

```

> #First generator curve of the first surface (S1:=C1\oplus C2)
> restart:tt:=time():
> C1:=[t/(t^6+1),t^3/(t^6+1),t^5/(t^6+1)];#This is C1
C1 := [t/(t^6+1), t^3/(t^6+1), t^5/(t^6+1)]
> with(LinearAlgebra):
> #Matrix A defining the affine transformation
> A:=Matrix(3,3,[[sqrt(2),-1,2*sqrt(2)],[1,-2,sqrt(2)],[-1,-sqrt
(2),2]]);
A := Matrix(3, 3, {(1, 1) = 2^(1/2), (1, 2) = -1, (1, 3) = 2*2^
(1/2), (2, 1) = 1, (2, 2) = -2, (2, 3) = 2^(1/2), (3, 1) = -1,
(3, 2) = -2^(1/2), (3, 3) = 2}, datatype = anything, storage =
rectangular, order = Fortran_order, shape = [])
> Cmat:=Matrix(3,1,[C1[1],C1[2],C1[3]]):
> CA:=MatrixMatrixMultiply(A,Cmat):
> CA:=simplify(CA):
> CA:=simplify(subs(t=t-1,CA)):

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> D1:=[simplify(CA[1]),simplify(CA[2]),simplify(CA[3])];
D1 := [Vector[row](1, {1 = (t-1)*(2*2^(1/2)*t^4-8*2^(1/2)*
t^3+12*2^(1/2)*t^2-8*2^(1/2)*t-t^2+3*2^(1/2)+2*t-1)/(t^6-6*
t^5+15*t^4-20*t^3+15*t^2-6*t+2)}, datatype = anything, storage =
rectangular, order = Fortran_order, shape = []), Vector[row](1,
{1 = (t-1)*(2^(1/2)*t^4-4*2^(1/2)*t^3+6*2^(1/2)*t^2-4*2^(1/2)*
t-2*t^2+2^(1/2)+4*t-1)/(t^6-6*t^5+15*t^4-20*t^3+15*t^2-6*t+2)},
datatype = anything, storage = rectangular, order =
Fortran_order, shape = []), Vector[row](1, {1 = -(-2*t^4+2^(1/2)
*t^2+8*t^3-2*2^(1/2)*t-12*t^2+2^(1/2)+8*t-1)*(t-1)/(t^6-6*
t^5+15*t^4-20*t^3+15*t^2-6*t+2)}, datatype = anything, storage =
rectangular, order = Fortran_order, shape = [])]

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> #First generator curve of the second translation surface (S2=
D1\oplus D2)
> D1:=[(t-1)*(2*2^(1/2)*t^4-8*2^(1/2)*t^3+12*2^(1/2)*t^2-8*2^(1/2)*
t-t^2+3*2^(1/2)+2*t-1)/(t^6-6*t^5+15*t^4-20*t^3+15*t^2-6*t+2),
(t-1)*(2^(1/2)*t^4-4*2^(1/2)*t^3+6*2^(1/2)*t^2-4*2^(1/2)*t-2*
t^2+2^(1/2)+4*t-1)/(t^6-6*t^5+15*t^4-20*t^3+15*t^2-6*t+2),-(t-1)*
(-2*t^4+2^(1/2)*t^2+8*t^3-2*2^(1/2)*t-12*t^2+2^(1/2)+8*t-1)/(t^6
-6*t^5+15*t^4-20*t^3+15*t^2-6*t+2)];#This is D2
D1 := [(t-1)*(2*2^(1/2)*t^4-8*2^(1/2)*t^3+12*2^(1/2)*t^2-8*2^
