EECS 391 Written Assignment 5

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2. The posterior distribution for θ is the expected distribution of θ given y and n.

We can express this using Bayes’ Rule as follows:

This represents that

We already know the likelihood:

The prior is a constant because we are assuming a uniform prior distribution for the probability of a heads. The normalization constant is necessary to make the distribution sum to 1. The normalization constant is expressed as:

Again, the prior is a constant which can be pulled out of the integral. The prior then cancels from the numerator and the denominator. From the problem statement, we are provided:

The expression for the posterior after some rearranging becomes:

Leading to the final answer:



Given n = 4 and θ = ¾

We can use R to plot the range of possible y values:

**R Implementation**

# Define possible values of y

y <- seq(0, 4, 1)

# Function that takes in a value of y and returns the probability of y

# given n = 4 and theta = 0.74

prob <- function(y) {

first\_term <- (factorial(4) / (factorial((4 - y)) \* factorial(y)))

second\_term <- (0.75 ^ y) \* (0.25 ^ (4 - y))

return(first\_term \* second\_term)

}

# The probability distribution of y

prob\_y <- sapply(y, prob)

# Created y for smooth lines

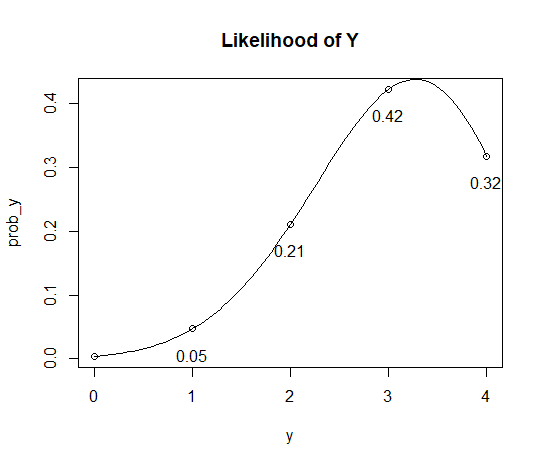
y\_smooth <- seq(0, 4, by = 4/100)

prob\_y\_smooth <- sapply(y\_smooth, prob)

# Plot the likelihood of y under the assumption theta = 0.75 and n = 4

plot(y, prob\_y, main = 'Likelihood of y') + lines(y\_smooth, prob\_y\_smooth) +

text(y, prob\_y - 0.04, labels = round(prob\_y, 2))



1. From part A, the posterior distribution of θ is

The four cases are y = 1, n = 1; y = 2, n = 2; y = 2, n = 3; y = 3, n = 4

**R Implementation**

# Calculates posterior probability for theta given n and y

calc\_posterior <- function(theta, y, n) {

first\_term <- factorial(n + 1) / (factorial(n - y) \* factorial(y))

second\_term <- (theta ^ y) \* ((1 - theta) ^ (n - y))

return(first\_term \* second\_term)

}

y <- c(1, 2, 2, 3)

n <- c(1, 2, 3, 4)

theta <- seq(0, 1, 1/1000)

posterior\_df <- as.data.frame(matrix(nrow = length(theta) \* 4, ncol = 4))

names(posterior\_df) <- c('n', 'y', 'theta', 'posterior')

i <- 1

all\_posteriors <- c()

for (head\_count in y) {

posteriors <- sapply(theta, calc\_posterior, y = head\_count, n = i)

all\_posteriors <- c(all\_posteriors, posteriors)

i <- i + 1

}

posterior\_df$n <- c(rep(1, 1001), rep(2, 1001),

rep(3, 1001), rep(4, 1001))

posterior\_df$y <- c(rep(1, 1001), rep(2, 2002),

rep(3, 1001))

posterior\_df$n\_y <- c(rep(c('n = 1, y = 1'), 1001), rep(c('n = 2, y = 2'), 1001),

rep(c('n = 3, y = 2'), 1001), rep(c('n = 4, y = 3'), 1001))

posterior\_df$theta <- c(rep(theta, 4))

posterior\_df$posterior <- all\_posteriors

ggplot(posterior\_df, aes(theta, posterior)) + geom\_point(col = 'darkgreen') +

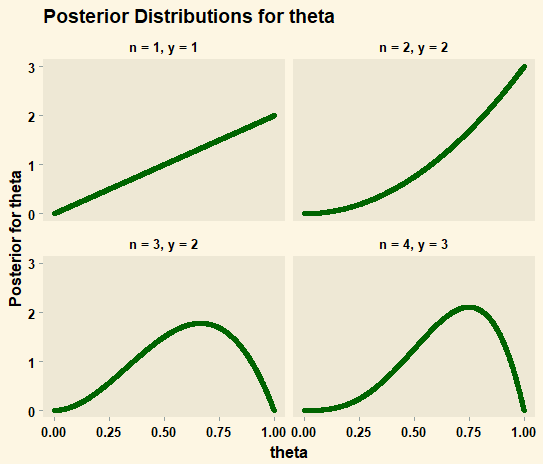
geom\_line(col = 'darkgreen') +

facet\_wrap(~n\_y) + ylab('Posterior for theta') +

ggtitle('Posterior Distributions for theta') + theme\_solarized\_2() +

theme(panel.grid = element\_blank(), text = element\_text(face = 'bold', color = 'black'),

title = element\_text(face = 'bold', color = 'black'))



In each case, the maximum posterior for theta occurs at y/n. For the first two plots, this is at theta = 1, for the third plot at theta = 0.667, and at thetat = 0.75 for the fourth plot.

