (4) - (5) which is obtained by combining a small-time analytic solution with a finite-element method. Our application is the maximal likelihood estimation

ADD OUT APPLICATION (ESTIMATION)

2 Approximate Numerical Solutions

Before considering any solutions to (4) - (5), we simplify the PDE in (4) by using the fact that parameters $(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$ are constant and solving for the exponential decomposition

$$p(x,y,t) = \exp(\alpha x + \beta y + \gamma t) q(x,y,t).$$

Plugging the above equation into (4) and setting the advection terms for $q(x,y,t)$ to zero, we arrive at the PDE

$$\frac{\partial}{\partial t} q(x,y,t') = \frac{1}{2}\sigma_x^2 \frac{\partial^2}{\partial x^2} q(x,y,t') + \rho\sigma_x\sigma_y \frac{\partial^2}{\partial x \partial y} q(x,y,t') + \frac{1}{2}\sigma_y^2 \frac{\partial^2}{\partial y^2} q(x,y,t'). \tag{7}$$

We will consider the solution to this PDE, as it encompasses the original problem.

2.1 Fourier Expansion

The formal Fourier (sinusoidal) expansion for the problem is given by

$$\begin{aligned} q(x,y,t) &= \lim_{K,L\to\infty} \sum_{k=1}^{K} \sum_{l=1}^{L} c_{k,l}(t) \sin\left(2\pi \cdot k \frac{x-a_x}{b_x-a_x}\right) \sin\left(2\pi \cdot l \frac{y-a_y}{b_y-a_y}\right) \\ \hat{q}(x,y,t) &= \sum_{k=1}^{K} \sum_{l=1}^{L} c_{k,l}(t) \sin\left(2\pi \cdot k \frac{x-a_x}{b_x-a_x}\right) \sin\left(2\pi \cdot l \frac{y-a_y}{b_y-a_y}\right) \qquad \text{for some } K,L \end{aligned}$$

With $\rho = 0$, the sinusoidal functions are the eigenvectors for the differential operator in (7), and we would proceed by substituting $\hat{q}$ into (7) and deriving a system of ODEs whose solution is the vector $(c_{1,1}(t), \ldots, c_{K,L}(t))$. In this case the system matrix is diagonal so that each $c_{k,l}(t)$ can be written down analytically.

We can proceed in the same manner in the case where $\rho \neq 0$. However, the mixing terms

$$\frac{\partial^2}{\partial x \partial y} \sin\left(2\pi \cdot k \frac{x-a_x}{b_x-a_x}\right) \sin\left(2\pi \cdot l \frac{y-a_y}{b_y-a_y}\right) = \frac{2\pi \cdot k}{b_x-a_x} \cos\left(2\pi \cdot k \frac{x-a_x}{b_x-a_x}\right) \cos\left(2\pi \cdot l \frac{y-a_y}{b_y-a_y}\right)$$

2.2 Finite Difference

2.3 Finite Element

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