Problem class sheet 3

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Exercise 1. $(\star\star)$

Inventory of useful formulas.

[Normal distr. conditioning] Let $x_1 \in \mathbb{R}^{d_1}$, and $x_2 \in \mathbb{R}^{d_2}$. If

$$
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N_{d_1+d_2} \left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_1 & \Sigma_{21}^\top \\ \Sigma_{21} & \Sigma_2 \end{bmatrix} \right)
$$

then it is

$$
x_2|x_1 \sim N_{d_2} \left(\mu_{2|1}, \Sigma_{2|1} \right)
$$

where

$$
\mu_{2|1} = \mu_2 + \Sigma_{21}\Sigma_1^{-1} \left(x_1 - \mu_1 \right) \text{ and } \Sigma_{2|1} = \Sigma_2 - \Sigma_{21}\Sigma_1^{-1}\Sigma_{21}^\top
$$

Consider the Bayesian Kriging from your lecture notes:

$$
Z\left(s\right) = Y\left(s\right) + \varepsilon\left(s\right), \ \ s \in \mathcal{S}
$$

where

$$
\varepsilon\left(\cdot\right) \sim GP\left(0, c_\varepsilon\left(\cdot,\cdot|\tau\right)\right)
$$

with $c_\varepsilon\left(s, s'|\tau\right) = \tau^2 1_{\{0\}}\left(\|s-s'\|\right)$ and

$$
Y\left(\cdot\right)|\beta, \theta \sim GP\left(\mu\left(\cdot|\beta\right), c_Y\left(\cdot,\cdot|\sigma^2, \phi\right)\right)
$$

with mean function $\mu\left(\cdot|\beta\right)$ (to be specified later) labeled by unknown parameter β and covariance function $c_Y\left(\cdot,\cdot|\sigma^2, \phi\right)$.

Assume there is available a dataset $\{(s_i, Z_i)\}_{i=1}^n$ where $Z_i = Z\left(s_i\right)$ is a realization of a stochastic process (Z_s).

(1) Write the hierarchical spatial model $Z\left(\cdot\right)|Y\left(\cdot\right), \beta, \varphi$ and $Y\left(\cdot\right)|\beta, \varphi$ where $\varphi = \left(\sigma^2, \phi, \tau\right)^\top$.

(2) Write the marginal process $Z\left(\cdot\right)|\beta, \varphi$ where $\varphi = \left(\sigma^2, \phi, \tau\right)^\top$, its mean function denoted as $\mu\left(\cdot|\cdot\right)$, and its covariance function denoted as $c\left(\cdot|\cdot\right)$.

Hint:: Let Y and X be independent random variables with $X \sim N\left(\mu_X, \Sigma_X\right)$, $Y \sim N\left(\mu_Y, \Sigma_Y\right)$. Let A and B be fixed matrices. Let c be a fixed vector. Then

$$
AX + BY + c \sim N\left(A\mu_X + B\mu_Y + c, A\Sigma_X A^\top + B\Sigma_Y B^\top\right)
$$

(3) Compute the predictive process $Z(\cdot)\,|Z,\beta,\varphi$ as

$$Z(\cdot)\,|Z,\beta,\varphi \sim \mathrm{GP}\left(\mu_1\left(\cdot|\beta,\varphi\right),c_1\left(\cdot,\cdot|\varphi\right)\right)$$

with

$$c_1\left(s,s'|\varphi\right) = c\left(s,s|\varphi\right) + \left(C\left(S,s|\varphi\right)\right)^\top \left(C\left(S,S|\varphi\right)\right)^{-1} C\left(S,s'|\varphi\right)$$
$$\mu_1\left(s|\beta,\varphi\right) = \mu\left(s|\beta\right) - \left(C\left(S,s|\varphi\right)\right)^\top \left(C\left(S,S|\varphi\right)\right)^{-1} \left(\mu\left(S|\beta\right) - Z\right)$$

Hint: See the Conditional Normal formula above.

(4) Assume $\mu\left(s|\beta\right) = \psi\left(s\right)^\top \beta$. Consider a conjugate prior $\beta \sim \mathrm{N}\left(b,B\right)$ on β where $B > 0$.

