LESSON 1 Introduction of Conic Sections and the Circle

What I Need to Know

Upon completion of this lesson, you should be able to:

- illustrate the different types of conic sections: parabola, ellipse, circle, hyperbola, and degenerate cases;
- determine the type of conic section defined by a given 2^{nd} degree equation in x and y;
- define a circle;
- determine the standard form of equation of circle;
- Graph a circle in a rectangular coordinate system;
- Derive and illustrate the equation of the circle;
- Find the center and the radius of the circle of an equation;
- Convert the general equation of circle to standard form and vice versa;

What's In

Activity 1: Recall

You had learned in your previous mathematics in junior high school about solving a quadratic equation by completing the squares. Let us recall your knowledge about the subject using these following examples.

In order to find the roots of a certain quadratic equation, the following steps will be used using completing the square method.

1. Rewrite the equation in the form $x^2 + bx = c$.
2. Add to both sides the term needed to complete the square.
3. Factor the perfect square trinomial.
4. Solve the resulting equation by using the square root property.

1. Solve the equation $x^2 + 8x + 5 = 0$ by completing the square.

Solution: $x^2 + 8x + 5 = 0$

$x^2 + 8x = -5$ *Rewrite the equation in the form $x^2 + bx = c$.*

$x^2 + 8x + 16 = -5 + 16$ *Add the appropriate constant to complete the square.*

$(x + 4)^2 = 11$ *Factoring the perfect square trinomial*

$x+4=\pm\sqrt{11}$ *Solve using the square root method.*
$x = -4 \pm \sqrt{11}$
$x_1=0.68$
$x_2= -7.38$

2. Find the roots of $x^2 + 10x - 4 = 0$ using completing the square method.

$x^2 + 10x = 4$ *Rewrite the equation in the form $x^2 + bx = c$.*

$x^2+10x+25=4+25$ *Add the appropriate constant to complete the square.*

$(x+5)^2=29$ *Factoring the perfect square trinomial*

$(x+5)=\pm\sqrt{29}$ *Solve using the square root method.*
$x = -5 \pm \sqrt{29}$
$X_1=0.39$
$X_2= -10.39$

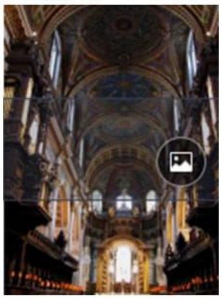
Completing the square method is useful in discussing conic sections specially the equation of a circle.

What's New

Geometric Figures or shapes are use in architectural designs. For this activity, identify the following shapes as circle, parabola, ellipse, or hyperbola as shown in the pictures being used in real-life. Write your answer on the space provided.




1) ___________ 2)__________ 3) ____________ 4) ____________



5) ___________ 6)__________ 7) ____________ 8) ____________

Photo source: file:///C:/Users/admin/Desktop/conic-sections.pdf and https://bit.ly/3iLqvXz

Does the activity ignite your interest to study more about geometric shapes particularly different conic sections like your answers in the activity? Can you name other architectural designs not in the pictures that used the idea of geometrical shapes? Does the shape matters on the durability, functionality and artistic designs?

Studying this module will help you appreciate nature and man's creation that would help daily life activities.

What is It

We present the conic sections, a particular class of curves which sometimes appear in nature and which have applications in other fields. In this lesson, we first illustrate how each of these curves is obtained from the intersection of a plane and a cone, and then discuss the first of their kind, circles. The other conic sections will be covered in the next lessons.

Conic sections (or conics), is a curved formed by a plane passing through a double-napped circular cone. One of the first shapes we learned, a circle, is a conic. When you throw a ball, the trajectory it takes is a parabola. The orbit taken by each planet around the sun is an ellipse. Properties of hyperbolas have been used in the design of certain telescopes and navigation systems. We will discuss circles in this lesson, leaving parabolas, ellipses, and hyperbolas for subsequent lessons.

