

# The Basic Theory of Wave Gradiometry

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Various methods of analyzing seismological data are available that allow the observation of seismic wave characteristics drawing on fundamental relationships between propagating waves and the medium through which they travel. Exploiting these relationships using time and phase properties across an arrays of sensors, are among others, reflectivity and refraction, providing complementary information on the earth's internal structure through using inversion techniques. There is however, a method using the differences of waves that is able to obtain expressions for geometrical spreading and the horizontal-slowness both as a function of position, available to provide additional insight to the nature of these waves and the earth's interior. Known as Wave Gradiometry, the displacement field is obtained over a single or multi-dimensional array of sensors and then the field's spatial gradient is obtained and manipulated to yield information on the disturbance. In this brief overview, the basic principles of Gradiometry will be established beginning with the seismic wave equation and its resulting displacement solution being used to build the main equation describing the spatial gradient field from which the geometrical spreading and wave-slowness are extracted in the form of coefficients in one dimension. An example of calculating these parameters using a synthesized source wave-train will be used to demonstrate the theory analytically and its straight forward nature as well as to highlight some issues to consider with this method that may occur in its application. It will be shown that the same process used to obtain and extract seismic information across a 1D array can be ported to a 2D model using the same single dimensional computations across all dimensions with appropriate considerations.

Seismic sensor arrays are necessary for a variety of analysis techniques. It is common for these arrays to reside in a 2D geometry on the earth's surface with additional probes above and below the ground to collect data to then take modeling into 3D with necessary additional processing. The methods using reflection and refraction calculations commonly select only portions of the array during processing to be able to reduce the complexity of solving for the system by making certain assumptions of a wave's nature given the virtual sensor geometry presented to the wave, a linear array for example. The same can be said for an array's ability to resolve a range of frequencies through the use of not only sensor element spacing but by interleaving certain sensors within the array to augment the array's apparent size to correspond to wavelengths of interest for a given disturbance. Ultimately, a time or frequency domain approach is used on the resultant data to observe phase delay across the extent of the array then to use that information to determine particle displacement associated with the wave and its velocity through differentiation. This can be done through application of the familiar 1D seismic wave equation for a planar source with sheer velocity given by:

$$\frac{1}{c^2} \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} \quad (1)$$

In practice this presents a problem where to have a reliable value for the displacement as a result from which to extract the desired parameters, all three of these components in equation (1) must be measured reliably and more than likely this expression is not accounting for the full physics of the system making it not very robust. It has been proposed that the first-order derivative of the displacement, the displacement gradient, be used to relate the wave equation to a general wave motion solution of the form [1]:

$$u(t, x) = G(x)f(t - p(x - x_0)) \quad (2)$$

which incorporates the properties of geometrical spreading  $G(x)$  and horizontal slowness  $p(x)$  for a vertically inhomogeneous medium with large-scale anisotropy with respect to the disturbance wavelength  $\lambda$ .

Addressing the significance of the parameters included in equation (2) is also necessary in the development and understanding of how this method can provide useful seismic information. Geometrical spreading is a direct consequence of the conservation of the amount of energy in a propagating wave disturbance as a function of distance from the source with respect to some observation point. For a 1D or plane wave this point will be  $x_0$  and essentially describes the wave's amplitude behavior as a function of its position. Common models for spreading depend on the dimensionality of the system where the 1D or plane wave model has a planar source for which  $G(x)$  is constant, a 2D or cylindrical wave originates from a line source and  $G(x) = x^{-1/2}$ , and for a 3D or spherical wave that emanates from a point source  $G(x) = x^{-1}$  where for the latter two models  $x$  is viewed as a radial distance from the source. The horizontal slowness or ray parameter is a function of the wave velocity  $v$  and the initial incidence angle  $\theta$  of the disturbance generated by the source measured from the vertical as:

$$p(x) = \frac{\sin\theta}{v} \quad (3)$$

This shows how a wave's velocity will vary proportionally with its angle of entry into a horizontally homogeneous media and inversely proportional to its phase speed which itself is subject to a velocity gradient occurring as a function of position (this would be a vertical dependence given the specified model). In more complex media the geometrical spreading can be related to the velocity gradient which results in a wave slowness that would vary with distance so an appropriate spreading parameter needs to be selected as  $G(x) = x^{-n}$  for example, where  $n$  would be determined through experimentation.

