Spatio-temporal statistics (MATH4341) Michaelmas term, 2024

Homework 3: Geostatistics (Change of support)

Lecturer: Georgios P. Karagiannis georgios.karagiannis@durham.ac.uk

---

Exercise 1. ($\star$) Assume we wish to estimate the average value in a domain V

$$Z_V = \frac{1}{|V|} \int_V Z(s)\,\mathrm{d}s$$

with the average of n sample points $\{s_i; i = 1, ..., n\}$.

$$\hat{Z} = \frac{1}{n} \sum_{i=1}^{n} Z(s_i)$$

Show that the estimation variance(or else extension variance)

$$\mathrm{Var}\left(\hat{Z} - Z_V\right) = -\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma(s_i - s_j) + \frac{1}{n|V|} \sum_{i=1}^{n} \int_V \gamma(s_i - x)\,\mathrm{d}x - \frac{1}{|V|^2} \int_{x \in B} \int_{y \in B} \gamma(x - y)\,\mathrm{d}x\mathrm{d}y$$

Hint:: Consider as known that

$$\mathrm{Cov}\left(Z(t) - Z(s), Z(v) - Z(u)\right) = \gamma(t - u) + \gamma(s - v) - \gamma(s - u) - \gamma(t - v)$$

Exercise 2. ($\star$) Consider a statistical model which is a stochastic process $(Z_s)_{s \in \mathbb{R}}$ (so s has dimension 1), where $Z(\cdot) \sim \mathrm{GP}(\mu(\cdot), c(\cdot, \cdot))$ with mean function $\mu(s) = 1$ and covariance function $c(s, t) = \exp\left(-(s - t)^2\right)$ for any $s \in \mathbb{R}$ and $t \in \mathbb{R}$. Assume there is available a dataset $\{(Z_i, s_i)\}_{i=1}^{n}$ where $Z_i = Z(s_i)$ and $s_i \in \mathbb{R}$ are point sites.

(1) Compute the length $|v|$ of the block $v = [a, b] \subset \mathbb{R}$.

(2) Compute the block mean $\mu(v)$ for some block $v = [a, b] \subset \mathbb{R}$ and point $s \in \mathbb{R}$.

(3) Compute the block covariance function $c(v, s)$ for some block $v = [a, b] \subset \mathbb{R}$ and point $s \in \mathbb{R}$.

(4) Compute the block covariance function $c(v, v')$ for some blocks $v = [a, b] \subset \mathbb{R}$ and $v' = [a', b'] \subset \mathbb{R}$.

(5) Denote $Z = (Z_1, ..., Z_n)^\top$, and $S = \{s_1, ..., s_n\}$. Let $v = [a, b] \subset \mathbb{R}$ and $v' = [a', b'] \subset \mathbb{R}$ be two intervals. Compute the joint distribution of $(Z(v), Z(v'), Z)^\top$ as a function of $c(\cdot, \cdot)$, S, v, v', Z, and $\mu(\cdot)$. What is the name of the distribution and what are the parameter functions defining it?

(6) (Bayesian Kriging) Compute the predictive stochastic process $[Z(v)\,|Z]$ at blocks $v = [a, b] \subset \mathbb{R}$ with $|v| > 0$.

Hint-1:: Let $x_1 \in \mathbb{R}^{d_1}$, and $x_2 \in \mathbb{R}^{d_2}$. If

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim \mathrm{N}_{d_1+d_2}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_1 & \Sigma_{21}^\top \\ \Sigma_{21} & \Sigma_2 \end{bmatrix}\right)$$

then it is

$$x_2 | x_1 \sim \mathrm{N}_{d_2}\left(\mu_{2|1}, \Sigma_{2|1}\right)$$

where

$$\mu_{2|1} = \mu_2 + \Sigma_{21}\Sigma_1^{-1}\left(x_1 - \mu_1\right) \text{ and } \Sigma_{2|1} = \Sigma_2 - \Sigma_{21}\Sigma_1^{-1}\Sigma_{21}^\top$$

Hint-2: You can use that $\int \mathrm{erf}\left(x\right) \mathrm{d}x = x\mathrm{erf}\left(x\right) + \frac{\exp\left(-x^2\right)}{\sqrt{\pi}} + \text{const}$, when $\mathrm{erf}\left(x\right) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2\right) \mathrm{d}t$
.
