EECS 391 Written Assignment 4

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1. The general version of the product rule is (1):

This can be extended to any number of events (2):

The conditional version of the product rule can be written as (3):

Using (2) from above, this becomes (4):

The probability of the union of Y and **e** can be written as (5):

This leaves us with a factor of, the prior probability of event **e,** in both the numerator and denominator. These are cancelled out and we arrive at the solution (6):

B. The general version of Bayes’ rule is (1):

The conditional version of Bayes’ rule is thus (2):

From part A (6), the right side of (2) can be rewritten as (3):

Also using the result from part A (6), the far right term of (3) becomes (4):

Equating this expression to the far left hand term of (3) results in the desired solution (5):

1. A. **Network C.** The statement indicates the genes of the father, mother, and child are absolutely independent of one another. The only network in Figure 14.20 for which this is true is C because there are no connections linking the genes in this network indicating independence.

B. **Network A and Network B.** The problem states the gene itself is equally likely to be inherited from either of an individual’s parents. Network A represents this hypothesis as does network B which both show the gene for handedness is dependent only on the parents’ genes for handedness with equal weight given to each parent.

C. **Network A** is the best description of the hypothesis. The hypothesis claims the gene for handedness solely determines the handedness of an individual (with a probability of s) which means the only parent of handedness should be the individual’s gene. The gene itself is inherited from either the mother or the father. The actual handedness of the child is conditionally independent of both the genes of the parents and the handedness of the parents. Networks B and C represent that the handedness of the child is directly dependent on the handedness of the parents.

D. Conditional Probability Table ofin terms of s and m where s is the probability the handedness is the same as the gene, and m is the probability of a random mutation flipping the handedness of the gene. The gene of the child does not depend on s because s relates the gene to the handedness, and the handedness is a child of the gene node. For the cases where the genes of the parent are different, the probability for the gene of the child is evenly split between the two options.

|  |  |  |
| --- | --- | --- |
| **Gene father** | **Gene mother** | **Gene Child <left, right>** |
| L | L | <1-m, m> |
| L | R | <1/2,1/2> |
| R | L | <1/2, 1/2> |
| R | R | <m, 1-m> |

E. We want to find the marginal probability of the child having the left hand gene. To derive this expression, we must hold the variable of interest (query variable) constant and sum across all possible values for the conditioning variables, in this case, the genes of the mother and the father. This can be written as an equation:

We can use the conditional probability table from above and the knowledge to simply the expression.

This simplifies to the expression:

F. Given the prior probability of left-handedness is the same for everyone, that would mean q = 0.5. This can be proved by setting and solving the above equation for q.

However, this contrasts with the actual state of the world because far fewer than 50% of individuals are left handed. According to research, about [5-30% of the world population is left-handed (this can vary by country).](https://www.scientificamerican.com/article/why-are-more-people-right/) This means the hypothesis that genetics is the sole determinant of handiness is incorrect and there must be hidden (latent) variables not accounted for by the network. These could include environmental factors like societal pressures/traditions. [Studies conducted in twins show](https://www.ncbi.nlm.nih.gov/pubmed/16611467) genetic factors can only explain 25% of the variance in handedness with the rest due to environmental factors. Another explanation is the right hand allele (a version of a gene) may be dominant meaning that if both a left hand and right hand allele are present in a gene for handedness, the right hand gene is expressed in the phenotype (physical manifestation of a genotype). In order to account for these factors, the Bayes Network must be expanded to include all factors which influence handedness. Each of these factors must then have a prior probability and a probability reflecting the factor’s relative influence on handedness in order to specify the full joint distribution.

1. The full Bayes net is pictured below for reference.



1. **(i): FALSE,** this statement claims the probability an individual broke the election law, was indicted, and that the prosecutor is politically motivated are absolutely independent of one another. This is not shown in the network because the probability of indictment is conditionally dependent on both the node broke election law and a politically motivated prosecutor.

