

Problem class sheet 1

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Exercise 1. If $c : \mathbb{R}^d \rightarrow \mathbb{R}$ is the covariogram of a weakly stationary random field $Z = (Z_s)_{s \in \mathbb{R}^d}$ then $c(\cdot)$ is semi-positive definite; i.e. for all $n \in \mathbb{N}$, $a \in \mathbb{R}^n$, and $\{s_1, \dots, s_n\} \subseteq S$

$$\sum_{i=1}^n \sum_{j=1}^n a_i a_j c(s_i - s_j) \geq 0$$

Solution. To show that $c(\cdot)$ is semi-positive definite, I need to show that $\forall a \in \mathbb{R}^n - \{0\}$ it is

$$\sum_{i=1}^n \sum_{j=1}^n a_i a_j c(s_i - s_j) \geq 0$$

Well it is

$$\begin{aligned} 0 &\leq \text{Var} \left(\sum_{i=1}^n a_i Z(s_i) \right) = \text{Cov} \left(\sum_{i=1}^n a_i Z(s_i), \sum_{j=1}^n a_j Z(s_j) \right) \\ &= \sum_{i=1}^n a_i \sum_{j=1}^n a_j \text{Cov}(Z(s_i), Z(s_j)) \\ &= \sum_{i=1}^n a_i \sum_{j=1}^n a_j a_j c(s_i, s_j) = \sum_{i=1}^n a_i \sum_{j=1}^n a_j c(s_i - s_j) \end{aligned}$$

Exercise 2. Show that if $c_1(\cdot, \cdot)$ and $c_2(\cdot, \cdot)$ are covariance functions (are non-negative definite) then so are $c_3(\cdot, \cdot) = bc_1(\cdot, \cdot) + dc_2(\cdot, \cdot)$ and $c_4(\cdot, \cdot) = c_1(\cdot, \cdot) c_2(\cdot, \cdot)$.

Solution. For all $n \in \mathbb{N}$ and a_1, \dots, a_n

$$\begin{aligned} \sum_{i=1}^n a_i \sum_{j=1}^n a_j c_3(s_i, s_j) &= \sum_{i=1}^n a_i \sum_{j=1}^n a_j (bc_1(s_i, s_j) + dc_2(s_i, s_j)) \\ &= \underbrace{\sum_{i=1}^n a_i \sum_{j=1}^n a_j bc_1(s_i, s_j)}_{\geq 0} + \underbrace{\sum_{i=1}^n a_i \sum_{j=1}^n a_j dc_2(s_i, s_j)}_{\geq 0} \\ &\geq 0 \end{aligned}$$

Regarding c_4 , assume independent stochastic processes $(Y_s)_{s \in S}$ and $(X_s)_{s \in S}$ with mean zero and covariance functions $c_1(\cdot, \cdot)$ and $c_2(\cdot, \cdot)$ correspondingly. Let stochastic processes $(Z_s)_{s \in S}$

with $Z_s = Y_s X_s$. Then

$$\begin{aligned}
\text{Cov}(Z_s, Z_t) &= \text{Cov}(Y_s X_s, Y_t X_t) \\
&= \text{E}(Y_s X_s Y_t X_t) \\
&= \text{E}(Y_s Y_t X_s X_t), \text{ but } Y_s \perp X_s \\
&= \text{E}(X_s X_t) \text{E}(Y_s Y_t) \\
&= \text{Cov}(X_s, X_t) \text{Cov}(Y_s, Y_t) \\
&= c_1(s, t) c_2(s, t) = c_4(s, t)
\end{aligned}$$

that is $c_4(\cdot, \cdot)$ is a covariance function of a stochastic processes.

The following is some theory from your Lecture notes: Handout 3: Point referenced data modeling / Geostatistics

Theorem. (Bochner's theorem) *A continuous even real-valued function $c : \mathbb{R}^d \rightarrow \mathbb{R}$ is a covariance function of a weakly stationary random process if and only if it can be represented as*

$$c(h) = \int_{\mathbb{R}^d} \exp(i\omega^\top h) dF(\omega)$$

where $dF(\omega)$ is a symmetric positive finite measure on \mathbb{R}^d .

