Thesis: Sixth meeting

All models completed (almost) and Model Assessment

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Models so far

- In order to compare all models using model assessment all models were rewritten into similar form.
- All models (except for some type 3 models) are implemented

Model 0

No association

$i = 1, ..., N; j = 1, .., n$

$$\begin{cases} y^1_{i,j} = (\beta^1_0 + u^1_{0,i}) + \beta^1_x \cdot x_i + \left(\beta^1_t + u^1_{t,i}\right) \cdot Period_{i,j} + \epsilon^1_{i,j} \\ y^2_{i,j} = (\beta^2_0 + u^2_{0,i}) + \beta^2_x \cdot x_i + \left(\beta^2_t + u^2_{t,i}\right) \cdot Period_{i,j} + \epsilon^2_{i,j} \end{cases}$$

with

$$\begin{bmatrix} u^1_{0,i} \\ u^1_{t,i} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma^2_{1,0} & \sigma_{1,(0,t)} \\ \sigma_{1,(t,0)} & \sigma^2_{1,t} \end{pmatrix} \right]; \quad \begin{bmatrix} u^2_{0,i} \\ u^2_{t,i} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma^2_{2,0} & \sigma_{2,(0,t)} \\ \sigma_{2,(t,0)} & \sigma^2_{2,t} \end{pmatrix} \right]$$

$$\begin{bmatrix} \epsilon^1_i \\ \epsilon^2_i \end{bmatrix} \sim \mathcal{N}_{2j}(\mathbf{0}, \mathbf{I}_{2j})$$

Model 1A

Association via residual errors

$i = 1, ..., N; j = 1, 2$

$$\begin{cases} y^1_{i,j} = \beta^1_0 + \beta^1_x \cdot x_i + \beta^1_t \cdot Period_{i,j} + \epsilon^1_{i,j} \\ y^2_{i,j} = \beta^2_0 + \beta^2_x \cdot x_i + \beta^2_t \cdot Period_{i,j} + \epsilon^2_{i,j} \end{cases}$$

with

$$\begin{bmatrix} \epsilon^1_{i,1} \\ \epsilon^1_{i,2} \\ \epsilon^2_{i,1} \\ \epsilon^2_{i,2} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma^2_{1,1} & ... & ... & ... \\ ... & \sigma^2_{1,2} & ... & ... \\ ... & ... & \sigma^2_{2,1} & ... \\ ... & ... & ... & \sigma^2_{2,2} \end{pmatrix} \right]$$

Model 2A

Random Intercept only

$i = 1, ..., N; j = 1, .., n$

$$\begin{cases} y^1_{i,j} = (\beta^1_0 + u^1_{0,i}) + \beta^1_x \cdot x_i + \beta^1_t \cdot Period_{i,j} + \epsilon^1_{i,j} \\ y^2_{i,j} = (\beta^2_0 + u^2_{0,i}) + \beta^2_x \cdot x_i + \beta^2_t \cdot Period_{i,j} + \epsilon^2_{i,j} \end{cases}$$

with

$$\begin{bmatrix} u^1_{0,i} \\ u^2_{0,i} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma^2_{1,0} & \sigma_{(1,2),0} \\ \sigma_{(2,1),0} & \sigma^2_{2,0} \end{pmatrix} \right]; \qquad \begin{bmatrix} \epsilon^1_i \\ \epsilon^2_i \end{bmatrix} \sim \mathcal{N}_{2j}(\mathbf{0}, \mathbf{I}_{2j})$$

Model 2B

Random Intercept & correlated residuals

$i = 1, ..., N; j = 1, 2$

$$\begin{cases} y^1_{i,j} = (\beta^1_0 + u^1_{0,i}) + \beta^1_x \cdot x_i + \beta^1_t \cdot Period_{i,j} + \epsilon^1_{i,j} \\ y^2_{i,j} = (\beta^2_0 + u^2_{0,i}) + \beta^2_x \cdot x_i + \beta^2_t \cdot Period_{i,j} + \epsilon^2_{i,j} \end{cases}$$

with

$$\begin{bmatrix} u^1_{0,i} \\ u^2_{0,i} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma^2_{1,0} & \sigma_{(1,2),0} \\ \sigma_{(2,1),0} & \sigma^2_{2,0} \end{pmatrix} \right]; \quad \begin{bmatrix} \epsilon^1_{i,1} \\ \epsilon^1_{i,2} \\ \epsilon^2_{i,1} \\ \epsilon^2_{i,2} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma^2_{1,1} & \dots & \dots & \dots \\ \dots & \sigma^2_{1,2} & \dots & \dots \\ \dots & \dots & \sigma^2_{2,1} & \dots \\ \dots & \dots & \dots & \sigma^2_{2,2} \end{pmatrix} \right]$$

Model 2C1

Random Slope & Intercept: Independent

$i = 1, ..., N; j = 1, .., n$

$$\begin{cases} y^1_{i,j} = (\beta^1_0 + u^1_{0,i}) + \beta^1_x \cdot x_i + \left(\beta^1_t + u^1_{t,i}\right) \cdot Period_{i,j} + \epsilon^1_{i,j} \\ y^2_{i,j} = (\beta^2_0 + u^2_{0,i}) + \beta^2_x \cdot x_i + \left(\beta^2_t + u^2_{t,i}\right) \cdot Period_{i,j} + \epsilon^2_{i,j} \end{cases}$$

with

$$\begin{bmatrix} u^1_{0,i} \\ u^2_{0,i} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma^2_{1,0} & \sigma_{(1,2),0} \\ \sigma_{(2,1),0} & \sigma^2_{2,0} \end{pmatrix} \right]; \quad \begin{bmatrix} u^1_{t,i} \\ u^2_{t,i} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma^2_{1,t} & \sigma_{(1,2),t} \\ \sigma_{(2,1),t} & \sigma^2_{2,t} \end{pmatrix} \right]$$

$$\begin{bmatrix} \epsilon^1_i \\ \epsilon^2_i \end{bmatrix} \sim \mathcal{N}_{2j}(\mathbf{0}, \mathbf{I}_{2j})$$

Model 2C2

Random Slope & Intercept: Dependent

$i = 1, ..., N; j = 1, .., n$

$$\begin{cases} y^1_{i,j} = (\beta^1_0 + u^1_{0,i}) + \beta^1_x \cdot x_i + \left(\beta^1_t + u^1_{t,i}\right) \cdot Period_{i,j} + \epsilon^1_{i,j} \\ y^2_{i,j} = (\beta^2_0 + u^2_{0,i}) + \beta^2_x \cdot x_i + \left(\beta^2_t + u^2_{t,i}\right) \cdot Period_{i,j} + \epsilon^2_{i,j} \end{cases}$$

with

$$\begin{bmatrix} u^1_{0,i} \\ u^2_{0,i} \\ u^1_{t,i} \\ u^2_{t,i} \end{bmatrix} \sim \mathcal{N}_4 \left[\mathbf{0}, \begin{pmatrix} \sigma^2_{1,0} & \dots & \dots & \dots \\ \dots & \sigma^2_{1,t} & \dots & \dots \\ \dots & \dots & \sigma^2_{2,0} & \dots \\ \dots & \dots & \dots & \sigma^2_{2,2} \end{pmatrix} \right]; \qquad \begin{bmatrix} \epsilon^1_i \\ \epsilon^2_i \end{bmatrix} \sim \mathcal{N}_{2j}(\mathbf{0}, \mathbf{I}_{2j})$$

