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EXTENDING CSG TO PROJECTIONS AND PARAMETRIC OBJECTS

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ABSTRACT

We extend traditional CSG expression trees consisting of geometric primitives with the projection operator. We show how existing algorithms for topology computation (of CSG sets) can benefit from our new representation, allowing them to deal with more types of sets. A geometric primitive may be defined in terms of a characteristic function, which can be seen as the zero-set of a corresponding system along with inequality constraints. To handle projections, we exploit the DNF form, since projection distributes over union. To handle intersections, we introduce the notion of dominant sets. Finally, we introduce the join operator as a means to deal with primitives consisting of more than one manifold. This allows us to handle parametric objects as well. Our approach, based on a traversal of the final expression tree, generates a set of simpler systems that express the same set, avoiding extra unknowns like Lagrange multipliers. We conclude with implementation notes and an extensions as future work.

KEYWORDS

Geometric modeling, CSG, projection, constraint solving, DNF form

1. INTRODUCTION

Let A be a geometric set, subset of a d-dimensional workspace $\mathbb{R}^d$. Then

$$A : F(\mathbf{x}; s)$$

with s being the characteristic variable of the set. F is a set of equations and inequalities that describes the set.

Our goal is to provide a framework to model geometric solids in any dimension that are described not only by CSG (boolean algebra) operators, but also by more sophisticated operations such as projections, Minkowski sums, or extrusions sweeps. This may readily extend existing algorithms that compute the topology of geometric sets [4, 5].

A straightforward approach would be to start describing the set with a set of equations and inequalities, possibly introducing (nested) optimization problems with the aid of Lagrange multipliers in the case of projections. To demonstrate the complexity, consider n unit disks centered at (x_i, y_i), $i = 1 \ldots n$. Construct the $(n+1) \times (n+3)$ system consisting of the n equations $(x - x_i)^2 + (y - y_i^2) - 1 - s_i = 0$, $i = 1 \ldots n$ as well as the equation $\prod_{i=1}^{n} (s - s_i) = 0$. Now the zeroset of the system wrt x, y, s, s_i where $s \leq 0$ describes the union of the n disks. This naive approach yielded an $(n+1) \times (n+3)$ system, while the same set can be expressed by concatenating the solutions of n independent equations in 3 variables. Our experiments have shown that such an approach is non-practical because i) it introduces a lot of extra unknowns, ii) it includes spurious components of non-zero dimension. The benchmarking included existing solvers (C++ interval Newton solver, Quimper [3]) as well as a hand-crafted naive branch and bound solver in Python/SAGE [7]. The outcome of these benchmarks was that even very sophisticated solvers like Quimper have trouble dealing with a lot of variables (more than 8). The interested reader may find an elaborate report on those experiments in [9]. Part of that work was presented in [8].

We have extended upon those experiments by implementing a prototype Bernstein solver in Python/SAGE. The result was that systems with a few variables benefit a lot from the improved domain reduction of Bernstein solvers, but the problem still remains for a lot of variables. We also studied the idea of solving subsystems separately and combining the results using database inner join techniques. This didn't work very well because in the case of underconstrained systems, the solver "gets lost" in a high-dimensional set of solutions. Finally, we implemented a routine to do symbolic manipulation of the algebraic system (in the ring of continuous functions) and decompose it to simpler systems by exploiting factorizations and Gaussian elimination of the linear part (row echelon form). The smaller subsystems are much more efficient to solve, but the elimination/decomposition routine is non-trivial to implement in pure C++. An observation was made that when considering projections, the Lagrangian systems provide some one-dimensional spurious factors. Moreover, the fact that some factorizations are always possible indicates that one might be able to decompose the problem in smaller sub-problems.

All of the above led further research to a new direction: A careful study of the Lagrangian systems was made and it was discovered that in our context, the Lagrangian multipliers can be completely eliminated, thus introducing no extra unknowns. We were able to obtain the same results using differential calculus and wedge products. To our advantage, the new systems contain no spurious components. The idea presented in this work involves considering each node of the expression tree separately in order to compute a *contributing primitive* for each point in the set. Our approach is able to output a set of simple subsystems with inequality constraints by a proper traversal of the tree, introducing no extra unknowns.

Our framework exploits the idea of a characteristic function.

Definition 1. *Given a d-dimensional workspace $\mathbb{R}^d$, a value is defined for the characteristic variable s of a set A for every point $\mathbf{x} \in \mathbb{R}^d$. If $\mathbf{x}$ lies strictly in the interior of A then $s < 0$. If $\mathbf{x}$ lies strictly outside A then $s > 0$. Otherwise $s = 0$ and $\mathbf{x}$ lies on the boundary of A.*

We represent a geometric object as an expression tree consisting of CSG operators, as well as additional operations such as projection. These operators are applied on geometric primitives, which are in fact the leaves of the expression tree. We represent a geometric primitive in $\mathbb{R}^d$ as a manifold f in $(d+1)$-space $(x_1, \ldots, x_d, s)$ where s is the characteristic variable of the manifold. This way, the geometric primitive consists of interior points where the s-coordinate is negative and of exterior points where the s-coordinate is positive. The boundary case $s = 0$ describes the boundary of the object. Thus the geometric primitive is essentially a solid in d dimensions. For example, manifold $f(x, y, s) = x^2 + y^2 - 1 - s = 0$ describes the unit disk as a paraboloid in 3-space. For every point $(\hat{x}, \hat{y})$ inside the unit disk, there exists $\hat{s} < 0$ such that $f(\hat{x}, \hat{y}, \hat{s}) = \hat{x}^2 + \hat{y}^2 - 1 - \hat{s} = 0$. This approach can not only handle implicit solids, but also parametric ones.

A geometric set can be described in *Disjunctive Normal Form (DNF)* as a union of intersections of primitives. Having the union operator at the top level has several advantages as we show in the sequel.

The paper presents Tzoumas' representation of geometric sets and is organized as follows. Section 2 presents the computation of the DNF for CSG operators. Section 3 deals with projections and parametric objects. Finally, section 4 presents several applications of our approach, and section 5 our reference implementation, while section 6 concludes with future extensions.

2. CSG OPERATIONS

2.1. Disjunctive Normal Form

A CSG formula can be converted to DNF by applying De Morgan's laws and by distributing $\cap$ over $\cup$ as follows. Let A, B geometric primitives. Let P, Q_i geometric sets defined by an expression tree.

1. A and $\neg A$ are in DNF.
2. $A \cup B$ and $A \cap B$ are in DNF.
3. $\neg(Q_1 \cup Q_2) = \neg Q_1 \cap \neg Q_2$.
4. $\neg(Q_1 \cap Q_2) = \neg Q_1 \cup \neg Q_2$.
5. $P \cap (Q_1 \cup Q_2 \cup \ldots \cup Q_n) = (P \cap Q_1) \cup (P \cap Q_2) \cup \ldots \cup (P \cap Q_n)$

We also merge consecutive binary operators of the same kind as follows:

6. $(Q_1 \cap Q_2) \cap Q_3 = Q_1 \cap Q_2 \cap Q_3$.
7. $(Q_1 \cup Q_2) \cup Q_3 = Q_1 \cup Q_2 \cup Q_3$.