Equation (2) is a solution to the following forward-traveling component of the factored wave-equation of (1):

$$\frac{1}{c^2} \frac{\partial^2 u(x, t)}{\partial t^2} - \frac{\partial^2 u(x, t)}{\partial x^2} = u(x, t) \left( \frac{1}{c} \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) \left( \frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) = 0 \quad (4)$$

Equation (4) provides an intuitive view on a direct link of the general solution in (2) and the wave characteristics that can be inferred by the displacement field in the time-domain.

The basis for Wave Gradiometry is the observation of spatial variations of the displacement field and to do so, the spatial gradient is had from the first-order differentiation of the displacement function (2) as:

$$\frac{\partial u(x,t)}{\partial x} = \frac{G'(x)}{G(x)} u(x,t) - \left[ p + \frac{\partial p(x-x_0)}{\partial x} \right] \frac{\partial u(x,t)}{\partial t} \quad (5)$$

Where the displacement gradient serves to isolate the desired parameters in the form of two coefficients:

$$A(x) = \frac{G'(x)}{G(x)} \text{ and } B(x) = - \left[ p + \frac{\partial p(x-x_0)}{\partial x} \right] \quad (6)$$

Appropriate integration of these coefficients with respect to the spatial variable over the observation interval  $[x, x_0]$  yields geometrical spreading and horizontal slowness respectively:

$$\int_{x_0}^x A(x) dx = \ln \left( \frac{G(x)}{G(x_0)} \right) \text{ and } - \frac{1}{(x-x_0)} \int_{x_0}^x B(x) dx = p \quad (7)$$

At this point one may choose to obtain approximate solutions in either the time domain using a simple matrix inversion on the system generated from (5) using successive adjacent time-samples for a fixed  $x$ :

$$\begin{bmatrix} u(t_0, x) & \frac{\partial u(t_0, x)}{\partial t} \\ \vdots & \vdots \\ u(t_i, x) & \frac{\partial u(t_i, x)}{\partial t} \end{bmatrix} \cdot \begin{bmatrix} A(x) \\ B(x) \end{bmatrix} = \begin{bmatrix} \frac{\partial u(t_0, x)}{\partial x} \\ \vdots \\ \frac{\partial u(t_i, x)}{\partial x} \end{bmatrix} \quad (8)$$

A frequency domain approach is a little more direct after a Fourier transform is applied to (5):

$$\mathcal{F} \left[ \frac{\partial u(x,t)}{\partial x} \right] = \mathcal{F}[u(x,t)] A(x) + i\omega \mathcal{F}[u(x,t)] B(x) \quad (9)$$

It is easy to see that if both sides of (9) are divided by  $\mathcal{F}[u(x,t)]$ , the coefficients  $A(x)$  and  $B(x)$  are obtained by equating the resulting quotient of transforms to its real and imaginary parts respectively. The solution options thus far present concerns that arise with either the computational burden manifested by (8) when inversion of a matrix representing a highly tessellated or spatially large (or both) solution space is necessary, then there is the issue of sensitivities regarding dividing spectral transforms as required in (9) not to mention extensive filtering that may be necessary in the frequency domain given the nature of the displacement data. The temporal and spatial derivatives are also a possible source of error and require special attention regarding their optimization.

A more concise method involves the use of the analytical form of the time series of (2). The analytical signal is obtained by employing the Hilbert transform which allows, in this case, mathematical manipulations to circumvent some of the difficulties in handling spectral ratios previously mentioned and may allow for more efficient data handling than dealing with large matrices in the time domain method. The Hilbert transform produces a complex-valued function which discards the negative frequency components of the real-valued input signal thus making the result amenable to determining

the instantaneous phase and then instantaneous frequency which is a key term in the derivation to follow. In terms of the displacement, the analytical time series result is defined as [2]:

$$U = u(x, t) - i\mathbb{H}[u(x, t)] \quad (10)$$

Now equation (5) will take the analytical form of:

$$U_x = A(x)U + B(x)U_t \quad (11)$$

Where the subscripts of  $x$  and  $t$  represent the analytical forms of the spatial and temporal displacement derivatives respectively. The benefit of this method can best be seen as each signal can now be viewed as a phasor, accounting for amplitude and phase, so (11) is now cast in the form:

$$|U_x|e^{i\psi} = A(x)|U|e^{i\phi} + B(x)e^{i\phi} \left[ \frac{\partial|U|}{\partial t} + i|U| \frac{\partial\phi(t)}{\partial t} \right] \quad (12)$$

