Homework 3: Geostatistics (Change of support)

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Exercise 1. ($\star$) Assume we wish to estimate the average value in a domain V

$$Z_V = \frac{1}{|V|}\int_V Z(s)\,\mathrm{d}s$$

with the average of n sample points $\{s_i; i = 1, ..., n\}$.

$$\hat{Z} = \frac{1}{n}\sum_{i=1}^{n} Z(s_i)$$

Show that the estimation variance(or else extension variance)

$$\mathrm{Var}\left(\hat{Z} - Z_V\right) = -\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma(s_i - s_j) + \frac{1}{n|V|}\sum_{i=1}^{n}\int_V \gamma(s_i - x)\,\mathrm{d}x - \frac{1}{|V|^2}\int_{x\in B}\int_{y\in B}\gamma(x-y)\,\mathrm{d}x\mathrm{d}y$$

Hint:: Consider as known that

$$\mathrm{Cov}\left(Z(t) - Z(s), Z(v) - Z(u)\right) = \gamma(t-u) + \gamma(s-v) - \gamma(s-u) - \gamma(t-v)$$

Solution. It is

$$\begin{aligned}
\mathrm{Var}\left(\hat{Z} - Z_V\right) =& \mathrm{Cov}\left(\hat{Z} - Z_V, \hat{Z} - Z_V\right)\\
=& \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{Cov}\left(Z_i - Z_V, Z_j - Z_V\right)\\
=& \frac{1}{n^2}\frac{1}{|V|^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{x\in V}\int_{y\in V}\mathrm{Cov}\left(Z_i - Z(x), Z_j - Z(y)\right)\mathrm{d}x\mathrm{d}y\\
=& \frac{1}{n^2}\frac{1}{|V|^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{x\in V}\int_{y\in V}\left[\gamma(s_i - y) + \gamma(x - s_j) - \gamma(x-y) - \gamma(s_i - s_j)\right]\mathrm{d}x\mathrm{d}y\\
=& -\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma(s_i - s_j) + \frac{1}{n|V|}\sum_{i=1}^{n}\int_V \gamma(s_i - x)\,\mathrm{d}x - \frac{1}{|V|^2}\int_{x\in B}\int_{y\in B}\gamma(x-y)\,\mathrm{d}x\mathrm{d}y
\end{aligned}$$

Exercise 2. ($\star$) Consider a statistical model which is a stochastic process $(Z_s)_{s\in\mathbb{R}}$ (so s has dimension 1), where $Z(\cdot) \sim \mathrm{GP}(\mu(\cdot), c(\cdot,\cdot))$ with mean function $\mu(s) = 1$ and covariance function $c(s,t) = \exp\left(-(s-t)^2\right)$ for any $s \in \mathbb{R}$ and $t \in \mathbb{R}$. Assume there is available a dataset $\{(Z_i, s_i)\}_{i=1}^{n}$ where $Z_i = Z(s_i)$ and $s_i \in \mathbb{R}$ are point sites.

(1) Compute the length $|v|$ of the block $v=[a,b]\subset\mathbb{R}$.

(2) Compute the block mean $\mu(v)$ for some block $v=[a,b]\subset\mathbb{R}$ and point $s\in\mathbb{R}$.

(3) Compute the block covariance function $c(v,s)$ for some block $v=[a,b]\subset\mathbb{R}$ and point $s\in\mathbb{R}$.

(4) Compute the block covariance function $c(v,v')$ for some blocks $v=[a,b]\subset\mathbb{R}$ and $v'=[a',b']\subset\mathbb{R}$.

(5) Denote $Z=(Z_1,...,Z_n)^\top$, and $S=\{s_1,...,s_n\}$. Let $v=[a,b]\subset\mathbb{R}$ and $v'=[a',b']\subset\mathbb{R}$ be two intervals. Compute the joint distribution of $(Z(v),Z(v'),Z)^\top$ as a function of $c(\cdot,\cdot)$, S, v, v', Z, and $\mu(\cdot)$. What is the name of the distribution and what are the parameter functions defining it?

(6) (Bayesian Kriging) Compute the predictive stochastic process $[Z(v)\,|Z]$ at blocks $v=[a,b]\subset\mathbb{R}$ with $|v|>0$.

Hint-1:: Let $x_1\in\mathbb{R}^{d_1}$, and $x_2\in\mathbb{R}^{d_2}$. If

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim \mathrm{N}_{d_1+d_2}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_1 & \Sigma_{21}^\top \\ \Sigma_{21} & \Sigma_2 \end{bmatrix}\right)$$

then it is

$$x_2|x_1 \sim \mathrm{N}_{d_2}\left(\mu_{2|1},\Sigma_{2|1}\right)$$

where

$$\mu_{2|1}=\mu_2+\Sigma_{21}\Sigma_1^{-1}(x_1-\mu_1) \text{ and } \Sigma_{2|1}=\Sigma_2-\Sigma_{21}\Sigma_1^{-1}\Sigma_{21}^\top$$

Hint-2: You can use that $\int \mathrm{erf}(x)\,\mathrm{d}x = x\mathrm{erf}(x)+\frac{\exp(-x^2)}{\sqrt{\pi}}+\text{const}$, when $\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-t^2\right)\mathrm{d}t$.

