

### Homework 3: Geostatistics (Change of support)

Lecturer: Georgios P. Karagiannis

georgios.karagiannis@durham.ac.uk

**Exercise 1.** (★) Suppose a large volume  $V$  is partitioned into  $n$  smaller units  $v$  of equal size. Show that the dispersion variance  $\sigma^2(v|V) = \frac{1}{n} \sum_{j=1}^n \sigma_E^2(v_j, V)$  can be written in term of variogram integrals

$$\bar{\gamma}(v, V) = \frac{1}{|v||V|} \int_{s \in v} \int_{s' \in V} \gamma(s - s') \, ds ds'$$

as

$$\sigma^2(v|V) = \bar{\gamma}(V, V) - \bar{\gamma}(v, v)$$

**Solution.**

$$\begin{aligned} \sigma^2(v|V) &= \frac{1}{n} \sum_{j=1}^n \sigma_E^2(v_j, V) \\ &= \frac{1}{n} \sum_{j=1}^n [2\bar{\gamma}(v_j, V) - \bar{\gamma}(v_j, v_j) - \bar{\gamma}(V, V)] \\ &= \frac{2}{n} \sum_{j=1}^n \bar{\gamma}(v_j, V) - \frac{1}{n} \sum_{j=1}^n \bar{\gamma}(v_j, v_j) - \frac{1}{n} \sum_{j=1}^n \bar{\gamma}(V, V) \\ &= \frac{2}{n} \sum_{j=1}^n \frac{1}{|v_j||V|} \int_{s \in v_j} \int_{s' \in V} \gamma(s - s') \, ds ds' \\ &\quad - \frac{1}{n} \sum_{j=1}^n \bar{\gamma}(v_j, v_j) - \frac{1}{n} \sum_{j=1}^n \bar{\gamma}(V, V) \quad (\text{but all } v_j \text{ are of the same size as } v) \\ &= 2 \frac{1}{n|v||V|} \sum_{j=1}^n \int_{s \in v_j} \int_{s' \in V} \gamma(s - s') \, ds ds' - \bar{\gamma}(v, v) - \bar{\gamma}(V, V) \\ &= 2 \underbrace{\frac{1}{n|v||V|}}_{=|V|} \underbrace{\sum_{j=1}^n \int_{s \in v_j} \int_{s' \in V} \gamma(s - s') \, ds ds'}_{\int_{s \in V} \int_{s' \in V} \gamma(s - s') \, ds ds'} - \bar{\gamma}(v, v) - \bar{\gamma}(V, V) \\ &= 2 \frac{1}{n} \sum_{j=1}^n \bar{\gamma}(V, V) - \bar{\gamma}(v, v) - \bar{\gamma}(V, V) = \bar{\gamma}(V, V) - \bar{\gamma}(v, v) \end{aligned}$$

**Exercise 2.** (★) Consider a statistical model which is a stochastic process  $(Z_s)_{s \in \mathbb{R}}$  (so  $s$  has dimension 1), where  $Z(\cdot) \sim \text{GP}(\mu(\cdot), c(\cdot, \cdot))$  with mean function  $\mu(s) = 1$  and covariance function

$c(s, t) = \exp\left(-(s - t)^2\right)$  for any  $s \in \mathbb{R}$  and  $t \in \mathbb{R}$ . Assume there is available a dataset  $\{(Z_i, s_i)\}_{i=1}^n$  where  $Z_i = Z(s_i)$  and  $s_i \in \mathbb{R}$  are point sites.

- (1) Compute the length  $|v|$  of the block  $v = [a, b] \subset \mathbb{R}$ .
- (2) Compute the block mean  $\mu(v)$  for some block  $v = [a, b] \subset \mathbb{R}$  and point  $s \in \mathbb{R}$ .
- (3) Compute the block covariance function  $c(v, s)$  for some block  $v = [a, b] \subset \mathbb{R}$  and point  $s \in \mathbb{R}$ .
- (4) Compute the block covariance function  $c(v, v')$  for some blocks  $v = [a, b] \subset \mathbb{R}$  and  $v' = [a', b'] \subset \mathbb{R}$ .
- (5) Denote  $Z = (Z_1, \dots, Z_n)^\top$ , and  $S = \{s_1, \dots, s_n\}$ . Let  $v = [a, b] \subset \mathbb{R}$  and  $v' = [a', b'] \subset \mathbb{R}$  be two intervals. Compute the joint distribution of  $(Z(v), Z(v'), Z)^\top$  as a function of  $c(\cdot, \cdot)$ ,  $S$ ,  $v$ ,  $v'$ ,  $Z$ , and  $\mu(\cdot)$ . What is the name of the distribution and what are the parameter functions defining it?
- (6) (Bayesian Kriging) Compute the predictive stochastic process  $[Z(v) | Z]$  at blocks  $v = [a, b] \subset \mathbb{R}$  with  $|v| > 0$ .

