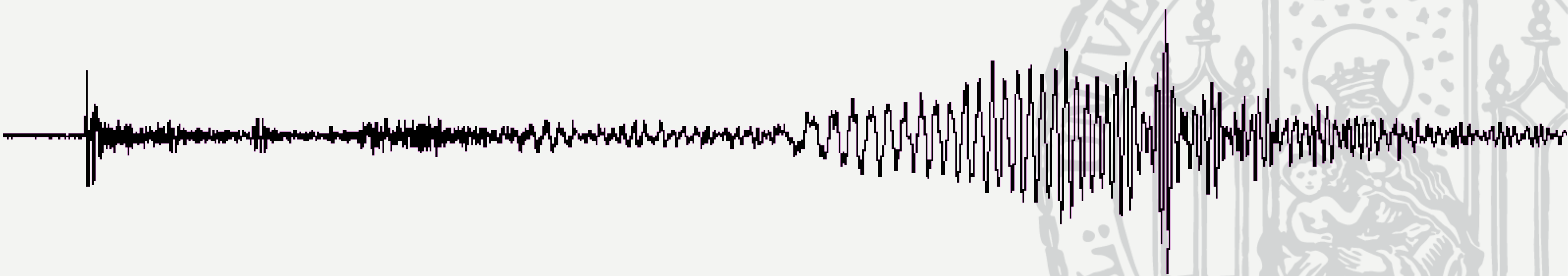
LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

Geophysical Data Analysis

Fourier series & transformation



Fourier series

Understand where the Fourier transform comes from.

Sequences and series

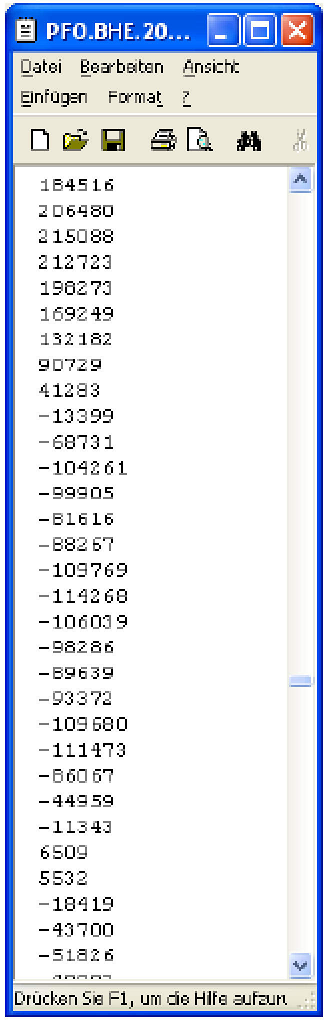
- *Sequence: an ordered collection of objects/numbers (repetition allowed)*
  e.g. prime numbers: 2, 3, 5, 7, 11, 13, ...

  OR samples at distinct sampling intervals Δt

$$a_k = a(k\Delta t); \quad k = 0, 1, 2, \ldots N$$

- *Series: partial sum of the terms of a sequence*

$$s_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \ldots + a_n$$



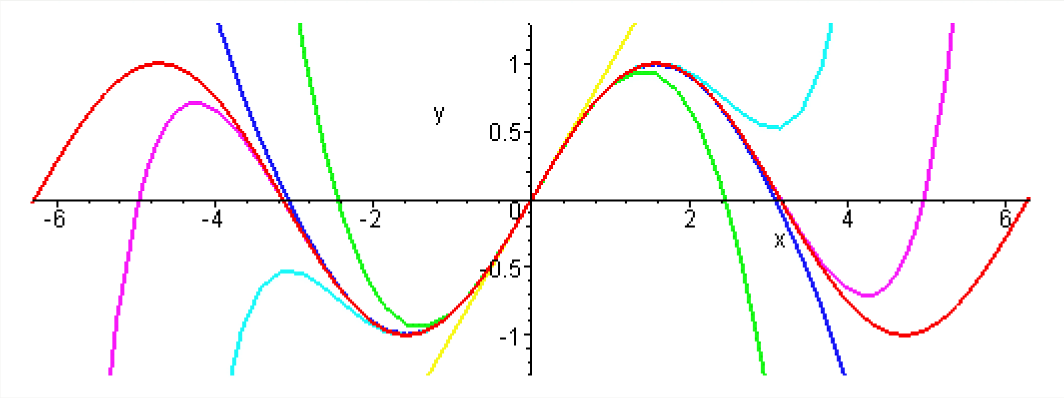
The power of series

- *Many (mildly or wildly non-linear) physical systems are transformed to a linear systems by using **Taylor series** …*

Why?

$$f(x+dx) = f(x) + f'dx + \frac{1}{2}f''dx^2 + \frac{1}{6}f'''dx^3 + \ldots$$

$$f(x+dx) = \sum_{i=1}^{\infty} \frac{f^i(x)}{i!}dx^i$$



e.g. deriving interpolation, source inversion, different weights, ….

The power of series

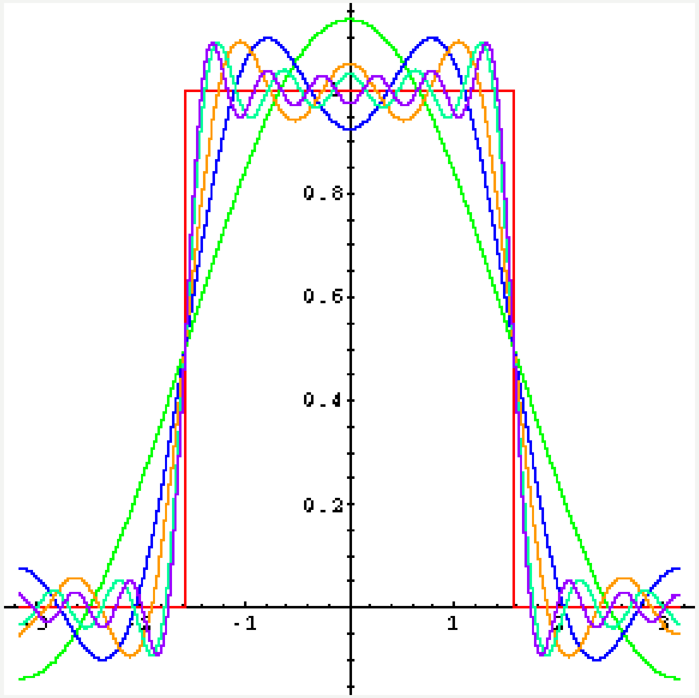
… and *Fourier series*

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1} a_n \cos\left(\frac{\pi n}{L}x\right) + b_n \sin\left(\frac{\pi n}{L}x\right)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$



Derive a Fourier Series

The Problem

We are trying to approximate a (unknown) function f(x) by another function $g_n(x)$ which consists of a sum over N basis functions Φ(x) weighted by some coefficients a_n.

$$f(x) \approx g_n(x) = \sum_{i=0}^{N} a_i \Phi_i(x)$$

At this stage, we consider continuous, periodic, and infinite functions.

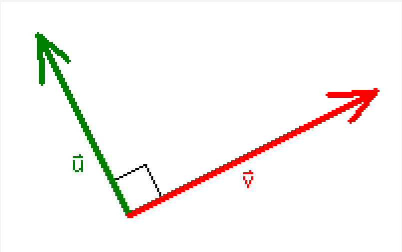
The problem

… a good choice for the basis Φ(x) are ***orthogonal*** *functions*

What are orthogonal functions?

Two functions f and g are said to be orthogonal in the interval [a,b] if

$$\int_a^b f(x)g(x)dx = 0$$



How is that related to the more conceivable concept of orthogonal vectors?

Question

Are these functions orthogonal?

$$\int_{-x}^{x} \cos(jx)\cos(kx)dx = \begin{cases} 0, & j \neq k \\ 2\pi, & j = k = 0 \\ \pi, & j = k > 0 \end{cases}$$

$$\int_{-x}^{x} \sin(jx)\sin(kx)dx = \begin{cases} 0, & j \neq k; j, k > 0 \\ \pi, & j = k > 0 \end{cases}$$

$$\int_{-x}^{x} \cos(jx)\sin(kx)dx = 0 \qquad j \geq 0, k > 0$$

Answer

Are these functions orthogonal?

$$\int_{-x}^{x} \cos(jx)\cos(kx)dx = \begin{cases} 0, & j \neq k \\ 2\pi, & j = k = 0 \\ \pi, & j = k > 0 \end{cases}$$

$$\int_{-x}^{x} \sin(jx)\sin(kx)dx = \begin{cases} 0, & j \neq k; j, k > 0 \\ \pi, & j = k > 0 \end{cases}$$

$$\int_{-x}^{x} \cos(jx)\sin(kx)dx = 0 \qquad j \geq 0, k > 0$$

YES! And these relations are valid for any interval of length 2π.