  (a) Write down the Bayesian statistical model involving layers $[Z|\beta,\varphi]$, and $[\beta|\varphi]$.

  (b) Compute the posterior distribution as

$$\beta|Z,\varphi \sim \mathrm{N}\left(b_n\left(\varphi\right),B_n\left(\varphi\right)\right)$$

with

$$B_n\left(\varphi\right) = \left(B^{-1} + \Psi^\top \left(C\left(S,S|\varphi\right)\right)^{-1}\Psi\right)^{-1}$$
$$b_n\left(\varphi\right) = B_n\left(\varphi\right)\left(B^{-1}b + \Psi^\top \left(C\left(S,S|\varphi\right)\right)^{-1} Z\right)$$

where $C\left(S,S|\varphi\right)$ is a matrix with $\left[C\left(S,S|\varphi\right)\right]_{i,j} = c\left(s_i,s_j|\varphi\right)$.

Hint: Use the following identity

$$\left(y - \Phi\beta\right)^\top \Sigma^{-1}\left(y - \Phi\beta\right) + \left(\beta - \mu\right)^\top V^{-1}\left(\beta - \mu\right) = \left(\beta - \mu^*\right)^\top \left(V^*\right)^{-1}\left(\beta - \mu^*\right) + S^*;$$
$$V^* = \left(V^{-1} + \Phi^\top \Sigma^{-1}\Phi\right)^{-1}; \qquad \mu^* = V^*\left(V^{-1}\mu + \Phi^\top \Sigma^{-1}y\right)$$
$$S^* = \mu^\top V^{-1}\mu - \left(\mu^*\right)^\top \left(V^*\right)^{-1}\left(\mu^*\right) + y^\top \Sigma^{-1}y;$$

  (c) Compute the (posterior) predictive process $Z(\cdot)\,|Z,\varphi$ given the data Z and given the parameters φ as

$$Z(\cdot)\,|Z,\varphi \sim \mathrm{GP}\left(\mu_2\left(\cdot|\varphi\right),c_2\left(\cdot,\cdot|\varphi\right)\right)$$

with

$$\mu_2\left(s|\varphi\right) = \left(\psi\left(s\right) - \Psi^\top C^{-1}C\left(s\right)\right)^\top \left(B^{-1} + \Psi^\top C^{-1}\Psi\right)^{-1}B^{-1}b$$
$$\quad + \left[\left(C\left(s\right)\right)^\top + \left(\psi\left(s\right) - \Psi^\top C^{-1}C\left(s\right)\right)^\top \left(B^{-1} + \Psi^\top C^{-1}\Psi\right)^{-1}\Psi\right]C^{-1}Z$$
$$c_2\left(s,s'|\varphi\right) = c\left(s,s'|\varphi\right) - \left(C\left(s\right)\right)^\top C^{-1}C\left(s'\right)$$
$$\quad + \left(\psi\left(s\right) - \Psi^\top C^{-1}C\left(s\right)\right)^\top \left(B^{-1} + \Psi^\top C^{-1}\Psi\right)^{-1}\left(\psi\left(s'\right) - \Psi^\top C^{-1}C\left(s'\right)\right)$$

with column vector $C\left(s\right) := \left(c\left(s,s_1|\varphi\right),...,c\left(s,s_n|\varphi\right)\right)^\top$, and matrix $C := C\left(S,S|\varphi\right)$.

(d) Compute the marginal likelihood $\Pr(Z|\varphi)$ in the form

$$\Pr(Z|\sigma^2, \varphi) = \mathrm{N}\left(Z|\Psi b, \left(C^{-1} - C^{-1}\Psi\left(B^{-1} + \Psi^\top B^{-1}\Psi\right)^{-1}\Psi^\top C^{-1}\right)^{-1}\right)$$

where Ψ is a matrix with $[\Psi]_{i,j} = \psi_j(s_i)$, and R is a matrix with $[C]_{i,j} = c(s_i, s_j|\varphi)$.

