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PHY 307

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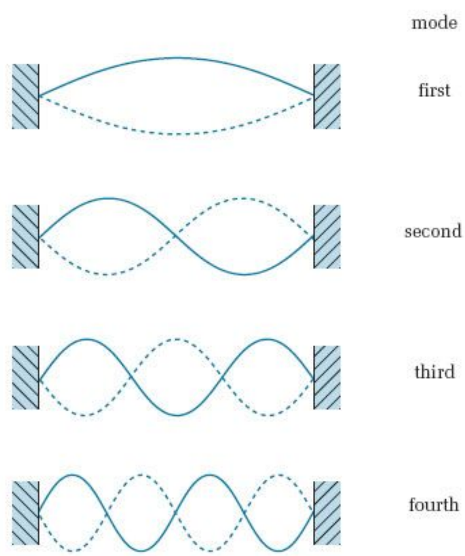
Project 7: Nonlinear Behavior in Vibrating Strings

Vibrating strings are very difficult to analyze in the real world. There are many factors that contribute to a vibrating string's frequency and the many interactions between all the particles in the string. Outside of very small amplitudes strings are essentially impossible to study analytically. The advent of computers allows very detailed approximations of vibrating strings, and this paper discusses one such method.

The premise of simulating a vibrating string in this project is to 'stretch' a series of nodes that are connected by springs to other nodes and holding the endpoints still while giving the other nodes some initial conditions (either some velocity or position). To ensure our simulation behaves accurately, we need to prescribe certain macroscopic parameters that we can measure in the real world and implement them into the simulation. For example, treating the string as one big spring, it's spring constant is given by α/L_0, where α is the stiffness of the string (which can be measured) and L_0 is the length of the unstretched (i.e. zero tension) string. The spring constant of each little spring connecting the nodes is given by $N*(\alpha/L_0) = N*k_{string}$, where N is the number of nodes in the string. The other parameters needed for the string are the linear mass density and the tension that are applied to the string. These can be used to solve for the mass of each node ($m_{node} = \mu*L_0/N$) and the stretched length of the spring using Hooke's Law ($L = k_{string}*T + L_0$).

With these tools, construction swiftly follows. Prescribing all the data, a string made of N nodes can be created with a spring connecting each node to the nodes adjacent to it. The force each node experiences can be determined using Hooke's Law. All that is left is to give the stretched system initial conditions and to analyze it's behavior. Two initial conditions are used in this paper, the first of which is a sine wave that can simulate any number of 'modes'. A mode contains two nodes that do not move as the string vibrates. The number of the mode describes how many of these regions there are.



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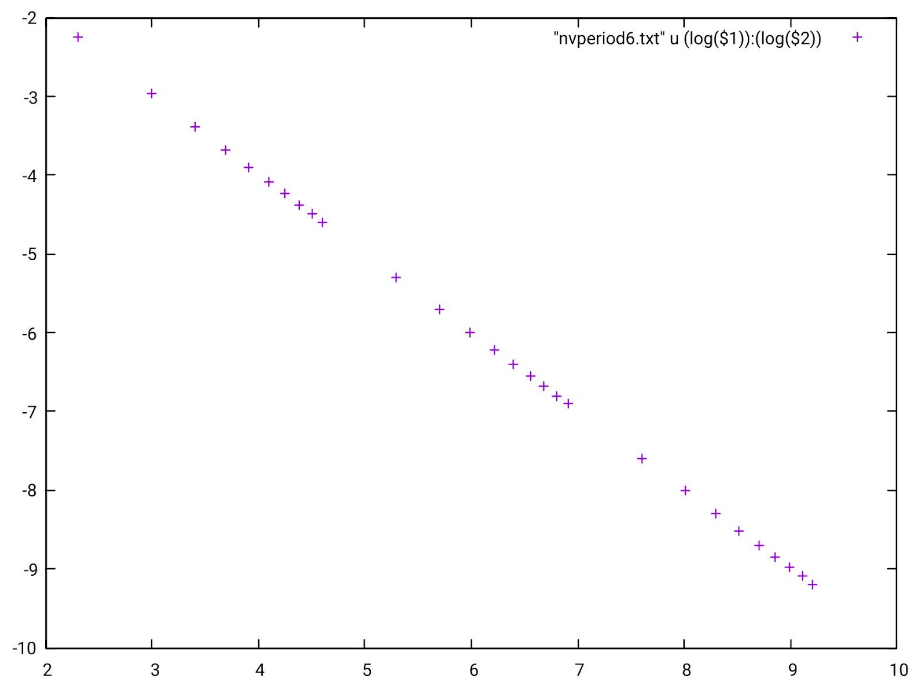
The other initial condition will be discussed later. In this paper, n will represent the mode.

