Problem class sheet 1

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Exercise 1. If $c : \mathbb{R}^d \to \mathbb{R}$ is the covariogram of a weakly stationary random field $Z = (Z_s)_{s \in \mathbb{R}^d}$ then $c(\cdot)$ is semi-positive definite; i.e. for all $n \in \mathbb{N}$, $a \in \mathbb{R}^n$, and $\{s_1, ..., s_n\} \subseteq S$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j c(s_i - s_j) \geq 0$$

Solution. To show that $c(\cdot)$ is semi-positive definite, I need to show that $\forall a \in \mathbb{R}^n - \{0\}$ it is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j c(s_i - s_j) \geq 0$$

Well it is

$$0 \leq \text{Var}\left(\sum_{i=1}^{n} a_i Z(s_i)\right) = \text{Cov}\left(\sum_{i=1}^{n} a_i Z(s_i), \sum_{j=1}^{n} a_j Z(s_j)\right)$$

$$= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j \text{Cov}(Z(s_i), Z(s_j))$$

$$= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j a_j c(s_i, s_j) = \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j c(s_i - s_j)$$

Exercise 2. Show that if $c_1(\cdot, \cdot)$ and $c_2(\cdot, \cdot)$ are are covariance functions (are non-negative definite) then so are $c_3(\cdot, \cdot) = bc_1(\cdot, \cdot) + dc_2(\cdot, \cdot)$ and $c_4(\cdot, \cdot) = c_1(\cdot, \cdot) c_2(\cdot, \cdot)$.

Solution. For all $n \in \mathbb{N}$ and $a_1, ... a_n$

$$\sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j c_3(s_i, s_j) = \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j (bc_1(s_i, s_j) + dc_2(s_i, s_j))$$

$$= \underbrace{\sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j bc_1(s_i, s_j)}_{\geq 0} + \underbrace{\sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j dc_2(s_i, s_j)}_{\geq 0}$$

$$\geq 0$$

Regarding c_4, assume independent stochastic processes $(Y_s)_{s \in S}$ and $(X_s)_{s \in S}$ with mean zero and covariance functions $c_1(\cdot, \cdot)$ and $c_2(\cdot, \cdot)$ correspondingly. Let stochastic processes $(Z_s)_{s \in S}$

with $Z_s = Y_s X_s$. Then

$$
\begin{aligned}
\mathrm{Cov}\left(Z_s, Z_t\right) &= \mathrm{Cov}\left(Y_s X_s, Y_t X_t\right) \\
&= \mathrm{E}\left(Y_s X_s Y_t X_t\right) \\
&= \mathrm{E}\left(Y_s Y_t X_s X_t\right), \ \text{ but } \ Y_s \perp X_s \\
&= \mathrm{E}\left(X_s X_t\right)\mathrm{E}\left(X_s X_t\right) \\
&= \mathrm{Cov}\left(X_s, X_t\right)\mathrm{Cov}\left(Y_s, Y_t\right) \\
&= c_1\left(s, t\right)c_2\left(s, t\right) = c_4\left(s, t\right)
\end{aligned}
$$

that is $c_4\left(\cdot, \cdot\right)$ is a covariance function of a stochastic processes.

The following is some theory from your Lecture notes: Handout 3: Point referenced data modeling / Geostatistics

Theorem. *(Bochner's theorem) A continuous even real-valued function $c : \mathbb{R}^d \to \mathbb{R}$ is a covariance function of a weakly stationary random process if and only if it can be represented as*

$$
c\left(h\right) = \int_{\mathbb{R}^d} \exp\left(i\omega^\top h\right) dF\left(\omega\right)
$$

where $\mathrm{d}F\left(\omega\right)$ is a symmetric positive finite measure on $\mathbb{R}^d$.