Hint-2:: It is

$$\int \mathrm{N}(Z|\Psi\beta, C)\,\mathrm{N}(\beta|b, B)\,\mathrm{d}\beta = \mathrm{N}\left(Z|\Psi b, C + \Psi B \Psi^\top\right)$$

Hint 3:: [Woodbury matrix identity]

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U\left(C^{-1} + VA^{-1}U\right)^{-1}VA^{-1}$$

(5) Consider non-informative prior $\Pr(\beta) \propto 1$ for β by specifying $b \to 0$ and letting $B^{-1} \to 0$. Argue whether such a prior can be used. Recompute the (asymptotic) quantities $\Pr(Z|\varphi)$, $[Z(\cdot)|Z, \varphi]$ under this new prior in the limit.

Solution.

(1) The hierarchical model is

$$Z(\cdot)|Y(\cdot), \tau \sim \mathrm{GP}\left(Y(\cdot), c_\varepsilon\left(\cdot, \cdot|\sigma^2, \xi\right)\right)$$
$$Y(\cdot)|\beta, \sim \mathrm{GP}\left(\mu(\cdot|\beta), c_Y\left(\cdot, \cdot|\sigma^2, \phi\right)\right)$$

(2) We use the additive property of the Gaussian distribution (in Hint) it is

$$Z(\cdot)|\beta, \varphi \sim \mathrm{GP}\left(\mu(\cdot|\beta), c(\cdot, \cdot|\varphi)\right)$$

where

$$c(s, s'|\varphi) = c_Y\left(s, s'|\sigma^2, \phi\right) + c_\varepsilon\left(s, s'|\sigma^2, \xi\right)$$

(3) Assume a vector of "unseen" sites $S_* = (s_{*,1}, ..., s_{*,q})^\top$ for any $q \in \mathbb{N}_0$. Let convenient notation $Z := Z(S)$, and $Z_* := Z(S_*)$. The join marginal distribution of $(Z_*, Z)^\top$ given β, $\varphi = (\sigma^2, \phi, \tau)^\top$ is

$$\binom{Z_*}{Z}|\beta, \varphi \sim \mathrm{N}\left(\binom{\mu(S_*; \beta)}{\mu(S; \beta)}, \begin{pmatrix} C(S_*, S_*|\varphi) & (C(S_*, S|\varphi))^\top \\ C(S_*, S|\varphi) & C(S, S|\varphi) \end{pmatrix}\right)$$

by using convenient notation $[C(S_*, S|\varphi)]_{i,j} = s(s_{*,i}, s_j|\varphi)$ and $[\mu(S; \beta)]_i = \mu(s_i; \beta)$. By conditioning the Normal distribution (see Hint), I get

$$Z_*|Z, \beta, \varphi \sim \mathrm{N}\left(\mu_*(S_*|\beta, \varphi), C_*(S_*, S_*|\varphi)\right)$$

where

$$C_1\left(S_*, S_*|\varphi\right) = C\left(S_*, S_*|\varphi\right) + \left(C\left(S, S_*|\varphi\right)\right)^\top \left(C\left(S, S|\varphi\right)\right)^{-1} C\left(S, S_*|\varphi\right)$$

$$\mu_1\left(S_*|\beta, \varphi\right) = \mu\left(S_*|\beta\right) - \left(C\left(S, S_*|\varphi\right)\right)^\top \left(C\left(S, S|\varphi\right)\right)^{-1} \left(\mu\left(S|\beta\right) - Z\right)$$

As it is for any length of of any vector S_*, then it is a Gaussian process

$$Z\left(\cdot\right)|Z, \beta, \varphi \sim \mathrm{GP}\left(\mu_1\left(\cdot|\beta, \varphi\right), c_1\left(\cdot, \cdot|\varphi\right)\right)$$

with

$$c_1\left(s, s'|\varphi\right) = c\left(s, s'|\varphi\right) + \left(C\left(S, s|\varphi\right)\right)^\top \left(C\left(S, S|\varphi\right)\right)^{-1} C\left(S, s'|\varphi\right)$$

$$\mu_1\left(s|\beta, \varphi\right) = \mu\left(s|\beta\right) - \left(C\left(S, s|\varphi\right)\right)^\top \left(C\left(S, S|\varphi\right)\right)^{-1} \left(\mu\left(S|\beta\right) - Z\right)$$

