Some Proofs of the Category Theory

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Notes and notation:
Ob_C and C are used interchangeable. (e.g. $\mathrm{Ob}_{\mathrm{Set}}$ and Set.)
Duality: $\mathrm{Ob}_C = \mathrm{Ob}_{C^{\mathrm{op}}}$ and $\forall x, y \in \mathrm{Ob}_C.\mathrm{Hom}_C(x, y) = \mathrm{Hom}_{C^{\mathrm{op}}}(y, x)$.
$\circ : \Pi(C \in \mathrm{Cat}).\Pi(X, Y, Z \in C).\mathrm{Hom}_C(X, Y) \times \mathrm{Hom}_C(Y, Z) \to \mathrm{Hom}_C(X, Z)$

1 Yoneda embedding lemma

Let C be a locally small category.
Let Set be a category of sets.
$[C^{op}, \mathrm{Set}]$ is some functor category.

Definition 1.1. Let C, D be the categories.
$(F, \widetilde{F})$ is a (covariant) functor iff

1. $F : C \to D$
2. $\widetilde{F} : \Pi(A\ B : C).\mathrm{Hom}_C(A, B) \to \mathrm{Hom}_D(F(A), F(B))$
3. $\forall A \in \mathrm{Ob}_C, \widetilde{F}^{A,A}(1_A) = 1_{F(A)}$
4. $\forall X, Y, Z \in \mathrm{Ob}_C.\forall f \in \mathrm{Hom}_C(Y, Z).\forall g \in \mathrm{Hom}_C(X, Y).\widetilde{F}^{X,Z}(f \circ_C^{X,Y,Z} g) = \widetilde{F}^{Y,Z}(f) \circ_D^{F(X),F(Y),F(Z)} \widetilde{F}^{X,Y}(g)$

Definition 1.2. Let C, D be the categories.
$[C, D]$ is the set of functors from C to D.

Definition 1.3. The morphism function of the Hom-functor.
Let $f \in \mathrm{Hom}_C(A, B)$.

$$\mathrm{Hom}_C : \mathrm{Ob}_C \times \mathrm{Ob}_C \to \mathrm{Set}$$

$$\mathrm{Hom}_C(-, x) : \mathrm{Ob}_C \to \mathrm{Set}$$

$$\widetilde{\mathrm{Hom}}_C^{A,B}(f, x) : \mathrm{Hom}_C(B, x) \to \mathrm{Hom}_C(A, x)$$

$$\widetilde{\mathrm{Hom}}_C^{A,B}(f, x) \overset{\mathrm{def}}{=} (g \in \mathrm{Hom}_C(B, x) \mapsto (g \circ_C f))$$

$$\widetilde{\mathrm{Hom}}_C^{A,B}(f, x) \overset{\mathrm{def}}{=} (g \in \mathrm{Hom}_C(B, x) \mapsto (g \circ_C f) \in \mathrm{Hom}_C(A, x))$$

$$\left(\text{another form is } \widetilde{\mathrm{Hom}_C(-, x)}^{A,B}(f) \overset{\mathrm{def}}{=} (g \in \mathrm{Hom}_C(B, x) \mapsto (g \circ_C f) \in \mathrm{Hom}_C(A, x)\right)$$

Lemma 1.1.

$$\mathrm{Hom}_C(-, x) \in [C^{\mathrm{op}}, \mathrm{Set}]$$

Proof.
Substitution:
$C \coloneqq C^{\mathrm{op}}$, $D \coloneqq \mathrm{Set}$, $F \coloneqq \mathrm{Hom}_C(-,x)$, $\widetilde{F} \coloneqq \widetilde{\mathrm{Hom}}_C(-,x)$

1. $\mathrm{Hom}(-,x) : C^{\mathrm{op}} \to \mathrm{Set}$ by definition of Hom.

