

## Exercise sheet

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### Part 1. Stochastic learning

**Exercise 1.** (★) Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $f(w) = g(\langle w, x \rangle + y)$  for some  $x \in \mathbb{R}^d$ ,  $y \in \mathbb{R}$ . Show that: If  $g$  is convex function then  $f$  is convex function.

**Exercise 2.** (★) Let functions  $g_1$  be  $\rho_1$ -Lipschitz and  $g_2$  be  $\rho_2$ -Lipschitz. Then, show that,  $f$  with  $f(x) = g_1(g_2(x))$  is  $\rho_1\rho_2$ -Lipschitz.

**Exercise 3.** (★) Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  with  $f(w) = g(\langle w, x \rangle + y)$   $x \in \mathbb{R}^d$  and  $y \in \mathbb{R}$ . Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a  $\beta$ -smooth function. Then show that  $f$  is a  $(\beta \|x\|^2)$ -smooth.

**Hint::** You may use Cauchy-Schwarz inequality  $\langle y, x \rangle \leq \|y\| \|x\|$

**Exercise 4.** (★) Show that  $f : S \rightarrow \mathbb{R}$  is  $\rho$ -Lipschitz over an open convex set  $S$  if and only if for all  $w \in S$  and  $v \in \partial f(w)$  it is  $\|v\| \leq \rho$ .

**Hint::** You may use Cauchy-Schwarz inequality  $\langle y, x \rangle \leq \|y\| \|x\|$

**Exercise 5.** (★) Let  $g_1(w), \dots, g_r(w)$  be  $r$  convex functions, and let  $f(\cdot) = \max_{j \in [r]} (g_j(\cdot))$ . Show that for some  $w$  it is  $\nabla g_k(w) \in \partial f(w)$  where  $k = \arg \max_j (g_j(w))$  is the index of function  $g_j(\cdot)$  presenting the greatest value at  $w$ .

**Exercise 6.** (★) Consider the regression learning problem  $(\mathcal{H}, \mathcal{Z}, \ell)$  with predictor rule  $h(x) = \langle w, x \rangle$  labeled by some unknown parameter  $w \in \mathcal{W}$ , loss function  $\ell(w, (x, y)) = (\langle w, x \rangle - y)^2$ , feature  $x \in \mathcal{X}$ , and target  $y \in \mathbb{R}$ . Let  $\mathcal{W} = \mathcal{X} = \{\omega \in \mathbb{R}^d : |\omega| \leq \rho\}$  for some  $\rho > 0$ .

- (1) Show that the resulting learning problem is Convex-Lipschitz-Bounded learning problem.
- (2) Specify the parameters of Lipschitzness.

The following is given as a homework (Formative assessment 1)

**Exercise 7.** (★) Consider the binary classification problem with inputs  $x \in \mathcal{X}$  where  $\mathcal{X} := \{x \in \mathbb{R}^d : \|x\|_2 \leq L\}$  for some given value  $L > 0$ , target  $y \in \mathcal{Y}$  where  $\mathcal{Y} := \{-1, +1\}$ , and prediction rule  $h_w : \mathbb{R}^d \rightarrow \{-1, +1\}$  with

$$(1) \quad h_w(x) = \text{sign}(w^\top x)$$

$$(2) \quad = \text{sign}\left(\sum_{j=1}^d w_j x_j\right)$$

Let the hypothesis class of prediction rules be

$$\mathcal{H} = \{x \rightarrow w^\top x : \forall w \in \mathbb{R}^d\}$$

In other words, the hypothesis  $h_w \in \mathcal{H}$  is parametrized by  $w \in \mathbb{R}^d$  it receives an input vector  $x \in \mathcal{X} := \mathbb{R}^d$  and it returns the label  $y = \text{sign}(w^\top x) \in \mathcal{Y} := \{\pm 1\}$ .

Consider a loss function  $\ell : \mathbb{R}^d \rightarrow \mathbb{R}_+$  with

$$(3) \quad \ell(w, z = (x, y)) = \max(0, 1 - yw^\top x) + \lambda \|w\|_2^2$$

for some given value  $\lambda > 0$ .

Assume there is available a dataset of examples  $S_n = \{z_i = (x_i, y_i) ; i = 1, \dots, n\}$  of size  $n$ .

Do the following tasks.

**Hint-1::** We denote

$$\text{sign}(\xi) = \begin{cases} -1, & \text{if } \xi < 0 \\ +1, & \text{if } \xi > 0 \end{cases}$$

**Hint-2::** The notation  $\pm 1$  means either  $-1$  or  $+1$ .

**Hint-3::** We define  $\mathbb{R}_+ := (0, +\infty)$

**Hint-4::** We denote  $\|x\|_2 := \sqrt{\sum_{j=1}^d (x_j)^2}$  the Euclidean distance.

- (1) Show that the function  $f : \mathbb{R} \rightarrow \mathbb{R}_+$  with  $f(x) = \max(0, 1 - x)$  is convex in  $\mathbb{R}$ ; and show that the loss (3) is convex.

