Thesis: Seventh meeting

Some minor additions

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Model 3A1

Y_1 scaled entirely with independent Slope & Intercept

$i = 1, ..., N; j = 1, .., n$

$$\begin{cases} m_{ij} = (\beta_0^1 + u_{0,i}^1) + \beta_x^1 \cdot x_i + \left(\beta_t^1 + u_{t,i}^1\right) \cdot Period_{i,j} \\ y_{i,j}^1 = m_{ij} + \epsilon_{i,j}^1 \\ y_{i,j}^2 = \gamma \cdot m_{ij} + (\beta_0^2 + u_{0,i}^2) + \beta_x^2 \cdot x_i + \left(\beta_t^2 + u_{t,i}^2\right) \cdot Period_{i,j} + \epsilon_{i,j}^2 \end{cases}$$

with

$$u_{0,i}^1 \sim \mathcal{N}(0, \sigma_{1,0}^2); \quad u_{t,i}^1 \sim \mathcal{N}(0, \sigma_{1,t}^2); \quad \begin{bmatrix} u_{0,i}^2 \\ u_{t,i}^2 \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma_{2,0}^2 & \sigma_{2,(0,t)} \\ \sigma_{2,(t,0)} & \sigma_{2,t}^2 \end{pmatrix} \right];$$

$$\begin{bmatrix} \epsilon_i^1 \\ \epsilon_i^2 \end{bmatrix} \sim \mathcal{N}_{2j}(\mathbf{0}, \mathbf{I}_{2j})$$

Model 3A2

Y_1 scaled entirely with dependent Slope & Intercept

$i = 1, ..., N; j = 1, .., n$

$$\begin{cases} m_{ij} = (\beta_0^1 + u_{0,i}^1) + \beta_x^1 \cdot x_i + \left(\beta_t^1 + u_{t,i}^1\right) \cdot Period_{i,j} \\ y_{i,j}^1 = m_{ij} + \epsilon_{i,j}^1 \\ y_{i,j}^2 = \gamma \cdot m_{ij} + (\beta_0^2 + u_{0,i}^2) + \beta_x^2 \cdot x_i + \left(\beta_t^2 + u_{t,i}^2\right) \cdot Period_{i,j} + \epsilon_{i,j}^2 \end{cases}$$

with

$$\begin{bmatrix} u_{0,i}^1 \\ u_{t,i}^1 \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma_{1,0}^2 & \sigma_{1,(0,t)} \\ \sigma_{1,(t,0)} & \sigma_{1,t}^2 \end{pmatrix} \right]; \quad \begin{bmatrix} u_{0,i}^2 \\ u_{t,i}^2 \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma_{2,0}^2 & \sigma_{2,(0,t)} \\ \sigma_{2,(t,0)} & \sigma_{2,t}^2 \end{pmatrix} \right];$$

$$\begin{bmatrix} \epsilon_i^1 \\ \epsilon_i^2 \end{bmatrix} \sim \mathcal{N}_{2j}(\mathbf{0}, \mathbf{I}_{2j})$$

Models 3A in INLA

- Although Models 3A1 and 3A2 are very similar and even nested, they are implemented in INLA very differently (One can implement models in a multitude of ways in INLA).
- I show the difference in implementation between the 2 models since the simulation results are very different.

Model 3A1 in INLA

- The copy trick is used and fixed effects are modelled as random effects with 2 levels: Each outcome has it's own level.
- Every term involved in each of the models is copied in this way.

```
$coefficients
         true      INLA true_combi INLA_combi
beta_0^1  2.0 1.961858         2.0   1.961858
beta_x^1  4.0 3.997196         4.0   3.997196
beta_t^1  2.5 2.501245         2.5   2.501245
beta_0^2  3.0 4.193585         5.4   5.341272
beta_x^2  1.5 3.981618         6.3   6.319978
beta_t^2  3.5 5.126604         6.5   6.589832
gamma     1.2 0.585000          NA         NA
```

```
$U1
$U1$true
            [,1]        [,2]
[1,]  2.06292431 -0.04536504
[2,] -0.04536504  2.71835906

$U1$INLA
         [,1]     [,2]
[1,] 2.134135 0.000000
[2,] 0.000000 2.744491
```

```
$U2
$U2$true
         [,1]     [,2]
[1,] 2.978606 1.420168
[2,] 1.420168 3.870316

$U2$INLA
           [,1]       [,2]
[1,]  5.0079391 -0.5584875
[2,] -0.5584875  7.6892306
```

Model 3A2 in INLA

- Data fit as 3 likelihood model
- First likelihood is outcome Y1
- Second likelihood is outcome Y2
- Third likelihood is a Gaussian with very low noise and outcome Y3=0. Only random effects u can be copied, so we need the linear predictors of model Y_1 as random effects u. For this we use that $\eta_1 = u \quad \rightarrow \eta_1 - u = 0$ to ensure the random effects u are the linear predictors for $Y1$, which we then copy to Y_2.

```
$coefficients
         true     INLA true_combi INLA_combi
beta_0^1  2.0 1.871538       2.00   1.871538
beta_x^1  4.0 4.000832       4.00   4.000832
beta_t^1  2.5 2.331216       2.50   2.331216
beta_0^2  3.0 3.175045       5.60   5.664190
beta_x^2  1.5 1.365655       6.70   6.686761
beta_t^2  3.5 3.396343       6.75   6.496861
gamma     1.3 1.330000         NA         NA
```

```
$U1
$U1$true
         [,1]     [,2]
[1,] 2.085740 1.600423
[2,] 1.600423 2.873966

$U1$INLA
          [,1]      [,2]
[1,] 2.5746415 0.6685351
[2,] 0.6685351 0.3646141
```

```
$U2
$U2$true
         [,1]     [,2]
[1,] 2.924897 1.572376
[2,] 1.572376 3.754722

$U2$INLA
          [,1]      [,2]
[1,] 3.6629548 0.9051673
[2,] 0.9051673 0.4174495
```

Included Unbalanced Design and Missing Data

- Model Parameters:
  - N: Number of of Patients
  - n: Maximum number of measurements per patient
  - p_1: probability that a measurement is observed
  - p_2: probability that covariate x is measured
- How does INLA deal with missing covariates?
  - INLA has no way to 'impute' or integrate-out missing covariates. You have to adjust your model to account for missing covariates. Sometimes, you can formulate a joint model for the data and the covariates, but this is case-specific.
  - If x[i] = NA this means that x[i] is not part of the linear predictor for y[i]. For fixed effects, this is equivalent to x[i]=0.