- **Circle** (Figure 1.1) – is a special case of ellipse in which the plane is perpendicular to the axis of the cone.
- **Ellipse** (Figure 1.1) - when the (tilted) plane intersects only one cone to form a bounded curve
- **Parabola** (Figure 1.2) – the plane is parallel to a generator line of the cone
- **Hyperbola** (Figure 1.3) – the intersection is an unbounded curve and the plane is not parallel to a generator line of the cone and the plane intersects both halves of the cone.

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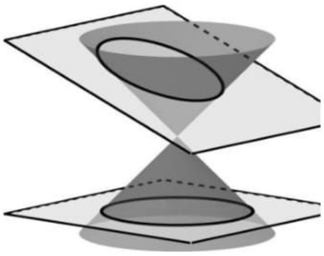


Figure 1.1

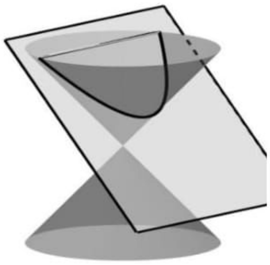


Figure 1.2

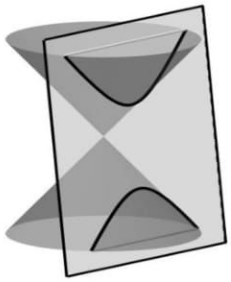


Figure 1.3

We can draw these conic sections on a rectangular coordinate plane and find their equations. To be able to do this, we will present equivalent definitions of these conic sections in subsequent sections, and use these to find the equations.

There are other ways for a plane and the cones to intersect, to form what are referred to as degenerate conics: a point, one line, and two lines. See Figures 1.4, 1.5, and 1.6.

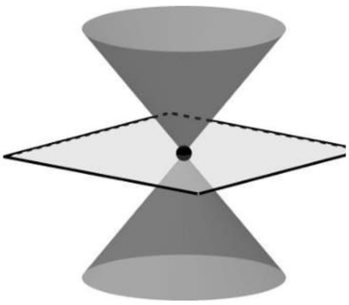


Figure 1.4

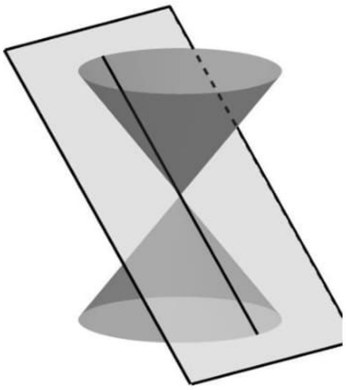


Figure 1.5

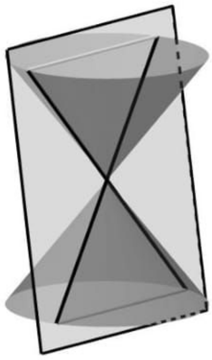


Figure 1.6

The graph of the second degree equation of the form $Ax^2 + Bx + Cy^2 + Dx + Ey + F = 0$ is determine by the values of $B^2 - 4ac$.

Table 1

Graphs of Quadratic Equations

Conic Section	**Value of $B^2 - 4ac$**	**Eccentricity**
Circle	$B^2 - 4ac$<0 or A=C	$e = 0$
Parabola	$B^2 - 4ac = 0$	$e = 1$
Ellipse	$B^2 - 4ac < 0, B \neq 0 \; or \; A \neq C$	$0 < e < 1$
Hyperbola	$B^2 - 4ac > 0$	$e > 1$

Example1.1 Determine the type conic section that each general equation will produce.

1. $2x^2 + 4xy + 3y^2 + 12y - 1 = 0$

2. $3x^2 - 2y^2 + 6x + 10y - 16 = 0$

3. $3x^2 + 3y^2 + 18x - 16y + 31 = 0$

4. $x^2 + 4xy - 4x + y^2 - 12y = 0$

Solutions: We will collect all the values of A, B, C in each equation. Then solve for the value of $B^2 - 4ac$. Interpret the result based on table 1.