(1/2)*t-t^2+3*2^(1/2)+2*t-1)/(t^6-6*t^5+15*t^4-20*t^3+15*t^2-6*
t+2), (t-1)*(2^(1/2)*t^4-4*2^(1/2)*t^3+6*2^(1/2)*t^2-4*2^(1/2)*
t-2*t^2+2^(1/2)+4*t-1)/(t^6-6*t^5+15*t^4-20*t^3+15*t^2-6*t+2), -
(-2*t^4+2^(1/2)*t^2+8*t^3-2*2^(1/2)*t-12*t^2+2^(1/2)+8*t-1)*
(t-1)/(t^6-6*t^5+15*t^4-20*t^3+15*t^2-6*t+2)]

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> #Affinities between the curves C1, D1

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> Eq[1]:=numer((m11*C1[1]+m12*C1[2]+m13*C1[3]+b1)-subs(t=(a*t+b),D1
[1])):
> Eq[2]:=numer((m21*C1[1]+m22*C1[2]+m23*C1[3]+b2)-subs(t=(a*t+b),D1
[2])):
> Eq[3]:=numer((m31*C1[1]+m32*C1[2]+m33*C1[3]+b3)-subs(t=(a*t+b),D1

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[3])):
> n:=max(degree(Eq[1],t),degree(Eq[2],t),degree(Eq[2],t));
n := 12
> lis:=[]:
> for i from 1 to 3 do
  for j from 0 to n do
    lis:=[op(lis),coeff(Eq[i],t,j)]:
  od:
od:
> with(SolveTools);
[AbstractRootOfSolution, Basis, CancelInverses, Combine,
Complexity, Engine, GreaterComplexity, Identity, Inequality,
Linear, Parametric, Polynomial, PolynomialSystem,
RationalCoefficients, SemiAlgebraic, SortByComplexity]
> PolynomialSystem(lis, [m11,m12,m13,m21,m22,m23,m31,m32,m33,b1,b2,
b3,a,b], engine=groebner);
{a = 0, b = b, b1 = (1/79)*(158*2^(1/2)*b^5-79*b^3-790*2^(1/2)*
b^4+237*b^2+1580*2^(1/2)*b^3-237*b-1580*2^(1/2)*b^2+79+869*2^
(1/2)*b-237*2^(1/2))/(b^6-6*b^5+15*b^4-20*b^3+15*b^2-6*b+2), b2
= -(1/79)*(-79*2^(1/2)*b^5+158*b^3+395*2^(1/2)*b^4-474*b^2-790*
2^(1/2)*b^3+395*b+790*2^(1/2)*b^2-79-395*2^(1/2)*b+79*2^(1/2))/
(b^6-6*b^5+15*b^4-20*b^3+15*b^2-6*b+2), b3 = -(-2*b^5+2^(1/2)*
b^3+10*b^4-3*2^(1/2)*b^2-20*b^3+3*2^(1/2)*b+20*b^2-2^(1/2)-9*
b+1)/(b^6-6*b^5+15*b^4-20*b^3+15*b^2-6*b+2), m11 = 0, m12 = 0,
m13 = 0, m21 = 0, m22 = 0, m23 = 0, m31 = 0, m32 = 0, m33 = 0},
{a = 1, b = 1, b1 = 0, b2 = 0, b3 = 0, m11 = 2^(1/2), m12 = -1,
m13 = 2*2^(1/2), m21 = 1, m22 = -2, m23 = 2^(1/2), m31 = -1, m32
= -2^(1/2), m33 = 2}, {a = -1, b = 1, b1 = 0, b2 = 0, b3 = 0,
