MATH3431 Machine Learning and Neural Networks III — Epiphany term 2023

Exercise sheet

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Part 1. Stochastic learning

Exercise 1. ($\star$)Let $f:\mathbb{R}^d\rightarrow\mathbb{R}$ such that $f\left(w\right)=g\left(<w,x>+y\right)$ or some $x\in\mathbb{R}^d$, $y\in\mathbb{R}$. If $g$is convex function then $f$is convex function.

Solution. Let $u,v\in\mathbb{R}^d$ and $a\in[0,1]$. It is

$$\begin{aligned}
f\left(\alpha u+\left(1-\alpha\right)v\right)= & g\left(<\alpha u+\left(1-\alpha\right)v,x>+y\right)\\
= & g\left(<\alpha u,x>+<\left(1-\alpha\right)v,x>+y\right)\\
= & g\left(\alpha\left(<u,x>+y\right)+\left(1-\alpha\right)\left(<v,x>+y\right)\right) && y=\alpha y+\left(1-\alpha\right)y\\
\leq & \alpha g\left(<u,x>+y\right)+\left(1-\alpha\right)g\left(<v,x>+y\right) && (g\text{ is convex})\\
= & \alpha f\left(u\right)+\left(1-\alpha\right)f\left(v\right)
\end{aligned}$$

Exercise 2. ($\star$)Let functions g_1 be ρ_1-Lipschitz and g_2 be ρ_2-Lipschitz. Then f with $f\left(x\right)=g_1\left(g_2\left(x\right)\right)$ is $\rho_1\rho_2$-Lipschitz.

Solution.

$$\begin{aligned}
\left|f\left(w_1\right)-f\left(w_2\right)\right|= & \left|g_1\left(g_2\left(w_1\right)\right)-g_1\left(g_2\left(w_2\right)\right)\right|\\
\leq & \rho_1\left|g_2\left(w_1\right)-g_2\left(w_2\right)\right|\\
\leq & \rho_1\rho_2\left|w_1-w_2\right|
\end{aligned}$$

Exercise 3. ($\star$)Let $f:\mathbb{R}^d\rightarrow\mathbb{R}$ with $f\left(w\right)=g\left(\langle w,x\rangle+y\right)$ $x\in\mathbb{R}^d$ and $y\in\mathbb{R}$. Let $g:\mathbb{R}\rightarrow\mathbb{R}$ be a β-smooth function. Then f is a $\left(\beta\left\Vert x\right\Vert ^2\right)$-smooth.

Hint:: You may use Cauchy-Schwarz inequality $\langle y,x\rangle\leq\left\Vert y\right\Vert \left\Vert x\right\Vert$

$$
\begin{aligned}
f(v) =& g(\langle w, x\rangle + y) \\
\leq& g(\langle w, x\rangle + y) + g'(\langle w, x\rangle + y)\langle v-w, x\rangle + \frac{\beta}{2}(\langle v-w, x\rangle)^2 && (g \text{ is smooth}) \\
\leq& g(\langle w, x\rangle + y) + g'(\langle w, x\rangle + y)\langle v-w, x\rangle + \frac{\beta}{2}(\|v-w\|\,\|x\|)^2 && (\text{Cauchy-Schwatz inequality}) \\
=& f(w) + \langle \nabla f(w), v-w\rangle + \frac{\beta\|x\|^2}{2}\|v-w\|^2
\end{aligned}
$$

Exercise 4. $(\star) f : S \to \mathbb{R}$ is ρ-Lipschitz over an open convex set S if and only if for all $w \in S$ and $v \in \partial f(w)$ it is $\|v\| \leq \rho$.

Hint:: You may use Cauchy-Schwarz inequality $\langle y, x\rangle \leq \|y\|\,\|x\|$

Solution. $\Longrightarrow$ Let $f : S \to \mathbb{R}$ be ρ-Lipschitz over convex set S, $w \in S$ and $v \in \partial f(w)$.

- Since S is open we get that there exist $\epsilon > 0$ such as $u := w + \epsilon\frac{v}{\|v\|}$ where $u \in S$. So $\langle u-w, v\rangle = \epsilon\|v\|$ and $\|u-w\| = \epsilon$.
- From the subgradient definition we get

$$f(u) - f(w) \geq \langle u-w, v\rangle = \epsilon\|v\|$$

- From the Lipschitzness of $f(\cdot)$ we get

$$f(u) - f(w) \geq \rho\|u-w\| = \rho\epsilon$$

Therefore $\|v\| \leq \rho$.

$\Longleftarrow$ It is for all $w \in S$ and $v \in \partial f(w)$ it is $\|v\| \leq \rho$.