2. $\widetilde{\mathrm{Hom}}(-,x) : \Pi(A\,B : C).\mathrm{Hom}_{C^{\mathrm{op}}}(A,B) \to \mathrm{Hom}_{\mathrm{Set}}(\mathrm{Hom}_C(A,x), \mathrm{Hom}_C(B,x))$
   $\widetilde{\mathrm{Hom}}_C^{A,B}(f \in \mathrm{Hom}_{C^{\mathrm{op}}}(A,B), x) = (g \in \mathrm{Hom}_C(A,x) \mapsto (g \circ_C f) \in \mathrm{Hom}_C(B,x))$ (this is a definition).
   In other words, fix the $A, B \in C$ and $f \in \mathrm{Hom}_{C^{\mathrm{op}}}(A,B)$.
   $\widetilde{\mathrm{Hom}}_C^{A,B}(f,x) = (g \mapsto (g \circ_C f))$
   $\widetilde{\mathrm{Hom}}_C^{A,B}(f,x) : \mathrm{Hom}_C(A,x) \to \mathrm{Hom}_C(B,x))$
   $\widetilde{\mathrm{Hom}}$ is total since $f : \mathrm{Hom}_{C^{\mathrm{op}}}(A,B), \Rightarrow f \in \mathrm{Hom}_C(B,A)$ by definition of dual category. (It follows that composition is defined, so it is total.)

3. $1_A \in \mathrm{Hom}_{C^{\mathrm{op}}}(A,A)$
   $\widetilde{\mathrm{Hom}}^{A,A}(1_A, x) = (g \in \mathrm{Hom}_C(A,x) \mapsto g \circ_C 1_A) =$
   $= (g \in \mathrm{Hom}_C(A,x) \mapsto g \in \mathrm{Hom}_C(A,x)) =$
   $= 1_{\mathrm{Hom}_C(A,x)} \in \mathrm{Hom}_{\mathrm{Set}}\left(\mathrm{Hom}_C(A,x), \mathrm{Hom}_C(A,x)\right).$
   since $1_A \circ_C g = g$

4. Now we need do a substitution in $\forall X,Y,Z \in \mathrm{Ob}_C.\forall f \in \mathrm{Hom}_C(Y,Z).\forall g \in \mathrm{Hom}_C(X,Y).\widetilde{F}^{X,Z}(f \circ_C^{X,Y,Z} g) = \widetilde{F}^{Y,Z}(f) \circ_D^{F(X),F(Y),F(Z)} \widetilde{F}^{X,Y}(g)$ and prove it.
   Again, $C \coloneqq C^{\mathrm{op}}$, $D \coloneqq \mathrm{Set}$, $F \coloneqq \mathrm{Hom}_C(-,x)$, $\widetilde{F} \coloneqq \widetilde{\mathrm{Hom}}_C(-,x)$
   So $\widetilde{F}^{X,Z}(f \circ_C^{X,Y,Z} g) = \widetilde{F}(f) \circ_D^{F(X),F(Y),F(Z)} \widetilde{F}(g)$ becames
   $\widetilde{\mathrm{Hom}}_C^{X,Z}(f \circ_C^{X,Y,Z} g, x) = \widetilde{\mathrm{Hom}}_C(f,x) \circ_D^{F(X),F(Y),F(Z)} \widetilde{\mathrm{Hom}}_C(g,x).$
   Equivalently, $\widetilde{\mathrm{Hom}}_C^{X,Z}(f \circ_{C^{\mathrm{op}}}^{X,Y,Z} g, x) =$
   $= \widetilde{\mathrm{Hom}}_C^{Y,Z}(f,x) \circ_{\mathrm{Set}}^{\widetilde{\mathrm{Hom}}_C(X,x),\widetilde{\mathrm{Hom}}_C(Y,x),\widetilde{\mathrm{Hom}}_C(Z,x)} \widetilde{\mathrm{Hom}}_C^{X,Y}(g,x).$
   We'll omit some notation for arrow composition below.
   $\widetilde{\mathrm{Hom}}_C^{X,Z}(f \circ_{C^{\mathrm{op}}}^{X,Y,Z} g, x) = \widetilde{\mathrm{Hom}}_C(f,x) \circ_{\mathrm{Set}} \widetilde{\mathrm{Hom}}_C(g,x),$
   where $f \in \mathrm{Hom}_{C^{\mathrm{op}}}(Y,Z)$ and $g \in \mathrm{Hom}_{C^{\mathrm{op}}}(X,Y)$.
   So $f \in \mathrm{Hom}_C(Z,Y)$ and $g \in \mathrm{Hom}_C(Y,X)$.
   $f \circ_{C^{\mathrm{op}}}^{X,Y,Z} g = g \circ_C^{Z,Y,X} f$
   So it's enough to prove that $\widetilde{\mathrm{Hom}}_C(g \circ_C f, x) = \widetilde{\mathrm{Hom}}_C(f,x) \circ_{\mathrm{Set}} \widetilde{\mathrm{Hom}}_C(g,x)$
   a) $\widetilde{\mathrm{Hom}}_C(g \circ_C f, x) = (h \in \mathrm{Hom}_C(Z,x) \mapsto (h \circ_C (g \circ_C f)) \in \mathrm{Hom}_C(X,x))$
   b) $\widetilde{\mathrm{Hom}}_C(f,x) = (u \in \mathrm{Hom}_C(Z,x) \mapsto (u \circ_C f) \in \mathrm{Hom}_C(Y,x))$
   c) $\widetilde{\mathrm{Hom}}_C(g,x) = (v \in \mathrm{Hom}_C(Y,x) \mapsto (v \circ_C g) \in \mathrm{Hom}_C(X,x))$