Solution.

(1) It is $|v|=b-a$

(2) It is

$$\mu(v)=\mu([a,b])=\frac{1}{|v|}\int_v \mu(s)\,\mathrm{d}s=\frac{1}{|v|}\int_v 1\mathrm{d}s=\frac{1}{|v|}|v|=1$$

(3) It is

$$\begin{aligned} c(v,s)=&\frac{1}{|v|}\int_v c(t,s)\,\mathrm{d}t=\frac{1}{b-a}\int_a^b \exp\left(-(t-s)^2\right)\mathrm{d}t \\ =&\frac{1}{b-a}\frac{\sqrt{\pi}}{2}\int_a^b \frac{2}{\sqrt{\pi}}\exp\left(-(t-s)^2\right)\mathrm{d}t=\frac{1}{b-a}\frac{\sqrt{\pi}}{2}\int_{a-s}^{b-s}\frac{2}{\sqrt{\pi}}\exp\left(-\xi^2\right)\mathrm{d}\xi \\ =&\frac{1}{b-a}\frac{\sqrt{\pi}}{2}\int_0^{b-s}\frac{2}{\sqrt{\pi}}\exp\left(-\xi^2\right)\mathrm{d}\xi-\frac{1}{b-a}\frac{\sqrt{\pi}}{2}\int_0^{a-s}\frac{2}{\sqrt{\pi}}\exp\left(-\xi^2\right)\mathrm{d}\xi \\ =&\frac{1}{b-a}\frac{\sqrt{\pi}}{2}\mathrm{erf}(b-s)-\frac{1}{b-a}\frac{\sqrt{\pi}}{2}\mathrm{erf}(a-s) \end{aligned}$$

(4) It is

$$
\begin{aligned}
c\left(v,v'\right) =&\frac{1}{|v'|}\frac{1}{|v|}\int_{a'}^{b'}\int_{a}^{b} c\left(t,s\right)\mathrm{d}t\mathrm{d}s = \frac{1}{|v'|}\frac{1}{|v|}\int_{a'}^{b'}\left[\int_{a}^{b} c\left(t,s\right)\mathrm{d}t\right]\mathrm{d}s = \frac{1}{b'-a'}\int_{a'}^{b'} c\left(v,s\right)\mathrm{d}s\\
=&\frac{1}{b'-a'}\int_{a'}^{b'}\left(\frac{1}{b-a}\frac{\sqrt{\pi}}{2}\mathrm{erf}\left(b-s\right)-\frac{1}{b-a}\frac{\sqrt{\pi}}{2}\mathrm{erf}\left(a-s\right)\right)\mathrm{d}s\\
=&\frac{1}{b'-a'}\frac{1}{b-a}\frac{\sqrt{\pi}}{2}\int_{a'}^{b'}\mathrm{erf}\left(b-s\right)\mathrm{d}s-\frac{1}{b'-a'}\frac{1}{b-a}\frac{\sqrt{\pi}}{2}\int_{a'}^{b'}\mathrm{erf}\left(a-s\right)\mathrm{d}s\\
=&\frac{1}{b'-a'}\frac{1}{b-a}\frac{\sqrt{\pi}}{2}\left(-1\right)\int_{b-a'}^{b-b'}\mathrm{erf}\left(\xi\right)\mathrm{d}\xi-\frac{1}{b'-a'}\frac{1}{b-a}\frac{\sqrt{\pi}}{2}\left(-1\right)\int_{a-a'}^{a-b'}\mathrm{erf}\left(\xi\right)\mathrm{d}\xi\\
=&\frac{1}{b'-a'}\frac{1}{b-a}\frac{\sqrt{\pi}}{2}\left(-1\right)\left[\xi\mathrm{erf}\left(\xi\right)+\frac{\exp\left(-\xi^2\right)}{\sqrt{\pi}}\right]_{b-a'}^{b-b'}-\frac{1}{b'-a'}\frac{1}{b-a}\frac{\sqrt{\pi}}{2}\left(-1\right)\left[\xi\mathrm{erf}\left(\xi\right)+\frac{\exp\left(-\xi^2\right)}{\sqrt{\pi}}\right]_{a-a'}^{a-b'}\\
=&-\frac{1}{b'-a'}\frac{1}{b-a}\frac{\sqrt{\pi}}{2}\left(b-b'\right)\mathrm{erf}\left(b-b'\right)-\frac{1}{b'-a'}\frac{1}{b-a}\frac{1}{2}\left(b-b'\right)\exp\left(-\left(b-b'\right)^2\right)\\
&+\frac{1}{b'-a'}\frac{1}{b-a}\frac{\sqrt{\pi}}{2}\left(b-a'\right)\mathrm{erf}\left(b-a'\right)+\frac{1}{b'-a'}\frac{1}{b-a}\frac{1}{2}\left(b-a'\right)\exp\left(-\left(b-a'\right)^2\right)\\
&+\frac{1}{b'-a'}\frac{1}{b-a}\frac{\sqrt{\pi}}{2}\left(a-b'\right)\mathrm{erf}\left(a-b'\right)+\frac{1}{b'-a'}\frac{1}{b-a}\frac{1}{2}\left(a-b'\right)\exp\left(-\left(a-b'\right)^2\right)\\
&-\frac{1}{b'-a'}\frac{1}{b-a}\frac{\sqrt{\pi}}{2}\left(a-a'\right)\mathrm{erf}\left(a-a'\right)-\frac{1}{b'-a'}\frac{1}{b-a}\frac{1}{2}\left(a-a'\right)\exp\left(-\left(a-a'\right)^2\right)
\end{aligned}
$$