- Here, we will focus on cases of the form $dF(\omega) = f(\omega) d\omega$ where $f(\cdot)$ is called spectral density of $c(\cdot)$ i.e.

$$c(h) = \int_{\mathbb{R}^d} \exp(i\omega^\top h) f(\omega) d\omega$$

In this case, $\lim_{h \rightarrow \infty} c(h) = 0$

Theorem. *If $c(\cdot)$ is integrable, $F(\cdot)$ is absolutely continuous with spectral density $f(\cdot)$ of $Z = (Z_s; s \in \mathcal{S})$ then by Fast Fourier transformation*

$$f(\omega) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \exp(-i\omega^\top h) c(h) dh$$

Exercise 3. Consider the Gaussian c.f. $c(h) = \sigma^2 \exp(-\beta \|h\|_2^2)$ for $\sigma^2, \beta > 0$ and $h \in \mathbb{R}^d$. Compute the spectral density from Bochner's theorem

Solution. It is

$$\begin{aligned}
f(\omega) &= \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \exp(-i\omega^\top h) \sigma^2 \exp(-\beta \|h\|_2^2) dh \\
&= \sigma^2 \left(\frac{1}{2\pi}\right)^d \prod_{j=1}^d \int_{\mathbb{R}} \exp(-i\omega_j h_j - \beta h_j^2) dh_j \\
&= \sigma^2 \left(\frac{1}{2\pi}\right)^d \prod_{j=1}^d \int_{\mathbb{R}} \exp(-\beta (h_j - (-i\omega_j / (2\beta)))^2) dh_j \\
&= \sigma^2 \left(\frac{1}{4\pi\beta}\right)^{d/2} \exp(-\|\omega\|_2^2 / (4\beta))
\end{aligned}$$

Exercise 4. Consider the Exponential c.f. $c(h) = \sigma^2 \exp(-\beta \|h\|_1^1)$ for $\sigma^2, \beta > 0$ and $h \in \mathbb{R}^d$. Compute the spectral density from Bochner's theorem

Solution. It is

$$\begin{aligned}
f(\omega) &= \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \exp(-i\omega^\top h) \sigma^2 \exp(-\beta \|h\|_1^1) dh \\
&= \sigma^2 \left(\frac{1}{2\pi}\right)^d \prod_{j=1}^d \int_{\mathbb{R}} \exp(-i\omega_j h_j - \beta |h_j|) dh_j
\end{aligned}$$

where

$$\begin{aligned}
\int_{\mathbb{R}} \exp(-i\omega_j h_j - \beta |h_j|) dh_j &= \int_{-\infty}^0 \exp(-i\omega_j h_j - \beta |h_j|) dh_j + \int_0^{\infty} \exp(-i\omega_j h_j - \beta |h_j|) dh_j \\
&= \int_{-\infty}^0 \exp(-i\omega_j h_j + \beta h_j) dh_j + \int_0^{\infty} \exp(-i\omega_j h_j - \beta h_j) dh_j \\
&= \int_{-\infty}^0 \exp(-(i\omega_j - \beta) h_j) dh_j + \int_0^{\infty} \exp(-(i\omega_j + \beta) h_j) dh_j \\
&= \int_0^{\infty} \exp(-(\beta - i\omega_j) h_j) dh_j + \int_0^{\infty} \exp(-(i\omega_j + \beta) h_j) dh_j \\
&= \frac{1}{(\beta - i\omega_j)} + \frac{1}{(\beta + i\omega_j)} = \frac{2\beta}{\beta^2 + \omega_j^2}
\end{aligned}$$

hence

$$f(\omega) = \sigma^2 \left(\frac{\beta}{\pi}\right)^d \prod_{j=1}^d \frac{1}{\beta^2 + \omega_j^2}$$