1. $2x^2 + 4xy + 3y^2 + 12y - 1 = 0$

   $A = 2, B = 4, C = 3$

   $B^2 - 4ac = 4^2 - 4(2)(3) = 16 - 24 = -8 < 0$

   Note that $B \neq 0$ and $A \neq C$. Thus, the conic section is an ellipse.

2. $3x^2 - 2y^2 + 6x + 10y - 16 = 0$

   $A = 3, B = 0, C = -2$

   $B^2 - 4ac = 0^2 - 4(3)(-2) = 0 + 24 = 24 > 0$

   Thus, the conic section is hyperbola.

3. $3x^2 + 3y^2 + 18x - 16y + 31 = 0$

   $A = 3, B = 0, C = 3$

   $B^2 - 4ac = 0^2 - 4(3)(3) = 0 - 36 = -36 < 0$

   Note that $B = 0$ and $A = C$. Thus, the conic section is a circle.

4. $4x^2 + 4xy - 4x + y^2 - 12y = 0$

   $A = 4, B = 4, C = 1$

   $B^2 - 4ac = 4^2 - 4(4)(1) = 16 - 16 = 0$

   The conic section is a parabola.

Definition and Equation of a Circle

A **circle** may also be considered a special kind of ellipse (for the special case when the tilted plane is horizontal). As we get to know more about a circle, we will also be able to distinguish more between these two conics.

See Figure 1.7, with the point $C(3,1)$shown. From the figure, the distance of $A(-2,1)$ from C is AC = 5. By the distance formula, the distance of $B(6,5)$ from C is BC = $\sqrt{(6-3)^2 + (5-1)^2} = 5$. There are other points P such that PC = 5. The collection of all such points which are 5 units away from C, forms a circle.

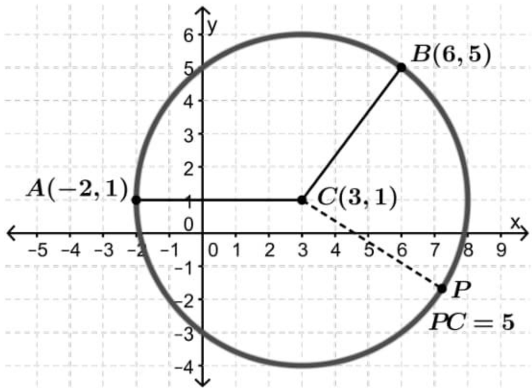


Figure 1.7

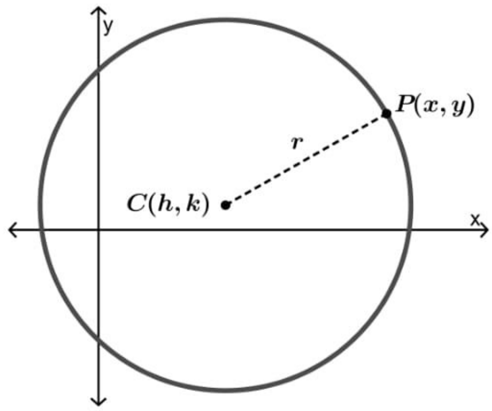


Figure 1.8

Let C be a given point. The set of all points P having the same distance from C is called a ***circle***. The point C is called the ***center*** of the circle, and the common distance its ***radius.***

The term radius is both used to refer to a segment from the center C to a point P on the circle, and the length of this segment.

See Figure 1.8, where a circle is drawn. It has center C(h, k) and radius r > 0. A point P(x, y) is on the circle if and only if PC = r. For any such point then, its coordinates should satisfy the following.

$$PC = r$$

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

$$(x-h)^2 + (y-k)^2 = r^2$$

This is the standard equation of the circle with center C(h, k) and radius r. If the center is the origin, then h = 0 and k = 0. The **standard equation** is then $x^2 + y^2 = r^2$.