**Hint-1::** Let  $x_1 \in \mathbb{R}^{d_1}$ , and  $x_2 \in \mathbb{R}^{d_2}$ . If

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N_{d_1+d_2} \left( \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_1 & \Sigma_{21}^\top \\ \Sigma_{21} & \Sigma_2 \end{bmatrix} \right)$$

then it is

$$x_2 | x_1 \sim N_{d_2}(\mu_{2|1}, \Sigma_{2|1})$$

where

$$\mu_{2|1} = \mu_2 + \Sigma_{21}\Sigma_1^{-1}(x_1 - \mu_1) \quad \text{and} \quad \Sigma_{2|1} = \Sigma_2 - \Sigma_{21}\Sigma_1^{-1}\Sigma_{21}^\top$$

**Hint-2:** You can use that  $\int \operatorname{erf}(x) dx = x \operatorname{erf}(x) + \frac{\exp(-x^2)}{\sqrt{\pi}} + \text{const}$ , when  $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt$ .

**Solution.**

- (1) It is  $|v| = b - a$
- (2) It is

$$\mu(v) = \mu([a, b]) = \frac{1}{|v|} \int_a^b \mu(s) ds = \frac{1}{|v|} \int_a^b 1 ds = \frac{1}{|v|} |v| = 1$$

- (3) It is

$$\begin{aligned} c(v, s) &= \frac{1}{|v|} \int_a^b c(t, s) dt = \frac{1}{b-a} \int_a^b \exp(-(t-s)^2) dt \\ &= \frac{1}{b-a} \frac{\sqrt{\pi}}{2} \int_a^b \frac{2}{\sqrt{\pi}} \exp(-(t-s)^2) dt = \frac{1}{b-a} \frac{\sqrt{\pi}}{2} \int_{a-s}^{b-s} \frac{2}{\sqrt{\pi}} \exp(-\xi^2) d\xi \\ &= \frac{1}{b-a} \frac{\sqrt{\pi}}{2} \int_0^{b-s} \frac{2}{\sqrt{\pi}} \exp(-\xi^2) d\xi - \frac{1}{b-a} \frac{\sqrt{\pi}}{2} \int_0^{a-s} \frac{2}{\sqrt{\pi}} \exp(-\xi^2) d\xi \\ &= \frac{1}{b-a} \frac{\sqrt{\pi}}{2} \operatorname{erf}(b-s) - \frac{1}{b-a} \frac{\sqrt{\pi}}{2} \operatorname{erf}(a-s) \end{aligned}$$

(4) It is

$$\begin{aligned}
c(v, v') &= \frac{1}{|v'|} \frac{1}{|v|} \int_{a'}^{b'} \int_a^b c(t, s) dt ds = \frac{1}{|v'|} \frac{1}{|v|} \int_{a'}^{b'} \left[ \int_a^b c(t, s) dt \right] ds = \frac{1}{b' - a'} \int_{a'}^{b'} c(v, s) ds \\
&= \frac{1}{b' - a'} \int_{a'}^{b'} \left( \frac{1}{b - a} \frac{\sqrt{\pi}}{2} \operatorname{erf}(b - s) - \frac{1}{b - a} \frac{\sqrt{\pi}}{2} \operatorname{erf}(a - s) \right) ds \\
&= \frac{1}{b' - a'} \frac{1}{b - a} \frac{\sqrt{\pi}}{2} \int_{a'}^{b'} \operatorname{erf}(b - s) ds - \frac{1}{b' - a'} \frac{1}{b - a} \frac{\sqrt{\pi}}{2} \int_{a'}^{b'} \operatorname{erf}(a - s) ds \\
&= \frac{1}{b' - a'} \frac{1}{b - a} \frac{\sqrt{\pi}}{2} (-1) \int_{b-a'}^{b-b'} \operatorname{erf}(\xi) d\xi - \frac{1}{b' - a'} \frac{1}{b - a} \frac{\sqrt{\pi}}{2} (-1) \int_{a-a'}^{a-b'} \operatorname{erf}(\xi) d\xi \\
&= \frac{1}{b' - a'} \frac{1}{b - a} \frac{\sqrt{\pi}}{2} (-1) \left[ \xi \operatorname{erf}(\xi) + \frac{\exp(-\xi^2)}{\sqrt{\pi}} \right]_{b-a'}^{b-b'} - \frac{1}{b' - a'} \frac{1}{b - a} \frac{\sqrt{\pi}}{2} (-1) \left[ \xi \operatorname{erf}(\xi) + \frac{\exp(-\xi^2)}{\sqrt{\pi}} \right]_{a-a'}^{a-b'} \\
&= -\frac{1}{b' - a'} \frac{1}{b - a} \frac{\sqrt{\pi}}{2} (b - b') \operatorname{erf}(b - b') - \frac{1}{b' - a'} \frac{1}{b - a} \frac{1}{2} (b - b') \exp(-(b - b')^2) \\
&\quad + \frac{1}{b' - a'} \frac{1}{b - a} \frac{\sqrt{\pi}}{2} (b - a') \operatorname{erf}(b - a') + \frac{1}{b' - a'} \frac{1}{b - a} \frac{1}{2} (b - a') \exp(-(b - a')^2) \\
&\quad + \frac{1}{b' - a'} \frac{1}{b - a} \frac{\sqrt{\pi}}{2} (a - b') \operatorname{erf}(a - b') + \frac{1}{b' - a'} \frac{1}{b - a} \frac{1}{2} (a - b') \exp(-(a - b')^2) \\
&\quad - \frac{1}{b' - a'} \frac{1}{b - a} \frac{\sqrt{\pi}}{2} (a - a') \operatorname{erf}(a - a') - \frac{1}{b' - a'} \frac{1}{b - a} \frac{1}{2} (a - a') \exp(-(a - a')^2)
\end{aligned}$$

