Spatio-temporal statistics (MATH4341) Michaelmas term, 2023

Problem class sheet 1

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Exercise 1. If $c : \mathbb{R}^d \rightarrow \mathbb{R}$ is the covariogram of a weakly stationary random field $Z = (Z_s)_{s\in\mathbb{R}^d}$ then $c(\cdot)$ is semi-positive definite; i.e. for all $n \in \mathbb{N}$, $a \in \mathbb{R}^n$, and $\{s_1, ..., s_n\} \subseteq S$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j c(s_i - s_j) \geq 0$$

Solution. To show that $c(\cdot)$ is semi-positive definite, I need to show that $\forall a \in \mathbb{R}^n - \{0\}$ it is

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j c(s_i - s_j) \geq 0$$

Well it is

$$\begin{aligned}
0 \leq \mathrm{Var}\left(\sum_{i=1}^{n} a_i Z(s_i)\right) &= \mathrm{Cov}\left(\sum_{i=1}^{n} a_i Z(s_i), \sum_{j=1}^{n} a_j Z(s_j)\right) \\
&= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j \mathrm{Cov}(Z(s_i), Z(s_j)) \\
&= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j a_j c(s_i, s_j) = \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j c(s_i - s_j)
\end{aligned}$$

Exercise 2. Show that if $c_1(\cdot,\cdot)$ and $c_2(\cdot,\cdot)$ are are covariance functions (are non-negative definite) then so are $c_3(\cdot,\cdot) = bc_1(\cdot,\cdot) + dc_2(\cdot,\cdot)$ with $b, d \geq 0$ and $c_4(\cdot,\cdot) = c_1(\cdot,\cdot)\, c_2(\cdot,\cdot)$.

Solution. For all $n \in \mathbb{N}$ and $a_1, ...a_n$

$$\begin{aligned}
\sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j c_3(s_i, s_j) &= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j \left(bc_1(s_i, s_j) + dc_2(s_i, s_j)\right) \\
&= \underbrace{\sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j bc_1(s_i, s_j)}_{\geq 0} + \underbrace{\sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j dc_2(s_i, s_j)}_{\geq 0} \\
&\geq 0
\end{aligned}$$

Regarding c_4, assume independent stochastic processes $(Y_s)_{s\in S}$ and $(X_s)_{s\in S}$ with mean zero and covariance functions $c_1(\cdot,\cdot)$ and $c_2(\cdot,\cdot)$ correspondingly. Let stochastic processes $(Z_s)_{s\in S}$

with $Z_s = Y_s X_s$. Then

$$
\begin{aligned}
\operatorname{Cov}(Z_s, Z_t) &= \operatorname{Cov}(Y_s X_s, Y_t X_t) \\
&= \mathrm{E}(Y_s X_s Y_t X_t) \\
&= \mathrm{E}(Y_s Y_t X_s X_t), \quad \text{but } \; Y_s \perp X_s \\
&= \mathrm{E}(X_s X_t)\, \mathrm{E}(X_s X_t) \\
&= \operatorname{Cov}(X_s, X_t) \operatorname{Cov}(Y_s, Y_t) \\
&= c_1(s,t)\, c_2(s,t) = c_4(s,t)
\end{aligned}
$$

that is $c_4(\cdot,\cdot)$ is a covariance function of a stochastic processes.

The following is some theory from your Lecture notes: Handout 3: Point referenced data modeling / Geostatistics

Theorem. *(Bochner's theorem) A continuous even real-valued function $c : \mathbb{R}^d \to \mathbb{R}$ is a covariance function of a weakly stationary random process if and only if it can be represented as*

$$c(h) = \int_{\mathbb{R}^d} \exp\left(i\omega^\top h\right) dF(\omega)$$

where $\mathrm{d}F(\omega)$ is a *symmetric positive finite measure on* $\mathbb{R}^d$.

