The MLE-$\hat{\theta}$ is an estimator of the true $\theta \equiv \theta_0$. Thus, in principle, it can be expressed as a function of $X_1, \ldots, X_n \Rightarrow$ MLE-$\hat{\theta}$ follows it own distribution. We are interested in the following:

i) How to calculate the MLE-$\hat{\theta}(X_1, \ldots X_n)$?

$\boxed{\text{likelihood score} \quad S_n(\theta) = \frac{d}{d\theta} l_n(\theta) = \frac{d}{d\theta}\left(\frac{1}{n}\sum_{i=1}^{n} \log f(X_i|\theta)\right) \equiv 0}$ AND $\boxed{\text{likelihood Hessian} \quad H_n(\theta) = -\frac{d^2}{d\theta^2} l_n(\theta) = -\frac{d^2}{d\theta^2}\left(\frac{1}{n}\sum_{1}^{n} \log f(X_i|\theta)\right) > 0}$

ii) Is the MLE-$\hat{\theta} \equiv \hat{\theta}(X_1, \ldots, X_n)$ unbiased?

It depends on each specific case. We much check if $E[\hat{\theta}(X_1, \ldots X_n)] \equiv \theta_0$

Find θ_0 as the θ that maximizes $l(\theta) = E[\log f(X|\theta)]$, $\theta_0 = \theta_0(X)$. Then check this

iii) What is the variance of MLE-$\hat{\theta}(X_1, \ldots, X_n)$?

2nd derivative of expected log-density!

efficient score $\quad S \equiv \frac{\partial}{\partial\theta} \log f(X|\theta)\Big|_{\theta=\theta_0}$ and $\boxed{\text{expected Hessian} \quad H_\theta \equiv -\frac{d^2}{d\theta^2}\left(E[\log f(X|\theta)]\right)\Big|_{\theta_0} = -\frac{d^2}{d\theta^2} l(\theta)\Big|_{\theta_0}}$

expected efficient score

variance of efficient score — Fisher Information

$E[S] = E\left[\frac{\partial}{\partial\theta} \log f(X|\theta)\Big|_{\theta_0}\right] \Rightarrow$

$\boxed{F_\theta \equiv var[S] = E[(S - E[S])^2] \stackrel{!}{=} E[S^2]}$

Information Matrix Equality $\quad F_\theta \equiv H_\theta$

$E[S] \equiv \frac{\partial}{\partial\theta} E[\log f(X|\theta)]\Big|_{\theta_0} \Rightarrow$

$E[S] \equiv \frac{\partial}{\partial\theta} l(\theta)\Big|_{\theta_0} \stackrel{①}{\Rightarrow}$

$E[S] \equiv 0$

1st derivative of expected log-density!

C-R Lower Bound CRLB : If $\tilde{\theta}$: unbiased estimator $\Rightarrow var[\tilde{\theta}] \geq (n \cdot F_\theta)^{-1}$

(*) An estimator-$\tilde{\theta}$ is C-R efficient if $\tilde{\theta}$: unbiased and $Var[\tilde{\theta}] \equiv (n \cdot F_\theta)^{-1}$

(*) CRLB for transformation : If $\check{\theta}$: unbiased estimator of $h(\theta) \Rightarrow var[\check{\theta}] \geq n^{-1} \cdot \left(\frac{d}{d\theta} h(\theta)\Big|_{\theta_0}\right)^2 \cdot F_\theta^{-1}$