*Now, we know that this is an orthogonal basis, **but** how can we obtain the coefficients for the basis function?*

Fourier coefficients

… from minimizing f(x) - g(x)

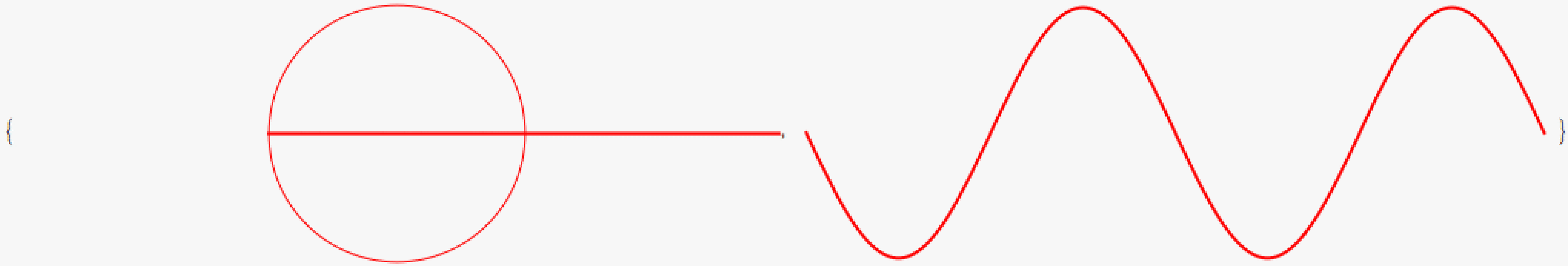
$$g_N(x) = \frac{1}{2} a_0 + \sum_{k=1}^{N} \left[a_k \cos(kx) + b_k \sin(kx) \right]$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx$$

$[-\pi \leq x \leq \pi]$

Fourier evolution



Example

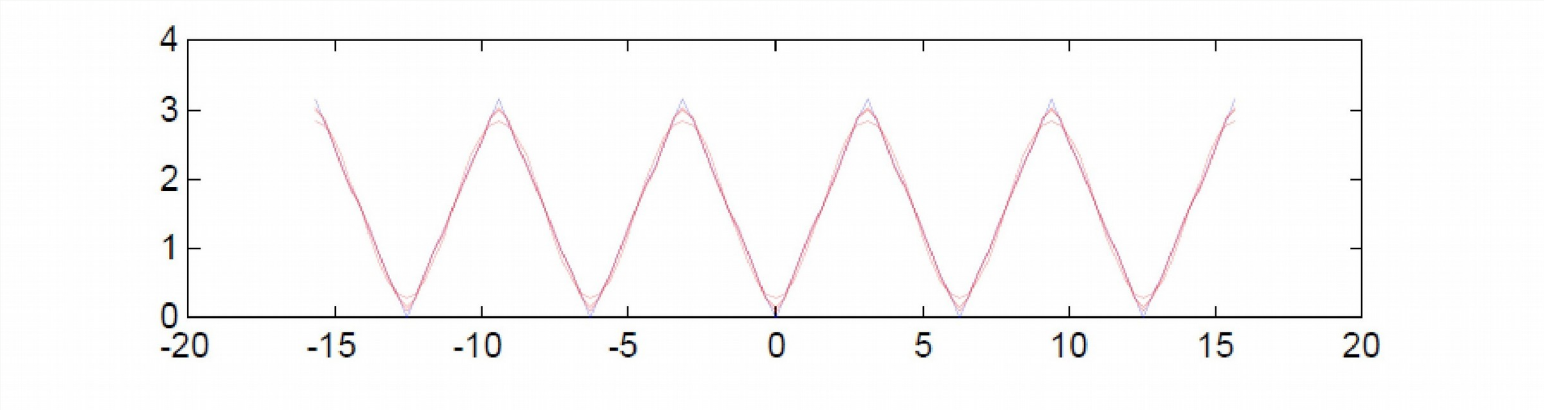
The Fourier approximation of …

$$f(x) = |x| \qquad -\pi \leq x \leq \pi$$

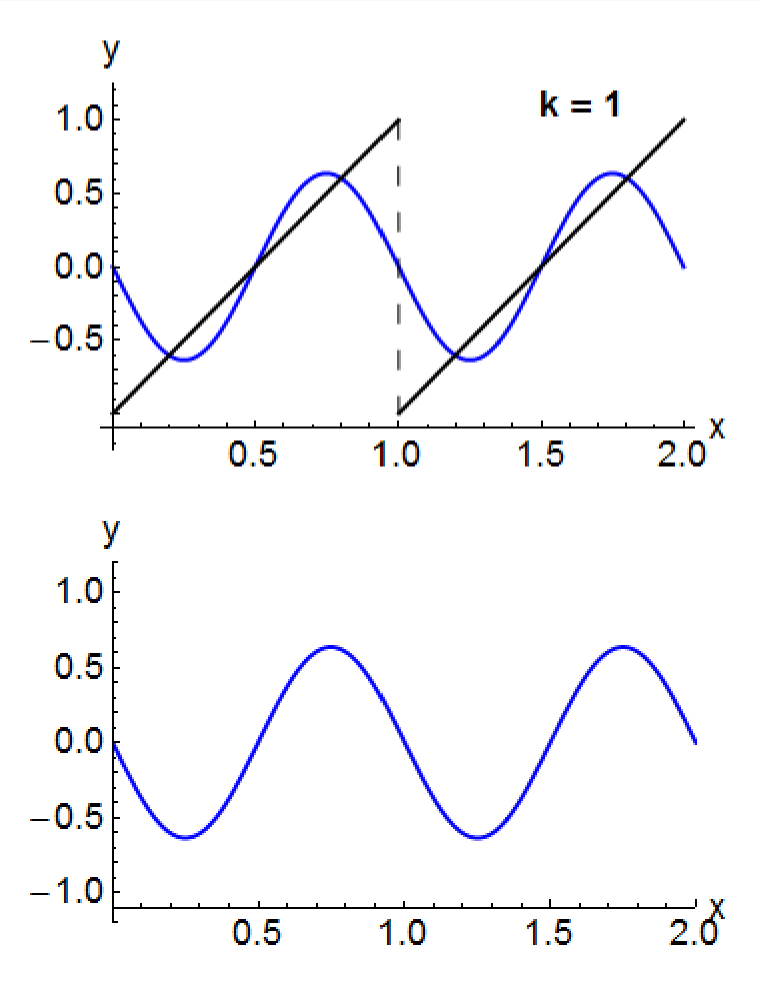
… leads to the Fourier series:

$$g(x) = \frac{1}{2}\pi - \frac{4}{\pi}\left[\frac{\cos(x)}{1^2} + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + \ldots\right]$$

And for n < 4 the function g(x) looks like:



Convergence



Does the Fourier series converge to the given/unknown function? How?

*The Fourier series of a function f(x) converges at a given point x, if the function is **differentiable at x**. That means f'(x) exists.*

*Discontinuity: If the function has left and right derivatives at x, then the Fourier series **converges to the average of the left and right limits**.*