By applying the above rules we end up having an expression tree where the topmost operator is $\cup$ and each operand subexpression contains only the $\cap$ operator applied to either a primitive or its complement. Note that conversion to DNF may result in an exponential explosion of the formula (*e.g.*, when computing the DNF of $(A_1 \cup B_1) \cap \cdots \cap (A_n \cup B_n)$). On the other hand, classical methods like propagation of bounding boxes [1, 2] can discard useless DNF clauses and reduce the number of nodes in the expression tree.

2.2. Complement

The simplest operation is the complement. If $A : f(\mathbf{x}; s) = 0$ then

$$\neg A : f(\mathbf{x}; -s) = 0$$

Note that in our framework we consider *regularized sets*. That is, a set and its complement share their boundary. In other words we consider inequalities $s \leq 0$ or $s \geq 0$ instead of their strict counterparts.

Remark 1. *It is important that the complement is still represented by points from the manifold.*

Recall that in Def. 1 we require the characteristic variable to be defined for all points in $\mathbb{R}^d$. For example, consider $f(x, y, s) = x^2 + y^2 - 1 + s^2 = 0$. Points outside the unit disk, like $(2, 2)$ are not represented at all, since $f(2, 2, s) = 0$ cannot be satisfied for any real s.

2.3. Union

Let $P = A_1 \cup \ldots \cup A_n$ a union of primitives. Since the formula is in DNF form, it suffices to consider each primitive separately, that is $A_i : f_i(\mathbf{x}; s_i)$, $i = 1 \ldots n$. Note that a point $\mathbf{x}$ may not be uniquely associated with a primitive, since it may belong to the intersection of many primitives. However, this does not create any problems. We could impose the uniqueness constraint by satisfying $s_i \leq s_j \forall i \neq j$, where A_j refers to each object in the considered intersection.

2.4. Intersection

Let $P = A_1 \cap \ldots \cap A_n$ an intersection of primitives. Contrary to the union, for the intersection we have to make sure that point $\mathbf{x}$ belongs to all A_i, $i = 1 \ldots n$ *at the same time*. We can choose the characteristic variable of the set to be $\max(s_1, \ldots, s_n)$. In order to represent $\max$, we consider the union of all possible cases for $\max$.

Definition 2. *Let A, B_i geometric sets and s_A and s_{B_i} their characteristic variables at point $\mathbf{x}$. Then:*

$$A|B_1, \ldots, B_n \ := \ \mathbf{x} \in \mathbb{R}^d : \ 0 \geq s_A \geq s_{B_i},$$

$i = 1, \ldots n$, and we say that A dominates *$B_1, \ldots, B_n$.*

The notion of dominant set allows us to express intersections as unions. Note that operator $|$ has lower precedence than the comma.

Property 1. $A \cap B \ = \ A|B \ \cup \ B|A$

Proof. ($\Rightarrow$.) Let $\mathbf{x} \in A \cap B$. Then $0 \geq s_A$ and $0 \geq s_B$. If $0 \geq s_A \geq s_B$, then $\mathbf{x} \in A|B$. Otherwise, we have $0 \geq s_B \geq s_A$ therefore $\mathbf{x} \in B|A$. ($\Leftarrow$.) Let $\mathbf{x} \in A|B$. Then $0 \geq s_A \geq s_B$ which means $\mathbf{x} \in A \cap B$. Similarly when $\mathbf{x} \in B|A$. □

Property 2. $(A|B)|C \ = \ A|B, C$

Proof. $\mathbf{x} \in (A|B)|C \Leftrightarrow 0 \geq s_A \geq s_C \wedge 0 \geq s_A \geq s_B \Leftrightarrow \mathbf{x} \in A|B, C$. □

Property 3. $A|(B|C) \ \cup \ A|(C|B) \ = \ A|B, C$

Proof. $\mathbf{x} \in A|(B|C) \cup A|(C|B) \Leftrightarrow 0 \geq s_A \geq s_B \geq s_C \vee 0 \geq s_A \geq s_C \geq s_B \Leftrightarrow 0 \geq s_A \geq s_B \wedge 0 \geq s_A \geq s_C \Leftrightarrow \mathbf{x} \in A|B, C$. □

Property 4. $A \cap B \cap C \ = \ (A|B, C) \cup (B|C, A) \cup (C|A, B)$

Proof. $A \cap B \cap C = (A|B \cup B|A) \cap C = (A|B \cap C) \cup (B|A \cap C) = (A|B)|C \cup C|(A|B) \cup (B|A)|C \cup C|(B|A) = (A|B, C) \cup (B|C, A) \cup (C|A, B)$. □

Property 5. $\neg(A|B) \ = \ \neg A \ \cup \ \neg B \ \cup \ B|A$

Proof. Points in $A|B$ satisfy $0 \geq s_A \geq s_B$, therefore, the complement of this set consists of points where $s_A \geq 0$, $s_B \geq 0$ or $0 \geq s_B \geq s_A$. See Fig.1. □

3. PROJECTION

The first non-trivial operation which concerns sets that cannot be described with CSG primitives is *projection*. Let $A : F(\mathbf{x}; s)$ where $\mathbf{x} = (x_1, \ldots, x_d)$, then the projection of A wrt x_i is denoted as $\pi_i(A)$. Let $\mathbf{x^i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$. Then $\pi_i(A) = F_\pi(\mathbf{x^i}; s)$. Projections wrt to more than one dimension are denoted with commas, *e.g.* $\pi_{i,j}(A) = F_\pi(\mathbf{x^{i,j}}; s) = \pi_j(\pi_i(A))$. When the particular dimension is not of importance we may simply write $\pi^2(A) = \pi(\pi(A))$. An interesting property of the projection is that it distributes over $\cup$. This is another argument in favor of the DNF form:

$$\pi(Q_1 \cup Q_2 \cup \ldots \cup Q_n) = \pi(Q_1) \cup \pi(Q_2) \cup \ldots \cup \pi(Q_n)$$

A naive way to deal with projections is to consider $F_{\pi_i} = F$, *i.e.*, just forget coordinate x_i. Doing so may allow for an 1-dimensional set of values for x_i and s, such that $F(\mathbf{x}; s) = 0$. This may slow down a solver (used to find a cover of the set for example) since an infinite set of solutions will have to be covered. We fix this problem by introducing extra constraints so as to limit the range of the characteristic variable to a 0-dimensional set of values for every projected point. It is still possible that the extra constraints fail to reduce the dimension of the solution set, but this does not arise in practice.

