Part I

Stochastic learning

Exercise 1. $(\star)$Let $f : \mathbb{R}^d \to \mathbb{R}$ such that $f(w) = g(<w, x> + y)$ or some $x \in \mathbb{R}^d$, $y \in \mathbb{R}$. If g is convex function then f is convex function.

Exercise 2. $(\star)$Let functions g_1 be ρ_1-Lipschitz and g_2 be ρ_2-Lipschitz. Then f with $f(x) = g_1(g_2(x))$ is $\rho_1\rho_2$-Lipschitz.

Exercise 3. $(\star)$Let $f : \mathbb{R}^d \to \mathbb{R}$ with $f(w) = g(\langle w, x \rangle + y)$ $x \in \mathbb{R}^d$ and $y \in \mathbb{R}$. Let $g : \mathbb{R} \to \mathbb{R}$ be a β-smooth function. Then f is a $\left(\beta \|x\|^2\right)$-smooth.

Hint: You may use Cauchy-Schwarz inequality $\langle y, x \rangle \leq \|y\| \|x\|$

Exercise 4. $(\star)f : S \to \mathbb{R}$ is ρ-Lipschitz over an open convex set S if and only if for all $w \in S$ and $v \in \partial f(w)$ it is $\|v\| \leq \rho$.

Hint: You may use Cauchy-Schwarz inequality $\langle y, x \rangle \leq \|y\| \|x\|$

Exercise 5. $(\star)$Let $g_1(w), ..., g_r(w)$ be r convex functions, and let $f(\cdot) = \max_{\forall j}(g_j(\cdot))$. Show that for some w it is $\nabla g_k(w) \in \partial f(w)$ where $k = \arg\max_j(g_j(w))$ is the index of function $g_j(\cdot)$ presenting the greatest value at w.

The following is given as a homework (Formative assessment 1)

Exercise 6. (⋆)Consider the binary classification problem with inputs $x \in \mathcal{X}$ where $\mathcal{X} := \left\{ x \in \mathbb{R}^d : \|x\|_2 \leq L \right\}$ for some given value $L > 0$, target $y \in \mathcal{Y}$ where $\mathcal{Y} := \{-1, +1\}$, and prediction rule $h_w : \mathbb{R}^d \to \{-1, +1\}$ with

$$h_w(x) = \text{sign}\left(w^\top x \right) \tag{1}$$

$$= \text{sign}\left(\sum_{j=1}^{d} w_j x_j \right) \tag{2}$$

Let the hypothesis class of prediction rules be

$$\mathcal{H} = \left\{ x \to w^\top x : \forall w \in \mathbb{R}^d \right\}$$

In other words, the hypothesis $h_w \in \mathcal{H}$ is parametrized by $w \in \mathbb{R}^d$ it receives an input vector $x \in \mathcal{X} := \mathbb{R}^d$ and it returns the label $y = \text{sign}\left(w^\top x \right) \in \mathcal{Y} := \{\pm 1\}$.

Consider a loss function $\ell : \mathbb{R}^d \to \mathbb{R}_+$ with

$$\ell(w, z = (x, y)) = \max\left(0, 1 - y w^\top x \right) + \lambda \|w\|_2^2 \tag{3}$$

for some given value $\lambda > 0$.

Assume there is available a dataset of examples $S_n = \{z_i = (x_i, y_i) ; i = 1, ..., n\}$ of size n.

Do the following tasks.

Hint-1: We denote

$$\text{sign}(\xi) = \begin{cases} -1, & \text{if } \xi < 0 \\ +1, & \text{if } \xi > 0 \end{cases}$$

Hint-2: The notation ± 1 means either -1 or $+1$.

HInt-3: We define $\mathbb{R}_+ := (0, +\infty)$

Hint-4: We denote $\|x\|_2 := \sqrt{\sum_{\forall j} (x_j)^2}$ the Euclidean distance.

1. Show that the function $f : \mathbb{R} \to \mathbb{R}_+$ with $f(x) = \max(0, 1 - x)$ is convex in $\mathbb{R}$; and show that the loss (3) is convex.

   Hint: You may use Example 13 from Handout 1.

2. Show that the loss $\ell(w, z)$ for $\lambda = 0$ (3) is L-Lipschitz (with respect to w) when $x \in \mathcal{X}$ where $\mathcal{X} := \left\{ x \in \mathbb{R}^d : \|x\|_2 \leq L \right\}$.

Hint: You may use the definition of Lipschitz function. Without loss of generality, you can consider any $w_1 \in \mathbb{R}^d$ and $w_2 \in \mathbb{R}^d$ such that $1 - yw_2^\top x \leq 1 - yw_1^\top x$, and then take cases $1 - yw_2^\top x > $ or < 0 and $1 - yw_1^\top x > $ or < 0 to deal with the max.