Model 3A1

Y_1 scaled entirely with independent Slope & Intercept

$i = 1, ..., N; j = 1, .., n$

$$\begin{cases} m_{ij} = (\beta_0^1 + u_{0,i}^1) + \beta_x^1 \cdot x_i + \left(\beta_t^1 + u_{t,i}^1\right) \cdot Period_{i,j} \\ y_{i,j}^1 = m_{ij} + \epsilon_{i,j}^1 \\ y_{i,j}^2 = \gamma \cdot m_{ij} + (\beta_0^2 + u_{0,i}^2) + \beta_x^2 \cdot x_i + \left(\beta_t^2 + u_{t,i}^2\right) \cdot Period_{i,j} + \epsilon_{i,j}^2 \end{cases}$$

with

$$u_{0,i}^1 \sim \mathcal{N}(0, \sigma_{1,0}^2); \quad u_{t,i}^1 \sim \mathcal{N}(0, \sigma_{1,t}^2); \quad \begin{bmatrix} u_{0,i}^2 \\ u_{t,i}^2 \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma_{2,0}^2 & \sigma_{2,(0,t)} \\ \sigma_{2,(t,0)} & \sigma_{2,t}^2 \end{pmatrix} \right];$$

$$\begin{bmatrix} \epsilon_i^1 \\ \epsilon_i^2 \end{bmatrix} \sim \mathcal{N}_{2j}(\mathbf{0}, \mathbf{I}_{2j})$$

Model 3B1

Random Y_1-effects scaled independently; independent Slope & Intercept

$i = 1, ..., N; j = 1, .., n$

$$\begin{cases} y^1_{i,j} = (\beta^1_0 + u^1_{0,i}) + \beta^1_x \cdot x_i + \left(\beta^1_t + u^1_{t,i}\right) \cdot Period_{i,j} + \epsilon^1_{i,j} \\ y^2_{i,j} = \gamma_1 \cdot u^1_{0,i} + \gamma_2 \cdot u^1_{t,i} + (\beta^2_0 + u^2_{0,i}) + \beta^2_x \cdot x_i + \left(\beta^2_t + u^2_{t,i}\right) \cdot Period_{i,j} + \epsilon^2_{i,j} \end{cases}$$

with

$$u^1_{0,i} \sim \mathcal{N}(0, \sigma^2_{1,0}); \quad u^1_{t,i} \sim \mathcal{N}(0, \sigma^2_{1,t}); \quad \begin{bmatrix} u^2_{0,i} \\ u^2_{t,i} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma^2_{2,0} & \sigma_{2,(0,t)} \\ \sigma_{2,(t,0)} & \sigma^2_{2,t} \end{pmatrix} \right];$$

$$\begin{bmatrix} \epsilon^1_i \\ \epsilon^2_i \end{bmatrix} \sim \mathcal{N}_{2j}(\mathbf{0}, \mathbf{I}_{2j})$$

Marginal Assessment methods in INLA

- Within INLA the following Model assessment criteria exist:
  - MLIK (Marginal Likelihood)
  - DIC (Deviance Information Criterion)
  - WAIC (Watanabe-Akaike Information Criterion)
  - CPO (Conditional Predictive Ordinates)
  - PIT (Predictive Integral Transform)

Marginal Likelihood $\pi(y)$

- Probability of observed data under given model
- In INLA approximated as:

$$\widetilde{\pi}(y) = \int \frac{\pi(\theta, x, y)}{\widetilde{\pi}_G(x|\theta, y)}\Big|_{x=x^*(\theta)} d\theta$$

- When considering set of M models $\{\mathcal{M}_m\}_{m=1}^{M}$, the marginal likelihoods are $\pi(y|\mathcal{M}_m)$.
- Posterior can be computed via model priors: $\pi(\mathcal{M}_m|y) \propto \pi(y|\mathcal{M}_m)\pi(\mathcal{M}_m)$
- Can be used to compute Bayes factor for models $\mathcal{M}_1$ and $\mathcal{M}_2$:

$$\frac{\pi(\mathcal{M}_1|y)}{\pi(\mathcal{M}_2|y)} = \frac{\pi(y|\mathcal{M}_1)\pi(\mathcal{M}_1)}{\pi(y|\mathcal{M}_2)\pi(\mathcal{M}_2)}$$

DIC & WAIC

- DIC
  - Given by:

$$DIC = D(\hat{x}, \hat{\theta}) + 2p_D$$

  - Takes into account goodness of fit $(D(\hat{x}, \hat{\theta}))$ and penalty for number of parameters $(2p_D)$.
  - D is the deviance with $\hat{x}$ the posterior mean and $\hat{\theta}$ the posterior mode (might be skewed).
  - $p_D = \mathbb{E}[D(x, \theta)] - D(\hat{x}, \hat{\theta})$
- WAIC is similar to DIC but p_D is calculated differently

CPO (Conditional Predictive Ordinates)

- Computed for each observation as:

$$CPO_i = \pi(y_i|y_{-i})$$

- Posterior probability of observing y_i when model is fit without y_i.
- Low value may indicate outlier
- Summarized over all data as:

$$CPO = -\sum_{i=1}^{N} ln(CPO_i)$$

PIT (Predictive Integral Transform)

- Computed for each observation as:

$$PIT_i = \pi(y_i^{new} \leq y_i | y_{-i})$$

- Measures probability for new observation y_i^{new} to be lower than y_i given all observations except for y_i.
- For a good model the PIT's should be approximately uniformly distributed on $[0, 1]$

Simulations

- Data was generated from each of the 8(6) models. In total 8(6) datasets were generated.
- Each dataset was fit by every model 8(6) (only in INLA). The coefficients and Model Assessment criteria were recorded.
- This was done 2 times:
  1. $N = 750$, $n = 2$. All models participated
  2. $N = 500$, $n = 4$. Models 1A and 2B did not participate,
     - ⋆ They model association via an unstructured variance-covariance matrix for the residual errors, which can only be modelled in INLA for $n = 2$.