A simple way to do that consistently is to choose the smallest value of the characteristic variable, among all possible values of coordinates $x_1, \ldots, x_k$ (the coordinates being eliminated). That is

$$\pi_{1,\ldots,k}(A) = \mathbf{x}^{\mathbf{1\ldots k}} \in \mathbb{R}^{d-k} : \begin{cases} \exists\, x_1, \ldots, x_k : \\ \mathbf{x} \in \mathbb{R}^d \cap A \\ s \text{ is minimal} \end{cases} \quad \Leftrightarrow$$

$$F_\pi(\mathbf{x}^{\mathbf{1\ldots k}}; s) = \begin{cases} \exists\, x_1, \ldots, x_k : \\ F(\mathbf{x}; s) \\ s \text{ is minimal} \end{cases}$$

This way we pick the point that lies "deepest" in the set to map to the projected set. Note that coordinates $x_1, \ldots, x_k$ are no longer free variables, but take a value and become parameters. The minimization constraint can be written in terms of an optimization problem with constraints those exactly in F and the objective function s. Typical approach involves considering a Lagrangian (*i.e.*, the Fritz John conditions). This is quite powerful a technique, but it has the disadvantage that it introduces extra equations and unknowns. A more direct approach exists, which is equivalent to solving the Lagrangian system by hand and getting rid of redundant solutions. See Appendix A for an analysis of several Lagrangian systems. Here we make use of differential calculus and wedge products. For an introduction to wedge products and their applications in optimization problems the reader may refer to [11]. With $\longrightarrow$ we denote the sufficient constraints to describe a geometric set. Note that the use of the term "minimal" in the above is abusive. We are actually looking for a *critical* point (without loss of generality). Thus, we don't have to perform extra computations to ensure that a critical point is actually a minimum. This is because we are interested in reducing the solution set to a hopefully 0-dimensional variety. It is perfectly acceptable for a point in the interior of the (projected) set to have not necessarily the smallest value of the characteristic variable, but some other (critical) value.

Theorem 1 (Projection of geometric primitive)**.** *Let $A : f(\mathbf{x}; s)$ be a geometric primitive. When projecting down k dimensions (eliminating $x_1 \ldots x_k$), the projection can be specified by:*

$$\pi^k(A) \longrightarrow \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \ldots = \frac{\partial f}{\partial x_k} = 0$$

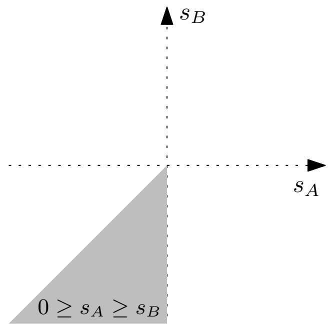

Figure 1 Graphical representation of $\neg(A|B)$. The shaded area corresponds to $A|B$

Proof. $0 = ds \wedge df = ds \wedge (\frac{\partial f}{\partial x_1} dx_1 + \ldots + \frac{\partial f}{\partial x_k} dx_k + \frac{\partial f}{\partial x_s} dx_s) = \frac{\partial f}{\partial x_1} ds \wedge dx_1 + \ldots + \frac{\partial f}{\partial x_k} ds \wedge dx_k \iff \frac{\partial f}{\partial x_1} = \ldots = \frac{\partial f}{\partial x_k} = 0.$ □

For an alternative proof using Lagrangians, see section A.

Note that we assume that there exists some critical value in the interior of the set, otherwise we would have to consider the boundary of the set. For complements, a critical value should exist in the exterior of the original set as well. We remedy this problem by considering the intersection with a big disk. This way we introduce a critical value where the characteristic variables of the set and the big disk are equal (to be explained in the sequel).

Definition 3. *Let A, B geometric sets and s_A and s_B their characteristic variables at point $\mathbf{x}$. Then:*

$$A \bowtie B \;:=\; \mathbf{x} \in \mathbb{R}^d : \; s_A = s_B \;\wedge\; s_A \leq 0$$

We define the precedence of the new operators to be: $\boxed{\bowtie} \succ \boxed{,} \succ \boxed{|}$. Observe the similarity with the join operator from relational algebra. Indeed, we join the two relations $A(\mathbf{x}; s_A)$ and $B(\mathbf{x}; s_B)$ on their characteristic variable.

Property 6. $\neg(A \bowtie B) \;=\; \neg A \bowtie \neg B$

Proof. $\mathbf{x} \in \neg(A \bowtie B) \Leftrightarrow s_A = s_B \;\wedge\; s_A \geq 0 \Leftrightarrow \mathbf{x} \in \neg A \bowtie \neg B.$ □

Definition 4. *We denote with $J_{i_1 i_2 \ldots i_n}(f_1, f_2, \ldots, f_n)$ the following $n \times n$ Jacobian determinant:*

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_{i_1}} & \frac{\partial f_1}{\partial x_{i_2}} & \cdots & \frac{\partial f_1}{\partial x_{i_n}} \\ \frac{\partial f_2}{\partial x_{i_1}} & \frac{\partial f_2}{\partial x_{i_2}} & \cdots & \frac{\partial f_2}{\partial x_{i_n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_{i_1}} & \frac{\partial f_n}{\partial x_{i_2}} & \cdots & \frac{\partial f_n}{\partial x_{i_n}} \end{vmatrix}$$

Lemma 1 (Projection of join sets)**.** *Let* $A : f_0(\mathbf{x}; s)$ *and* $B_i : f_i(\mathbf{x}; s), i = 1 \dots n$ *geometric primitives. Then* $\pi^k(A \bowtie B_1 \bowtie \cdots \bowtie B_n) \rightarrow$

$$\begin{cases} \emptyset, & k \leq n \\ J_{i_0 i_1 \dots i_n}(f_0, f_1, \dots, f_n) = 0, & k > n \end{cases},$$

where $1 \leq i_0 < i_1 < \dots < i_n \leq k$.

Proof. Assume without loss of generality that we are projecting wrt $x_1, x_2, \dots, x_k$. Considering the wedge product (to find the critical value of s) we have $ds \wedge df_0 \wedge df_1 \wedge df_2 \wedge \cdots \wedge df_n = ds \wedge (\sum_{i=1}^k \frac{\partial f_0}{\partial x_i} dx_i + \frac{\partial f_0}{\partial s} ds) \wedge (\sum_{i=1}^k \frac{\partial f_1}{\partial x_i} dx_i + \frac{\partial f_1}{\partial s} ds) \wedge \cdots \wedge (\sum_{i=1}^k \frac{\partial f_n}{\partial x_i} dx_i + \frac{\partial f_n}{\partial s} ds) = \epsilon_{x_{i_0} x_{i_1} \dots x_{i_n}} (\sum_{j=0}^n \frac{\partial f_j}{\partial x_{i_j}} dx_{i_j}) ds \wedge dx_{i_0} \wedge \cdots \wedge dx_{i_n}$. Symbol ϵ is the permutation sign determined by the number of inversions in the considered permutation, which appears in the combinatorial definition of the determinant [10]. Now, if $k \leq n$ the wedge product is identically zero, because of some dx_i being equal, due to the pigeonhole principle (we have a wedge product of $n+1$ factors with k choices for each factor, and we have that $dx_i \wedge dx_i = 0$). Otherwise, if $k > n$, the wedge product expands to $\binom{n+1}{k}$ coefficients which should all vanish. These coefficients are precisely $J_{i_0 i_1 \dots i_n}(f_0, f_1, \dots, f_n), 1 \leq i_0 < i_1 < \dots < i_n \leq k$. ☐

Since no extra condition is required to describe a projection of a join with respect to a single variable, we have that

Corollary 1. $\pi(A \bowtie B) = A \bowtie B$.