1. The graphs in Figure 20.1 were drawn under hypothesis 5, that 100% of candies are lime.



There are four other hypothesis to evaluate.

Under each of the hypotheses, we must plot the posterior distributions for all the hypotheses and the probability that the next candy is lime for a range of observations. I will plot the distributions over 100 observations with the number of limes unwrapped depending on a binomial distribution with the probability under a given hypothesis.

1. Posterior

Posterior probabilities are expressed in terms of Bayes’ rule:

Where **d** is a vector representing all of the observations. represents the likelihood of the data under the hypothesis, and is the prior for the hypothesis. The prior probabilities for each of the hypotheses are provided by the manufacturer as <0.1, 0.2, 0.4, 0.2, 0.1>. T

We can treat this problem as a binomial distribution, as in Problem 1. In this case, θ is the percentage of lime candy under a given hypothesis, y is the number of lime candies unwrapped, and n is the total number of candies unwrapped. For each assumed hypothesis, and for each value of y and n observed, we can then compute the posterior probability for a given hypothesis, θ:

In this case, the normalization constant is the total chance of the observed number of limes given the total number of observations summed across all hypothesis.

This normalization constant is needed so that the probability for all the hypothesis given a certain set of observations sums to one. The prior probabilities do not depend on the number of observations. The total equation for the posterior probability is thus expressed:

Where is the hypothesis (percentage of lime candies), y is the number of lime candies unwrapped, and n is the total number of candies unwrapped. is the prior for each hypothesis provided by the manufacturer, and the normalization constant appears in the denominator.

1. Probability of next candy being a lime

The Bayesian prediction for the next candy being lime is This can be expressed as

The posterior was derived above:

The conditional is the probability of a lime given the hypothesis. The final equation for the next candy selected being a lime is:

This expression is the conditional probability of a lime candy given a hypothesis multiplied by the posterior probability for the hypothesis given the observations. The probability of the next candy being a lime depends on the prior distribution of the number of lime candies.

Both the posterior for each hypothesis and the probability the next candy is a lime under each of the five hypotheses ( I will recreate hypothesis 5 for completeness) can be calculated using straightforward code. The results are then graphed for intuitive visualization. The basic procedure is to assume one hypothesis, calculate the distribution of observations under that hypothesis, and then iterate through all the observations calculating the posterior for each hypothesis and the probability the next candy is a lime at each iteration. This task can be performed five times, each time assuming a different one of the five hypotheses is true. This is implemented in R code as demonstrated.

**R Implementation**

# Prior probabilities and list of hypotheses expressed as probability of lime candy

priors <- list('h1' = 0.1, 'h2' = 0.2, 'h3' = 0.4, 'h4' = 0.2, 'h5' = 0.1)

hypotheses <- list('h1' = 0.0, 'h2' = 0.25, 'h3' = 0.5, 'h4' = 0.75, 'h5' = 1.0)

# The likelihood of a hypothesis (theta) given observations of successes, y,

# and total number of trials

calc\_likelihood <- function(theta, y, n) {

first\_term <- factorial(n) / (factorial(n - y) \* factorial(y))

second\_term <- (theta ^ y) \* ((1 - theta) ^ (n - y))

return(first\_term \* second\_term)

}

# Calculates the posterior for each of the five hypothesis under one true hypothesis

calc\_posteriors <- function(true\_hyp) {

# Set seed to ensure consistent results

set.seed(42)

# Create a dataframe to hold results

posts\_df <- as.data.frame(matrix(ncol = 8))

names(posts\_df) <- c('n', 'y', 'h1', 'h2', 'h3', 'h4', 'h5', 'next\_lime')

# Vector of observations under the true hypothesis

limes <- c(0, rbinom(n = 99, size = 1, prob = true\_hyp))

total\_limes <- cumsum(limes)

# Priors for each hypothesis and the conditional probability of a lime candy

# for each hypothesis

priors <- list('h1' = 0.1, 'h2' = 0.2, 'h3' = 0.4, 'h4' = 0.2, 'h5' = 0.1)

lime\_pct <- list('h1' = 0.0, 'h2' = 0.25, 'h3' = 0.5, 'h4' = 0.75, 'h5' = 1.0)