PIT (Probability Integral Transform

- Computed for each observation as:

$$PIT_i = \pi(y_i^{new} \leq y_i | y_{-i})$$

- Measures probability for new observation y_i^{new} to be lower than y_i given all observations except for y_i.
- For a good model the PIT's should be approximately uniformly distributed on $[0, 1]$
- Kolmogorov Smirnov non-parametric test has been used to test whether the PIT's are uniformly distributed

Simulations

- $N = 75$, $n = 7$, $p_1 = 0.7$, $p_2 = 0.9$, $CV = 5$
- Models 0, 2A, 2C1, 2C2, 3A1, 3A2, 3B1
- 3 times INLA failed: 2* model 2A, 1* model 3A2

Simulation 3: Models 0 & 2A

```
> round(Model_comparison_big$Model_0$outcomes, 3)
            true   Model_0 Model_1A  Model_2A Model_2B Model_2C1 Model_2C2 Model_3A1 Model_3A2 Model_3B1
beta_0^1     2.0     2.054       NA     1.657       NA     2.188     2.177     1.984     2.047     2.028
beta_x^1     4.0     3.971       NA     3.208       NA     3.880     3.892     3.977     3.950     3.988
beta_t^1     2.5     2.503       NA     1.969       NA     2.466     2.484     2.451     2.485     2.509
beta_0^2     3.0     2.978       NA     2.264       NA     2.924     2.893     2.947     2.996     2.981
beta_x^2     1.5     1.449       NA     1.101       NA     1.248     1.238     1.387     1.481     1.450
beta_t^2     3.5     3.536       NA     2.880       NA     3.571     3.576     3.497     3.555     3.535
MLIK          NA -1748.083       NA -1886.484       NA -2237.132 -2303.754 -1801.186 -3279.242 -1768.688
DIC_approx    NA  3496.165       NA  3772.969       NA  4474.264  4607.507  3602.372  6558.485  3537.376
DIC           NA  2714.622       NA  3480.144       NA  3961.460  4006.514  2775.814   430.750  2770.717
WAIC          NA  2718.874       NA  3487.437       NA  4016.850  4063.332  2769.720   316.461  2765.140
PIT           NA     0.043       NA     0.051       NA     0.063     0.057     0.040     0.176     0.040
CPO           NA  1398.959       NA  1748.647       NA  2055.452  2071.026  1421.023   537.682  1418.437
> round(Model_comparison_big$Model_2A$outcomes, 3)
            true   Model_0 Model_1A  Model_2A Model_2B Model_2C1 Model_2C2 Model_3A1 Model_3A2 Model_3B1
beta_0^1     2.0     2.058       NA     2.038       NA     2.048     2.048     2.001     2.024     2.065
beta_x^1     4.0     3.974       NA     3.977       NA     3.969     3.971     3.982     3.964     3.986
beta_t^1     2.5     2.506       NA     2.510       NA     2.509     2.507     2.511     2.490     2.503
beta_0^2     3.0     2.939       NA     2.930       NA     2.939     2.937     2.935     2.872     2.940
beta_x^2     1.5     1.453       NA     1.466       NA     1.460     1.460     1.442     1.405     1.454
beta_t^2     3.5     3.491       NA     3.492       NA     3.489     3.489     3.481     3.473     3.491
MLIK          NA -1476.651       NA -1453.937       NA -1479.779 -1487.310 -1514.907 -3065.777 -1483.718
DIC_approx    NA  2953.301       NA  2907.874       NA  2959.559  2974.619  3029.814  6131.554  2967.435
DIC           NA  2615.436       NA  2599.689       NA  2622.528  2626.535  2638.587   555.853  2646.145
WAIC          NA  2620.082       NA  2607.338       NA  2628.554  2632.216  2639.295   432.081  2646.777
PIT           NA     0.045       NA     0.048       NA     0.044     0.046     0.048     0.173     0.045
CPO           NA  1324.910       NA  1308.418       NA  1327.216  1327.770  1331.194   559.197  1334.704
```

Simulation 3: Models 2C1 & 2C2

```
> round(Model_comparison_big$Model_2C1$outcomes, 3)
            true   Model_0 Model_1A  Model_2A Model_2B Model_2C1 Model_2C2 Model_3A1 Model_3A2 Model_3B1
beta_0^1     2.0     2.058       NA     2.096       NA     2.105     2.132     1.987     2.047     2.028
beta_x^1     4.0     3.970       NA     3.858       NA     3.822     3.850     3.974     3.950     3.985
beta_t^1     2.5     2.471       NA     2.470       NA     2.478     2.474     2.430     2.452     2.477
beta_0^2     3.0     2.935       NA     2.872       NA     2.877     2.854     2.897     2.934     2.934
beta_x^2     1.5     1.445       NA     1.380       NA     1.242     1.212     1.367     1.433     1.446
beta_t^2     3.5     3.532       NA     3.582       NA     3.573     3.560     3.487     3.524     3.532
MLIK          NA -1750.133       NA -2343.481       NA -2203.346 -2234.194 -1798.279 -3270.808 -1773.908
DIC_approx    NA  3500.266       NA  4686.961       NA  4406.692  4468.389  3596.559  6541.616  3547.817
DIC           NA  2720.114       NA  4344.374       NA  3913.100  3928.159  2779.528   444.036  2779.793
WAIC          NA  2722.445       NA  4354.656       NA  3968.779  3985.730  2772.889   330.819  2771.551
PIT           NA     0.041       NA     0.068       NA     0.057     0.056     0.038     0.176     0.040
CPO           NA  1404.303       NA  2182.770       NA  2029.339  2032.891  1424.032   542.822  1423.468
> round(Model_comparison_big$Model_2C2$outcomes, 3)
            true   Model_0 Model_1A  Model_2A Model_2B Model_2C1 Model_2C2 Model_3A1 Model_3A2 Model_3B1
beta_0^1     2.0     2.054       NA     2.081       NA     2.133     2.150     1.992     2.049     2.030
beta_x^1     4.0     3.963       NA     3.870       NA     3.806     3.817     3.976     3.949     3.985
beta_t^1     2.5     2.516       NA     2.507       NA     2.505     2.495     2.460     2.495     2.522
beta_0^2     3.0     2.935       NA     2.888       NA     2.928     2.903     2.932     2.930     2.933
beta_x^2     1.5     1.445       NA     1.408       NA     1.226     1.220     1.420     1.425     1.447
beta_t^2     3.5     3.554       NA     3.594       NA     3.572     3.572     3.541     3.543     3.554
MLIK          NA -1749.952       NA -2372.049       NA -2244.561 -2290.603 -1809.352 -3285.456 -1781.830
DIC_approx    NA  3499.903       NA  4744.098       NA  4489.122  4581.206  3618.704  6570.912  3563.661
DIC           NA  2706.094       NA  4384.055       NA  3987.776  4016.442  2779.284   438.455  2774.774
WAIC          NA  2709.358       NA  4395.450       NA  4044.784  4073.223  2772.288   324.355  2768.259
PIT           NA     0.042       NA     0.072       NA     0.068     0.066     0.040     0.176     0.041
CPO           NA  1394.794       NA  2203.476       NA  2068.135  2076.706  1422.851   540.641  1420.717
```

