Spatio-temporal statistics (MATH4341) Easter term, 2024

Revision sheet

Lecturer & author: Georgios P. Karagiannis georgios.karagiannis@durham.ac.uk

Part 1. Point referenced data / Geostatistics

Exercise 1. Consider the Gaussian c.f. $c(h) = \sigma^2 \exp\left(-\beta \|h\|_2^2\right)$ for $\sigma^2, \beta > 0$ and $h \in \mathbb{R}^d$. Compute the spectral density from Bochner's theorem

Solution. It is

$$
\begin{aligned}
f(\omega) &= \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \exp\left(-i\omega^\top h\right) \sigma^2 \exp\left(-\beta \|h\|_2^2\right) \mathrm{d}h \\
&= \sigma^2 \left(\frac{1}{2\pi}\right)^d \prod_{j=1}^d \int_{\mathbb{R}} \exp\left(-i\omega_j h_j - \beta h_j^2\right) \mathrm{d}h \\
&= \sigma^2 \left(\frac{1}{2\pi}\right)^d \prod_{j=1}^d \int_{\mathbb{R}} \exp\left(-\beta\left(h_j - \left(-i\omega / (2\beta)\right)\right)^2\right) \mathrm{d}h_j \\
&= \sigma^2 \left(\frac{1}{4\pi\beta}\right)^{d/2} \exp\left(-\|\omega\|_2^2 / (4\beta)\right)
\end{aligned}
$$

Exercise 2. Let $(Z_s)_{s\in\mathcal{S}}$ be a specified statistical model. Assume that process $(Z_s)_{s\in\mathcal{S}}$ has known mean $\mu(s) = \mathrm{E}(Z(s))$ and known covariance function $c(\cdot,\cdot)$. Assume there is available a dataset $\{(s_i, Z_i := Z(s_i))\}_{i=1}^n$. Assume that the matrix C such as $[C]_{i,j} = c(s_i, s_j)$ has an inverse. Consider the "Kriging" estimator μ_{SK} Consider the "Kriging" estimator $Z_{\mathrm{SK}}(s_0)$ of $Z(s_0)$ at an unseen spatial location s_0 as the BLUE (Best Linear Unbiased Estimator)

$$
Z_{\mathrm{SK}}(s_0) = w_{n+1} + \sum_{i=1}^n w_i Z(s_i) = w_{n+1} + w^\top Z,
$$

for some unknown $\{w_i\}$ that we need to learn, and $Z = (Z_1, ..., Z_n)^\top$. Let $w = (w_1, ..., w_n)^\top$.

(1) Find sufficient conditions on $w = (w_1, ..., w_n)^\top$ so that the Kriging estimator $Z_{\mathrm{SK}}(s_0)$ to be unbiased.

(2) Derive the MSE of $Z_{\mathrm{SK}}(s_0)$ as

$$
\mathrm{E}\left(Z_{\mathrm{SK}}(s_0) - Z(s_0)\right)^2 = w^\top C w + c(s_0, s_0) - 2w^\top C_0
$$

where C_0 is a vector such as $[C_0]_i = c(s_0, s_i)$.

(3) Derive the Kriging estimator of $Z(s_0)$ as

$$Z_{\text{SK}}(s_0) = \mu(s_0) + C_0^\top C^{-1}[Z - \mu(s_{1:n})]$$

where $\mu(s_{1:n})$ is a vector such as $[\mu(s_{1:n})]_i = \mu(s_i)$.

(4) Compute the Kriging standard error $\sigma_{\text{SK}} = \sqrt{\text{E}(Z_{\text{SK}}(s_0) - Z(s_0))^2}$.

Solution. The method is called Simple Kriging, and hence we denote it as SK.

(1) It is

$$Z_{\text{SK}}(s_0) = w_{n+1} + \sum_{i=1}^{n} w_i Z(s_i) = w_{n+1} + w^\top Z,$$

where $\{w_i\}$ is a set of unknown weights to be learned.