   $u \xmapsto{\widetilde{\mathrm{Hom}}_C(g,x)} (u \circ_C g) \xmapsto{\widetilde{\mathrm{Hom}}_C(g,x)} ((u \circ_C g) \circ_C f))$
   $\mathrm{Hom}_C(Z,x) \xrightarrow{\widetilde{\mathrm{Hom}}_C(g,x)} \mathrm{Hom}_C(Y,x) \xrightarrow{\widetilde{\mathrm{Hom}}_C(f,x)} \mathrm{Hom}_C(X,x)$

Qed.

Definition 1.4. Yoneda functor between categories C and $[C^{\mathrm{op}}, \mathrm{Set}]$.

$$h_x = \mathrm{Hom}_C(-, x)$$

Definition 1.5.

$$\eta : \Pi(w \in C^{\mathrm{op}}).\mathrm{Hom}_C(w, x) \to \mathrm{Hom}_C(w, y)$$

$$\eta = \lambda(w \in C^{\mathrm{op}}).\lambda g \in \mathrm{Hom}_C(w, x).(f \circ_C^w, x, yg)$$

Now our aim is to define a natural transformation of the elements of $[C^{\mathrm{op}}, \mathrm{Set}]$.

Lemma 1.2.
We have:
$F = \mathrm{Hom}_C(-, x) \in [C^{\mathrm{op}}, \mathrm{Set}]$
$G = \mathrm{Hom}_C(-, y) \in [C^{\mathrm{op}}, \mathrm{Set}]$
$f \in \mathrm{Hom}_C(x, y)$
We want:
Find a natural tranformation η from $\mathrm{Hom}_C(-, x)$ to $\mathrm{Hom}_C(-, x)$.
(define and prove)

Fix $w \in C^{\mathrm{op}}$. $\eta_w =?$.
$\eta_w \mathrm{Hom}_{\mathrm{Set}}(\mathrm{Hom}_C(w, x), \mathrm{Hom}_C(w, y))$
$\eta_w : \mathrm{Hom}_C(w, x) \to \mathrm{Hom}_C(w, y)$

Substitution:
$C := C^{\mathrm{op}}$, $D := \mathrm{Set}$, $F := \mathrm{Hom}_C(-, x)$, $G := \mathrm{Hom}_C(-, y)$, $x := a$, $y := b$, $f := m$

Lemma 1.3.

$$\forall a, b \in C^{\mathrm{op}}.\forall m \in \mathrm{Hom}_C\mathrm{op}(a, b).\eta_b \circ_{\mathrm{Set}} \mathrm{Hom}_C(m, x) = \mathrm{Hom}_C(m, y) \circ_{\mathrm{Set}} \eta_a$$