**Hint::** You may use Example 13 from Handout 1.

- (2) Show that the loss  $\ell(w, z)$  for  $\lambda = 0$  (3) is  $L$ -Lipschitz (with respect to  $w$ ) when  $x \in \mathcal{X}$  where  $\mathcal{X} := \{x \in \mathbb{R}^d : \|x\|_2 \leq L\}$ .

**Hint::** You may use the definition of Lipschitz function. Without loss of generality, you can consider any  $w_1 \in \mathbb{R}^d$  and  $w_2 \in \mathbb{R}^d$  such that  $1 - yw_2^\top x \leq 1 - yw_1^\top x$ , and then take cases  $1 - yw_2^\top x > \text{or} < 0$  and  $1 - yw_1^\top x > \text{or} < 0$  to deal with the max.

- (3) Construct the set of sub-gradients  $\partial f(x)$  for  $x \in \mathbb{R}$  of the function  $f : \mathbb{R} \rightarrow \mathbb{R}_+$  with  $f(x) = \max(0, 1 - x)$ . Show that the vector  $v$  with

$$v = \begin{cases} 2\lambda w, & yw^\top x > 1 \\ 2\lambda w, & yw^\top x = 1 \\ -yx + 2\lambda w, & yw^\top x < 1 \end{cases}$$

is  $v \in \partial_w \ell(w, z = (x, y))$ , aka a sub-gradient of  $\ell(w, z = (x, y))$  at  $w$ , for any  $w \in \mathbb{R}^d$ .

- (4) Write down the algorithm of online AdaGrad (Adaptive Stochastic Gradient Descent) with learning rate  $\eta_t > 0$ , batch size  $m$ , and termination criterion  $t > T_{\max}$  for some  $T_{\max} > 0$  in order to discover  $w^*$  such as

$$(4) \quad w^* = \arg \min_{w: h_w \in \mathcal{H}} (\mathbb{E}_{z \sim g} (\ell(w, z = (x, y))))$$

The formulas in your algorithm have to be tailored to 3.

- (5) Use the R code given below in order to generate the dataset of observed examples  $S_n = \{z_i = (x_i, y_i)\}_{i=1}^n$  that contains  $n = 10^6$  examples with inputs  $x$  of dimension  $d = 2$ . Consider  $\lambda = 0$ . Use a seed  $w^{(0)} = (0, 0)^\top$ .
- (a) By using appropriate values for  $m$ ,  $\eta_t$  and  $T_{\max}$ , code in R the algorithm you designed in part 4, and run it.
  - (b) Plot the trace plots for each of the dimensions of the generated chain  $\{w^{(t)}\}$  against the iteration  $t$ .
  - (c) Report the value of the output  $w_{\text{adaGrad}}^*$  (any type) of the algorithm as the solution to (4).
  - (d) To which cluster  $y$  (i.e.,  $-1$  or  $1$ )  $x_{\text{new}} = (1, 0)^\top$  belongs?