Simulation 2

$N = 500$, $n = 4$
Models 0, 2A, 2C1, 2C2, 3A1, 3B1

Simulation 2: Models 0 & 2A

$Model_0$Model_0_df

	true	Model_0	Model_2A	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	2.129866	2.127995	2.129451	2.129671	2.113251	2.129382
beta_x^1	4.0	4.015218	3.986687	4.008675	4.012158	4.008108	4.007623
beta_t^1	2.5	2.599241	2.600256	2.599472	2.599348	2.594592	2.599510
beta_0^2	3.0	3.112763	3.119732	3.111930	3.112714	3.110197	3.112676
beta_x^2	1.5	1.520176	1.629831	1.507061	1.519394	1.515059	1.518827
beta_t^2	3.5	3.684672	3.680893	3.685136	3.684700	3.681410	3.684719
MLIK	NA	-8381.279613	-10622.260731	-8456.818098	-8421.715350	-8462.436164	-8442.100242
DIC	NA	12865.935243	19833.113670	12862.875267	12878.800789	12859.426281	12865.640575
WAIC	NA	12843.868420	19877.082468	12803.035691	12856.227218	12817.404958	12823.729900
CPO	NA	6773.535911	9977.461558	6828.089741	6773.182060	6799.374018	6802.389827

$Model_2A$Model_2A_df

	true	Model_0	Model_2A	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	2.097733	2.097733	2.097734	2.097732	2.090256	2.097732
beta_x^1	4.0	4.002521	4.000894	4.001353	4.000890	4.000405	4.000966
beta_t^1	2.5	2.480511	2.480513	2.480512	2.480513	2.480947	2.480513
beta_0^2	3.0	2.954926	2.954938	2.954935	2.954935	2.952751	2.954934
beta_x^2	1.5	1.453171	1.456217	1.455051	1.455108	1.440928	1.454644
beta_t^2	3.5	3.474924	3.474920	3.474921	3.474922	3.468346	3.474922
MLIK	NA	-6897.256372	-6847.404247	-6871.094009	-6879.428477	-6903.710711	-6877.569841
DIC	NA	12113.799456	12143.023504	12112.394116	12108.052727	12123.662912	12122.342153
WAIC	NA	12147.357605	12184.018332	12146.199373	12145.191976	12159.782075	12159.436708
CPO	NA	6156.133611	6131.611256	6147.268539	6146.035832	6140.935794	6140.782805

