Spatio-temporal statistics (MATH4341) Michaelmas term, 2023

Homework 4: Aerial data modeling

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Exercise 1. ($\star$) Show that the local characteristics

$$\begin{aligned}\text{pr}_1\left(x_1|x_2\right) =&\sqrt{\frac{1}{2\pi}}\exp\left(-\frac{1}{2}\left(x_1-x_2\right)^2\right)\\ \text{pr}_2\left(x_2|x_1\right) =&\sqrt{\frac{1}{2\pi}}\exp\left(-\frac{1}{2}\left(x_2-x_1\right)^2\right)\end{aligned}$$

do not define a proper joint distribution on $\mathbb{R}^{\{1,2\}}$.

Definition. (Pseudo-likelihood) The pseudo likelihood pseudo-$L\left(Z;\theta\right)$ of observables $Z=\left(Z_1,...,Z_n\right)^\top$ given parameters θ is an approximation of the likelihood $L\left(Z;\theta\right)$ of observables $Z=\left(Z_1,...,Z_n\right)^\top$ given parameters θ which is equal to

$$\text{pseudo-}L\left(Z;\theta\right)=\prod_i\text{pr}\left(Z_i|Z_{-i},\theta\right)$$

where $\text{pr}\left(Z_i|Z_{-i},\theta\right)$ are the conditionals of the joint pdf/pmf of the sampling distribution $\text{pr}\left(Z|\theta\right)$ of Z given parameter θ.

Definition. (Maximum Pseudo-Likelihood Estimator) The Maximum Pseudo-Likelihood Estimator (MPLE) $\tilde{\theta}$ of θ is the maximizer of the pseudo likelihood function pseudo-$L\left(Z;\theta\right)$ where the parameter θ is the argument and the observables $Z=\left(Z_1,...,Z_n\right)^\top$ are fixed values.

$$\tilde{\theta}=\arg\max_{\theta}\left(\text{pseudo-}L\left(Z;\theta\right)\right)$$

Exercise 2. ($\star$) Let $Z\in\mathcal{Z}^{\mathcal{S}}$ where $\mathcal{S}=\{1,...,n\}$ and $\mathcal{Z}=\mathbb{R}$. Consider the model

$$Z=X\beta+B\left(Z-X\beta\right)+E$$

where X is a $n\times p$ design matrix X, $\beta\in\mathbb{R}^p$, B is an $n\times n$ symmetric positive definite matrix with $[B]_{i,i}=0$, $E\sim\text{N}\left(0,\sigma^2\left(I-B\right)\right)$, and $\sigma^2>0$.

Hint The following formulas are provided for your information

- $\partial(XY) = (\partial X)Y + X(\partial Y)$
- $\partial\left(X^\top\right) = (\partial X)^\top$
- $\frac{\partial}{\partial x}\left(x^\top Bx\right) = \left(B + B^\top\right)x$
- $\frac{\partial}{\partial x}\left((s - Ax)^\top W(s - Ax)\right) = -2AW(s - Ax)$

1. This is a multiple choice question, choose any number of correct answers.

   (a) Z follows a simultaneous autoregressive (SAR) with Gaussian joint distribution with mean $X\beta$ and covariance matrix $\sigma^2(I - B)^{-1}$

   (b) Ising model

   (c) Conditional autoregressive (CAR) with Gaussian joint distribution with mean $X\beta$ and covariance matrix $\sigma^2 I$

   (d) Bernoulli regression

2. Show that the minus two log Pseudo-Likelihood is such as

$$-2\log\left(\text{pseudo-}L\left(Z;\beta,\sigma^2\right)\right) = n\log\left(\sigma^2\right) + \frac{1}{\sigma^2}(Z - X\beta)^\top(I - B)^2(Z - X\beta) + \text{const.}$$

3. Compute the Maximum Pseudo-Likelihood Estimators (MPLE) $\tilde{\beta}$ and $\tilde{\sigma}^2$ of β and σ^2