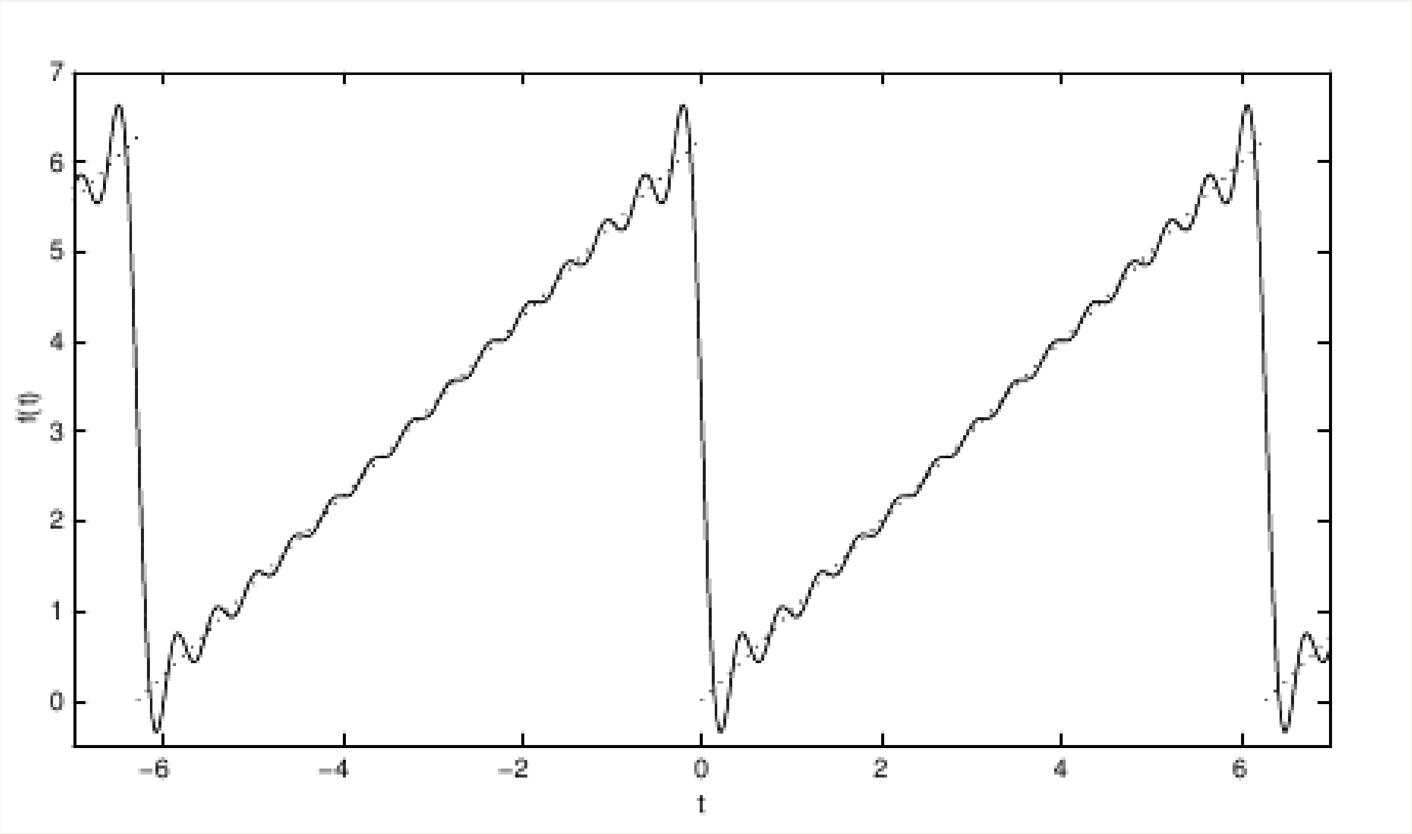
⇒ *point-wise convergence*

Do you see the problem?

Gibbs phenomenon

(Strong) undulations close to the discontinuities!

→ additional (very high frequency) upper tones to the main frequency



Fourier transformation

What is it doing?

Fourier series to transformation

... change the interval from [-π; π] to [-T/2; T/2], then:

(We are still dealing with ***periodic*** *functions!)*

$$f(t) = \frac{1}{2}a_0 + \sum_{k=1}^{N}\left[a_k \cos\left(\frac{2\pi kt}{T}\right) + b_k \sin\left(\frac{2\pi kt}{T}\right)\right]$$

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi kt}{T}\right)dt$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi kt}{T}\right)dt$$

$[-T/2; T/2]$

Fourier series to transformation

... substitute cos and sin by the complex exponential function ...

$$\cos x = \Re\left(e^{ix}\right) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \Im\left(e^{ix}\right) = \frac{e^{ix} - e^{-ix}}{2i}$$

... and $$\omega_k = \frac{2\pi k t}{T}$$

Thus, we get the complex Fourier series:

(How do the coefficients f_k look like?)

$$f(t) = \sum_{k=-\infty}^{\infty} f_k\, e^{i\omega_k t}$$

What happens, when we deal with ***non-periodic*** *functions?*

Fourier series to transformation

... Limes T → ∞ ; i.e. interval of periodicity is:

$$[-T/2; T/2]_{T\to\infty} \qquad \Delta\omega = 2\pi/T \to 0$$

That means: Steps between neighbouring frequencies become smaller and smaller.

The infinite sum of the Fourier series turns into an integral.

Fourier transformation

… splits a continuous, aperiodic signal in a continuous spectrum.

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

Forward transformation

$$f(t) = \int_{-\infty}^{\infty} f(\omega) e^{i\omega t} d\omega$$

Inverse transformation

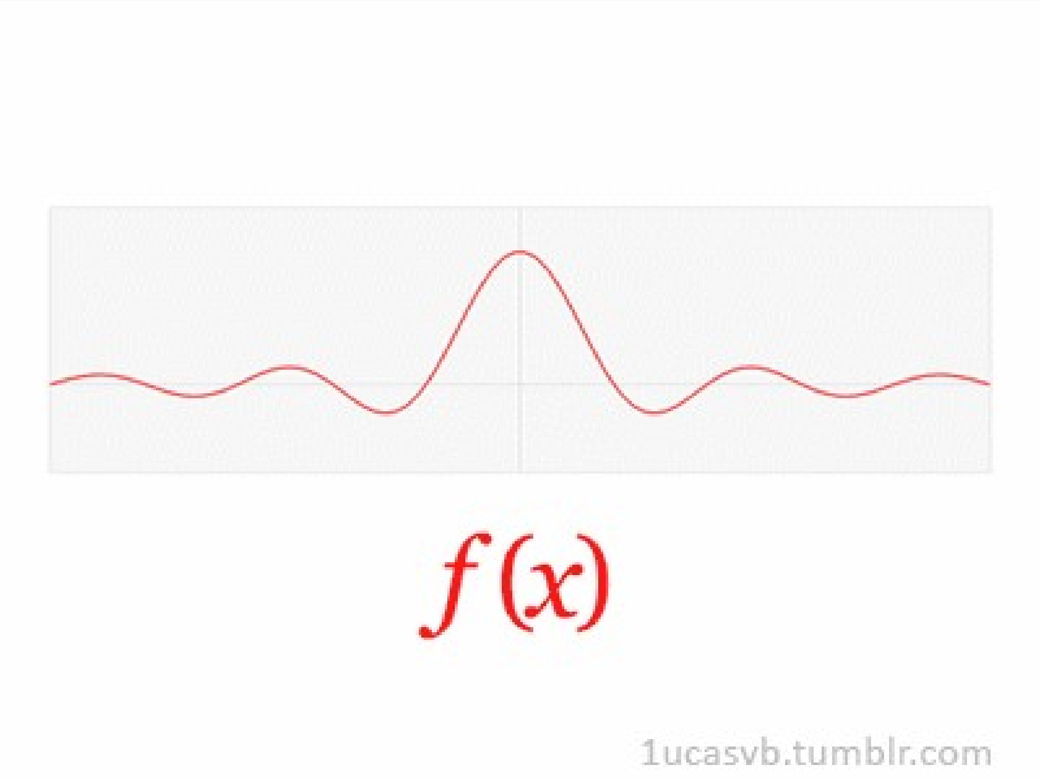
F(ω) is called the complex spectrum of f(t)

Be careful with sign conventions!!!!

