

**Homework 1: Geostatistics (building concepts)**

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As formative assessment 1, submit the solutions to all the Exercises

**Exercise 1.** (★) Let  $Z = (Z_s)_{s \in \mathbb{R}^d}$  be an intrinsically stationary stochastic process, and let  $\gamma : \mathbb{R}^d \rightarrow \mathbb{R}$  be its semivariogram. Assume  $a \in \mathbb{R}^n$  s.t.  $\sum_{i=1}^n a_i = 0$ .

(1) Let  $a \in \mathbb{R}^n$  be a vector of constants. Show that

$$\text{Var} \left( \sum_{i=1}^n a_i Z(s_i) \right) = \sum_{i=1}^n \sum_{j=1}^n a_i a_j c_Y(s_i, s_j)$$

where  $c_Y(s, t) = \text{E}(Y(s)Y(t))$ , and  $Y_s = Z_s - Z_0$ .

(2) Show that

$$c_Y(s, t) = \gamma(s) + \gamma(t) - \gamma(s - t)$$

(3) Show that for all  $\forall \{s_1, \dots, s_n\} \subseteq S$  it is

$$\sum_{i=1}^n \sum_{j=1}^n a_i a_j \gamma(s_i - s_j) \leq 0$$

**Solution.** Assume  $0 \in S$ , and a random variable  $Z(0)$ . Let  $Y_s = Z_s - Z_0$ .

(1) It is

$$\begin{aligned} \text{Var} \left( \sum_{i=1}^n a_i Z(s_i) \right) &= \text{Var} \left( \sum_{i=1}^n a_i Z(s_i) - \overbrace{\sum_{i=1}^n a_i Z(0)}^{0=} \right) = \text{Var} \left( \sum_{i=1}^n a_i Y(s_i) \right) \\ &= \sum_{i=1}^n \sum_{j=1}^n a_i a_j \text{E}(Y(s_i)Y(s_j)) = c_Y(s, t) \end{aligned}$$

(2) For  $\text{E}(Y(s_i)) = 0$  it is

$$\begin{aligned} \gamma(s - t) &= \frac{1}{2} \text{E}(Z(s) - Z(0) + Z(t) - Z(0))^2 \\ &= \frac{1}{2} (2\gamma(s) + 2\gamma(t) - 2c_Y(s, t)) \\ \implies c_Y(s, t) &= \gamma(s) + \gamma(t) - \gamma(s - t) \end{aligned}$$

(3) It is

$$\begin{aligned}
0 &\leq \text{Var} \left( \sum_{i=1}^n a_i Z(s_i) \right) = \sum_{i=1}^n \sum_{j=1}^n a_i a_j c_Y(s_i, s_j) \\
&= \sum_{i=1}^n \sum_{j=1}^n a_i a_j (\gamma(s_i) + \gamma(s_j) - \gamma(s_i - s_j)) \\
&= \sum_{i=1}^n a_i \gamma(s_i) \sum_{j=1}^n a_j + \sum_{j=1}^n a_j \gamma(s_j) \sum_{i=1}^n a_i - \sum_{i=1}^n \sum_{j=1}^n a_i a_j \gamma(s_i - s_j)
\end{aligned}$$

hence

$$\sum_{i=1}^n \sum_{j=1}^n a_i a_j \gamma(s_i - s_j) \leq 0$$

**Exercise 2.** (★) Consider the zero-mean geostatistical process  $Z = (Z_s)_{s \in \mathbb{R}^d}$  with a weakly stationary and isotropic covariance function given by

$$c(h) = \begin{cases} \xi^2 (1 + \rho \|h\|) \exp(-\rho \|h\|), & h > 0 \\ \nu^2 + \xi^2, & h = 0 \end{cases}$$

- (1) Compute the semi-variogram for the geostatistical process  $(Z_s)$
- (2) What are the nugget, sill and partial sill for this covariance model? Justify your answer.
- (3) Would the slightly altered covariance function defined below be a good model for spatial data for  $\phi > 0$ ? Justify your answer.

$$c(h) = \begin{cases} \xi^2 (1 + \rho \|h\|) \exp(-\rho \|h\|) + \phi, & h > 0 \\ \nu^2 + \xi^2 + \phi, & h = 0 \end{cases}$$

**Solution.**

- (1) For all  $h \neq 0$ , it is

$$\begin{aligned}
\gamma(h) &= c(0) - c(h), \\
&= \nu^2 + \xi^2 - \xi^2 (1 + \rho \|h\|) \exp(-\rho \|h\|) \\
&= \nu^2 + \xi^2 (1 - (1 + \rho \|h\|) \exp(-\rho \|h\|))
\end{aligned}$$

then

$$\gamma(h) = \begin{cases} \nu^2 + \xi^2 (1 - (1 + \rho \|h\|) \exp(-\rho \|h\|)) & h > 0 \\ 0 & h = 0 \end{cases}$$

- (2)

- The sill is the covariance function at distance 0, that is  $c(0) = \nu^2 + \xi^2$ . Or since analogously, it is  $\lim_{\|h\| \rightarrow \infty} \gamma(h)$ . So,

$$\begin{aligned} \lim_{\|h\| \rightarrow \infty} (\|h\| \exp(-\rho \|h\|)) &= \lim_{\|h\| \rightarrow \infty} (\|h\| / \exp(\rho \|h\|)) \\ &= \lim_{\|h\| \rightarrow \infty} (\|h\| / \exp(\rho \|h\|)) = \lim_{\|h\| \rightarrow \infty} (\exp(-\rho \|h\|)) = 0 \end{aligned}$$

then

$$\lim_{\|h\| \rightarrow \infty} \gamma(h) = \nu^2 + \xi^2$$

- The nugget effect is the limiting value of the semi-variogram as  $h \rightarrow 0$  from above, hence it is  $\gamma(h) \rightarrow \nu^2$  as  $h \rightarrow 0^+$ .
  - The partial sill is the sill minus the nugget and is hence  $\xi^2$ .
- (3) No, it would be unrealistic because if  $\phi > 0$  then the covariance is always positive for infinitely large distances  $h$ . In practical terms this means that two points will always be correlated however far apart they are, it would be unrealistic.