Example 1.2. Graph the circle $x^2 + y^2 = 9$.

The given equation is in standard form with center at the origin C(0,0) and radius. We can rewrite the equation into this form $x^2 + y^2 = 3^2$.following the standard equation $x^2 + y^2 = r^2$. Thus, r=4. To be able to graph the circle, we take all the points that are 4 units from the center (0,0) to all directions along the plane. See Figure 1.9 below.

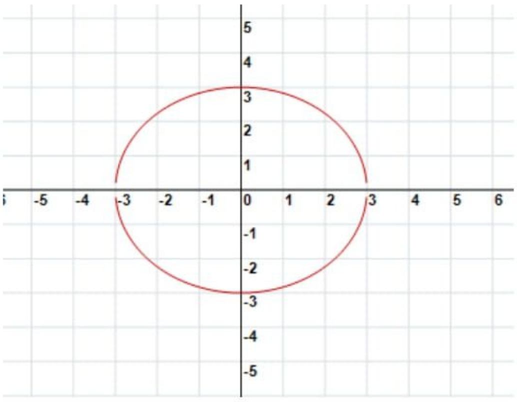


Figure 1.9

Example1.2. In each item, give the standard equation of the circle satisfying the given conditions.

(1) center at the origin, radius 4

(2) center (−4, 3), radius $\sqrt{7}$

(3) circle in Figure 1.7

(4) circle A in Figure 1.10

(5) circle B in Figure 1.10

(6) center (5,–6), tangent to the y-axis

(7) center (5, −6), tangent to the x-axis

(8) It has a diameter with endpoints A(−1, 4) and B(4, 2).

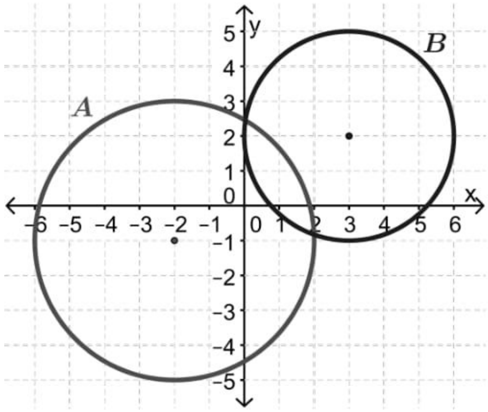


Figure 1.10

Solution:

(1) Since the center of the circle is the origin, then h=0 and k=0, the standard equation of the circle given radius (r=4) is $x^2 + y^2 = 16$.

(2) Since the center of the circle is not the origin, we will use the standard equation $(x-h)^2 + (y-k)^2 = r^2$

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x-(-4))^2 + (y-3) = (\sqrt{7})^2$$

$$(x+4)^2 + (y-3)^2 = 7$$

(3) The center is (3, 1) and the radius is 5

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x-3)^2 + (y-1) = (5)^2$$

$$(x-3)^2 + (y-1)^2 = 25$$

(4) By inspection, the center is (-2, -1) and the radius is 4. The equation is $(x + 2)^2 + (y + 1)^2 = 16$.

(5) Similarly by inspection, we have $(x - 3)^2 + (y - 2)^2 = 9$.

(6) The center is 5 units away from the y-axis, so the radius is r= 5 (you can make a sketch to see why). The equation is

$(x-5)^2 + (y+6)^2 = 25$.

(7) Similarly, since the center is 6 units away from the x-axis, the equation is $(x-5)^2 + (y+6)^2 = 36$.

(8) The center C is the midpoint of A and B: C= $\frac{-1+4}{2}, \frac{4+2}{2} = (\frac{3}{2}, 3)$. The radius is then r=AC= $\sqrt{(-1-\frac{3}{2}) + (4-3)^2} = \sqrt{\frac{29}{4}}$. The circle has an equation $(x-\frac{3}{2})^2 + (y-3)^2 = \frac{29}{4}$.