**(ii): TRUE,** this statement claims the probability an individual was jailed is conditionally independent of the probability that the individual was indicted given that the individual was found guilty. This is true as indicated by the network. Indictment indirectly is a cause of jailed, but the outcome of jailed is conditionally independent of indictment once the information that the individual was found guilty is known.

**(iii) TRUE,** this statements claims the probability the prosecutor is politically motivated is conditionally independent of the individual being jailed given that the individual broke the law, was indicted, and was found guilty. The network does not have any direct connections between the politically motivated prosecutor and jailed and hence they are conditionally independent.

1. In order to reason with a Bayes Net, the probability postulate must be broken down into smaller chunks that can be evaluated from the known conditionals and priors. The known prior probabilities are . The probability postulate can be expressed in English as the probability that the individual was jailed given that he broke the election law, was indicted, was found guilty, and the prosecutor was not politically motivated.

The probability of guilty verdict is not a prior and must be expressed as a conditional.

Likewise, the probability of an indictment is not known a priori. It must also be expressed as a conditional.

The probability statement is now in terms of known conditionals and priors. To solve, the numbers can be substituted:

Given the information, **we can conclude the probability an individual will be jailed given that he broke the law, was indicted, was found guilty, and the prosecutor is not politically motivated is 29.16%**. This probability of a jailed outcome may seem low, but as Professor Lewicki discussed in class, the more specific the situation, the less likely that exact situation will occur. There are 25 possible combinations of variables in this network, and each one in isolation is unlikely, which contrasts with our intuition because we only ever see one version of reality.

1. The probability an individual goes to jail given he broke the law, has been indicted, and faces a politically motivated prosecutor can be expressed in a postulate as:

Unlike Part B, we are not given the value of one of the evidence variables, found guilty. Therefore, in order to evaluate the postulate, we need to sum over all possible values of the unobserved variable, found guilty, while holding the query variable and evidence variables constant. In order to express this as a valid probability, we then need to normalize (divide) this by summing over all possible values of both the unknown variable and the query variable. In effect, we are summing out the possible values of the unknown variable.

Now, this equation needs to be expressed in terms of the known priors and conditionals

Starting with the numerator:

The probability an individual is jailed given they were not found guilty is zero and therefore this can be removed from the equation:

Moving to the denominator:

Again, the probability an individual is jailed given they were not found guilty is 0.

We have a common factor of in the numerator and denominator which can be cancelled yielding the equation:

At this point, the conditional and prior probabilities can be substituted and the equation solved:

**To conclude, the probability an individual went to jail given he broke the election law, was indicted, and faces a politically motivated prosecutor is 0.81 or 81%.**

1. A presidential pardon is when an individual is found guilty, but has his/her sentence shortened by the President. In the United States, a presidential pardon can only be used for an individual [found guilty of a federal crime.](https://www.pbs.org/newshour/politics/presidents-pardon-power-works) The presidential pardon will only be connected to the found guilty and jailed nodes. A pardon can only occur after an individual has been found guilty, thus guilty is a parent of presidential pardon. A presidential pardon nullifies a jail sentence and therefore the pardon is a parent of the jailed node. The presidential pardon is represented by the following addition to the Bayes Net:



P P(J|P)

P = T 0.0

P = F 0.8

P

The conditional probabilities that need to be specified are that for a presidential pardon given the individual was found guilty, and that for jail given knowledge of the pardon and guilty verdict. The base rate (prior) probability of a pardon is extremely low, and thus is represented by a 1/1000 chance given that the individual was found guilty. If the individual was not found guilty, there is no chance of a presidential pardon.

|  |  |
| --- | --- |
| **Guilty** | **P(Pardon | Guilty)** |
| T | 0.001 |
| F | 0.0 |

If the presidential pardon is granted, the individual will not go to jail given the verdict was guilty (or not guilty). The only situation in which the individual will go to jail is if he is found guilty and there is no pardon, with the same posterior probability of jail as in the original graph.