- For any $u \in S$, it is

$$
\begin{aligned}
f(w) - f(u) \leq& \langle v, w-u\rangle && (\text{ because } v \in \partial f(w) \text{ }) \\
\leq& \|v\|\,\|w-u\| && \text{by Cauchy-Schwarz inequality} \\
\leq& \rho\|w-u\| && \text{because } \|v\| \leq \rho
\end{aligned}
\tag{1}
$$

- Similarly it results $u, w \in S$

$$f(w) - f(u) \leq \langle v, u-w\rangle\|v\| \leq \|v\|\,\|u-w\| \leq \rho\|u-w\|$$

from (1) because w, u can be swaped in (1) as they both are any values in S.

Exercise 5. $(\star)$Let $g_1(w), ..., g_r(w)$ be r convex functions, and let $f(\cdot) = \max_{\forall j}(g_j(\cdot))$. Show that for some w it is $\nabla g_k(w) \in \partial f(w)$ where $k = \arg\max_j(g_j(w))$ is the index of function $g_j(\cdot)$ presenting the greatest value at w.

Since g_k is convex, for all u

$$g_k(u) \geq g_k(w) + \langle u - w, \nabla g_k(w) \rangle$$

However $f(u) = \max_{\forall j}(g_j(u)) \geq g_k(u)$ for any j, and $f(w) = g_k(w)$ at w. Then

$$\begin{aligned} f(u) \geq & g_k(u) \\ \geq & g_k(w) + \langle u - w, \nabla g_k(w) \rangle \\ = & f(w) + \langle u - w, \nabla g_k(w) \rangle \end{aligned}$$

Then by the definition of the sub-gradient $\nabla g_k(w) \in \partial f(w)$

The following is given as a homework (Formative assessment 1)

Exercise 6. ($\star$)Consider the binary classification problem with inputs $x \in \mathcal{X}$ where $\mathcal{X} := \left\{x \in \mathbb{R}^d : \|x\|_2 \leq L\right\}$ for some given value $L > 0$, target $y \in \mathcal{Y}$ where $\mathcal{Y} := \{-1, +1\}$, and prediction rule $h_w : \mathbb{R}^d \to \{-1, +1\}$ with

$$h_w(x) = \text{sign}\left(w^\top x\right) \tag{2}$$

$$= \text{sign}\left(\sum_{j=1}^{d} w_j x_j\right) \tag{3}$$

Let the hypothesis class of prediction rules be

$$\mathcal{H} = \left\{x \to w^\top x : \forall w \in \mathbb{R}^d\right\}$$

In other words, the hypothesis $h_w \in \mathcal{H}$ is parametrized by $w \in \mathbb{R}^d$ it receives an input vector $x \in \mathcal{X} := \mathbb{R}^d$ and it returns the label $y = \text{sign}\left(w^\top x\right) \in \mathcal{Y} := \{\pm 1\}$.

Consider a loss function $\ell : \mathbb{R}^d \to \mathbb{R}_+$ with

$$\ell(w, z = (x, y)) = \max\left(0, 1 - y w^\top x\right) + \lambda \|w\|_2^2 \tag{4}$$

for some given value $\lambda > 0$.

Assume there is available a dataset of examples $S_n = \{z_i = (x_i, y_i); i = 1, ..., n\}$ of size n.

Do the following tasks.

Hint-1:: We denote

$$\text{sign}(\xi) = \begin{cases} -1, & \text{if} \, \xi < 0 \\ +1, & \text{if} \, \xi > 0 \end{cases}$$

Hint-2:: The notation ± 1 means either -1 or $+1$.

HInt-3:: We define $\mathbb{R}_+ := (0, +\infty)$

Hint-4:: We denote $\|x\|_2 := \sqrt{\sum_{\forall j}(x_j)^2}$ the Euclidean distance.

(1) Show that the function $f : \mathbb{R} \to \mathbb{R}_+$ with $f(x) = \max(0, 1 - x)$ is convex in $\mathbb{R}$; and show that the loss (4) is convex.

Hint:: You may use Example 13 from Handout 1.

(2) Show that the loss $\ell(w,z)$ for $\lambda = 0$ (4) is L-Lipschitz (with respect to w) when $x \in \mathcal{X}$ where $\mathcal{X} := \left\{x \in \mathbb{R}^d : \|x\|_2 \leq L\right\}$.

Hint:: You may use the definition of Lipschitz function. Without loss of generality, you can consider any $w_1 \in \mathbb{R}^d$ and $w_2 \in \mathbb{R}^d$ such that $1 - yw_2^\top x \leq 1 - yw_1^\top x$, and then take cases $1 - yw_2^\top x$ > or < 0 and $1 - yw_1^\top x$ > or < 0 to deal with the max.