After expanding the standard equation, say for example the standard form in example 1.8, $(x-\frac{3}{2})^2 + (y-3)^2 = \frac{29}{4}$, can be written as $x^2 + y^2 - 3x - 6y + 4 = 0$, an equation of the circle in **general form.** If the equation of a circle is given in the general form

$$Ax^2 + By^2 + Cx + Dy + E = 0, \qquad A \neq 0,$$

$$x^2 + y^2 + Cx + Dy + E = 0,$$

we can determine the standard form by completing the square in both variables. Steps below show the knowledge we had in our previous activity about completing the square.

$$(x-\frac{3}{2})^2 + (y-3)^2 = \frac{29}{4}$$

$$-3x + \frac{9}{4} + Y^2 - 6y + 9 = \frac{29}{4}$$

$$x^2 + Y^2 - 3x - 6y = \frac{29}{4} - \frac{9}{4} - 9$$

$$x^2 + Y^2 - 3x - 6y = -4$$

$$x^2 + Y^2 - 3x - 6y + 4 = 0$$

In completing the square like the expression $(x^2 + 14x)$, means determining the term to be added that will produce a perfect polynomial square. Since the coefficient of x^2 is already 1, we take half the coefficient of x and square it, and we get 49. Indeed, $x^2 + 14x + 49 = (x+7)^2$ is a perfect square. To complete the square in, say, $3x^2 + 18x$, we factor the coefficient of x^2 from the expression: $3(x^2 + 6x)$, then add 9 inside. When completing a square in an equation, any extra term introduced on one side should also be added to the other side.

Example 1.3. Identify the center and radius of the circle with the given equation in each item. Sketch its graph, and indicate the center.

(1) $x^2 - 6x + y^2$=7

(2) $x^2 + y^2 - 14x + 2y = -14$

(3) $16x^2 + 16y^2 + 96x - 40y = 315$

Solution. The first step is to rewrite each equation in standard form by completing the square in x and in y. From the standard equation, we can determine the center and radius.

(1)

$x^2 - 6x + y^2 = 7$ (Given)

$x^2 - 6x + 9 + y^2 = 7 + 9$ (Adding 9 both sides)

$x^2 - 6x + 9 + y^2 = 16$ (Simplify right side of the equation)

$(x-3)^2 + y^2 = 16$ (Factoring into perfect square binomial)

$(x-3)^2 + y^2 = 4^2$

Center (3,0), r=4, see Figure 1.11

(2)

$x^2 + y^2 - 14x + 2y = -14$ (Given)

$x^2 - 14x + y^2 + 2y = -14$ (Rearrange by terms)

$x^2 - 14x+49 + y^2 + 2y + 1 = -14 + 49 + 1$ (Adding 49 & 1 both sides)

$x^2 - 14x+49 + y^2 + 2y + 1 = 36$ (Simplify right side of the equation)

$(x-7)^2 + (y+1)^2 = 36$ (Factoring into perfect square binomial)

$(x-7)^2 + (y+1)^2 = 6^2$

Center (7,-1), r=6, see Figure 1.12

(3)

$16x^2 + 16y^2 + 96x - 40y = 315$ (Given)

$16x^2 + 96x + 16y^2 - 40y = 315$ (Rearrange by terms)

$16(x^2 + 6x) + 16(y^2 - \frac{5}{2}y) = 315$ Applying common monomial factor

$$16(x^2+6x+9)+16\left(y^2-\frac{5}{2}y+\frac{25}{16}\right)=315+16(9)+16(\frac{25}{16})$$ (Adding $16(9)+16(\frac{25}{16})$ both sides)

$$16(x+3)^2+16(y-\frac{5}{4})^2=484$$ (Simplify & factoring into perfect square binomial)

$$\frac{1}{16}\cdot\left\{16(x+3)^2+16(y-\frac{5}{4})^2\right\}=484\cdot\frac{1}{16}$$ (Multiplying $\frac{1}{16}$ both sides)

