Topics in statistics III/IV (MATH3361/4071) Michaelmas term, 2018-2019

Exercises: Likelihood methods for large samples [a, b]

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Exercise 1

($\star\star$) From Fatou-Lesbeque Lemma, prove Monotone Convergence theorem. (Hint: Use $Y \equiv 0$, use $\limsup_{n\to\infty} f_n$and $\liminf_{n\to\infty} f_n$)

Exercise 2

($\star\star$) From Fatou-Lesbeque Lemma, prove Lesbeque Dominant Convergence theorem. (Hint: Use that $-Y \le -X_n$ and $-Y \le X_n$, use $\limsup_{n\to\infty} f_n$and $\liminf_{n\to\infty} f_n$)

Exercise 3

($\star\star$) Let μ be a constant. Show that $X_n \xrightarrow{\text{qm}} \mu$ if and only if $\mathrm{E}X_n \to \mu$ and $\mathrm{Var}(X_n) \to 0$, both in uni-variate and multivariate case.

Exercise 4

($\star\star$) Consider that $\sqrt{n}(X_n - \mu) \underset{D}{\longrightarrow} Z$, where $Z \sim \mathrm{N}(0, \Sigma)$ for $\Sigma > 0$ (positive definite). Show that $X_n \xrightarrow{P} \mu$. (Hint: Use the concept 'bounded in probability)'

Exercise 5

($\star\star$) Consider a sequence of discrete r.v. $\{X_n\}$ with probability $P(X_n = k) = \frac{1}{n}$, for $k = 1/n, 2/n, ..., n/n$. Show that $X_n \xrightarrow{D} X$ where $X \sim \mathrm{U}(0, 1)$. (Hint: Just use the definition.)

Exercise 6

($\star$)

1. Show that

$$\mathrm{E}_\pi (X - \theta)^T (X - \theta) = \mathrm{Var}_\pi(X) + (\mathrm{E}_\pi(X) - \theta)^T (\mathrm{E}_\pi(X) - \theta)$$

, where is a constant point, and X is a random variable $X \sim \mathrm{d}\pi(\cdot)$.

2. Show that

$$\mathrm{E}_{\pi}|X-\theta|^2 = \mathrm{Var}_{\pi}(X) + |\mathrm{E}_{\pi}(X)-\theta|^2$$

, where is a constant point, X is a random variable $X \sim \mathrm{d}\pi(\cdot)$, and $|X| = \sqrt{X_1^2 + ...X_d^2}$ is the Euclidean norm.

Exercise 7

$(\star\star)$

1. If $X_1, X_2, ...$ are IID in $\mathbb{R}^2$ with distribution giving probability

$$\mathrm{P}(X=x) = \begin{cases} \theta_1 & \text{, if } x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \theta_2 & \text{, if } x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \theta_1+\theta_2 & \text{, if } x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases}$$

there $\theta_1 + \theta_2 \leq 1$.. What is the asymptotic distribution of $\bar{X}_n$ given the CLT?

2. If $X_1, X_2, ...$ are IID from a Poisson distribution $\mathrm{Poi}(\theta)$ distribution as

$$\mathrm{P}(x|\theta) = \frac{e^{-\theta}\theta^x}{x!} 1(x \in \{0, 1, 2, ...\})$$

Let Z_n be the proportion of zeros observed $Z_n = \frac{1}{n}\sum_{j=1}^{n} 1(X_j = 0)$. What is the joint asymptotic distribution of $(\bar{X}_n, Z_n)$

Exercise 8

$(\star\star\star\star$Super difficult) (The autoregressive model) Consider that $\{\epsilon_n\}$ are IID, with mean $\mathrm{E}(\epsilon_n) = \mu$, and variance $\mathrm{Var}(\epsilon_n) = \sigma^2$, $\forall n$. A time series $\{X_n\}_{n\geq 1}$ is modeled as $X_n \sim \mathrm{AR}(\beta)$ where $\beta \in (-1, 1)$ if

$$\begin{aligned} X_n =& \beta X_{n-1} + \epsilon_n; \text{ for } n \geq 2 \\ X_1 =& \epsilon_1 \end{aligned}$$