(3) Construct the set of sub-gradients $\partial f(x)$ for $x \in \mathbb{R}$ of the function $f : \mathbb{R} \to \mathbb{R}_+$ with $f(x) = \max(0, 1-x)$. Show that the vector v with

$$v = \begin{cases} 2\lambda w, & yw^\top x > 1 \\ 2\lambda w, & yw^\top x = 1 \\ -yx + 2\lambda w, & yw^\top x < 1 \end{cases}$$

is $v \in \partial_w \ell(w, z = (x,y))$, aka a sub-gradient of $\ell(w, z = (x,y))$ at w, for any $w \in \mathbb{R}^d$.

(4) Write down the algorithm of online AdaGrad (Adaptive Stochastic Gradient Descent) with learning rate $\eta_t > 0$, batch size m, and termination criterion $t > T_{\max}$ for some $T_{\max} > 0$ in order to discover w^* such as

$$w^* = \arg \min_{\forall w : h_w \in \mathcal{H}} \left(\mathrm{E}_{z \sim g}\left(\ell(w, z = (x,y))\right)\right) \tag{5}$$

The formulas in your algorithm have to be tailored to 4.

(5) Use the R code given below in order to generate the dataset of observed examples $S_n = \{z_i = (x_i, y_i)\}_{i=1}^n$ that contains $n = 10^6$ examples with inputs x of dimension $d = 2$. Consider $\lambda = 0$. Use a seed $w^{(0)} = (0,0)^\top$.

(a) By using appropriate values for m, η_t and $T_{\max}$, code in R the algorithm you designed in part 4, and run it.

(b) Plot the trace plots for each of the dimensions of the generated chain $\left\{w^{(t)}\right\}$ against the iteration t.

(c) Report the value of the output w^*_{adaGrad} (any type) of the algorithm as the solution to (5).

(d) To which cluster y (i.e., -1 or 1) $x_{\text{new}} = (1,0)^\top$ belongs?

```
# R code. Run it before you run anything else
#
data_generating_model <- function(n,w) {
z <- rep( NaN, times=n*3 )
z <- matrix(z, nrow = n, ncol = 3)
z[,1] <- rep(1,times=n)
z[,2] <- runif(n, min = -10, max = 10)
p <- w[1]*z[,1] + w[2]*z[,2] p <- exp(p) / (1+exp(p))
z[,3] <- rbinom(n, size = 1, prob = p)
ind <- (z[,3]==0)
z[ind,3] <- -1
x <- z[,1:2]
y <- z[,3]
return(list(z=z, x=x, y=y))
}
n_obs <- 1000000
w_true <- c(-3,4)
set.seed(2023)
out <- data_generating_model(n = n_obs, w = w_true)
set.seed(0)
z_obs <- out$z #z=(x,y)
x <- out$x
y <- out$y
#z_obs2=z_obs
#z_obs2[z_obs[,3]==-1,3]=0
#w_true <- as.numeric(glm(z_obs2[,3]~ 1+ z_obs2[,2],family = "binomial"
)$coefficients)
```

Solution.

(1) $f_1(x) = 0$ is convex, $f_2(x) = 1 - x$ is convex, hence from the example in Handout 1, $f(x) = \max(f_1(x), f_2(x))$ is convex as well. Regarding the loss function, we just have $f_2(w) = 1 - yx^\top w$ which is convex as a composition due to linearity.

(2) Given a fixed example $(x, y) \in \left\{x \in \mathbb{R}^d : \|x'\|_2 \leq R\right\} \times \{-1, 1\}$.

Assume $w_1, w_2 \in \mathbb{R}^d$. Let $\ell_i = \max\left\{0, 1 - yx^\top w_i\right\}$, for $i = 1, 2$. It suffices to show that $|\ell_1 - \ell_2|_2 \leq R\,|w_1 - w_2|_2$. I take cases

Case-1: Assume $yx^\top w_1 \geq 1$ and $yx^\top w_2 \geq 1$ then $|\ell_1 - \ell_2|_2 = 0 \leq R\,|w_1 - w_2|_2$

Case-2: Assume that at least one of $yx^\top w_1 < 1$ or $yx^\top w_2 < 1$ but not both is true. Assume without loss of generality that $1 - yx^\top w_1 < 1 - yx^\top w_2$. Then

$$\begin{aligned}
|\ell_1 - \ell_2|_2 =& \ell_1 - \ell_2 \\
=& 1 - yx^\top w_1 - \max\left(0, 1 - yx^\top w_2\right) \\
\leq& 1 - yx^\top w_1 - \left(1 - yx^\top w_2\right) \\
=& yx^\top \left(w_2 - w_1\right) \\
\leq& y \left\|x^\top\right\|_2 \|w_1 - w_2\|_2 \quad \text{because} \quad a^\top b \leq \|a\| \, \|b\|
\end{aligned}$$