$$(x+3)^2+(y-\frac{5}{4})^2=\frac{121}{4}$$ (Simplify)

$$(x+3)^2+(y-\frac{5}{4})^2=\left(\frac{11}{2}\right)^2$$

Center (-3, $\frac{5}{4}$), r=$\frac{11}{2}$ or 5.5, see Fig. 1.13

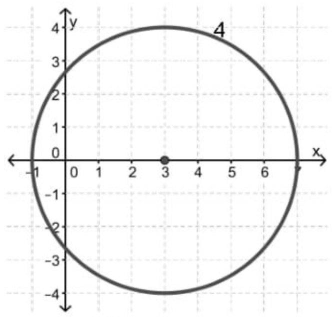


Figure 1.11

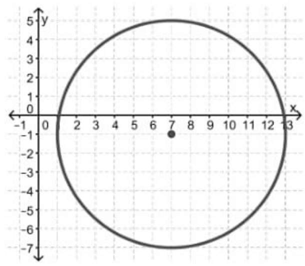


Figure 1.12

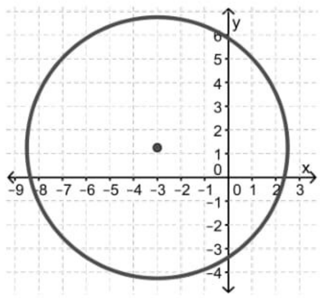
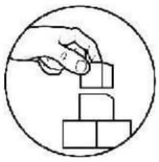


Figure 1.13

In the standard equation $(x-h)^2+(y-k)^2=r^2$ both the two squared terms on the left side have coefficient 1. This is the reason why the preceding example, we multiplied $\frac{1}{16}$ at the last equation.



What's More

Activity 1.1: Let Me try!

Let us find out if you really understood the discussed concept by answering these follow-up exercises.

1. Determine the type of conic section that each general equation will produce. Show your solution.

a. $x^2+y^2-2x-4y+1=0$	b. $x^2+y^2+8x-4y-2=0$
c. $5x^2+5y^2-9x-14y+26=0$	d. $9x^2+16y^2-54x+4xy-64y+1=0$

2. In each item, give the standard equation of the circle satisfying the given conditions.

a. center at the origin, contains (0, 3)	b. center (1, 5), diameter 8
c. circle A in Figure 1.10	d. circle B in Figure 1.10
e. circle C in Figure 1.10	f. center (-2,-3), tangent to the y- axis
g. center (-2,-3), tangent to the x-axis	h. contains the points (-2,0) and (8, 0), radius 5.

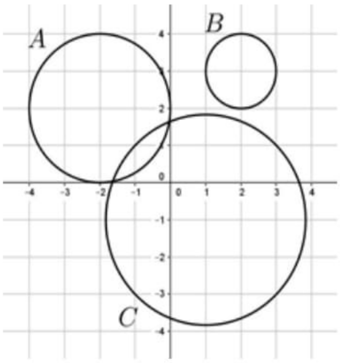


Figure1.10

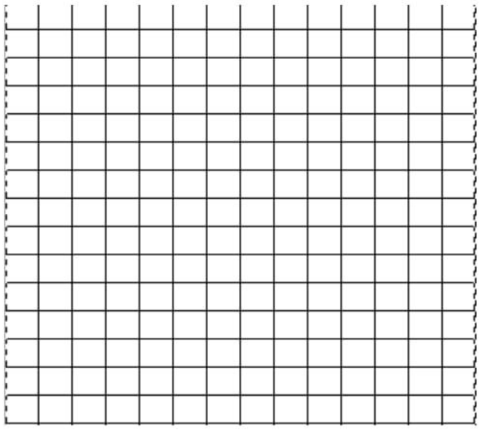
3. Identify the center and radius of the circle with the given equation in each item. Sketch its graph, and indicate the center.