Simulation 2: Models 2C1 & 2C2

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$Model_2C1$Model_2C1_df
          true      Model_0      Model_2A     Model_2C1     Model_2C2     Model_3A1     Model_3B1
beta_0^1   2.0     2.097733      2.097618      2.097731      2.097731      2.089620      2.097734
beta_x^1   4.0     4.002780      3.980313      4.000695      4.000915      4.001743      4.001821
beta_t^1   2.5     2.399231      2.399276      2.399225      2.399224      2.391553      2.399231
beta_0^2   3.0     2.954909      2.954914      2.954923      2.954922      2.950497      2.954916
beta_x^2   1.5     1.448278      1.472289      1.451368      1.451349      1.437566      1.449916
beta_t^2   3.5     3.323889      3.323887      3.323878      3.323878      3.317111      3.323887
MLIK        NA -8406.869297 -10450.343520  -8349.866095  -8393.631079  -8450.716190  -8418.578527
DIC         NA 12922.015663  19776.053749  12895.802346  12909.911187  12918.496001  12904.086970
WAIC        NA 12871.429198  19818.415774  12848.019719  12867.785330  12859.350676  12853.248810
CPO         NA  6843.523154   9941.981285   6819.326053   6821.665613   6838.080212   6830.165684
$Model_2C2$Model_2C2_df
          true      Model_0      Model_2A     Model_2C1     Model_2C2     Model_3A1     Model_3B1
beta_0^1   2.0     2.097729      2.097649      2.097733      2.097724      2.088923      2.097735
beta_x^1   4.0     4.003316      3.996762      4.001333      4.002722      4.001716      4.002305
beta_t^1   2.5     2.457149      2.457178      2.457145      2.457143      2.449463      2.457155
beta_0^2   3.0     2.954909      2.954935      2.954923      2.954924      2.942093      2.954916
beta_x^2   1.5     1.448775      1.481918      1.451457      1.451896      1.420161      1.450891
beta_t^2   3.5     3.370851      3.370839      3.370842      3.370836      3.352603      3.370846
MLIK        NA -8378.421490 -10554.654633  -8373.751226  -8345.385805  -8460.479877  -8435.561154
DIC         NA 12848.046465  19806.962333  12828.597370  12830.850387  12838.283823  12831.525107
WAIC        NA 12808.011012  19850.700034  12766.348677  12798.189518  12771.364590  12772.385227
CPO         NA  6764.142319   9961.687452   6783.885676   6731.922986   6784.445274   6780.338331
```

Simulation 2: Models 3A1 & 3B1

$Model_3A1$Model_3A1_df

	true	Model_0	Model_2A	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	2.097735	2.097606	2.097727	2.097727	2.089939	2.097729
beta_x^1	4.0	4.003397	3.980725	4.002853	4.003129	4.002232	4.003454
beta_t^1	2.5	2.441049	2.441096	2.441041	2.441040	2.433857	2.441050
beta_0^2	3.0	2.982998	2.982961	2.982995	2.982988	2.979523	2.982987
beta_x^2	1.5	1.455043	1.476305	1.452601	1.452923	1.453170	1.452393
beta_t^2	3.5	3.306117	3.306132	3.306121	3.306113	3.304296	3.306120
MLIK	NA	-8485.784612	-10695.676210	-8455.652933	-8470.112374	-8525.897037	-8480.428140
DIC	NA	12915.629262	20175.325401	12874.087816	12893.804790	12904.742217	12887.659605
WAIC	NA	12866.076356	20218.309156	12815.501148	12852.513669	12855.291505	12840.831961
CPO	NA	6832.886151	10143.175916	6817.070009	6798.283022	6831.933536	6803.806640

$Model_3B1$Model_3B1_df

	true	Model_0	Model_2A	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	2.097735	2.097585	2.097724	2.097751	2.089780	2.097729
beta_x^1	4.0	4.003409	3.983855	4.001807	4.014631	4.002131	4.003408
beta_t^1	2.5	2.441049	2.441091	2.441027	2.441008	2.434081	2.441052
beta_0^2	3.0	2.982996	2.982820	2.982996	2.982976	2.979989	2.982981
beta_x^2	1.5	1.454259	1.456724	1.452645	1.451898	1.447676	1.451559
beta_t^2	3.5	3.276508	3.276577	3.276500	3.276479	3.275273	3.276509
MLIK	NA	-8637.959223	-11115.931909	-8471.794750	-8526.312867	-8678.940036	-8629.961380
DIC	NA	12933.011639	21187.604996	12861.907384	13414.745128	12926.460545	12896.425117
WAIC	NA	12880.063870	21227.207876	12804.146016	13366.191114	12870.271504	12848.227873
CPO	NA	6856.409569	10639.158957	6801.257805	6954.081126	6857.957757	6816.796811

Simulation 1

$N = 750$, $n = 2$
Models 0, 1A, 2A, 2B, 2C1, 2C2, 3A1, 3B1

Simulation 1: Models 0 & 1A

$Model_0$Model_0_df

	true	Model_0	Model_1A	Model_2A	Model_2B	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	2.062555	2.062551	2.062406	2.062524	2.062554	2.062547	2.058499	2.062551
beta_x^1	4.0	3.994878	3.995314	4.013629	3.998129	3.994915	3.995740	3.994106	3.995384
beta_t^1	2.5	2.529890	2.529896	2.530182	2.529940	2.529890	2.529903	2.522164	2.529898
beta_0^2	3.0	3.138011	3.138012	3.138134	3.138012	3.138044	3.138011	3.138730	3.138004
beta_x^2	1.5	1.481725	1.481527	1.466400	1.481810	1.477839	1.481640	1.484109	1.482418
beta_t^2	3.5	3.529583	3.529580	3.529364	3.529584	3.529526	3.529581	3.530829	3.529594
MLIK	NA	-6473.577169	-6517.192965	-6758.467139	-6540.734366	-6468.938935	$\boxed{-6423.443748}$	-6483.441570	-6493.339563
DIC	NA	11061.126033	-33486.369224	12434.664974	-33486.369217	-18794.330685	11148.574887	-13208.705298	-2850.634205
WAIC	NA	10978.342196	-34406.926848	12459.289241	-34406.926809	-18120.952881	11053.801535	-14225.961044	-3013.651656
CPO	NA	5962.388187	-14505.925676	6361.865738	-14505.925265	-6297.703708	5961.902173	-4414.884958	107.657495

$Model_1A$Model_1A_df

	true	Model_0	Model_1A	Model_2A	Model_2B	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	1.947935	1.947892	1.947931	1.947934	1.947919	1.947934	1.940167	1.947929
beta_x^1	4.0	4.032881	4.038846	4.035294	4.033802	4.038607	4.033792	4.034340	4.035810
beta_t^1	2.5	2.545497	2.545544	2.545502	2.545506	2.545513	2.545506	2.543746	2.545504
beta_0^2	3.0	3.041790	3.041723	3.041807	3.041801	3.041817	3.041804	3.046431	3.041805
beta_x^2	1.5	1.519116	1.535394	1.514392	1.515780	1.510932	1.514896	1.520036	1.515016
beta_t^2	3.5	3.449960	3.450000	3.449947	3.449956	3.449938	3.449953	3.453001	3.449949
MLIK	NA	-6413.163102	-7395.154646	-6379.644607	$\boxed{-6236.679591}$	-6374.434150	-6311.517456	-6420.020553	-6398.109654
DIC	NA	12276.846445	-33486.368549	12307.525312	-33486.368669	12122.223687	8850.825051	12271.401998	12286.438357
WAIC	NA	12308.244717	-34406.925124	12333.652038	-34406.926257	12133.540017	8346.635439	12307.466984	12315.622331
CPO	NA	6247.565414	-14505.919316	6202.012515	-14505.923884	6153.812560	6009.408817	6197.178540	6202.414744

Simulation 1: Models 2A & 2B

$Model_2A$Model_2A_df

	true	Model_0	Model_1A	Model_2A	Model_2B	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	1.955696	1.955707	1.955709	1.955706	1.955706	1.955708	1.950829	1.955707
beta_x^1	4.0	4.020166	4.017974	4.017217	4.017997	4.017823	4.017529	4.016688	4.017643
beta_t^1	2.5	2.531249	2.531243	2.531240	2.531243	2.531242	2.531241	2.530182	2.531242
beta_0^2	3.0	2.980848	2.980857	2.980860	2.980857	2.980860	2.980859	2.967714	2.980859
beta_x^2	1.5	1.519238	1.517028	1.516225	1.516982	1.516246	1.516328	1.486431	1.516397