|  |  |  |
| --- | --- | --- |
| **Pardon** | **Guilty** | **P(J| Pardon, Guilty)** |
| T | T | 0.0 |
| F | T | 0.9 |
| T | F | 0.0 |
| F | F | 0.0 |

1. A. The Bayes Net for this situation is represented below as well as the prior probabilities of each of the coins being selected. The S stands for selected coin:

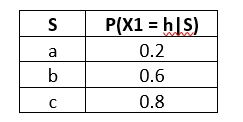
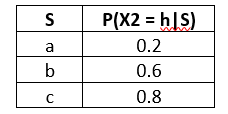
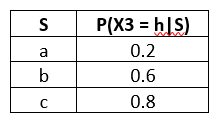
|  |
| --- |
| **P(S = a): 1/3** |
| **P(S = b): 1/3** |
| **P(S = c): 1/3** |

S

X1

X2

X3



The S node represents the selection of the code. Each coin is equally likely to be selected and there are three coins yielding the a priori probabilities of 1/3 for a, b, and c. The conditional probability tables relate the probability of the variables X1, X2, and X3 coming up heads given the selection of a particular coin. These rates represent the biased probability of the coin given in the problem statement.

B. The outcome of each coin is independent of the others, and the assignment of the coins does not matter for the problem formulation (in other words, X1 = h, X2 = h, and X3 = t is the same as X1 = t, X2 = h, X3 = h). The three expressions we need to solve for to determine the most likely coin are:

These equations provide a chance to put Bayes’ Rule into practice. I will demonstrate with just the first equation, but the second two follow the same process:

The probability in the denominator can be expanded using the total probability rule:

The denominator is the same for all three cases as is the prior probability in the numerator. The denominator can be determined by substituting in the known values:

The results of the coin flips are conditionally independent of one another which means the first term in the numerator can be expressed in terms of the known conditionals and priors. The equation for the first coin is shown below with a similar expression for the other coins:

This expression can be evaluated for all the coins:

The denominator is the same for all coins, so there is no need to figure out the actual probabilities to declare that Coin 2 is the most likely to have come out of the bag given the results 2 heads and 1 tail. However, for curiosity (and to check my answer), I will go ahead and calculate the percentages by normalizing with the denominator and multiplying by the prior:

**Given the result of 2 heads and 1 tail, Coin 2 is the most likely to have been selected at 47.4%.**

1. A. The outcome of a given match depends only on the difference in quality between the teams. Therefore, in order to express the probabilities for the outcome of a match, the optimal method is to express the probability of an outcome based on the differences in quality. The following shows the conditional probability table for the outcome of a match conditioned on the quality of the two teams in the match. There are seven possibilities for

**Conditional Probabilities for Outcome of Match Given**

|  |  |  |
| --- | --- | --- |
|  | **Probability (1 win|)** | **Probability (Tie|)** |
| -3 | 0.05 | 0.05 |
| -2 | 0.1 | 0.15 |
| -1 | 0.2 | 0.2 |
| 0 | 0.3 | 0.4 |
| 1 | 0.6 | 0.2 |
| 2 | 0.75 | 0.15 |
| 3 | 0.9 | 0.05 |

The other probabilities to define are the posteriors, which in this case are the quality of the teams.

The following prior probabilities apply to all teams:

**Prior Probabilities for Quality of a Team**

|  |  |
| --- | --- |
| **Quality** | **Probability (Q = q)** |
| 0 | 0.25 |
| 1 | 0.25 |
| 2 | 0.25 |
| 3 | 0.25 |

These prior probabilities are the used to calculate the probability of any difference in quality between two teams.