Proof.
Fix a, b and $m \in \mathrm{Hom}_C(b, a)$.
Aim: $(\lambda g_1 \in \mathrm{Hom}_C(b, x).f \circ_C^{b,x,y} g_1) \circ_{\mathrm{Set}} \widetilde{\mathrm{Hom}_C}^{B,A}(m, x) =$
$= \widetilde{\mathrm{Hom}_C}^{B,A}(m, y) \circ_{\mathrm{Set}} (\lambda g_2 \in \mathrm{Hom}_C(a, x).f \circ_C^{a,x,y} g_2)$.
Note that $(f \circ_C^{a,x,y} g_2) \in \mathrm{Hom}_C(a, y)$
$\widetilde{\mathrm{Hom}_C}^{B,A}(m, y) \overset{\mathrm{def}}{=} (\lambda g_3 \in \mathrm{Hom}_C(a, y).g_3 \circ_C^{b,a,y} m)$
$\widetilde{\mathrm{Hom}_C}^{B,A}(m, y) : \mathrm{Hom}_C(a, y) \to \mathrm{Hom}_C(b, y)$
RightHandSide $= \lambda g_2 \in \mathrm{Hom}_C(a, x).((f \circ_C^{a,x,y} g_2) \circ_C^{b,a,y} m)$
RHS:$\mathrm{Hom}_C(a, x) \to \mathrm{Hom}_C(b, y)$
$\widetilde{\mathrm{Hom}_C}^{B,A}(m, x) \overset{\mathrm{def}}{=} (\lambda g_4 \in \mathrm{Hom}_C(a, y).g_4 \circ_C^{b,a,y} m)$
LeftHandSide $= \lambda g_2 \in \mathrm{Hom}_C(a, x).(f \circ_C^{b,x,y} (g_2 \circ_C^{b,a,x} m))$
LHS:$\mathrm{Hom}_C(a, x) \to \mathrm{Hom}_C(b, y)$
LHS=RHS
Qed.

Lemma 1.4. $h_- : C \to [C^{op}, \mathrm{Set}]$ is a fully faithful functor.

Proof.

- h_- is a functor.
  $\widetilde{h_-} = ...$
  Substitution: $C \coloneqq C$, $D \coloneqq [C^{\mathrm{op}}, \mathrm{Set}]$, $F \coloneqq h_-$,

  1. $F : C \to D$
  2. $\widetilde{F} : \Pi(A\ B : C).\mathrm{Hom}_C(A, B) \to \mathrm{Hom}_D(F(A), F(B))$
  3. $\forall A \in \mathrm{Ob}_C, \widetilde{F}^{A,A}(1_A) = 1_{F(A)}$
  4. $\forall X, Y, Z \in \mathrm{Ob}_C.\forall f \in \mathrm{Hom}_C(Y, Z).\forall g \in \mathrm{Hom}_C(X, Y).\widetilde{F}^{X,Z}(f \circ_C^{X,Y,Z} g) = \widetilde{F}^{Y,Z}(f) \circ_D^{F(X),F(Y),F(Z)} \widetilde{F}^{X,Y}(g)$

- h_- is full.
- h_- is faithfull.
- h_- is injective on morphisms.

Qed.

Lemma 1.5. by N. Yoneda. (Contravariant version) $G : C^{\mathrm{op}} \to \mathrm{Set}$
$h_A = \mathrm{Hom}_C(-, A)$
$h_A : C^{\mathrm{op}} \to \mathrm{Set}$
$Nat(h_A, G) \sim G(A)$
(there is a bijection between value of functior G and natural transformations from h_A to G)

Proof.
Let $u \in G(A)$. Let's define a natural transformation Φ from h_A to G.
$\Phi : \Pi(x \in C^{\mathrm{op}}).h_A(x) \to G(x)$ (because it's a Hom in Set)
1) Let $\Phi_A^u(\mathrm{id}_A) \overset{\mathrm{def}}{=} u$.
Now we need to define $\Phi_x(f)$ for arbitrary $x \in C$ and
$f \in h_A(x). = \mathrm{Hom}_C(x, A). = \mathrm{Hom}_{C^{\mathrm{op}}}(A, x). \Longrightarrow$
$\Longrightarrow$ therefore $\widetilde{G}^{A,x}(f) : G(A) \to G(x)$ is defined.
So the definition will be $\Phi_x^u(f) \overset{\mathrm{def}}{=} (\widetilde{G}^{A,x}(f))(u)$.
Now we need to show the correctness of the definition.
$\Phi_A^u(\mathrm{id}_A) \overset{\mathrm{def}}{=} (\widetilde{G}^{A,A}(f))(\mathrm{id}_A) = (\mathrm{id}_{G(A)})(u) = u$. The correctness is proved. 2) The next aim is proving commutativity of natural transformation with functor applied to morphisms.
$\forall x, y \in C^{\mathrm{op}}.\forall f \in \mathrm{Hom}_{C^{\mathrm{op}}}(x, y).(G(f) \circ_{\mathrm{Set}} \Phi_x) = (\Phi_y \circ_{\mathrm{Set}} (\mathrm{Hom}_C(f, A)))$ Fix $x, y \in C^{\mathrm{op}}$ and $f \in \mathrm{Hom}_{C^{\mathrm{op}}}(x, y)$.
Aim: $(G(f) \circ_{\mathrm{Set}} \Phi_x) = (\Phi_y \circ_{\mathrm{Set}} (\mathrm{Hom}_C(f, A)))$. Fix $g \in \mathrm{Hom}_C(x, A)$. Recall that $\mathrm{Hom}_C(f, A) = (- \circ^{y,x,a})$.
Aim: $(G(f) \circ_{\mathrm{Set}} \Phi_x)(g) = (\Phi_y \circ_{\mathrm{Set}} (\mathrm{Hom}_C(f, A)))(g)$.
Equivalently, $(G(f) \circ_{\mathrm{Set}} \Phi_x)(g) = \Phi_y(g \circ_C^{y,x,A} f)$.

