Deadlocking in Queueing Networks

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This section will define and discuss the properties and detection of deadlock in queueing networks. Throughout the section, when discussing queueing networks, it is assumed that the queueing network is open and connected. Open queueing networks are those networks that have at least one node to which customers arrive from the exterior, and and least one node from which customers leave to the exterior.

1 Introduction

Definition 1. *When a simulation is in a situation where at least one service station, despite having arrivals, ceases to finish any more services due to recursive upstream blocking the system is said to be in deadlock.*

Deadlock can be experienced in any open queueing network that experiences blocking, with at least once cycle containing all service stations with resticted queueing capacity.

Deadlock occurs when a customer finishes service at node i and is blocked from transitioning to node j; however the individuals in node j are all blocked, directly or indirectly, by the blocked individual in node i. That is, deadlock occurs if every individual blocking individual X, directly or indirectly, are also blocked.

In Figure 1 a simple two node queueing network is shown in a deadlocked state. Customer occupying server A_1 has finished service at node A, but remains there as there is not enough queueing space at node B to accept them. We say the customer at server A_1 is blocked by the customer at server B_1, as he is waiting for that customer to be released. Similarly, the customer occupying server B_1 has finished service at node B, but remains there as there is not enough queueing space at node A, and so the customer at server B_1 is blocked by the customer at server A_1.

When there are multiple servers, individuals become blocked by all customers in service at the destination service station. Figure 2a shows two nodes in deadlock, the customer occupying server A_1 is blocked by customers at both B_1 and B_2, who are both blocked by the customer at A_1. However in 2b, customer at A_1 is blocked by customers at both B_1 and B_2, but the customer at B_2 isn't blocked, and so there is no deadlock.

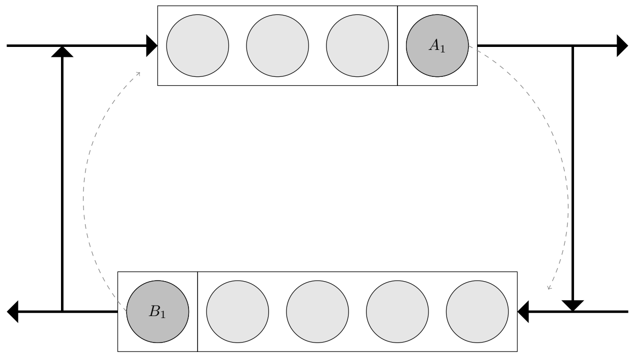
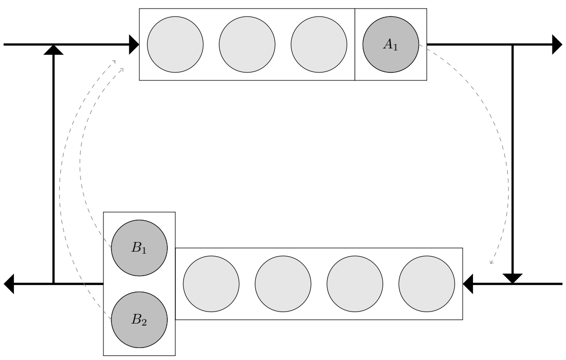
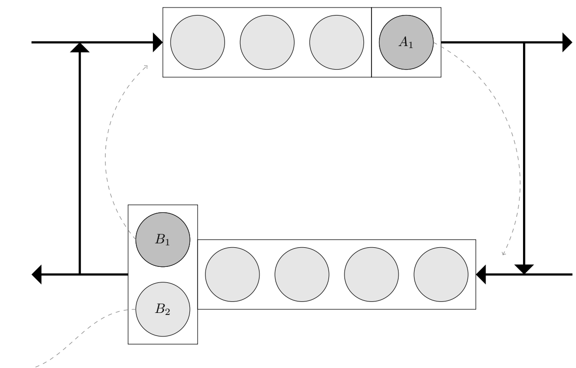


Figure 1: Two nodes in deadlock.



(a) Two nodes in deadlock.



(b) Two nodes not deadlocked

Figure 2: Two nodes: a) in deadlock and b) not in deadlock.

Note that the whole queueing network need not be deadlocked, only a part of it. If one section of the network is in deadlock, then the system is deadlocked, even though customers may still be able to have services and transitions in other areas of the network. An example is shown in Figure 3a. This idea is expanded on in Section 3.

2 Literature Review

Most of the literature on blocking either does not consider networks with feedback loops, ignores the possibility of deadlock, or conveniently assumes the networks are deadlock-free. For closed networks of K customers with only one class of customer, [3] proves the following condition to ensures no deadlock: for each minimum cycle C, $K < \sum_{j \in C} B_j$, the total number of customers cannot exceed the total queueing capacity of each minimum subcycle of the network. The paper also presents algorithms for finding the minimum queueing space required to ensure deadlock never occurs, for closed cactus networks, where no two cycles have more than one node in common. This result is extended to multiple classes of customer in [4], with more restrictions such as single servers and each class having the same service time distribution. Here a integer linear program is formulated to find the minimum queueing space assignment that prevents deadlock. The literature does not discuss deadlock properties in open restricted queueing networks.

General deadlock situations that are not specific to queueing networks are discussed in [2]. Conditions for this type of deadlock, also referred to as deadly embraces, to potentially occur are given:

- Mutual exclusion: Tasks have exclusive control over resources.
- Wait for: Tasks do not release resources while waiting for other resources.
- No preemption: Resources cannot be removed until they have been used to completion.
- Circular wait: A circular chain of tasks exists, where each task requests a resource from another task in the chain.

Dynamic state-graphs are defined, with resources as vertices and requests as edges. For scenarios where there is only one type of each resource, deadlock arises if and only if the state-graph contains a cylce.

In [1] the vertices and edges of the state graph are given labels in relation to a reference node. Using these labels *simple bounded circuits* are defined whose existance within the state graph is sufficient to detect deadlock.

3 Types of Deadlock

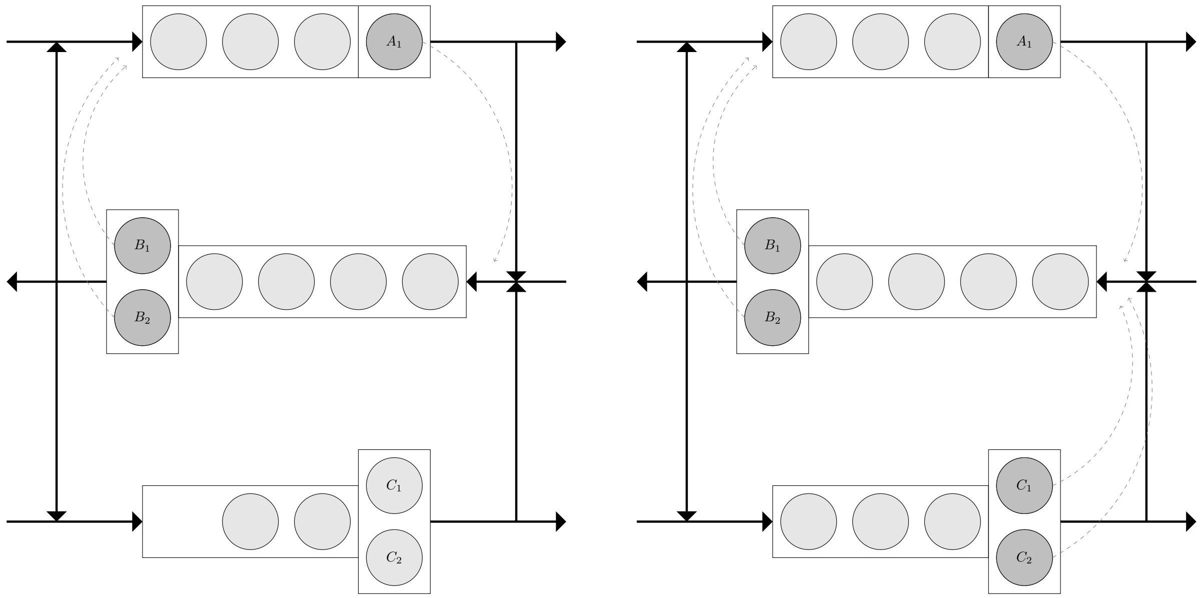
In a previous section an idea was introduced that parts of a queueing network can be in a deadlocked state, although other parts will continue to flow. The different configurations of which nodes experience deadlock can be thought of as different types of deadlock. The amount of different types of deadlock that a queueing network can experience is equal to the number of directed cycles in in the queueing network's routing matrix.

