

## Homework 3: Geostatistics (Change of support)

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**Exercise 1.** (★) Suppose a large volume  $V$  is partitioned into  $n$  smaller units  $v$  of equal size. Show that the dispersion variance  $\sigma^2(v|V) = \frac{1}{n} \sum_{j=1}^n \sigma_E^2(v_j, V)$  can be written in term of variogram integrals

$$\bar{\gamma}(v, V) = \frac{1}{|v||V|} \int_{s \in v} \int_{s' \in V} \gamma(s - s') \, ds ds'$$

as

$$\sigma^2(v|V) = \bar{\gamma}(V, V) - \bar{\gamma}(v, v)$$

**Exercise 2.** (★) Consider a statistical model which is a stochastic process  $(Z_s)_{s \in \mathbb{R}}$  (so  $s$  has dimension 1), where  $Z(\cdot) \sim \text{GP}(\mu(\cdot), c(\cdot, \cdot))$  with mean function  $\mu(s) = 1$  and covariance function  $c(s, t) = \exp\left(-(s - t)^2\right)$  for any  $s \in \mathbb{R}$  and  $t \in \mathbb{R}$ . Assume there is available a dataset  $\{(Z_i, s_i)\}_{i=1}^n$  where  $Z_i = Z(s_i)$  and  $s_i \in \mathbb{R}$  are point sites.

1. Compute the length  $|v|$  of the block  $v = [a, b] \subset \mathbb{R}$ .
2. Compute the block mean  $\mu(v)$  for some block  $v = [a, b] \subset \mathbb{R}$  and point  $s \in \mathbb{R}$ .
3. Compute the block covariance function  $c(v, s)$  for some block  $v = [a, b] \subset \mathbb{R}$  and point  $s \in \mathbb{R}$ .
4. Compute the block covariance function  $c(v, v')$  for some blocks  $v = [a, b] \subset \mathbb{R}$  and  $v' = [a', b'] \subset \mathbb{R}$ .
5. Denote  $Z = (Z_1, \dots, Z_n)^\top$ , and  $S = \{s_1, \dots, s_n\}$ . Let  $v = [a, b] \subset \mathbb{R}$  and  $v' = [a', b'] \subset \mathbb{R}$  be two intervals. Compute the joint distribution of  $(Z(v), Z(v'), Z)^\top$  as a function of  $c(\cdot, \cdot)$ ,  $S$ ,  $v$ ,  $v'$ ,  $Z$ , and  $\mu(\cdot)$ . What is the name of the distribution and what are the parameter functions defining it?
6. (Bayesian Kriging) Compute the predictive stochastic process  $[Z(v)|Z]$  at blocks  $v = [a, b] \subset \mathbb{R}$  with  $|v| > 0$ .

**Hint-1:** Let  $x_1 \in \mathbb{R}^{d_1}$ , and  $x_2 \in \mathbb{R}^{d_2}$ . If

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N_{d_1+d_2} \left( \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_1 & \Sigma_{21}^\top \\ \Sigma_{21} & \Sigma_2 \end{bmatrix} \right)$$

then it is

$$x_2|x_1 \sim N_{d_2}(\mu_{2|1}, \Sigma_{2|1})$$

where

$$\mu_{2|1} = \mu_2 + \Sigma_{21}\Sigma_1^{-1}(x_1 - \mu_1) \quad \text{and} \quad \Sigma_{2|1} = \Sigma_2 - \Sigma_{21}\Sigma_1^{-1}\Sigma_{21}^\top$$

**Hint-2** Assume known function  $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt$ . Then  $\int \operatorname{erf}(x) dx = x \operatorname{erf}(x) + \frac{\exp(-x^2)}{\sqrt{\pi}} + \text{const}$

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