We assume that assume zero systematic error (unbiasness), hence

$$\text{E}(Z_{\text{SK}}(s_0) - Z(s_0)) = \text{E}\left(w_{n+1} + \sum_{i=1}^{n} w_i Z(s_i) - Z(s_0)\right) = w_{n+1} + \sum_{i=1}^{n} w_i \mu(s_i) - \mu(s_0)$$

which is satisfied given the assumption

$$w_{n+1} = \mu(s_0) - \sum_{i=1}^{n} w_i \mu(s_i) \Longleftrightarrow w_{n+1} = \mu(s_0) - w^\top \mu(s_{1:n})$$

where $w = (w_1, ..., w_n)^\top$.

(2) It is

$$\begin{aligned}
\text{E}(Z_{\text{SK}}(s_0) - Z(s_0))^2 =& \text{Var}(Z_{\text{SK}}(s_0) - Z(s_0)) = \text{Var}\left(w_{n+1} + w^\top Z - Z(s_0)\right) \\
=& \text{Var}\left(w_{n+1} + w^\top Z\right) + \text{Var}(Z(s_0)) - 2\text{Cov}\left(w_{n+1} + w^\top Z, Z(s_0)\right) \\
=& w^\top C w + c(s_0, s_0) - 2w^\top \text{Cov}(Z, Z(s_0)) \\
=& w^\top C w + c(s_0, s_0) - 2w^\top C_0
\end{aligned}$$

where $C_0 = \text{Cov}(Z, Z(s_0))$, i.e. $[C_0]_j = c(s_j, s_0)$.

(3) To learn the unknown weights $\{w_i\}$ we need to solve

$$w^{\text{SK}} = \arg\min_{w} \text{E}(Z_{\text{SK}}(s_0) - Z(s_0))^2, \text{ subject to } w_{n+1} = \mu(s_0) - w^\top \mu(s_{1:n})$$

As $\text{E}(\mu_{\text{SK}} - Z(s_0))^2$ does not depend on w_{n+1} we minimize

$$\begin{aligned}
0 =& \nabla_w \text{E}(Z_{\text{SK}}(s_0) - Z(s_0))^2 = \nabla_w \text{Var}(Z_{\text{SK}}(s_0) - Z(s_0)) \\
=& 2Cw - 2C_0
\end{aligned}$$

So I get

$$w_{\text{SK}} = C^{-1} C_0$$

So

$$
\begin{aligned}
Z_{\mathrm{SK}}(s_0) &= w_{n+1} + C^{-1}C_0 Z \\
&= \mu(s_0) - \left(C^{-1}C_0\right)^{\top}\mu(s_{1:n}) + \left(C^{-1}C_0\right)^{\top} Z \\
&= \mu(s_0) + C_0^{\top}C^{-1}\left[Z - \mu(s_{1:n})\right]
\end{aligned}
$$

(4) It is

$$
\begin{aligned}
\sigma_{\mathrm{SK}} &= \sqrt{\mathrm{E}\left(Z_{\mathrm{SK}}(s_0) - Z(s_0)\right)^2} \\
&= \sqrt{w_{\mathrm{SK}}^{\top} C w_{\mathrm{SK}} + c(s_0, s_0) - 2w_{\mathrm{SK}}^{\top} C_0} \\
&= \sqrt{c(s_0, s_0) - C_0^{\top} C^{-1} C_0}
\end{aligned}
$$

Part 2. Aerial unit data / spatial data on lattices

Exercise 3. The conditionals $x|y \sim \mathrm{N}\left(a + by, \sigma^2 + \tau^2 y^2\right)$ and $y|x \sim \mathrm{N}\left(c + dx, \tilde{\sigma}^2 + \tilde{\tau}^2 x^2\right)$ are compatible if $\tau^2 = \tilde{\tau}^2 = 0$ and $d/\tilde{\sigma}^2 = b/\sigma^2$.