Simulation 3: Models 3A1 & 3A2

```
> round(Model_comparison_big$Model_3A1$outcomes, 3)
            true   Model_0 Model_1A  Model_2A Model_2B Model_2C1 Model_2C2 Model_3A1 Model_3A2 Model_3B1
beta_0^1     2.0     2.049       NA     1.834       NA     2.180     2.143     1.988     2.049     2.054
beta_x^1     4.0     3.970       NA     3.230       NA     3.885     3.923     3.971     3.948     3.961
beta_t^1     2.5     2.468       NA     1.961       NA     2.439     2.454     2.422     2.445     2.466
beta_0^2     3.0     3.071       NA     2.691       NA     3.176     3.214     3.034     3.085     3.078
beta_x^2     1.5     1.451       NA     1.196       NA     1.143     1.127     1.445     1.470     1.453
beta_t^2     3.5     3.471       NA     2.793       NA     3.480     3.485     3.467     3.481     3.469
MLIK          NA -1803.768       NA -1968.144       NA -2319.771 -2299.474 -1841.867 -3282.387 -1807.930
DIC_approx    NA  3607.535       NA  3936.289       NA  4639.543  4598.949  3683.734  6564.774  3615.860
DIC           NA  2730.578       NA  3689.090       NA  4183.360  4210.751  2740.338   424.577  2707.425
WAIC          NA  2732.059       NA  3697.635       NA  4234.121  4259.236  2738.879   311.833  2709.207
PIT           NA     0.042       NA     0.052       NA     0.075     0.070     0.042     0.166     0.041
CPO           NA  1415.781       NA  1852.172       NA  2150.883  2155.218  1418.732   534.786  1399.975
> round(Model_comparison_big$Model_3A2$outcomes, 3)
            true   Model_0 Model_1A  Model_2A Model_2B Model_2C1 Model_2C2 Model_3A1 Model_3A2 Model_3B1
beta_0^1     2.0     2.053       NA     2.122       NA     2.196     2.190     1.999     2.052     2.059
beta_x^1     4.0     3.969       NA     3.998       NA     3.867     3.923     3.978     3.947     3.965
beta_t^1     2.5     2.524       NA     2.502       NA     2.482     2.515     2.469     2.503     2.523
beta_0^2     3.0     3.071       NA     3.070       NA     3.170     3.173     3.028     3.086     3.077
beta_x^2     1.5     1.452       NA     1.360       NA     1.105     1.075     1.423     1.471     1.452
beta_t^2     3.5     3.545       NA     3.575       NA     3.537     3.570     3.537     3.555     3.542
MLIK          NA -1793.906       NA -2481.395       NA -2349.244 -2315.303 -1839.229 -3290.847 -1807.798
DIC_approx    NA  3587.813       NA  4962.789       NA  4698.488  4630.606  3678.458  6581.695  3615.597
DIC           NA  2715.076       NA  4635.182       NA  4237.194  4277.798  2764.952   418.409  2705.713
WAIC          NA  2717.952       NA  4645.334       NA  4290.033  4330.753  2755.146   304.195  2707.171
PIT           NA     0.043       NA     0.077       NA     0.076     0.074     0.044     0.169     0.044
CPO           NA  1401.715       NA  2326.916       NA  2179.236  2188.504  1418.904   530.093  1398.057
```

Simulation 3: Models 3B1

```
> round(Model_comparison_big$Model_3B1$outcomes, 3)
            true  Model_0 Model_1A  Model_2A Model_2B Model_2C1 Model_2C2 Model_3A1 Model_3A2 Model_3B1
beta_0^1     2.0    2.049       NA     2.127       NA     2.172     2.161     1.979     1.656     2.059
beta_x^1     4.0    3.970       NA     3.971       NA     3.901     3.910     3.966     3.172     3.962
beta_t^1     2.5    2.468       NA     2.453       NA     2.439     2.441     2.421     1.978     2.466
beta_0^2     3.0    3.076       NA     2.829       NA     2.828     2.779     2.989     2.508     3.082
beta_x^2     1.5    1.456       NA     1.389       NA     1.379     1.350     1.432     1.131     1.456
beta_t^2     3.5    3.599       NA     3.665       NA     3.668     3.678     3.585     2.806     3.599
MLIK          NA -1801.974      NA -2464.282       NA -2323.225 -2366.542 -1831.292 -2660.278 -1799.014
DIC_approx    NA  3603.948      NA  4928.564       NA  4646.450  4733.085  3662.584  5320.556  3598.028
DIC           NA  2729.300      NA  4600.511       NA  4171.442  4202.531  2723.047   391.176  2704.466
WAIC          NA  2730.985      NA  4611.220       NA  4221.125  4248.222  2724.711   300.086  2707.422
PIT           NA     0.042      NA     0.069       NA     0.072     0.064     0.043     0.139     0.042
CPO           NA  1414.722      NA  2310.537       NA  2146.458  2152.619  1406.626   456.698  1396.219
```

To-do

- Complete model configuration in INLA
  - Model 3
    - Implement model 3A2 in INLA using both methods ('copy' trick and 'extra likelihood' trick)
    - Compare INLA implementation with JMBayes2 (ask Ed)
  - Model 1: Inspect other residual error covariance structures
- Model assessment
  - Look at posterior probabilities of fitted values
    - Calculate CPO, PIT manually
    - Calculate MSE per outcome using posterior means
    - Let INLA fit new data
- Theoretical results
  - Give Likelihood for every model
  - Try to write them as LGM
- Simulating Data
  - Change unbalanced design such that both outcomes are not measured at same time-points
  - Inspect simulated data more closely using e.g. Spaghetti plots

Models so far

- In order to compare all models using model assessment all models were rewritten into similar form.
- All models (except for some type 3 models) are implemented