The derivative of the instantaneous phase with respect to time in the second term above is the instantaneous frequency:

$$\frac{\partial\phi(t)}{\partial t} = \omega(t) = \frac{\frac{\partial u(x,t)}{\partial t} \mathbb{H}[u(x,t)] - u(x,t) \mathbb{H}\left[\frac{\partial u(x,t)}{\partial t}\right]}{|U|^2} \quad (13)$$

Minor manipulations of (12) while substituting in (13) then separating the resulting equation into its real and imaginary parts allows one to obtain the following expressions for the desired aforementioned coefficients:

$$A(x) = \frac{|U_x|}{|U|} \left[ \cos(\psi - \phi) - \frac{\partial|U|/\partial t}{\omega(t)|U|} \sin(\psi - \phi) \right] \quad (14a)$$

$$B(x) = \frac{|U_x|}{|U|\omega(t)} \sin(\psi - \phi) \quad (14b)$$

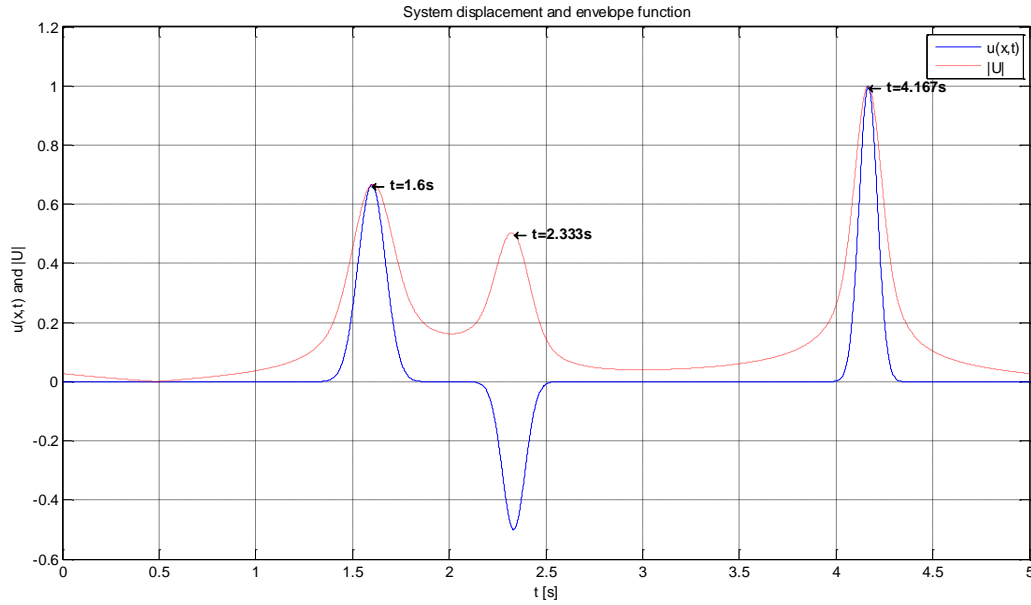
The straight forward integration from (7) still applies and allows extraction of the geometrical spreading and ray parameter terms as functions of position.

To demonstrate the functionality of Wave Gradiometry requires an example which will be recreated in detail from [2]. As the main focus of this paper is the theory involved, all attention here will be on analytical inputs and results with some discussion of issues that may be encountered when certain scenarios are presented as inputs. A wave train of summed Gaussian-type disturbances is synthesized, the time history results manipulated, and then the coefficients produced will be shown to yield the wave parameters contained in the unique waves. First, each wave is defined by the Gaussian equation:

$$u(x, t) = A \frac{e^{-\alpha^2(t-px-\tau)^2}}{x} \quad (15)$$

With  $A$  being amplitude,  $\alpha$  describing the pulse-width,  $\tau$  is the time-delay, and the rest previously described. This implies that the spreading term is inversely proportional to the distance traveled as in the spherical case and among three waves the slowness will be specified  $p = \frac{2}{5}, -\frac{1}{3},$  and  $\frac{2}{3}$ . The other

wave parameters are chosen as to give the total displacement time-series and  $|U|$  or envelope function of the following figure:



**Figure 1 – Gaussian wave-train displacement function from the sum of 3 waves using equation (15) and overlaid envelope function or  $|U|$  with annotated times at corresponding peaks**

As seen in the formulations from above, the envelope function is of particular importance given its repeated use. As it is used solely in the denominator of terms present in the calculation of  $A(x)$  and  $B(x)$ , those terms are then prone to singularities where the envelope function has a zero. One must be cautious in implementing an algorithm to handle such a mathematical anomaly especially in this analytical example as it is clear to see from Figure 1 that around  $t = 0.5$  seconds one will occur and similarly problems of very large calculated values will may arise for arbitrarily small values of  $|U|$ . An interesting characteristic of the envelope function is that as a result of the analytical signal  $U$ , one can see that given the waves' proximity to one another, especially waves 1 & 2 between 1 and 3 seconds, there is significant amplitude occurring between the waves as a consequence. These interactions between the waves from their spectral components can propagate into the coefficient solutions thus pointing out a substantial weakness in this method. The interaction of waves with very little separation in time, especially with similar frequency content and relative amplitudes can greatly interfere with the outcome of both  $A(x)$  and  $B(x)$ . This is due to the initial displacement given by equation (2) being inadequate to handle wave interference of this nature [2]. In the interest of merely illustrating the proper function of using Wave Gradiometry without too much discourse on particular contingencies and in conjunction with the issue of closely interacting waves, it will be said that this method is ideally suited to waves largely separated in time or those with a substantial difference in frequency content.

Otherwise, if this method is to be employed in such an environment where such wave interference is expected, the physical model for the waves as given by equation (2) must be recast to better represent the physical circumstances of such a system while still ultimately satisfying the seismic wave motion equation constrained by the boundary conditions present.

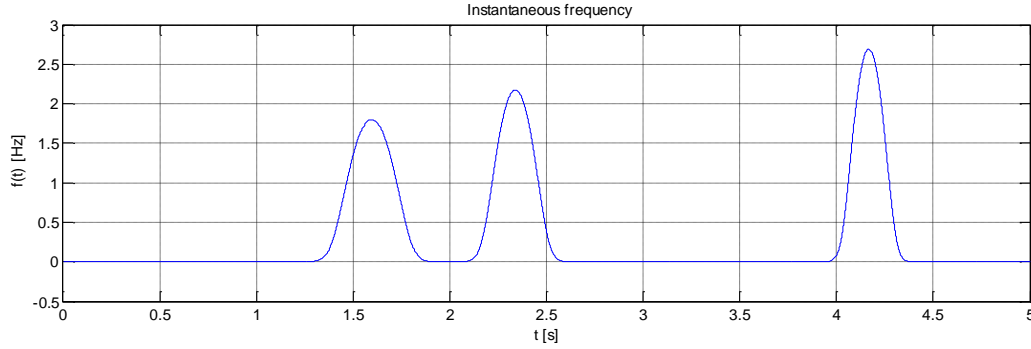


Figure 2 – Instantaneous frequency

Continuing with the determination of the coefficients, the instantaneous frequency  $f(t) = -\omega(t)/2\pi$  [Hz], is shown in Figure 2 with peaks corresponding to those of the displacement and envelope function in time and is functionally well behaved in the vicinity of the waves respectively. A key component is the spatial derivative of the displacement function shown for the analytical solution:

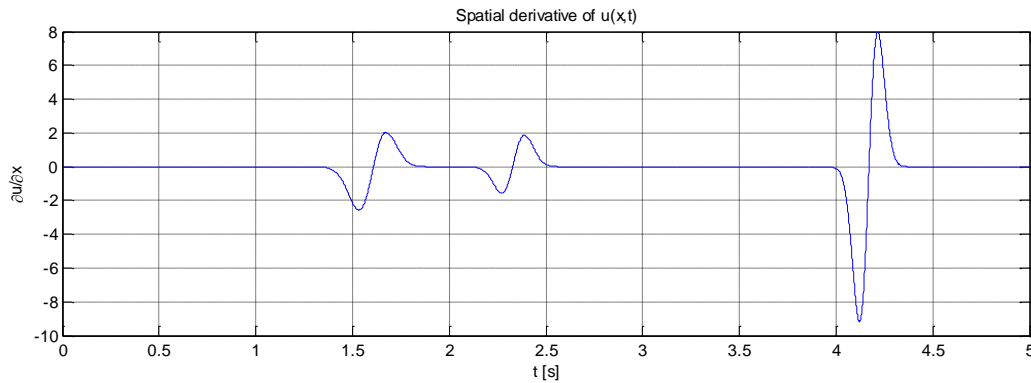
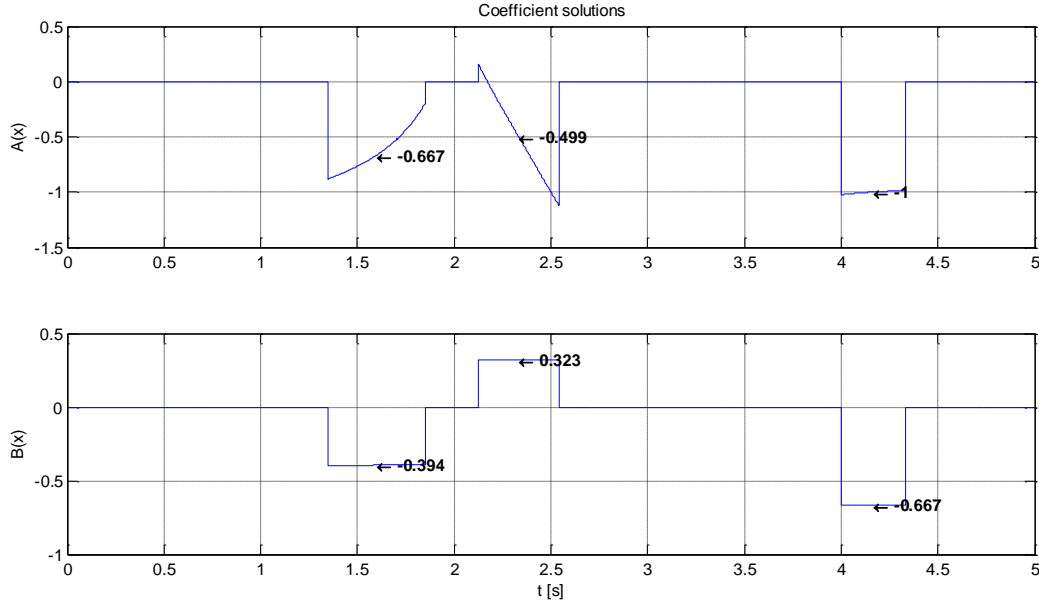


Figure 3 – exact  $\partial u(x, t)/\partial x$

The result here is predictable given the nature of  $u(x, t)$  shown in Figure 1. However, this step is also subject to scrutiny in the applied sense. It is necessary to point out that when taking the numerical derivative as would be required on actual data, care must be taken to not introduce significant errors as a result of using a finite-difference operator, this can be done by adjusting the sampling interval  $\Delta x$  to optimize the calculation in terms of observed disturbance wavelength, for example [1].

Performing the complete calculation on the synthesized waves while windowing only pertinent data in order to circumvent any singularities that would occur elsewhere as previously described, the estimated coefficients are acquired:



**Figure 4 – Coefficient solutions using analytical inputs with annotated solutions occurring at times of interest as marked in the total displacement of Figure 1**