For connected queueing networks, these deadlocks can be classified into transient deadlocked states and the absorbing deadlocked state.

Definition 2. *A transient deadlock state is when there are still some changes of state whilst a subgraph of the queueing network is itself in deadlock.*

Definition 3. *The absorbing deadlock state is when all subgraphs of the queueing network are in deadlock.*

Figure 3 shows a three nodes network in a transient deadlocked state, and an absorbing deadlocked state. In Figure 3a the occupants of servers B_1 and B_2 are blocked from entering node A; and the occipant of server A_1 is blocked from entering node B, and so these two nodes are in deadlock. However, node C can continue with regular services, until the occupants of every server of C attempt to join a deadlocked node. At which point, the whole system is deadlocked, and so has reached absorbing deadlock, show in Figure 3b.



(a) Transient deadlock.

(b) Absorbing deadlock.

Figure 3: Types of deadlock: a) transient and b) absorbing.

For a queueing network Q with N service stations, the absorbing deadlocked state corresponds to $(i_1 = -1, ..., i_N = -1)$, the state where all service stations experience deadlock. It should be clear that if the queueing network is connected, then there is a non-zero probability that once one part of the network is in deadlock, the whole system will fall into a deadlocked state, simply by the individuals in the non-deadlocked nodes attempting to transition into a deadlocked node. That is, once Q falls into one of the transient deadlocked states, it will eventually transition, either directly or through other transient deadlocked states, into the absorbing deadlocked state.

If the routing matrix of Q is complete, that is there is a possible route from every service station to every other service station, then there are $\sum_{i=1}^{N} \binom{N}{i}$ possible deadlock types.

4 Dynamically Detecting Deadlock

In the following subsections, a method will be presented to dynamically detect deadlock in discrete event simulations of queueing networks. The following definitions will be required in this section:

$\lvert V(D)\rvert$	The order of the directed graph D is its number of vertices.
Weakly connected component	A weakly connected component of a digraph containing X is the set of all nodes that can be reached from X if we ignore the direction of the edges.
Direct successor	If a directed graph contains an edge from X_i to X_j , then we say that X_j is a direct successor of X_i.
Ancestors	If a directed graph contains a path from X_i to X_j , then we say that X_i is an ancestor of X_j.
Descendants	If a directed graph contains a path from X_i to X_j , then we say that X_j is a descendant of X_i.
$\deg^{\text{out}}(X)$	The out-degree of X is the number of outgoing edges emanating from that vertex.
Subgraph	A subgraph H of a graph G is a graph whose vertices are a subset of the vertex set of G, and whose edges are a subset of the edge set of G.
Sink vertex	A sink vertex is a vertex in a directed graph that has out-degree of zero.
Knot	In a directed graph, a knot is a set of vertices with out-edges such that while traversing the directed edges of that directed graph, once a vertex in the knot is reached, you cannot rech any vertex that is not in the knot.

4.1 State Digraph

Presented is a method of detecting when deadlock occurs in an open queueing network Q with N nodes, using a dynamic directed graph, the state graph.

Let the number of servers in node i be denoted by c_i. Define $D(t) = (V(t), E(t))$ as the state graph of Q at time t.

The vertices at time t, $V(t)$ correspond to servers in the queueing system. Thus, $|V(D(t))| = \sum_{i=1}^{N} c_i$ for all $t \geq 0$.

The edges at time t, $E(t)$ correspond to a blocking relationship. There is a directed edge at time t from vertex $X_a \in V(t)$ to vertex $X_b \in V(t)$ if and only if an individual occupying the server corresponding to vertex X_a is being blocked by an individual occupying the server corresponding to vertex X_b.

The state graph $D(t)$ can be partitioned into N service-station subgraphs, $D(t) = \bigcup_{i=1}^{N} d_i(t)$, where the vertices of $d_i(t)$ represent the servers of node i. The vertex set of each subgraph is static over time, however their edge sets may change.

The state graph is dynamically built up as follows. When an individual finishes service at node i, and this individual's next destination is node j, but there is not enough queueing capacity for j to accept that individual, then that individual remains at node i and becomes blocked. At this point c_j directed edges between this individual's server and the vertices of $d_j(t)$ are created in $D(t)$.

When an individual is released and another customer who wasn't blocked occupies their server, that servers out-edges are removed. When an individual is released and another customer who was previously blocked occupies their server, that server's out-edges are removed along with the in-edge from the server who that previously blocked customer occupied. When an individual is released and there isn't another customer to occupy that server, then all edges incident to that server are removed.

This general process of building up the state graph as the queueing network is simulated will now be shown. Customers are labelled (i, j, k) where i denotes the server that customer is occupying, j denotes that individuals i.d. number, and k denotes the service station that customer is waiting to enter. As an example, a customer labelled $(A_2, 10, C)$ would have an i.d. number of 2, is occupying server A_2 and is currently waiting to join node C. If a customer isn't occupying a server the notation $\emptyset$ is used. Similarly for customers occupying a server and still in service, their next destination is yet undecided, so $\emptyset$ is used.

The simulation starts with full queues, and every server occupied by a customer in service. This is shown in Figure 4.

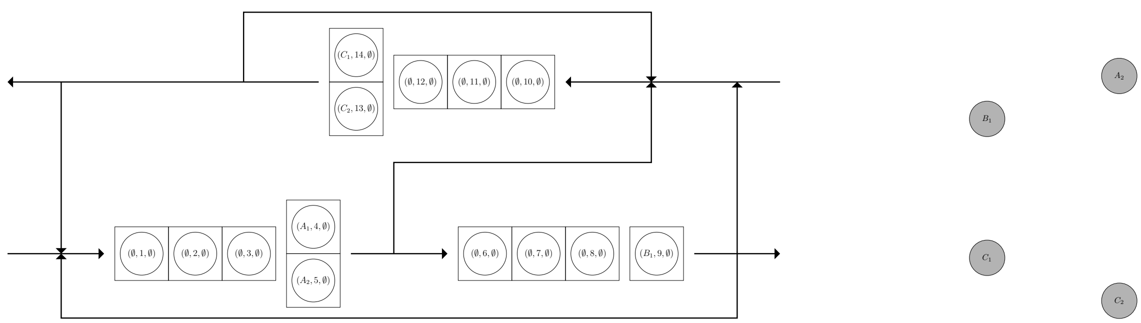


Figure 4

Customer 13 finishes service, and is blocked from entering node A. Figure 5.

