The random vector $X \equiv (X_1 \cdots X_m)^T \in \mathbb{R}^m$ is considered random variable as a whole because of the joint PDF $f \equiv f(x_1, \dots, x_m)$ that is used to calculate expectation values with respect to each component X_i. In other words, the components $X_1, \dots, X_m$ of the random vector $X \in \mathbb{R}^m$ are in general not-independend. This is reflected in the Σ as $\sigma_{ij}^2 \neq 0$ in general. Thus,

Let $f(x_1, \dots, x_m)$ the PDF of a random $X \in \mathbb{R}^m$

- $\to$ If $f(x_1, \dots, x_m) \equiv f_1(x_1) \cdots f_m(x_m) \Longleftrightarrow$ $X_1, \dots, X_m$ mutually independent $\Longleftrightarrow$ $\forall X_i, X_j$ $\nexists$ relation $X_i \equiv g(X_j)$ $\Longrightarrow$ $X_1, \dots, X_m$ mutually uncorrelated $\Longleftrightarrow$ $\forall X_i, X_j$ $\sigma_{ij}^2 \equiv 0$ $\Longleftrightarrow$ Σ is diagonal
- $\to$ If $X_1, \dots, X_m$: not mutually independend then $X_1, \dots, X_m$ can still be mutually uncorrelated $\{\sigma_{ij}^2 = 0 \ \forall i, j = 1, \dots, m\}$ $\Longleftrightarrow$ It can still be $\forall X_i, X_j$ $X_i \neq aX_j + b$ BUT $\forall X_i, X_j$ $\nexists$ relation $X_i \equiv g(X_j)$ It cannot be

(*) The latter is in particular relevant in regression theory $Y = h(X_1, \dots, X_m) + \varepsilon \dots$

What about the correlation or independence between two random vectors $X \in \mathbb{R}^{m_X}$, $Y \in \mathbb{R}^{m_Y}$

joint CPF : $F(x, y) \equiv P[X \leq x, Y \leq y] \equiv P[X_1 \leq x_1, \dots, X_{m_X} \leq x_{m_X}, Y_1 \leq y_1, \dots, Y_{m_Y} \leq y_{m_Y}]$

joint PDF : $f(x, y) \equiv P[X \in [x, x+dx], Y \in [y, y+dy]] \equiv \partial^{m_X + m_Y} F(x,y) / \partial x_1 \cdots \partial x_{m_X} \partial y_1 \cdots \partial y_{m_Y} = f(x_1, \dots, x_{m_X}, y_1, \dots, y_{m_Y})$

marginal PDF: $f_X(x) \equiv P[X \in [x, x+dx], Y \in \mathbb{R}] = \int_{-\infty}^{\infty} f(x,y)\, dy_1 \cdots dy_{m_Y}$

$f_Y(y) = P[Y \in [y, y+dy], X \in \mathbb{R}] \equiv \int_{-\infty}^{\infty} f(x,y)\, dx_1 \cdots dx_{m_X}$

conditional PDF: $f_{X/Y}(x/y) \equiv P[X \in [x, x+dx] / Y \equiv y] \equiv f(x,y)/f_Y(y)$

$f_{Y/X}(y/x) \equiv P[Y \in [y, y+dy] / X \equiv x] \equiv f(x,y)/f_X(x)$

2 random vectors $X \in \mathbb{R}^{m_X}$, $Y \in \mathbb{R}^{m_Y}$ are independent iff $f(x,y) \equiv f_X(x) \cdot f_Y(y)$

(*1) the random vectors X, Y do NOT need to have the same dimension to define joint PF or examine independence!

(*2) It can be that
- $\to$ $X_1, \dots, X_{m_X}$: not mutually independend, $Y_1, \dots, Y_{m_Y}$: not mutually independent BUT X, Y: independent
- $\to$ $X_1, \dots, X_{m_X}$: mutually independent, $Y_1, \dots, Y_{m_Y}$: mutually independent BUT X, Y: dependent