Spatio-temporal statistics (MATH4341) Michaelmas term, 2024

Homework 1: Point referenced data (building concepts)

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Exercise 1. ($\star$) Let $Z = \left(Z(s) : s \in \mathbb{R}^d\right)$ be an intrinsic random field with $\mathrm{E}\left(Z(s) - Z(t)\right) = 0$ and let $\gamma : \mathbb{R}^d \to \mathbb{R}$ be its semivariogram.

1. Let $a \in \mathbb{R}^n$ be a vector of constants. Consider sites $\left\{s_1, ..., s_n \subseteq \mathbb{R}^d\right\}$ Show that

$$\mathrm{Var}\left(\sum_{i=1}^{n} a_i Z(s_i)\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j c_I(s_i, s_j)$$

   where $c_I(s,t) = \gamma(s - s_0) + \gamma(t - s_0) - \gamma(s - t)$ at some additional $s_0 \in \mathbb{R}^d$.

2. Show that for all $n \in \mathbb{N}$, $(a_1, ..., a_n) \subseteq \mathbb{R}^n$ s.t. $\sum_{i=1}^{n} a_i = 0$, and for all $(s_1, ..., s_n) \subseteq S^n$, it is

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \gamma(s_i - s_j) \leq 0$$

Exercise 2. ($\star$) Consider the zero-mean random field $Z = \left(Z(s) : s \in \mathbb{R}^d\right)$ with covariogram function given by

$$c(h) = \begin{cases} \xi^2 (1 + \rho \|h\|) \exp(-\rho \|h\|), & h > 0 \\ \nu^2 + \xi^2, & h = 0 \end{cases}$$

1. Compute the semivariogram for the random field $\left(Z(s) : s \in \mathbb{R}^d\right)$

2. What are the nugget, sill and partial sill for this covariance model? Justify your answer.

3. Would the slightly altered covariance function defined below be a good model for spatial data for $\phi > 0$? Justify your answer.

$$c(h) = \begin{cases} \xi^2 (1 + \rho \|h\|) \exp(-\rho \|h\|) + \phi, & h > 0 \\ \nu^2 + \xi^2 + \phi, & h = 0 \end{cases}$$