Show that $\bar{X}_n \xrightarrow{\text{qm}} \mu/(1-\beta)$

1. Show that $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} \epsilon_j (1-\beta^{n-j+1})/(1-\beta)$
2. Find $\lim_{n\to\infty} \mathrm{E}(\bar{X}_n) =?$
3. Show that $\lim_{n\to\infty} \mathrm{Var}(\bar{X}_n) = 0$
4. Show that $\bar{X}_n \xrightarrow{\text{qm}} \mu/(1-\beta)$

[Hint] (1.) Show that $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} \epsilon_j (1-\beta^{n-j+1})/(1-\beta)$ (2) Find $\lim_{n\to\infty} \mathrm{E}(\bar{X}_n) = \mu/(1-\beta)$; (3) Show that $\lim_{n\to\infty} \mathrm{Var}(\bar{X}_n) = 0$, (4.) ...

Exercise 9

($\star\star$) Prove that:

1. if $Z \sim \mathrm{N}(0, I)$ then $\varphi_Z(t) = \exp(-\frac{1}{2}t^T t)$, where $Z \in \mathbb{R}^d$
2. if $X \sim \mathrm{N}(\mu, \Sigma)$ then $\varphi_X(t) = \exp(it^T\mu - \frac{1}{2}t^T \Sigma t)$, where $X \in \mathbb{R}^d$

Hint: Assume as known that if $Z \sim \mathrm{N}(0, 1)$ then $\varphi_Z(t) = \exp(-\frac{1}{2}t^2)$, where $Z \in \mathbb{R}$

Exercise 10

($\star\star$) Let $X_i \overset{\text{IID}}{\sim} F_X$ for $i = 1, ..., n$, and $F_X = P(X \leq x)$. Show that the empirical distribution function $\hat{F}_X(x) = \frac{1}{n}\sum_{i=1}^{n} 1(x \in [x_i, \infty))$ is a strongly consistent estimator of F_X.

The next exercise is from Problem Class 2

Exercise 11

Consider random variables $X, X_1, X_2, ...$, where $\mu_n = \mathrm{E}(X-\mu)^n$, and $\mu = \mathrm{E}(X)$

1. Show that,
$$\sqrt{n}(\begin{bmatrix}\bar{X}\\ s_x^2\end{bmatrix} - \begin{bmatrix}\mu\\ \sigma^2\end{bmatrix}) \xrightarrow{D} \mathrm{N}(\begin{bmatrix}0\\ 0\end{bmatrix}, \begin{bmatrix}\sigma^2 & \mu_3\\ \mu_3 & \mu_4-\sigma^4\end{bmatrix})$$
2. Show that the asymptotic distribution of the coefficient of variation $\mathrm{cv} = \frac{s_x}{\bar{X}}$, is
$$\sqrt{n}(\frac{s_x}{\bar{X}} - \frac{\sigma}{\mu}) \xrightarrow{D} \mathrm{N}(0, \frac{\mu_4-\sigma^4}{4\mu^2\sigma^2} - \frac{\mu_3}{\mu^3} + \frac{\sigma^4}{\mu^4})$$

3. Show that the asymptotic distribution of the 3rd central moment $m_3 = \frac{1}{n}\sum_{i=1}^{n}(X_j - \bar{X})^3$ is

$$\sqrt{n}(m_3 - \mu_3) \xrightarrow{D} \mathrm{N}(0, \mu_6 - \mu_3^2 - 6\sigma^2\mu_4 + 9\sigma^6)$$

Exercise 12

($\star\star$) Assume X_1,X_2,X_3 independent from Uniform distribution $\mathrm{U}(0,1)$. Compare the exact, Normal approximation, and Edgeworth approximation.