(4)

(a) The Bayesian model is

(0.1)
$$\begin{cases} Z|\beta, \varphi \sim \mathrm{N}\left(\Psi\beta, C\left(S, S|\varphi\right)\right) \\ \beta \sim \mathrm{N}\left(b, B\right) \end{cases}$$

(b) Let $C := C\left(S, S|\varphi\right)$. The posterior distribution (by using Bayes theorem) is

$$\Pr\left(\beta|Z, \varphi\right) \propto \Pr\left(Z|\beta, \varphi\right) \Pr\left(\beta|\varphi\right)$$

$$= \mathrm{N}\left(Z|\Psi\beta, C\right) \mathrm{N}\left(\beta|b, B\right)$$

$$\propto \exp\left(-\frac{1}{2}\left(Z - \Psi\beta\right)^\top C^{-1}\left(Z - \Psi\beta\right)\right) \exp\left(-\frac{1}{2}\left(\beta - b\right)^\top B^{-1}\left(\beta - b\right)\right)$$

$$= \exp\left(-\frac{1}{2}\left[\left(Z - \Psi\beta\right)^\top C^{-1}\left(Z - \Psi\beta\right) + \left(\beta - b\right)^\top B^{-1}\left(\beta - b\right)\right]\right)$$

By using the Hint I have

$$\left(Z - \Psi\beta\right)^\top C^{-1}\left(Z - \Psi\beta\right) + \left(\beta - b\right)^\top B^{-1}\left(\beta - b\right) = \left(\beta - b_n\right)^\top \left(B_n\right)^{-1}\left(\beta - b_n\right) + R_n$$

where by denoting $B_n := B_n\left(\varphi\right)$, and $b_n := b_n\left(\varphi\right)$ I get

$$B_n = \left(B^{-1} + \Psi^\top C^{-1}\Psi\right)^{-1}$$

$$b_n = B_n\left(B^{-1}b + \Psi^\top C^{-1}Z\right)$$

and R_n is a "constant" quantity that does not contain any β. Hence

$$\Pr\left(\beta|Z, \varphi\right) \propto \exp\left(-\frac{1}{2}(\beta - b_n)^\top \left(B_n\right)^{-1}\left(\beta - b_n\right) - \frac{1}{2}R_n\right)$$

$$\propto \exp\left(-\frac{1}{2}(\beta - b_n)^\top \left(B_n\right)^{-1}\left(\beta - b_n\right)\right)$$

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Well, from the above, I recognize the kernel of the Multivariate Normal distribution, as

$$\beta | Z, \varphi \sim \mathrm{N}\left(b_n\left(\varphi\right), B_n\left(\varphi\right)\right)$$

(c) Assume a vector of "unseen" sites $S_* = \left(s_{*,1}, ..., s_{*,q}\right)^\top$ for any $q \in \mathbb{N} - \{0\}$. Let convenient notation $Z := Z\left(S\right)$, and $Z_* := Z\left(S_*\right)$. I have already computed

$$\mathrm{Pr}\left(Z_* | Z, \beta, \varphi\right) = \mathrm{N}\left(Z_* | \mu_1\left(S_* | \beta, \varphi\right), C_1\left(S_*, S_* | \varphi\right)\right)$$

from the previous part. It is

$$\mathrm{Pr}\left(Z_* | Z, \varphi\right) = \int \mathrm{Pr}\left(Z_* | Z, \beta, \varphi\right) \mathrm{Pr}\left(\beta | Z, \varphi\right) \mathrm{d}\beta$$

$$= \int \mathrm{N}\left(Z_* | \mu_1\left(S_* | \beta, \varphi\right), C_1\left(S_*, S_* | \varphi\right)\right) \mathrm{N}\left(\beta | b_n, B_n\right) \mathrm{d}\beta$$

