

Homework 4: Aerial data modeling

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Exercise 1. (★) Show that the local characteristics

$$\begin{aligned}\text{pr}_1(x_1|x_2) &= \sqrt{\frac{1}{2\pi}} \exp\left(-\frac{1}{2}(x_1 - x_2)^2\right) \\ \text{pr}_2(x_2|x_1) &= \sqrt{\frac{1}{2\pi}} \exp\left(-\frac{1}{2}(x_2 - x_1)^2\right)\end{aligned}$$

do not define a proper joint distribution on $\mathbb{R}^{\{1,2\}}$.

Definition. (Pseudo-likelihood) The pseudo likelihood pseudo- $L(Z; \theta)$ of observables $Z = (Z_1, \dots, Z_n)^\top$ given parameters θ is an approximation of the likelihood $L(Z; \theta)$ of observables $Z = (Z_1, \dots, Z_n)^\top$ given parameters θ which is equal to

$$\text{pseudo-}L(Z; \theta) = \prod_i \text{pr}(Z_i | Z_{-i}, \theta)$$

where $\text{pr}(Z_i | Z_{-i}, \theta)$ are the conditionals of the joint pdf/pmf of the sampling distribution $\text{pr}(Z | \theta)$ of Z given parameter θ .

Definition. (Maximum Pseudo-Likelihood Estimator) The Maximum Pseudo-Likelihood Estimator (MPLE) $\tilde{\theta}$ of θ is the maximizer of the pseudo likelihood function pseudo- $L(Z; \theta)$ where the parameter θ is the argument and the observables $Z = (Z_1, \dots, Z_n)^\top$ are fixed values.

$$\tilde{\theta} = \arg \max_{\theta} (\text{pseudo-}L(Z; \theta))$$

Exercise 2. (★) Let $Z \in \mathcal{Z}^{\mathcal{S}}$ where $\mathcal{S} = \{1, \dots, n\}$ and $\mathcal{Z} = \mathbb{R}$. Consider the model

$$Z = X\beta + B(Z - X\beta) + E$$

where X is a $n \times p$ design matrix X , $\beta \in \mathbb{R}^p$, $I - B$ is an $n \times n$ symmetric positive definite matrix with $[B]_{i,i} = 0$, $E \sim N(0, \sigma^2(I - B))$, and $\sigma^2 > 0$.

Hint The following formulas are provided for your information

- $\partial (XY) = (\partial X) Y + X (\partial Y)$
- $\partial (X^\top) = (\partial X)^\top$
- $\frac{\partial}{\partial x} (x^\top Bx) = (B + B^\top) x$
- $\frac{\partial}{\partial x} \left((s - Ax)^\top W (s - Ax) \right) = -2AW (s - Ax)$

1. This is a multiple choice question, choose any number of correct answers.

- (a) Z follows a simultaneous autoregressive (SAR) with Gaussian joint distribution with mean $X\beta$ and covariance matrix $\sigma^2 (I - B)^{-1}$
- (b) Ising model
- (c) Conditional autoregressive (CAR) with Gaussian joint distribution with mean $X\beta$ and covariance matrix $\sigma^2 I$
- (d) Bernoulli regression

2. Show that the minus two log Pseudo-Likelihood is such as

$$-2 \log (\text{pseudo-}L(Z; \beta, \sigma^2)) = n \log (\sigma^2) + \frac{1}{\sigma^2} (Z - X\beta)^\top (I - B)^2 (Z - X\beta) + \text{const.}$$

3. Compute the Maximum Pseudo-Likelihood Estimators (MPLE) $\tilde{\beta}$ and $\tilde{\sigma}^2$ of β and σ^2
