Spatio-temporal statistics (MATH4341) Michaelmas term, 2023

Homework 1: Regional statistical & probabilistic concepts

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As formative assessment, submit the solutions to Exercise ...

Exercise 1. ($\star\star\star$) Let $Z = (Z_s)_{s\in\mathbb{R}^d}$ be an intrinsically stationary stochastic process, and let $\gamma : \mathbb{R}^d \to \mathbb{R}$ be its semivariogram. Assume $a \in \mathbb{R}^n$s.t. $\sum_{i=1}^n a_i = 0$.

(1) Let $a \in \mathbb{R}^n$ be a vector of constants. Show that

$$\text{Var}\left(\sum_{i=1}^n a_i Z(s_i)\right) = \sum_{i=1}^n \sum_{j=1}^n a_i a_j c_Y(s_i, s_j)$$

where $c_Y(s,t) = \text{E}(Y(s)Y(t))$, and $Y_s = Z_s - Z_0$.

(2) Show that

$$c_Y(s,t) = \gamma(s) + \gamma(t) - \gamma(s-t)$$

(3) Show that for all $\forall \{s_1, ..., s_n\} \subseteq S$ it is

$$\sum_{i=1}^n \sum_{j=1}^n a_i a_j \gamma(s_i - s_j) \le 0$$

Solution. Assume $0 \in S$, and a random variable $Z(0)$. Let $Y_s = Z_s - Z_0$.

(1) It is

$$\begin{aligned}\text{Var}\left(\sum_{i=1}^n a_i Z(s_i)\right) =& \text{Var}\left(\sum_{i=1}^n a_i Z(s_i) - \overbrace{\sum_{i=1}^n a_i Z(0)}^{0=}\right) = \text{Var}\left(\sum_{i=1}^n a_i Y(s_i)\right)\\ =& \sum_{i=1}^n \sum_{j=1}^n a_i a_j \text{E}(Y(s_i)Y(s_j)) = c_Y(s,t)\end{aligned}$$

(2) For $\text{E}(Y(s_i)) = 0$ it is

$$\begin{aligned}\gamma(s-t) &= \frac{1}{2}\text{E}(Z(s) - Z(0) + Z(t) - Z(0))^2\\ &= \frac{1}{2}(2\gamma(s) + 2\gamma(t) - 2c_Y(s,t))\\ \implies c_Y(s,t) &= \gamma(s) + \gamma(t) - \gamma(s-t)\end{aligned}$$

(3) It is

$$
\begin{aligned}
0 \leq & \operatorname{Var}\left(\sum_{i=1}^{n} a_i Z\left(s_i\right)\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j c_Y\left(s_i, s_j\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j\left(\gamma\left(s_i\right) + \gamma\left(s_j\right) - \gamma\left(s_i - s_j\right)\right) \\
&= \sum_{i=1}^{n} a_i \gamma\left(s_i\right) \cancelto{0}{\sum_{j=1}^{n} a_j} + \sum_{j=1}^{n} a_j \gamma\left(s_j\right) \cancelto{0}{\sum_{i=1}^{n} a_i} - \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \gamma\left(s_i - s_j\right)
\end{aligned}
$$

hence

$$
\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \gamma\left(s_i - s_j\right) \leq 0
$$