A) Test with simple H_0: for H_0 true, $\theta=\theta_0$, $\exists$ unique null distribution of X, $F(x)$.

- t-test for Normal Sampling: $\boxed{\begin{array}{l} X\sim N(\mu,\sigma^2) \text{ unknown} \\ H_0:\mu\equiv\mu_0 \end{array} \Rightarrow t\equiv\frac{\bar{X}_n-\mu_0}{\sqrt{s^2/n}}\sim G_0\equiv t_{n-1}}$ (replace with σ^2 if known) $\Rightarrow$ rejects H_0 in favor
  - $\to H_1:\mu>\mu_0$ if $t>q_{1-\alpha}$
  - $\to H_1:\mu<\mu_0$ if $t<q_\alpha$
  - $\to H_1:\mu\neq\mu_0$ if $|t|>q_{1-\alpha/2}$
- asymptotic t-test: $\boxed{\begin{array}{l} X\sim \text{unknown} \\ H_0:\mu\equiv\mu_0 \end{array} \Rightarrow t=\frac{\bar{X}_n-\mu_0}{\sqrt{s^2/n}}\xrightarrow[d]{n\to\infty}N(0,1)}\Rightarrow$ rejects H_0 in favor
  - $\to H_1:\mu>\mu_0$ if $t>z_{1-\alpha}$
  - $\to H_1:\mu<\mu_0$ if $t<z_\alpha$
  - $\to H_1:\mu\neq\mu_0$ if $|t|>z_{1-\alpha/2}$
- general asymptotic t-test: Let $\boxed{\begin{array}{l} X\sim\text{unknown} \\ \theta: \text{parameter of interest} \\ \hat{\theta}: \text{estimator of } \theta \\ s(\hat{\theta}): \text{standard error} \end{array} \xrightarrow[\text{standard conditions}]{\text{under}} \frac{\hat{\theta}-\theta}{s(\hat{\theta})}\xrightarrow[d]{n\to\infty}N(0,1)}\Rightarrow$ We can perform an asymptotic t-test under the null hypothesis $H_0:\theta=\theta_0$
  $$T\equiv\frac{\hat{\theta}-\theta_0}{s(\hat{\theta})}\xrightarrow[d]{n\to\infty}G_0=N(0,1)$$

- Likelihood Ratio Test

  Let $X\sim f(x/\theta)$, $H_0:\theta=\theta_0$, $H_1:\theta\neq\theta_0$

  $\boxed{T=\frac{L_n(\hat{\theta})}{L_n(\theta_0)} \text{ or equivalently } LR_n\equiv 2(\ell_n(\hat{\theta})-\ell_n(\theta_0))}$ with MLE ($\hat{\theta}\equiv\max \ell_n(\theta)$); LR_n: likelihood ratio statistic $\Rightarrow$ rejects H_0 if $LR_n>c$ with $c: P[LR_n>c|\theta_0]\equiv\alpha$

  LR_n does not follow a "known" distribution. We can use asymptotic!

- asymptotic likelihood Ratio test: $\boxed{LR_n \overset{\text{Taylor}}{=} \frac{(\hat{\theta}-\theta_0)^2}{\hat{V}}+O_p(1)\xrightarrow[d]{n\to\infty}X^2_m}$ where X^2_m: chi-distribution, $\hat{V}\equiv\left[-\frac{1}{n}\frac{\partial^2}{\partial\theta^2}\ell_n(\theta)|_{\hat{\theta}}\right]^{-1}$

- Connection of LR_n-test and t-test

i) As $n\to\infty$ the tests "Reject H_0 if $LR_n>c$" and "Reject H_0 if $|T|>c$" are asymptotically equivalent

ii) If $X\sim N(\theta,\sigma^2)$ with σ^2: known for the test $H_0:\theta=\theta_0$ and $H_1:\theta>\theta_0$ ($H_1:\theta<\theta_0$) the LR-test an t-test are equivalent

iii) Among all simple H_0 tests the LR is the most powerful. Thus in some case the t-test is also

B) Test with composite H_0: for H_0 true, $\theta=\theta_0$, $\nexists$ unique distribution $F_0(x)$ $\{X\sim F(x/\mu,\sigma^2)$, $H_0:\mu=\mu_0 \Rightarrow f_0(x)=f(x/\mu_0,\sigma^2)$ many$\}$

Let X with known parametric $F(x/\beta=(\beta_1,..,\beta_k))$. $\theta=h(\beta)$: scalar parameter of interest $\Rightarrow\{H_0:\theta=\theta_0, H_1:\theta\neq\theta_0\}$:

$\boxed{LR_n\equiv 2(\ell_n(\hat{\beta})-\ell_n(\tilde{\beta})) \text{ where } \hat{\beta}\equiv\max_{\forall\beta}\ell_n(\beta) \text{ and } \tilde{\beta}\equiv\max_{h(\beta)=\theta_0}\ell_n(\beta)\Rightarrow \text{reject } H_0 \text{ if } LR_n>c \text{ with } c: P[LR_n>c|\theta_0]\equiv\alpha}$

! If $X\sim N(\mu,\sigma^2)$: LR test $H_0:\mu=\mu_0$, $H_1:\mu\neq\mu_0$ equivalent to t-test $H_0:\mu=\mu_0$, $H_1\neq\mu_0$