

## Homework 2: Stochastic Gradient Descent

Lecturer: Georgios P. Karagiannis

georgios.karagiannis@durham.ac.uk

**Exercise 1.** <sup>1</sup>Consider the binary classification problem with inputs  $x \in \mathcal{X}$  where  $\mathcal{X} := \{x \in \mathbb{R}^d : \|x\|_2 \leq L\}$  for some given value  $L > 0$ , target  $y \in \mathcal{Y}$  where  $\mathcal{Y} := \{-1, +1\}$ , and prediction rule  $h_w : \mathbb{R}^d \rightarrow \{-1, +1\}$  with

$$(0.1) \quad h_w(x) = \text{sign}(w^\top x)$$

$$(0.2) \quad = \text{sign}\left(\sum_{j=1}^d w_j x_j\right)$$

Let the hypothesis class is

$$(0.3) \quad \mathcal{H} = \{x \rightarrow w^\top x : \forall w \in \mathbb{R}^d\}$$

In other words, the hypothesis  $h_w \in \mathcal{H}$  is parametrized by  $w \in \mathbb{R}^d$ , it receives an input vector  $x \in \mathcal{X} := \mathbb{R}^d$  and it returns the label  $y = \text{sign}(w^\top x) \in \mathcal{Y} := \{\pm 1\}$  where

$$\text{sign}(\xi) = \begin{cases} -1, & \text{if } \xi < 0 \\ +1, & \text{if } \xi > 0 \end{cases}$$

Consider a loss function  $\ell : \mathbb{R}^d \rightarrow \mathbb{R}_+$  with

$$(0.4) \quad \ell(w, z = (x, y)) = \max(0, 1 - yw^\top x) + \lambda \|w\|_2^2$$

for some given value  $\lambda > 0$ .

Assume there is available a dataset of examples  $S_n = \{z_i = (x_i, y_i) ; i = 1, \dots, n\}$  of size  $n$ . Do the following:

- (1) Show that the function  $f : \mathbb{R} \rightarrow \mathbb{R}_+$  with  $f(x) = \max(0, 1 - x)$  is convex in  $\mathbb{R}$ ; and show that the loss (0.4) is convex.

**Hint::** You may use Note ?? from Lecture notes ?? : Elements of convex learning problems.

- (2) Show that the loss  $\ell(w, z)$  for  $\lambda = 0$  (0.4) is  $L$ -Lipschitz (with respect to  $w$ ) when  $x \in \mathcal{X}$  where  $\mathcal{X} := \{x \in \mathbb{R}^d : \|x\|_2 \leq L\}$ .

<sup>1</sup>We use standard notation

$$\text{sign}(\xi) = \begin{cases} -1, & \text{if } \xi < 0 \\ +1, & \text{if } \xi > 0 \end{cases}$$

$\pm 1$  means either  $-1$  or  $+1$ ,  $\mathbb{R}_+ := (0, +\infty)$ , and  $\|x\|_2 := \sqrt{\sum_{j=1}^d (x_j)^2}$  for the Euclidean distance.

**Hint::** You may use the definition of Lipschitz function. Without loss of generality, you can consider any  $w_1 \in \mathbb{R}^d$  and  $w_2 \in \mathbb{R}^d$  such that  $1 - yw_2^\top x \leq 1 - yw_1^\top x$ , and then take cases  $1 - yw_2^\top x > \text{or} < 0$  and  $1 - yw_1^\top x > \text{or} < 0$  to deal with the max.

- (3) Construct the set of sub-gradients  $\partial f(x)$  for  $x \in \mathbb{R}$  of the function  $f : \mathbb{R} \rightarrow \mathbb{R}_+$  with  $f(x) = \max(0, 1 - x)$ . Show that the vector  $v$  with

$$v = \begin{cases} 2\lambda w, & yw^\top x > 1 \\ 2\lambda w, & yw^\top x = 1 \\ -yx + 2\lambda w, & yw^\top x < 1 \end{cases}$$

is  $v \in \partial_w \ell(w, z = (x, y))$ , aka a sub-gradient of  $\ell(w, z = (x, y))$  at  $w$ , for any  $w \in \mathbb{R}^d$ .

- (4) Write down the algorithm of online AdaGrad (Adaptive Stochastic Gradient Descent) with learning rate  $\eta_t > 0$ , batch size  $m$ , and termination criterion  $t > T_{\max}$  for some  $T_{\max} > 0$  in order to discover  $w^*$  such as

$$(0.5) \quad w^* = \arg \min_{w: h_w \in \mathcal{H}} (\mathbb{E}_{z \sim g} (\ell(w, z = (x, y))))$$

The formulas in your algorithm should be implemented for the above learning problem and tailored to 0.1, 0.3, and 0.4.

- (5) Use the R code given below in order to generate the dataset of observed examples  $S_n = \{z_i = (x_i, y_i)\}_{i=1}^n$  that contains  $n = 10^6$  examples with inputs  $x$  of dimension  $d = 2$ . Consider  $\lambda = 0$ . Use a seed  $w^{(0)} = (0, 0)^\top$ .
- By using appropriate values for  $m$ ,  $\eta_t$  and  $T_{\max}$ , code in R the algorithm you designed in part 4, and run it.
  - Plot the trace plots for each of the dimensions of the generated chain  $\{w^{(t)}\}$  against the iteration  $t$ .
  - Report the value of the output  $w_{\text{adaGrad}}^*$  (any type) of the algorithm as the solution to (0.5).
  - To which cluster  $y$  (i.e.,  $-1$  or  $1$ )  $x_{\text{new}} = (1, 0)^\top$  belongs?

```

# R code. Run it before you run anything else
#
data_generating_model <- function(n,w) {
  z <- rep( NaN, times=n*3 )
  z <- matrix(z, nrow = n, ncol = 3)
  z[,1] <- rep(1,times=n)
  z[,2] <- runif(n, min = -10, max = 10)
  p <- w[1]*z[,1] + w[2]*z[,2] p <- exp(p) / (1+exp(p))
  z[,3] <- rbinom(n, size = 1, prob = p)
  ind <- (z[,3]==0)
  z[ind,3] <- -1
  x <- z[,1:2]
  y <- z[,3]
  return(list(z=z, x=x, y=y))
}
n_obs <- 1000000
w_true <- c(-3,4)
set.seed(2023)
out <- data_generating_model(n = n_obs, w = w_true)
set.seed(0)
z_obs <- out$z #z=(x,y)
x <- out$x
y <- out$y
#z_obs2=z_obs
#z_obs2[z_obs[,3]==-1,3]=0
#w_true <- as.numeric(glm(z_obs2[,3]~ 1+ z_obs2[,2],family = "binomial"
)$coefficients)

```

**Solution.**