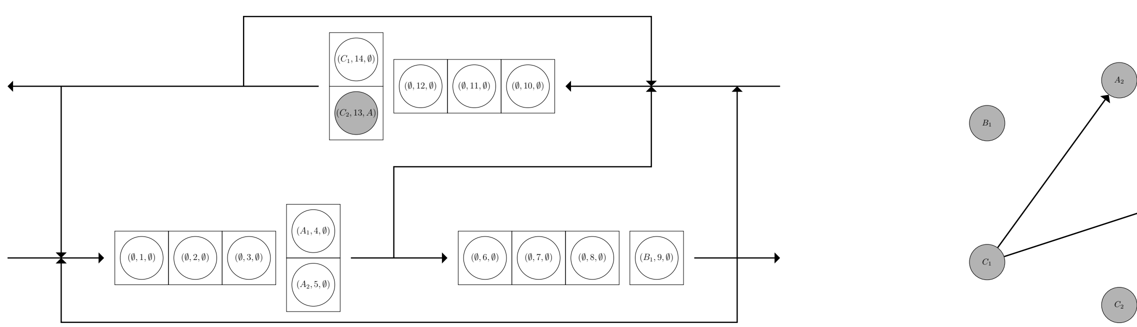


Figure 5

Then customer 4 finishes service and is blocked from entering node B. Figure 6.

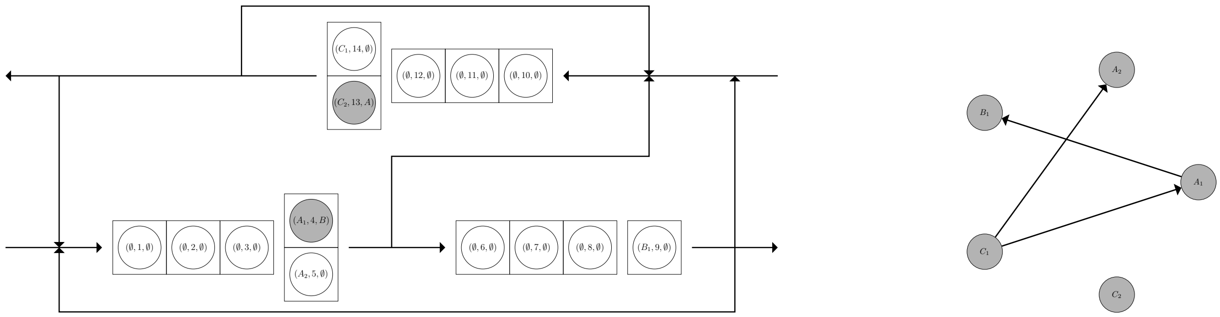


Figure 6

Then customer 9 finishes service and is blocked from entering node A. Figure 7.

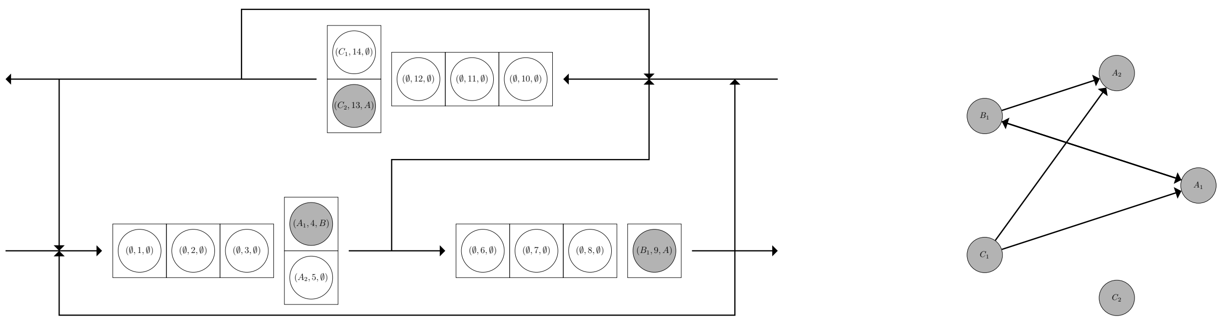


Figure 7

Finally, in Figure 8 customer 5 finishes service and wants to reenter the queue for node A but is blocked. A deadlock situation arises as customer 5 is waiting for customer 4 to move, who is waiting for customer 9 to move, who is waiting for either customer 4 or 5 to move.

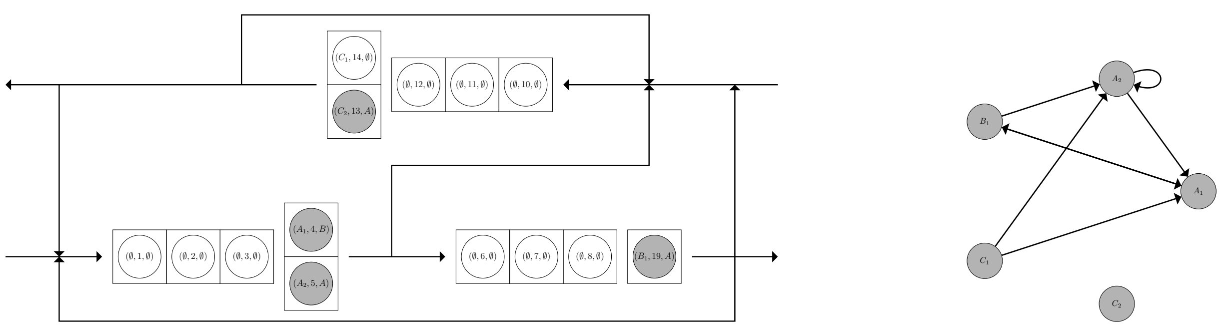


Figure 8

The rules on how edges are removed from the state graph will now be shown. For illustrative purposes the queueing network here is a different queueing network than discussed above.

Here the simulation begins with four customers occupying servers; those at node A blocked to node B, the customer at node C blocked to node A, and the customer at node B still in service. This is shown in Figure 9.

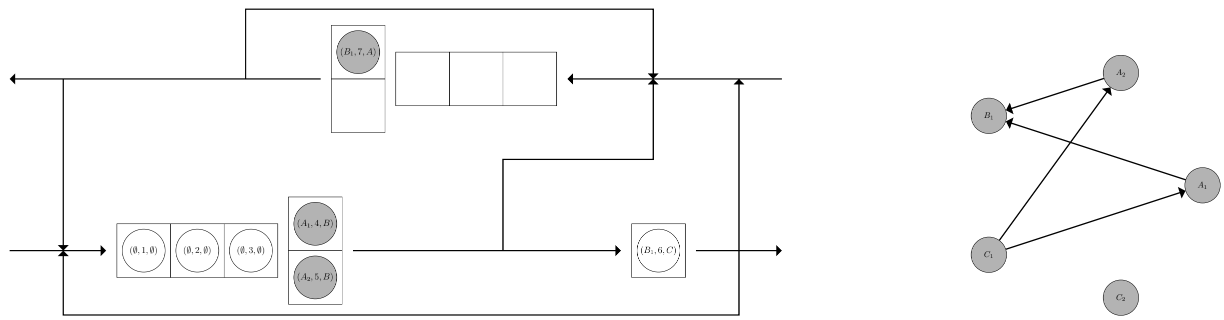


Figure 9

Customer 6 finishes service and immediately joins service at node C. Figure 10.