There are a few tests that can be run to ensure the simulation is behaving realistically. Analytically, in the limit of a small amplitude the period of each of these modes is given by $\tau = \lambda_n/v$, where λ_n is the wavelength of the nth normal mode and v is the velocity of the wave, which is also determined analytically and will not be covered in this paper. This can be rewritten as (2L/n)sqrt(μ/T). Using a n of 1, an α of 10 N/m, a tension of 5 N, a L_0 of 1 m, and a μ of 1 kg/m the analytic period is 1.095445s. In our simulation, we can track one antinode and determine when it's velocity changes sign. This time gives one half the period. Experimenting with a range of N varying from 10 to 10,000 and constructing a log-log graph of the fractional difference in period ($\Delta\tau = |\tau_{an}-\tau_{num}|/\tau_{num}$) vs N, we get a linear graph with a slope of -1:

Plot of ln(Δτ) vs ln(N)



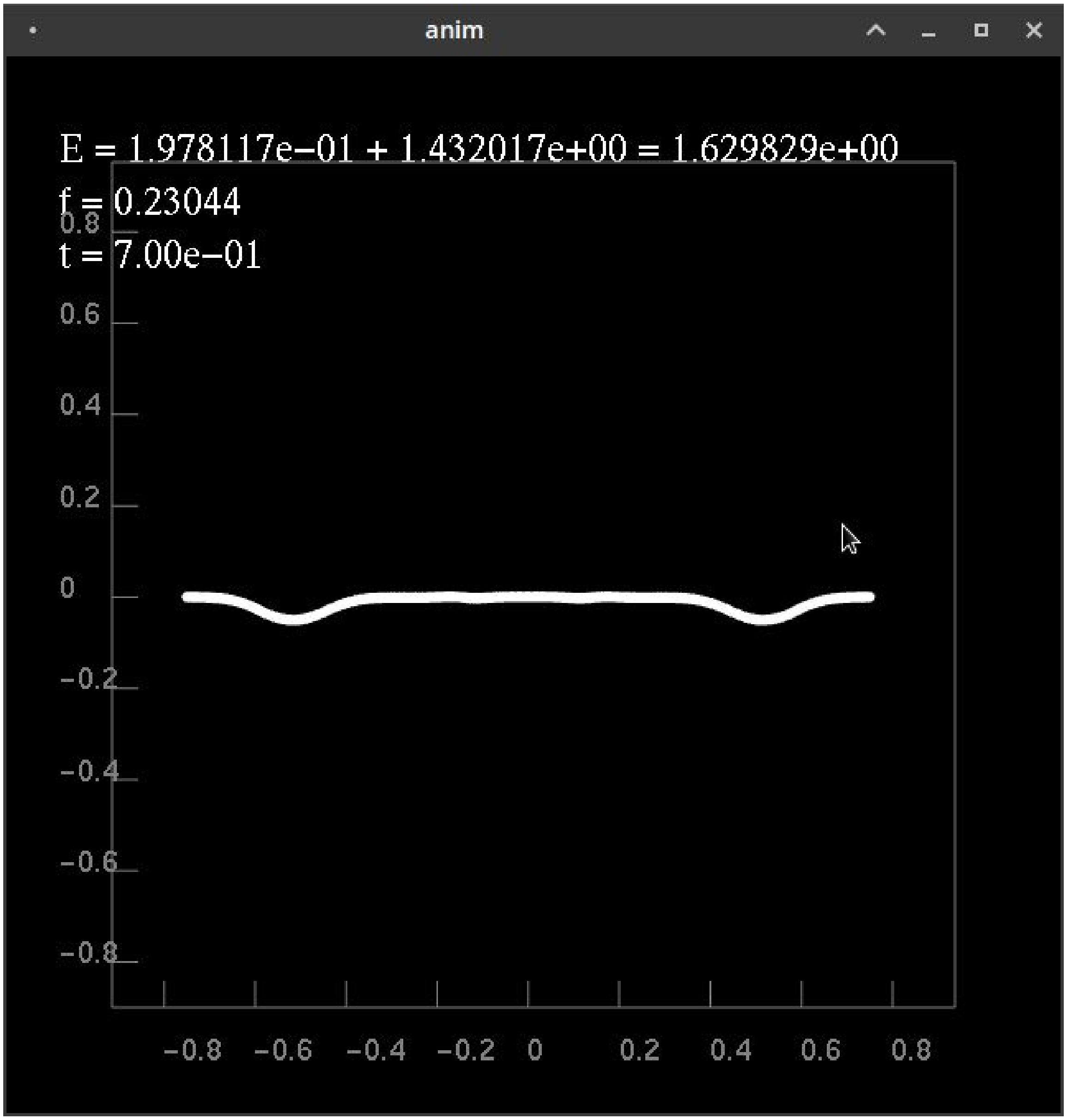
This tells two things; one is that the string behaves more realistically as the number of nodes increases (which makes conceptual sense) and that our method for simulating a string is accurate only to the first-order, i.e. the error in our simulation is proportional to 1/N, rather than $1/N^2$ or something greater. While this does mean we have to be conscious of computational error in our studies, with 10,000 nodes our fractional difference was negligible (0.000101 with a dt of 10^{-7}s). This simulation uses the leapfrog algorithm which is accurate to the second order however, allowing us to use a lower time step with similar results. The fractional difference with a dt of

