12 bivariate random variables

$(X,Y) \in H \subseteq R^2 \rightarrow$ joint density distribution: $\boxed{f(x,y)\,dx\,dy \equiv P[X \in [x, x+dx], Y \in [y, y+dy]]}$

$\hookrightarrow P(X,Y) \in H \equiv \iint_H f(x,y)\,dx\,dy \equiv 1$

marginal density distribution

$\rightarrow f_X(x)\cdot dx \equiv P[X \in (x, x+dx)] = P[X \in (x, x+dx), Y \in R] = \left(\int_{-\infty}^{\infty} f(x,y)\cdot dy\right) dx$

$\hookrightarrow f_Y(y)\,dy \equiv P[Y \in (y, y+dy)] = P[Y \in (y, y+dy), X \in R] = \left(\int_{-\infty}^{\infty} f(x,y)\,dx\right) dy$

conditional density distribution

$\rightarrow f_{X/Y}(x/y)\,dx \equiv P[X \in (x, x+dx) / Y = y] = \dfrac{f(x,y)\,dx}{f_Y(y)}$

$\hookrightarrow f_{Y/X}(y/x)\,dy \equiv P[Y \in (y, y+dy) / X = x] = \dfrac{f(x,y)\,dx}{f_X(x)}$

$\left\{ \begin{array}{l} \text{From} \\ P(x,y) = P(y)\cdot P(x/y) \\ \Rightarrow P(x/y) = P(x,y)/P(y) \end{array} \right\}$

$\boxed{\text{Bayes Rule}}$: $\begin{array}{l} P(x,y) = P(y)\cdot P(x/y) \\ P(x,y) = P(x)\cdot P(y/x) \end{array} \Rightarrow$ $P(x)\cdot P(y/x) = P(y)\cdot P(x/y)$ $\boxed{P(y/x) = P(x/y)\cdot P(y)/P(x)}$ $\Rightarrow$ $f_X(x)\cdot f_{Y/X}(y/x) = f_Y(y)\cdot f_{X/Y}(x/y)$ $\boxed{f_{Y/X}(y/x) = f_{X/Y}(x/y)\cdot \dfrac{f_Y(y)}{f_X(x)}}$

expectation and variance

$$\begin{cases} \mu_X \equiv E[X] = \iint x\cdot f(x,y)\,dx\,dy = \int x\cdot f_X(x)\,dx \\ \sigma_X^2 \equiv var[X] \equiv E[(x-\mu_X)^2] = \iint (x-\mu_X)^2\cdot f(x,y)\,dx\,dy = \iint (x-\mu_X)^2\cdot f_X(x)\,dx \end{cases}$$

$$\begin{cases} \mu_Y \equiv E[Y] = \iint y\cdot f(x,y)\,dx\,dy = \int y\cdot f_Y(y)\,dy \\ \sigma_Y^2 \equiv var[Y] \equiv E[(y-\mu_Y)^2] = \iint (y-\mu_Y)^2\cdot f(x,y)\,dx\,dy = \iint (y-\mu_Y)^2\cdot f_Y(y)\,dy \end{cases}$$

conditional expectation and conditional variance

$$\begin{cases} m_X(y) \equiv E[X/Y=y] = \int x\cdot f_{X/Y}(x/y)\,dx = \int x\cdot f(x,y)\,dx / f_Y(y) \\ \sigma_X^2(y) \equiv E[(x-m_X(y))^2] = \int (x-m_X(y))^2\cdot f_{X/Y}(x,y)\,dx = \int (x-m_X(y))^2\cdot f(x,y)\,dx / f_Y(y) \end{cases}$$

$$\begin{cases} m_Y(x) \equiv E[Y/X=x] = \int y\cdot f_{Y/X}(y/x)\,dy = \int y\cdot f(x,y)\,dy / f_X(x) \\ \sigma_Y^2(x) \equiv E[y-m_Y(x)/X=x] = \int (y-m_Y(x))^2\cdot f_{Y/X}(y/x)\,dy = \int (y-m_Y(x))^2\cdot f(x,y)\,dy / f_X(x) \end{cases}$$

Law of Iterated Expectations

$E[X] = E[E[X|Y=y]]$ and $var[X] = E[var[X|Y=y]] + var[E[X|Y=y]]$

$E[Y] = E[E[Y|X=x]]$ and $var[Y] = E[var[Y|X=x]] + var[E[Y|X=x]]$