The Fourier transform pair is defined: $f(t) \Rightarrow F(\omega)$

Fourier transformation

Fourier transformation

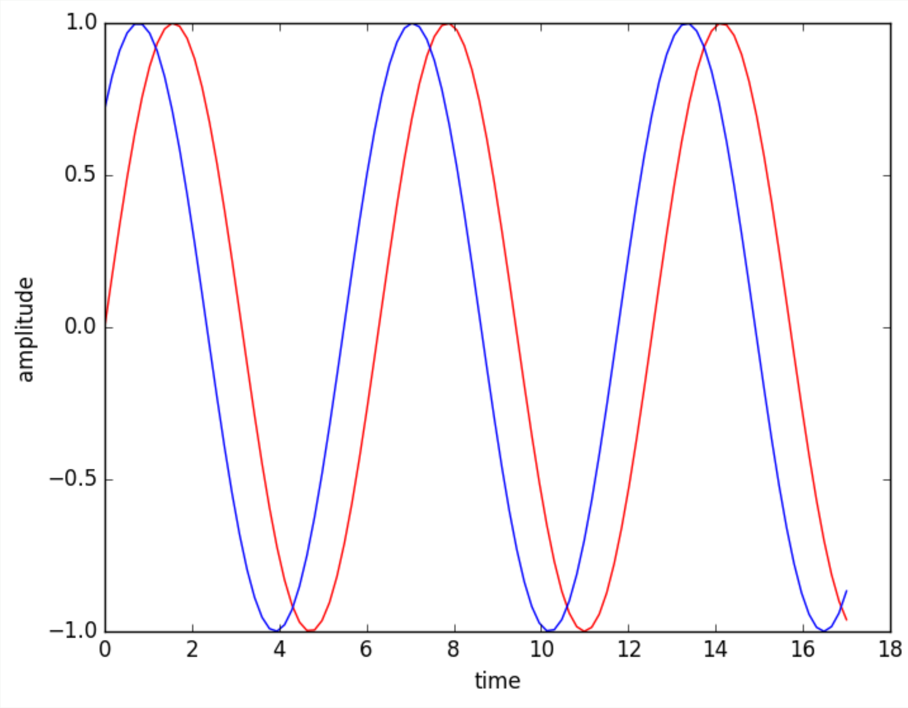


Amplitude & phase

$$F(\omega) = \Re(\omega) + i\Im(\omega) = A(\omega)e^{i\Phi(\omega)}$$

$$A(\omega) = |F(\omega)| = \sqrt{\Re^2(\omega) + \Im^2(\omega)}$$

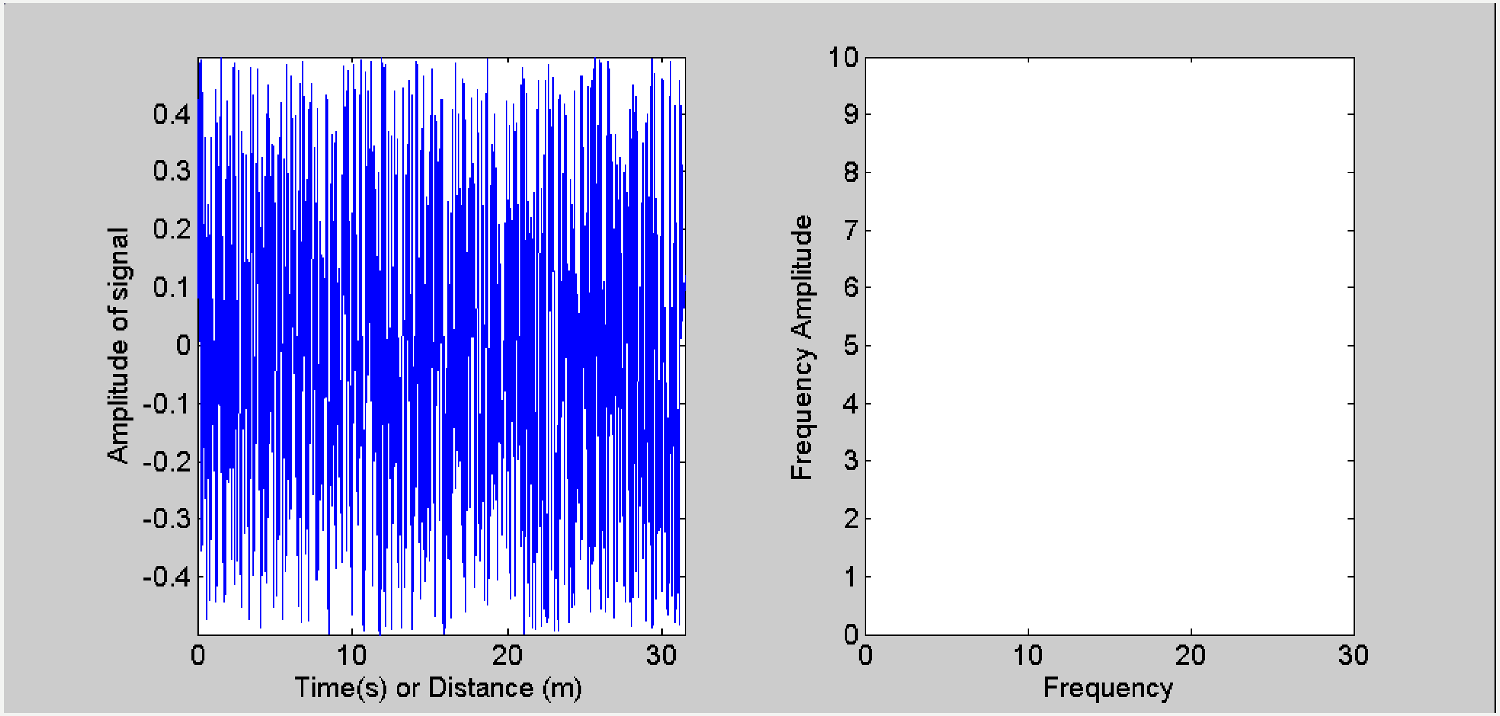
$$\Phi(\omega) = \arctan \frac{\Im(\omega)}{\Re(\omega)}$$



In most applications it is the amplitude spectrum that is of interest. However, there are cases where the phase spectrum plays an important role (resonance, seismometer).

Examples

Random signal

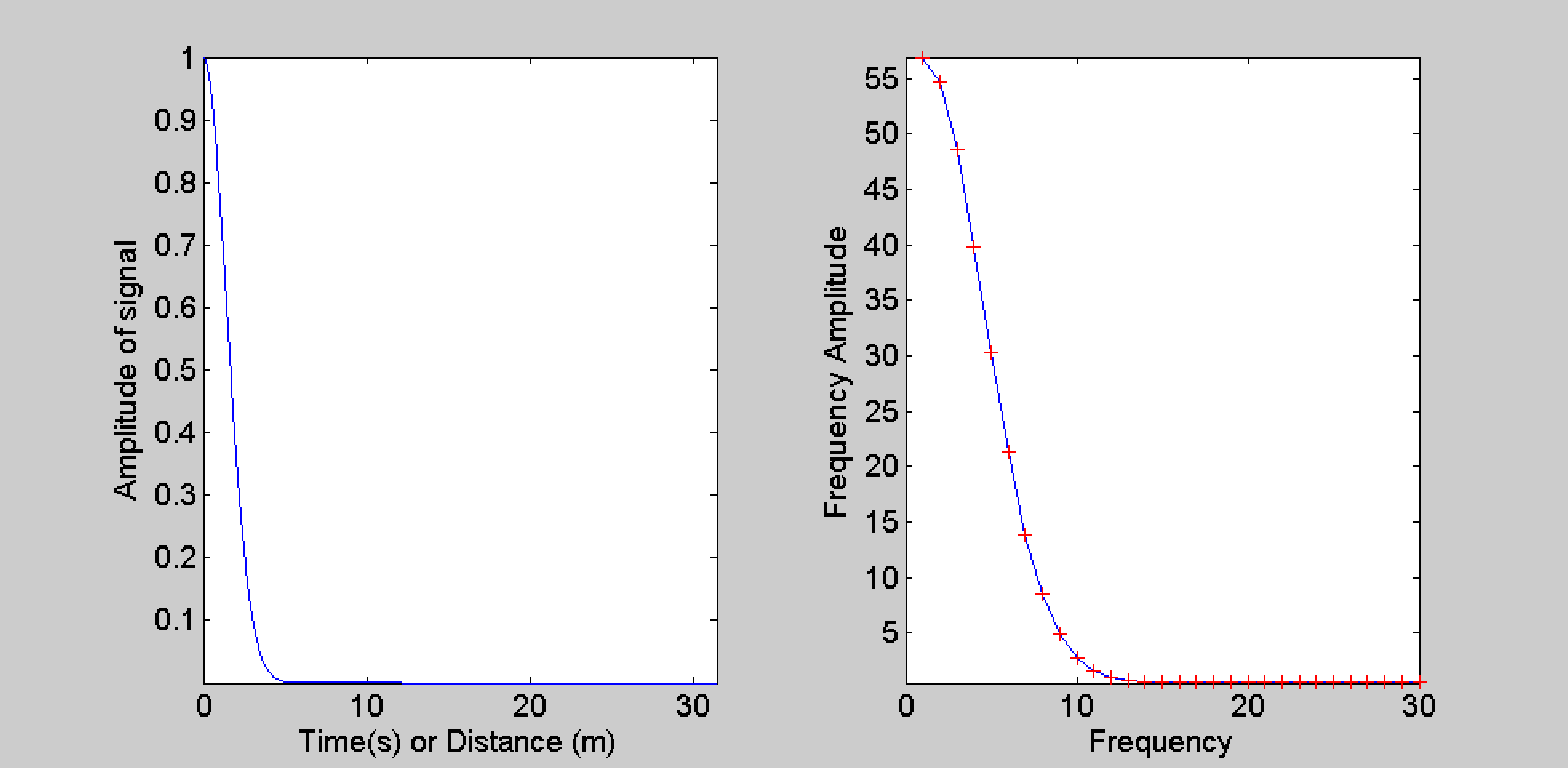


Random signals may contain all frequencies

→ Spectrum with constant contribution of all frequencies = white spectrum

Examples

Gaussian signal $$f(t) = e^{-at^2}$$ $$F(\omega) = \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$$



The spectrum of a Gaussian function is a Gaussian function again.

