4 LIMIT THEOREMS

We are interested in the behavior of estimators and their distributions as $n \to \infty$

A sequence of random $\{Z_1, \dots, Z_n\}$

- Converges in probability to c as $n \to \infty$ if $\lim_{n \to \infty} P[|Z_n - c| \leq \delta] = 1 \ \forall \delta > 0$

  $\operatorname{plim}_{n \to \infty} Z_n = c$ OR $Z_n \xrightarrow[p]{n \to \infty} c$

- An estimator $\hat{\theta}$ of θ is consistent if $\hat{\theta} \xrightarrow[p]{n \to \infty} \theta$

- Weak Law of Large Numbers (WLLN) {can be proven using Chebishev's inequality which has σ^2 as parameter}

  $\boxed{\text{If } \begin{array}{l} X_i: \text{iid} \\ E[X] \equiv \mu < \infty \end{array} \implies \begin{array}{l} \bar{X}_n : \text{consistent estimator of } \mu \equiv E[X] \\ \bar{X}_n \xrightarrow[p]{n \to \infty} \mu \equiv E[X] \end{array}}$

(*) The covergence of WLLN is NOT-uniform, as it depends on σ^2 of the distribution of X

As a consequence, $\forall n \neq \infty, \delta > 0 \ \exists$ a distribution of X s.t. $P[|\bar{X}_n - \mu| > \delta] \equiv 1$

So to make the WLLN-convergence uniform
- → restrict to bounded $f(X)$ with $\sigma^2 \leq B$
- OR
- → apply WLLN to rescaled sample means: $\bar{X}^*_n = \frac{1}{n} \sum_i X^*_i$, $X^*_i = X_i / \sigma^2$

- Continuous Maping Theorem (CMT): $\boxed{\text{If } \begin{array}{l} Z_n \xrightarrow[p]{n \to \infty} c \\ \text{AND} \\ h(\,): \text{continuous at } c \end{array} \implies h(Z_n) \xrightarrow[p]{n \to \infty} h(c)}$

  {If Z_n consistent for $c \implies h(Z_n)$: consistent for $h(c)$}