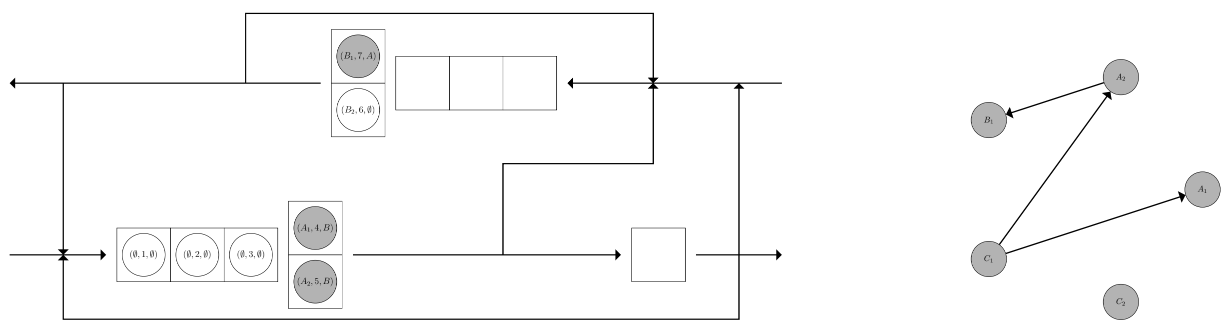


Figure 10

Now there is room for customer 4 to move into service at node B. Figure 11. Notice that the edge $A_2 \longrightarrow B_1$ remains in the state graph, as customer 5 is still blocked by that server.

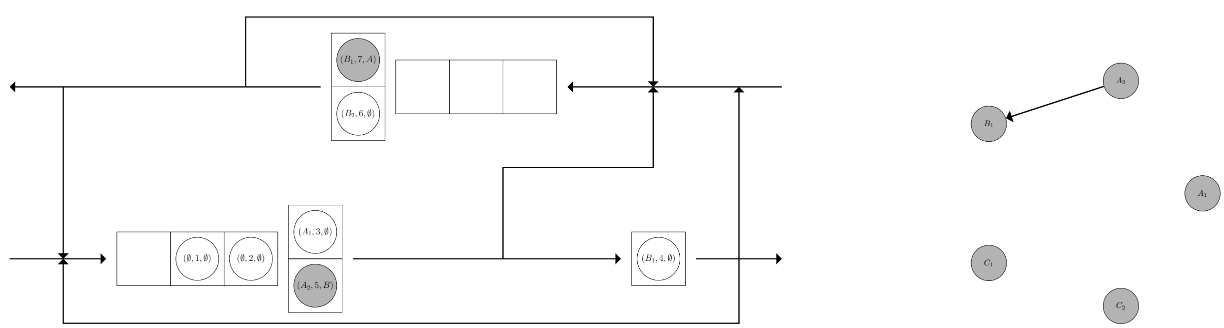


Figure 11

The customers queueing at node A move along the queue, with customer 3 begining service. This leaves enough room for customer 7 to join the back of the queue at A. Figure 12.

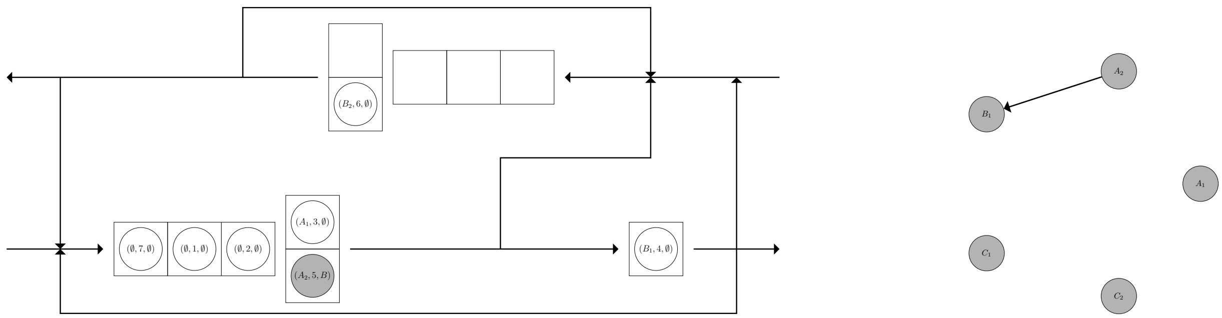


Figure 12

Observations

Consider one weakly connected component $G(t)$ of $D(t)$. Consider the node $X_a \in G(t)$. If X_a is unoccupied, then X_a has no incident edges. Consider the case when X_a is occupied by individual a, whose next destination is node j. Then X_a's direct successors are the servers occupied by individuals who are blocked or in service at node j. We can interpret all X_a's decendents as the servers whose occupants are directly or indirectly blocking a, and we can interpret all X_a's ancestors as those servers whose individuals who are being blocked directly or indirectly by a.

Note that the only possibilities for $\deg^{\text{out}}(X_a)$ are being 0 or c_j. If $\deg^{\text{out}}(X_a) = c_j$ then a is blocked by all its direct successors. The only other situation is that a is not blocked, and $X_a \in G(t)$ because a is in service at X_a and blocking other individuals, in which case $\deg^{\text{out}}(X_a) = 0$.

It is clear that if all of X_a's descendents are occupied by blocked individuals, then the system is deadlocked at time t. We also know that by definition all of X_a's ancestors are occupied by blocked individuals.

Also note that if a service-station subgraph $d_i(t)$ contains edges, then there is an individual in $X_a \in d_i(t)$ that is being blocked by himself. This does not necassarily mean there is deadlock.

4.2 Results on the State Digraph

Theorem 1. *A deadlocked state arises at time t if and only if $D(t)$ contains a knot.*

Proof. Consider one weakly connected component $G(t)$ of $D(t)$ at time t. All vertices of $G(t)$ are either decendents of another vertex and so are occupied by an individual who is blocking someone; or are ancestors of another vertex, and so are occupied by someone who is blocked.

Assume that $G(t)$ contains a vertex X such that $\deg^{\text{out}}(X) = 0$, and there is a path from every other non-sink vertix to X. This imples that X's occupant is not blocked and is a descendent of another vertrex. Therefore Q is not deadlocked as there does not exist a vertex whose descendents are all blocked.

Now assume that we have deadlock. For a vertex X who is deadlocked, all decendents of X are are occupied by individuals who are blocked, and so must have out-degrees greater than 0. And so there is no path from

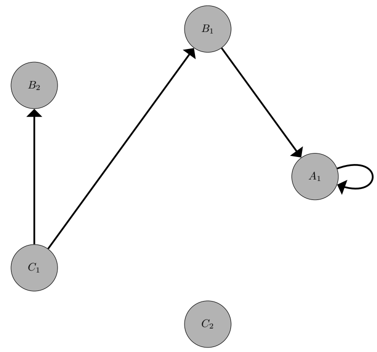


Figure 13: A Counter-Example State Digraph.

X to a vertex with out-degree of 0. ∎

Lemma 1. *For a queueing network with two nodes or less, a deadlocked state arises if and only if there exists a weakly connected component without a sink node.*

Proof. Consider a one node queueing network Q_1.

If there is deadlock, then all servers are occupied by blocked individuals, and so all servers have an out-edge.

Consider a two node queueing network Q_2.

If both nodes are involved in the deadlock, so there is a customer in node 1 blocked from entering node 2, and a customer from node 2 blocked from entering node 1, then all servers in node 1 and node 2 in $D(t)$ will have out edges as they are occupied by a blocked individual. The servers of node 1 and 2 consist of the entirety of $D(t)$, and so there is no sink nodes.

