Spatio-temporal statistics (MATH4341) Michaelmas term, 2023

Homework 3: Geostatistics (Change of support)

Lecturer: Georgios P. Karagiannis georgios.karagiannis@durham.ac.uk

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Exercise 1. ($\star$) Suppose a large volume V is partitioned into n smaller units v of equal size. Show that the dispersion variance $\sigma^2(u|V) = \frac{1}{n}\sum_{j=1}^{n}\sigma_E^2(v_j, V)$ can be written in term of variogram integrals

$$\bar{\gamma}(v, V) = \frac{1}{|v|\,|V|}\int_{s\in v}\int_{s\in V}\gamma\left(s - s'\right)\mathrm{d}s\mathrm{d}s'$$

as

$$\sigma^2(v|V) = \bar{\gamma}(V, V) - \bar{\gamma}(v, v)$$

Exercise 2. ($\star$) Consider a statistical model which is a stochastic process $(Z_s)_{s\in\mathbb{R}}$ (so s has dimension 1), where $Z(\cdot) \sim \text{GP}(\mu(\cdot), c(\cdot,\cdot))$ with mean function $\mu(s) = 1$ and covariance function $c(s,t) = \exp\left(-(s-t)^2\right)$ for any $s \in \mathbb{R}$ and $t \in \mathbb{R}$. Assume there is available a dataset $\{(Z_i, s_i)\}_{i=1}^{n}$ where $Z_i = Z(s_i)$ and $s_i \in \mathbb{R}$ are point sites.

1. Compute the length $|v|$ of the block $v = [a, b] \subset \mathbb{R}$.

2. Compute the block mean $\mu(v)$ for some block $v = [a, b] \subset \mathbb{R}$ and point $s \in \mathbb{R}$.

3. Compute the block covariance function $c(v, s)$ for some block $v = [a, b] \subset \mathbb{R}$ and point $s \in \mathbb{R}$.

4. Compute the block covariance function $c(v, v')$ for some blocks $v = [a, b] \subset \mathbb{R}$ and $v' = [a', b'] \subset \mathbb{R}$.

5. Denote $Z = (Z_1, ..., Z_n)^\top$, and $S = \{s_1, ..., s_n\}$. Let $v = [a, b] \subset \mathbb{R}$ and $v' = [a', b'] \subset \mathbb{R}$ be two intervals. Compute the joint distribution of $(Z(v), Z(v'), Z)^\top$ as a function of $c(\cdot,\cdot)$, S, v, v', Z, and $\mu(\cdot)$. What is the name of the distribution and what are the parameter functions defining it?

6. (Bayesian Kriging) Compute the predictive stochastic process $[Z(v)\,|Z]$ at blocks $v = [a, b] \subset \mathbb{R}$ with $|v| > 0$.

Hint-1: Let $x_1 \in \mathbb{R}^{d_1}$, and $x_2 \in \mathbb{R}^{d_2}$. If

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim \mathrm{N}_{d_1+d_2}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_1 & \Sigma_{21}^\top \\ \Sigma_{21} & \Sigma_2 \end{bmatrix}\right)$$

then it is

$$x_2|x_1 \sim \mathrm{N}_{d_2}\left(\mu_{2|1}, \Sigma_{2|1}\right)$$

where

$$\mu_{2|1} = \mu_2 + \Sigma_{21}\Sigma_1^{-1}\left(x_1 - \mu_1\right) \text{ and } \Sigma_{2|1} = \Sigma_2 - \Sigma_{21}\Sigma_1^{-1}\Sigma_{21}^\top$$

Hint-2 Assume known function $\mathrm{erf}\left(x\right) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-t^2\right)\mathrm{d}t$. Then $\int \mathrm{erf}\left(x\right)\mathrm{d}x = x\mathrm{erf}\left(x\right) + \frac{\exp\left(-x^2\right)}{\sqrt{\pi}} + \text{const}$