Denote $\Psi_* = \Psi\left(S_*\right)$, $C_* = C\left(S_*, S | \varphi\right)$, and $C_{**} = C\left(S_*, S_* | \varphi\right)$. Notice that

$$\mu_1\left(S_*\right) = \Psi_* \beta - C_* C^{-1}\left(\Psi \beta - Z\right)$$
$$= \left[\Psi_* - C_* C^{-1}\Psi\right]\beta + C_* C^{-1} Z$$

Hence, for given/fixed Z, φ, it is

$$Z_* = C_* C^{-1} Z + \left[\Psi_* - C_* C^{-1}\Psi\right]\beta + \zeta, \quad \zeta \sim \mathrm{N}\left(0, C_1\left(S_*, S_*\right)\right)$$

Hence, because $\beta \sim \mathrm{N}\left(b_n, B_n\right)$, and because $Z_* | Z, \varphi$ is a linear combination of the Normally distributed random vector $\beta \sim \mathrm{N}\left(b_n, B_n\right)$, $Z_* | Z, \varphi$ follows a Normal distribution, with mean

$$\mu_2\left(S_*\right) = \mathrm{E}_{\beta \sim \mathrm{N}(b_n, B_n)}\left(Z_* | \mu_1\left(S_*\right), C_1\left(S_*, S_*\right)\right)$$
$$= \left(\Psi_* - C_* C^{-1}\Psi\right) \mathrm{E}_{\beta \sim \mathrm{N}(b_n, B_n)}\left(\beta\right) + C_* C^{-1} Z$$
$$= \left(\Psi_* - C_* C^{-1}\Psi\right) b_n + C_* C^{-1} Z$$
$$= \left(\Psi_* - C_* C^{-1}\Psi\right)\left(B^{-1} + \Psi^\top C^{-1}\Psi\right)^{-1}\left(B^{-1} b + \Psi^\top C^{-1} Z\right) + C_* C^{-1} Z$$
$$= \left(\Psi_* - C_* C^{-1}\Psi\right)\left(B^{-1} + \Psi^\top C^{-1}\Psi\right)^{-1} B^{-1} b$$
$$\quad + \left[\left(\Psi_* - C_* C^{-1}\Psi\right)\left(B^{-1} + \Psi^\top C^{-1}\Psi\right)^{-1}\Psi^\top + C_*\right] C^{-1} Z$$

and with covariance matrix

$$
\begin{aligned}
C_2\left(S_*, S_*\right) =& \operatorname{Var}_{\beta \sim N\left(b_n, B_n\right)}\left(Z_* | \mu_1\left(S_*\right), C_1\left(S_*, S_*\right)\right) \\
=& \operatorname{Var}_{\beta \sim N\left(b_n, B_n\right)}\left(\left[\Psi_* - C_* C^{-1} \Psi\right] \beta\right) + \operatorname{Var}_{\zeta \sim N\left(0, C_1\left(S_*, S_*\right)\right)}(\zeta) \\
=& \left[\Psi_* - C_* C^{-1} \Psi\right] B_n\left[\Psi_* - C_* C^{-1}\right]^{\top} + C_1\left(S_*, S_*\right) \\
=& \left[\Psi_* - C_* C^{-1} \Psi\right]\left(B^{-1} + \Psi^{\top} C^{-1} \Psi\right)^{-1}\left[\Psi_* - C_* C^{-1} \Psi\right]^{\top} \\
& + C_{**} + C_* C^{-1}\left(C_*\right)^{\top}
\end{aligned}
$$