Hint: The exact result is $P(X_1 + X_2 + x_3 \le 2) = 0.8333$

The next exercise is from Homework 3

Exercise 13

($\star\star\star$) Consider an M-way contingency table and consider the quantities obs. cell counts, cell probabilities, cell proportions in their vectorised forms as

$$\underset{\sim}{n} = (n_1, ..., n_N)^T \; ; \qquad \underset{\sim}{\pi} = (\pi_1, ..., \pi_N)^T \; ; \qquad \underset{\sim}{p} = (p_1, ..., p_N)^T$$

where $n = \sum_{j=1}^{N} n_j$, and $p_j = n_j/n$.

1. Consider a constant matrix $C \in \mathbb{R}^{k\times N}$, and show that

$$\sqrt{n}(C\log(\underset{\sim}{p}) - C\log(\underset{\sim}{\pi})) \xrightarrow{D} \mathrm{N}(0, C\mathrm{diag}(\pi)^{-1}C^T - C11^TC^T) \tag{1}$$

2. Consider a 3×3 contingency table with probabilities $(\pi_{i,j})$. Find the joint asymptotic distribution of the vector of different log odd ratios

$$\log(\underset{\sim}{\theta}^C) = \begin{bmatrix} \log(\frac{\pi_{11}\pi_{22}}{\pi_{21}\pi_{12}}) \\ \log(\frac{\pi_{22}\pi_{33}}{\pi_{23}\pi_{32}}) \end{bmatrix}$$

Exercise 14

($\star\star\star$) Consider a random sample $X, X_1, X_2, ...$ an IID sample with finite moments $E(X) = 0$, and $\mathrm{E}(X^4) < \infty$.

1. Show that if $m_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $m_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ then

$$\sqrt{n}(\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} - \begin{bmatrix} 0 \\ \sigma^2 \end{bmatrix}) \xrightarrow{D} \mathrm{N}(0, \Sigma)$$

   where $\Sigma = \begin{bmatrix} Var(X) & Cov(X^2, X) \\ Cov(X^2, X) & Var(X^2) \end{bmatrix}$

2. Find an $(1-a)\%$ asymptotic confidence interval for S_n^2.

The next exercise is from Homework 3

Exercise 15

($\star\star\star$) Consider an IID sample $X, X_1, X_2, ...$ with $EX = 0$, $EX^4 < \infty$. Consider that

$$\sqrt{n}\frac{S_n^2 - \sigma^2}{\sqrt{EX^4 - \sigma^4}} \xrightarrow{D} \mathrm{N}(0, 1) \qquad (2)$$

where $S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

1. Find the asymptotic distribution of $\log(S_n^2)$.
2. Produce the $1-a$ asymptotic confidence interval for $\log(\sigma_n^2)$; by performing suitable calculations, so that the boundaries of the confidence interval do not depend on any unknown moments of the real distribution.

Exercise 16

($\star\star\star$) Let function $g : \mathbb{R} \to \mathbb{R}$ such that $\dot{g}(x)$ and $\ddot{g}(x)$ are continuous in a neighborhood of $\mu \in \mathbb{R}$, and $\dot{g}(\mu) = 0$. Prove the following statement:

- If $X_n \in \mathbb{R}$ is a sequence of random vectors such that $\sqrt{n}(X_n - \mu) \xrightarrow{D} \mathrm{N}(0, \sigma^2)$ then

$$n(g(X_n) - g(\mu)) \xrightarrow{D} \frac{\sigma^2 \ddot{g}(\mu)}{2}\chi_1^2$$

Hint-1. Use Taylor expansion of 2nd order.

Hint-2. The Taylor expansion of function $f : \mathbb{R} \to \mathbb{R}$ around point x_0 is:

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!}(x - x_0) f^{(k)}(x_0) + R_n(x)$$

where $R_n(x) = \frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n)}(x_0) = o((x-x_0)^n)$ as $x \to x_0$, provided that the n-th derivative $f^{(n)}(x)$ exists in some interval containing x_0.