(5) It is

$$\begin{bmatrix} Z(v) \\ Z(v') \\ Z \end{bmatrix} \sim N \left( \begin{bmatrix} \mu(v) \\ \mu(v') \\ \mu(S) \end{bmatrix}, \begin{bmatrix} c(v, v) & c(v, v') & c(v, S) \\ c(v', v) & c(v', v') & c(v', S) \\ c(S, v) & c(S, v') & c(S, S) \end{bmatrix} \right)$$

(6) Taking a better look at part 1, I can see

$$\begin{bmatrix} \begin{bmatrix} Z(v_1) \\ Z(v_2) \end{bmatrix} \\ [Z] \end{bmatrix} \sim N \left( \begin{bmatrix} \begin{bmatrix} \mu(v) \\ \mu(v_2) \end{bmatrix} \\ [\mu(S)] \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} c(v_1, v_1) & c(v_1, v_2) \\ c(v_2, v_1) & c(v_2, v_2) \end{bmatrix} \\ \begin{bmatrix} c(S, v_1) & c(S, v_2) \end{bmatrix} \\ [c(S, S)] \end{bmatrix} \right)$$

From the hint I can see, I can see that

$$\begin{bmatrix} Z(v_1) \\ Z(v_2) \end{bmatrix} | Z \sim N(\mu^\dagger, C^\dagger)$$

with

$$C^\dagger = \begin{bmatrix} C_{11}^\dagger & C_{12}^\dagger \\ C_{21}^\dagger & C_{22}^\dagger \end{bmatrix} = \begin{bmatrix} c(v_1, v_1) & c(v_1, v_2) \\ c(v_2, v_1) & c(v_2, v_2) \end{bmatrix} - \begin{bmatrix} c(v_1, S) \\ c(v_2, S) \end{bmatrix} [c(S, S)]^{-1} \begin{bmatrix} c(S, v_1) & c(S, v_2) \end{bmatrix}$$

and

$$\mu^\dagger = \mu(v_1) + c(v_1, S) [c(S, S)]^{-1} (Z - \mu(S))$$

As this is consistent for any vector of blocks with any size, not only  $V = \{v_1, v_2\}$ , but also  $V = \{v_1, v_2, \dots, v_q\}$  then the predictive stochastic process is a Gaussian Process

$$Z(\cdot) | Z \sim \text{GP}(\mu^*(\cdot), c^*(\cdot, \cdot))$$

with mean function at block  $v$

$$\mu^*(v) = \mu(v) + c(v, S) [c(S, S)]^{-1} (Z - \mu(S))$$

by looking at  $\mu^\dagger$ , and with covariance function at any pair of blocks  $v$  and  $v'$

$$c^*(v, v') = c(v, v') - c(v, S) [c(S, S)]^{-1} c(S, v')$$

looking at any off-diagonal element of  $C^\dagger$  e.g. the  $(1, 2)$  element marked in red.

---