$\boxed{\text{3 ESTIMATORS OF } \mu \text{ AND } \sigma^2}$

Let distribution $f(X)$ (unknown) with unknown $\mu \equiv E[X]$, $\sigma^2 \equiv var[X]$. Take random n-sample.

- to estimate μ: Define the analog-estimator[(*1)] $\boxed{\hat{\mu} \equiv \bar{X}_n = \frac{1}{n}\sum_1^n X_i}$. For this we get
  - $E[\bar{X}_n] \equiv \mu \Rightarrow bias(\bar{X}_n) = 0$ $\{E[\bar{X}_n] = E[\frac{1}{n}\sum_1^n X_i] \overset{linear}{=} \frac{1}{n}\sum_1^n E[X_i] = \frac{1}{n}\cdot(n\cdot\mu) = \mu,\ bias(\bar{X}_n) = E[\bar{X}_n] - \mu = \mu - \mu = 0\}$
  - $var[\bar{X}_n] \equiv \sigma^2/n$ $\{var[\bar{X}_n] = var[\frac{1}{n}\sum_1^n X_i] \overset{linear}{=} \frac{1}{n^2}\cdot var[\sum_1^n X_i] \overset{X_i:\ independ/uncorrelated}{=} \frac{1}{n^2}\cdot\sum_1^n var[X_i] = \frac{1}{n^2}\cdot n\cdot\sigma^2 = \sigma^2/n\}$ (*2)
  - $mse[\bar{X}_n] \equiv \sigma^2/n$ $\{mse[\bar{X}_n] = bias^2(\bar{X}_n) + var[\bar{X}_n] = 0 + \sigma^2/n = \sigma^2/n\}$

(*1) $\bar{X}_n$: Best Linear Unbiased Estimator (BLUE) of μ. In fact it can be proven that $\bar{X}_n$: BUE of μ.

$\left\{ \text{let } \hat{\mu} \begin{array}{l} \text{linear: } \hat{\mu} = \sum w_i \cdot X_i \\ \text{AND} \\ \text{unbiased: } E[\hat{\mu}] = \mu \end{array} \Rightarrow \begin{array}{l} \hat{\mu} = \sum w_i \cdot X_i \\ \sum w_i = 1 \end{array} \xrightarrow[var[\hat{\mu}]:\ min\ (Lagrange..)]{find\ w_1,\dots,w_n\ s.t.} w_1 = \dots = w_n = 1/n \Rightarrow \hat{\mu} \equiv \frac{1}{n}\sum X_i : \text{BLUE} \right\}$

(*2) As we don't know the actual σ^2 we have to use an appropriate estimate $\hat{\sigma}^2$ instead. Thus instead of the actual variance of $\bar{X}_n$ we use the estimator $\hat{\sigma}^2/n$

- to estimate σ^2: Define the analog estimator $\hat{\sigma}^2 \equiv \frac{1}{n}\cdot\sum_1^n (X_i - \bar{X}_n)^2$
  - $E[\hat{\sigma}^2] \equiv \frac{n-1}{n}\cdot\sigma^2 \Rightarrow bias(\hat{\sigma}^2) = -\sigma^2/n \neq 0$ $\{E[\hat{\sigma}^2] = E[\frac{1}{n}\sum_1^n (X_i - \bar{X}_n)^2] \overset{algebra}{=} E[\frac{1}{n}\underbrace{(X_i-\mu)^2}_{n\cdot\sigma^2}] - \underbrace{E[(\bar{X}_n-\mu)^2]}_{\sigma^2/n} = (1-\frac{1}{n})\sigma^2\}$

The above estimator $\hat{\sigma}^2$ is BIASED. So, we define instead the bias-corrected variance estimator: $\boxed{S^2 \equiv \frac{n}{n-1}\hat{\sigma}^2 = \frac{1}{n-1}\sum_1^n (X_i - \bar{X}_n)^2}$. Now

  - $E[S^2] \equiv \sigma^2 \Rightarrow bias(S^2) \equiv 0$ $\{E[S^2] = E[\frac{n}{n-1}\hat{\sigma}^2] = \frac{n}{n-1}E[\hat{\sigma}^2] = \frac{n}{n-1}\cdot\frac{n-1}{n}\sigma^2 = \sigma^2\}$
  - $var[S^2] = E[(S^2 - E[S^2])^2] = E[(S^2-\sigma^2)^2] = \dots$ not-closed form
  - $mse[S^2] = var[S^2] = \dots$ not-closed form

For a general unknown distribution $f(X)$ with unknown μ, σ^2: Take random n-sample and get
- estimate of $\mu \simeq \bar{X}_n$ with standard error $s(\bar{X}_n) = S/\sqrt{n}$
- estimate of $\sigma^2 \simeq S^2$ with standard error $s(S^2) = \sqrt{var[S^2]} = \dots$ not-closed form

It would be way more informative to have the actual distribution of an estimator such as $\bar{X}_n$ or S^2, as we could answer very accurately to interesting probabilistic questions. The distribution of an estimator relies heavily on the distribution $f(X)$. For instance:

If $\boxed{X \sim N(\mu, \sigma^2)}$ then taking random n-sample with X_i: iid $N(\mu, \sigma^2)$ $\Rightarrow$

- $\bar{X}_n \sim N(\mu, \sigma^2/n)$ → the two statistics are independent!
- $\frac{n\hat{\sigma}^2}{\sigma^2} \equiv \frac{(n-1)\cdot S^2}{\sigma^2} \sim \chi^2_{n-1}$
- $T \equiv \sqrt{n}(\bar{X}_n - \mu)/S \sim t_{n-1}$ {studentized ratio or t-statistic, z-statistic}

However, we rarely know the $f(X)$. We need extra tools to retrieve extra knowledge!