Spatio-temporal statistics (MATH4341) Michaelmas term, 2023

Problem class sheet 1

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Exercise 1. If $c : \mathbb{R}^d \rightarrow \mathbb{R}$ is the covariogram of a weakly stationary random field $Z = (Z_s)_{s\in\mathbb{R}^d}$ then $c(\cdot)$ is semi-positive definite; i.e. for all $n \in \mathbb{N}$, $a \in \mathbb{R}^n$, and $\{s_1, ..., s_n\} \subseteq S$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j c(s_i - s_j) \geq 0$$

Solution. To show that $c(\cdot)$ is semi-positive definite, I need to show that $\forall a \in \mathbb{R}^n - \{0\}$ it is

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j c(s_i - s_j) \geq 0$$

Well it is

$$\begin{aligned}
0 \leq \text{Var}\left(\sum_{i=1}^{n} a_i Z(s_i)\right) &= \text{Cov}\left(\sum_{i=1}^{n} a_i Z(s_i), \sum_{j=1}^{n} a_j Z(s_j)\right) \\
&= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j \text{Cov}(Z(s_i), Z(s_j)) \\
&= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j a_j c(s_i, s_j) = \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j c(s_i - s_j)
\end{aligned}$$

Exercise 2. Show that if $c_1(\cdot,\cdot)$ and $c_2(\cdot,\cdot)$ are are covariance functions (are non-negative definite) then so are $c_3(\cdot,\cdot) = bc_1(\cdot,\cdot) + dc_2(\cdot,\cdot)$ and $c_4(\cdot,\cdot) = c_1(\cdot,\cdot)\, c_2(\cdot,\cdot)$.

Solution. For all $n \in \mathbb{N}$ and $a_1, ...a_n$

$$\begin{aligned}
\sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j c_3(s_i, s_j) &= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j \left(bc_1(s_i, s_j) + dc_2(s_i, s_j)\right) \\
&= \underbrace{\sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j bc_1(s_i, s_j)}_{\geq 0} + \underbrace{\sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j dc_2(s_i, s_j)}_{\geq 0} \\
&\geq 0
\end{aligned}$$

Regarding c_4, assume independent stochastic processes $(Y_s)_{s\in S}$ and $(X_s)_{s\in S}$ with mean zero and covariance functions $c_1(\cdot,\cdot)$ and $c_2(\cdot,\cdot)$ correspondingly. Let stochastic processes $(Z_s)_{s\in S}$

with $Z_s = Y_s X_s$. Then

$$\begin{aligned}
\operatorname{Cov}(Z_s, Z_t) &= \operatorname{Cov}(Y_s X_s, Y_t X_t) \\
&= \operatorname{E}(Y_s X_s Y_t X_t) \\
&= \operatorname{E}(Y_s Y_t X_s X_t), \quad \text{but } \; Y_s \perp X_s \\
&= \operatorname{E}(X_s X_t) \operatorname{E}(X_s X_t) \\
&= \operatorname{Cov}(X_s, X_t) \operatorname{Cov}(Y_s, Y_t) \\
&= c_1(s,t)\, c_2(s,t) = c_4(s,t)
\end{aligned}$$

that is $c_4(\cdot,\cdot)$ is a covariance function of a stochastic processes.

The following is some theory from your Lecture notes: Handout 3: Point referenced data modeling / Geostatistics

Theorem. *(Bochner's theorem) A continuous even real-valued function $c : \mathbb{R}^d \to \mathbb{R}$ is a covariance function of a weakly stationary random process if and only if it can be represented as*

$$c(h) = \int_{\mathbb{R}^d} \exp\left(i\omega^\top h\right) dF(\omega)$$

where $\mathrm{d}F(\omega)$ is a *symmetric positive finite measure on* $\mathbb{R}^d$.