The next exercise is from Homework 3

Exercise 17

($\star\star\star$) Consider random sample $X, X_1, X_2, ...$ IID from a Bernoulli distribution with probability of success p. Find the variance stabilization transformation for the estimator average $\hat{p}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

Exercise 18

Prove the Information inequality theorem:

Let $x \in \mathbb{R}^d$ random vector following distribution $\mathrm{d}f_\theta(\cdot)$ labeled by an parameter $\theta \in \Theta \subset \mathbb{R}^r$ and admitting PDF $f(\cdot|\theta)$. Consider an estimator $\hat{\theta}_n := \hat{\theta}_n(x) \in \Theta \subset \mathbb{R}^r$ such that $g(\theta) = \mathrm{E}_{f_\theta}(\hat{\theta}_n)$ exists on Θ. Assume that, $\frac{\mathrm{d}}{\mathrm{d}\theta}f(x|\theta)$ exists ; $\frac{\mathrm{d}}{\mathrm{d}\theta}$ can pass under the integral sign in $\int f(x|\theta)\mathrm{d}x$ and $\int \hat{\theta}_n(x)f(x|\theta)\mathrm{d}x$. Then

$$\mathrm{var}_{f_\theta}(\hat{\theta}_n(x)) \geq \frac{1}{n}\dot{g}(\theta)\mathcal{I}(\theta)^{-1}\dot{g}(\theta)^T \tag{3}$$

where $\mathcal{I}(\theta)$ is the Fisher's information matrix.

- The quantity $\frac{1}{n}\dot{g}(\theta)\mathcal{I}(\theta)^{-1}\dot{g}(\theta)^T$ is called Cramer-Rao lower bound (CRLB).

Hint-1: Use $0 \leq \mathrm{var}_{f_\theta}(\hat{\theta}_n - \dot{g}(\theta)\mathcal{I}(\theta)^{-1}\Psi(x,\theta)) = ...$

Hint-2: Use $\mathrm{var}_{f_\theta}(A+B) = \mathrm{var}_{f_\theta}(A) + \mathrm{var}_{f_\theta}(B) + 2\mathrm{cov}_{f_\theta}(A,B)$

Exercise 19

Consider random sample $x_1, ..., x_n \overset{IID}{\sim} \mathrm{G}(a,b)$, $a>0$, $b>0$ with PDF

$$f(x|a,b) = \frac{1}{\Gamma(a)b^a}x^a e^{-x\frac{1}{b}}1(x>0)$$

1. Find the moment estimator $\tilde{\theta}$ of $\theta = (a,b)^T$ by using the first raw moment and the first central moment
2. Is the moment estimator $\tilde{\theta}$ consistent and asymptotically Normal?

3. Find the one step estimator by Fisher scoring algorithm.

Hint-1 Digamma function $\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x} \log \Gamma(x)$

Hint-2 Trigamma function $\psi_1(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} \log \Gamma(x)$

Hint-3 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Exercise 20

Prove the following statement: Given that the assumptions of Cramer Theorem (for the Normality of MLE) are satisfied, and that $\mathcal{I}(\theta)$ and $\mathcal{J}_n(\theta)$ are continuous on θ, then

$$\sqrt{n}\mathcal{I}(\theta_0)^{1/2}(\hat{\theta}_n - \theta_0) \xrightarrow{D} \mathrm{N}(0, I) \tag{4}$$

$$\sqrt{n}\mathcal{I}(\hat{\theta}_n)^{1/2}(\hat{\theta}_n - \theta_0) \xrightarrow{D} \mathrm{N}(0, I) \tag{5}$$

$$\mathcal{J}_n(\hat{\theta}_n)^{1/2}(\hat{\theta}_n - \theta_0) \xrightarrow{D} \mathrm{N}(0, I) \tag{6}$$

where $\hat{\theta}_n$ denotes the MLE, θ_0 denotes the true value of θ, and $A^{1/2}$ denotes the lower triangular matrix of the Cholesky decomposition of A; i.e., $A = A^{1/2}(A^{1/2})^T$.

