MATH3431 Machine Learning and Neural Networks III Epiphany term 2023

Homework 2: Artificial Neural Networks

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Instructions: For Formative assessment, submit the solutions all of the problems

Exercise 1. ($\star$)Consider the multi-class classification problem, with a predictive rule $h_w : \mathbb{R}^d \to \mathcal{P}$, as a classification probability i.e, $h_{w,k}(x) = \Pr(x \text{ belongs to class } k)$, that receives values $x \in \mathbb{R}^d$ returns vales in $\mathcal{P} = \left\{p \in (0,1)^q : \sum_{j=1}^q p_j = 1\right\}$. We assume $h_w = (h_{w,1}, ..., h_{w,q})^\top$, and modeled as an ANN

$$h_k(x) = \sigma_2\left(\sum_{j=1}^{c} w_{2,k,j}\sigma_1\left(\sum_{i=1}^{d} w_{1,j,i}x_i\right)\right)$$

for $k = 1, ..., q$, with activation functions softmax function

$$\sigma_2(a_k) = \frac{\exp(a_k)}{\sum_{k'=1}^{q}\exp(a_{k'})}, \text{ for } k = 1, ..., q$$

and $\sigma_1(a) = \arctan(a)$. Consider a loss

$$\ell(w, z = (x, y)) = -\sum_{k=1}^{q} y_k \log(h_{w,k}(x))$$

at w and example $z = (x, y)$, where $x \in \mathbb{R}^d$ is the input vector (features), and $y = (y_1, ..., y_q)$ is the output vector (labels) with $y \in \{0,1\}^q$ and $\sum_{k=1}^{q} y_k = 1$. Consider that d, c, and q are known quantities.

Hint: You may use

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan(x) = \frac{1}{1+x^2}$$

(1) Perform the forward pass of the back-propagation procedure to compute the activations which may be denoted as $\{a_{t,i}\}$ and outputs which may be denoted as $\{o_{t,i}\}$ at each layer t.

(2) Show that

$$\frac{\mathrm{d}}{\mathrm{d}a_k}\sigma_2(a_j) = \sigma_2(a_j)\left(1(j=k) - \sigma_2(a_k)\right)$$

for $k = 1, ..., q$. Let $1(j = k) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$.

(3) Perform the backward pass of the back-propagation procedure in order to compute the elements of the gradient $\nabla_w \ell(w, (x, y))$.

Solution.