- Here, we will focus on cases of the form $\mathrm{d}F(\omega) = f(\omega)\,\mathrm{d}\omega$ where $f(\cdot)$ is called spectral density of $c(\cdot)$ i.e.

$$c(h) = \int_{\mathbb{R}^d} \exp\left(i\omega^\top h\right) f(\omega)\,\mathrm{d}\omega$$

  In this case, $\lim_{h\to\infty} c(h) = 0$

Theorem. *If $c(\cdot)$ is integrable, $F(\cdot)$ is absolutely continuous with spectral density $f(\cdot)$ of $Z = (Z_s; s \in \mathcal{S})$ then by Fast Fourier transformation*

$$f(\omega) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \exp\left(-i\omega^\top h\right) c(h)\, dh$$

Exercise 3. Consider the Gaussian c.f. $c(h) = \sigma^2 \exp\left(-\beta \|h\|_2^2\right)$ for $\sigma^2, \beta > 0$ and $h \in \mathbb{R}^d$. Compute the spectral density from Bochner's theorem

Solution. It is

$$\begin{aligned} f\left(\omega\right) &= \left(\frac{1}{2\pi}\right)^{d}\int_{\mathbb{R}^d}\exp\left(-i\omega^{\top}h\right)\sigma^2\exp\left(-\beta\left\|h\right\|_2^2\right)\mathrm{d}h \\ &= \sigma^2\left(\frac{1}{2\pi}\right)^{d}\prod_{j=1}^{d}\int_{\mathbb{R}}\exp\left(-i\omega_j h_j-\beta h_j^2\right)\mathrm{d}h \\ &= \sigma^2\left(\frac{1}{2\pi}\right)^{d}\prod_{j=1}^{d}\int_{\mathbb{R}}\exp\left(-\beta\left(h_j-\left(-i\omega/\left(2\beta\right)\right)\right)^2\right)\mathrm{d}h_j \\ &= \sigma^2\left(\frac{1}{4\pi\beta}\right)^{d/2}\exp\left(-\left\|\omega\right\|_2^2/\left(4\beta\right)\right) \end{aligned}$$

Exercise 4. Consider the Exponential c.f. $c\left(h\right)=\sigma^2\exp\left(-\beta\left\|h\right\|_1^1\right)$ for $\sigma^2,\beta>0$ and $h\in\mathbb{R}^d$. Compute the spectral density from Bochner's theorem

Solution. It is

$$\begin{aligned} f\left(\omega\right) &= \left(\frac{1}{2\pi}\right)^{d}\int_{\mathbb{R}^d}\exp\left(-i\omega^{\top}h\right)\sigma^2\exp\left(-\beta\left\|h\right\|_1^1\right)\mathrm{d}h \\ &= \sigma^2\left(\frac{1}{2\pi}\right)^{d}\prod_{j=1}^{d}\int_{\mathbb{R}}\exp\left(-i\omega_j h_j-\beta\left|h_j\right|\right)\mathrm{d}h_j \end{aligned}$$

where

$$\begin{aligned} \int_{\mathbb{R}}\exp\left(-i\omega_j h_j-\beta\left|h_j\right|\right) &= \int_{-\infty}^{0}\exp\left(-i\omega_j h_j-\beta\left|h_j\right|\right)\mathrm{d}h_j+\int_{0}^{\infty}\exp\left(-i\omega_j h_j-\beta\left|h_j\right|\right)\mathrm{d}h_j \\ &= \int_{-\infty}^{0}\exp\left(-i\omega_j h_j+\beta h_j\right)\mathrm{d}h_j+\int_{0}^{\infty}\exp\left(-i\omega_j h_j-\beta h_j\right)\mathrm{d}h_j \\ &= \int_{-\infty}^{0}\exp\left(-\left(i\omega_j-\beta\right)h_j\right)\mathrm{d}h_j+\int_{0}^{\infty}\exp\left(-\left(i\omega_j+\beta\right)h_j\right)\mathrm{d}h_j \\ &= \int_{0}^{\infty}\exp\left(-\left(\beta-i\omega_j\right)h_j\right)\mathrm{d}h_j+\int_{0}^{\infty}\exp\left(-\left(i\omega_j+\beta\right)h_j\right)\mathrm{d}h_j \\ &= \frac{1}{\left(\beta-i\omega_j\right)}+\frac{1}{\left(\beta+i\omega_j\right)}=\frac{2\beta}{\beta^2+\omega_j^2} \end{aligned}$$

hence

$$f\left(\omega\right)=\sigma^2\left(\frac{\beta}{\pi}\right)^{d}\prod_{j=1}^{d}\frac{1}{\beta^2+\omega_j^2}$$