Theorem 2 (Projection of dominant set)**.**

$$\pi^k(A|B) = \pi^k(A)|B \cup \pi^k(A \bowtie B)$$

Proof.

First proof Without loss of generality we assume that we project wrt $x_1, \dots, x_k$. Since we have a constrained optimization problem, the critical value can be attained either when a constraint is active or not.

$\pi^k(A|B) = \mathbf{x}^{\mathbf{1} \dots \mathbf{k}} \in \mathbb{R}^{d-k} : \exists\, x_1, \dots, x_k : (\mathbf{x} \in \mathbb{R}^d \cap A) \wedge (s_A \text{ is critical}) \wedge (s_A \geq s_B)$. This means that s_A takes its critical value on the critical points of $\pi(A)$ that happen to satisfy $s_A \geq s_B$, which is precisly $\pi^k(A)|B$ or somewhere where $s_A = s_B$, which is $\pi^k(A \bowtie B)$.

Second proof (Wedge product.) Let $A : f_0(\mathbf{x}; s)$ and $B : f_1(\mathbf{x}; s_1)$. We have a constrained optimization problem: Find $x_1 \dots x_k, s, s_1, u_1$ that optimize s (so that it achieves a critical value) subject to $f_0(\mathbf{x}; s) = f_1(\mathbf{x}; s_1) = g_1(s, s_1, u_1) = 0$, where $g_1(s, s_1, u_1) = s - s_1 - u_1^2$. Function g_1 enforces constraint $s \geq s_1$. Considering the wedge product to optimize s we have $ds \wedge df_0 \wedge df_1 \wedge dg_1 = 0 \Leftrightarrow ds \wedge (\sum_{i=1}^k \frac{\partial f_0}{\partial x_i} dx_i + \frac{\partial f_0}{\partial s} ds) \wedge (\sum_{i=1}^k \frac{\partial f_1}{\partial x_i} dx_i + \frac{\partial f_1}{\partial s_1} ds_1) \wedge (ds - ds_1 - 2u_1 du_1) = 0$. Now, if $k < 2$, we have that $0 = (\sum_{i=1}^k -2u_1 \frac{\partial f_0}{\partial x_i} \frac{\partial f_1}{\partial s_1} ds \wedge dx_i \wedge ds_1 \wedge du_1)$ which implies that $u_1 = 0$ or $\frac{\partial f_1}{\partial s_1} = 0$ or $\frac{\partial f_0}{\partial x_i} = 0, i = 1 \dots k$. If $k \geq 2$, then we have that $0 = (\sum_{i=1}^k -2u_1 \frac{\partial f_0}{\partial x_i} \frac{\partial f_1}{\partial s_1} ds \wedge dx_i \wedge ds_1 \wedge du_1) + (\sum_{\substack{i,j=1 \\ i \neq j}}^k -2u_1 \frac{\partial f_0}{\partial x_i} \frac{\partial f_1}{\partial x_j} ds \wedge dx_i \wedge dx_j \wedge du_1) + (\sum_{\substack{i,j=1 \\ i \neq j}}^k -\frac{\partial f_0}{\partial x_i} \frac{\partial f_1}{\partial x_j} ds \wedge dx_i \wedge dx_j \wedge ds_1) = 0 \Leftrightarrow u_1 \frac{\partial f_0}{\partial x_1} \frac{\partial f_1}{\partial s_1} = u_1 \frac{\partial f_0}{\partial x_2} \frac{\partial f_1}{\partial s_1} = \cdots = u_1 \frac{\partial f_0}{\partial x_k} \frac{\partial f_1}{\partial s_1} = u_1 \sum_{\substack{i,j=1 \\ i<j}}^k [J_{ij}(f_0, f_1)]^2 = \sum_{\substack{i,j=1 \\ i<j}}^k [J_{ij}(f_0, f_1)]^2 = 0$. If $\frac{\partial f_1}{\partial s_1} = 0$, then we obtain $J_{ij}(f_0, f_1) = 0, 1 \leq i < j \leq k$. Otherwise, if $u_1 = 0$, then again $J_{ij}(f_0, f_1) = 0, 1 \leq i < j \leq k$. Otherwise we obtain that $\frac{\partial f_0}{\partial x_i} = 0, i = 1 \dots k$. In this case $J_i j(f_0, f_1)$ vanishes trivially.

The union of all previous cases is precisely $\pi^k(A)|B \cup \pi^k(A \bowtie B)$.

Third proof (Lagrangian.) See Appendix A.

☐

Care has to be taken here that the set B in the expression $\pi^k(A)|B$ lies in a lower dimension. That is we consider points in $\pi^k(A)$ that happen to lie in B. We could denote B in this case as $B_{/\pi^k(A)}$ but we avoid so due to abuse of notation.

Let $[B_m]^{1:n}$ denote sequence $B_m, m = 1 \dots n$.

Theorem 3 (Projection of dominant sets, generalized)**.** $\pi^k(A|[B_m]^{1:n}) =$

$$\begin{array}{ll} & \pi^k(A)|[B_m]^{1:n} \\ \bigcup_{i=1}^n & \pi^k(A \bowtie B_i)|[B_m]^{1:n}_{m \neq i} \\ \bigcup_{\substack{i,j=1 \\ i<j}}^n & \pi^k(A \bowtie B_i \bowtie B_j)|[B_m]^{1:n}_{m \neq i, m \neq j} \\ \bigcup & \dots \\ \bigcup & \pi^k(A \bowtie B_1 \bowtie \cdots \bowtie B_n) \end{array}$$

Proof. This comes as a generalization of Theorem 2. The same proof methods can be applied in this case with n constraints. ☐

3.1. Projection and parametric sets

Definition 3 implies that the join operator can be used to express the intersection of manifolds. In this case the join is performed on the common variables. For example $X : x - cos(t) = 0$, $Y : y - sin(t) = 0$. Now $X \bowtie Y$ expresses points (x, y) that lie on the unit circle. This way, the join variable t is implicitly eliminated. This is beacuse we project wrt t, due to the join operator being applied. Lemma 1 shows that projection of joins may be trivial if the number of variables projected is less than the number of terms in the join expression. Here $X \bowtie Y$ has 2 terms therefore, the resulting subspace consists of x and y only. More generically, given a parametric solid G in $\mathbb{R}^d$ defined by $X_i = f_i(\mathbf{x}; t_1, \ldots, t_d)$, $i = 1 \ldots d$, we can represent this set as the join of the defining manifolds. That is $G = X_1 \bowtie X_2 \bowtie \ldots \bowtie X_d$. Assume that t_d is the characteristic variable. Now projection in the first $d-1$ dimensions eliminates the corresponding variables, and we have from Lemma 1 that $\pi^{d-1}(X_1 \bowtie \ldots \bowtie X_d) = X_1 \bowtie X_2 \bowtie \ldots \bowtie X_d$, since $k = d - 1 < d$. See sec. 4.3 for an example.