The next exercise is from Homework 4

Exercise 21

(Log likelihood ratio statistic) Let $x_1, x_2, ..., x_n$ be IID random variables generated from a distribution f_θ labeled by a d-dimensional parameter $\theta \in \Theta \subset \mathbb{R}^d$, and admitting PDF $f(\cdot|\theta)$. Assume the conditions from the Cramér Theorem are satisfied, and that θ_0 is the true value. Prove that

$$W_{\mathrm{LR}}(\theta_0) = -2(\ell_n(\theta_0) - \ell_n(\hat{\theta}_n)) \xrightarrow{D} \chi^2_d$$

it is where $\hat{\theta}_n$ is the MLE of θ.

Hint-1 Expand $\ell_n(\theta_0)$ around $\hat{\theta}_n$ by Taylor expansion

Hint-2 Prove that $W_{\mathrm{LR}}(\theta_0) \xrightarrow{a.s} n(\theta_0 - \hat{\theta}_n)^T \mathcal{I}(\theta_0)(\theta_0 - \hat{\theta}_n)$

Hint-3 Prove that $W_{\mathrm{LR}}(\theta_0) \xrightarrow{D} \chi^2_d$

The next exercise is from Homework 4

Exercise 22

Let $x_1, ..., x_n \overset{IID}{\sim} f_\theta$ with unknown parameter $\theta \in (0, \infty)$ and PDF

$$f(x|\theta) = \begin{cases} \theta \exp(-x) + (1-\theta) x \exp(-x) & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

1. Calculate the moment estimator $\tilde{\theta}_n$ of θ, (I give you a bit of freedom here)
2. Calculate the asymptotic distribution of the $\tilde{\theta}_n$
3. Find the 1-step estimator $\breve{\theta}_n$ of θ such that it can be asymptotically efficient.

Hint: Recall that $\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} \mathrm{d}x$, and $\Gamma(a) = (a-1)\Gamma(a-1)$

The next exercise is from Homework 4

Exercise 23

Let

$$y_i \overset{\text{ind}}{\sim} \text{Bin}(n, \pi_i)$$

where $i = 1, ..., N$. Consider that the probability of success is modeled such as

$$\text{logit}(\pi_i) = x_i^T \theta \tag{7}$$

where $\text{logit}(\pi_i) = \log(\frac{\pi_i}{1-\pi_i})$. Here $x_i = (x_{i,1}, ..., x_{i,d})^T$ are known vertors containing the values of the d regessions at the i-th observation, and $\theta \in \mathbb{R}^d$.

1. Show that
$$\pi_i = \frac{e^{x_i^T \theta}}{1 + e^{x_i^T \theta}}$$
2. Assume that the MLE $\hat{\theta}$ of θ is known/calculated. Show that the $(1-a)$ Wald confidence interval for the unknown parameter θ, by using the observed information matrix, is
$$\text{C.I.} : \{\, \theta \in \mathbb{R}^d \,:\, (\hat{\theta}_n - \theta)^T X^T (\text{diag}_{\forall i}(n\hat{\pi}_i(1-\hat{\pi}_i))) X (\hat{\theta}_n - \theta) \leq \chi^2_{d,1-a} \,\}$$
where
$$\hat{\pi}_i = \frac{e^{x_i^T \hat{\theta}}}{1 + e^{x_i^T \hat{\theta}}}$$

X is the so called design matrix from the regression

$$\begin{bmatrix} \text{logit}(\pi_1) \\ \vdots \\ \text{logit}(\pi_N) \end{bmatrix} = \underbrace{\begin{bmatrix} \longleftarrow x_1^T \longrightarrow \\ \vdots \\ \longleftarrow x_N^T \longrightarrow \end{bmatrix}}_{=X} \theta$$

and $\text{diag}_{\forall i}(\heartsuit_i) = \begin{bmatrix} \heartsuit_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \heartsuit_N \end{bmatrix}$.

3. Find the score statistic rejection area for the hypothesis test $\text{H}_0 : \theta = \theta_*$ versus $H_1 : \theta \neq \theta_*$.

