Homework 3: Geostatistics (Change of support)

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Exercise 1. $(\star)$ Suppose a large volume V is partitioned into n smaller units v of equal size. Show that the dispersion variance $\sigma^2\left(u|V\right) = \frac{1}{n}\sum_{j=1}^{n}\sigma_E^2\left(v_j, V\right)$ can be written in term of variogram integrals

$$\bar{\gamma}\left(v, V\right) = \frac{1}{|v|\,|V|}\int_{s\in v}\int_{s\in V}\gamma\left(s - s'\right)\mathrm{d}s\mathrm{d}s'$$

as

$$\sigma^2\left(v|V\right) = \bar{\gamma}\left(V, V\right) - \bar{\gamma}\left(v, v\right)$$

Solution.

Exercise 2. $(\star)$ Consider a statistical model which is a stochastic process $(Z_s)_{s\in\mathbb{R}}$ (so s has dimension 1), where $Z\left(\cdot\right)\sim\mathrm{GP}\left(\mu\left(\cdot\right), c\left(\cdot, \cdot\right)\right)$ with mean function $\mu\left(s\right) = 1$ and covariance function $c\left(s, t\right) = \exp\left(-\left(s - t\right)^2\right)$ for any $s\in\mathbb{R}$ and $t\in\mathbb{R}$. Assume there is available a dataset $\{(Z_i, s_i)\}_{i=1}^{n}$ where $Z_i = Z\left(s_i\right)$ and $s_i\in\mathbb{R}$ are point sites.

(1) Compute the length $|v|$ of the block $v = [a, b]\subset\mathbb{R}$.

(2) Compute the block mean $\mu\left(v\right)$ for some block $v = [a, b]\subset\mathbb{R}$ and point $s\in\mathbb{R}$.

(3) Compute the block covariance function $c\left(v, s\right)$ for some block $v = [a, b]\subset\mathbb{R}$ and point $s\in\mathbb{R}$.

(4) Compute the block covariance function $c\left(v, v'\right)$ for some blocks $v = [a, b]\subset\mathbb{R}$ and $v' = [a', b']\subset\mathbb{R}$.

(5) Denote $Z = (Z_1, ..., Z_n)^\top$, and $S = \{s_1, ..., s_n\}$. Let $v = [a, b]\subset\mathbb{R}$ and $v' = [a', b']\subset\mathbb{R}$ be two intervals. Compute the joint distribution of $\left(Z\left(v\right), Z\left(v'\right), Z\right)^\top$ as a function of $c\left(\cdot, \cdot\right)$, S, v, v', Z, and $\mu\left(\cdot\right)$. What is the name of the distribution and what are the parameter functions defining it?

(6) (Bayesian Kriging) Compute the predictive stochastic process $[Z\left(v\right)|Z]$ at blocks $v = [a, b]\subset\mathbb{R}$ with $|v| > 0$.

Hint-1:: Let $x_1\in\mathbb{R}^{d_1}$, and $x_2\in\mathbb{R}^{d_2}$. If

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}\sim\mathrm{N}_{d_1+d_2}\left(\begin{bmatrix}\mu_1\\\mu_2\end{bmatrix}, \begin{bmatrix}\Sigma_1 & \Sigma_{21}^\top\\\Sigma_{21} & \Sigma_2\end{bmatrix}\right)$$

then it is

$$x_2|x_1\sim\mathrm{N}_{d_2}\left(\mu_{2|1}, \Sigma_{2|1}\right)$$

where

$$\mu_{2|1} = \mu_2 + \Sigma_{21}\Sigma_1^{-1}(x_1 - \mu_1) \quad \text{and} \quad \Sigma_{2|1} = \Sigma_2 - \Sigma_{21}\Sigma_1^{-1}\Sigma_{21}^\top$$

Hint-2: Assume known function $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2\right) \mathrm{d}t$. Then $\int \mathrm{erf}(x)\,\mathrm{d}x = x\,\mathrm{erf}(x) + \frac{\exp\left(-x^2\right)}{\sqrt{\pi}} + \mathrm{const}$

Solution.