

## Homework 4: Aerial data modeling

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**Exercise 1.** (★) Show that the local characteristics

$$\begin{aligned}\text{pr}_1(x_1|x_2) &= \sqrt{\frac{1}{2\pi}} \exp\left(-\frac{1}{2}(x_1 - x_2)^2\right) \\ \text{pr}_2(x_2|x_1) &= \sqrt{\frac{1}{2\pi}} \exp\left(-\frac{1}{2}(x_2 - x_1)^2\right)\end{aligned}$$

do not define a proper joint distribution on  $\mathbb{R}^{\{1,2\}}$ .

**Definition.** (Pseudo-likelihood) The pseudo likelihood pseudo- $L(Z; \theta)$  of observables  $Z = (Z_1, \dots, Z_n)^\top$  given parameters  $\theta$  is an approximation of the likelihood  $L(Z; \theta)$  of observables  $Z = (Z_1, \dots, Z_n)^\top$  given parameters  $\theta$  which is equal to

$$\text{pseudo-}L(Z; \theta) = \prod_i \text{pr}(Z_i | Z_{-i}, \theta)$$

where  $\text{pr}(Z_i | Z_{-i}, \theta)$  are the conditionals of the joint pdf/pmf of the sampling distribution  $\text{pr}(Z | \theta)$  of  $Z$  given parameter  $\theta$ .

**Definition.** (Maximum Pseudo-Likelihood Estimator) The Maximum Pseudo-Likelihood Estimator (MPLE)  $\tilde{\theta}$  of  $\theta$  is the maximizer of the pseudo likelihood function pseudo- $L(Z; \theta)$  where the parameter  $\theta$  is the argument and the observables  $Z = (Z_1, \dots, Z_n)^\top$  are fixed values.

$$\tilde{\theta} = \arg \max_{\theta} (\text{pseudo-}L(Z; \theta))$$

**Exercise 2.** (★) Let  $Z \in \mathcal{Z}^{\mathcal{S}}$  where  $\mathcal{S} = \{1, \dots, n\}$  and  $\mathcal{Z} = \mathbb{R}$ . Consider the model

$$Z = X\beta + B(Z - X\beta) + E$$

where  $X$  is a  $n \times p$  design matrix  $X$ ,  $\beta \in \mathbb{R}^p$ ,  $B$  is an  $n \times n$  symmetric positive definite matrix with  $[B]_{i,i} = 0$ ,  $E \sim N(0, \sigma^2(I - B))$ , and  $\sigma^2 > 0$ .

**Hint** The following formulas are provided for your information

- $\partial (XY) = (\partial X) Y + X (\partial Y)$
- $\partial (X^\top) = (\partial X)^\top$
- $\frac{\partial}{\partial x} (x^\top Bx) = (B + B^\top) x$
- $\frac{\partial}{\partial x} \left( (s - Ax)^\top W (s - Ax) \right) = -2AW (s - Ax)$

1. This is a multiple choice question, choose any number of correct answers.

- (a)  $Z$  follows a simultaneous autoregressive (SAR) with Gaussian joint distribution with mean  $X\beta$  and covariance matrix  $\sigma^2 (I - B)^{-1}$
- (b) Ising model
- (c) Conditional autoregressive (CAR) with Gaussian joint distribution with mean  $X\beta$  and covariance matrix  $\sigma^2 I$
- (d) Bernoulli regression

2. Show that the minus two log Pseudo-Likelihood is such as

$$-2 \log (\text{pseudo-}L(Z; \beta, \sigma^2)) = n \log (\sigma^2) + \frac{1}{\sigma^2} (Z - X\beta)^\top (I - B)^2 (Z - X\beta) + \text{const.}$$

3. Compute the Maximum Pseudo-Likelihood Estimators (MPLE)  $\tilde{\beta}$  and  $\tilde{\sigma}^2$  of  $\beta$  and  $\sigma^2$

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