$\boxed{\text{7 ESTIMATORS of RANDOM VECTORS}}$

Let a random vector $X \equiv \begin{bmatrix} X_{,1} \\ \vdots \\ X_{,m} \end{bmatrix} \in R^m$ with PDF $f(x)$ that has $\mu \equiv E[X] \in R^m$ and $\Sigma \equiv var[X] \in R^m \times R^m$.

To estimate the true μ, Σ or even the true multivariate distribution of X $\{f(x)\}$, we take:

a sample $\{X_1, \dots, X_n\}$ where $X_i \in R^m$ and are i.i.d. and define the analog estimators:

$$\boxed{\hat{\mu} \equiv \bar{X}_n \equiv \frac{1}{n}\sum_{1}^{n} X_k \equiv \begin{pmatrix} \bar{X}_{n,1} \\ \bar{X}_{n,2} \\ \vdots \\ \bar{X}_{n,m} \end{pmatrix}}, \quad \boxed{\hat{\Sigma} \equiv \frac{1}{n-1}\sum_{1}^{n} (X_k - \bar{X}_n)(X_k - \bar{X}_n)^T},$$

In particular:

- $\mu_i \equiv E[X_{,i}] \longrightarrow \hat{\mu}_i \equiv \bar{X}_{n,i}$, $\sigma_i^2 \equiv var[X_{,i}] \equiv \Sigma_{ii} \longrightarrow \hat{\sigma}_i^2 \equiv \hat{\Sigma}_{ii}$, $\sigma_{ij}^2 \equiv cov[X_{,i}, X_{,j}] \equiv \Sigma_{ij} \longrightarrow \hat{\sigma}_{ij}^2 \equiv \hat{\Sigma}_{ij}$

- $E[\bar{X}_n] \equiv \mu$ $\{\bar{X}_n$: unbiased$\}$, also $E[\bar{X}_{n,i}] = \mu_i$ $\{\bar{X}_{n,i}$: unbiased$\}$
- $E[\hat{\Sigma}] \equiv \Sigma$ $\{\hat{\Sigma}$: unbiased$\}$, thus $E[\hat{\Sigma}_{ij}] = \Sigma_{ij} \equiv \sigma_{ij}^2$ $\{\hat{\Sigma}_{ij}$: unbiased$\}$
- $var[\bar{X}_n] \equiv E[(\bar{X}_n - E[\bar{X}_n])(\bar{X}_n - E[\bar{X}_n])^T] = \Sigma/n$

Also the MSE-matrix of $\bar{X}_n$, $mse[\bar{X}_n] \equiv E[(\bar{X}_n - \mu)(\bar{X}_n - \mu)^T] = \Sigma/n$

and we can prove that $\bar{X}_n$: best linear unbiased estimator of μ

If $X \in R^m \sim N(\mu, \Sigma) \implies \bar{X}_n \sim N(\mu, \Sigma/n)$ and $\hat{\Sigma} \sim$ Wishart $\{$multivariate generalization of $\chi^2_{n-1}\}$