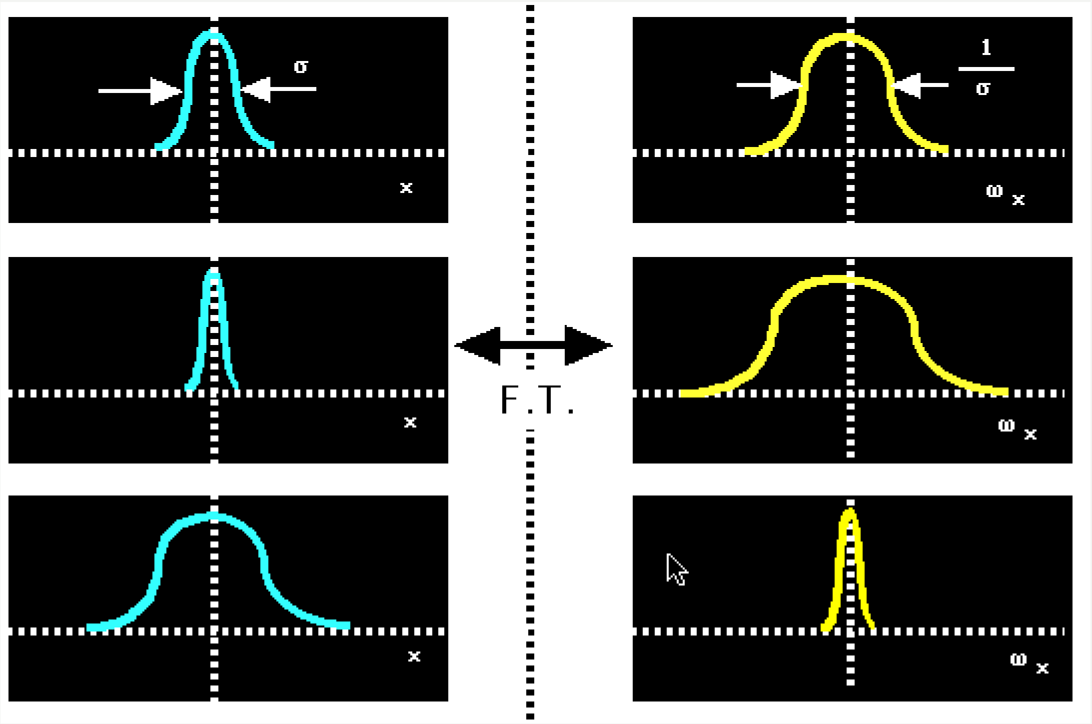
What happens when the Gaussian is made narrower?

Examples

Gaussian signal

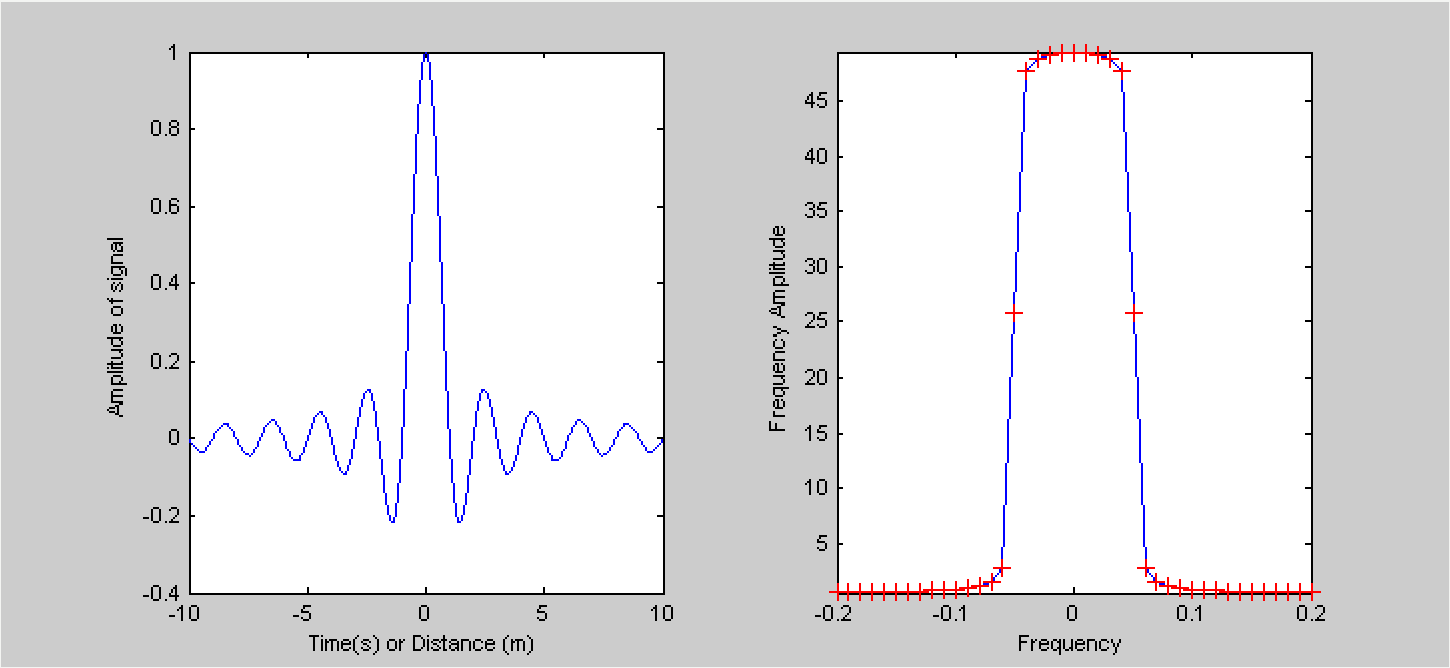
$$f(t) = e^{-at^2}$$

$$F(\omega) = \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$$



Examples

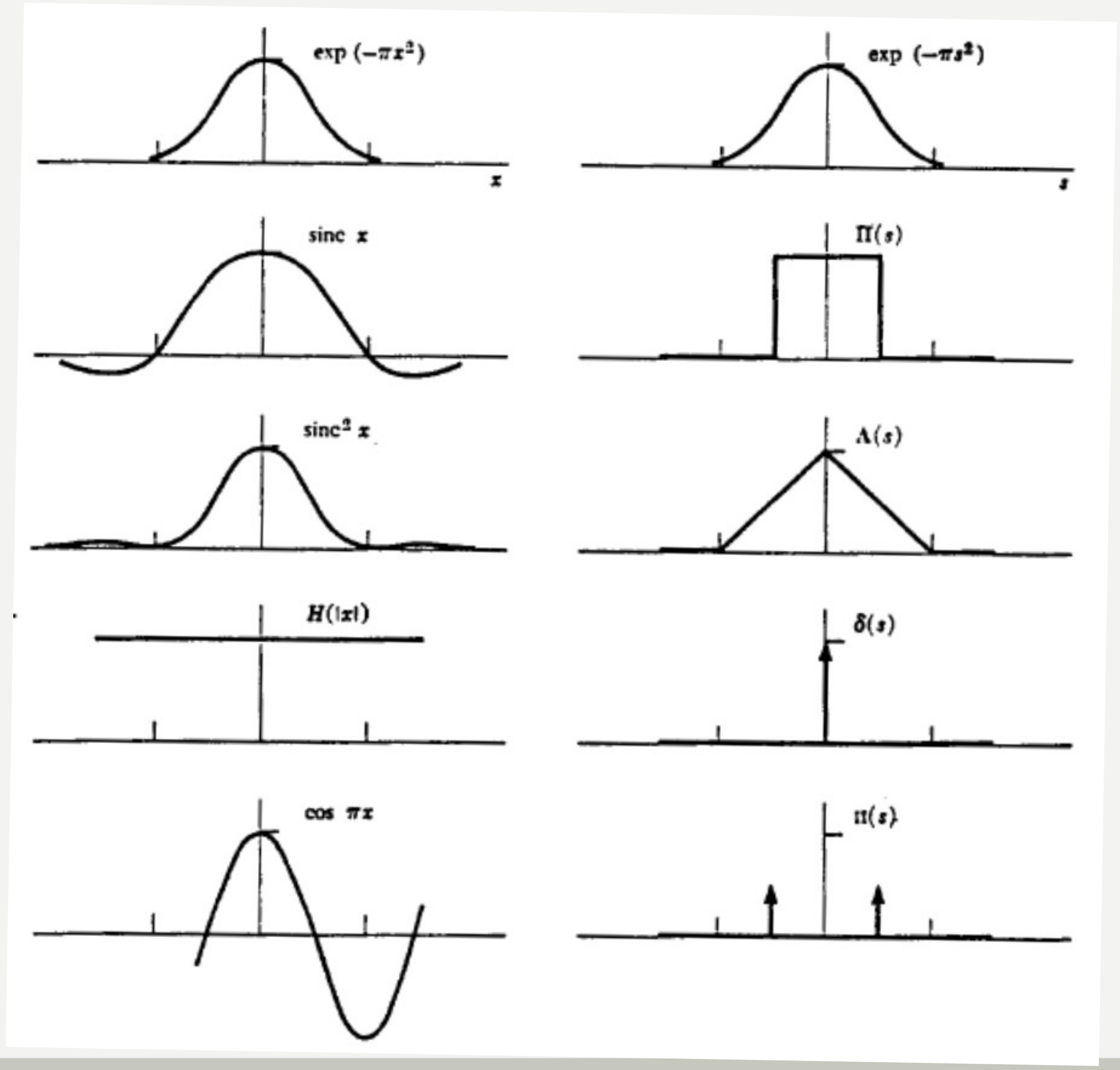
Transient waveform



… is a waveform limited in time (or space) in comparison with infinite harmonic waveforms

Examples

f(t) *F(ω)*



Joke

*What says the Fourier transform of the triangle pulse to
the Fourier transform of the sinc function?*

What says the Fourier transform of the triangle pulse to the Fourier transform of the sinc function?