Model 0

No association

$i = 1, ..., N; j = 1, .., n$

$$\begin{cases} y^1_{i,j} = (\beta^1_0 + u^1_{0,i}) + \beta^1_x \cdot x_i + \left(\beta^1_t + u^1_{t,i}\right) \cdot Period_{i,j} + \epsilon^1_{i,j} \\ y^2_{i,j} = (\beta^2_0 + u^2_{0,i}) + \beta^2_x \cdot x_i + \left(\beta^2_t + u^2_{t,i}\right) \cdot Period_{i,j} + \epsilon^2_{i,j} \end{cases}$$

with

$$\begin{bmatrix} u^1_{0,i} \\ u^1_{t,i} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma^2_{1,0} & \sigma_{1,(0,t)} \\ \sigma_{1,(t,0)} & \sigma^2_{1,t} \end{pmatrix} \right]; \quad \begin{bmatrix} u^2_{0,i} \\ u^2_{t,i} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma^2_{2,0} & \sigma_{2,(0,t)} \\ \sigma_{2,(t,0)} & \sigma^2_{2,t} \end{pmatrix} \right]$$

$$\begin{bmatrix} \epsilon^1_i \\ \epsilon^2_i \end{bmatrix} \sim \mathcal{N}_{2j}(\mathbf{0}, \mathbf{I}_{2j})$$

Model 1A

Association via residual errors

$i = 1, ..., N; j = 1, 2$

$$\begin{cases} y^1_{i,j} = \beta^1_0 + \beta^1_x \cdot x_i + \beta^1_t \cdot Period_{i,j} + \epsilon^1_{i,j} \\ y^2_{i,j} = \beta^2_0 + \beta^2_x \cdot x_i + \beta^2_t \cdot Period_{i,j} + \epsilon^2_{i,j} \end{cases}$$

with

$$\begin{bmatrix} \epsilon^1_{i,1} \\ \epsilon^1_{i,2} \\ \epsilon^2_{i,1} \\ \epsilon^2_{i,2} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma^2_{1,1} & ... & ... & ... \\ ... & \sigma^2_{1,2} & ... & ... \\ ... & ... & \sigma^2_{2,1} & ... \\ ... & ... & ... & \sigma^2_{2,2} \end{pmatrix} \right]$$

Model 2A

Random Intercept only

$i = 1, ..., N; j = 1, .., n$

$$\begin{cases} y^1_{i,j} = (\beta^1_0 + u^1_{0,i}) + \beta^1_x \cdot x_i + \beta^1_t \cdot Period_{i,j} + \epsilon^1_{i,j} \\ y^2_{i,j} = (\beta^2_0 + u^2_{0,i}) + \beta^2_x \cdot x_i + \beta^2_t \cdot Period_{i,j} + \epsilon^2_{i,j} \end{cases}$$

with

$$\begin{bmatrix} u^1_{0,i} \\ u^2_{0,i} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma^2_{1,0} & \sigma_{(1,2),0} \\ \sigma_{(2,1),0} & \sigma^2_{2,0} \end{pmatrix} \right]; \qquad \begin{bmatrix} \boldsymbol{\epsilon}^1_i \\ \boldsymbol{\epsilon}^2_i \end{bmatrix} \sim \mathcal{N}_{2j}(\mathbf{0}, \mathbf{I}_{2j})$$

Model 2B

Random Intercept & correlated residuals

$i = 1, ..., N; j = 1, 2$

$$\begin{cases} y^1_{i,j} = (\beta^1_0 + u^1_{0,i}) + \beta^1_x \cdot x_i + \beta^1_t \cdot Period_{i,j} + \epsilon^1_{i,j} \\ y^2_{i,j} = (\beta^2_0 + u^2_{0,i}) + \beta^2_x \cdot x_i + \beta^2_t \cdot Period_{i,j} + \epsilon^2_{i,j} \end{cases}$$

with

$$\begin{bmatrix} u^1_{0,i} \\ u^2_{0,i} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma^2_{1,0} & \sigma_{(1,2),0} \\ \sigma_{(2,1),0} & \sigma^2_{2,0} \end{pmatrix} \right]; \quad \begin{bmatrix} \epsilon^1_{i,1} \\ \epsilon^1_{i,2} \\ \epsilon^2_{i,1} \\ \epsilon^2_{i,2} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma^2_{1,1} & \dots & \dots & \dots \\ \dots & \sigma^2_{1,2} & \dots & \dots \\ \dots & \dots & \sigma^2_{2,1} & \dots \\ \dots & \dots & \dots & \sigma^2_{2,2} \end{pmatrix} \right]$$

Model 2C1

Random Slope & Intercept: Independent

$i = 1, ..., N; j = 1, .., n$

$$\begin{cases} y^1_{i,j} = (\beta^1_0 + u^1_{0,i}) + \beta^1_x \cdot x_i + \left(\beta^1_t + u^1_{t,i}\right) \cdot Period_{i,j} + \epsilon^1_{i,j} \\ y^2_{i,j} = (\beta^2_0 + u^2_{0,i}) + \beta^2_x \cdot x_i + \left(\beta^2_t + u^2_{t,i}\right) \cdot Period_{i,j} + \epsilon^2_{i,j} \end{cases}$$

with

$$\begin{bmatrix} u^1_{0,i} \\ u^2_{0,i} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma^2_{1,0} & \sigma_{(1,2),0} \\ \sigma_{(2,1),0} & \sigma^2_{2,0} \end{pmatrix}\right]; \quad \begin{bmatrix} u^1_{t,i} \\ u^2_{t,i} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma^2_{1,t} & \sigma_{(1,2),t} \\ \sigma_{(2,1),t} & \sigma^2_{2,t} \end{pmatrix}\right]$$

$$\begin{bmatrix} \epsilon^1_i \\ \epsilon^2_i \end{bmatrix} \sim \mathcal{N}_{2j}(\mathbf{0}, \mathbf{I}_{2j})$$

