Spatio-temporal statistics (MATH4341) Michaelmas term, 2023

Homework 4: Aerial data modeling

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Exercise 1. ($\star$) Show that the local characteristics

$$\Pr_1(x_1|x_2) = \sqrt{\frac{1}{2\pi}}\exp\left(-\frac{1}{2}(x_1-x_2)^2\right)$$
$$\Pr_2(x_2|x_1) = \sqrt{\frac{1}{2\pi}}\exp\left(-\frac{1}{2}(x_2-x_1)^2\right)$$

do not define a proper joint distribution on $\mathbb{R}^{\{1,2\}}$

Definition. (Pseudo-likelihood) The pseudo likelihood pseudo-$L(Z;\theta)$ of observables $Z=(Z_1,...,Z_n)^\top$ given parameters θ is an approximation of the likelihood $L(Z;\theta)$ of observables $Z=(Z_1,...,Z_n)^\top$ given parameters θ which is equal to

$$\text{pseudo-}L(Z;\theta) = \prod_i \Pr(Z_i|Z_{-i},\theta)$$

where $\Pr(Y|\theta)$ is the joint pdf/pmf of the sampling distribution of Y given parameter θ.

Definition. (Maximum Pseudo-Likelihood Estimator) The Maximum Pseudo-Likelihood Estimator (MPLE) $\tilde{\theta}$ of θ is the maximizer of the pseudo likelihood function pseudo-$L(Z;\theta)$ where the parameter θ is the argument and the observables $Z=(Z_1,...,Z_n)^\top$ are fixed values.

$$\tilde{\theta} = \arg\max_\theta(\text{pseudo-}L(Z;\theta))$$

Exercise 2. ($\star$) Let $Z\in\mathcal{Z}^{\mathcal{S}}$ where $\mathcal{S}=\{1,...,n\}$ and $\mathcal{Z}=\mathbb{R}$. Consider the model

$$Z = X\beta + B(Z-X\beta)+E$$

where X is a $n\times p$ design matrix X, $\beta\in\mathbb{R}^p$, B is an $n\times n$ symmetric positive definite matrix with $[B]_{i,i}=0$, $E\sim\mathrm{N}\left(0,\sigma^2(I-B)\right)$, and $\sigma^2>0$.

1. This is a multiple choice question, choose any number of correct answers.

   (a) Z follows a simultaneous autoregressive (SAR) with Gaussian joint distribution with mean $X\beta$ and covariance matrix $\sigma^2 (I - B)^{-1}$

   (b) Ising model

   (c) Conditional autoregressive (CAR) with Gaussian joint distribution with mean $X\beta$ and covariance matrix $\sigma^2 I$

   (d) Bernoulli regression

2. Show that the minus two log Pseudo-Likelihood is such as

$$-2 \log \left(\text{pseudo-}L\left(Z; \beta, \sigma^2\right)\right) = n \log\left(\sigma^2\right) + \frac{1}{\sigma^2} (Z - X\beta)^{\top} (I - B)^2 (Z - X\beta) + \text{const.}$$

3. Compute the Maximum Pseudo-Likelihood Estimators (MPLE) $\tilde{\beta}$ and $\tilde{\sigma}^2$ of β and σ^2