Now consider the case when only one node is involved in the deadlock. Without loss of generality, let's say that node 1 is in deadlock with itself, then the servers of node 1 have out-edges. For the servers of node 2 to be part of that weakly conneced component, there either needs to be an edge from a server in node 1 to a server in node 2, or and edge from a server in node 2 to a server in node 1. An edge from a server in node 1 to a server in node 2 implies that a customer from node 1 is blocked from entering node 2, and so node 1 is not in deadlock with itself. An edge from a server in node 2 to a server in node 1 implies that a customer in node 2 is blocked from entering node 1. In this case the server in node 2 has an out-edge, and so there is still no sink.

For the case of a queueing network with more than two nodes, the following counter-example proves the claim:

Begin with all servers occupied by customers in service. The customer at server B_1 is blocked from entering node A. Then the customer at server C_1 is blocked from entering node B. Then the customer at server A_1 is blocked from entering node A. The resulting state digraph in Figure 13 has a weakly connected component with a sink. ∎

Lemma 2. *An absorbing deadlocked state arises at time t if $D(t)$ doesn't contain a sink vertex.*

Proof. A vertex with out-degree greater than zero represents an occupied server whose occupant has finished service and is blocked. If all vertices have out-degree greater than zero, then all servers are occupied by blocked individuals. A release at vertex X_a can only be triggered by one of X_a's descendents finishing service. As all servers are occupied by blocked individuals, no server can finish service, and so no server can release their occupant, implying an absorbing deadlocked state. □

4.3 Finding Knots

By definition knots are strongly connected subgraphs where every member's descendants belong to that subgraph.

Using the Python package NetworkX, finding strongly connected components, and finding a vertex's decendants are built-in methods. The following algoithm, taken from the NetworkX developer zone ticket #663, is sufficient to identify knots in a directed graph.

```
Find the strongly connected subgraphs of D(t)
for strongly connected subgraph SCS in list of strongly connected subgraphs of D(t) do
    for vertex v in SCS do
        if number of v's decendants > number of vertices in SCS then
            Add SCS to list of knots
            break
        end
    end
end
```

5 Markov Chain Model

It is interesting to build an analytical model of the system's behaviour to deadlock. As a Markov chain model, the deadlocking state is an absorbing state, and so any queueing network that can experience deadlock is guaranteed to experience deadlock.

We can however find the expected time until deadlock is reached. It is shown in [5] that for a discrete transition matrix of the form $P = \begin{pmatrix} T & U \\ 0 & I \end{pmatrix}$ then the expected number of time steps until absorbtion starting from state i is the ith element of the vector

$$(I - T)^{-1}e \tag{1}$$

where e is a vector of 1s.

Another interesting analytical measure to find is the median time to deadlock.

5.1 One Node Network

Consider the one node network with feedback loop shown in Figure 14. There is room for n customers to queue at any one time, customers arrive at a rate of Λ and served at a rate μ. Once a customer has finished service he rejoins the queue with probability r_{11}, and so exits the system with probability $1 - r_{11}$.

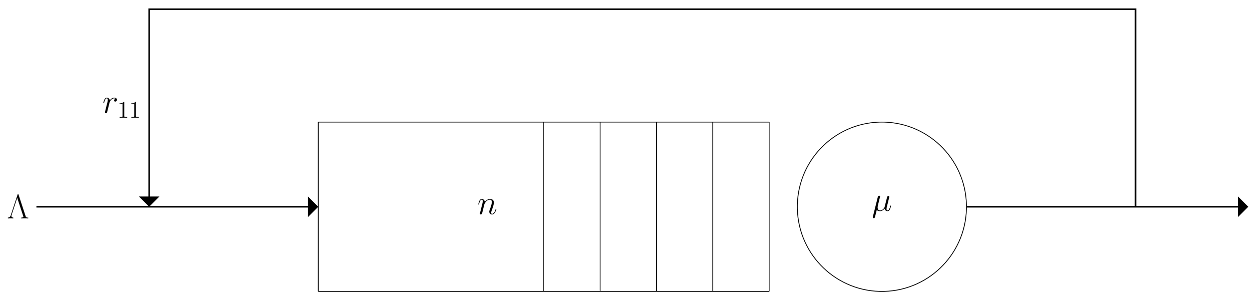


Figure 14: A one node queueing network.

State space:

$$S = \{i \in \mathbb{N} \mid 0 \leq i \leq n+1\} \cup \{-1\}$$

where i denotes number of individuals in service or waiting.

If we define $\delta = i_2 - i_1$ for all $i_k \geq 0$ then the transitions are given by:

$$q_{i_1,i_2} = \begin{cases} \left.\begin{matrix} \Lambda & \text{if } i < n+1 \\ 0 & \text{otherwise} \end{matrix}\right\} & \text{if } \delta = 1 \\ (1 - r_{11})\mu & \text{if } \delta = -1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$q_{i,-1} = \begin{cases} r_{11}\mu & \text{if } i = n+1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and

$$q_{-1,i} = 0 \tag{4}$$

The Markov chain is shown in Figure 15.

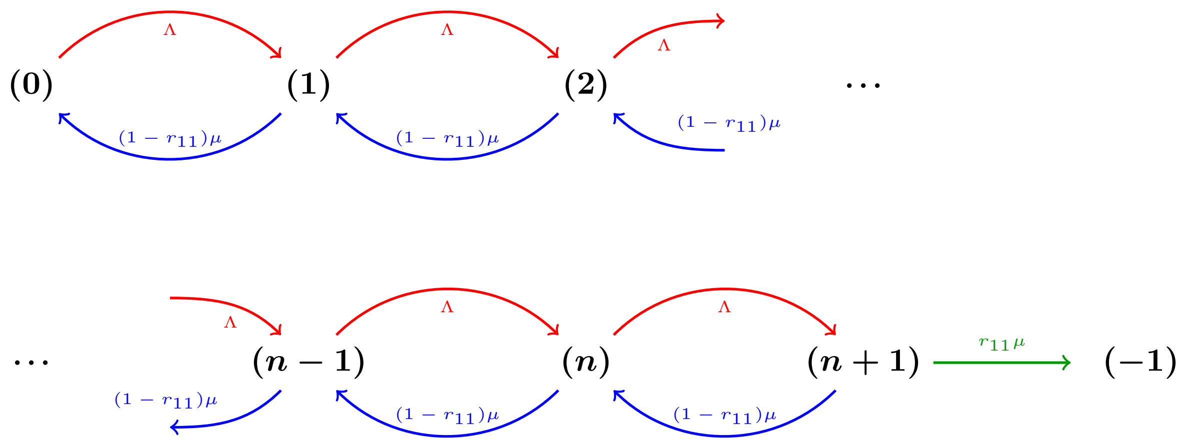


Figure 15: Markov chain of the one node system.

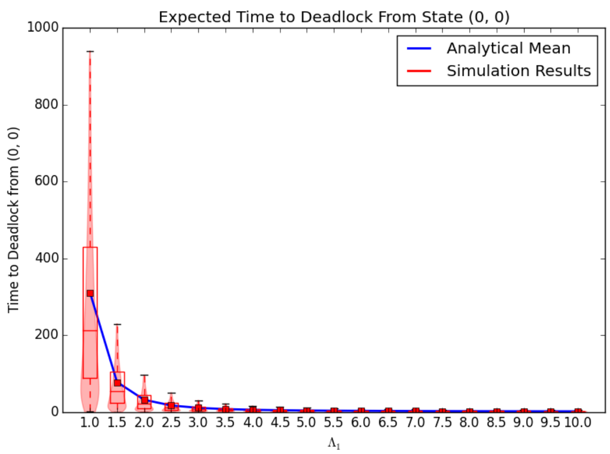
Figure 16 shows the effect of varying the parameters of the queueing network on times to deadlock. Base parameters of $\Lambda = 10$, $n = 3$, $\mu = 5$ and $r_{11} = 0.25$ were used.