Model 2C2

Random Slope & Intercept: Dependent

$i = 1, ..., N; j = 1, .., n$

$$\begin{cases} y_{i,j}^1 = (\beta_0^1 + u_{0,i}^1) + \beta_x^1 \cdot x_i + \left(\beta_t^1 + u_{t,i}^1\right) \cdot Period_{i,j} + \epsilon_{i,j}^1 \\ y_{i,j}^2 = (\beta_0^2 + u_{0,i}^2) + \beta_x^2 \cdot x_i + \left(\beta_t^2 + u_{t,i}^2\right) \cdot Period_{i,j} + \epsilon_{i,j}^2 \end{cases}$$

with

$$\begin{bmatrix} u_{0,i}^1 \\ u_{0,i}^2 \\ u_{t,i}^1 \\ u_{t,i}^2 \end{bmatrix} \sim \mathcal{N}_4 \left[\mathbf{0}, \begin{pmatrix} \sigma_{1,0}^2 & \dots & \dots & \dots \\ \dots & \sigma_{1,t}^2 & \dots & \dots \\ \dots & \dots & \sigma_{2,0}^2 & \dots \\ \dots & \dots & \dots & \sigma_{2,2}^2 \end{pmatrix} \right] ; \qquad \begin{bmatrix} \epsilon_i^1 \\ \epsilon_i^2 \end{bmatrix} \sim \mathcal{N}_{2j}(\mathbf{0}, \mathbf{I}_{2j})$$

Model 3A1

Y_1 scaled entirely with independent Slope & Intercept

$i = 1, ..., N; j = 1, .., n$

$$\begin{cases} m_{ij} = (\beta_0^1 + u_{0,i}^1) + \beta_x^1 \cdot x_i + \left(\beta_t^1 + u_{t,i}^1\right) \cdot Period_{i,j} \\ y_{i,j}^1 = m_{ij} + \epsilon_{i,j}^1 \\ y_{i,j}^2 = \gamma \cdot m_{ij} + (\beta_0^2 + u_{0,i}^2) + \beta_x^2 \cdot x_i + \left(\beta_t^2 + u_{t,i}^2\right) \cdot Period_{i,j} + \epsilon_{i,j}^2 \end{cases}$$

with

$$u_{0,i}^1 \sim \mathcal{N}(0, \sigma_{1,0}^2); \quad u_{t,i}^1 \sim \mathcal{N}(0, \sigma_{1,t}^2); \quad \begin{bmatrix} u_{0,i}^2 \\ u_{t,i}^2 \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma_{2,0}^2 & \sigma_{2,(0,t)} \\ \sigma_{2,(t,0)} & \sigma_{2,t}^2 \end{pmatrix} \right];$$

$$\begin{bmatrix} \epsilon_i^1 \\ \epsilon_i^2 \end{bmatrix} \sim \mathcal{N}_{2j}(\mathbf{0}, \mathbf{I}_{2j})$$

Model 3B1

Random Y_1-effects scaled independently; independent Slope & Intercept

$i = 1, ..., N; j = 1, .., n$

$$\begin{cases} y^1_{i,j} = (\beta^1_0 + u^1_{0,i}) + \beta^1_x \cdot x_i + \left(\beta^1_t + u^1_{t,i}\right) \cdot Period_{i,j} + \epsilon^1_{i,j} \\ y^2_{i,j} = \gamma_1 \cdot u^1_{0,i} + \gamma_2 \cdot u^1_{t,i} + (\beta^2_0 + u^2_{0,i}) + \beta^2_x \cdot x_i + \left(\beta^2_t + u^2_{t,i}\right) \cdot Period_{i,j} + \epsilon^2_{i,j} \end{cases}$$

with

$$u^1_{0,i} \sim \mathcal{N}(0, \sigma^2_{1,0}); \quad u^1_{t,i} \sim \mathcal{N}(0, \sigma^2_{1,t}); \quad \begin{bmatrix} u^2_{0,i} \\ u^2_{t,i} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma^2_{2,0} & \sigma_{2,(0,t)} \\ \sigma_{2,(t,0)} & \sigma^2_{2,t} \end{pmatrix} \right];$$

$$\begin{bmatrix} \epsilon^1_i \\ \epsilon^2_i \end{bmatrix} \sim \mathcal{N}_{2j}(\mathbf{0}, \mathbf{I}_{2j})$$

Marginal Assessment methods in INLA

- Within INLA the following Model assessment criteria exist:
  - MLIK (Marginal Likelihood)
  - DIC (Deviance Information Criterion)
  - WAIC (Watanabe-Akaike Information Criterion)
  - CPO (Conditional Predictive Ordinates)
  - PIT (Predictive Integral Transform)

Marginal Likelihood $\pi(y)$

- Probability of observed data under given model
- In INLA approximated as:

$$\tilde{\pi}(y) = \int \frac{\pi(\theta, x, y)}{\tilde{\pi}_G(x|\theta, y)}\Big|_{x=x^*(\theta)} d\theta$$

- When considering set of M models $\{\mathcal{M}_m\}_{m=1}^{M}$, the marginal likelihoods are $\pi(y|\mathcal{M}_m)$.
- Posterior can be computed via model priors: $\pi(\mathcal{M}_m|y) \propto \pi(y|\mathcal{M}_m)\pi(\mathcal{M}_m)$
- Can be used to compute Bayes factor for models $\mathcal{M}_1$ and $\mathcal{M}_2$:

$$\frac{\pi(\mathcal{M}_1|y)}{\pi(\mathcal{M}_2|y)} = \frac{\pi(y|\mathcal{M}_1)\pi(\mathcal{M}_1)}{\pi(y|\mathcal{M}_2)\pi(\mathcal{M}_2)}$$

DIC & WAIC

- DIC
  - Given by:

$$DIC = D(\hat{x}, \hat{\theta}) + 2p_D$$

  - Takes into account goodness of fit ($D(\hat{x}, \hat{\theta})$) and penalty for number of parameters ($2p_D$).
  - D is the deviance with $\hat{x}$ the posterior mean and $\hat{\theta}$ the posterior mode (might be skewed).
  - $p_D = \mathbb{E}[D(x, \theta)] - D(\hat{x}, \hat{\theta})$
- WAIC is similar to DIC but p_D is calculated differently

CPO (Conditional Predictive Ordinates)

- Computed for each observation as:

$$CPO_i = \pi(y_i | y_{-i})$$

- Posterior probability of observing y_i when model is fit without y_i.
- Low value may indicate outlier
- Summarized over all data as:

$$CPO = -\sum_{i=1}^{N} ln(CPO_i)$$

PIT (Predictive Integral Transform)

- Computed for each observation as:

$$PIT_i = \pi(y_i^{new} \leq y_i | y_{-i})$$

- Measures probability for new observation y_i^{new} to be lower than y_i given all observations except for y_i.
- For a good model the PIT's should be approximately uniformly distributed on $[0, 1]$

Simulations

- Data was generated from each of the 8(6) models. In total 8(6) datasets were generated.
- Each dataset was fit by every model 8(6) (only in INLA). The coefficients and Model Assessment criteria were recorded.
- This was done 2 times:
  1. $N = 750$, $n = 2$. All models participated
  2. $N = 500$, $n = 4$. Models 1A and 2B did not participate,
     - ★ They model association via an unstructured variance-covariance matrix for the residual errors, which can only be modelled in INLA for $n = 2$.

