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iv) Asymptotic consistency and distribution of MLE-$\hat{\theta}$

- Under some weak assumptions with most important: $\forall \theta \neq \theta_0 : l(\theta) < l(\theta_0)$) $\Longrightarrow$ $\boxed{\text{MLE-}\hat{\theta} \text{ is consistent estimator of } \theta_0 : \hat{\theta} \xrightarrow[p]{n\to\infty} \theta_0}$

- Under some appropriate weak assumptions : $\boxed{\sqrt{n}\,(\hat{\theta}-\theta_0) \xrightarrow[d]{n\to\infty} N(0, F_\theta^{-1})}$ thus asymptotically the MLE
  - → converges at rate $n^{-1/2}$ with radius to θ_0
  - → follows a normal distribution with
  - → variance $V \equiv F_\theta^{-1}$.

thus approximately (as $n\to\infty$) $\boxed{\text{MLE-}\hat{\theta} \underset{a}{\sim} N(\theta_0, F_\theta^{-1}/n)}$

- asymptotical CR Lower Bound : (If $\tilde{\theta}$: unbiased estimator $\Longrightarrow var[\sqrt{n}(\tilde{\theta}-\theta_0)] \geq F_\theta^{-1}$) $\overset{!}{\Longrightarrow}$ The asymptotic variance of MLE-$\hat{\theta}$ is the best possible among all the unbiased estimators of θ_0

But the variance $\boxed{V \equiv F_\theta^{-1} \equiv H_\theta^{-1}}$ is non-calculable as it requires the knowledge of θ_0 (that we are looking)!

Thus we must replace it with appropriate consistent plug-in estimator-$\hat{V}$:

(*) $H_\theta \equiv -\frac{d^2}{d\theta^2} l(\theta)\Big|_{\theta=\theta_0} \longrightarrow {}_0\hat{H}_\theta \equiv -\frac{d^2}{d\theta^2} l(\theta)\Big|_{\theta=\hat{\theta}}$, so $\hat{V}_0 \equiv {}_0\hat{H}_\theta^{-1}$

(*) $H_\theta \equiv -\frac{d^2}{d\theta^2}\left(E[\log f(X|\theta)]\right)_{\theta=\theta_0} = E\left[-\frac{d^2}{d\theta^2}\log f(X|\theta)\Big|_{\theta=\theta_0}\right] \longrightarrow {}_1\hat{H}_\theta \equiv \frac{1}{n}\sum_{1}^{n} -\frac{d^2}{d\theta^2}\log f(X_i|\theta)\Big|_{\theta=\hat{\theta}} = -\frac{1}{n}\frac{d^2}{d\theta^2} l_n(\theta)\Big|_{\theta=\hat{\theta}}$, so $\hat{V}_1 \equiv {}_1\hat{H}_\theta^{-1}$

(*) $F_\theta \equiv E[S^2] \equiv E\left[\left(\frac{d}{d\theta}\log f(X|\theta)\Big|_{\theta=\theta_0}\right)^2\right] \longrightarrow \hat{F}_\theta \equiv \frac{1}{n}\sum_{1}^{n}\left(\frac{d}{d\theta}\log f[X_i|\theta]\Big|_{\theta=\hat{\theta}}\right)^2$, so $\hat{V}_2 \equiv \hat{F}_\theta^{-1}$

All three estimators of V are consistent
- → $\hat{V}_0 \xrightarrow[p]{n\to\infty} V$
- → $\hat{V}_1 \xrightarrow[p]{n\to\infty} V$
- → $\hat{V}_2 \xrightarrow[p]{n\to\infty} V$