We can see that increasing the arrival rate Λ and the transition probability r_{11} results in reaching deadlock faster. This is intuitive as increasing these parameters results in the first node's queue filling up quicker. Increasing the queueing capacity n results in reaching deadlock slower. Again this are intuitive, as increasing the queueing capacity allows more customers in the system before becoming deadlock.

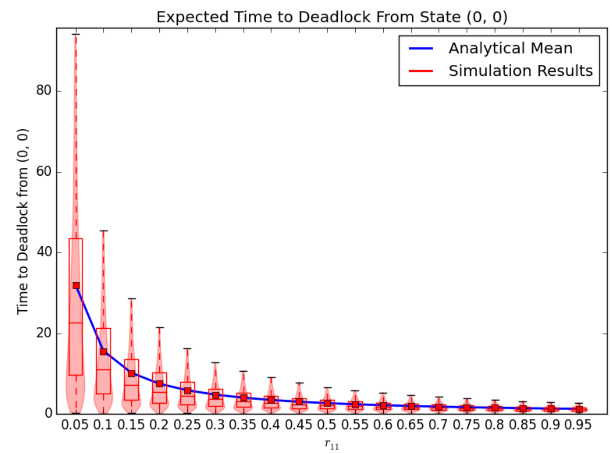
We get interesting behaviour as the service rate μ varies, as the service rate contributed towards bothmoving customers from the system and allowing customers to rejoin the queue, causing blockages and deadlock. This behavious can be interperated as follows:

- At low service rates below a certain threshold, the arrival rate is relatively large compared to the service rate, and we can assume a saturated system. At this point services where a customer exits the system does not have much of an effect, as we can assume another arrival immediately. However services where a customer wishes to rejoin the queue results in a blockage as the system is saturated. Therefore, increasing the service rate here increases the chance of a blockage, and so the chance of deadlock.
- Above this threshold the service rate is large enough that we cannot assume a saturated system, and so services where the customer exits the system does have an affect on the number of customers in the system. Thus increasing the service rate removes people from the system, and as sush there is less chance of getting blocked and deadlocked.

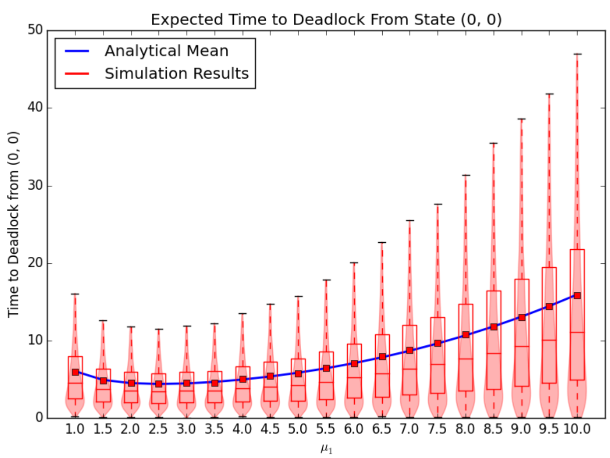
This effect is closely related to the transition rate r_{11}, as the rate at which the system enters deadlock from a full queue is $r_{11}\mu$. Figure 17 shows the effect of the transition rate on the behaviour of varying the service rate.



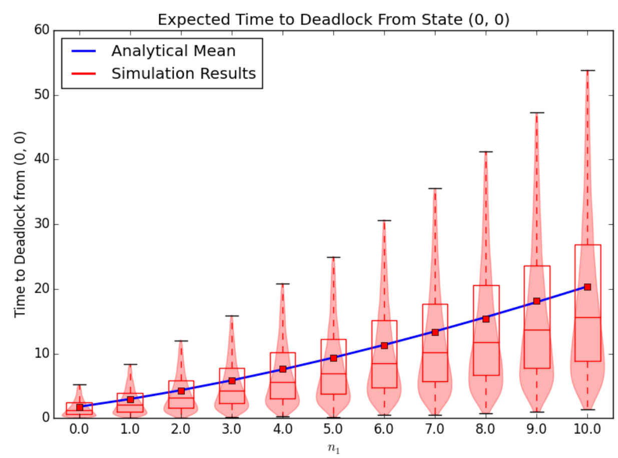
(a) Varying Λ



(b) Varying r_{11}



(c) Varying μ



(d) Varying n

Figure 16: Analytical & Simulation Results of Times to Deadlock (10,000 iterations)

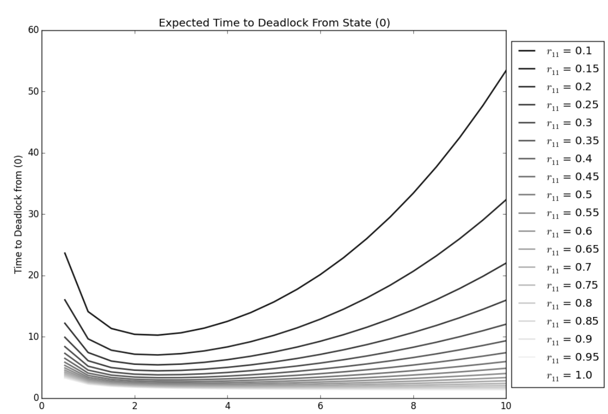


Figure 17: The effect of r_{11} and μ on times to deadlock.

5.2 Two Node Network without Self Loops

Consider the queueing network shown in Figure 18. This shows two $M/M/1$ queues, with n_i queueing capacity at at each service station and service rates μ_i. Λ_i is the external arrival rates to each service station. All routing possibilities except self loops are possible, where the routing probability from node i to node j is denoted by r_{ij}.

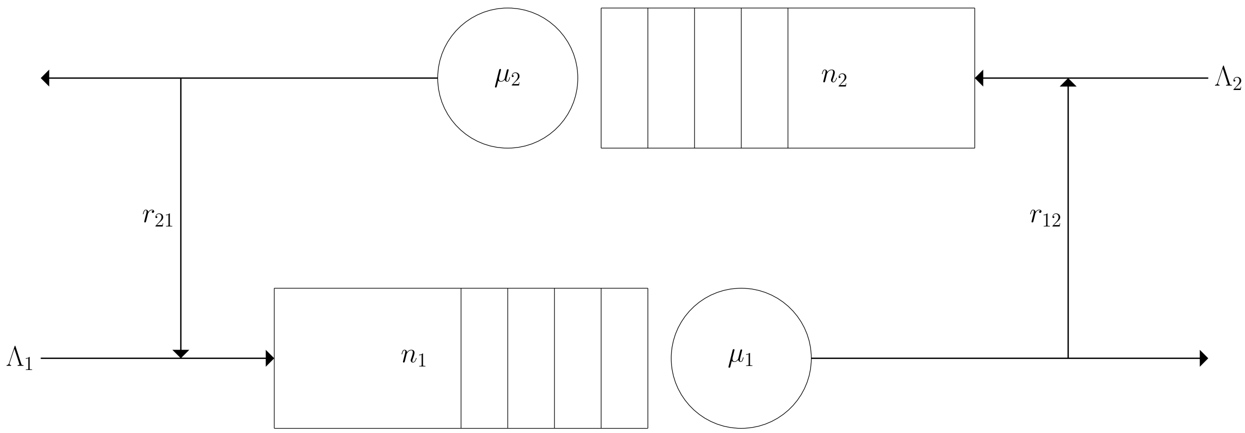


Figure 18: A two node queueing network.