10^{-6}s was 0.000114, close enough to that with a dt of 10^{-7}s. For the rest of the paper a timestep of 10^{-6}s and an N of 10,000 nodes is used.

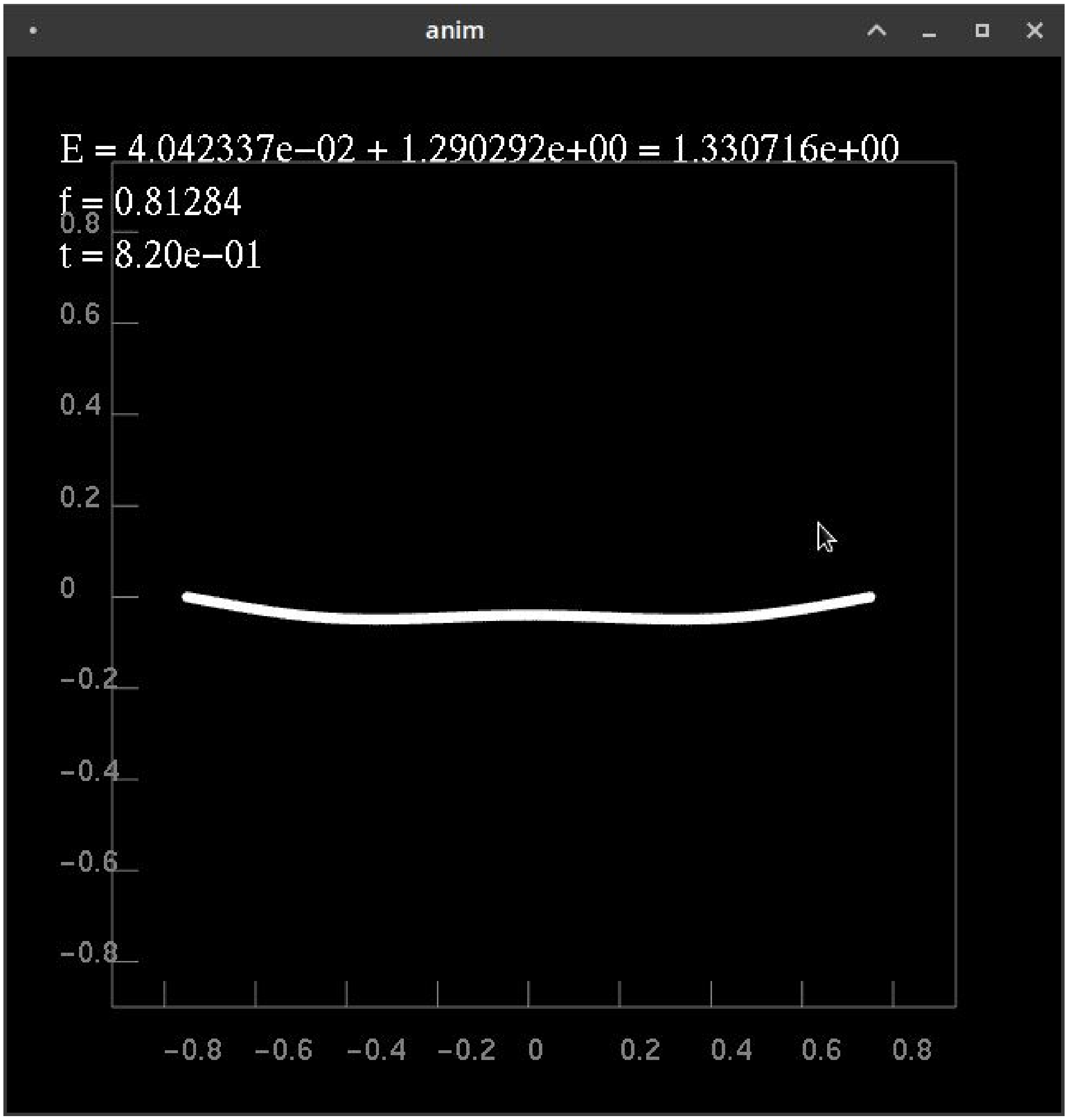
Another check to ensure accuracy is to change the parameters and ensure the program behaves appropriately. Following the process described above and adjusting the unstretched length, tension, and linear mass density individually, the fractional difference in period was never more than 0.000120, indicating our system is behaving as intended.