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# R code. Run it before you run anything else
#
data_generating_model <- function(n,w) {
z <- rep( NaN, times=n*3 )
z <- matrix(z, nrow = n, ncol = 3)
z[,1] <- rep(1,times=n)
z[,2] <- runif(n, min = -10, max = 10)
p <- w[1]*z[,1] + w[2]*z[,2] p <- exp(p) / (1+exp(p))
z[,3] <- rbinom(n, size = 1, prob = p)
ind <- (z[,3]==0)
z[ind,3] <- -1
x <- z[,1:2]
y <- z[,3]
return(list(z=z, x=x, y=y))
}
n_obs <- 1000000
w_true <- c(-3,4)
set.seed(2023)
out <- data_generating_model(n = n_obs, w = w_true)
set.seed(0)
z_obs <- out$z #z=(x,y)
x <- out$x
y <- out$y
#z_obs2=z_obs
#z_obs2[z_obs[,3]==-1,3]=0
#w_true <- as.numeric(glm(z_obs2[,3]~ 1+ z_obs2[,2],family = "binomial"
)$coefficients)

```

**Exercise 8.** (★) Assume a Bayesian model

$$\begin{cases} z_i|w & \stackrel{\text{ind}}{\sim} f(z_i|w), \quad i = 1, \dots, n \\ w & \sim f(w) \end{cases}$$

and consider that our objective is the discovery of MAP estimate  $w^*$  i.e.

$$w^* = \arg \min_{w \in \Theta} (-\log(L_n(w)) - f(w)) = \arg \min_{w \in \Theta} \left( -\sum_{i=1}^n \log(f(z_i|w)) - \log(f(w)) \right)$$

by using SGD with update

$$w^{(t+1)} = w^{(t)} + \eta_t \left( \frac{n}{m} \sum_{j \in \mathcal{J}^{(t)}} \nabla_w \log(f(z_j|w^{(t)})) + \nabla_w \log(f(w^{(t)})) \right)$$

for some randomly selected set  $\mathcal{J}^{(t)} \subseteq \{1, \dots, n\}^m$  of  $m$  integers from 1 to  $n$  via simple random sampling (SRS) with replacement. Show that

$$\mathbb{E}_{\mathcal{J}^{(t)} \sim \text{simple-random-sampling}} \left( \frac{n}{m} \sum_{j \in \mathcal{J}^{(t)}} \nabla_w \log \left( f \left( z_j | w^{(t)} \right) \right) \right) = \sum_{i=1}^n \nabla_w \log \left( f \left( z_i | w^{(t)} \right) \right)$$


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## Part 2. Artificial Neural Networks

**Exercise 9.** (★) Consider the regression problem, with a predictive rule  $h_w : \mathbb{R}^d \rightarrow \mathbb{R}$ , as a classification probability, that receives values  $x \in \mathbb{R}^d$  returns values in  $\mathbb{R}$ . Let  $h_w(x)$  be modeled as an ANN

$$h(x) = \sigma_2 \left( \sum_{j=1}^c w_{2,1,j} \sigma_1 \left( \sum_{i=1}^d w_{1,j,i} x_i \right) \right)$$

and let the associated activation function be

$$\sigma_2(a) = x \Phi(x)$$

where  $\Phi(x) = \int_{-\infty}^x \phi(t) dt$  is considered as known function, and  $\phi(t) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}t^2)$  and

$$\sigma_1(a) = \exp(-a^2)$$

Consider a loss

$$\ell(w, z = (x, y)) = \frac{1}{2} (y - h_w(x))^2$$

at  $w$  and example  $z = (x, y)$ , where  $x \in \mathbb{R}^d$  is the input vector (features), and  $y$  is the output vector (targets) with  $y \in \mathbb{R}$ . Consider that  $d$ ,  $c$ , and  $q$  are known integers.

- (1) Perform the forward pass of the back-propagation procedure to compute the activations which may be denoted as  $\{a_{t,i}\}$  and outputs which may be denoted as  $\{o_{t,i}\}$  at each layer  $t$ .
  - (2) Perform the backward pass of the back-propagation procedure in order to compute the elements of the gradient  $\nabla_w \ell(w, (x, y))$ .
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**Exercise 10.** (★) Students are encouraged to practice on the Exercises 5.1-5.28 from the textbook

- Bishop, C. M. (2006). Pattern recognition and machine learning (Vol. 4, No. 4, p. 738). New York: Springer.

available from

- <https://www.microsoft.com/en-us/research/uploads/prod/2006/01/Bishop-Pattern-Recognition-and-Machine-Learning-2006.pdf>
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- <https://www.maths.dur.ac.uk/users/georgios.karagiannis/temp/Pattern%20recognition%20and%20machine%20learning-solutions-2.pdf>

The following is given as a homework (Formative assessment 2)

**Exercise 11.** (★) Consider the multi-class classification problem, with a predictive rule  $h_w : \mathbb{R}^d \rightarrow \mathcal{P}$ , as a classification probability i.e,  $h_{w,k}(x) = \Pr(x \text{ belongs to class } k)$ , that receives values  $x \in \mathbb{R}^d$  returns vales in  $\mathcal{P} = \left\{ p \in (0, 1)^q : \sum_{j=1}^q p_j = 1 \right\}$ . Let  $h_w = (h_{w,1}, \dots, h_{w,q})^\top$ , let  $h_w(x)$  be modeled as an ANN

$$h_k(x) = \sigma_2 \left( \sum_{j=1}^c w_{2,k,j} \sigma_1 \left( \sum_{i=1}^d w_{1,j,i} x_i \right) \right)$$