- State space:

$$S = \{(i,j) \in \mathbb{N}^{(n_1+2\times n_2+2)} \mid 0 \leq i+j \leq n_1+n_2+2\} \cup \{(-1)\}$$

  where i denotes number of individuals:

  - In service or waiting at the first node.
  - Occupying a server but having finished service at the second node waiting to join the first

  where j denotes number of individuals:

  - In service or waiting at the second node.
  - Occupying a server but having finished service at the first node waiting to join the first

  and the state (-1) denotes the deadlocked state.

If we define $\delta = (i_2, j_2) - (i_1, j_1)$ for all $(i_k, j_k) \in S$, then the transitions are given by:

$$q_{(i_1,j_1),(i_2,j_2)} = \begin{cases} \left.\begin{array}{rr} \Lambda_1 & \text{if } i_1 < n_1 + 1 \\ 0 & \text{otherwise} \end{array}\right\} & \text{if } \delta = (1,0) \\ \left.\begin{array}{rr} \Lambda_2 & \text{if } j_1 < n_2 + 1 \\ 0 & \text{otherwise} \end{array}\right\} & \text{if } \delta = (0,1) \\ \left.\begin{array}{rr} (1 - r_{12})\mu_1 & \text{if } j_1 < n_1 + 2 \\ 0 & \text{otherwise} \end{array}\right\} & \text{if } \delta = (-1,0) \\ \left.\begin{array}{rr} (1 - r_{21})\mu_2 & \text{if } i_1 < n_1 + 2 \\ 0 & \text{otherwise} \end{array}\right\} & \text{if } \delta = (0,-1) \\ \left.\begin{array}{rr} r_{12}\mu_1 & \text{if } j_1 < n_2 + 2 \text{ and } (i_1, j_1) \neq (n_1 + 2, n_2) \\ 0 & \text{otherwise} \end{array}\right\} & \text{if } \delta = (-1,1) \\ \left.\begin{array}{rr} r_{21}\mu_2 & \text{if } i_1 < n_1 + 2 \text{ and } (i_1, j_1) \neq (n_1, n_2 + 2) \\ 0 & \text{otherwise} \end{array}\right\} & \text{if } \delta = (1,-1) \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$q_{(i_1,j_1),(-1)} = \begin{cases} r_{21}\mu_2 & \text{if } (i,j) = (n_1, n_2 + 2) \\ r_{12}\mu_1 & \text{if } (i,j) = (n_1 + 2, n_2) \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

and

$$q_{-1,s} = 0 \tag{7}$$

For $n_1 = 1$ and $n_2 = 2$, the resulting Markov chain is shown in Figure 19.

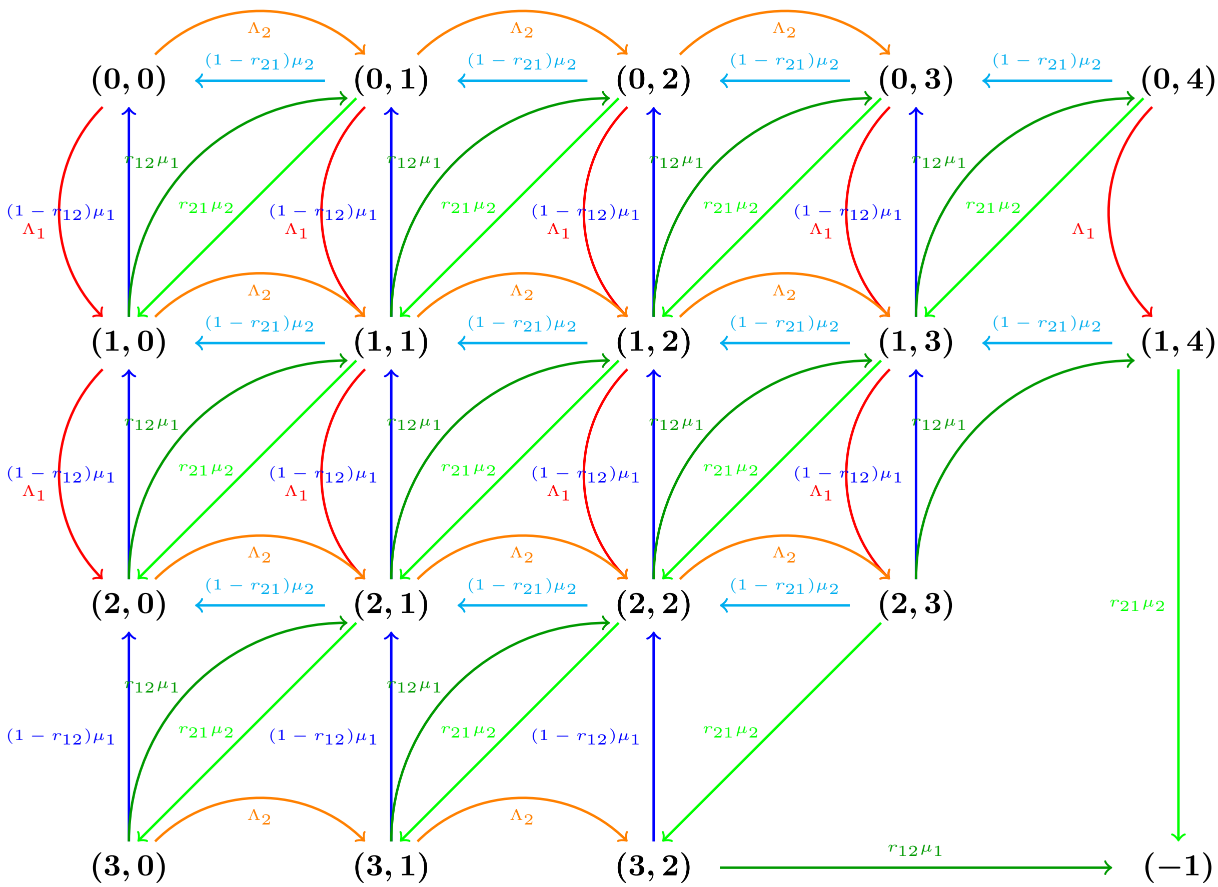
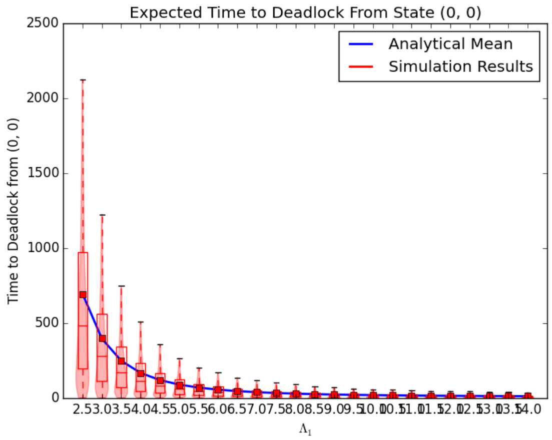
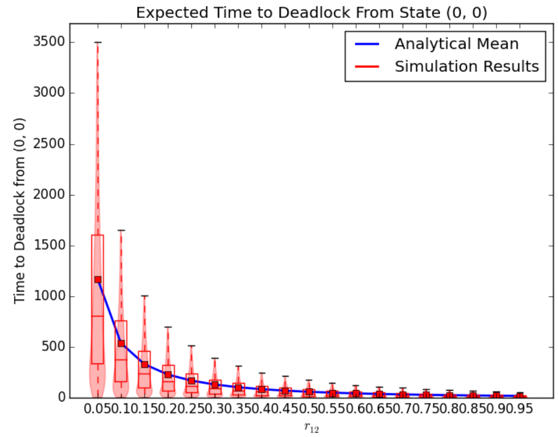


Figure 19: Markov chain of the two node system without self loops, $n_1 = 1$ and $n_2 = 2$.