Recall that $C\left(s\right) = \left(c\left(s_1, s | \varphi\right), ..., c\left(s_n, s | \varphi\right)\right)^{\top}$ is a column vector.
Since this is for any vector S_* of any length, then

$$
Z\left(\cdot\right) | Z, \varphi \sim \operatorname{GP}\left(\mu_2\left(\cdot | \varphi\right), c_2\left(\cdot, \cdot | \varphi\right)\right)
$$

with mean function at s

$$
\begin{aligned}
\mu_2\left(s | \varphi\right) =& \left(\psi\left(s\right) - \left(C\left(s\right)\right)^{\top} C^{-1} \Psi\right)\left(B^{-1} + \Psi^{\top} C^{-1} \Psi\right)^{-1} B^{-1} b \\
& + \left[\left(\psi\left(s\right) - \left(C\left(s\right)\right)^{\top} C^{-1} \Psi\right)\left(B^{-1} + \Psi^{\top} C^{-1} \Psi\right)^{-1} \Psi^{\top} + \left(C\left(s\right)\right)^{\top}\right] C^{-1} Z
\end{aligned}
$$

and covariance function as s, s'

$$
\begin{aligned}
c_2\left(s, s' | \varphi\right) =& \left[\psi\left(s\right) - \left(C\left(s\right)\right)^{\top} C^{-1} \Psi\right]\left(B^{-1} + \Psi^{\top} C^{-1} \Psi\right)^{-1}\left[\psi\left(s'\right) - \left(C\left(s'\right)\right)^{\top} C^{-1} \Psi\right]^{\top} \\
& + c\left(s, s' | \varphi\right) + \left(C\left(s\right)\right)^{\top} C^{-1} C\left(s'\right)
\end{aligned}
$$

Recall that $C\left(s\right) = \left(c\left(s_1, s | \varphi\right), ..., c\left(s_n, s | \varphi\right)\right)^{\top}$ is a column vector.

(d) It is, from Hint-2,

$$
\begin{aligned}
\operatorname{Pr}\left(Z | \varphi\right) =& \int \operatorname{Pr}\left(Z | \beta, \varphi\right) \operatorname{Pr}\left(\beta\right) \mathrm{d}\beta \\
=& \int \mathrm{N}\left(Z | \Psi \beta, C\left(S, S | \varphi\right)\right) \mathrm{N}\left(\beta | b, B\right) \mathrm{d}\beta \\
=& \int \mathrm{N}\left(Z | \Psi \beta, C\left(S, S | \varphi\right)\right) \mathrm{N}\left(\Psi \beta | \Psi b, \Psi B \Psi^{\top}\right) \mathrm{d}\beta \\
=& \mathrm{N}\left(Z | \Psi b, C\left(S, S | \varphi\right) + \Psi B \Psi^{\top}\right)
\end{aligned}
$$

By letting $C := C\left(S, S | \varphi\right)$ and using the Hint I get

$$
\left(\left(C + \Psi B \Psi^{\top}\right)^{-1}\right)^{-1} = \left(C^{-1} - C^{-1} \Psi\left(B^{-1} + \Psi^{\top} C^{-1} \Psi\right)^{-1} \Psi^{\top} C^{-1}\right)^{-1}
$$

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(5) Denote $C = C(S, S|\varphi)$. It is

$$\lim_{\substack{B^{-1} \to 0 \\ b \to 0}} \Pr(Z|\varphi) = \lim_{\substack{B^{-1} \to 0 \\ b \to 0}} \mathrm{N}\left(Z|\Psi b, C + \Psi B \Psi^\top\right)$$

$$\propto \mathrm{N}\left(Z|0, \left(C^{-1} - C^{-1}\Psi\left(\Psi^\top C^{-1}\Psi\right)^{-1}\Psi^\top C^{-1}\right)^{-1}\right)$$

$$< \infty$$

namely the bottom part of the fraction of the posterior of $\beta|Z, \varphi$ is bounded (finite); this implies that the posterior is proper. The posterior of $\beta|Z, \varphi$ has density such as

$$\Pr(\beta|Z, \varphi) \propto \exp\left(-\frac{1}{2}(\beta - b_n)^\top B_n^{-1}(\beta - b_n)\right)$$

then by computing the limit

$$\lim_{\substack{B^{-1} \to 0 \\ b \to 0}} \exp\left(-\frac{1}{2}(\beta - b_n)^\top B_n^{-1}(\beta - b_n)\right) =$$

$$\exp\left(-\frac{1}{2}\left(\beta - \left(\Psi^\top C^{-1}\Psi\right)^{-1}\Psi^\top C^{-1}Z\right)^\top \left(\Psi^\top C^{-1}\Psi\right)\left(\beta - \left(\Psi^\top C^{-1}\Psi\right)^{-1}\Psi^\top C^{-1}Z\right)\right)$$