Now to calculate the actual values for geometrical spreading and horizontal slowness corresponding to the values noted in Figure 4 we look back to the integrations of equation (7). For ease of computation we are only interested single points occurring at the specific positions input into the original displacement function of  $x_0 = 1.5, 2.0, 1.0$  in kilometers. As we are using only these points, the required integrations from (7) collapse to very simply reading the values of interest from Figure 4 from the respective coefficient data with very little other manipulation to produce the desired parameters for each wave as follows. Only negative values are obtained from  $A(x_0)$  where the physical result will be of the opposite sign and for the other coefficient, as  $x \rightarrow x_0$ , slowness is obtained by  $-B(x_0) = p(x_0)$  [1]. Obviously not all extractions will be this straight forward, but this simple example on synthesized data serves to demonstrate that the parameter estimation provided by the calculated coefficients does indeed return the expected values to a fairly high degree of accuracy. First for geometrical spreading, recall that equation (15) specifies spherical spreading or  $G(x_0) = x_0^{-1}$  so using the positions  $x_0$  it is immediately evident that the results noted for  $A(x_0)$  are related to  $x_0$  values as  $G(x_0) = -A(x_0)$ . The horizontal slowness in this case is similarly read by simply inverting the sign of the  $B(x_0)$  from the figure. The calculated values versus the  $x_0$  and  $p = \frac{2}{5}, -\frac{1}{3}$ , and  $\frac{2}{3}$  inputs demonstrate accuracies to within 0.01 of exact values.

Exercising Wave Gradiometry in two dimensions is just an extension of the base theory discussed earlier in this report for the one dimensional system as shown in [3]. When ported into two dimensions equation (2) becomes:

$$u(t, x, y) = G(x, y) f\left(t - p_x(x - x_0) - p_y(y - y_0)\right) \quad (16)$$

Now the geometrical spreading term can be composed of radial and angular information as would be defined in the cylindrical coordinate system confined to a horizontal plane. This is evident when expressing the new two dimensional equation (16) in terms of a radial component  $r$  and azimuthal component  $\theta$  as:

$$u(t, r, \theta) = G(r)R(\theta)f(t - p(r - r_0)) \quad (17)$$

The new term when compared to the one dimensional derivation yields additional information on the radiation pattern of the field as a function of  $\theta$ . Derivatives with respect to the introduced independent variables give first the familiar form of:

$$\frac{\partial u(t, r, \theta)}{\partial r} = A(r)u(t, r, \theta) + B(r)\frac{\partial u(t, r, \theta)}{\partial t} \quad (18)$$

And an additional differential with respect to the azimuth:

$$\frac{\partial u(t, r, \theta)}{\partial \theta} = \mathcal{R}(\theta)u(t, r, \theta) \quad (19)$$

The coefficients  $A(r)$  and  $B(r)$  maintain the same form as those of equation (6) but with radial dependence and in the new coordinate system via equation (18) they become:

$$A(r) = A_x \sin \theta + A_y \cos \theta \text{ and } B(r) = B_x \sin \theta + B_y \cos \theta \quad (20)$$

$\mathcal{R}(\theta)$  is the radiation pattern term and behaves very much like  $A(r)$  and the radial spreading term contained within it where:

$$\int_{\theta_0}^{\theta} \mathcal{R}(\theta) d\theta = \ln \left( \frac{R(\theta)}{R(\theta_0)} \right) \quad (21)$$

The additional spreading term is related to the Cartesian components of the  $A$  coefficient by:

$$\mathcal{R}(\theta) = \sqrt{x^2 + y^2} [A_x \cos \theta - A_y \sin \theta] \quad (22)$$

And finally the association to the  $B$  coefficient to obtain the azimuthal angle:

$$\theta = \tan^{-1} \left( \frac{B_x}{B_y} \right) = \tan^{-1} \left( \frac{p_x}{p_y} \right) \quad (23)$$

From this quick formulation moving the one dimensional solution methods into the second dimension, it is evident that if necessary effects are accounted for that solutions may be obtained for three dimensional models by implementing the 2D method at varied depths.

In this brief and strictly theoretical development of Wave Gradiometry, one can see that a physical model utilizing the differences between waves by observing their displacement field can be used to extract the geometrical spreading and horizontal slowness parameters from an ideal disturbance field model. The use of the Hilbert transform to obtain the analytical signal from the displacement function as a time series allows for a fast method to analyze a system as opposed to



analysis residing solely in the time domain or attempting to use spectral data alone. As with other models enabling the decomposition of wave solutions into pertinent data on the earth's internal structure, limitations present themselves in the form of mathematical anomalies such as singularities to be mitigated by the use of custom algorithms to remove them while maintaining the fidelity of the representative physical model. The model for Gradiometry also lends itself quite well to a simple solution occurring in one dimension which can then be carried out in other dimensions specified by a given system to allow for 2D or 3D analysis so long as the conditions for a basic wave solution are either met or the displacement solution is augmented to properly model the given conditions for the overall system.

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