(5) It is

$$
\begin{bmatrix} Z\left(v\right)\\ Z\left(v'\right)\\ Z \end{bmatrix} \sim \mathrm{N}\left(\begin{bmatrix}\mu\left(v\right)\\ \mu\left(v'\right)\\ \mu\left(S\right)\end{bmatrix}, \begin{bmatrix} c\left(v,v\right) & c\left(v,v'\right) & c\left(v,S\right)\\ c\left(v',v\right) & c\left(v',v'\right) & c\left(v',S\right)\\ c\left(S,v\right) & c\left(S,v'\right) & c\left(S,S\right)\end{bmatrix}\right)
$$

(6) Taking a better look at part 1, I can see

$$
\begin{bmatrix} \begin{bmatrix} Z\left(v_1\right)\\ Z\left(v_2\right)\end{bmatrix}\\ \left[Z\right] \end{bmatrix} \sim \mathrm{N}\left(\begin{bmatrix}\begin{bmatrix}\mu\left(v\right)\\ \mu\left(v_2\right)\end{bmatrix}\\ \left[\mu\left(S\right)\right]\end{bmatrix}, \begin{bmatrix} \begin{bmatrix} c\left(v_1,v_1\right) & c\left(v_1,v_2\right)\\ c\left(v_2,v_1\right) & c\left(v_2,v_2\right)\end{bmatrix} & \begin{bmatrix} c\left(v_1,S\right)\\ c\left(v_2,S\right)\end{bmatrix}\\ \begin{bmatrix} c\left(S,v_1\right) & c\left(S,v_2\right)\end{bmatrix} & \left[c\left(S,S\right)\right]\end{bmatrix}\right)
$$

From the hint I can see, I can see that

$$
\begin{bmatrix} Z\left(v_1\right)\\ Z\left(v_2\right)\end{bmatrix}|Z \sim \mathrm{N}\left(\mu^\dagger, C^\dagger\right)
$$

with

$$
C^\dagger = \begin{bmatrix} C_{11}^\dagger & C_{12}^\dagger\\ C_{21}^\dagger & C_{22}^\dagger\end{bmatrix} = \begin{bmatrix} c\left(v_1,v_1\right) & c\left(v_1,v_2\right)\\ c\left(v_2,v_1\right) & c\left(v_2,v_2\right)\end{bmatrix} - \begin{bmatrix} c\left(v_1,S\right)\\ c\left(v_2,S\right)\end{bmatrix}\left[c\left(S,S\right)\right]^{-1}\begin{bmatrix} c\left(S,v_1\right) & c\left(S,v_2\right)\end{bmatrix}
$$

and

$$
\mu^\dagger = \mu\left(v_1\right) + c\left(v_1,S\right)\left[c\left(S,S\right)\right]^{-1}\left(Z-\mu\left(S\right)\right)
$$

As this is consistent for any vector of blocks with any size, not only $V = \{v_1, v_2\}$, but also $V = \{v_1, v_2, ..., v_q\}$ then the predictive stochastic process is a Gaussian Process

$$Z(\cdot) | Z \sim \mathrm{GP}(\mu^*(\cdot), c^*(\cdot, \cdot))$$

with mean function at block v

$$\mu^*(v) = \mu(v) + c(v, S)[c(S, S)]^{-1}(Z - \mu(S))$$

by looking at $\mu^\dagger$, and with covariance function at any pair of blocks v and v'

$$c^*(v, v') = c(v, v') - c(v, S)[c(S, S)]^{-1} c(S, v')$$

looking at any off-diagonal element of $C^\dagger$ e.g. the $(1, 2)$ element marked in red.