(3) It is

$$f(x) = \max(0, 1 - x) = \begin{cases} 0 & x > 1 \\ 0 & x = 1 \\ 1 - x & x < 1 \end{cases}$$

- For $x > 1$, f is differentiable so $\partial f(x) = \{f'(x)\} = \{0\}$.
- For $x < 1$, f is differentiable so $\partial f(x) = \{f'(x)\} = \{-1\}$.
- For $x = 1$, f is not differentiable. By definition I have that v is subgradient of $f(x)$ at $x = 0 \in S$ if

  $$\forall u \in \mathbb{R}, \;\; f(u) \geq f(x) + \langle u - x, v \rangle$$

  So, for $u \geq 1$, it is $0 \geq (u - 1) v \implies v \leq 0$, and for $u < 1$ it is $(1 - u) \geq (u - 1) v \implies v \geq -1$. Hence the common space is $v \in [0, 1]$ So $\partial f(x) = [0, 1]$. Hence,

  $$\partial f(x) = \begin{cases} 0, & x > 1 \\ [0, 1], & x = 1 \\ -1, & x < 1 \end{cases}$$

Now regarding the loss $\partial_w \ell(w, z = (x, y))$

- for $yw^\top x > 1$ it is differentiable so $\nabla_w \ell(w, z = (x, y)) = \nabla_w \left(0 + \lambda \sum_{j=1}^d w_j^2\right) = 2\lambda w$; as

  $$\frac{\mathrm{d}}{\mathrm{d}w_j} \sum_{j'=1}^{d} w_{j'}^2 = 2\lambda w_j$$

- for $yw^\top x > 1$ it is differentiable so $\nabla_w \ell(w, z = (x, y)) = \nabla_w \left(1 - yw^\top x + \lambda \sum_{j=1}^d w_j^2\right) = yx + 2\lambda w$ as

  $$\frac{\mathrm{d}}{\mathrm{d}w_j} \left(1 - yw^\top x\right) = \frac{\mathrm{d}}{\mathrm{d}w_j} \left(1 - y \sum_{j'=1}^{d} w_{j'} x_{j'}\right) = -yx_j$$

- for $yw^\top x = 1$, $v = 0$ satisfies the definition of the sub-gradient

$$
\begin{aligned}
\forall u, \quad f(u) &\geq \cancelto{0}{f(w)} + \langle u - w, v \rangle \\
\max\left(0, 1 - yu^\top x\right) &\geq 0 + (u - w)^\top 0
\end{aligned}
$$

So

$$
\begin{aligned}
\partial \ell\left(w, z = (x, y)\right) &= \partial\left(\max\left(0, 1 - yw^\top x\right) + \lambda \left\|w\right\|_2^2\right) \\
&= \partial\left(\max\left(0, 1 - yw^\top x\right)\right) + \partial\left(\lambda \left\|w\right\|_2^2\right) \\
&= \partial\left(\max\left(0, 1 - yw^\top x\right)\right) + \nabla\left(\lambda \left\|w\right\|_2^2\right) \\
& 0 + 2\lambda w
\end{aligned}
$$

but $\partial\left(\lambda \left\|w\right\|_2^2\right) = \left\{\nabla\left(\lambda \left\|w\right\|_2^2\right)\right\}$ because $\lambda \left\|w\right\|_2^2$ is differentiable. Hence

