Spatio-temporal statistics (MATH4341)

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## Problem class sheet 1

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**Exercise 1.** If  $c: \mathbb{R}^d \to \mathbb{R}$  is the covariogram of a weakly stationary random field  $Z = (Z_s)_{s \in \mathbb{R}^d}$  then  $c(\cdot)$  is semi-positive definite; i.e. for all  $n \in \mathbb{N}$ ,  $a \in \mathbb{R}^n$ , and  $\{s_1, ..., s_n\} \subseteq S$ 

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j c(s_i - s_j) \ge 0$$

**Solution.** To show that  $c(\cdot)$  is semi-positive definite, I need to show that  $\forall a \in \mathbb{R}^n - \{0\}$  it is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j c\left(s_i - s_j\right) \ge 0$$

Well it is

$$0 \le \operatorname{Var}\left(\sum_{i=1}^{n} a_{i} Z\left(s_{i}\right)\right) = \operatorname{Cov}\left(\sum_{i=1}^{n} a_{i} Z\left(s_{i}\right), \sum_{j=1}^{n} a_{j} Z\left(s_{j}\right)\right)$$

$$= \sum_{i=1}^{n} a_{i} \sum_{j=1}^{n} a_{j} \operatorname{Cov}\left(Z\left(s_{i}\right), Z\left(s_{j}\right)\right)$$

$$= \sum_{i=1}^{n} a_{i} \sum_{j=1}^{n} a_{j} a_{j} c\left(s_{i}, s_{j}\right) = \sum_{i=1}^{n} a_{i} \sum_{j=1}^{n} a_{j} c\left(s_{i} - s_{j}\right)$$

**Exercise 2.** Show that if  $c_1(\cdot, \cdot)$  and  $c_2(\cdot, \cdot)$  are are covariance functions (are non-negative definite) then so are  $c_3(\cdot, \cdot) = bc_1(\cdot, \cdot) + dc_2(\cdot, \cdot)$  and  $c_4(\cdot, \cdot) = c_1(\cdot, \cdot) c_2(\cdot, \cdot)$ .

**Solution.** For all  $n \in \mathbb{N}$  and  $a_1, ... a_n$ 

$$\sum_{i=1}^{n} a_{i} \sum_{j=1}^{n} a_{j} c_{3} (s_{i}, s_{j}) = \sum_{i=1}^{n} a_{i} \sum_{j=1}^{n} a_{j} (bc_{1} (s_{i}, s_{j}) + dc_{2} (s_{i}, s_{j}))$$

$$= \sum_{i=1}^{n} a_{i} \sum_{j=1}^{n} a_{j} bc_{1} (s_{i}, s_{j}) + \sum_{i=1}^{n} a_{i} \sum_{j=1}^{n} a_{j} dc_{2} (s_{i}, s_{j})$$

$$\geq 0$$

Regarding  $c_4$ , assume independent stochastic processes  $(Y_s)_{s\in S}$  and  $(X_s)_{s\in S}$  with mean zero and covariance functions  $c_1(\cdot,\cdot)$  and  $c_2(\cdot,\cdot)$  correspondingly. Let stochastic processes  $(Z_s)_{s\in S}$ 

with  $Z_s = Y_s X_s$ . Then

$$Cov (Z_s, Z_t) = Cov (Y_s X_s, Y_t X_t)$$

$$= E (Y_s X_s Y_t X_t)$$

$$= E (Y_s Y_t X_s X_t), \text{ but } Y_s \bot X_s$$

$$= E (X_s X_t) E (X_s X_t)$$

$$= Cov (X_s, X_t) Cov (Y_s, Y_t)$$

$$= c_1 (s, t) c_2 (s, t) = c_4 (s, t)$$

that is  $c_4(\cdot, \cdot)$  is a covariance function of a stochastic processes.

The following is some theory from your Lecture notes: Handout 3: Point referenced data modeling / Geostatistics

**Theorem.** (Bochner's theorem) A continuous even real-valued function  $c : \mathbb{R}^d \to \mathbb{R}$  is a covariance function of a weakly stationary random process if and only if it can be represented as

$$c(h) = \int_{\mathbb{R}^d} \exp\left(i\omega^{\top} h\right) dF(\omega)$$

where  $dF(\omega)$  is a symmetric positive finite measure on  $\mathbb{R}^d$ .