4. EXAMPLES

4.1. Projection of the unit sphere

Consider the unit sphere: $A : f(x, y, z, s) = x^2 + y^2 + z^2 - 1 - s = 0$. Then from Thm. 1 we have $F_\pi(x, y, s) =$

$$\begin{cases} f & = & x^2 + y^2 + z^2 - 1 - s & = & 0 \\ \frac{\partial f}{\partial z} & = & 2z & = & 0 \end{cases}$$

$$\Rightarrow \begin{cases} x^2 + y^2 - 1 - s & = & 0 \\ z & = & 0 \end{cases}$$

which is effectively the unit disk. Note that with the Lagrangian we have a bigger system: $F_\pi(x, y, s) =$

$$\begin{cases} x^2 + y^2 + z^2 - 1 - s & = & 0 \\ u_0 - v_1 & = & 0 \\ 2v_1 z & = & 0 \\ u_0 + v_1^2 & = & 1 \end{cases}$$

$$\Rightarrow \begin{cases} x^2 + y^2 - 1 - s & = & 0 \\ z & = & 0 \\ u_0 & = & v_1 \\ v_1 & = & \frac{-1+\sqrt{5}}{2} \end{cases}$$

4.2. Intersection of projections of intersection of spheres

Let E_1, E_2, E_3 be three spheres in $\mathbb{R}^3$, we want to express the object G defined as

$$G = \pi(E_1 \cap E_2) \ \cap \ \pi(E_1 \cap E_3).$$

The above expression will be transformed in DNF. We have that: $G = [\pi(E_1|E_2) \cup \pi(E_2|E_1)] \cap [\pi(E_1|E_3) \cup \pi(E_3|E_1)] = [\pi(E_1|E_2) \cap \pi(E_1|E_3)] \cup [\pi(E_1|E_2) \cap \pi(E_3|E_1)] \cup [\pi(E_2|E_1) \cap \pi(E_1|E_3)] \cup [\pi(E_2|E_1) \cap \pi(E_3|E_1)] = [\pi(E_1)|E_2 \cup E_1 \bowtie E_2] \cap [\pi(E_1)|E_3 \cup E_1 \bowtie E_3] \cup \cdots = [\pi(E_1)|E_2 \cap \pi(E_1)|E_3] \cup \cdots = [\pi(E_1)|E_2, \pi(E_1)|E_3)] \cup [\pi(E_1)|E_3, \pi(E_1)|E_2] \cup \cdots = \bigcup_{i=1}^{18} S_i$. That is G is equal to the union of 18 sets $S_1, \ldots, S_{18}$ which in fact can be grouped into five sets depending on the contributing set being $\pi(E_1), \pi(E_2), \pi(E_3), E_1 \bowtie E_2, E_1 \bowtie E_3$, as shown in the following table:

set	contributing set	formula
S_1	$\pi(E_1)$	$\pi(E_1)\|E_2, \pi(E_1)\|E_3$
S_2		$\pi(E_1)\|E_3, \pi(E_1)\|E_2$
S_3		$\pi(E_1)\|E_2, E_1 \bowtie E_3$
S_4		$\pi(E_1)\|E_3, E_1 \bowtie E_2$
S_5		$\pi(E_1)\|E_2, \pi(E_3)\|E_1$
S_6		$\pi(E_1)\|E_3, \pi(E_2)\|E_1$
S_7	$\pi(E_2)$	$\pi(E_2)\|E_1, \pi(E_1)\|E_3$
S_8		$\pi(E_2)\|E_1, E_1 \bowtie E_3$
S_9		$\pi(E_2)\|E_1, \pi(E_3)\|E_1$
S_{10}	$\pi(E_3)$	$\pi(E_3)\|E_1, \pi(E_1)\|E_2$
S_{11}		$\pi(E_3)\|E_1, E_1 \bowtie E_2$
S_{12}		$\pi(E_3)\|E_1, \pi(E_2)\|E_1$
S_{13}	$E_1 \bowtie E_2$	$E_1 \bowtie E_2\|(\pi(E_1)\|E_3)$
S_{14}		$E_1 \bowtie E_2\|E_1 \bowtie E_3$
S_{15}		$E_1 \bowtie E_2\|(\pi(E_3)\|E_1)$
S_{16}	$E_1 \bowtie E_3$	$E_1 \bowtie E_3\|(\pi(E_1)\|E_2)$
S_{17}		$E_1 \bowtie E_3\|E_1 \bowtie E_2$
S_{18}		$E_1 \bowtie E_3\|(\pi(E_2)\|E_1)$

Let (x, y, z, r) denote a sphere centered at (x, y, z) with radius r. If $E_1 = (0, 0, 0, 1)$, $E_2 = (\frac{1}{2}, 0, \frac{1}{2}, \sqrt{\frac{3}{2}})$ and $E_3 = (-\frac{3}{2}, 0, \frac{3}{2}, \frac{3}{2})$ then $\pi(E_1 \cap E_2)$, $\pi(E_1 \cap E_3)$ and $\pi(E_1 \cap E_2) \cap \pi(E_1 \cap E_3)$ are shown in Fig. 2 and Fig. 3, bottom. Also visible are $E_1 \bowtie E_2$, $\pi(E_1)|E_2$ and $E_1 \bowtie E_3$ (Fig. 3, top).