- Here, we will focus on cases of the form $\mathrm{d}F(\omega) = f(\omega)\,\mathrm{d}\omega$ where $f(\cdot)$ is called spectral density of $c(\cdot)$ i.e.

  $$c(h) = \int_{\mathbb{R}^d} \exp\left(i\omega^\top h\right) f(\omega)\,\mathrm{d}\omega$$

  In this case, $\lim_{h\to\infty} c(h) = 0$

Theorem. *If $c(\cdot)$ is integrable, $F(\cdot)$ is absolutely continuous with spectral density $f(\cdot)$ of $Z = (Z_s; s \in \mathcal{S})$ then by Fast Fourier transformation*

$$f(\omega) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \exp\left(-i\omega^\top h\right) c(h)\, dh$$

Exercise 3. Consider the Gaussian c.f. $c(h) = \sigma^2 \exp\left(-\beta \|h\|_2^2\right)$ for $\sigma^2, \beta > 0$ and $h \in \mathbb{R}^d$. Compute the spectral density from Bochner's theorem

Solution. It is

$$
\begin{aligned}
f(\omega) =& \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \exp\left(-i\omega^\top h\right) \sigma^2 \exp\left(-\beta \|h\|_2^2\right) \mathrm{d}h \\
=&\sigma^2 \left(\frac{1}{2\pi}\right)^d \prod_{j=1}^d \int_{\mathbb{R}} \exp\left(-i\omega_j h_j - \beta h_j^2\right) \mathrm{d}h \\
=&\sigma^2 \left(\frac{1}{2\pi}\right)^d \prod_{j=1}^d \left(\int_{\mathbb{R}} \exp\left(-\beta\left(h_j - \left(-i\omega_j / \left(2\beta\right)\right)\right)^2\right) \mathrm{d}h_j \exp\left(-\omega_j^2 / \left(4\beta\right)\right)\right) \\
=&\sigma^2 \left(\frac{1}{4\pi\beta}\right)^{d/2} \exp\left(-\|\omega\|_2^2 / \left(4\beta\right)\right)
\end{aligned}
$$

Exercise 4. Consider the Exponential c.f. $c(h) = \sigma^2 \exp\left(-\beta \|h\|_1^1\right)$ for $\sigma^2, \beta > 0$ and $h \in \mathbb{R}^d$. Compute the spectral density from Bochner's theorem

Solution. It is

$$
\begin{aligned}
f(\omega) &= \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \exp\left(-i\omega^\top h\right) \sigma^2 \exp\left(-\beta \|h\|_1^1\right) \mathrm{d}h \\
&= \sigma^2 \left(\frac{1}{2\pi}\right)^d \prod_{j=1}^d \int_{\mathbb{R}} \exp\left(-i\omega_j h_j - \beta |h_j|\right) \mathrm{d}h_j
\end{aligned}
$$

where

$$
\begin{aligned}
\int_{\mathbb{R}} \exp\left(-i\omega_j h_j - \beta |h_j|\right) &= \int_{-\infty}^0 \exp\left(-i\omega_j h_j - \beta |h_j|\right) \mathrm{d}h_j + \int_0^\infty \exp\left(-i\omega_j h_j - \beta |h_j|\right) \mathrm{d}h_j \\
&= \int_{-\infty}^0 \exp\left(-i\omega_j h_j + \beta h_j\right) \mathrm{d}h_j + \int_0^\infty \exp\left(-i\omega_j h_j - \beta h_j\right) \mathrm{d}h_j \\
&= \int_{-\infty}^0 \exp\left(-\left(i\omega_j - \beta\right) h_j\right) \mathrm{d}h_j + \int_0^\infty \exp\left(-\left(i\omega_j + \beta\right) h_j\right) \mathrm{d}h_j \\
&= \int_0^\infty \exp\left(-\left(\beta - i\omega_j\right) h_j\right) \mathrm{d}h_j + \int_0^\infty \exp\left(-\left(i\omega_j + \beta\right) h_j\right) \mathrm{d}h_j \\
&= \frac{1}{\left(\beta - i\omega_j\right)} + \frac{1}{\left(\beta + i\omega_j\right)} = \frac{2\beta}{\beta^2 + \omega_j^2}
\end{aligned}
$$

hence

$$
f(\omega) = \sigma^2 \left(\frac{\beta}{\pi}\right)^d \prod_{j=1}^d \frac{1}{\beta^2 + \omega_j^2}
$$