# Iterate through the observations and calculate posteriors and

# chance of next lime

for (i in seq(1, 100, 1)) {

n\_lime <- total\_limes[i]

# Calculate the unnormalized posterior for each hypothesis

h1\_post <- calc\_likelihood(lime\_pct$h1, n\_lime, i - 1) \* priors$h1

h2\_post <- calc\_likelihood(lime\_pct$h2, n\_lime, i - 1) \* priors$h2

h3\_post <- calc\_likelihood(lime\_pct$h3, n\_lime, i - 1) \* priors$h3

h4\_post <- calc\_likelihood(lime\_pct$h4, n\_lime, i - 1) \* priors$h4

h5\_post <- calc\_likelihood(lime\_pct$h5, n\_lime, i - 1) \* priors$h5

# Calculate the normalization constant and normalize the posteriors

posts <- c(h1\_post, h2\_post, h3\_post, h4\_post, h5\_post)

norm\_constant <- sum(posts)

norm\_posts <- posts / norm\_constant

# The probability of a lime next

next\_lime <- sum(unlist(lime\_pct) \* norm\_posts)

# Explicitly add results to a dataframe

posts\_df <- add\_row(posts\_df, n = i - 1, y = n\_lime, h1 = norm\_posts[1], h2 = norm\_posts[2],

h3 = norm\_posts[3], h4 = norm\_posts[4], h5 = norm\_posts[5],

next\_lime = next\_lime)

}

return(posts\_df[-1, ])

}

# Assume each hypothesis is true and calculate the corresponding distributions

h1\_true <- calc\_posteriors(0.0)

h2\_true <- calc\_posteriors(0.25)

h3\_true <- calc\_posteriors(0.5)

h4\_true <- calc\_posteriors(0.75)

h5\_true <- calc\_posteriors(1.0)

# Function to graph posterior distributions for each hypothesis and the

# probability the next candy is a lime

# Takes in a dataframe with posteriors and the string of the true hypothesis

# that was used to calculate the posteriors

graph\_posts <- function(post\_df, true\_hyp\_str) {

# Titles for figures

main1 <- sprintf('Posteriors under %s', true\_hyp\_str)

main2 <- sprintf('Probability of next lime under %s', true\_hyp\_str)

# Put results in long format with each row and observation and each column

# a variable

long\_post <- gather(post\_df, key = 'hyp', value = 'post', -n, -y, -next\_lime)

color\_vector <- c('blue', 'red', 'darkgreen', 'darkorange', 'black')

th <- theme(axis.text = element\_text(color = 'black'))

# Graph the posteriors for each hypothesis under the true hypothesis

p1 <- ggplot(long\_post, aes(n, post, col = hyp, group = hyp, shape = hyp)) + geom\_line() +

geom\_point() + xlab('Number of Observations') + ylab('Posterior') +

ggtitle(main1) + theme\_economist(12) + th + scale\_color\_manual(values = color\_vector)

# Graph the probability of the next candy being a lime

p2 <- ggplot(post\_df, aes(n, next\_lime)) + geom\_line(col = 'red') + geom\_point(col = 'red') +

xlab('Number of Observations') + ylab('Probability') + ggtitle(main2) +

coord\_cartesian(ylim = c(0, 1)) + scale\_y\_continuous(breaks = seq(0, 1, 0.1)) +

theme\_economist(12) + th

# Display the graphs

print(p1)

print(p2)

}

# Graph results for each assumed hypothesis

graph\_posts(h1\_true, 'h1')

graph\_posts(h2\_true, 'h2')

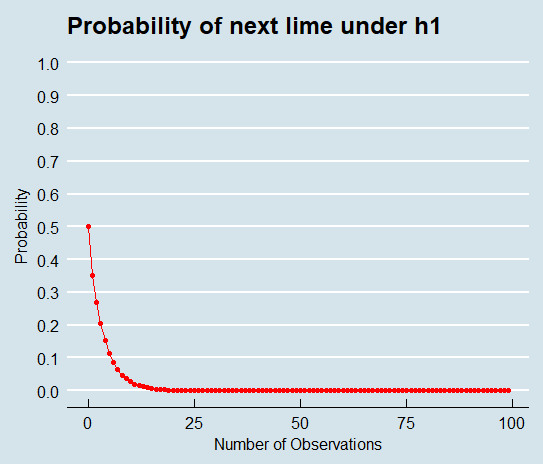
graph\_posts(h3\_true, 'h3')