Exercise 24

For $i = 1, ..., k$, let $x_{i,1}, ..., x_{i,n} \stackrel{\text{IID}}{\sim} \text{Poi}(\theta_i)$. Find the asymptotic likelihood ratio rejection area for teasting the hypothesis

$$\text{H}_0 : \theta_1 = ... = \theta_k$$

Hint: It is

$$f(x|\theta) = \frac{\theta^x e^{-\theta}}{x!} 1(x \in \mathbb{N})$$

Exercise 25

Let $x = (x_1, ..., x_c) \sim \text{Mult}(\pi_1, ..., \pi_c)$, with $\pi_i \in (0, \infty)$ and $\sum_{i=1}^{c} \pi_i = 1$. Find the asymptotic likelihood ratio rejection area for teasting the hypothesis

$$\text{H}_0 : \pi_1 = ... = \pi_c = \frac{1}{c}$$

Hint: It is

$$f(x|\theta) = \binom{n}{x_1 ... x_c} \prod_{i=1}^{c} \pi_i^{x_i}$$

The next exercise was addressed in the last Lecture in Term 1

Exercise 26

(Very difficult) Consider a contigency table with N cells. Consider a Multimomial sampling scheme was used to collect n observations. Let $y = (y_1, ..., y_N)^T$ be the observed counts, and $\pi = (\pi_1, ..., \pi_N)^T$ be the expected probabilities in N cells of a contingency table. Let the total number of observations be $n = \sum_{i=1}^N y_i$. Assume that

$$y \sim \text{Mult}(n, \pi) \tag{8}$$

where

$$f(y|n, \pi) = \binom{n}{y_1 ... y_N} \prod_{i=1}^{n} \pi_i^{y_i}$$

Consider a log-linear model

$$\pi_i = \pi_i(\theta) = \frac{\exp(x_i^T \theta)}{\sum_{\forall k} \exp(x_i^T \theta)} \tag{9}$$

$\theta \in \Theta$ is a d-dimensional vector of unknown coefficients, and $x_i = (x_{i,1}, ..., x_{i,d})^T$ are the values of d regressors.

In a matrix form

$$\pi = \frac{\exp(X\theta)}{1_d^T \exp(X\theta)}$$

where

$$X = \begin{bmatrix} \longleftarrow & x_1^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & x_N^T & \longrightarrow \end{bmatrix}$$

Assume that Cramer's Theorem conditions are satisfied. Consider that the MLE $\hat{\theta}_n$ of θ is computed/calculated, and that θ_0 is the unknown true value of θ. Then

1. Show that
$$\frac{\mathrm{d}\pi}{\mathrm{d}\theta} = (\text{diag}(\pi) - \pi\pi^T)X$$

2. Show that the likelihood equations to find the MLE $\hat{\theta}$ of θ are such as
$$X^T y = nX^T \pi(\hat{\theta}_n)$$
Does it ring a bell?

3. Consider the j-th single observation $\xi_j = (\xi_{j,1}, ..., \xi_{j,N})^T$ where $\xi_{j,i} = 1$ if it falls in cell i and $\xi_{j,i} = 0$ if it does not fall in cell i. Write the probability distribution $f(\xi_i|...) =?$ in the form of the Multinomial distribution.

4. Calculate the asymptotic distribution of the MLE $\hat{\theta}$ of θ.

Hint: Use the fact that a single observation falls in only one cell, and use its probability.

5. Calculate the asymptotic distribution of cell probability estimators $\hat{\pi}$ of π.

6. Calculate the Wald's $(1-a)$ CI for θ, that results as an ellipsoid easy to compute or plot in 2D on 3D.

Exercise 27

Show that

$$\lim_{n\to\infty}(1+\frac{1}{n}a_n)^n = \exp(\lim_{n\to\infty} a_n)$$

provided that $\frac{1}{n}a_n \to 0$, as $n \to \infty$.

Hint: From Taylor expansion, it is

$$\log(1+x) = x + o(x), \text{ as } x \to 0.$$