Asymptotic distribution of $\bar{X}_n \equiv \frac{1}{n}\sum_1^n X_i$

From WLLN we know that $\bar{X}_n \xrightarrow[p]{} \mu$. Thus from previous (*) comment: $\boxed{\bar{X}_n \xrightarrow[d]{n\to\infty} \mu}$

The asymptotic distribution of $\bar{X}_n$ is degenerate. To obtain a non-degenerate result we need to rescale $\bar{X}_n$, defining the normalized sample mean $Z_n \equiv \sqrt{n}(\bar{X}_n - \mu)$

Now we can prove {using the concept of characteristic function}:

Lindeberg-Lévy Central Limit Theorem (CLT) {or asymptotic distribution of $Z_n \equiv \sqrt{n}(\bar{X}_n - \mu)$}

$$\boxed{\text{IF } \begin{array}{l} X_i: \text{i.i.d} \\ E[X^2] < \infty \end{array} \Longrightarrow \sqrt{n}(\bar{X}_n - \mu) \xrightarrow[d]{} Z \sim N(0, \sigma^2) \text{ where } \begin{array}{l} \mu \equiv E[X] \\ \sigma^2 \equiv E[(X-\mu)^2] \end{array}}$$

(*1) We knew that $E[\bar{X}_n] \equiv \mu$ and $var[\bar{X}_n] \equiv \frac{\sigma^2}{n}$, thus for $Z_n \equiv \sqrt{n}(\bar{X}_n - \mu)$ $E[Z_n] \equiv 0$ and $var[Z_n] \equiv \sigma^2$

Now with CLT we also know that as $n \to \infty$ the distribution of Z_n approaches normal $N[0, \sigma^2]$

(*2) The result is independend of the populations distribution $F(X)$! For example we saw

For $X \sim N(\mu, \sigma^2)$: $\bar{X}_n \sim N(0, \sigma^2)$ and for $X \sim \chi_1^2$: $\bar{X}_n \sim \sqrt{n}\left(\frac{\chi_n^2}{n} - 1\right) \neq N(0, \sigma^2)$. As $n \to \infty$ it approximates $N[0, \sigma^2]$

(*3) Making use of the property $\{Y \sim N(\mu, \sigma^2) \Rightarrow Y' \equiv a \cdot Y + b \sim N(a\mu + b, a^2\sigma^2)\}$ we can use the CMT to approximate the distribution of the unnormalized statistic $\bar{X}_n$ as

$$\bar{X}_n \underset{a}{\sim} N(\mu, \sigma^2/n) \quad \text{for } n \gg 1 \text{ but } n \neq \infty$$

↑ means asymptotically upon rescaling