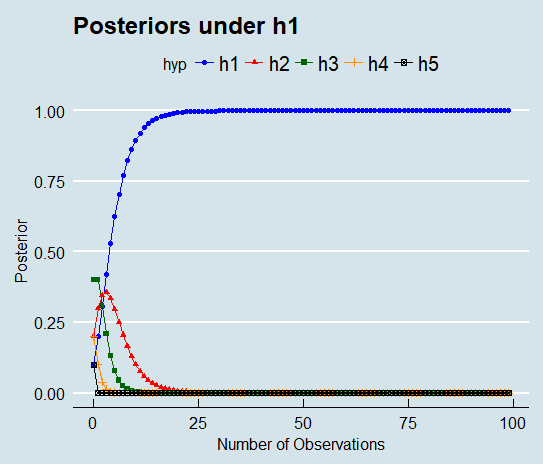
graph\_posts(h4\_true, 'h4')

graph\_posts(h5\_true, 'h5')

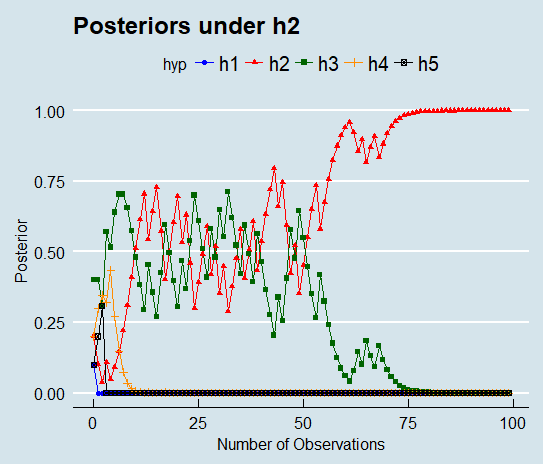
**Graphical Results**

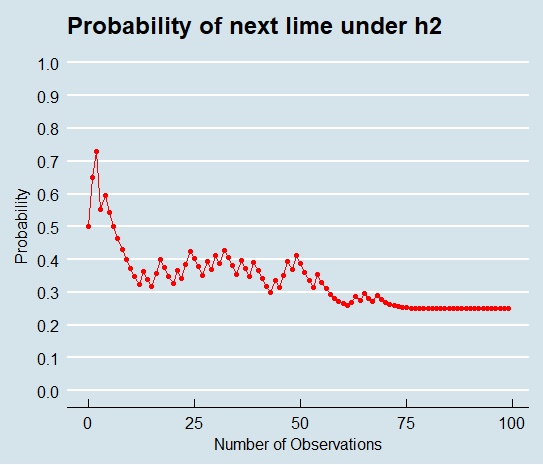
**Hypothesis 1: 0% Lime**



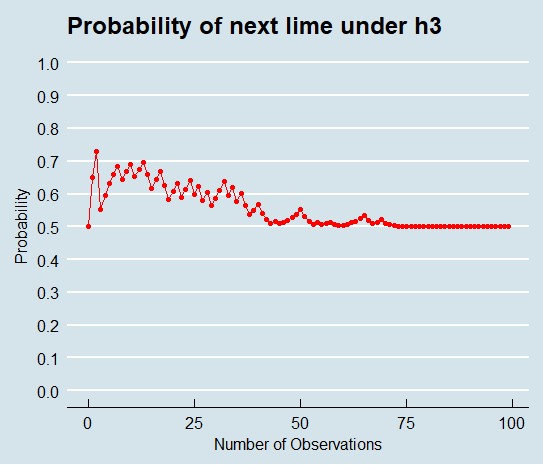


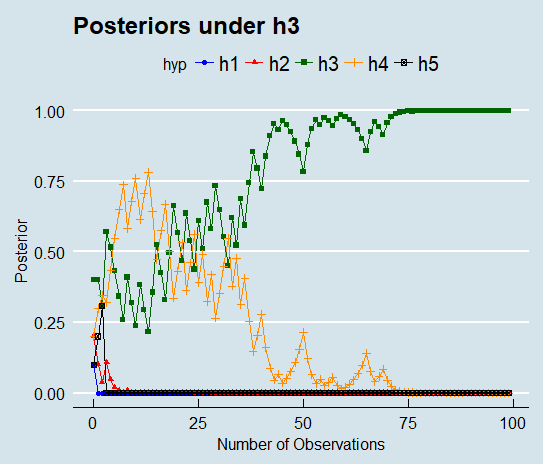
**Hypothesis 2: 25% Lime**

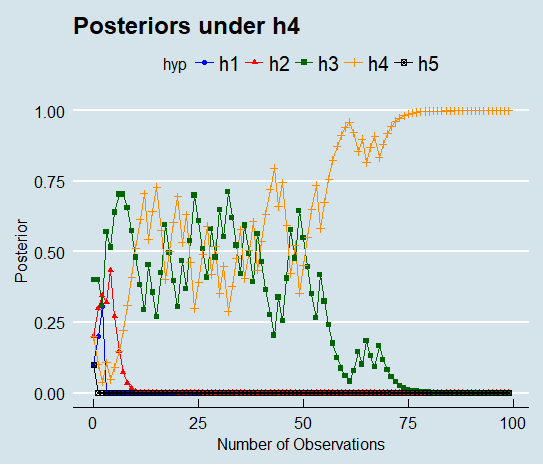
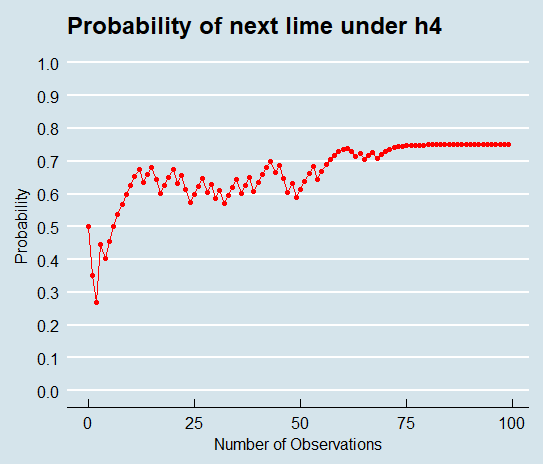




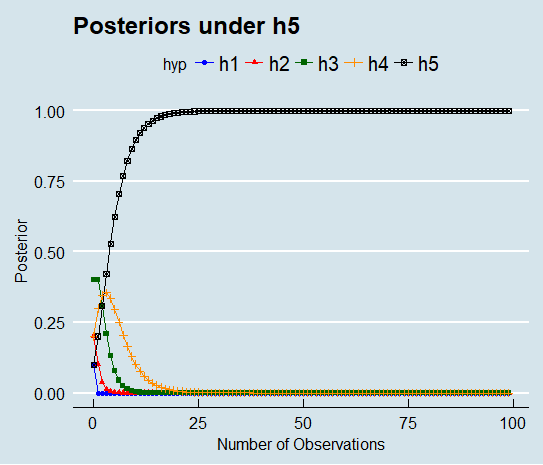
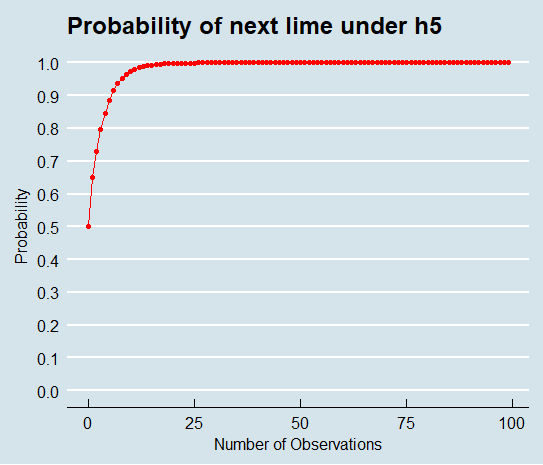
**Hypothesis 3: 50% Lime**





H**ypothesis 4: 75% Lime**

**Hypothesis 5: 100% Lime**



**Discussion**

These graphs are insightful as they show how under each hypothesis, the posterior probability for each hypothesis and the probability the next candy is a lime converge on the true values over enough observations. In all graphs, the results are exactly as would be expected from Bayesian Inference. Hypotheses 1 and 5 are mirror images of each other and they correctly show the probability of the next candy being a lime converges to 0% and 100% respectively. Hypotheses 2 and 4 are also mirror images of each other with the probability the next candy is a lime converging to 25% and 75% respectively as indicated by the hypotheses. H3 shows the probability of the next candy a lime converges to 50%. In all cases, by 50 observations, the most likely hypothesis as indicated by the posteriors is the correct hypothesis. In H1 and H5 this is arrived at after 4 observations. The exact results under H2, H3, H4 will vary because the observations are randomly generated from a binomial distribution with the probability of success set by the true hypothesis.

B. The posterior probability for a hypothesis is given by the following equation. θ is the hypothesis (probability of a lime candy being drawn next), y is the number of successes (lime candies), and n is the total number of observations.

To solve for the minimum number of observations needed to be more than 90% sure of the hypothesis, the posterior expression should be set equal to 0.9 and then solved for n. The exact number of observations at which this will occur depends on the true hypothesis, which affects the actual sequence of observations. For example, we can use the following code to determine the number of observations at which the posterior probability for the correct hypothesis exceeds 90%.

**R Implementation**

# Find the minimum number of observations for a given confidence level

# Posterior df is generated from the specified true hypothesis (lime pct)

find\_minimum\_obs <- function(confidence\_level, post\_df, true\_hyp\_str) {

# Filter dataframe to observations with true posterior greater than confidence level

above\_cl <- post\_df[post\_df[true\_hyp\_str] > confidence\_level, ]

minimum\_obs <- above\_cl[[which.min(above\_cl$n), 'n']]

# Return the minimum observations for the specified confidence level

return(minimum\_obs)

