

Speaking more general...

Let a parameter θ of $F(X)$ { such as $\mu = E[X]$, $\sigma^2 = E[(X-\mu)^2]$, ... }
 If an estimator $\hat{\theta} = \hat{\theta}(n, x_1, \dots, x_n)$ such that $\hat{\theta} \xrightarrow{P} \theta$ {consistent} and $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} Z$, then

$$\boxed{\begin{aligned} &\hat{\theta} \xrightarrow{P} \theta \quad \text{AND} \quad \sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} Z \sim \mathcal{Z} \\ &\text{that delta method} \Rightarrow \sqrt{n}(h(\hat{\theta}) - h(\theta)) \xrightarrow{d} h'(\theta) \cdot Z \quad \forall \text{ continuous } h \end{aligned}}$$

If we consider a population parameter $\theta = E[g(X)]$ with $g(X)$ s.t. $E[g(X)]$, $E[g(X)^2] < \infty$, then

• WLLN: the plug-in estimator $\hat{\theta} = \frac{1}{n} \sum_{i=1}^n g(X_i) \xrightarrow[n \rightarrow \infty]{P} E[g(X)] = \theta$

• CLT: the plug-in estimator $\hat{\theta} = \frac{1}{n} \sum_{i=1}^n g(X_i)$, $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow[n \rightarrow \infty]{d} Z \sim N(0, V_g)$, $V_g = \text{Var}(g(X)) = E[(g(X) - \theta)^2]$

> delta method: $\sqrt{n}(h(\hat{\theta}) - h(\theta)) \xrightarrow{d} Z \sim N(0, V_h) = N(0, h'(\theta)^2 \cdot V_g) = N(0, V_h)$, $V_h = h'(\theta)^2 \cdot V_g$

*1) To practically use the asymptotic distribution we need an estimation for V_h . It turns out that the plug-in estimator $\hat{V}_h = h'(\hat{\theta})^2 \cdot \hat{V}_g = h'(\hat{\theta})^2 \cdot \hat{V}_g = h'(\hat{\theta})^2 \cdot \left(\frac{1}{n} \sum_{i=1}^n g(X_i) - \frac{1}{n} \sum_{i=1}^n g(X_i) \right)^2 \xrightarrow{P} V_h$ ✓

*2) $h(\hat{\theta}) \sim N(h(\theta), V_h/h)$ and $t = \frac{\sqrt{n}(\hat{\theta} - \theta)}{\sqrt{V_g}} = \frac{\sqrt{n}(h(\hat{\theta}) - h(\theta))}{\sqrt{V_h}} \sim N(0, 1)$