~~Continuous Mapping~~ Theorem (CMT) {or asymptotic distribution of function of $Z_n \equiv \sqrt{n}(\bar{X}_n - \mu)$}

$$\boxed{\text{If } \begin{array}{l} Z_n \xrightarrow{d} Z \\ h(): \text{continuous} \end{array} \Longrightarrow h(Z_n) \xrightarrow{d} h(Z)}$$

(*1) If we know the asymptotic distribution of Z_n the with CMT we find the asymptotic distribution of $h(Z_n)$

(*2) A special case of CMT is the Slutsky's Theorem:

$$\boxed{\text{If } \begin{array}{l} Z_n \xrightarrow{d} Z \\ C_n \xrightarrow{p} c \end{array} \Longrightarrow \begin{array}{l} Z_n + C_n \xrightarrow{d} Z + c \\ Z_n \cdot C_n \xrightarrow{d} Z \cdot c \\ Z_n / C_n \xrightarrow{d} Z / c \end{array}}$$

Delta Method {or asymptotic distribution of function of $\bar{X}_n$}

We cannot use CMT to calculate directly the $h(\bar{X}_n) \xrightarrow{d}$ as we don't know $\bar{X}_n \xrightarrow{d}$!

What we know is only $Z_n \equiv \sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$. Thus we can prove(*) that:

$$\boxed{\begin{array}{l} \text{For } Z_n \equiv \sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} Z \sim N(0, \sigma^2) \\ \text{if } h(): \text{continuous} \end{array} \Longrightarrow \sqrt{n}\left(h(\bar{X}_n) - h(\mu)\right) \xrightarrow{d} h' \cdot Z \sim N(0, \underbrace{h'^2 \cdot \sigma^2}_{V_h}) \quad \text{where } h' \equiv \frac{\partial h}{\partial u}\Big|_\mu}$$

and thus we can say that $\boxed{h(\bar{X}_n) \overset{a}{\sim} N(h(\mu), h'^2 \cdot \sigma^2 / n)}$

(*) {CMT $\forall \mu, \bar{X}_n$ $\exists \tilde{X}_n$ with $|\tilde{X}_n - \mu| < |\bar{X}_n - \mu|$ s.t. $h(\bar{X}_n) - h(\mu) = h'(\tilde{X}_n) \cdot (\bar{X}_n - \mu) \Longrightarrow \sqrt{n}(h(\bar{X}_n) - h(\mu)) = h'(\tilde{X}_n) \cdot \sqrt{n} \cdot (\bar{X}_n - \mu)$; $\Longrightarrow \tilde{X}_n \xrightarrow{p} \mu \overset{CMT}{\Longrightarrow} h'(\tilde{X}_n) \xrightarrow{p} h'(\mu) \Longrightarrow \sqrt{n}(h(\bar{X}_n) - h(\mu)) \xrightarrow{d} h'(\mu) \cdot Z$}

(*1) To practically use the asymptotic distribution we need an estimation of the variation $V_h \equiv h'^2 \cdot \sigma^2$

We choose the plug-in estimator: $\hat{V}_h \equiv \widehat{h'^2(\mu) \cdot \sigma^2} \equiv \hat{h}'^2(\mu) \cdot \hat{\sigma}^2 \equiv h'^2(\hat{\mu}) \cdot \hat{\sigma}^2 \equiv h'^2(\bar{X}_n) \cdot s^2$ which can be proven to be consistent {WLLN: $\bar{X}_n \xrightarrow{p} \mu$, $s^2 \xrightarrow{p} \sigma^2$ AND CMT: $h'(\bar{X}_n) \xrightarrow{p} h'(\mu)$, Thus CMT: $h'^2(\bar{X}_n) \cdot s^2 \xrightarrow{p} h'^2(\mu) \cdot \sigma^2$}

(*2) Asymptotic distribution of t-statistic $\boxed{t \equiv \frac{Z_n}{\sqrt{\hat{V}_h}} \equiv \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sqrt{\hat{V}_h}} \underset{\text{Slutsky}}{\xrightarrow{d}} \frac{N(0, V_h)}{\sqrt{V_h}} \underset{\text{gaussian property}}{\equiv} N(0,1)}$