The other initial condition is known as the 'Gaussian bump' and is given by $A\exp((x_i-x_c)^2/\sigma^2)$ where A is the amplitude, x_i is the x position of the node, x_c is the center of the bump, and σ describes the width of the bump. This bump can be used to simulate a wide variety of real-world interactions with a string, like a finger or guitar pick plucking a string. Animating the string allows us to visualize what goes on in a string under these different bumps.

These bumps can be classified into two different categories; small bumps and large bumps. This is not in reference to the amplitude, but instead describes the width of the bump. Large bumps can be seen affecting the entire string all at once while smaller bumps appear to affect only small portions at a time. Acoustically, large bumps create darker tones while small bumps create brighter tones. This can be tested yourself if you have a guitar or other string instrument lying around. Pluck the guitar string with your finger, and then pluck it with a guitar pick. Your finger, which applies the force over a large area creating a larger bump, sounds darker and fuller than when the string is plucked with the pick, which applies the force to a very small area. This can also be visualized through animation of the string. Exciting the string with a small bump(s) creates many small wiggles throughout the string:



While larger bumps tend to only create distinct bumps throughout the string:

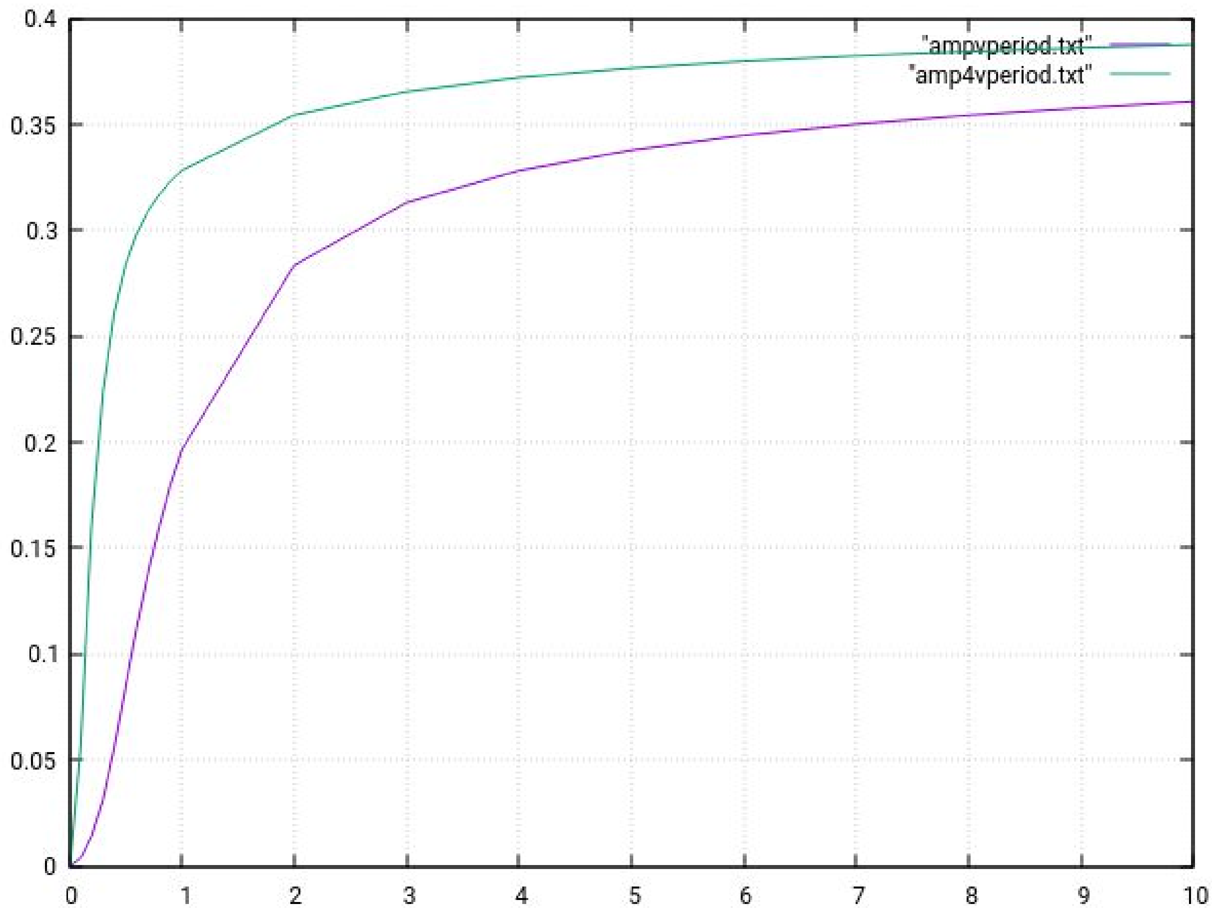


This follows what was said earlier; small bumps create little wiggles which have more energy in higher modes leading to a brighter sound than the large bumps which have more energy in the lower modes and as such have a deeper tone, which we observed with the guitar string.

Now we begin to go beyond the limit of a small amplitude and analyze the nonlinear behavior exhibited by vibrating strings at higher amplitudes; the stuff we can not analyze in the real world. At higher amplitudes, the different modes are visibly very distinct and separate from one another and vibrate with the same frequency. Using an amplitude ranging from 1e-12 up to