Simulation 2

$N = 500$, $n = 4$
Models 0, 2A, 2C1, 2C2, 3A1, 3B1

Simulation 2: Models 0 & 2A

$Model_0$Model_0_df

	true	Model_0	Model_2A	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	2.129866	2.127995	2.129451	2.129671	2.113251	2.129382
beta_x^1	4.0	4.015218	3.986687	4.008675	4.012158	4.008108	4.007623
beta_t^1	2.5	2.599241	2.600256	2.599472	2.599348	2.594592	2.599510
beta_0^2	3.0	3.112763	3.119732	3.111930	3.112714	3.110197	3.112676
beta_x^2	1.5	1.520176	1.629831	1.507061	1.519394	1.515059	1.518827
beta_t^2	3.5	3.684672	3.680893	3.685136	3.684700	3.681410	3.684719
MLIK	NA	-8381.279613	-10622.260731	-8456.818098	-8421.715350	-8462.436164	-8442.100242
DIC	NA	12865.935243	19833.113670	12862.875267	12878.800789	12859.426281	12865.640575
WAIC	NA	12843.868420	19877.082468	12803.035691	12856.227218	12817.404958	12823.729900
CPO	NA	6773.535911	9977.461558	6828.089741	6773.182060	6799.374018	6802.389827

$Model_2A$Model_2A_df

	true	Model_0	Model_2A	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	2.097733	2.097733	2.097734	2.097732	2.090256	2.097732
beta_x^1	4.0	4.002521	4.000894	4.001353	4.000890	4.000405	4.000966
beta_t^1	2.5	2.480511	2.480513	2.480512	2.480513	2.480947	2.480513
beta_0^2	3.0	2.954926	2.954938	2.954935	2.954935	2.952751	2.954934
beta_x^2	1.5	1.453171	1.456217	1.455051	1.455108	1.440928	1.454644
beta_t^2	3.5	3.474924	3.474920	3.474921	3.474922	3.468346	3.474922
MLIK	NA	-6897.256372	-6847.404247	-6871.094009	-6879.428477	-6903.710711	-6877.569841
DIC	NA	12113.799456	12143.023504	12112.394116	12108.052727	12123.662912	12122.342153
WAIC	NA	12147.357605	12184.018332	12146.199373	12145.191976	12159.782075	12159.436708
CPO	NA	6156.133611	6131.611256	6147.268539	6146.035832	6140.935794	6140.782805

Simulation 2: Models 2C1 & 2C2

$Model_2C1$Model_2C1_df

	true	Model_0	Model_2A	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	2.097733	2.097618	2.097731	2.097731	2.089620	2.097734
beta_x^1	4.0	4.002780	3.980313	4.000695	4.000915	4.001743	4.001821
beta_t^1	2.5	2.399231	2.399276	2.399225	2.399224	2.391553	2.399231
beta_0^2	3.0	2.954909	2.954914	2.954923	2.954922	2.950497	2.954916
beta_x^2	1.5	1.448278	1.472289	1.451368	1.451349	1.437566	1.449916
beta_t^2	3.5	3.323889	3.323887	3.323878	3.323878	3.317111	3.323887
MLIK	NA	-8406.869297	-10450.343520	-8349.866095	-8393.631079	-8450.716190	-8418.578527
DIC	NA	12922.015663	19776.053749	12895.802346	12909.911187	12918.496001	12904.086970
WAIC	NA	12871.429198	19818.415774	12848.019719	12867.785330	12859.350676	12853.248810
CPO	NA	6843.523154	9941.981285	6819.326053	6821.665613	6838.080212	6830.165684

$Model_2C2$Model_2C2_df

	true	Model_0	Model_2A	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	2.097729	2.097649	2.097733	2.097724	2.088923	2.097735
beta_x^1	4.0	4.003316	3.996762	4.001333	4.002722	4.001716	4.002305
beta_t^1	2.5	2.457149	2.457178	2.457145	2.457143	2.449463	2.457155
beta_0^2	3.0	2.954909	2.954935	2.954923	2.954924	2.942093	2.954916
beta_x^2	1.5	1.448775	1.481918	1.451457	1.451896	1.420161	1.450891
beta_t^2	3.5	3.370851	3.370839	3.370842	3.370836	3.352603	3.370846
MLIK	NA	-8378.421490	-10554.654633	-8373.751226	-8345.385805	-8460.479877	-8435.561154
DIC	NA	12848.046465	19806.962333	12828.597370	12830.850387	12838.283823	12831.525107
WAIC	NA	12808.011012	19850.700034	12766.348677	12798.189518	12771.364590	12772.385227
CPO	NA	6764.142319	9961.687452	6783.885676	6731.922986	6784.445274	6780.338331

Simulation 2: Models 3A1 & 3B1

$Model_3A1$Model_3A1_df

	true	Model_0	Model_2A	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	2.097735	2.097606	2.097727	2.097727	2.089939	2.097729
beta_x^1	4.0	4.003397	3.980725	4.002853	4.003129	4.002232	4.003454
beta_t^1	2.5	2.441049	2.441096	2.441041	2.441040	2.433857	2.441050
beta_0^2	3.0	2.982998	2.982961	2.982995	2.982988	2.979523	2.982987
beta_x^2	1.5	1.455043	1.476305	1.452601	1.452923	1.453170	1.452393
beta_t^2	3.5	3.306117	3.306132	3.306121	3.306113	3.304296	3.306120
MLIK	NA	-8485.784612	-10695.676210	-8455.652933	-8470.112374	-8525.897037	-8480.428140
DIC	NA	12915.629262	20175.325401	12874.087816	12893.804790	12904.742217	12887.659605
WAIC	NA	12866.076356	20218.309156	12815.501148	12852.513669	12855.291505	12840.831961
CPO	NA	6832.886151	10143.175916	6817.070009	6798.283022	6831.933536	6803.806640