}

# Find minimum observations required under each hypothesis

h1\_min <- find\_minimum\_obs(0.9, h1\_true, 'h1')

h2\_min <- find\_minimum\_obs(0.9, h2\_true, 'h2')

h3\_min <- find\_minimum\_obs(0.9, h3\_true, 'h3')

h4\_min <- find\_minimum\_obs(0.9, h4\_true, 'h4')

h5\_min <- find\_minimum\_obs(0.9, h5\_true, 'h5')

# Vectors for plotting results

hyps <- c(0.0, 0.25, 0.5, 0.75, 1.0)

min\_obs <- c(h1\_min, h2\_min, h3\_min, h4\_min, h5\_min)

# Plot the results

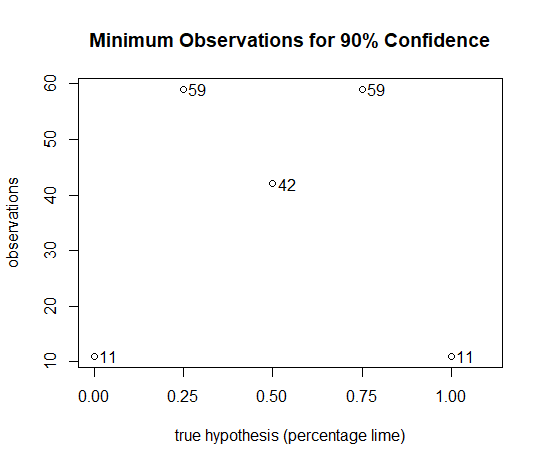
plot(x = hyps, y = min\_obs, xaxt = 'n', xlim = c(0, 1.1),

xlab = 'true hypothesis (percentage lime)',

ylab = 'observations', main = 'Minimum Observations for 90% Confidence') +

text(x = hyps + 0.04, y = min\_obs, labels = min\_obs) + axis(side = 1, at = hyps)

**Plot for Minimum Number of Observations for 90% Confidence in True Hypothesis**

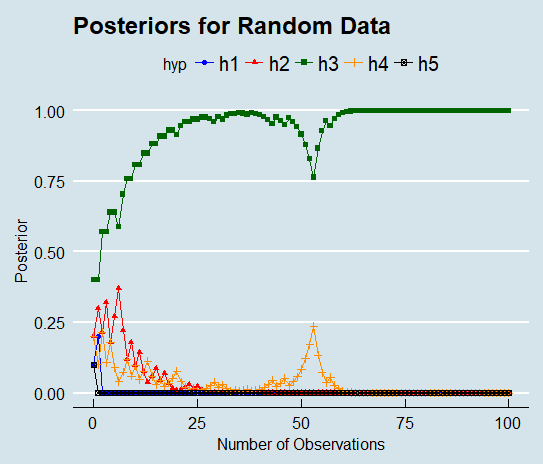


C. First, I will need to create a random dataset of 100 observations. To do this, I will generate 100 datasets of length 100, 20 under each of the hypotheses. The resulting observations can then be averaged to arrive at a random dataset of observations weighted by the prior probability (as specified by the manufacturer) for each hypothesis.

I will define the uncertainty for a hypothesis as:

Where is the posterior for the hypothesis given an unlimited number of observations. This is the steady-state value for the posterior of the hypothesis, the value to which the posterior for a given hypothesis will converge. In order to find this value, we can use the randomly generated data and plot the posteriors for all hypotheses versus the number of observations. Then, based on this plot, we can determine the steady state posterior for the hypothesis if we were to perform an infinite number of experiments. We can then use the convergence value to calculate the uncertainty in each hypothesis at each observation and find the difference from one observation to the next. All of these values will be plotted to provide an intuitive explanation.

As the data has been randomly generated and the most likely hypothesis is h3 at 0.4, we would expect the final posteriors would go to 100% for h3 and 0% for all other hypothesis with randomly generated data. The plot below shows this to be true with a dataset of length 100.



The random data and the plot was created by the following code:

**R Implementation**

# Generate 100 random datasets of 100 observations and average

generate\_random <- function() {

set.se

observations <- c(rep(0, 100))

i <- 1

for (prob in c(0.0, 0.25, 0.5, 0.75, 1.0)) {

for (m in 1:20) {

observations <- (observations + rbinom(n = 100, size = 1, prob = prob))

i <- i + 1

}

}

# Take mean and round to either a 0 or 1

return(round(observations / i))

}

# Generate the random dataset

random\_obs <- generate\_random()

# Calculates the posterior for each of the five hypothesis given a set of observations

calc\_posteriors\_given\_obs <- function(obs) {

# Create a dataframe to hold results

posts\_df <- as.data.frame(matrix(ncol = 8))

names(posts\_df) <- c('n', 'y', 'h1', 'h2', 'h3', 'h4', 'h5', 'next\_lime')

# Vector of observations provided

limes <- c(0, obs)

total\_limes <- cumsum(limes)