3. Construct the set of sub-gradients $\partial f(x)$ for $x \in \mathbb{R}$ of the function $f : \mathbb{R} \to \mathbb{R}_+$ with $f(x) = \max(0, 1 - x)$. Show that the vector v with

$$
v = \begin{cases} 2\lambda w, & yw^\top x > 1 \\ 2\lambda w, & yw^\top x = 1 \\ -yx + 2\lambda w, & yw^\top x < 1 \end{cases}
$$

is $v \in \partial_w \ell(w, z = (x, y))$, aka a sub-gradient of $\ell(w, z = (x, y))$ at w, for any $w \in \mathbb{R}^d$.

4. Write down the algorithm of online AdaGrad (Adaptive Stochastic Gradient Descent) with learning rate $\eta_t > 0$, batch size m, and termination criterion $t > T_{\max}$ for some $T_{\max} > 0$ in order to discover w^* such as

$$
w^* = \arg \min_{\forall w : h_w \in \mathcal{H}} (\mathbb{E}_{z \sim g} (\ell(w, z = (x, y)))) \tag{4}
$$

The formulas in your algorithm have to be tailored to 3.

5. Use the R code given below in order to generate the dataset of observed examples $S_n = \{z_i = (x_i, y_i)\}_{i=1}^n$ that contains $n = 10^6$ examples with inputs x of dimension $d = 2$. Consider $\lambda = 0$. Use a seed $w^{(0)} = (0, 0)^\top$.

   (a) By using appropriate values for m, η_t and $T_{\max}$, code in R the algorithm you designed in part 4, and run it.

   (b) Plot the trace plots for each of the dimensions of the generated chain $\{w^{(t)}\}$ against the iteration t.

   (c) Report the value of the output w^*_{adaGrad} (any type) of the algorithm as the solution to (4).

   (d) To which cluster y (i.e., -1 or 1) $x_{\text{new}} = (1, 0)^\top$ belongs?

3

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# R code. Run it before you run anything else
#
data_generating_model <- function(n,w) {
z <- rep( NaN, times=n*3 )
z <- matrix(z, nrow = n, ncol = 3)
z[,1] <- rep(1,times=n)
z[,2] <- runif(n, min = -10, max = 10)
p <- w[1]*z[,1] + w[2]*z[,2] p <- exp(p) / (1+exp(p))
z[,3] <- rbinom(n, size = 1, prob = p)
ind <- (z[,3]==0)
z[ind,3] <- -1
x <- z[,1:2]
y <- z[,3]
return(list(z=z, x=x, y=y))
}
n_obs <- 1000000
w_true <- c(-3,4)
set.seed(2023)
out <- data_generating_model(n = n_obs, w = w_true)
set.seed(0)
z_obs <- out$z #z=(x,y)
x <- out$x
y <- out$y
#z_obs2=z_obs
#z_obs2[z_obs[,3]==-1,3]=0
#w_true <- as.numeric(glm(z_obs2[,3]~ 1+ z_obs2[,2],family = "binomial"
)$coefficients)
```

Exercise 7. $(\star)$Assume a Bayesian model

$$
\begin{cases}
z_i|w & \overset{\text{ind}}{\sim} f\left(z_i|w\right), \ i = 1,...,n \\
w & \sim f\left(w\right)
\end{cases}
$$

and consider that our objective is the discovery of MAP estimate w^* i.e.

$$
w^* = \arg\min_{\forall w \in \Theta} \left(-\log\left(L_n\left(w\right)\right) - f\left(w\right)\right) = \arg\min_{\forall w \in \Theta} \left(-\sum_{i=1}^{n} \log\left(f\left(w|z_i\right)\right) - \log\left(f\left(w\right)\right)\right)
$$

by using SGD with update

$$w^{(t+1)} = w^{(t)} + \eta_t \left(\frac{n}{m} \sum_{j \in \mathcal{J}^{(t)}} \nabla_w \log \left(f \left(w^{(t)} | z_j \right) \right) + \nabla_w \log \left(f \left(w^{(t)} \right) \right) \right)$$

for some randomly selected set $\mathcal{J}^{(t)} \subseteq \{1, ..., n\}^m$ of m integers from 1 to n via simple random sampling (SRS) with replacement. Show that

$$\mathrm{E}_{\mathcal{J}^{(t)} \sim \text{simple-random-sampling}} \left(\frac{n}{m} \sum_{j \in \mathcal{J}^{(t)}} \nabla_w \log \left(f \left(w^{(t)} | z_j \right) \right) \right) = \sum_{i=1}^{n} \nabla_w \log \left(f \left(w^{(t)} | z_i \right) \right)$$

5