$Model_3B1$Model_3B1_df

	true	Model_0	Model_2A	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	2.097735	2.097585	2.097724	2.097751	2.089780	2.097729
beta_x^1	4.0	4.003409	3.983855	4.001807	4.014631	4.002131	4.003408
beta_t^1	2.5	2.441049	2.441091	2.441027	2.441008	2.434081	2.441052
beta_0^2	3.0	2.982996	2.982820	2.982996	2.982976	2.979989	2.982981
beta_x^2	1.5	1.454259	1.456724	1.452645	1.451898	1.447676	1.451559
beta_t^2	3.5	3.276508	3.276577	3.276500	3.276479	3.275273	3.276509
MLIK	NA	-8637.959223	-11115.931909	-8471.794750	-8526.312867	-8678.940036	-8629.961380
DIC	NA	12933.011639	21187.604996	12861.907384	13414.745128	12926.460545	12896.425117
WAIC	NA	12880.063870	21227.207876	12804.146016	13366.191114	12870.271504	12848.227873
CPO	NA	6856.409569	10639.158957	6801.257805	6954.081126	6857.957757	6816.796811

Simulation 1

$N = 750$, $n = 2$
Models 0, 1A, 2A, 2B, 2C1, 2C2, 3A1, 3B1

Simulation 1: Models 0 & 1A

$Model_0$Model_0_df

	true	Model_0	Model_1A	Model_2A	Model_2B	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	2.062555	2.062551	2.062406	2.062524	2.062554	2.062547	2.058499	2.062551
beta_x^1	4.0	3.994878	3.995314	4.013629	3.998129	3.994915	3.995740	3.994106	3.995384
beta_t^1	2.5	2.529890	2.529896	2.530182	2.529940	2.529890	2.529903	2.522164	2.529898
beta_0^2	3.0	3.138011	3.138012	3.138134	3.138012	3.138044	3.138011	3.138730	3.138004
beta_x^2	1.5	1.481725	1.481527	1.466400	1.481810	1.477839	1.481640	1.484109	1.482418
beta_t^2	3.5	3.529583	3.529580	3.529364	3.529584	3.529526	3.529581	3.530829	3.529594
MLIK	NA	-6473.577169	-6517.192965	-6758.467139	-6540.734366	-6468.938935	-6423.443748	-6483.441570	-6493.339563
DIC	NA	11061.126033	-33486.369224	12434.664974	-33486.369217	-18794.330685	11148.574887	-13208.705298	-2850.634205
WAIC	NA	10978.342196	-34406.926848	12459.289241	-34406.926809	-18120.952881	11053.801535	-14225.961044	-3013.651656
CPO	NA	5962.388187	-14505.925676	6361.865738	-14505.925265	-6297.703708	5961.902173	-4414.884958	107.657495

$Model_1A$Model_1A_df

	true	Model_0	Model_1A	Model_2A	Model_2B	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	1.947935	1.947892	1.947931	1.947934	1.947919	1.947934	1.940167	1.947929
beta_x^1	4.0	4.032881	4.038846	4.035294	4.033802	4.038607	4.033792	4.034340	4.035810
beta_t^1	2.5	2.545497	2.545544	2.545502	2.545506	2.545513	2.545506	2.543746	2.545504
beta_0^2	3.0	3.041790	3.041723	3.041807	3.041801	3.041817	3.041804	3.046431	3.041805
beta_x^2	1.5	1.519116	1.535394	1.514392	1.515780	1.510932	1.514896	1.520036	1.515016
beta_t^2	3.5	3.449960	3.450000	3.449947	3.449956	3.449938	3.449953	3.453001	3.449949
MLIK	NA	-6413.163102	-7395.154646	-6379.644607	-6236.679591	-6374.434150	-6311.517456	-6420.020553	-6398.109654
DIC	NA	12276.846445	-33486.368549	12307.525312	-33486.368669	12122.223687	8850.825051	12271.401998	12286.438357
WAIC	NA	12308.244717	-34406.925124	12333.652038	-34406.926257	12133.540017	8346.635439	12307.466984	12315.622331
CPO	NA	6247.565414	-14505.919316	6202.012515	-14505.923884	6153.812560	6009.408817	6197.178540	6202.414744

Simulation 1: Models 2A & 2B

$Model_2A$Model_2A_df

	true	Model_0	Model_1A	Model_2A	Model_2B	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	1.955696	1.955707	1.955709	1.955706	1.955706	1.955708	1.950829	1.955707
beta_x^1	4.0	4.020166	4.017974	4.017217	4.017997	4.017823	4.017529	4.016688	4.017643
beta_t^1	2.5	2.531249	2.531243	2.531240	2.531243	2.531242	2.531241	2.530182	2.531242
beta_0^2	3.0	2.980848	2.980857	2.980860	2.980857	2.980860	2.980859	2.967714	2.980859
beta_x^2	1.5	1.519238	1.517028	1.516225	1.516982	1.516246	1.516328	1.486431	1.516397
beta_t^2	3.5	3.475404	3.475399	3.475396	3.475398	3.475396	3.475396	3.456185	3.475396
MLIK	NA	-5623.561814	-5619.900372	-5587.065983	-5480.472438	-5590.733690	-5603.848759	-5633.463099	-5605.276716
DIC	NA	9620.788523	-33486.369651	9784.697084	-33486.369667	9654.370843	9731.584817	9705.418215	9684.344051
WAIC	NA	9637.660147	-34406.927145	9802.080993	-34406.927165	9661.558879	9736.742625	9690.361150	9687.131617
CPO	NA	5083.410842	-14505.927214	5058.243902	-14505.927262	5061.136981	5061.755725	5060.561054	5060.202948

$Model_2B$Model_2B_df

	true	Model_0	Model_1A	Model_2A	Model_2B	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	1.926926	1.926926	1.926934	1.926926	1.926917	1.926926	1.919334	1.926931
beta_x^1	4.0	4.034246	4.035865	4.033621	4.035664	4.038539	4.035717	4.031976	4.034335
beta_t^1	2.5	2.545501	2.545511	2.545497	2.545511	2.545515	2.545511	2.540654	2.545500
beta_0^2	3.0	3.006339	3.006346	3.006360	3.006346	3.006361	3.006348	3.000786	3.006358
beta_x^2	1.5	1.514451	1.513085	1.509271	1.513016	1.508440	1.512585	1.492554	1.510016
beta_t^2	3.5	3.449947	3.449948	3.449933	3.449948	3.449932	3.449947	3.438864	3.449935
MLIK	NA	-6849.963356	-6811.244393	-6819.210427	-6684.695111	-6806.489301	-6771.620631	-6864.370687	-6840.018325
DIC	NA	12610.671284	-33486.368685	12681.946793	-33486.368694	12285.088363	11899.246840	12648.105376	12652.690583
WAIC	NA	12637.450311	-34406.926271	12710.735544	-34406.926280	12257.484441	11839.618817	12680.988466	12676.876908
CPO	NA	6487.543230	-14505.924018	6463.083064	-14505.923997	6381.703660	6307.621430	6465.126718	6464.026157

