MATH3431 Machine Learning and Neural Networks III — Epiphany term 2025

Exercise sheet

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Part 1. Convex learning problems

Exercise 1. ($\star$)Let $f:\mathbb{R}^d\rightarrow\mathbb{R}$ such that $f(w)=g(<w,x>+y)$ or some $x\in\mathbb{R}^d$, $y\in\mathbb{R}$. Show that: If g is convex function then $f$is convex function.

Exercise 2. ($\star$)Let functions g_1 be ρ_1-Lipschitz and g_2 be ρ_2-Lipschitz. Then, show that, f with $f(x)=g_1(g_2(x))$ is $\rho_1\rho_2$-Lipschitz.

Exercise 3. ($\star$)Let $f:\mathbb{R}^d\rightarrow\mathbb{R}$ with $f(w)=g(\langle w,x\rangle+y)$ $x\in\mathbb{R}^d$ and $y\in\mathbb{R}$. Let $g:\mathbb{R}\rightarrow\mathbb{R}$ be a β-smooth function. Then show that f is a $\left(\beta\|x\|^2\right)$-smooth.

Hint:: You may use Cauchy-Schwarz inequality $\langle y,x\rangle\leq\|y\|\|x\|$

Exercise 4. ($\star$)Show that $f:S\rightarrow\mathbb{R}$ is ρ-Lipschitz over an open convex set S if and only if for all $w\in S$ and $v\in\partial f(w)$ it is $\|v\|\leq\rho$.

Hint:: You may use Cauchy-Schwarz inequality $\langle y,x\rangle\leq\|y\|\|x\|$

Exercise 5. ($\star$)Let $g_1(w),...,g_r(w)$ be r convex functions, and let $f(\cdot)=\max_{\forall j}(g_j(\cdot))$. Show that for some w it is $\nabla g_k(w)\in\partial f(w)$ where $k=\arg\max_j(g_j(w))$ is the index of function $g_j(\cdot)$ presenting the greatest value at w.

Exercise 6. ($\star$)Consider the regression learning problem $(\mathcal{H},\mathcal{Z},\ell)$ with predictor rule $h(x)=\langle w,x\rangle$ labeled by some unknown parameter $w\in\mathcal{W}$, loss function $\ell(w,(x,y))=(\langle w,x\rangle-y)^2$, feature $x\in\mathcal{X}$, and target $y\in\mathbb{R}$. Let $\mathcal{W}=\mathcal{X}=\left\{\omega\in\mathbb{R}^d:|\omega|\leq\rho\right\}$ for some $\rho>0$.

(1) Show that the resulting learning problem is Convex-Lipschitz-Bounded learning problem.
(2) Specify the parameters of Lipschitnzess.

Exercise 7. ($\star$)If $f$is λ-strongly convex and u is a minimizer of f then for any w

$$f(w) - f(u) \geq \frac{\lambda}{2} \|w - u\|^2$$

Hint:: Use the definition, and set $\alpha \to 0$.

Exercise 8. ($\star$) Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex and β-smooth function.

(1) Show that for $v, w \in \mathbb{R}^d$

$$f(v) - f(w) \in \left(\langle \nabla f(w), v - w \rangle, \langle \nabla f(w), v - w \rangle + \frac{\beta}{2} \|v - w\|^2 \right)$$

(2) Show that for $v, w \in \mathbb{R}^d$ such that $v = w - \frac{1}{\beta} \nabla f(w)$, it is

$$\frac{1}{2\beta} \|\nabla f(w)\|^2 \leq f(w) - f(v)$$

(3) Additionally assume that $f(x) > 0$ for all $x \in \mathbb{R}^d$. Show that for $w \in \mathbb{R}^d$,

$$\|\nabla f(w)\| \leq \sqrt{2\beta f(w)}$$

Exercise 9. ($\star$)Let $f : \mathbb{R}^d \to \mathbb{R}$ be a λ-strongly convex function. Assume that w^* is a minimizer of f i.e.

$$w^* = \arg\min_w \{f(w)\}$$

Show that for any $w \in \mathbb{R}^d$ it holds

$$f(w) - f(w^*) \geq \frac{\lambda}{2} \|w - w^*\|^2$$

Hint: Use the definition of λ-strongly convex function, properly rearrange it, and ...

Exercise 10. ($\star$)Show that the function $J(x; \lambda) = \lambda \|x\|^2$ is 2λ-strongly convex

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