4.3. Parametric disk in $\mathbb{R}^2$

Let $X(\theta, r) = r \ \cos\theta$ and $Y(\theta, r) = r \ \sin\theta$. With $\theta \in [-\pi, \pi)$ and $r \in [0, 1]$ we obtain the unit disk. However, we would like one parameter to be a characteristic variable. That is, negative values correspond to points in the interior of the solid, and positive values correspond to points in the exterior of the solid (and should cover the complement of the solid). We set $r = \sqrt{1 + s}$. Now our solid becomes

$$\begin{bmatrix} X(x, y, \theta, s) \\ Y(x, y, \theta, s) \end{bmatrix} = \begin{bmatrix} x - \sqrt{1+s} \ \cos\theta \\ y - \sqrt{1+s} \ \sin\theta \end{bmatrix}$$

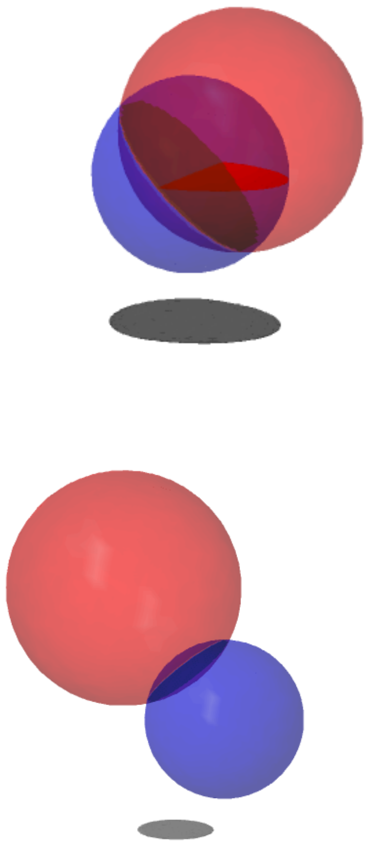

Figure 2 Top: $\pi(E_1 \cap E_2)$ in 3D. Bottom: $\pi(E_1 \cap E_3)$ in 3D

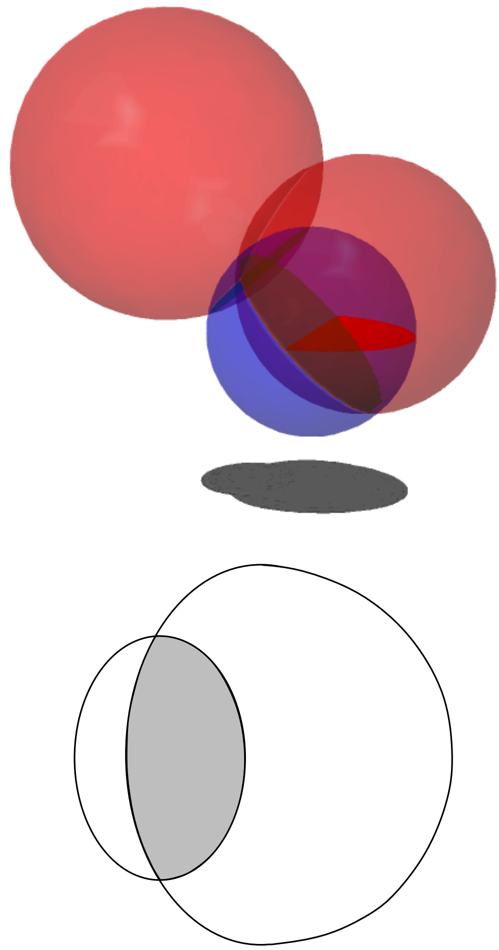

Figure 3 Top: $\pi(E_1 \cap E_2) \cup \pi(E_1 \cap E_3)$ in 3D; Bottom: $\pi(E_1 \cap E_2) \cap \pi(E_1 \cap E_3)$ in 2D

which is represented by $X \bowtie Y$ and yields the following constraints:

$$\begin{cases} x - \sqrt{1+s}\ \cos\theta & = 0 \\ y - \sqrt{1+s}\ \sin\theta & = 0 \end{cases}$$

The use of square roots may lead to problems, depending on the nature of the solver used. Moreover it leads to problems when one tries to compute the projection of the above set, because the square root function is not defined in $\mathbb{R}$, but only in $[0, \infty)$ (*e.g.*, where does the critical point of s lie?). It is possible to avoid using square root expressions at the cost of introducing one extra equation and one unknown. We consider $R(r, s) = r^2 - 1 - s$. Now our solid becomes $R \bowtie X \bowtie Y$. The join expression contains three terms, therefore it is equivalent to the projection wrt to two variables r, θ yielding a parametric solid in (x, y, s), according to Lemma 1.

5. IMPLEMENTATION

We have implemented the approach presented in this paper in Pyhon/SAGE. It automatically generates the necessary conditions (as subsystems) to describe the geometric set, which are then passed to Quimper for solving. The routine to transform the expression tree in DNF turns out to be non-trivial to implement, if one allows for simplifications and cancellations. Therefore the DNF may not computed automatically for arbitrary complex trees, but it is easy to manually specify the corresponding subexpressions in DNF. The expression tree is described by object constructors. For example, given spheres A and B, $\pi(A \cap B)$ is expressed as:

```
x,y,z = SR.var('x,y,z')
A=PrimitiveSet((x-3/4)^2+(y-3/4)^2+(z-3/4)^2<2/3,
   {x:RIF(-2,2), y:RIF(-2,2),z:RIF(-2,2)})
B=PrimitiveSet((x-1/4)^2+(y-1/4)^2+(z-1/4)^2<1,
       {x:RIF(-2,2), y:RIF(-2,2),z:RIF(-2,2)})
G=ProjectionSet(IntersectionSet(A,B),set([z]))
```

The output is the DNF expression: $\pi(A)|B \cup \pi(A \bowtie B) \cup \pi(B)|A \cup \pi(B \bowtie A)$. Note that although $\pi(B \bowtie A)$ is identical to $\pi(A \bowtie B)$ it still appears in the expression. We hope to allow for such optimiza-

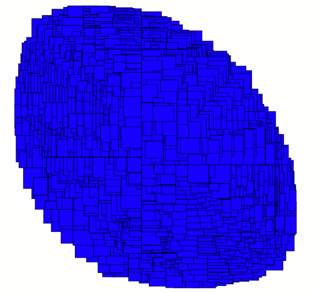

Figure 4 Visualization of $\pi(A \cap B)$ in (x, y)

tions in future versions. The four systems generated are:

1. $$\begin{cases} \frac{1}{16}(4z-3)^2 + \frac{1}{16}(4y-3)^2 + \\ \quad +\frac{1}{16}(4x-3)^2 - s_0 - \frac{2}{3} & = & 0 \\ \frac{1}{16}(4z-1)^2 + \frac{1}{16}(4y-1)^2 + \\ \quad +\frac{1}{16}(4x-1)^2 - s_1 - 1 & = & 0 \\ 2z - \frac{3}{2} & = & 0 \end{cases}$$

2. $$\begin{cases} \frac{1}{16}(4z-3)^2 + \frac{1}{16}(4y-3)^2 + \\ \quad +\frac{1}{16}(4x-3)^2 - s_0 - \frac{2}{3} & = & 0 \\ \frac{1}{16}(4z-1)^2 + \frac{1}{16}(4y-1)^2 + \\ \quad +\frac{1}{16}(4x-1)^2 - s_1 - 1 & = & 0 \\ s_0 - s_1 & = & 0 \end{cases}$$

3. $$\begin{cases} \frac{1}{16}(4z-3)^2 + \frac{1}{16}(4y-3)^2 + \\ \quad +\frac{1}{16}(4x-3)^2 - s_0 - \frac{2}{3} & = & 0 \\ \frac{1}{16}(4z-1)^2 + \frac{1}{16}(4y-1)^2 + \\ \quad +\frac{1}{16}(4x-1)^2 - s_1 - 1 & = & 0 \\ 2z - \frac{1}{2} & = & 0 \end{cases}$$

4. Identical to 2.

Finally, the systems are solved with Quimper and the results are merged and visualized (Fig. 4).

