MULTIVARIATE NORMAL

Univariate standard normal distribution $N(0,1)$: $\varphi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$, $E[X] = 0$, $var[X] = 1$

Univariate normal distribution $N(\mu, \sigma^2)$: $\varphi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$, $E[X] = \mu$, $var[X] = \sigma^2$

If $X \sim N[0,1]$ then $Y \equiv \mu + \sigma \cdot X \sim N(\mu, \sigma^2)$. If $X \sim N(\mu, \sigma^2)$ then $Y = a + bX \sim N(a + b\mu, b^2\sigma^2)$

Multivariate standard normal of $X \overset{\in R^m}{\sim} N(0, I_{m\times m})$:

$f(x) \equiv f(x_1, \dots, x_m) \equiv f(x_1) \cdot \ldots \cdot f(x_m) \equiv \prod_1^m \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x_i^2}{2}\right) \equiv \frac{1}{(2\pi)^{m/2}} \exp\left(-\frac{x^T \cdot x}{2}\right)$, $E[X] = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ and $\Sigma \equiv var[X] = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I_{m\times m}$

Multivariate normal of $X \overset{\in R^m}{\sim} N(\mu, \Sigma)$:

$f(x) \equiv f(x_1, \dots, x_m) \equiv \frac{1}{(2\pi)^{m/2} \cdot (\det \Sigma)^{1/2}} \exp\left(-\frac{(x-\mu)^T \cdot \Sigma^{-1} \cdot (x-\mu)}{2}\right)$ where

$E[X] \equiv \mu \equiv \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix}$ and $var[X] \equiv \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12}^2 & \cdots \\ \vdots & \ddots & \\ & & \sigma_m^2 \end{bmatrix}$

(*1) In general $\sigma_{ij}^2 \equiv cov(x_i, x_j) \neq 0 \ \forall i,j \Rightarrow x_1, \dots, x_m$: correlated $\Rightarrow x_1, \dots, x_m$: NOT independent $\Rightarrow f(x) \neq f(x_1) \cdot \ldots \cdot f(x_m)$

(*2) If $\sigma_{ij}^2 \equiv 0 \ \forall i,j \Rightarrow x_1, \dots, x_m$: uncorrelated $\xrightarrow[\text{normal}]{\text{only for}}$ $x_1, \dots, x_m$: independent $\Rightarrow f(x) \equiv \prod_1^m f(x_i) = \prod_1^m \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(x-\mu)^T(x-\mu)}{2\sigma_i^2}\right)$

as $\Sigma \equiv diag(\sigma_1^2, \dots, \sigma_m^2) \Rightarrow \Sigma^{-1} \equiv diag(1/\sigma_1^2, \dots, 1/\sigma_m^2)$ and $\det \Sigma = \prod_1^m \sigma_i^2$

(*3) If $X \sim N(0, I_m) \Longrightarrow Y \equiv \mu + K \cdot X \sim N(\mu, \Sigma \equiv K K^T)$

(*3') If $X \sim N(\mu, \Sigma) \Longrightarrow Y \equiv A + BX \sim N(A + B\mu, B \Sigma B^T)$

(*4) Let $X \in R^m$, $Y \in R^k$ random vectors. Define the $\begin{pmatrix} X \\ Y \end{pmatrix} \in R^{m+k}$, then

$\Sigma \equiv var\begin{bmatrix} X \\ Y \end{bmatrix} \equiv E\left[\left(\begin{pmatrix} x \\ y \end{pmatrix} - E\begin{bmatrix} x \\ y \end{bmatrix}\right)\left(\begin{pmatrix} x \\ y \end{pmatrix} - E\begin{pmatrix} x \\ y \end{pmatrix}\right)^T\right] \equiv \begin{array}{c|c} & \begin{matrix} m & & k \end{matrix} \\ \begin{matrix} m \\ \\ k \end{matrix} & \left[\begin{array}{cc|cc} \sigma_{x_1}^2 & & \sigma_{x_1 y_1}^2 & \cdots \\ & \sigma_{x_m}^2 & & \\ \hline \sigma_{y_1 x_1}^2 & & \sigma_{y_1}^2 & \\ & & & \sigma_{y_k}^2 \end{array}\right] \end{array} = \left[\begin{array}{c|c} \Sigma_{XX} & \Sigma_{XY} \\ \hline \Sigma_{YX} & \Sigma_{YY} \end{array}\right]$ $\to cov(X,Y)$, $\to cov(Y,X)$

If $\Sigma_{XY} = \Sigma_{YX} \equiv cov(X,Y) = 0 \Rightarrow X, Y$: uncorrelated $\xrightarrow[X,Y:\ \text{normal}]{\text{if}}$ X, Y : independend $f(x,y) = f_X(x) \cdot f_Y(y)$

(*5) Many important distributions are derived as transformations of multi normal...