m11 = -2^(1/2), m12 = 1, m13 = -2*2^(1/2), m21 = -1, m22 = 2,
m23 = -2^(1/2), m31 = 1, m32 = 2^(1/2), m33 = -2}, {a = RootOf
(_Z^2-_Z+1), b = 1, b1 = 0, b2 = 0, b3 = 0, m11 = 2^(1/2)*RootOf
(_Z^2-_Z+1), m12 = 1, m13 = -2*2^(1/2)*RootOf(_Z^2-_Z+1)+2*2^
(1/2), m21 = RootOf(_Z^2-_Z+1), m22 = 2, m23 = -2^(1/2)*RootOf
(_Z^2-_Z+1)+2^(1/2), m31 = -RootOf(_Z^2-_Z+1), m32 = 2^(1/2),
m33 = 2-2*RootOf(_Z^2-_Z+1)}, {a = RootOf(_Z^2+_Z+1), b = 1, b1
= 0, b2 = 0, b3 = 0, m11 = 2^(1/2)*RootOf(_Z^2+_Z+1), m12 = -1,
m13 = -2*2^(1/2)*RootOf(_Z^2+_Z+1)-2*2^(1/2), m21 = RootOf(_Z^2+
_Z+1), m22 = -2, m23 = -2^(1/2)*RootOf(_Z^2+_Z+1)-2^(1/2), m31 =
-RootOf(_Z^2+_Z+1), m32 = -2^(1/2), m33 = -2-2*RootOf(_Z^2+_Z+1)
}

> #Second generator curve of the first surface
> C2:=[t^2,t/(t^2+1),t^4-t]:#This is C2
> Cmat2:=Matrix(3,1,[C2[1],C2[2],C2[3]]):
> CA2:=MatrixMatrixMultiply(A,Cmat2):
> CA2:=simplify(CA2):
> CA2:=simplify(subs(t=t-1,CA2)):

> D2:=[simplify(CA2[1]),simplify(CA2[2]),simplify(CA2[3])];
D2 := [Vector[row](1, {1 = (t-1)*(2*2^(1/2)*t^5-10*2^(1/2)*
t^4+23*2^(1/2)*t^3-31*2^(1/2)*t^2+24*2^(1/2)*t-10*2^(1/2)-1)/
(t^2-2*t+2)}, datatype = anything, storage = rectangular, order
= Fortran_order, shape = []), Vector[row](1, {1 = (t-1)*(2^(1/2)
*t^5-5*2^(1/2)*t^4+11*2^(1/2)*t^3-14*2^(1/2)*t^2+t^3+10*2^(1/2)*

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t-3*t^2-4*2^(1/2)+4*t-4)/(t^2-2*t+2)}, datatype = anything,
storage = rectangular, order = Fortran_order, shape = []),
Vector[row](1, {1 = -(t-1)*(-2*t^5+10*t^4-21*t^3+25*t^2+2^(1/2)
-16*t+6)/(t^2-2*t+2)}, datatype = anything, storage =
rectangular, order = Fortran_order, shape = [])]
> #The second generator curve of the second curve is A*C2+[1,0,-1]
^T
> #Second generator curve of the second translation surface
> D2:=[(t-1)*(2*2^(1/2)*t^5-10*2^(1/2)*t^4+23*2^(1/2)*t^3-31*2^(
1/2)*t^2+24*2^(1/2)*t-10*2^(1/2)-1)/(t^2-2*t+2)+1, (t-1)*(2^(1/2)
*t^5-5*2^(1/2)*t^4+11*2^(1/2)*t^3-14*2^(1/2)*t^2+t^3+10*2^(1/2)*
t-3*t^2-4*2^(1/2)+4*t-4)/(t^2-2*t+2), -(t-1)*(-2*t^5+10*t^4-21*