**Prior Probabilities for**

|  |  |
| --- | --- |
|  | **Probability ()** |
| -3 | 1/16 = 0.0625 |
| -2 | 2/16 = 0.125 |
| -1 | 3/16 = 0.1875 |
| 0 | 4/16 = 0.25 |
| 1 | 3/16 = 0.1875 |
| 2 | 2/16 = 0.125 |
| 3 | 1/16 = 0.0625 |

The outcome of a match is completely dependent on the difference in quality of the two teams in the match. Therefore, the conditionals relating the outcome of a match to the difference in quality between the two teams and the prior probabilities of the quality of the teams is enough to specify the complete joint distribution.

B. The Bayes’ Net can be drawn as follows:

QC

QB

QA

A vs B

A vs C

B vs C

As mentioned in the problem statement, the outcome of a match is probabilistically determined by the difference in quality between the teams. Therefore, each match has two parents: the quality values of each of the teams in the match. The quality of the teams and the outcomes of a match are independent of one another.

C. I will solve for the conditional probability of the outcome of the third match using **inference by enumeration.** This starts with the general expression for inference:

where α is a normalization constant to makes the probabilities sum to 1, X is the query event, e is the list of evidence variables, and y is the possible values of the unobserved variables.

From the information given in the problem (A beats B and draws with C), we need to calculate the posterior probability of the outcome of the third game. This requires assessing the probability of the difference in quality based on the evidence.

We need to determine the probability of the quality of both B and C given the evidence. The results of the games are independent from one another, and the quality of a team not involved in the match is independent of the outcome of the match (b refers to beats and t refers to ties).

This can be expressed in terms of Bayes’ Rule:

The same expression can be written for any quality value of B or C:

In the case of a tie game, that information does not given us any indication of the quality of the teams involved. In other words, we can gain no information from a tie, so the conditional probability for the quality of C is identical to the prior probability for the quality of C.

The conditional probability of , given the results of the games can be expressed by the general inference procedure:

Here, the normalization constant will be:

The chance of a given quality of a team is always 0.25, and the probability in the denominator can be worked out in general for any two teams 1 and 2 given the conditional and prior probabilities by summing over the possible values of the difference in quality:

With the result that

We can now compute the probability of any quality of B

The results are the following two tables for the quality of B and C

**Inferred Probabilities for the Quality of B and C Given the Match Results**

|  |  |  |
| --- | --- | --- |
| **Quality** |  |  |
| 0 | 0.408 | 0.25 |
| 1 | 0.296 | 0.25 |
| 2 | 0.192 | 0.25 |
| 3 | 0.104 | 0.25 |

To make sure we are on the right track, we can sum the columns for the quality of B and C and confirm that both of them add up to 1 as expected. The next step is to calculate the probabilities of the difference between teams B and C.

|  |  |
| --- | --- |
|  | **Probability ()** |
| -3 | (0.104) \* 0.25= **0.026** |
| -2 | (0.104 + 0.192) \* 0.25 = **0.074** |
| -1 | (0.104 + 0.192 + 0.296) \* 0.25 = **0.148** |
| 0 | (0.408 + 0.296 + 0.192+ 0.104) \* 0.25 = **0.25** |
| 1 | (0.408 + 0.296 + 0.192) \* 0.25 = **0.224** |
| 2 | (0.296 + 0.408) \* 0.25 = **0.176** |
| 3 | (0.408) \* 0.25 = **0.102** |

Again, these probabilities must sum to 1 which is confirmed. The final step is to determine the probabilities of each of the possible outcomes of the match given the probabilities for the difference in quality:

The final posterior distribution for the outcome of B vs C is as follows:

**Probability C beats B = 0.4715 = 47.15%**

**Probability C ties B = 0.2183 = 21.83%**

**Probability B beats C = 0.3102 = 31.02%**

**The results include the somewhat surprising conclusion that the probability that B beats C is greater than the probability of a tie. However, this is easier to realize once the priors are taken into account. The background rate of a tie is much lower at 21.875% than that of a win for either team at 39.0625%.**