# Priors for each hypothesis and the conditional probability of a lime candy

# for each hypothesis

priors <- list('h1' = 0.1, 'h2' = 0.2, 'h3' = 0.4, 'h4' = 0.2, 'h5' = 0.1)

lime\_pct <- list('h1' = 0.0, 'h2' = 0.25, 'h3' = 0.5, 'h4' = 0.75, 'h5' = 1.0)

# Iterate through the observations and calculate posteriors and

# chance of next lime

for (i in seq(1, 101, 1)) {

n\_lime <- total\_limes[i]

# Calculate the unnormalized posterior for each hypothesis

h1\_post <- calc\_likelihood(lime\_pct$h1, n\_lime, i - 1) \* priors$h1

h2\_post <- calc\_likelihood(lime\_pct$h2, n\_lime, i - 1) \* priors$h2

h3\_post <- calc\_likelihood(lime\_pct$h3, n\_lime, i - 1) \* priors$h3

h4\_post <- calc\_likelihood(lime\_pct$h4, n\_lime, i - 1) \* priors$h4

h5\_post <- calc\_likelihood(lime\_pct$h5, n\_lime, i - 1) \* priors$h5

# Calculate the normalization constant and normalize the posteriors

posts <- c(h1\_post, h2\_post, h3\_post, h4\_post, h5\_post)

norm\_constant <- sum(posts)

norm\_posts <- posts / norm\_constant

# The probability of a lime next

next\_lime <- sum(unlist(lime\_pct) \* norm\_posts)

# Explicitly add results to a dataframe

posts\_df <- add\_row(posts\_df, n = i - 1, y = n\_lime, h1 = norm\_posts[1], h2 = norm\_posts[2],

h3 = norm\_posts[3], h4 = norm\_posts[4], h5 = norm\_posts[5],

next\_lime = next\_lime)

}

return(posts\_df[-1, ])

}

# Calculate the posterior odds given the randomly generated data

random\_post <- calc\_posteriors\_given\_obs(random\_obs)

# Put dataframe in long format

random\_long <- gather(random\_post, key = 'hyp', value = 'post', -n, -y, -next\_lime)

# Theme for axis text and color vector

th <- theme(axis.text = element\_text(color = 'black'))

color\_vector <- c('blue', 'red', 'darkgreen', 'darkorange', 'black')

# Graph the posteriors for each hypothesis with the random data

p1 <- ggplot(random\_long, aes(n, post, col = hyp, group = hyp, shape = hyp)) + geom\_line() +

geom\_point() + xlab('Number of Observations') + ylab('Posterior') +

ggtitle('Posteriors for Random Data') + theme\_economist(12) + th +

scale\_color\_manual(values = color\_vector)

# Graph the probability of the next candy being a lime with the random data

p2 <- ggplot(random\_post, aes(n, next\_lime)) + geom\_line(col = 'red') + geom\_point(col = 'red') +

xlab('Number of Observations') + ylab('Probability') +

ggtitle("Probability of Next Lime with Random Data") +

coord\_cartesian(ylim = c(0, 1)) + scale\_y\_continuous(breaks = seq(0, 1, 0.1)) +

theme\_economist(12) + th

print(p1)

print(p2)

To summarize thus far, the steady-state posteriors are as follows:

|  |  |
| --- | --- |
| **Hypothesis** | **with random data** |
|  | 0.0 |
|  | 0.0 |
|  | 1.0 |
|  | 0.0 |
|  | 0.0 |

Now, I need to calculate the uncertainty at each observation and the reduction in uncertainty from the previous observation.

The following code accomplishes this.

**R Implementation**

# Create uncertainty columns for each hypothesis

random\_unc <- random\_post %>% mutate(h1\_un = h1 - 0,

h2\_un = h2 - 0,

h3\_un = 1 - h3,

h4\_un = h4 - 0,

h5\_un = h5 - 0)

# Create a long format dataframe for plotting

random\_unc\_long <- random\_unc %>% dplyr::select(n, h1\_un:h5\_un) %>%

gather(key = 'hyp', value = 'unc', -n)

# Clean up the names for plotting

random\_unc\_long <- random\_unc\_long %>%

mutate(hyp = stringr::str\_split\_fixed(hyp, pattern = '\_', n = 2)[, 1])

# Plot the uncertainties

ggplot(random\_unc\_long, aes(n, unc \* 100, col = hyp, shape = hyp, group = hyp)) +

geom\_line() + geom\_point() + xlab('Number of Observations') +

ylab('Uncertainty for posterior (%)') + ggtitle('Uncertainty vs Obs for Random Data') +

scale\_color\_manual(values = color\_vector) + theme\_economist(12) + th

# Create columns with reduction in uncertainty

random\_reduc <- random\_unc %>% mutate(h1\_reduc = c(0, diff(h1\_un)),

h2\_reduc = c(0, diff(h2\_un)),

h3\_reduc = c(0, diff(h3\_un)),

h4\_reduc = c(0, diff(h4\_un)),

h5\_reduc = c(0, diff(h5\_un)))