$$
\partial \ell\left(w, z = (x, y)\right) = 0 + 2\lambda w
$$

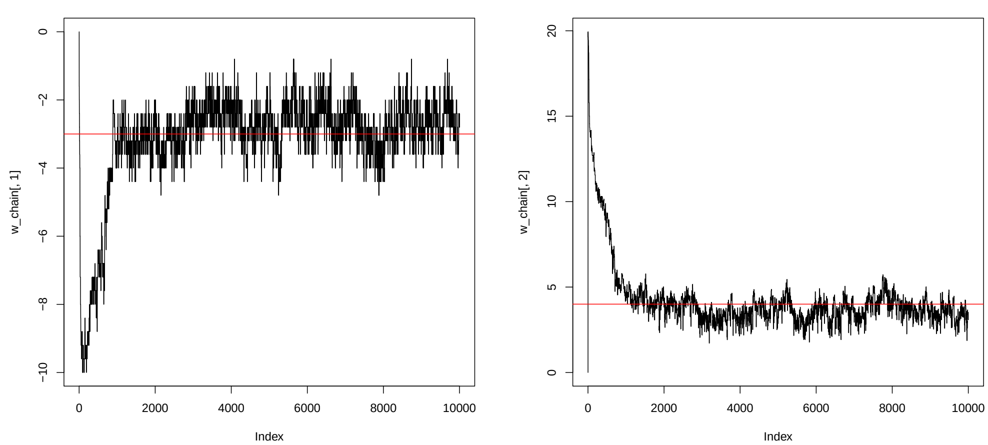
Hence

$$
v = \begin{cases} 2\lambda w, & yw^\top x > 1 \\ 2\lambda w, & yw^\top x = 1 \\ -yx + 2\lambda w, & yw^\top x < 1 \end{cases}
$$

(4)

(a) The R code can be found in the link `https://raw.githubusercontent.com/georgios-stats/Machine_Learning_and_Neural_Networks_III_Epiphany_2023/main/Exercises/supplementary/q6.R`

(b) The figures are presented below



(A) $w_1^{(t)}$ (B) $w_2^{(t)}$

FIGURE 1. trace plots

(c) I found $w = (-2.674615, 3.205785)$

(d) It belongs to -1

Exercise 7. ($\star$)Assume a Bayesian model

$$\begin{cases} z_i|w & \overset{\text{ind}}{\sim} f\left(z_i|w\right),\ i=1,...,n \\ w & \sim f\left(w\right) \end{cases}$$

and consider that our objective is the discovery of MAP estimate w^* i.e.

$$w^* = \arg\min_{\forall w\in\Theta}\left(-\log\left(L_n\left(w\right)\right) - f\left(w\right)\right) = \arg\min_{\forall w\in\Theta}\left(-\sum_{i=1}^{n}\log\left(f\left(z_i|w\right)\right) - \log\left(f\left(w\right)\right)\right)$$

by using SGD with update

$$w^{(t+1)} = w^{(t)} + \eta_t\left(\frac{n}{m}\sum_{j\in\mathcal{J}^{(t)}}\nabla_w\log\left(f\left(z_j|w^{(t)}\right)\right) + \nabla_w\log\left(f\left(w^{(t)}\right)\right)\right)$$

for some randomly selected set $\mathcal{J}^{(t)} \subseteq \{1,...,n\}^m$ of m integers from 1 to n via simple random sampling (SRS) with replacement. Show that

$$\mathrm{E}_{\mathcal{J}^{(t)}\sim\text{simple-random-sampling}}\left(\frac{n}{m}\sum_{j\in\mathcal{J}^{(t)}}\nabla_w\log\left(f\left(z_j|w^{(t)}\right)\right)\right) = \sum_{i=1}^{n}\nabla_w\log\left(f\left(z_i|w^{(t)}\right)\right)$$

Solution. It is

$$\begin{aligned}
\mathrm{E}_{\mathcal{J}^{(t)}\sim\text{SRS}}\left(\frac{n}{m}\sum_{j\in\mathcal{J}^{(t)}}\nabla_w\log\left(f\left(z_j|w^{(t)}\right)\right)\right) =& \frac{n}{m}\sum_{j\in\mathcal{J}^{(t)}}\mathrm{E}_{\mathcal{J}^{(t)}\sim\text{SRS}}\left(\nabla_w\log\left(f\left(z_j|w^{(t)}\right)\right)\right) \\
=& \frac{n}{m}\sum_{j\in\mathcal{J}^{(t)}}\mathrm{E}_{\mathcal{J}^{(t)}\sim\text{SRS}}\left(\nabla_w\log\left(f\left(z_j|w^{(t)}\right)\right)\right) \\
=& \frac{n}{m}\sum_{j\in\mathcal{J}^{(t)}}\frac{1}{n}\sum_{i=1}^{n}\nabla_w\log\left(f\left(z_i|w^{(t)}\right)\right) \\
=& \sum_{i=1}^{n}\nabla_w\log\left(f\left(z_i|w^{(t)}\right)\right)
\end{aligned}$$

It is $\mathrm{E}_{\mathcal{J}^{(t)}\sim\text{SRS}}\left(\nabla_w\log\left(f\left(z_j|w^{(t)}\right)\right)\right) = \frac{1}{n}\sum_{i=1}^{n}\nabla_w\log\left(f\left(z_i|w^{(t)}\right)\right)$ because the expectation is under the probability I get randomly an integer and for the jth on the probability is $1/n$ due to the random scheme. Also $\left|\mathcal{J}^{(t)}\right| = m$.