Solution. It is

$$
\begin{aligned}
\frac{g(x|y)}{q(y|x)} =& \frac{\mathrm{N}\left(x|a + by, \sigma^2 + \tau^2 y^2\right)}{\mathrm{N}\left(y|c + dx, \tilde{\sigma}^2 + \tilde{\tau}^2 x^2\right)} \\
=& \frac{\sqrt{\tilde{\sigma}^2 + \tilde{\tau}^2 x^2}}{\sqrt{\sigma^2 + \tau^2 y^2}} \exp\left(-\frac{1}{2}\left(\frac{(x - a - by)^2}{\sigma^2 + \tau^2 y^2} - \frac{(y - c - dx)^2}{\tilde{\sigma}^2 + \tilde{\tau}^2 x^2}\right)\right) \text{ (set } \tau^2 = \tilde{\tau}^2 = 0) \\
=& \frac{\sqrt{\tilde{\sigma}^2}}{\sqrt{\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{(x - a - by)^2}{\sigma^2} - \frac{(y - c - dx)^2}{\tilde{\sigma}^2}\right)\right) \\
=& \frac{\sqrt{\tilde{\sigma}^2}}{\sqrt{\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{x^2}{\sigma^2} + \frac{a^2}{\sigma^2} + \frac{b^2 y^2}{\sigma^2} - \frac{2xa}{\sigma^2} - \frac{xby}{\sigma^2} - \frac{y^2}{\tilde{\sigma}^2}\right.\right. \\
&\left.\left. + \frac{d^2}{\tilde{\sigma}^2} + \frac{c^2 x^2}{\tilde{\sigma}^2} - \frac{2yd}{\tilde{\sigma}^2} - \frac{ydx}{\tilde{\sigma}^2}\right)\right)
\end{aligned}
$$

If $d/\tilde{\sigma}^2 = b/\sigma^2$ (and $\tau^2 = \tilde{\tau}^2 = 0$)

$$
\frac{g(x|y)}{q(y|x)} \propto \underbrace{\exp\left(-\frac{1}{2}\left(\frac{x^2}{\sigma^2} + \frac{c^2 x^2}{\tilde{\sigma}^2} - \frac{2xa}{\sigma^2}\right)\right)}_{u(x)} \underbrace{\exp\left(+\frac{1}{2}\left(\frac{b^2 y^2}{\sigma^2} - \frac{y^2}{\tilde{\sigma}^2} - \frac{2yd}{\tilde{\sigma}^2}\right)\right)}_{v(y)}
$$

for $N_g = N_q = N = \mathbb{R}$. Also it is

$$\int u(x)\, g(x|y)\, \mathrm{d}x =$$

$$\int_{\mathbb{R}} \exp\left(-\frac{1}{2}\left(\frac{x^2}{\sigma^2} + \frac{c^2x^2}{\tilde{\sigma}^2} - \frac{2xa}{\sigma^2}\right)\right) \mathrm{N}\left(x|a+by,\sigma^2\right) \mathrm{d}x < \infty$$

as $u(x)$ is a Gaussian PDF wrt x up to a finite multiplicative constant and hence bounded.

Exercise 4. Consider that $Z(s)$ represents presence or absence of a characteristic at location $s \in \mathcal{S}$. Mathematically, assume random field Z taking values on a set of indices $\mathcal{S}$ in $\mathcal{Z} = \{0,1\}$ on $\mathcal{S} = \{1,...,n\}$, $n \in \mathbb{N} - \{0\}$. Consider that for a given z_{-i} it is

$$\begin{cases} z_i|z_{-i} & \sim \text{Logistic}\left(\theta_i\left(z_{-i}\right)\right), \quad i \in \mathcal{S} \\ \theta_i\left(z_{-i}\right) & = \alpha_i + \sum_{j:j\sim i} \beta_{i,j} z_j \end{cases}$$

(1) Show that the conditionals $z_i|z_{-i}$ are compatible as a Besag's auto-model when $\{\alpha_i\}$ and $\{\beta_{i,j}\}$ satisfy certain conditions, and specify these conditions.