# Put into long format

random\_reduc\_long <- random\_reduc %>% dplyr::select(n, h1\_reduc:h5\_reduc) %>%

gather(key = 'hyp', value = 'reduc', -n)

# Clean up the names for plotting

random\_reduc\_long <- random\_reduc\_long %>%

mutate(hyp = stringr::str\_split\_fixed(hyp, pattern = '\_', n = 2)[, 1])

# Plot the reductions in uncertainty

ggplot(random\_reduc\_long, aes(n, reduc \* 100, col = hyp, shape = hyp, group = hyp)) +

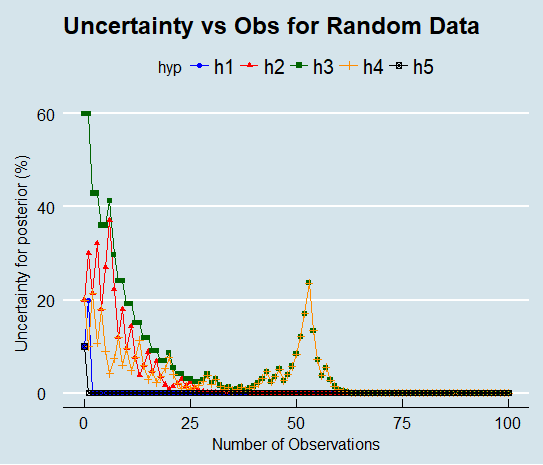
geom\_line() + geom\_point() + xlab('Number of Observations') +

ylab('Reduction in Uncertainty (%)') + ggtitle('Reduction in Uncertainty vs Obs') +

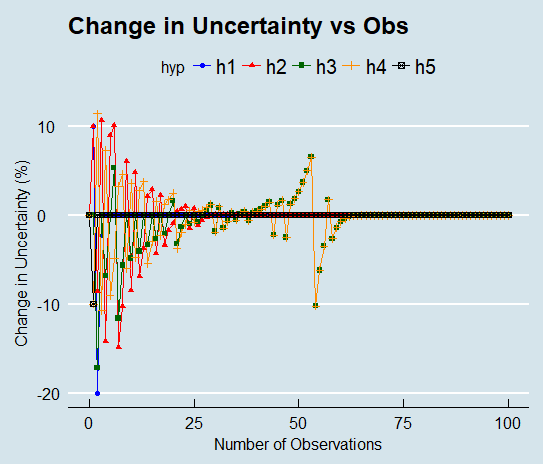
scale\_color\_manual(values = color\_vector) + theme\_economist(12) + th

**Visualization of Results**

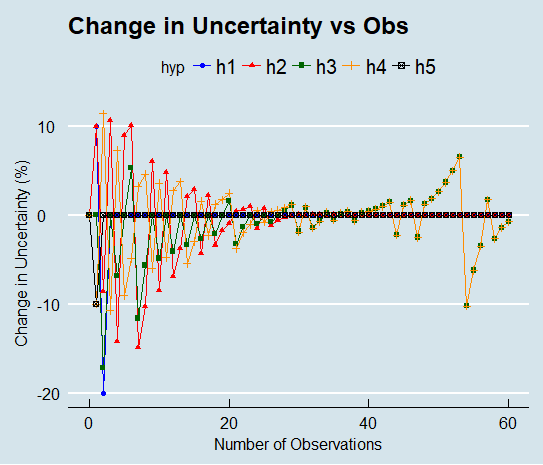
The first plot shows the raw uncertainty for each posterior versus the number of observations.



The next graph shows the final answer, the change in uncertainty versus the number of observations for each type of bag with the randomly generated data. A negative value indicates the observation decreased the uncertainty for that bag, while a positive value indicated the observation increased the uncertainty for that type of bag.



Most of the reduction in uncertainty has been accomplished by the 60th observation, so we can segment to that part of the plot to better visualize what is occurring.



**Discussion**

After enough observations, the posteriors converge on the steady-state value for the hypothesis, the type of bag. The initial observations have a much larger effect on the uncertainty of the bag than do later observations because the earlier observations change the proportion of limes more than do later observations as greater amounts of data is collected. Eventually, after 60 observations, each observation no longer has an effect on the uncertainty for the bag because the posteriors have converged to their steady state values. This plot illustrates the pitfalls of reasoning with small datasets: the fewer total observations, the greater the effect of any single observation. In a random process, this can lead to erroneous conclusions because randomly generated data will only converge to the true distribution with enough trials. Some observations can be seen to actually increase the uncertainty in a type of bag. For instance around observation 50, there is a long string of limes in a row, which makes h4 more likely and h3 correspondingly less likely. There are several other stretches such as this which show that even random data appears to exhibit trends over a small enough sample size. With enough observations though, the posteriors converge on the true value which shows the most likely bag is that split evenly between the two flavors.