- Here, we will focus on cases of the form $\mathrm{d}F\left(\omega\right) = f\left(\omega\right)\mathrm{d}\omega$ where $f\left(\cdot\right)$ is called spectral density of $c\left(\cdot\right)$ i.e.

$$
c\left(h\right) = \int_{\mathbb{R}^d} \exp\left(i\omega^\top h\right) f\left(\omega\right)\mathrm{d}\omega
$$

  In this case, $\lim_{h\to\infty} c\left(h\right) = 0$

Theorem. *If $c\left(\cdot\right)$ is integrable, $F\left(\cdot\right)$ is absolutely continuous with spectral density $f\left(\cdot\right)$ of $Z = \left(Z_s; s \in \mathcal{S}\right)$ then by Fast Fourier transformation*

$$
f\left(\omega\right) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \exp\left(-i\omega^\top h\right) c\left(h\right) dh
$$

Exercise 3. Consider the Gaussian c.f. $c\left(h\right) = \sigma^2 \exp\left(-\beta \left\|h\right\|_2^2\right)$ for $\sigma^2, \beta > 0$ and $h \in \mathbb{R}^d$. Compute the spectral density from Bochner's theorem

Solution. It is

$$f\left(\omega\right) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \exp\left(-i\omega^\top h\right) \sigma^2 \exp\left(-\beta \|h\|_2^2\right) \mathrm{d}h$$

$$= \sigma^2 \left(\frac{1}{2\pi}\right)^d \prod_{j=1}^d \int_{\mathbb{R}} \exp\left(-i\omega_j h_j - \beta h_j^2\right) \mathrm{d}h$$

$$= \sigma^2 \left(\frac{1}{2\pi}\right)^d \prod_{j=1}^d \int_{\mathbb{R}} \exp\left(-\beta\left(h_j - \left(-i\omega/\left(2\beta\right)\right)\right)^2\right) \mathrm{d}h_j$$

$$= \sigma^2 \left(\frac{1}{4\pi\beta}\right)^{d/2} \exp\left(-\|\omega\|_2^2 / \left(4\beta\right)\right)$$

Exercise 4. Consider the Exponential c.f. $c\left(h\right) = \sigma^2 \exp\left(-\beta \|h\|_1^1\right)$ for $\sigma^2, \beta > 0$ and $h \in \mathbb{R}^d$. Compute the spectral density from Bochner's theorem

Solution. It is

$$f\left(\omega\right) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \exp\left(-i\omega^\top h\right) \sigma^2 \exp\left(-\beta \|h\|_1^1\right) \mathrm{d}h$$

$$= \sigma^2 \left(\frac{1}{2\pi}\right)^d \prod_{j=1}^d \int_{\mathbb{R}} \exp\left(-i\omega_j h_j - \beta |h_j|\right) \mathrm{d}h_j$$

where

$$\int_{\mathbb{R}} \exp\left(-i\omega_j h_j - \beta |h_j|\right) = \int_{-\infty}^0 \exp\left(-i\omega_j h_j - \beta |h_j|\right) \mathrm{d}h_j + \int_0^\infty \exp\left(-i\omega_j h_j - \beta |h_j|\right) \mathrm{d}h_j$$

$$= \int_{-\infty}^0 \exp\left(-i\omega_j h_j + \beta h_j\right) \mathrm{d}h_j + \int_0^\infty \exp\left(-i\omega_j h_j - \beta h_j\right) \mathrm{d}h_j$$

$$= \int_{-\infty}^0 \exp\left(-\left(i\omega_j - \beta\right) h_j\right) \mathrm{d}h_j + \int_0^\infty \exp\left(-\left(i\omega_j + \beta\right) h_j\right) \mathrm{d}h_j$$

$$= \int_0^\infty \exp\left(-\left(\beta - i\omega_j\right) h_j\right) \mathrm{d}h_j + \int_0^\infty \exp\left(-\left(i\omega_j + \beta\right) h_j\right) \mathrm{d}h_j$$

$$= \frac{1}{\left(\beta - i\omega_j\right)} + \frac{1}{\left(\beta + i\omega_j\right)} = \frac{2\beta}{\beta^2 + \omega_j^2}$$

hence

$$f\left(\omega\right) = \sigma^2 \left(\frac{\beta}{\pi}\right)^d \prod_{j=1}^d \frac{1}{\beta^2 + \omega_j^2}$$

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