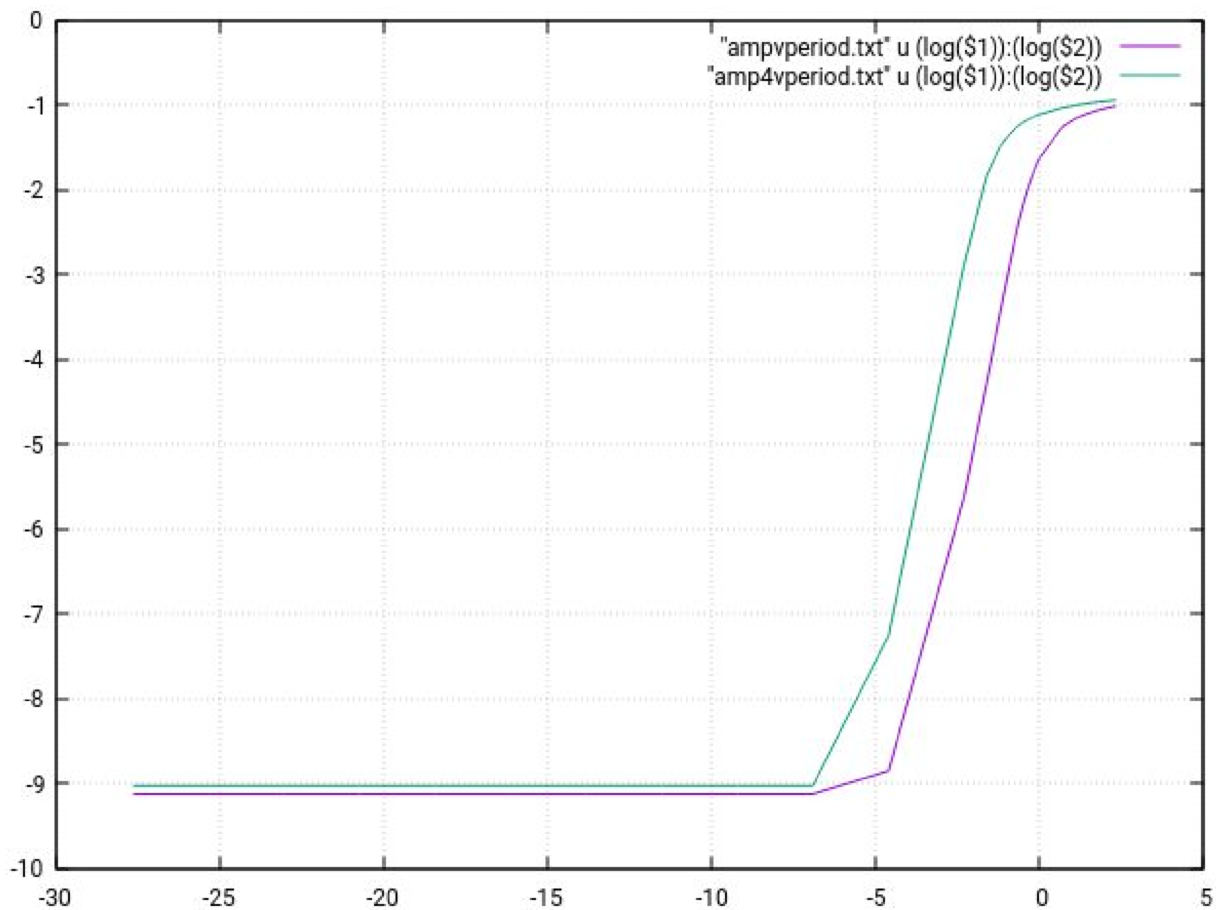
10 (L_0 = 1, L = 2) for a mode of 1 <u>and</u> 4 and then plotting Δτ vs amplitude produces the following:

Δτ vs amplitude



The log-log plot below gives more information as well:

ln(Δτ) vs ln(amplitude)