LHS $:= (G(f) \circ_{\text{Set}} \Phi_x)(g) = (\text{def of } \Phi) = \widetilde{G}^{x,y}(f)\left(\left(\widetilde{G}^{A,x}(g)\right)(u)\right) =$
$= \widetilde{G}^{A,y}(g \circ_C f)(u)$
Then since $g \circ_C f = f \circ_{C^{\text{op}}} g$ and $\widetilde{G}(f \circ_{C^{\text{op}}} g) = \widetilde{G}(f) \circ_{\text{Set}} \widetilde{G}(g)$ we have:
RHS $:= \Phi_y^u(g \circ f) = (\text{def of } \Phi) = \widetilde{G}^{A,y}(g \circ_C f)(u)$
This LHS=RHS holds therefore we obtained a natural transformation $\Phi \in \text{Nat}(h_A, G)$ using the $u \in G(A)$.
We already know that the relation between $G(A)$ and $\Phi \in \text{Nat}(h_A, G)$ is functional and total on $G(A)$. The next step is to prove its injectivity and surjectivity on $\Phi \in \text{Nat}(h_A, G)$.
Injectivity: $u_1, u_2 \in G(A)$, $u_1 \neq u_2$.
$\Phi_{1,A}(\text{id}(A)) = u_1 \neq u_2 = \Phi_{2,A}(\text{id}(A)) \implies \Phi_1 \neq \Phi_2$. It's proved.
Surjectivity on $\text{Nat}(h_A, G)$.:
Let $\Psi \in \text{Nat}(h_A, G)$. Let's show that $\exists u \in G(A).\Phi^u = \Psi$.
$u := \Psi_A(\text{id}_A)$
$\Psi : \Pi(x \in C).h_A(x) \to G(x)$
$\Phi_x^u(f) \overset{\text{def}}{=} (\widetilde{G}^{A,x}(f))(u)$.
Fix $x \in C$ and $f \in \text{Hom}_C(x, A)$. $\Psi_x(f) = ?$. In other words, there is a $\Psi \in \text{Nat}(h_A, G)$, and it is known that $\Psi_A(\text{id}_A) = u$. Find $\Psi_x(f)$.
Firstly, $\text{id}_A \in \text{Hom}_C(A, A)$. $\implies (\text{id}_A \circ_C^{x,A,A} f) \in \text{Hom}_C(x, A)$. $\implies \Psi_x(\text{id}_A \circ_C^{x,A,A} f) \in G(A)$. $\implies \Psi_x(f) \in G(A)$.
Secondly, $\text{id}_A \in \text{Hom}_C(A, A)$. $\implies \Psi_A(\text{id}_A) \in G(A)$. Since $\Psi_A(\text{id}_A) = u$ we have $G(f)(u) \in G(x)$.
Since Ψ is a natural transformation we obtained that $\Psi_x(f) = G(f)(u)$, which is equal to $\Phi_x^u(f)$.
This equality holds for any argument $f \in Hom_C(x, A)$ then $\forall x \in C.\Psi_x = \Phi_x^u$ by functional extensionality. Therefore the relation is surjective on $\text{Nat}(h_A, G)$.
Qed.