• Here, we will focus on cases of the form  $dF(\omega) = f(\omega) d\omega$  where  $f(\cdot)$  is called spectral density of  $c(\cdot)$  i.e.

$$c(h) = \int_{\mathbb{R}^d} \exp\left(i\omega^\top h\right) f(\omega) d\omega$$

In this case,  $\lim_{h\to\infty} c(h) = 0$ 

**Theorem.** If  $c(\cdot)$  is integrable,  $F(\cdot)$  is absolutely continuous with spectral density  $f(\cdot)$  of  $Z = (Z_s; s \in \mathcal{S})$  then by Fast Fourier transformation

$$f(\omega) = \left(\frac{1}{2\pi}\right)^{d} \int_{\mathbb{R}^{d}} \exp\left(-i\omega^{\top}h\right) c(h) dh$$

**Exercise 3.** Consider the Gaussian c.f.  $c(h) = \sigma^2 \exp(-\beta \|h\|_2^2)$  for  $\sigma^2, \beta > 0$  and  $h \in \mathbb{R}^d$ . Compute the spectral density from Bochner's theorem

**Solution.** It is

$$f(\omega) = \left(\frac{1}{2\pi}\right)^{d} \int_{\mathbb{R}^{d}} \exp\left(-i\omega^{\top}h\right) \sigma^{2} \exp\left(-\beta \|h\|_{2}^{2}\right) dh$$

$$= \sigma^{2} \left(\frac{1}{2\pi}\right)^{d} \prod_{j=1}^{d} \int_{\mathbb{R}} \exp\left(-i\omega_{j}h_{j} - \beta h_{j}^{2}\right) dh$$

$$= \sigma^{2} \left(\frac{1}{2\pi}\right)^{d} \prod_{j=1}^{d} \int_{\mathbb{R}} \exp\left(-\beta \left(h_{j} - \left(-i\omega/\left(2\beta\right)\right)\right)^{2}\right) dh_{j}$$

$$= \sigma^{2} \left(\frac{1}{4\pi\beta}\right)^{d/2} \exp\left(-\|\omega\|_{2}^{2}/\left(4\beta\right)\right)$$

**Exercise 4.** Consider the Exponential c.f.  $c(h) = \sigma^2 \exp(-\beta \|h\|_1^1)$  for  $\sigma^2, \beta > 0$  and  $h \in \mathbb{R}^d$ . Compute the spectral density from Bochner's theorem

**Solution.** It is

$$f(\omega) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \exp\left(-i\omega^\top h\right) \sigma^2 \exp\left(-\beta \|h\|_1^1\right) dh$$
$$= \sigma^2 \left(\frac{1}{2\pi}\right)^d \prod_{j=1}^d \int_{\mathbb{R}} \exp\left(-i\omega_j h_j - \beta |h_j|\right) dh_j$$

where

$$\int_{\mathbb{R}} \exp\left(-i\omega_{j}h_{j} - \beta |h_{j}|\right) = \int_{-\infty}^{0} \exp\left(-i\omega_{j}h_{j} - \beta |h_{j}|\right) dh_{j} + \int_{0}^{\infty} \exp\left(-i\omega_{j}h_{j} - \beta |h_{j}|\right) dh_{j}$$

$$= \int_{-\infty}^{0} \exp\left(-i\omega_{j}h_{j} + \beta h_{j}\right) dh_{j} + \int_{0}^{\infty} \exp\left(-i\omega_{j}h_{j} - \beta h_{j}\right) dh_{j}$$

$$= \int_{-\infty}^{0} \exp\left(-\left(i\omega_{j} - \beta\right)h_{j}\right) dh_{j} + \int_{0}^{\infty} \exp\left(-\left(i\omega_{j} + \beta\right)h_{j}\right) dh_{j}$$

$$= \int_{0}^{\infty} \exp\left(-\left(\beta - i\omega_{j}\right)h_{j}\right) dh_{j} + \int_{0}^{\infty} \exp\left(-\left(i\omega_{j} + \beta\right)h_{j}\right) dh_{j}$$

$$= \frac{1}{(\beta - i\omega_{j})} + \frac{1}{(\beta + i\omega_{j})} = \frac{2\beta}{\beta^{2} + \omega_{j}^{2}}$$

hence

$$f(\omega) = \sigma^2 \left(\frac{\beta}{\pi}\right)^d \prod_{j=1}^d \frac{1}{\beta^2 + \omega_j^2}$$