6. CONCLUSION

This article presented Tzoumas' representation for geometric sets. We exploited the DNF form, the join operator and the notion of dominant set. This is based on the observation that in general, only one geometric primitive should be contributing to each point of a geometric set described by a complex expression. The join operator deals effectively with boundary conditions where more primitives contribute to that point. Such a representation is essential to extend geometric and topological algorithms.

Other types of sets such as extrusions or sweeps should be fairly easy to be expressed in our framework, since extrusions and sweeps are parametric objects. For Minkowski sums, things are a bit more complicated, as we currently know of no easy way to describe the characteristic function (simply considering the sum of the characteristic variables is not enough).

Another application of Tzoumas' representation (and a basic motivation behind this unified approach) is to extend the HIA method (Homotopy via Interval type Analysis) [4] to objects more general than CSG like the aforementioned ones. This is a topic that we are currently investigating.

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A. ALTERNATIVE PROOFS WITH LAGRANGE MULTIPLIERS

Assume that F contains k intermediate variables s_1, $s_2, \ldots, s_k$, n equations $f_j(\mathbf{x}; s_1 \ldots s_k, s)$, $j = 1 \ldots n$ and m inequalities $g_j(\mathbf{x}; s_1 \ldots s_k, s) \leq 0$, $j = 1 \ldots m$. Then, by using the Fritz John conditions (*e.g.*, [6]) we obtain $F_\pi(\mathbf{x^i}; s) =$

$$\begin{cases} u_0 \nabla s + \sum_{j=1}^m u_j \nabla g_j(x_i, s_1 \ldots s_k, s) + \\ \quad + \sum_{j=1}^n v_j \nabla f_j(x_i, s_1 \ldots s_k, s) & = & 0 \\ u_j g_j(\mathbf{x}; s_1 \ldots s_k, s) & = & 0 \\ (j = 1 \ldots m) \\ f_j(\mathbf{x}; s_1 \ldots s_k, s) & = & 0 \\ (j = 1 \ldots n) \\ u_j & \geq & 0 \\ (j = 0 \ldots m) \\ g_j(\mathbf{x}; s_1 \ldots s_k, s) & \leq & 0 \\ \sum_{j=0}^m u_j + \sum_{j=1}^n v_j^2 & = & 1 \end{cases}$$

The last condition is a normalization condition which implies that $u_j \in [0, 1]$ (since $u_j \geq 0$) and $v_j \in [-1, 1]$.

Theorem 1 (Projection of geometric primitive)**.** *Let $A : f(\mathbf{x}; s)$ be a geometric primitive. When projecting down k dimensions (eliminating $x_1 \ldots x_k$), the projection can be specified by:*

$$\pi^k(A) \longrightarrow \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \ldots = \frac{\partial f}{\partial x_k} = 0$$

Proof. Let $L = us + vf(\mathbf{x}; s)$. Then $F_\pi(\mathbf{x^{1\ldots k}}; s) =$

$$\begin{cases} u + v\frac{\partial f}{\partial s} & = & 0 & (L'_s) \\ v\frac{\partial f}{\partial x_1} & = & 0 & (L'_{x_1}) \\ & \vdots & & \\ v\frac{\partial f}{\partial x_k} & = & 0 & (L'_{x_k}) \\ u + v^2 & = & 1 \\ f(\mathbf{x}; s) & = & 0 \end{cases} \iff$$

$$\begin{cases} v^2 - \frac{\partial f}{\partial s} v - 1 &= 0 \\ \frac{\partial f}{\partial x_1} &= 0 \\ &\vdots \\ \frac{\partial f}{\partial x_k} &= 0 \\ v &\neq 0 \\ u &= 1 - v^2 \\ f(\mathbf{x}; s) &= 0 \end{cases}$$

Therefore, to optimize s one has to look at the critical points where each derivative wrt x_i vanishes, *i.e.*,

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \cdots = \frac{\partial f}{\partial x_k} = 0.$$

□

Theorem 2 (Projection of dominant set)**.**

$$\pi^k(A|B) \;=\; \pi^k(A)|B \;\cup\; \pi^k(A \bowtie B)$$

Proof. Consider $A : f_0(\mathbf{x}; s), B : f_1(\mathbf{x}; s_1)$. Let $L = u_0 s + u_1(s_1 - s) + v_0 f_0(\mathbf{x}; s) + v_1 f_1(\mathbf{x}; s_1)$. Then $F_\pi(\mathbf{x^{1\ldots k}}; s) =$

$$\begin{cases} u_0 - u_1 + v_0 \frac{\partial f_0}{\partial s} &= 0 \quad (L'_s) \\ u_1 + v_1 \frac{\partial f_1}{\partial s_1} &= 0 \quad (L'_{s_1}) \\ v_0 \frac{\partial f_0}{\partial x_1} + v_1 \frac{\partial f_1}{\partial x_1} &= 0 \quad (L'_{x_1}) \\ &\vdots \\ v_0 \frac{\partial f_0}{\partial x_k} + v_1 \frac{\partial f_1}{\partial x_k} &= 0 \quad (L'_{x_k}) \\ u_0 + u_1 + v_0^2 + v_1^2 &= 1 \\ u_1(s_1 - s) &= 0 \\ u_j &\geq 0 \quad (j = 0, 1) \\ f_0(\mathbf{x}; s) &= 0 \\ f_1(\mathbf{x}; s_1) &= 0 \end{cases}$$

Case $s > s_1$**.** It follows that $u_1 = 0$. Then

$$\begin{cases} u_0 + v_0 \frac{\partial f_0}{\partial s} &= 0 \\ v_1 \frac{\partial f_1}{\partial s_1} &= 0 \\ v_0 \frac{\partial f_0}{\partial x_1} + v_1 \frac{\partial f_1}{\partial x_1} &= 0 \\ &\vdots \\ v_0 \frac{\partial f_0}{\partial x_k} + v_1 \frac{\partial f_1}{\partial x_k} &= 0 \\ u_0 + v_0^2 + v_1^2 &= 1 \\ f_0(\mathbf{x}; s) &= 0 \\ f_1(\mathbf{x}; s_1) &= 0 \end{cases}$$

- $v_1 = 0$.

$$\begin{cases} u_0 + v_0 \frac{\partial f_0}{\partial s} &= 0 \\ v_0 \frac{\partial f_0}{\partial x_1} &= 0 \\ &\vdots \\ v_0 \frac{\partial f_0}{\partial x_k} &= 0 \\ u_0 + v_0^2 &= 1 \\ f_0(\mathbf{x}; s) &= 0 \\ f_1(\mathbf{x}; s_1) &= 0 \end{cases}$$