beta_t^2	3.5	3.475404	3.475399	3.475396	3.475398	3.475396	3.475396	3.456185	3.475396
MLIK	NA	-5623.561814	-5619.900372	-5587.065983	-5480.472438	-5590.733690	-5603.848759	-5633.463099	-5605.276716
DIC	NA	9620.788523	-33486.369651	9784.697084	-33486.369667	9654.370843	9731.584817	9705.418215	9684.344051
WAIC	NA	9637.660147	-34406.927145	9802.080993	-34406.927165	9661.558879	9736.742625	9690.361150	9687.131617
CPO	NA	5083.410842	-14505.927214	5058.243902	-14505.927262	5061.136981	5061.755725	5060.561054	5060.202948

$Model_2B$Model_2B_df

	true	Model_0	Model_1A	Model_2A	Model_2B	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	1.926926	1.926926	1.926934	1.926926	1.926917	1.926926	1.919334	1.926931
beta_x^1	4.0	4.034246	4.035865	4.033621	4.035664	4.038539	4.035717	4.031976	4.034335
beta_t^1	2.5	2.545501	2.545511	2.545497	2.545511	2.545515	2.545511	2.540654	2.545500
beta_0^2	3.0	3.006339	3.006346	3.006360	3.006346	3.006361	3.006348	3.000786	3.006358
beta_x^2	1.5	1.514451	1.513085	1.509271	1.513016	1.508440	1.512585	1.492554	1.510016
beta_t^2	3.5	3.449947	3.449948	3.449933	3.449948	3.449932	3.449947	3.438864	3.449935
MLIK	NA	-6849.963356	-6811.244393	-6819.210427	-6684.695111	-6806.489301	-6771.620631	-6864.370687	-6840.018325
DIC	NA	12610.671284	-33486.368685	12681.946793	-33486.368694	12285.088363	11899.246840	12648.105376	12652.690583
WAIC	NA	12637.450311	-34406.926271	12710.735544	-34406.926280	12257.484441	11839.618817	12680.988466	12676.876908
CPO	NA	6487.543230	-14505.924018	6463.083064	-14505.923997	6381.703660	6307.621430	6465.126718	6464.026157

Simulation 1: Models 2C1 & 2C2

$Model_2C1$Model_2C1_df

	true	Model_0	Model_1A	Model_2A	Model_2B	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	1.955868	1.955875	1.955943	1.955877	1.955878	1.955873	1.951153	1.955865
beta_x^1	4.0	3.966042	3.964373	3.942856	3.963773	3.963621	3.965017	3.943641	3.967733
beta_t^1	2.5	2.607244	2.607234	2.607184	2.607232	2.607232	2.607236	2.603580	2.607248
beta_0^2	3.0	2.980952	2.980922	2.980967	2.980924	2.980929	2.980924	2.981361	2.980937
beta_x^2	1.5	1.486335	1.496275	1.482183	1.495729	1.494285	1.495670	1.483777	1.491427
beta_t^2	3.5	3.415953	3.415976	3.415949	3.415974	3.415970	3.415974	3.414693	3.415967
MLIK	NA	-6436.800701	-6434.293272	-6543.466389	-6419.819028	-6398.947179	-6388.958380	-6497.078009	-6449.312907
DIC	NA	11880.936110	-33486.368935	12445.207333	-33486.368935	10758.427525	11690.009414	12009.445033	11443.256350
WAIC	NA	11845.161995	-34406.926554	12479.318102	-34406.926546	10576.818072	11650.886131	11935.768859	11340.267668
CPO	NA	6185.831802	-14505.925115	6314.207829	-14505.925051	6092.884596	6084.854690	6195.681026	6157.586279

$Model_2C2$Model_2C2_df

	true	Model_0	Model_1A	Model_2A	Model_2B	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	1.955796	1.955816	1.955897	1.955812	1.955814	1.955818	1.949278	1.955802
beta_x^1	4.0	3.987327	3.980462	3.956155	3.981061	3.982928	3.979929	3.983381	3.986389
beta_t^1	2.5	2.581123	2.581100	2.581044	2.581098	2.581106	2.581099	2.573173	2.581121
beta_0^2	3.0	2.980923	2.980898	2.980930	2.980877	2.980903	2.980902	2.980561	2.980919
beta_x^2	1.5	1.495044	1.503940	1.493211	1.506495	1.501500	1.502642	1.488733	1.496805
beta_t^2	3.5	3.382313	3.382331	3.382317	3.382327	3.382325	3.382327	3.379775	3.382317
MLIK	NA	-6471.173412	-6448.167363	-6679.595361	-6452.594694	-6439.453617	-6323.306268	-6503.568057	-6478.627661
DIC	NA	11470.117769	-33486.369206	12501.519073	-33486.369199	6396.364226	11217.180452	-1628.298306	-1861.511312
WAIC	NA	11407.622734	-34406.926862	12531.072933	-34406.926745	6439.324195	11176.406842	-2606.110799	-2243.681927
CPO	NA	6097.650133	-14505.925416	6370.785045	-14505.925079	5958.275713	5969.357302	212.923448	418.585437

Simulation 1: Models 3A1 & 3B1

$Model_3A1$Model_3A1_df

	true	Model_0	Model_1A	Model_2A	Model_2B	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	1.955716	1.955703	1.955722	NA	1.955702	1.955694	1.951377	1.955701
beta_x^1	4.0	4.013707	4.016508	4.010417	NA	4.016969	4.019107	4.021408	4.017387
beta_t^1	2.5	2.480880	2.480884	2.480875	NA	2.480885	2.480892	2.475474	2.480891
beta_0^2	3.0	3.083040	3.083038	3.083150	NA	3.083050	3.083031	3.075170	3.083028
beta_x^2	1.5	1.506024	1.506122	1.470687	NA	1.502925	1.508065	1.498191	1.509141
beta_t^2	3.5	3.393346	3.393342	3.393263	NA	3.393335	3.393349	3.385820	3.393355
MLIK	NA	-6545.895839	-6548.561624	-6705.595984	NA	-6505.596862	-6541.381528	-6579.721299	-6547.787680
DIC	NA	11698.771557	-33486.369077	12577.732176	NA	-3355.980072	11673.875652	11355.651221	11267.087081
WAIC	NA	11630.697743	-34406.926731	12608.341609	NA	-3304.003118	11617.742379	11246.386888	11132.139616
CPO	NA	6167.753134	-14505.925633	6395.637957	NA	48.978159	6126.917631	6177.789901	6152.630117

$Model_3B1$Model_3B1_df

	true	Model_0	Model_1A	Model_2A	Model_2B	Model_2C1	Model_2C2	Model_3A1	Model_3B1
beta_0^1	2.0	1.955716	1.956119	1.955716	1.955698	1.955696	1.955694	1.950879	1.955699
beta_x^1	4.0	4.013723	3.897429	4.011802	4.017805	4.018933	4.019703	4.022540	4.017945
beta_t^1	2.5	2.480880	2.480469	2.480879	2.480878	2.480881	2.480883	2.474315	2.480892
beta_0^2	3.0	3.083033	3.083536	3.083178	3.083046	3.083057	3.083044	3.079319	3.083025
beta_x^2	1.5	1.508197	1.365191	1.461668	1.503214	1.500601	1.504111	1.508898	1.510566
beta_t^2	3.5	3.355581	3.355086	3.355474	3.355557	3.355552	3.355560	3.354027	3.355590
MLIK	NA	-6703.762707	-7332.338520	-6876.675075	-6603.582489	-6576.167425	-6558.040635	-6735.277017	-6698.642638
DIC	NA	12054.506284	-33486.368881	13085.770921	-33486.368919	-3203.768879	11176.560685	11762.491301	11656.606677
WAIC	NA	11973.207654	-34406.925965	13118.930882	-34406.926552	-3444.098004	11046.685107	11637.582631	11493.347321
CPO	NA	6390.051954	-14505.922939	6621.167449	-14505.924678	-122.309984	6115.235178	6411.125399	6374.150934

To-do

- Complete model configuration in INLA
  - Complete model 3: dependent copied random effects
  - Model 1: Inspect other residual error covariance structures
- Theoretical results
  - Give Likelihood for every model
  - Try to write them as LGM
- Implement models on Dataset
  - Open dataset
    - Back to PBC?
  - Duchenne
  - COVID

Simulated Data

- To better test the models simulated data was used for every setting

Model 1: Simulated Data

Correlated errors

i=1, ..., 1000; j=1,2

$$\begin{cases} y^1_{i,j} = 2 + 3 \cdot x_i + \epsilon^1_{i,j} \\ y^2_{i,j} = 4 + 2 \cdot x_i + \epsilon^2_{i,j} \end{cases}$$

with

$$\begin{bmatrix} \epsilon^1_{i,1} \\ \epsilon^1_{i,2} \\ \epsilon^2_{i,1} \\ \epsilon^2_{i,2} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} 5.16 & 2.06 & -0.93 & -0.39 \\ 2.06 & 4.22 & -0.48 & -1.42 \\ -0.93 & -0.48 & 4.01 & 1.04 \\ -0.39 & -1.42 & 1.04 & 3.86 \end{pmatrix} \right]$$

Model 1: Correlated Errors