Hence the limiting case is

$$\beta|Z, \varphi \overset{\mathrm{approx}}{\sim} \mathrm{N}\left(\left(\Psi^\top C^{-1}\Psi\right)^{-1}\Psi^\top C^{-1}Z, \ \left(\Psi^\top C^{-1}\Psi\right)^{-1}\right)$$

Hence the predictive process becomes

$$Z(\cdot)|Z, \varphi \overset{\mathrm{approx}}{\sim} \mathrm{GP}\left(\mu_3(\cdot|\varphi), c_3(\cdot, \cdot|\varphi)\right)$$

$$\mu_3(s|\varphi) = \left[\left(\psi(s) - (C(s))^\top C^{-1}\Psi\right)\left(\Psi^\top C^{-1}\Psi\right)^{-1}\Psi^\top + (C(s))^\top\right]C^{-1}Z$$

$$c_3(s, s'|\varphi) = \left(\psi(s) - (C(s))^\top C^{-1}\Psi\right)\left(\Psi^\top C^{-1}\Psi\right)^{-1}\left(\psi(s') - (C(s'))^\top C^{-1}\Psi\right)^\top$$
$$+ c(s, s'|\varphi) + (C(s))^\top C^{-1}C(s')$$

Exercise 2. ($\star$) Show that the extension variance $\sigma_E^2(v, V)$ of a small volume v to a larger volume V is obtained by

$$\sigma_E^2(v, V) = 2\bar{\gamma}(v, V) - \bar{\gamma}(v, v) - \bar{\gamma}(V, V)$$

where

$$\bar{\gamma}(v, V) = \frac{1}{|v||V|}\int_{s \in v}\int_{s \in V}\gamma(s - s')\,\mathrm{d}s\mathrm{d}s'$$

Solution. Essentially I need to show that that

$$\mathrm{Var}\left(Z\left(A\right)-Z\left(B\right)\right)=\frac{1}{\left|A\right|\left|B\right|}\int_{x\in A}\int_{y\in B}\gamma\left(x-y\right)\mathrm{d}x\mathrm{d}y$$

$$-\frac{1}{\left|A\right|\left|A\right|}\int_{x\in A}\int_{y\in A}\gamma\left(x-y\right)\mathrm{d}x\mathrm{d}y$$

$$-\frac{1}{\left|B\right|\left|B\right|}\int_{x\in B}\int_{y\in B}\gamma\left(x-y\right)\mathrm{d}x\mathrm{d}y$$

where I use A, B instead of v, V and x, y instead of s, s' for clarity on notation.

It is

$$\mathrm{Var}\left(Z\left(A\right)-Z\left(B\right)\right)=\mathrm{Cov}\left(Z\left(A\right)-Z\left(B\right),Z\left(A\right)-Z\left(B\right)\right)$$

$$=\mathrm{Cov}\left(Z\left(A\right),Z\left(A\right)\right)+\mathrm{Cov}\left(Z\left(B\right),Z\left(B\right)\right)-2\mathrm{Cov}\left(Z\left(A\right),Z\left(B\right)\right)$$

$$=\frac{1}{\left|A\right|\left|A\right|}\int_{x\in A}\int_{y\in A}\mathrm{Cov}\left(Z\left(x\right),Z\left(y\right)\right)\mathrm{d}x\mathrm{d}y$$

$$+\frac{1}{\left|B\right|\left|B\right|}\int_{x\in B}\int_{y\in B}\mathrm{Cov}\left(Z\left(x\right),Z\left(y\right)\right)\mathrm{d}x\mathrm{d}y$$

$$-2\frac{1}{\left|A\right|\left|B\right|}\int_{x\in A}\int_{y\in B}\mathrm{Cov}\left(Z\left(x\right),Z\left(y\right)\right)\mathrm{d}x\mathrm{d}y$$