t^3+25*t^2+2^(1/2)-16*t+6)/(t^2-2*t+2)-1];#This is D2
D2 := [(t-1)*(2*2^(1/2)*t^5-10*2^(1/2)*t^4+23*2^(1/2)*t^3-31*2^(
1/2)*t^2+24*2^(1/2)*t-10*2^(1/2)-1)/(t^2-2*t+2)+1, (t-1)*(2^(
1/2)*t^5-5*2^(1/2)*t^4+11*2^(1/2)*t^3-14*2^(1/2)*t^2+t^3+10*2^(
1/2)*t-3*t^2-4*2^(1/2)+4*t-4)/(t^2-2*t+2), -(t-1)*(-2*t^5+10*
t^4-21*t^3+25*t^2+2^(1/2)-16*t+6)/(t^2-2*t+2)-1]

[> #Affinities between the curves C2, D2

> Eq2[1]:=numer((m11*C2[1]+m12*C2[2]+m13*C2[3]+b1)-subs(t=(a*t+b),
D2[1])):
> Eq2[2]:=numer((m21*C2[1]+m22*C2[2]+m23*C2[3]+b2)-subs(t=(a*t+b),
D2[2])):
> Eq2[3]:=numer((m31*C2[1]+m32*C2[2]+m33*C2[3]+b3)-subs(t=(a*t+b),
D2[3])):
> n:=max(degree(Eq2[1],t),degree(Eq2[2],t),degree(Eq2[3],t));
n := 8
> lis:=[]:
> for i from 1 to 3 do
  for j from 0 to n do
    lis:=[op(lis),coeff(Eq2[i],t,j)]:
  od:
od:
> with(SolveTools);
[AbstractRootOfSolution, Basis, CancelInverses, Combine,
Complexity, Engine, GreaterComplexity, Identity, Inequality,
Linear, Parametric, Polynomial, PolynomialSystem,
RationalCoefficients, SemiAlgebraic, SortByComplexity]
> PolynomialSystem(lis, [m11,m12,m13,m21,m22,m23,m31,m32,m33,b1,b2,
b3,a,b], engine=groebner);
{a = 0, b = b, b1 = -(1/2)*(-4*2^(1/2)*b^6+24*2^(1/2)*b^5-66*2^(
1/2)*b^4+108*2^(1/2)*b^3+6*b-110*2^(1/2)*b^2-6+68*2^(1/2)*b-2*
b^2-20*2^(1/2))/(b^2-2*b+2), b2 = -(-2^(1/2)*b^6+6*2^(1/2)*b^5
-16*2^(1/2)*b^4+25*2^(1/2)*b^3-b^4+8*b-24*2^(1/2)*b^2+4*b^3
-4+14*2^(1/2)*b-7*b^2-4*2^(1/2))/(b^2-2*b+2), b3 = -(-2*b^6+12*
b^5-31*b^4+46*b^3+2^(1/2)*b-40*b^2-2^(1/2)+20*b-4)/(b^2-2*b+2),
m11 = 0, m12 = 0, m13 = 0, m21 = 0, m22 = 0, m23 = 0, m31 = 0,
m32 = 0, m33 = 0}, {a = 1, b = 1, b1 = 1, b2 = 0, b3 = -1, m11 =
2^(1/2), m12 = -1, m13 = 2*2^(1/2), m21 = 1, m22 = -2, m23 = 2^(
1/2), m31 = -1, m32 = -2^(1/2), m33 = 2}

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> #Checking the solution
> X:=C1+subs(t=s,C2);
X := [s^2+t/(t^6+1), s/(s^2+1)+t^3/(t^6+1), s^4-s+t^5/(t^6+1)]
> X:=1/2*X;#Parametrization of S1