```
          true  gls MCMCglmm INLA
beta_0^1     2 2.06     2.05 2.06
beta_x^1     3 2.98     2.99 2.98
beta_0^2     4 4.02     4.02 4.02
beta_x^2     2 1.95     1.96 1.95
```

```
$INLA
      [,1]  [,2]  [,3]  [,4]
[1,]  5.11  2.04 -0.92 -0.37
[2,]  2.04  4.18 -0.47 -1.41
[3,] -0.92 -0.47  3.96  1.03
[4,] -0.37 -1.41  1.03  3.82
```

```
$MCMCglmm
      [,1]  [,2]  [,3]  [,4]
[1,]  5.20  2.23 -0.92 -2.30
[2,]  2.23  4.23  0.02 -1.42
[3,] -0.92  0.02  4.03 -1.74
[4,] -2.30 -1.42 -1.74  3.85
```

```
$true
      [,1]  [,2]  [,3]  [,4]
[1,]  5.16  2.06 -0.93 -0.39
[2,]  2.06  4.22 -0.48 -1.42
[3,] -0.93 -0.48  4.01  1.04
[4,] -0.39 -1.42  1.04  3.86
```

```
$gls
Marginal variance covariance matrix
      [,1]  [,2]  [,3]  [,4]
[1,]  5.17  2.06 -0.93 -0.39
[2,]  2.06  4.23 -0.48 -1.42
[3,] -0.93 -0.48  4.00  1.04
[4,] -0.39 -1.42  1.04  3.86
```

Model 2: Simulated Data

Only random intercept

i=1, ..., 1000; j=1,..., 3

$$\begin{cases} y_{i,j}^1 = 3 + 2 \cdot x_i + \mathbf{u}_i^1 + \epsilon_{i,j}^1 \\ y_{i,j}^2 = 4 + 3 \cdot x_i + \mathbf{u}_i^2 + \epsilon_{i,j}^2 \end{cases}$$

with

$$\begin{bmatrix} \mathbf{u}_i^1 \\ \mathbf{u}_i^2 \end{bmatrix} \sim \mathcal{N}\left[\mathbf{0}, \begin{pmatrix} 1.98 & -0.9 \\ -0.9 & 2.93 \end{pmatrix}\right]; \quad \begin{bmatrix} \epsilon_i^1 \\ \epsilon_i^2 \end{bmatrix} \sim \mathcal{N}_6(\mathbf{0}, \mathbf{I}_6)$$

Model 2: Only random intercept

	true	lmer	nlme	INLA	MCMCglmm
β_0^1	3.00	3.00	3.00	3.00	3.00
β_x^1	2.00	2.00	2.00	2.00	2.02
β_0^2	4.00	3.97	3.97	3.97	3.97
β_x^2	3.00	3.05	3.05	3.05	3.07
u_1	1.98	1.98	1.98	1.99	1.76
u_2	2.93	2.96	2.96	2.94	2.84
u_{12}	-0.90	-0.90	-0.90	-0.90	-0.88

Model 2: Simulated Data

Random intercept & Random Errors

i=1, ..., 1000; j=1,2

$$\begin{cases} y^1_{i,j} = 3 + 2 \cdot x_i + \mathbf{u}^1_i + \epsilon^1_{i,j} \\ y^2_{i,j} = 4 + 3 \cdot x_i + \mathbf{u}^2_i + \epsilon^2_{i,j} \end{cases}$$

with

$$\begin{bmatrix} \mathbf{u}^1_i \\ \mathbf{u}^2_i \end{bmatrix} \sim \mathcal{N}\left[\mathbf{0}, \begin{pmatrix} 1.98 & -0.9 \\ -0.9 & 2.93 \end{pmatrix}\right]; \quad \begin{bmatrix} \epsilon^1_{i,1} \\ \epsilon^1_{i,2} \\ \epsilon^2_{i,1} \\ \epsilon^2_{i,2} \end{bmatrix} \sim \mathcal{N}_2\left[\mathbf{0}, \begin{pmatrix} 5.16 & 2.06 & -0.93 & -0.39 \\ 2.06 & 4.22 & -0.48 & -1.42 \\ -0.93 & -0.48 & 4.01 & 1.04 \\ -0.39 & -1.42 & 1.04 & 3.86 \end{pmatrix}\right]$$

Model 2: Random intercept & Random Errors

	true	nlme	INLA
beta_0^1	3.00	3.04	3.04
beta_x^1	2.00	2.06	2.06
beta_0^2	4.00	4.18	4.18
beta_x^2	3.00	3.02	3.02
u_1	2.14	4.06	2.58
u_2	3.23	4.23	1.91
u_12	-1.06	-1.90	1.26

$true

	[,1]	[,2]	[,3]	[,4]
[1,]	4.98	2.06	-1.02	-0.43
[2,]	2.06	4.19	-0.44	-1.46
[3,]	-1.02	-0.44	4.21	0.96
[4,]	-0.43	-1.46	0.96	4.05

$nlme

	1	2	3	4
1	2.90	0.04	-0.20	0.33
2	0.04	2.22	0.50	-0.58
3	-0.20	0.50	3.04	-0.16
4	0.33	-0.58	-0.16	2.99

$INLA

	[,1]	[,2]	[,3]	[,4]
[1,]	4.28	1.49	-3.41	-2.85
[2,]	1.49	3.67	-2.72	-3.69
[3,]	-3.41	-2.72	5.42	2.28
[4,]	-2.85	-3.69	2.28	5.32

Model 2: Simulated Data

Random intercept & Random Slope

i=1, ..., 1000; j=1,2

$$\begin{cases} y^1_{i,j} = (\beta^1_0 + u^1_{0,i}) + \beta^1_x \cdot x_i + \left(\beta^1_t + u^1_{t,i}\right) \cdot Period_{i,j} + \epsilon^1_{i,j} \\ y^2_{i,j} = (\beta^2_0 + u^2_{0,i}) + \beta^2_x \cdot x_i + \left(\beta^2_t + u^2_{t,i}\right) \cdot Period_{i,j} + \epsilon^2_{i,j} \end{cases}$$

with

$$\begin{bmatrix} u^1_{0,i} \\ u^2_{0,i} \\ u^1_{t,i} \\ u^2_{t,i} \end{bmatrix} \sim \mathcal{N}_{2j}\left[\mathbf{0}, \begin{pmatrix} \sigma^2_{0,1} & \sigma_{0,(1,2)} & 0 & 0 \\ \sigma_{0,(2,1)} & \sigma^2_{0,2} & 0 & 0 \\ 0 & 0 & \sigma^2_{t,1} & \sigma_{t,(1,2)} \\ 0 & 0 & \sigma_{t,(2,1)} & \sigma^2_{t,2} \end{pmatrix}\right]; \quad \begin{bmatrix} \epsilon^1_i \\ \epsilon^2_i \end{bmatrix} \sim \mathcal{N}_{2j}(\mathbf{0}, \mathbf{I}_{2j})$$

Pain dataset

- Dataset given as example in Weiss [4].
  - Measurements on pain tolerance and pain rating on 63 children
  - 4 measurements per child
  - 2 outcomes: Pain tolerance & Pain rating
  - 3 covariates: Treatment, sex & ses (socio-economic status)

	id	ses	sex	treatment	pain_tolerance	pain_rating
2.1	2	74.70	female	distract	3.34	6
2.2	2	74.70	female	distract	3.19	5
2.3	2	74.70	female	distract	2.76	4
2.4	2	74.70	female	distract	3.01	6
3.1	3	81.50	female	none	2.48	7
3.2	3	81.50	female	none	2.30	2

Method 2: Joint Mixed Model

Joint Mixed Model

$$\begin{cases} y_i(t_{ij}) = \mathbf{v}_{yi}^{\top}(t_{ij})\beta_{\mathbf{y}} + \mathbf{z}_{yi}^{\top}(t_{ij})\mathbf{b}_{yi} + \epsilon_{yi}(t_{ij}) \\ x_i(t_{ij}) = \mathbf{v}_{xi}^{\top}(t_{ij})\beta_{\mathbf{x}} + \mathbf{z}_{xi}^{\top}(t_{ij})\mathbf{b}_{xi} + \epsilon_{xi}(t_{ij}) \end{cases}$$

with

$$\begin{bmatrix} \mathbf{b}_{yi} \\ \mathbf{b}_{xi} \end{bmatrix} \sim \mathcal{N}(\mathbf{0}, \mathbf{D}); \quad \begin{bmatrix} \epsilon_{yi} \\ \epsilon_{xi} \end{bmatrix} \sim \mathcal{N}_{n_i}(\mathbf{0}, \Sigma_i)$$

$$\epsilon_{yi}(t_{ij}) \perp\!\!\!\perp \mathbf{b}_{yi}, \epsilon_{xi}(t_{ij}) \perp\!\!\!\perp \mathbf{b}_{xi}$$

- Association measured via $\mathbf{D}$ and Σ_i.

Model 2 on Pain Data

Random Intercept Model

$$\begin{cases} Tolerance_i(t_{ij}) = \beta_0^t + \beta_1^t \cdot sex_i + \beta_2^t \cdot ses_i + b_{i0}^t + \epsilon_{ij}^t \\ Rating_i(t_{ij}) = \beta_0^r + \beta_1^r \cdot sex_i + \beta_2^r \cdot ses_i + b_{i0}^r + \epsilon_{ij}^r \end{cases}$$

with

$$\begin{bmatrix} \mathbf{b}_{yi} \\ \mathbf{b}_{xi} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \begin{pmatrix} \sigma_y^2 & \sigma_{y,x} \\ \sigma_{y,x} & \sigma_x^2 \end{pmatrix} \right]; \quad \begin{bmatrix} \epsilon_{yi} \\ \epsilon_{xi} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \mathbf{I}_2 \right]$$

$$\epsilon_{yi}(t_{ij}) \perp\!\!\!\perp \mathbf{b}_{yi}, \epsilon_{xi}(t_{ij}) \perp\!\!\!\perp \mathbf{b}_{xi}$$

Model 2 on Pain Data: LMER+NLME