OK, now I need to write all these Cov as γ ; I know that

$$\gamma\left(x-y\right)=\frac{1}{2}\mathrm{Var}\left(Z\left(x\right)-Z\left(y\right)\right)$$

$$=\frac{1}{2}\mathrm{Var}\left(Z\left(x\right)\right)+\frac{1}{2}\mathrm{Var}\left(Z\left(y\right)\right)-\mathrm{Cov}\left(Z\left(x\right),Z\left(y\right)\right)$$

that is

$$\mathrm{Cov}\left(Z\left(x\right),Z\left(y\right)\right)=\frac{1}{2}\mathrm{Var}\left(Z\left(x\right)\right)+\frac{1}{2}\mathrm{Var}\left(Z\left(y\right)\right)-\gamma\left(x-y\right)$$

Now I 'll gonna put all these in the quantity of interest, one by one

$$\frac{1}{\left|A\right|\left|A\right|}\int_{x\in A}\int_{y\in A}\mathrm{Cov}\left(Z\left(x\right),Z\left(y\right)\right)\mathrm{d}x\mathrm{d}y=\frac{1}{\left|A\right|\left|A\right|}\int_{x\in A}\int_{y\in A}\frac{1}{2}\mathrm{Var}\left(Z\left(x\right)\right)\mathrm{d}x\mathrm{d}y$$

$$+\frac{1}{\left|A\right|\left|A\right|}\int_{x\in A}\int_{y\in A}\frac{1}{2}\mathrm{Var}\left(Z\left(y\right)\right)\mathrm{d}x\mathrm{d}y$$

$$-\frac{1}{\left|A\right|\left|A\right|}\int_{x\in A}\int_{y\in A}\gamma\left(x-y\right)\mathrm{d}x\mathrm{d}y$$

$$=\frac{1}{\left|A\right|}\int_{x\in A}\mathrm{Var}\left(Z\left(x\right)\right)\mathrm{d}x$$

$$-\frac{1}{\left|A\right|^{2}}\int_{x\in A}\int_{y\in A}\gamma\left(x-y\right)\mathrm{d}x\mathrm{d}y$$

and by symmetry

$$\frac{1}{|B|\,|B|}\int_{x\in B}\int_{y\in B}\mathrm{Cov}\left(Z\left(x\right),Z\left(y\right)\right)\mathrm{d}x\mathrm{d}y = \frac{1}{|B|}\int_{x\in B}\mathrm{Var}\left(Z\left(x\right)\right)\mathrm{d}x$$

$$-\frac{1}{|B|^2}\int_{x\in B}\int_{y\in B}\gamma\left(x-y\right)\mathrm{d}x\mathrm{d}y$$

and finally,

$$\frac{1}{|A|\,|B|}\int_{x\in A}\int_{y\in B}\mathrm{Cov}\left(Z\left(x\right),Z\left(y\right)\right)\mathrm{d}x\mathrm{d}y = \frac{1}{|A|\,|B|}\int_{x\in A}\int_{y\in B}\frac{1}{2}\mathrm{Var}\left(Z\left(x\right)\right)\mathrm{d}x\mathrm{d}y$$

$$+\frac{1}{|A|\,|B|}\int_{x\in A}\int_{y\in B}\frac{1}{2}\mathrm{Var}\left(Z\left(y\right)\right)\mathrm{d}x\mathrm{d}y$$

$$-\frac{1}{|A|\,|B|}\int_{x\in A}\int_{y\in B}\gamma\left(x-y\right)\mathrm{d}x\mathrm{d}y$$

$$=\frac{1}{2}\frac{1}{|A|}\int_{x\in A}\mathrm{Var}\left(Z\left(x\right)\right)\mathrm{d}x$$

$$+\frac{1}{2}\frac{1}{|B|}\int_{x\in B}\mathrm{Var}\left(Z\left(x\right)\right)\mathrm{d}x$$

$$-\frac{1}{|A|\,|B|}\int_{x\in A}\int_{y\in B}\gamma\left(x-y\right)\mathrm{d}x\mathrm{d}y$$

Putting all these together, we get the result.

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