(a) $x^2 + y^2 + 8y = 33$
b.$4x^2 + 4y^2 - 16x + 40y + 67 = 0$
c. $4x^2 + 12x + 4y^2 + 16y - 11 = 0$

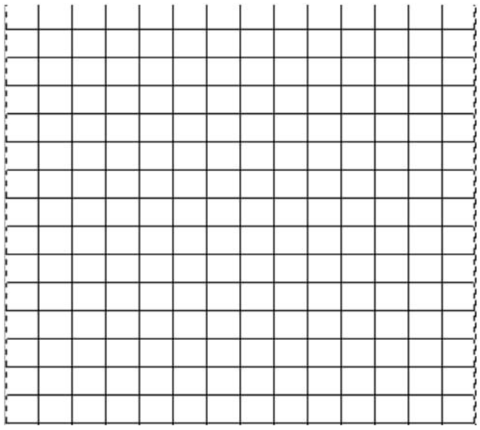
d. $x^2 + y^2 - 6x + 4y + 4 = 0$

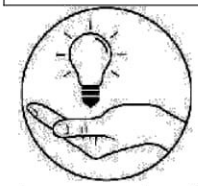
e. $x^2 + y^2 - 4x - 8y + 20 = 0$

f. What is the equation of a circle having a diameter with endpoints at A(4,5) and B (-2,3). Sketch the graph.



g. What is the general equation of a circle whose center is at M(5,-3) and whose radius is 4 units. Sketch the graph.





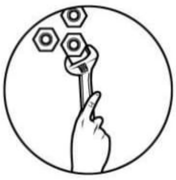
What I Have Learned

Let me check your knowledge by filling the blanks with a correct symbols/letter or terms in order to complete the statement/s.

There are four types of conic sections. When the plane is perpendicular to the axis of the cone and intersects each generator, a/an (1).__________is formed. However, when the plane is tilted slightly so that it intersects each generator, but only intersects one nappe of the cone, a/an (2)___________ is formed. When the plane is tilted further so that it is parallel to one and only one generator and intersects only one nappe of the cone, a /an(3) ____________ is formed. A hyperbola is generated when plane intersects both nappes.

Given a general equation of the conic sections, we can determine what type of conics by collecting the values (4)___, (5)___ and (6)___. Then solve the value of $B^2 - 4ac$ and interpret the result based on the table of the graphs of quadratic equations.

The first type of conic section is circle. It is defined as a set of all points in a plane equidistant from a fixed point called (7)_______ of the circle and the constant equal distance is called (8)________. The standard form of the equation of a circle is $((x-h)^2 + (y-k)^2 = r^2$ with the center: (9)_____, and radius: (10)____. However, when the circle has a center at origin: C(0,0), the standard equation would be (11)__________. This equation of the circle $Ax^2 + By^2 + Cx + Dy + E = 0$, is called (12)_____________________. This equation can be converted into standard form using completing of (13) _______. To graph the equation of the circle into the coordinate plane, use the center represented by (14) _____. After locating the center, use the value of the (15)_______ to move in all directions and then connect the dots to form a circle.



What I Can Do

Performance Task: Let's do this!

Materials: Grid paper, Philippine new coins (1, 5 and 10 peso coins), ruler, and pen.

Procedures:

A. Center (0,0)

1. Use the grid paper below and draw 3 sets of Cartesian coordinate plane. Use 1 unit in labelling the x- and y- axes.
2. Locate the center (0,0) of each 3 sets of Cartesian plane by putting a visible dots.
3. Place each coin (designated set) at the center or on the dot in each Cartesian plane. Using pen, draw a circle by tracing the circumference (edge) of the each coin.
4. After you draw, remove the coins. From the center (dot), draw a line to any point of the circle.
5. Get your ruler, measure in centimeters the line (radius) you created in each Cartesian plane and record the values.
6. Solve the equation of each circle using the obtained value of the radius.
7. Compare the equation obtained and make an observation note of the activity.