```
model_lmer<-lmer(value~ variable+variable:sex+variable:ses-1+
            (variable-1|id), na.action=na.omit,
             data=data_long_melted)

model_lme<-lme(value~variable+variable:sex+variable:ses-1,
           random = ~0+variable | id, data=data_long_melted,
            na.action=na.omit)
```

Model 2 on Pain Data: MCMCglmm

```
prior = list(R = list(V = diag(2), n = 4),
             G = list(G1=list(V = matrix(c(10,-0.50,-0.50, 0.45),
             nrow=2), n = 1)))

m_pain<-MCMCglmm(cbind(pain_tolerance, pain_rating) ~
        trait+trait:sex+trait:ses - 1,
        random = ~ us(trait):units, rcov = ~ idh(trait):units,
        family = rep("gaussian", 2),prior=prior, nitt = 10000,
        burnin = 1000, thin=25, data = data_long)
```

Model 2 on Pain Data: INLA

- One can indeed calculate transformation of hyperparameter
- I did not understand the prior specification properly

```
formula.model_2=Y~-1+f(Intercept1, model='linear')+
  f(Intercept2, model='linear')+f(ses1, model='linear')+
  f(ses2, model='linear')+  f(sex1, model='linear')+
  f(sex2, model='linear')+
  f(ID1, model="iid2d", n=2*212, hyper = list(theta1 =
  list(prior = "wishart2d", param =c(4,0.1,2.0, 0))))
final_model_2<-inla(formula.model_2, family =
                    c("gaussian","gaussian"),data = final_data)
```

Model 2 on Pain Data: Results

	lmer	lmer_sd	nlme	nlme_sd	INLA	INLA_sd	MCMC	MCMC_sd
beta_0^t	3.20	0.16	3.20	0.16	3.01	0.32	3.20	0.10
beta_sf^t	0.18	0.20	0.18	0.20	0.18	0.19	0.19	0.13
beta_ses^t	-0.23	0.10	-0.23	0.10	-0.23	0.09	-0.23	0.07
beta_0^r	5.68	0.38	5.68	0.38	6.04	0.63	5.68	0.24
beta_sf^r	-0.37	0.48	-0.37	0.48	-0.37	0.37	-0.37	0.30
beta_ses^r	-0.24	0.23	-0.24	0.23	-0.24	0.18	-0.24	0.15
u_t^2	0.15		0.15		0.39		0.70	
u_r^2	2.50		2.50		2.84		2.15	
rho_rt	-0.43		-0.42		-0.77		-0.21	

Method 2: Sources

- Books
  - G. Fitzmaurice, M. Davidian, G. Verbeke, Longitudinal Data Analysis, 2009, Chapters 13 & 16
- Papers
  - S. Fieuws, G. Verbeke, Pairwise Fitting of Mixed Models for the Joint Modeling of Multivariate Longitudinal Profiles, 2006, Biometrics
  - G. Verbeke, S. Fieuws, The analysis of multivariate longitudinal data: A review, 2012, Stat Methods Med Res

Method 3: Joint Mixed Model with latent variable

Joint Mixed Model with latent variable

$$\begin{cases} x_i(t_{ij}) = m_i(t_{ij}) + \epsilon_{xi}(t_{ij}) \\ y_i(t_{ij}) = \mathbf{w}_{yi}^{\mathsf{T}}\alpha_{\mathbf{y}} + \gamma \cdot m_i(t_{ij}) + \mathbf{v}_{yi}^{\mathsf{T}}(t_{ij})\beta_{\mathbf{y}} + \mathbf{z}_{yi}^{\mathsf{T}}(t_{ij})\mathbf{b}_{yi} + \epsilon_{yi}(t_{ij}) \end{cases}$$

with

$$m_i(t_{ij}) = \mathbf{w}_{xi}^{\mathsf{T}}\alpha_{\mathbf{x}} + \mathbf{v}_{xi}^{\mathsf{T}}(t_{ij})\beta_{\mathbf{x}} + \mathbf{z}_{xi}^{\mathsf{T}}(t_{ij})\mathbf{b}_{xi}$$

and

$$\mathbf{b}_{xi} \sim \mathcal{N}(\mathbf{0}, \mathbf{D}_x), \mathbf{b}_{yi} \sim \mathcal{N}(\mathbf{0}, \mathbf{D}_y)$$

$$\epsilon_{yi}(t_{ij}) \sim \mathcal{N}_{n_i}(\mathbf{0}, \sigma_y^2), \epsilon_{xi}(t_{ij}) \sim \mathcal{N}_{n_i}(\mathbf{0}, \sigma_x^2)$$

$$\epsilon_{yi}(t_{ij}) \perp\!\!\!\perp \mathbf{b}_{yi}, \epsilon_{xi}(t_{ij}) \perp\!\!\!\perp \mathbf{b}_{xi}$$

Model 3 on Pain Data

Random ses + Intercept Model

$$\begin{cases} Tolerance_i(t_{ij}) = \beta_0^t + b_1^t \cdot ses_i + \epsilon_{ij}^t \\ Rating_i(t_{ij}) = \beta_0^r + \gamma_1 \cdot b_1^t \cdot ses_i + \epsilon_{ij}^r \end{cases}$$

Model 3 on Pain Data: LM, LME & INLA

```
model_lm_3<-lm(value~variable+variable:ses -1,
               data=data_long_melted, na.action=na.omit)

model_lme_3<-lme(value~variable -1,random = ~0+ses | variable,
                 data=data_long_melted, na.action=na.omit)