You are such a square!

Recommendation (with interactive content):

betterexplained.com

www.fourier-series.com

Question

Can you think of three situations where considering the spectrum would be more useful than the time domain?

Why?

Answer

Can you think of three situations where considering the spectrum would be more useful than the time domain?

Why?

1. *Removing instrument response. – Mathematically more easy.*
2. *Analysing seismic noise. - Different noise sources have different frequencies.*
3. *Calculate seismic moment and moment magnitude. - Mathematically more easy.*
4. *Attenuation studies. - Mathematically more easy.*
5. *Studying seismic phases. - Clear phase separation possible.*

Conditions to the transform

- *f(x) needs to be integrable (at least point-wise, with finite numbers of jumps)*

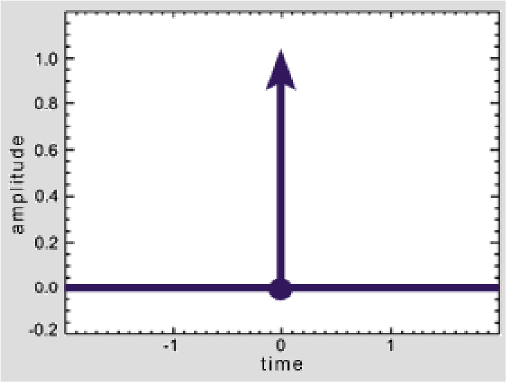
$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$$

- *limits need to exist from both sides of a jump*

For real-world signals, there is never a problem of existence.

Idealized signals (e.g. sinusoids – infinite!) have a normalization difficulty.

→ *use Dirac's delta function*



Dirac's delta function

… so weird but so useful …

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

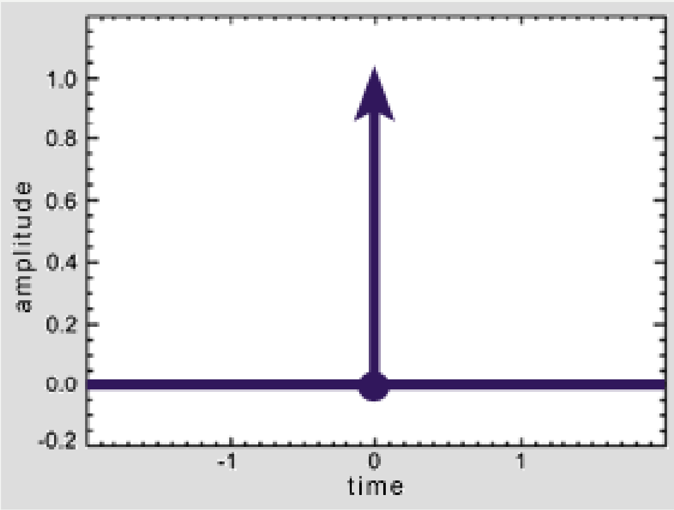
$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

$$\int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0)$$

$$f(t)\delta(t-a) = f(a)$$

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

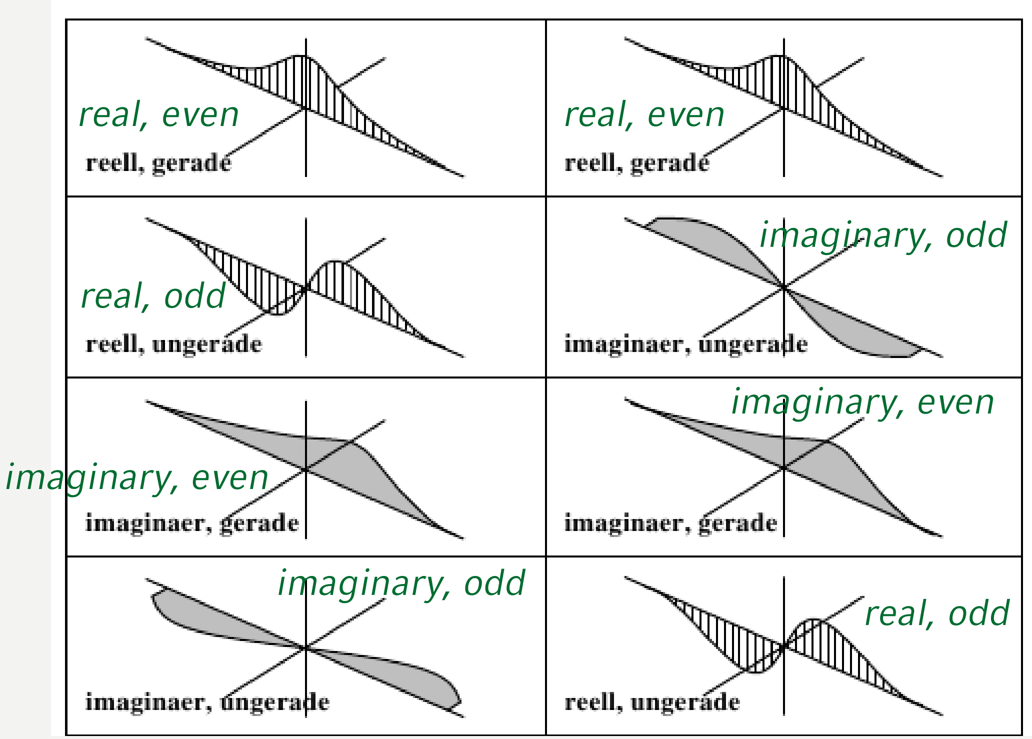
$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} d\omega$$