Simulation 1: Models 2C1 & 2C2

$Model_2C1$Model_2C1_df

	true	Model_0	Model_1A	Model_2A	Model_2B	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	1.955868	1.955875	1.955943	1.955877	1.955878	1.955873	1.951153	1.955865
beta_x^1	4.0	3.966042	3.964373	3.942856	3.963773	3.963621	3.965017	3.943641	3.967733
beta_t^1	2.5	2.607244	2.607234	2.607184	2.607232	2.607232	2.607236	2.603580	2.607248
beta_0^2	3.0	2.980952	2.980922	2.980967	2.980924	2.980929	2.980924	2.981361	2.980937
beta_x^2	1.5	1.486335	1.496275	1.482183	1.495729	1.494285	1.495670	1.483777	1.491427
beta_t^2	3.5	3.415953	3.415976	3.415949	3.415974	3.415970	3.415974	3.414693	3.415967
MLIK	NA	-6436.800701	-6434.293272	-6543.466389	-6419.819028	-6398.947179	-6388.958380	-6497.078009	-6449.312907
DIC	NA	11880.936110	-33486.368935	12445.207333	-33486.368935	10758.427525	11690.009414	12009.445033	11443.256350
WAIC	NA	11845.161995	-34406.926554	12479.318102	-34406.926546	10576.818072	11650.886131	11935.768859	11340.267668
CPO	NA	6185.831802	-14505.925115	6314.207829	-14505.925051	6092.884596	6084.854690	6195.681026	6157.586279

$Model_2C2$Model_2C2_df

	true	Model_0	Model_1A	Model_2A	Model_2B	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	1.955796	1.955816	1.955897	1.955812	1.955814	1.955818	1.949278	1.955802
beta_x^1	4.0	3.987327	3.980462	3.956155	3.981061	3.982928	3.979929	3.983381	3.986389
beta_t^1	2.5	2.581123	2.581100	2.581044	2.581098	2.581106	2.581099	2.573173	2.581121
beta_0^2	3.0	2.980923	2.980898	2.980930	2.980877	2.980903	2.980902	2.980561	2.980919
beta_x^2	1.5	1.495044	1.503940	1.493211	1.506495	1.501500	1.502642	1.488733	1.496805
beta_t^2	3.5	3.382313	3.382331	3.382317	3.382327	3.382325	3.382327	3.379775	3.382317
MLIK	NA	-6471.173412	-6448.167363	-6679.595361	-6452.594694	-6439.453617	-6323.306268	-6503.568057	-6478.627661
DIC	NA	11470.117769	-33486.369206	12501.519073	-33486.369199	6396.364226	11217.180452	-1628.298306	-1861.511312
WAIC	NA	11407.622734	-34406.926862	12531.072933	-34406.926745	6439.324195	11176.406842	-2606.110799	-2243.681927
CPO	NA	6097.650133	-14505.925416	6370.785045	-14505.925079	5958.275713	5969.357302	212.923448	418.585437

Simulation 1: Models 3A1 & 3B1

$Model_3A1$Model_3A1_df

	true	Model_0	Model_1A	Model_2A	Model_2B	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	1.955716	1.955703	1.955722	NA	1.955702	1.955694	1.951377	1.955701
beta_x^1	4.0	4.013707	4.016508	4.010417	NA	4.016969	4.019107	4.021408	4.017387
beta_t^1	2.5	2.480880	2.480884	2.480875	NA	2.480885	2.480892	2.475474	2.480891
beta_0^2	3.0	3.083040	3.083038	3.083150	NA	3.083050	3.083031	3.075170	3.083028
beta_x^2	1.5	1.506024	1.506122	1.470687	NA	1.502925	1.508065	1.498191	1.509141
beta_t^2	3.5	3.393346	3.393342	3.393263	NA	3.393335	3.393349	3.385820	3.393355
MLIK	NA	-6545.895839	-6548.561624	-6705.595984	NA	-6505.596862	-6541.381528	-6579.721299	-6547.787680
DIC	NA	11698.771557	-33486.369077	12577.732176	NA	-3355.980072	11673.875652	11355.651221	11267.087081
WAIC	NA	11630.697743	-34406.926731	12608.341609	NA	-3304.003118	11617.742379	11246.386888	11132.139616
CPO	NA	6167.753134	-14505.925633	6395.637957	NA	48.978159	6126.917631	6177.789901	6152.630117

$Model_3B1$Model_3B1_df

	true	Model_0	Model_1A	Model_2A	Model_2B	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	1.955716	1.956119	1.955716	1.955698	1.955696	1.955694	1.950879	1.955699
beta_x^1	4.0	4.013723	3.897429	4.011802	4.017805	4.018933	4.019703	4.022540	4.017945
beta_t^1	2.5	2.480880	2.480469	2.480879	2.480878	2.480881	2.480883	2.474315	2.480892
beta_0^2	3.0	3.083033	3.083536	3.083178	3.083046	3.083057	3.083044	3.079319	3.083025
beta_x^2	1.5	1.508197	1.365191	1.461668	1.503214	1.500601	1.504111	1.508898	1.510566
beta_t^2	3.5	3.355581	3.355086	3.355474	3.355557	3.355552	3.355560	3.354027	3.355590
MLIK	NA	-6703.762707	-7332.338520	-6876.675075	-6603.582489	-6576.167425	-6558.040635	-6735.277017	-6698.642638
DIC	NA	12054.506284	-33486.368881	13085.770921	-33486.368919	-3203.768879	11176.560685	11762.491301	11656.606677
WAIC	NA	11973.207654	-34406.925965	13118.930882	-34406.926552	-3444.098004	11046.685107	11637.582631	11493.347321
CPO	NA	6390.051954	-14505.922939	6621.167449	-14505.924678	-122.309984	6115.235178	6411.125399	6374.150934

To-do

- Complete model configuration in INLA
  - Complete model 3: dependent copied random effects
  - Model 1: Inspect other residual error covariance structures
- Theoretical results
  - Give Likelihood for every model
  - Try to write them as LGM
- Implement models on Dataset
  - Open dataset
    - Back to PBC?
  - Duchenne
  - COVID

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