Which implies that $v_0 \neq 0$. In this case, $\frac{\partial f_1}{\partial x_1} = \ldots = \frac{\partial f_1}{\partial x_k} = 0$. Finally

$$\begin{cases} u_0 &= 1 - v_0^2 \\ 0 &= v_0^2 - v_0 \frac{\partial f_0}{\partial s} - 1 \end{cases}$$

The discriminant of the quadratic polynomial wrt v_0 equals $(\frac{\partial f_0}{\partial s})^2 + 4 > 0$ which means there always exists a real solution wrt v_0. This solution lies in $[-1, 1]$ as required. The interesting constraint we obtained is that

$$\frac{\partial f_1}{\partial x_1} = \frac{\partial f_1}{\partial x_2} = \ldots = \frac{\partial f_1}{\partial x_k} = 0.$$

- $\frac{\partial f_1}{\partial s_1} = 0 \Rightarrow v_1 \in [-1, 1]$ and $u_0 > 0$.

$$\begin{cases} u_0 + v_0 \frac{\partial f_0}{\partial s} &= 0 \\ v_0 \frac{\partial f_0}{\partial x_1} + v_1 \frac{\partial f_1}{\partial x_1} &= 0 \\ &\vdots \\ v_0 \frac{\partial f_0}{\partial x_k} + v_1 \frac{\partial f_1}{\partial x_k} &= 0 \\ u_0 + v_0^2 + v_1^2 &= 1 \\ f_0(\mathbf{x}; s) &= 0 \\ f_1(\mathbf{x}; s_1) &= 0 \end{cases}$$

This solution set contains the previous case when $v_1 = 0$. Nevertheless we solve this system to obtain more general conditions. We set $\chi_{ik} = \frac{\partial f_i}{\partial x_k}$ and $\sigma_0 = \frac{\partial f_0}{\partial s}$:

$$\begin{cases} \left(v_0 - \frac{\sigma_0}{2}\right)^2 + v_1^2 &= 1 + \frac{\sigma_0^2}{4} \\ v_0 \chi_{01} + v_1 \chi_{11} &= 0 \\ &\vdots \\ v_0 \chi_{0k} + v_1 \chi_{1k} &= 0 \\ u_0 &= 1 - v_0^2 - v_1^2 \\ f_0(\mathbf{x}; s) &= 0 \\ f_1(\mathbf{x}; s_1) &= 0 \end{cases}$$

The solution set wrt (v_0, v_1) lies at the intersection of a circle centered at $\left(\frac{\sigma_0}{2}, 0\right)$ and k lines passing through the origin, the slope of which is determined by $(\chi_{0j}, \chi_{1j}),\ j = 1 \ldots k$. If $k = 1$ then the line intersects the circle in 2 points in general, otherwise the set of lines has to be coincident (since we have a homogeneous system). That is there exist $\binom{k}{2}$ extra constraints which force each pair of lines to be parallel. These can be expressed in terms of the Jacobian determinant: $\forall i, j,\ 1 \leq i < j \leq k$

$$\begin{vmatrix} \chi_{0i} & \chi_{0j} \\ \chi_{1i} & \chi_{1j} \end{vmatrix} = 0 \iff J_{ij}(f_0, f_1) = 0.$$

Case $s = s_1$**.** We have:

$$\begin{cases} u_0 - u_1 + v_0 \frac{\partial f_0}{\partial s} & = 0 \\ u_1 + v_1 \frac{\partial f_1}{\partial s_1} & = 0 \\ v_0 \frac{\partial f_0}{\partial x_1} + v_1 \frac{\partial f_1}{\partial x_1} & = 0 \\ & \vdots \\ v_0 \frac{\partial f_0}{\partial x_k} + v_1 \frac{\partial f_1}{\partial x_k} & = 0 \\ u_0 + u_1 + v_0^2 + v_1^2 & = 1 \\ u_1 & \geq 0 \\ f_0(\mathbf{x}; s) & = 0 \\ f_1(\mathbf{x}; s_1) & = 0 \end{cases}$$

There are many solutions, but at least one is considered in the previous case, when $u_1 = v_1 = 0$.

A better approach is to consider a different Lagrangian when $s = s_1$ in order to avoid simultaneous vanishing of all constraints. In this case we have $L = u_0 s + v_0 f_0(\mathbf{x}; s) + v_1 f_1(\mathbf{x}; s)$ which leads to:

$$F\pi(\mathbf{x}^{\mathbf{1}\ldots\mathbf{k}}; s) = \begin{cases} u_0 + v_0 \frac{\partial f_0}{\partial s} + v_1 \frac{\partial f_1}{\partial s} & = 0 \\ v_0 \frac{\partial f_0}{\partial x_1} + v_1 \frac{\partial f_1}{\partial x_1} & = 0 \\ & \vdots \\ v_0 \frac{\partial f_0}{\partial x_k} + v_1 \frac{\partial f_1}{\partial x_k} & = 0 \\ u_0 + v_0^2 + v_1^2 & = 1 \\ u_0 & \geq 0 \\ f_0(\mathbf{x}; s) & = 0 \\ f_1(\mathbf{x}; s) & = 0 \end{cases}$$

We set $\chi_{ik} = \frac{\partial f_i}{\partial x_k}$ and $\sigma_i = \frac{\partial f_i}{\partial s}$:

$$\begin{cases} \left(v_0 - \frac{\sigma_0}{2}\right)^2 + \left(v_1 - \frac{\sigma_1}{2}\right)^2 & = & 1 + \frac{\sigma_0^2}{4} + \frac{\sigma_1^2}{4} \\ v_0 \chi_{01} + v_1 \chi_{11} & = & 0 \\ & \vdots & \\ v_0 \chi_{0k} + v_1 \chi_{1k} & = & 0 \\ u_0 & = & 1 - v_0^2 - v_1^2 \\ u_0 & \geq & 0 \\ f_0(\mathbf{x}; s) & = & 0 \\ f_1(\mathbf{x}; s) & = & 0 \end{cases}$$

The solution set wrt (v_0, v_1) lies at the intersection of a circle centered at $\left(\frac{\sigma_0}{2}, \frac{\sigma_1}{2}\right)$ and k lines passing through the origin, the slope of which is determined by $(\chi_{0j}, \chi_{1j}),\ j = 1 \ldots k$. If $k = 1$ then the line intersects the circle in 2 points in general, otherwise the set of lines has to be coincident (since we have a homogeneous system). That is there exist $\binom{k}{2}$ extra constraints which force each pair of lines to be parallel. These can be expressed in terms of the Jacobian determinant: $\forall i, j,\ 1 \leq i < j \leq k \begin{vmatrix} \chi_{0i} & \chi_{0j} \\ \chi_{1i} & \chi_{1j} \end{vmatrix} = 0 \iff J_{ij}(f_0, f_1) = 0$. $\square$

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