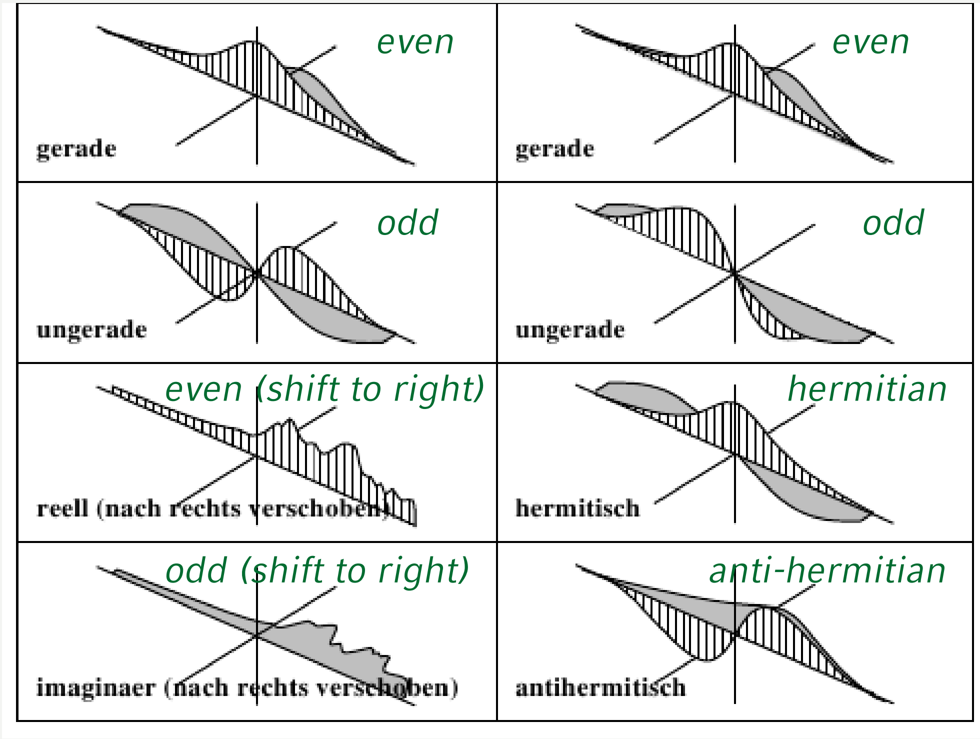


Properties of the Fourier Transform

Linearity: $$a f_1(t) + b f_2(t) \Rightarrow a F_1(\omega) + b F_2(\omega)$$

Symmetry: $$f(-t) \Rightarrow 2\pi\, F(-\omega)$$

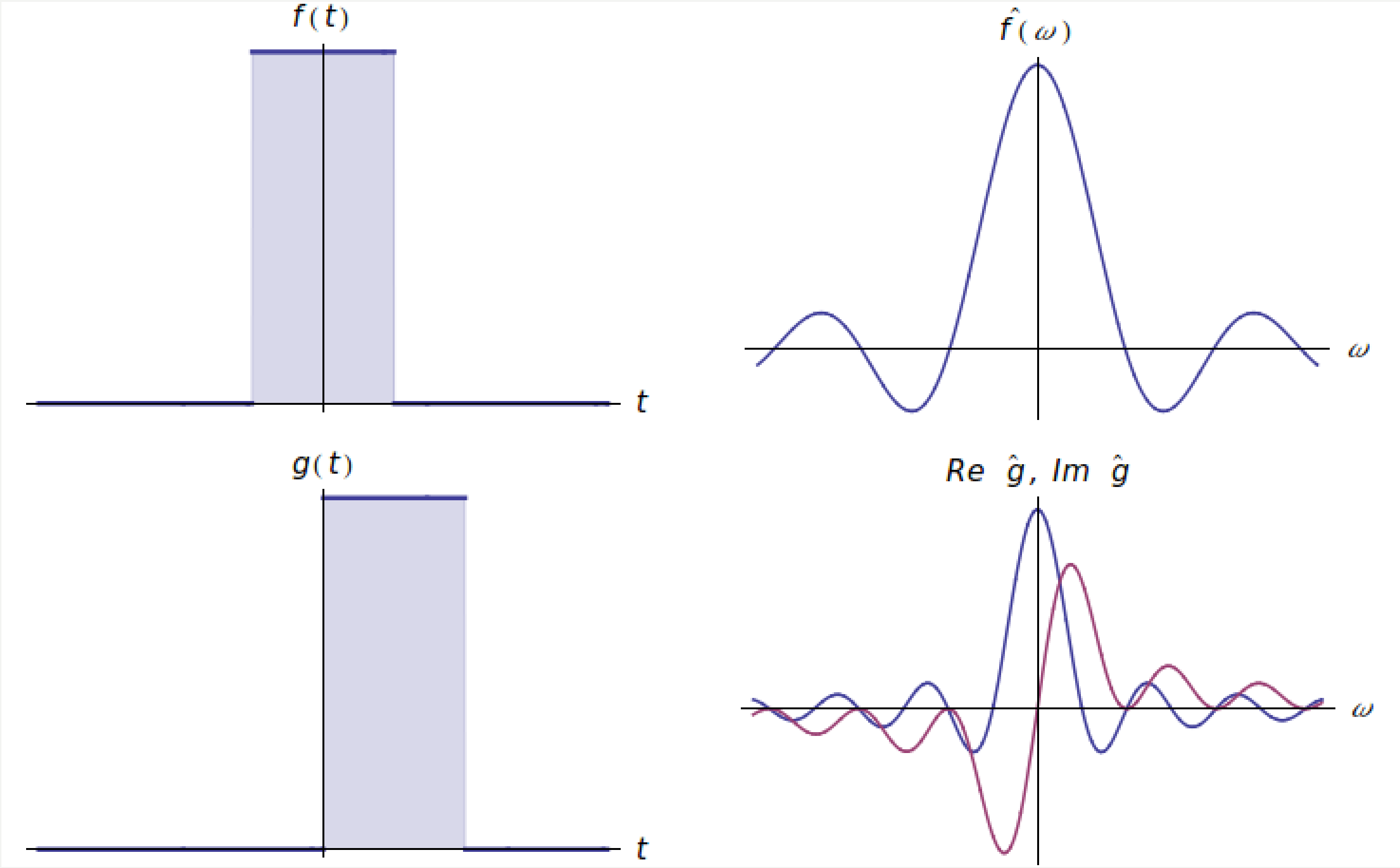




Properties of the Fourier Transform

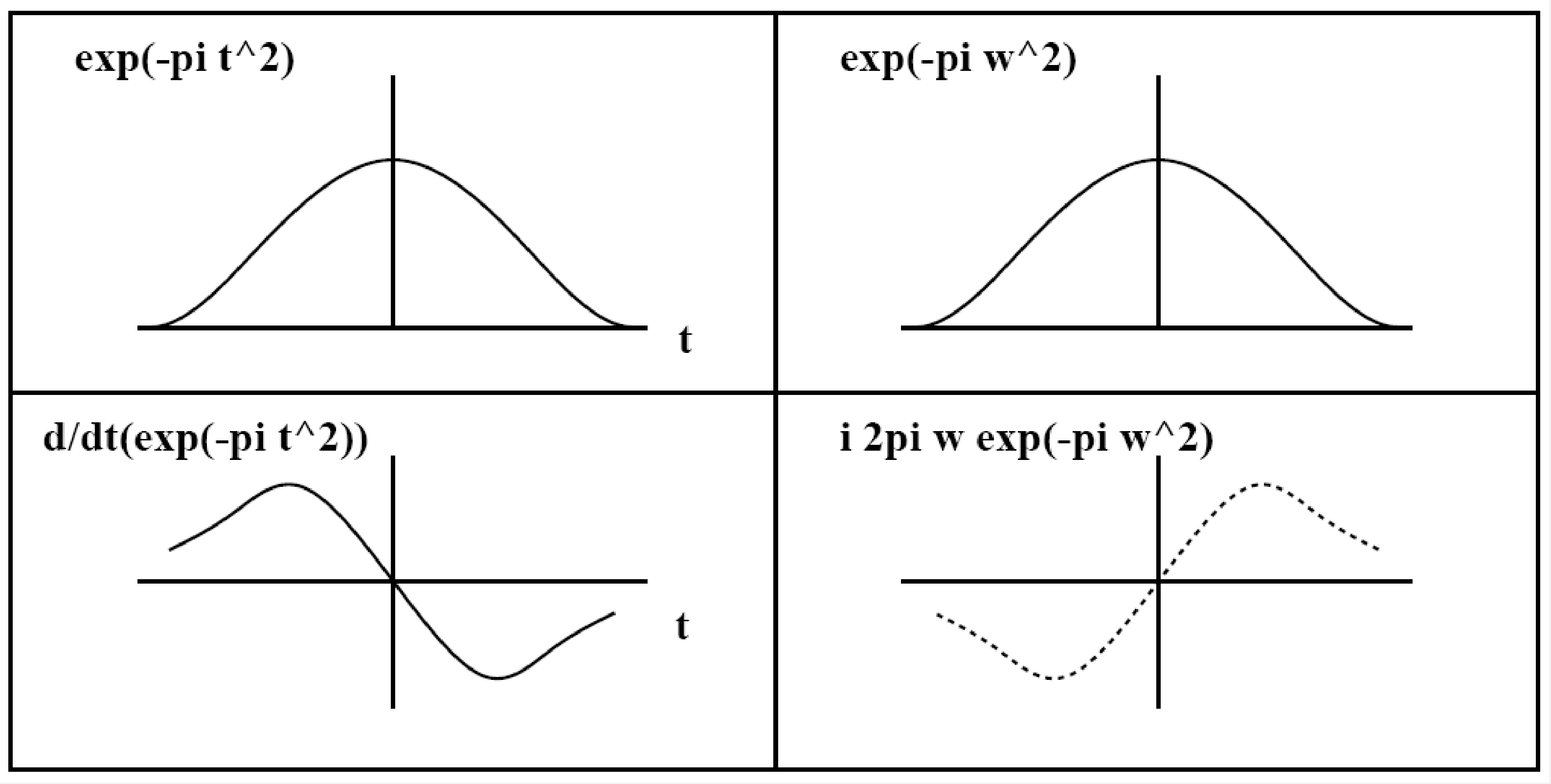
Time shifting:

$$f(t+\Delta t) \Rightarrow e^{i\omega\Delta t} F(\omega)$$



Properties of the Fourier Transform

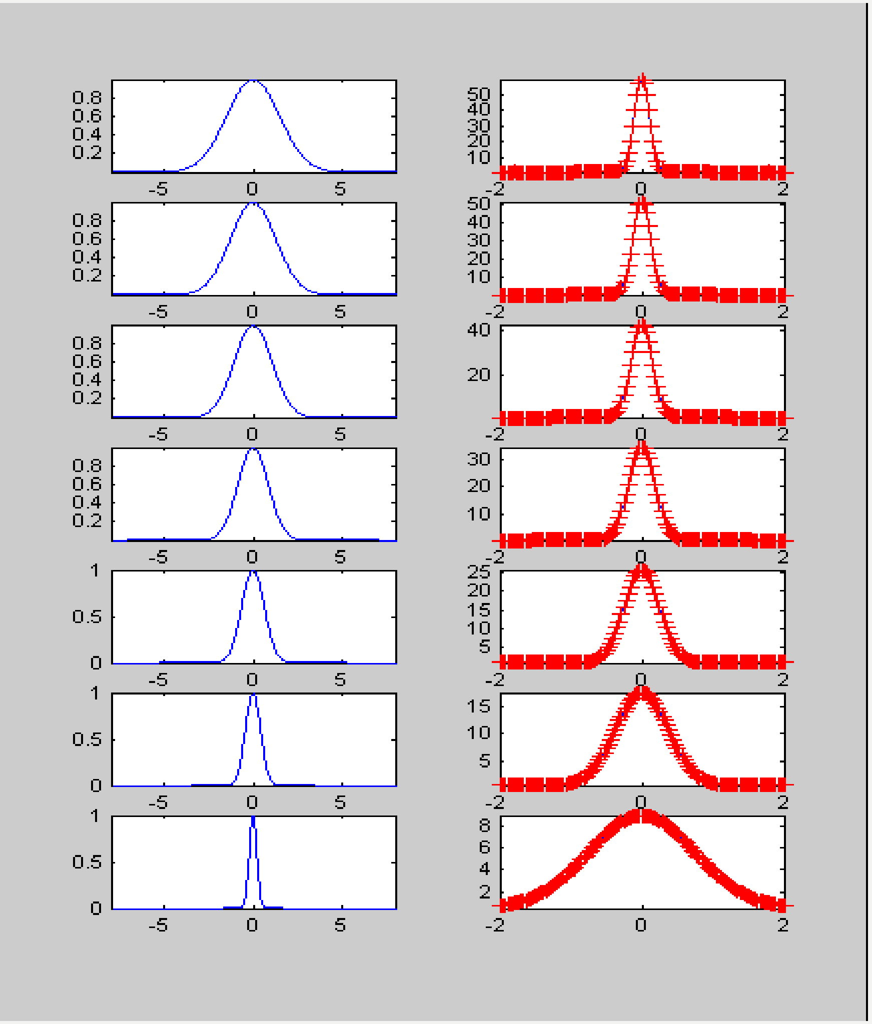
Time differentiation: $$\frac{\partial^n f(t)}{\partial t^n} \Rightarrow (-i\omega)^n F(\omega)$$



Pulse-width & Frequency Bandwidth

time(space) *spectrum*

Narrowing physical signal

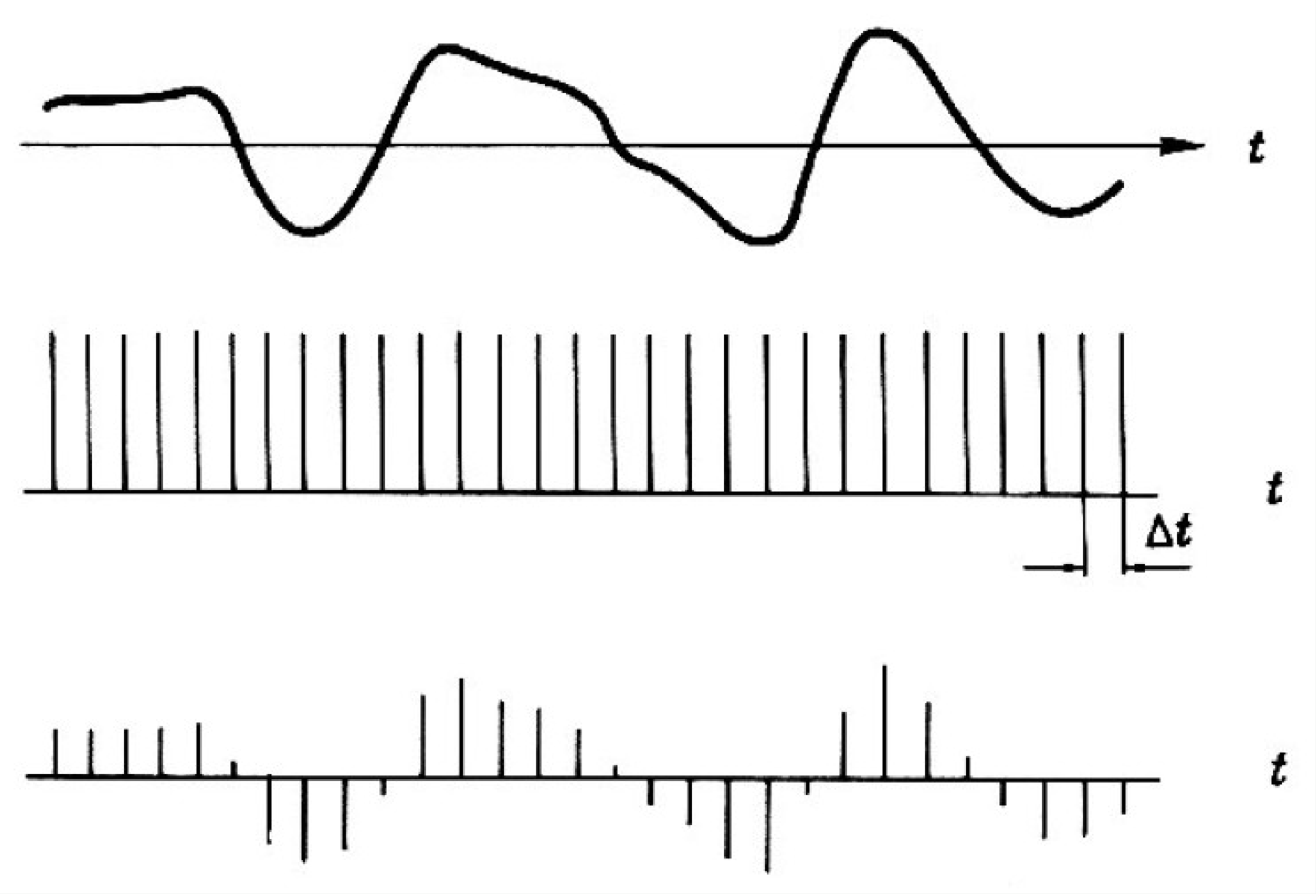


Widening frequency band

The real, digital world

... is NEITHER periodic NOR continuous!

$$g_S(t) = g(t) \sum_{j=-\infty}^{\infty} \delta(t - j\,dt)$$



$g_s(t)$ is the digitized version of $g(t)$

The sum is called the comb function.

The Fourier transform

Whatever we do on the computer with data will be based on the discrete Fourier transform.

discrete

$$F_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j \, e^{-2\pi i k j / N}$$

$$f_k = \sum_{j=0}^{N-1} F_j \, e^{2\pi i k j / N}$$

$$k = 0, 1, ..., N-1$$

continuous

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$f(t) = \int_{-\infty}^{\infty} f(\omega) e^{i\omega t} d\omega$$

Discrete Fourier Transform

- *j measures time in units of sampling interval*

$j = k\Delta t$ *up to maximum time* $T = N\Delta t$

- *k measures frequency in intervals of the sampling frequency*

$\Delta f = 1/T$ *up to maximum of sampling frequency*

$$f_{max} = N\Delta f = 1/\Delta t$$

- *angular frequency:* $\omega_k = \frac{2\pi k}{T} = \frac{2\pi k}{N\Delta t} = 2\pi k\Delta f$

$$F_k = \frac{1}{N}\sum_{j=0}^{N-1} f_j\, e^{-2\pi ikj/N}$$

$$f_k = \sum_{j=0}^{N-1} F_j\, e^{2\pi ikj/N}$$

$$k = 0, 1, ..., N-1$$

⇨ *increase sampling density by zero padding*

⇨ *increase frequency resolution by using longer signal*

Fast Fourier Transformation

Spectral analysis became interesting for computing with the introduction of the Fast Fourier Transform – FFT – developed in 1965.

Written as matrix-vector product the inverse discrete Fourier transform looks like:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \dots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \dots & \omega^{2N-2} \\ \dots & & & & & \dots \\ 1 & \omega^{N-1} & \dots & \dots & \dots & \omega^{(N-1)^2} \end{bmatrix} \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ \dots \\ F_{N-1} \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \dots \\ f_{N-1} \end{bmatrix}$$

with

$$\omega = e^{2\pi i/N}$$

Fast Fourier Transformation

The full matrix-vector multiplication can be written as a few sparse matrix-vector multiplications.

Number of multiplications

full matrix	FFT
N^2	$2N \log_2 N$

Problem	full matrix	FFT	ratio full / FFT
1D (nx = 512)	2.6×10^5	9.2×10^3	28
1D (nx = 2096)			95
1D (nx = 8384)			313

Exercise

Show that the discrete Fourier Transform of the sequence

$$(0,\ 1,\ -1,\ 0)$$

is

$$(A0, A1, A2, A3) = [\ 0,\ \ 1/4(1-i),\ \ -1/2,\ \ 1/4(1+i)\ \ ]$$