X := [(1/2)*s^2+(1/2)*t/(t^6+1), (1/2)*s/(s^2+1)+(1/2)*t^3/(t^6+1), (1/2)*s^4-(1/2)*s+(1/2)*t^5/(t^6+1)]

> Y:=D1+subs(t=s,D2);
Y := [(s-1)*(2*2^(1/2)*s^5-10*2^(1/2)*s^4+23*2^(1/2)*s^3-31*2^(1/2)*s^2+24*2^(1/2)*s-10*2^(1/2)-1)/(s^2-2*s+2)+1+(t-1)*(2*2^(1/2)*t^4-8*2^(1/2)*t^3+12*2^(1/2)*t^2-8*2^(1/2)*t-t^2+3*2^(1/2)+2*t-1)/(t^6-6*t^5+15*t^4-20*t^3+15*t^2-6*t+2), (s-1)*(2^(1/2)*s^5-5*2^(1/2)*s^4+11*2^(1/2)*s^3-14*2^(1/2)*s^2+s^3+10*2^(1/2)*s-3*s^2-4*2^(1/2)+4*s-4)/(s^2-2*s+2)+(t-1)*(2^(1/2)*t^4-4*2^(1/2)*t^3+6*2^(1/2)*t^2-4*2^(1/2)*t-2*t^2+2^(1/2)+4*t-1)/(t^6-6*t^5+15*t^4-20*t^3+15*t^2-6*t+2), -(s-1)*(-2*s^5+10*s^4-21*s^3+25*s^2+2^(1/2)-16*s+6)/(s^2-2*s+2)-1-(-2*t^4+2^(1/2)*t^2+8*t^3-2*2^(1/2)*t-12*t^2+2^(1/2)+8*t-1)*(t-1)/(t^6-6*t^5+15*t^4-20*t^3+15*t^2-6*t+2)]
> Y:=1/2*Y;#Parametrization of S2
Y := [(1/2)*(s-1)*(2*2^(1/2)*s^5-10*2^(1/2)*s^4+23*2^(1/2)*s^3-31*2^(1/2)*s^2+24*2^(1/2)*s-10*2^(1/2)-1)/(s^2-2*s+2)+1/2+(1/2)*(t-1)*(2*2^(1/2)*t^4-8*2^(1/2)*t^3+12*2^(1/2)*t^2-8*2^(1/2)*t-t^2+3*2^(1/2)+2*t-1)/(t^6-6*t^5+15*t^4-20*t^3+15*t^2-6*t+2), (1/2)*(s-1)*(2^(1/2)*s^5-5*2^(1/2)*s^4+11*2^(1/2)*s^3-14*2^(1/2)*s^2+s^3+10*2^(1/2)*s-3*s^2-4*2^(1/2)+4*s-4)/(s^2-2*s+2)+(1/2)*(t-1)*(2^(1/2)*t^4-4*2^(1/2)*t^3+6*2^(1/2)*t^2-4*2^(1/2)*t-2*t^2+2^(1/2)+4*t-1)/(t^6-6*t^5+15*t^4-20*t^3+15*t^2-6*t+2), -(1/2)*(s-1)*(-2*s^5+10*s^4-21*s^3+25*s^2+2^(1/2)-16*s+6)/(s^2-2*s+2)-1/2-(1/2)*(-2*t^4+2^(1/2)*t^2+8*t^3-2*2^(1/2)*t-12*t^2+2^(1/2)+8*t-1)*(t-1)/(t^6-6*t^5+15*t^4-20*t^3+15*t^2-6*t+2)]

> #Affinity found
> m11:= 2^(1/2);m12:=-1;m13:= 2*2^(1/2);m21:= 1;m22:=-2;m23:=2^(1/2); m31:= -1;m32:=-2^(1/2);m33:= 2;
m11 := 2^(1/2)
m12 := -1
m13 := 2*2^(1/2)
m21 := 1
m22 := -2
m23 := 2^(1/2)
m31 := -1
m32 := -2^(1/2)
m33 := 2
> B1:=1/2;B2:=0;B3=-1/2;
B1 := 1/2
B2 := 0
B3 := -1/2

> #Let us check the solution
> P1[1]:=m11*X[1]+m12*X[2]+m13*X[3]+B1;
P1[1] := 2^(1/2)*((1/2)*s^2+(1/2)*t/(t^6+1))-(1/2)*s/(s^2+1)-(1/2)*t^3/(t^6+1)+2*2^(1/2)*((1/2)*s^4-(1/2)*s+(1/2)*t^5/(t^6+1))+1/2

```

```
> P2[1]:=simplify(subs({t=t+1,s=s+1},Y[1]));
P2[1] := (1/2)*(2*2^(1/2)*s^6*t^6+3*2^(1/2)*s^4*t^6-2*2^(1/2)*