formula_comparison_INLA_2=Y~-1+
    f(Intercept1, model='linear')+f(Intercept2, model='linear')+
    f(sestol, ses, model = "iid")+
    f(sesrat,ses, copy="sestol",hyper = list(
    beta = list(fixed=FALSE)))
```

Model 3 on Pain Data: Results

	lm	nlme	INLA
beta_0^t	3.313	3.313	3.313
beta_0^r	5.448	5.448	5.448
u_ses^t	-0.223	-0.172	-0.180
u_ses^r	-0.241	-0.186	-0.184

- Did not manage to combine multiple parameters with one γ.

Method 3: Sources

- https://groups.google.com/g/r-inla-discussion-group/c/ClNVlx1lgwY
- https://arxiv.org/pdf/1210.0333.pdf

Method 1: Multivariate Normal Model

Multivariate Normal Model

$$\begin{cases} y_i(t_{ij}) = \mathbf{v}_{yi}^{\mathsf{T}}(t_{ij})\beta_{\mathbf{y}} + \epsilon_{yi}(t_{ij}) \\ x_i(t_{ij}) = \mathbf{v}_{xi}^{\mathsf{T}}(t_{ij})\beta_{\mathbf{x}} + \epsilon_{xi}(t_{ij}) \end{cases}$$

with

$$\begin{bmatrix} \epsilon_{yi} \\ \epsilon_{xi} \end{bmatrix} \sim \mathcal{N}_{n_i}(\mathbf{0}, \Sigma_i)$$

- Association measured via Σ_i

Model 1 on Pain Data

Random Intercept Model

$$\begin{cases} Tolerance_i(t_{ij}) = \beta_0^t + \beta_1^t \cdot sex_i + \beta_2^t \cdot ses_i + \epsilon_{ij}^t \\ Rating_i(t_{ij}) = \beta_0^r + \beta_1^r \cdot sex_i + \beta_2^r \cdot ses_i + \epsilon_{ij}^r \end{cases}$$

with

$$\begin{bmatrix} \epsilon_{yi} \\ \epsilon_{xi} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \text{Unstructured}_8 \right]$$

Model 1 on Pain Data: INLA

```
gls_unstructured<-gls(value~variable+variable:ses+variable:sex-1,
                    correlation=corSymm(form=~1|id),
                    weights= varIdent(form=~1|Period),
                    data=data_long_melted, na.action=na.omit)

m_pain<-MCMCglmm(cbind(pain_tolerance, pain_rating) ~
        trait+trait:sex+trait:ses - 1,
        rcov = ~ us(Period:trait):units,
        family = rep("gaussian", 2),prior=prior,
        nitt = 10000, burnin = 1000,
              thin=25, data = data_long)
```

Model 1 on Pain Data: Results

	gls	gls_sd	MCMCglmm	MCMCglmm_sd
beta_0^t	5.25	0.00	5.30	0.03
beta_sf^t	0.82	0.01	0.24	0.01
beta_ses^t	-0.03	0.06	-0.03	0.00
beta_0^r	7.50	0.00	6.70	0.32
beta_sf^r	-0.54	0.07	-0.62	0.05
beta_ses^r	-0.02	0.25	-0.01	0.01

Model 1 Results

Marginal variance covariance matrix

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]
[1,]	1.000000	0.9646700	0.934000	0.880790	-0.146390	0.31915	0.011592	0.0085110
[2,]	0.964670	1.0000000	0.915690	0.897470	-0.084704	0.31812	0.077383	0.0020768
[3,]	0.934000	0.9156900	1.000000	0.923760	-0.183270	0.28127	-0.034945	-0.0365370
[4,]	0.880790	0.8974700	0.923760	1.000000	-0.135120	0.24249	0.020602	-0.0749610
[5,]	-0.146390	-0.0847040	-0.183270	-0.135120	1.000000	0.53617	0.435130	0.2272000
[6,]	0.319150	0.3181200	0.281270	0.242490	0.536170	1.00000	0.373270	0.2487200
[7,]	0.011592	0.0773830	-0.034945	0.020602	0.435130	0.37327	1.000000	0.3341800
[8,]	0.008511	0.0020768	-0.036537	-0.074961	0.227200	0.24872	0.334180	1.0000000

Standard Deviations: 1 1 1 1 1 1 1 1

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]
[1,]	1.0000000	0.7923632234	0.2757604	-0.16859774	-0.32375237	0.3394998084	0.7253773	0.53763412
[2,]	0.7923632	1.0000000000	0.4762324	0.06301609	-0.58650226	-0.0005261689	0.6880497	0.60665062
[3,]	0.2757604	0.4762324384	1.0000000	0.76345228	-0.47261506	0.3155500984	-0.1407778	0.44914344
[4,]	-0.1685977	0.0630160916	0.7634523	1.00000000	-0.01470108	0.1720782933	-0.4947750	-0.06233257
[5,]	-0.3237524	-0.5865022575	-0.4726151	-0.01470108	1.00000000	0.1274419714	-0.3406734	-0.74698798
[6,]	0.3394998	-0.0005261689	0.3155501	0.17207829	0.12744197	1.0000000000	0.1558983	0.16098986
[7,]	0.7253773	0.6880497328	-0.1407778	-0.49477504	-0.34067340	0.1558983051	1.0000000	0.58234240
[8,]	0.5376341	0.6066506239	0.4491434	-0.06233257	-0.74698798	0.1609898619	0.5823424	1.00000000

Method 1: Correlation of residuals in INLA

- Correlation of residuals seems not to be available in INLA (Spatial models also use random effects with multivariate normal distribution)
- I asked on the INLA forum whether correlated residuals can be implemented:
    - INLA only support conditional independent data, but for Gaussian response, you can always just expand the latent field (or linear predictor) to have define dependent data. sometimes this involve setting the likelihood to have fixed high precision, like having AR1 dependenency in the observations. In these cases, you'll end up with a 'fake' linear predictor, for which a subset of it, is the real one.
      There are also generic iid models in dimension up to 10, with general covariance structure, that can be used for this (see inla.doc("iidkd") for a new testing version)
      Best H

Model 1 on Simulated data

Random Intercept Model

$$\begin{cases} y_{ij}^1 = \beta_0^t + \beta_1^t \cdot sex_i + \beta_2^t \cdot ses_i + \epsilon_{ij}^t \\ y = \beta_0^r + \beta_1^r \cdot sex_i + \beta_2^r \cdot ses_i + \epsilon_{ij}^r \end{cases}$$

with

$$\begin{bmatrix} \epsilon_{yi} \\ \epsilon_{xi} \end{bmatrix} \sim \mathcal{N}_2 \left[\mathbf{0}, \text{Unstructured}_8\right]$$

Method 1: Sources

- Usually mentioned briefly as introduction
- Books
  - G. Fitzmaurice, M. Davidian, G. Verbeke, Longitudinal Data Analysis, 2009, Chapters 13
- Papers
  - H. Cho, The analysis of multivariate longitudinal data using multivariate marginal models, Journal of Multivariate analysis, 2016
  - G. Verbeke, S. Fieuws, The analysis of multivariate longitudinal data: A review, 2012, Stat Methods Med Res

To-do

- I need to write a Thesis proposal: What will be the main direction of research?
  1. Correlated Residuals
     - ⋆ Do we need correlated residuals? Many Marginal Models (correlated residuals) can be implemented by using Mixed Models (random effects)
     - ⋆ Do we want to implement correlated residuals in INLA? Maybe with generic package?
  2. Apply methods on COVID-dataset

- H. Rue, S. Martino, Approximate Bayesian inference for latent Gaussian models by using integrated nested Laplace approximations, 2009, JRSS B
- J. van Niekerk, H. Bakka, H. Rue, Joint models as latent Gaussian models - not reinventing the wheel, 2019, arXiv
- J. van Niekerk, H. Bakka, H. Rue, Competing risks joint models using R-INLA, 2021, Statistical Modelling
- R. Weiss, Modelling Longitudinal Data, 2005, Chapter 13