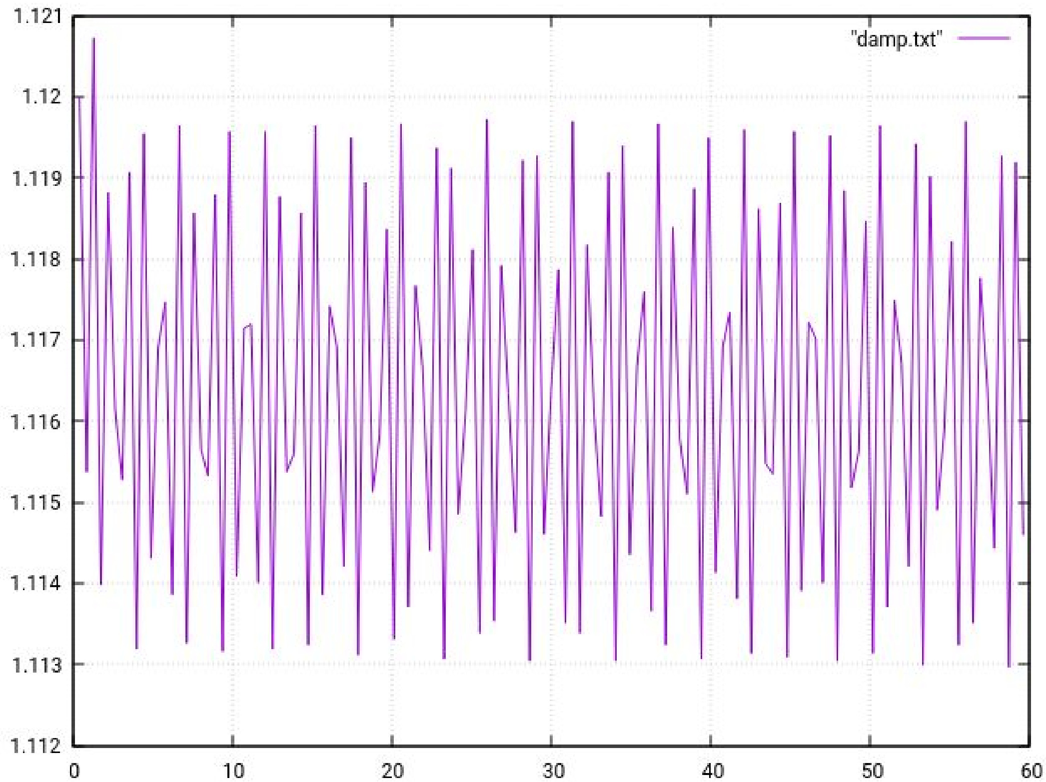


For very small amplitudes the period of the string hardly changes at all, until the amplitude is roughly equal to 0.01, then Δτ begins to increase rapidly, and then proceed to stabilize as the amplitude continues to increase. Interestingly, the steep portion of the log-log graph appears to have a slope of 2 regardless of the amplitude, meaning the period depends on the square of the amplitude for a certain range of amplitudes. Outside of that range the period begins to converge to a value that depends on which side of the range you are out of. The steep portion started sooner with the higher mode however, and the Δτ was always higher even as the periods began to converge. The higher mode was more subject to the effects of the higher amplitude.

The analysis of this string is somewhat comparable to the analysis of a pendulum. Periods of both objects can be approximated using the small angle approximation and they exhibit nonlinear behaviour outside of that limit. For example, the pendulum's period increased at a constant rate with the amplitude of its swing and began to grow rapidly after reaching a certain point. Likewise in this project, the string's period is comparable to the analytic solution for small amplitudes, but instead of increasing at a constant rate before growing rapidly, it grows rapidly to a constant rate and then begins to stabilize. Physical phenomena are often unpredictable, and in a lot of cases impossible to study without computers.

