Spatio-temporal statistics (MATH4341) Michaelmas term, 2024

Homework 4: Aerial data modeling

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Exercise 1. ($\star$) Show that the local characteristics

$$\mathrm{pr}_1(x_1|x_2) = \sqrt{\frac{1}{2\pi}}\exp\left(-\frac{1}{2}(x_1-x_2)^2\right)$$
$$\mathrm{pr}_2(x_2|x_1) = \sqrt{\frac{1}{2\pi}}\exp\left(-\frac{1}{2}(x_2-x_1)^2\right)$$

do not define a proper joint distribution on $\mathbb{R}^{\{1,2\}}$.

Solution.

Definition 2. (Pseudo-likelihood) The pseudo likelihood pseudo-$L(Z;\theta)$ of observables $Z = (Z_1,...,Z_n)^\top$ given parameters θ is an approximation of the likelihood $L(Z;\theta)$ of observables $Z = (Z_1,...,Z_n)^\top$ given parameters θ which is equal to

$$\text{pseudo-}L(Z;\theta) = \prod_i \mathrm{pr}(Z_i|Z_{-i},\theta)$$

where $\mathrm{pr}(Z_i|Z_{-i},\theta)$ are the conditionals of the joint pdf/pmf of the sampling distribution $\mathrm{pr}(Z|\theta)$ of Z given $Z_{-i} = (Z_1,...,Z_{i-1},Z_{i+1}...,Z_n)^\top$ and parameter θ.

Definition. (Maximum Pseudo-Likelihood Estimator) The Maximum Pseudo-Likelihood Estimator (MPLE) $\tilde{\theta}$ of θ is the maximizer of the pseudo likelihood function pseudo-$L(Z;\theta)$ where the parameter θ is the argument and the observables $Z = (Z_1,...,Z_n)^\top$ are fixed values.

$$\tilde{\theta} = \arg\max_\theta(\text{pseudo-}L(Z;\theta))$$

Exercise 3. ($\star$) Let $Z \in \mathcal{Z}^{\mathcal{S}}$ where $\mathcal{S} = \{1,...,n\}$ and $\mathcal{Z} = \mathbb{R}$. Consider the model

$$Z = X\beta + B(Z - X\beta) + E$$

where X is a $n\times p$ design matrix X, $\beta \in \mathbb{R}^p$, $I - B$ is an $n \times n$ symmetric positive definite matrix with $[B]_{i,i} = 0$, $E \sim \mathrm{N}\left(0, \sigma^2(I-B)\right)$, and $\sigma^2 > 0$.

Hint: The following formulas are provided for your information

- $\partial(XY) = (\partial X)Y + X(\partial Y)$
- $\partial\left(X^\top\right) = (\partial X)^\top$
- $\frac{\partial}{\partial x}\left(x^\top Bx\right) = \left(B + B^\top\right)x$
- $\frac{\partial}{\partial x}\left((s-Ax)^\top W(s-Ax)\right) = -2AW(s-Ax)$

(1) This is a multiple choice question, choose any number of correct answers.
   (a) Z follows a simultaneous autoregressive (SAR) with Gaussian joint distribution with mean $X\beta$ and covariance matrix $\sigma^2 (I - B)^{-1}$
   (b) Ising model
   (c) Conditional autoregressive (CAR) with Gaussian joint distribution with mean $X\beta$ and covariance matrix $\sigma^2 I$
   (d) Bernoulli regression

(2) Show that the minus two log Pseudo-Likelihood is such as

$$-2 \log \left(\text{pseudo-}L\left(Z; \beta, \sigma^2\right)\right) = n \log\left(\sigma^2\right) + \frac{1}{\sigma^2} (Z - X\beta)^\top (I - B)^2 (Z - X\beta) + \text{const.}$$

(3) Compute the Maximum Pseudo-Likelihood Estimators (MPLE) $\tilde{\beta}$ and $\tilde{\sigma}^2$ of β and σ^2

Solution.