for  $k = 1, \dots, q$ , and let the associated activation functions be

$$\sigma_2(a_k) = \frac{\exp(a_k)}{\sum_{k'=1}^q \exp(a_{k'})}, \text{ for } k = 1, \dots, q$$

(called softmax function) and  $\sigma_1(a) = \arctan(a)$ . Consider a loss

$$\ell(w, z = (x, y)) = - \sum_{k=1}^q y_k \log(h_{w,k}(x))$$

at  $w$  and example  $z = (x, y)$ , where  $x \in \mathbb{R}^d$  is the input vector (features), and  $y = (y_1, \dots, y_q)$  is the output vector (labels) with  $y \in \{0, 1\}^q$  and  $\sum_{k=1}^q y_k = 1$ . Consider that  $d$ ,  $c$ , and  $q$  are known integers.

**Hint:** You may use

$$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$$

- (1) Perform the forward pass of the back-propagation procedure to compute the activations which may be denoted as  $\{a_{t,i}\}$  and outputs which may be denoted as  $\{o_{t,i}\}$  at each layer  $t$ .
- (2) Show that

$$\frac{\partial}{\partial a_k} \sigma_2(a_j) = \sigma_2(a_j) (1(j=k) - \sigma_2(a_k))$$

$$\text{for } k = 1, \dots, q. \text{ Let } 1(j = k) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}.$$

- (3) Perform the backward pass of the back-propagation procedure in order to compute the elements of the gradient  $\nabla_w \ell(w, (x, y))$ .

### Part 3. Support Vector Machines

The following is given as a homework (Formative assessment 3)

**Exercise 12.** (★★) Consider a training data set  $\mathcal{D} = \{z_i = (x_i, y_i)\}_{i=1}^m$ . Consider the Soft-SVM Algorithm that requires the solution of the following quadratic minimization problem (in a slightly modified but equivalent form to what we have discussed)

**Primal problem:**

$$(5) \quad (w^*, b^*, \xi^*) = \arg \min_{(w, b, \xi)} \left( \frac{1}{2} \|w\|_2^2 + C \sum_{i=1}^m \xi_i \right)$$

$$(6) \quad \text{subject to: } y_i (\langle w, x_i \rangle + b) \geq 1 - \xi_i, \quad \forall i = 1, \dots, m$$

$$(7) \quad \xi_i \geq 0, \quad \forall i = 1, \dots, m$$

for some user-specified fixed parameter  $C > 0$ .

- (1) Specify the Lagrangian function  $L$  associated to the above primal quadratic minimization problem, where  $\{\alpha_i\}$  are the Lagrange coefficients wrt (6), and  $\{\beta_i\}$  are the Lagrange coefficients wrt (7). Write down any possible restrictions on the Lagrange coefficients.
- (2) Compute the dual Lagrangian function denoted as  $\tilde{L}$  as a function of the Lagrange coefficients and the data points  $\mathcal{D}$ .
- (3) Apply the Karush–Kuhn–Tucker (KKT) conditions to the above problem, and write them down.
- (4) Derive and write down the dual Lagrangian quadratic maximization problem, along with the inequality and equality constraints, where you seek to find  $\{\alpha_i\}$ .
- (5) Justify why the  $i$ -th point  $x_i$  lies on the margin boundary when  $\alpha_i \in (0, C)$  (beware it is  $\alpha_i \neq C$ ), and why the  $i$ -th point  $x_i$  lies inside the margin when  $\alpha_i = C$ .
- (6) Given optimal values  $\{\alpha_i^*\}$  for Lagrangian coefficients  $\{\alpha_i\}$  as they are derived by solving the dual Lagrangian maximization problem in part 4, derive the optimal values  $w^*$  and  $b^*$  for the parameters  $w$  and  $b$  as function of the support vectors. Regarding parameter  $b$  it should be derived in the form

$$b^* = \frac{1}{|\mathcal{M}|} \sum_{i \in \mathcal{M}} \left( y_i - \sum_{j \in \mathcal{S}} \alpha_j^* y_j \langle x_j, x_i \rangle \right)$$

where you determine the sets  $\mathcal{M}$  and  $\mathcal{S}$ .

- (7) Report the halfspace predictive rule  $h_{w,b}(x)$  of the above problem as a function of  $\alpha^*$  and  $b^*$ .
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**Exercise 13.** (\*\*) Show that  $K$  with

$$K(x, y) = \frac{\sin\left(2\pi\left(N + \frac{1}{2}\right)(x - y)\right)}{\sin(\pi(x - y))}$$

is a valid kernel.

**Hint-1:** You may use that  $\sum_{n=0}^r z^n = \frac{1-z^{r+1}}{1-z}$

**Hint-2:** You may use that  $e^{ix} = \cos(x) + i \sin(x)$

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**Exercise 14.** (\*) Students are encouraged to practice on the Exercises 6.1-6.19 from the textbook

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## Part 4. Gaussian process regression

**Exercise 15.** (\*) Students are encouraged to practice on the Exercises 6.19-6.27 from the textbook

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