Continuing further beyond data analysis, simulating the string can answer some real world questions. For example, does the decay of a string's amplitude affect the overall sound of an instrument? To answer this, a damping force is added to the simulation and we then plot frequency vs time as the vibration decays to see if there is any obvious correlation. Letting the simulation run for 60 seconds to be sure the initial amplitude of 0.1 (same lengths as before) completely dies down. The following graph is produced:

Frequency (Hz) vs Time (s)



To be entirely honest I am unsure why the frequency bounces around so much. I used linear regression to deal with this.

To see if there is a trend in the data it was fitted against a linear line in the form f(x) = m*x + b. The fitted line had a slope of -7.19293e-6 with an RMS of 0.00239985, which is fairly small for the data as seen above. This means a linear fit accurately represents the data. To see the difference between the initial frequency and the final frequency, we compare the b value of the fit and the final value (i.e. m*60 + b). The initial frequency was 1.11659 s and the final frequency was 1.11616 s. The fractional difference between these two values is 0.04%. Considering a deviation of 1% is noticeable, this would not affect the sound produced by the instrument drastically, especially since the initial amplitude is much higher than would be seen in regular playing.

In summation, simulation of real world objects can lead to better understanding that is not possible by direct observation in the real world. Taking quantities that we can measure and implementing them into our simulation ensures realistic results while allowing for easy analysis. Simulation can also answer questions outside of the scope of analysis. In this case, a simulation of a vibrating string was used to study the nonlinear behaviour it represents outside the limit of a small amplitude, as well as to determine whether or not the decaying of the amplitude would affect music.