s^3*t^6+2^(1/2)*s^2*t^6+2*2^(1/2)*s^2*t^5-2*2^(1/2)*s*t^6+s^2*
t^6+2*2^(1/2)*s^6-s*t^6+2*2^(1/2)*t^5+t^6+3*2^(1/2)*s^4-s^2*t^3
-2*2^(1/2)*s^3+2^(1/2)*s^2*t+2^(1/2)*s^2-t^3-2*2^(1/2)*s+2^(1/2)
*t+s^2-s+1)/((s^2+1)*(t^6+1))
```

```
> simplify(P1[1]-P2[1]);
```

```
0
```

```
> P1[2]:=m21*X[1]+m22*X[2]+m23*X[3]+B2;
```

```
P1[2] := (1/2)*s^2+(1/2)*t/(t^6+1)-s/(s^2+1)-t^3/(t^6+1)+2^(1/2)
*((1/2)*s^4-(1/2)*s+(1/2)*t^5/(t^6+1))
```

```
> P2[2]:=simplify(subs({t=t+1,s=s+1},Y[2]));
```

```
P2[2] := (1/2)*(2^(1/2)*s^6*t^6+2^(1/2)*s^4*t^6-2^(1/2)*s^3*t^6+
s^4*t^6+2^(1/2)*s^2*t^5-2^(1/2)*s*t^6+s^2*t^6+2^(1/2)*s^6-2*s*
t^6+2^(1/2)*t^5+2^(1/2)*s^4-2*s^2*t^3-2^(1/2)*s^3+s^4+s^2*t-2*
t^3-2^(1/2)*s+s^2-2*s+t)/((s^2+1)*(t^6+1))
```

```
> simplify(P1[2]-P2[2]);
```

```
0
```

```
> P1[3]:=m31*X[1]+m32*X[2]+m33*X[3]+B3;
```

```
P1[3] := -(1/2)*s^2-(1/2)*t/(t^6+1)-2^(1/2)*((1/2)*s/(s^2+1)+
(1/2)*t^3/(t^6+1))+s^4-s+t^5/(t^6+1)+B3
```

```
> P2[3]:=simplify(subs({t=t+1,s=s+1},Y[3]));
```

```
P2[3] := -(1/2)*(-2*s^6*t^6-s^4*t^6+2*s^3*t^6+2^(1/2)*s*t^6+2*
s^2*t^6-2*s^2*t^5+2*s*t^6+2^(1/2)*s^2*t^3-2*s^6+t^6-2*t^5+2*
(1/2)*t^3-s^4+2*s^3+s^2*t+2^(1/2)*s+2*s^2+2*s+t+1)/((s^2+1)*
(t^6+1))
```

```
> simplify(P1[1]-P2[1]);
```

```
0
```

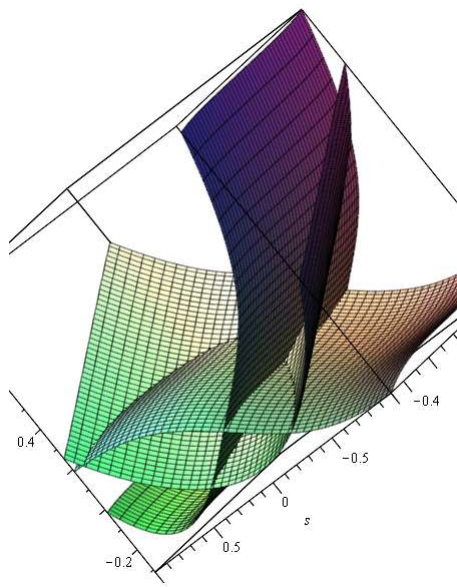
```
> #The solution is fine.
```

```
> time()-tt;#Total timing
```

```
17.000
```

```
> with(plots):
```

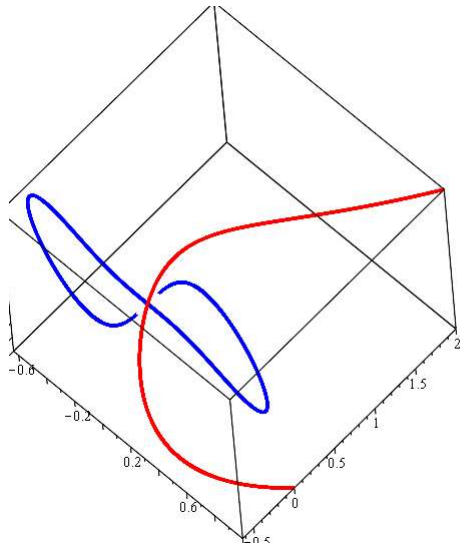
```
> plot3d({X[1],X[2],X[3]},t=-0.5..0.5,s=-1..1,numpoints=50000)
;#Plot of the surface (difficult to visualize)
```



```

> p1:=spacecurve(C1,t=-7..7,color=blue,thickness=3,numpoints=50000)
:
> p2:=spacecurve(C2,t=-1..1,color=red,thickness=3,numpoints=50000):
> display(p1,p2); #Plot of the generator curves of the first surface

```



```

> p3:=spacecurve(D1,t=-15..15,color=blue,thickness=3,numpoints=
50000):
> p4:=spacecurve(D2,t=0.1..2,color=red,thickness=3,numpoints=50000)
:
> display(p3,p4);#Plot of the generator curves of the second
surface

```