Set A (1 peso coin)

Set B (5 peso coin)

Set C (10 peso coin)

Observation Note:

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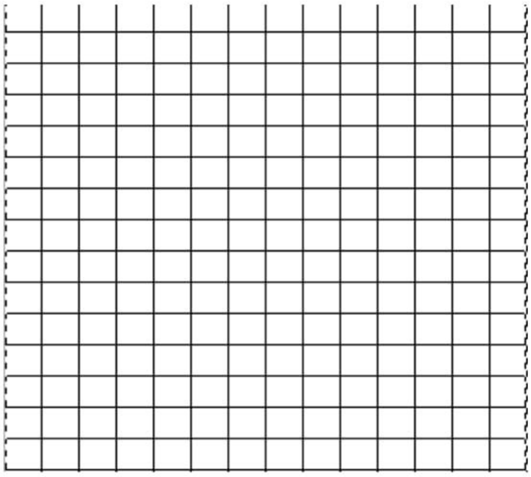
____.

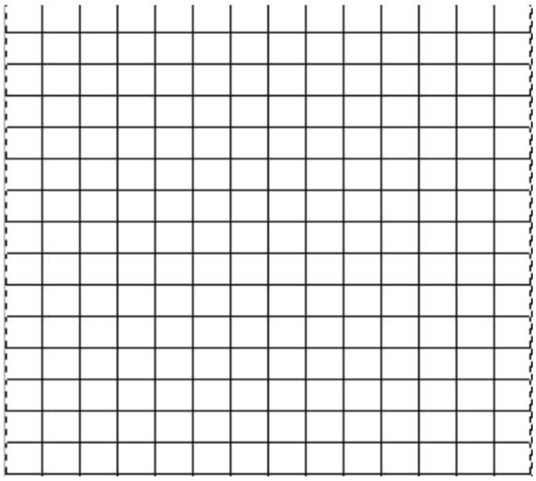
B. Center(h,k)

1. Use the grid paper below and draw 3 sets of Cartesian coordinate plane. Use 1 unit in labelling the x- and y- axes.
2. Locate the center (2,3) of each 3 sets of Cartesian plane by putting a visible dots.
3. Place each coin (designated set) at the center or on the dot in each Cartesian plane. Using pen, draw a circle by tracing the circumference (edge) of the each coin.
4. After you draw, remove the coins. From the center (dot), draw a line to any point of the circle.
5. Get your ruler, measure in centimeters the line (radius) you created in each Cartesian plane and record the values.
6. Solve the equation of each circle using the obtained value of the radius.
7. Compare the equation obtained and make an observation note of the activity.

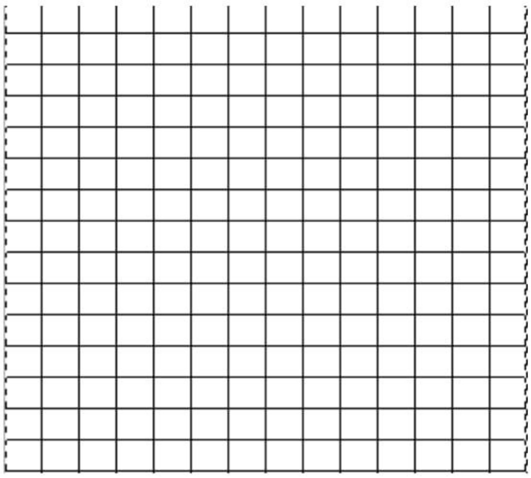
Set A (1 peso coin)



Set B (5 peso coin)



Set C (10 peso coin)



Observation Note:

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Online connect! *For additional knowledge and information about the topics please visit the link indicated below.*

1.https://www.youtube.com/watch?v=HO2zAU3Eppo

2. https://www.youtube.com/watch?v=auD46ZWZxQo

3. https://www.youtube.com/watch?v=JUvo3GrgWHk

4. shorturl.at/bKU67