Hint: The PMF of Logistic distribution $x|\theta \sim \text{Logistic}(\theta)$ can be written as

$$\Pr(x|\theta) = (1 - \exp(x\theta))^{-1}\, 1\,(x \in \{0,1\})$$

(2) Write down the marginal distribution of the associated random field.

(3) What would be the sign of $\{\beta_{i,j}\}$ if you wish to introduce competition between neighboring sites? What would be the sign of $\{\beta_{i,j}\}$ if you wish to introduce similarity between neighboring sites? What does α_i represent when $\beta_{i,j} = 0$?

Solution.

(1) Then the characteristics are

$$\Pr_i\left(z_i|z_{-i}\right) = \frac{\exp\left(z_i\theta_i\left(z_{-i}\right)\right)}{1+\exp\left(\theta_i\left(z_{-i}\right)\right)} 1\left(z_i \in \{0,1\}\right)$$

Now, we have parameterized $\{\theta_i(\cdot)\}$ as

$$\theta_i\left(z_{-i}\right) = \alpha_i + \sum_{j:j\sim i} \beta_{i,j} z_j$$

for $\{\alpha_i\}$ and $\{\beta_{i,j}\}$ with $\beta_{i,j} = \beta_{j,i}$. Hence, we've got

$$\log\left(\Pr_i\left(z_i|z_{-i}\right)\right) = \underbrace{z_i}_{B_i(z_i)} \underbrace{\left(\alpha_i + \sum_{j\sim i} \beta_{i,j} \overbrace{z_j}^{B_i(z_j)}\right)}_{A_i(z_{-i})} + \underbrace{0}_{C_i(z_i)} + \underbrace{\left(-\log\left(1+\exp\left(\alpha_i + \sum_{j:j\sim i}\beta_{i,j}z_j\right)\right)\right)}_{D_i(z_{-i})}$$

Notice that all the conditionals $z_i|z_{-i}$ follow an Exponential family with

$$\begin{aligned}
A_i\left(z_{-i}\right) =&\alpha_i + \sum_{j\sim i}\beta_{i,j}B_i\left(z_j\right)\\
B_i\left(z_i\right) =&z_i\\
C_i\left(z_i\right) =&0\\
D_i\left(z_{-i}\right) =&-\log\left(1+\exp\left(\alpha_i+\sum_{j:j\sim i}\beta_{i,j}z_j\right)\right)
\end{aligned}$$

I can get $C_i\left(\zeta\right)=0$ and $D_i\left(\zeta,...,\zeta\right)=0$ by considering a reference point $\zeta=0$. Hence the conditionals $z_i|z_{-i}$ are compatible as a Besag's auto-model for any $\{\alpha_i\}$ and $\{\beta_{i,j}\}$ with $\beta_{i,j}=\beta_{j,i}$ according to a Theorem discussed in the Lectures.

(2) The Besag auto-model has marginal distribution

$$\Pr_Z\left(z\right)\propto\exp\left(\overbrace{\sum_i\underbrace{\alpha_i\underbrace{z_i}_{B_i(z_i)}}_{V_i(z_i)}+\underbrace{\sum_i\sum_{j:j\sim i}}_{\sum_{\{i,j\}:j\sim i}}\beta_{i,j}z_iz_j}^{U(z)=}\right)$$

according to a Theorem discussed in the Lectures.

(3) I observe that: (1.) the model has spatially dependent coefficients $\{\alpha_i\}$ and $\{\beta_{i,j}\}$. (2.) when $\beta_{i,j}=0$, for all j such as $j\sim i$, it is $\Pr_i\left(z_i|z_{-i}\right)=\frac{\exp(\alpha_i)}{1+\exp(\alpha_i)}$ and (3.) Characteristic's present at site i is encouraged in neighboring site j when $\beta_{i,j}>0$, and discouraged when $\beta_{i,j}<0$.