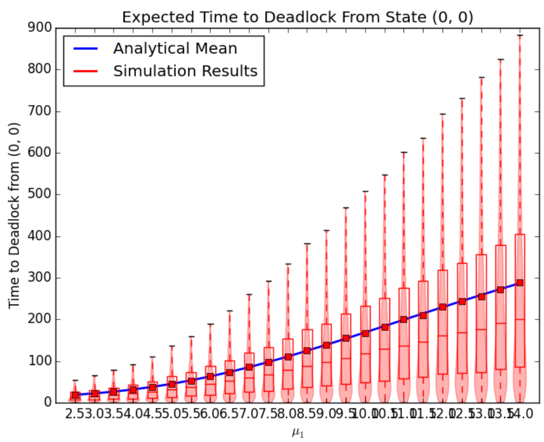
Figure 20 shows the effect of varying the parameters of the above Markov model. Base parameters of $\Lambda_1 = 4$, $\Lambda_2 = 5$, $n_1 = 3$, $n_2 = 2$, $\mu_1 = 10$, $\mu_2 = 8$, $r_{12} = 0.25$ and $r_{21} = 0.15$ were used. Only parameters for one node are shown, as the other node's parameters will have the same affect. We can see that we get similar behaviour as the 1 node network.



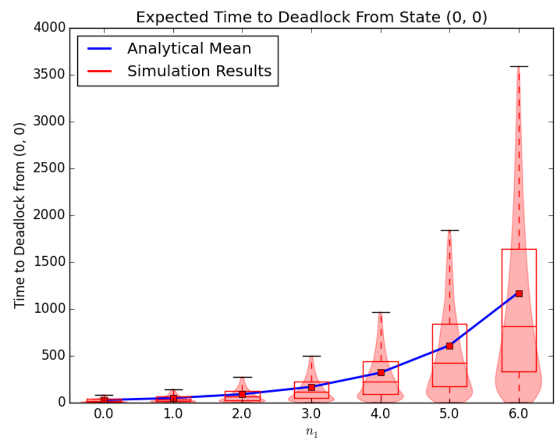
(a) Varying Λ_1



(b) Varying r_{12}



(c) Varying μ_1



(d) Varying n_1

Figure 20: Analytical & Simulation Results of Times to Deadlock (10,000 iterations)

5.3 Two Node Network each with Self Loops

Consider the queueing network shown in Figure 21. This shows two $M/M/1$ queues, with n_i queueing capacity at at each service station and service rates μ_i. Λ_i is the external arrival rates to each service station. All routing possibilities are possible, where the routing probability from node i to node j is denoted by r_{ij}.

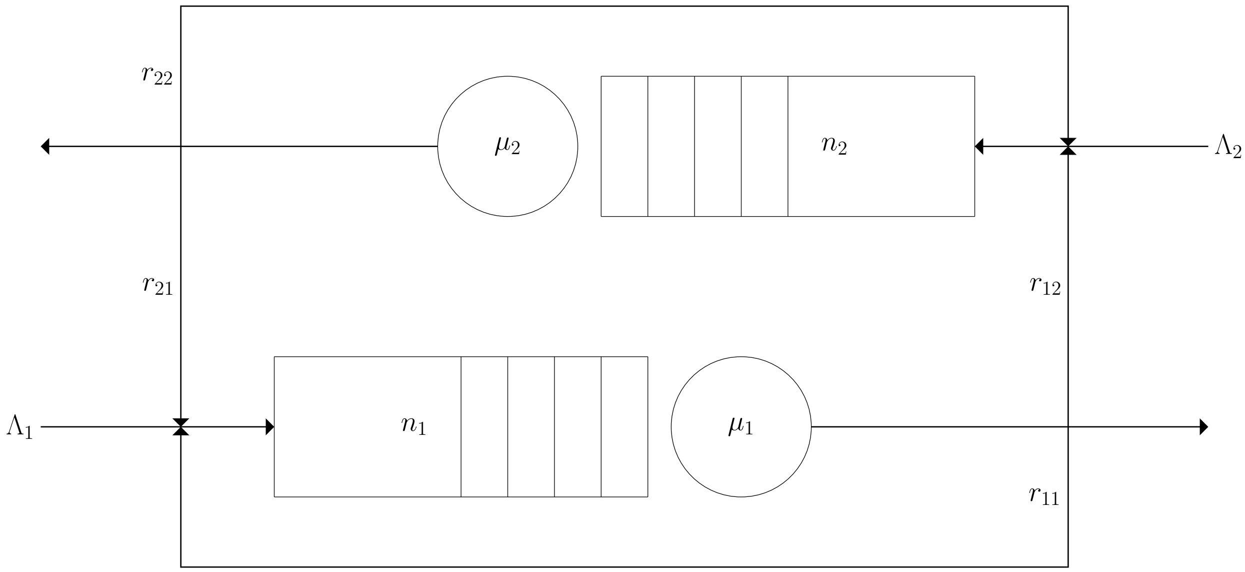


Figure 21: A two node queueing network, each with feedback loops.

- State space:

$$S = \{(i, j) \in \mathbb{N}^{(n_1+2\times n_2+2)} \mid 0 \leq i + j \leq n_1 + n_2 + 2\} \cup \{(-1), (-2), (-3)\}$$

  where i denotes number of individuals:

  - In service or waiting at the first node.
  - Occupying a server but having finished service at the second node waiting to join the first

  where j denotes number of individuals:

  - In service or waiting at the second node.
  - Occupying a server but having finished service at the first node waiting to join the first

  and the state (-3) denotes the deadlocked state cause by both nodes; (-1) denotes the deadlocked state caused by the first node only; and (-2) denotes the deadlocked state caused by the second node only.

If we define $\delta = (i_2, j_2) - (i_1, j_1)$ for all $(i_k, j_k) \in S$, then the transitions are given by:

$$q_{(i_1,j_1),(i_2,j_2)} = \begin{cases} \left.\begin{array}{rr} \Lambda_1 & \text{if } i_1 < n_1+1 \\ 0 & \text{otherwise} \end{array}\right\} & \text{if } \delta = (1,0) \\ \left.\begin{array}{rr} \Lambda_2 & \text{if } j_1 < n_2+1 \\ 0 & \text{otherwise} \end{array}\right\} & \text{if } \delta = (0,1) \\ \left.\begin{array}{rr} (1-r_{11}-r_{12})\mu_1 & \text{if } j_1 < n_1+2 \\ 0 & \text{otherwise} \end{array}\right\} & \text{if } \delta = (-1,0) \\ \left.\begin{array}{rr} (1-r_{21}-r_{22})\mu_2 & \text{if } i_1 < n_1+2 \\ 0 & \text{otherwise} \end{array}\right\} & \text{if } \delta = (0,-1) \\ \left.\begin{array}{lr} r_{12}\mu_1 & \text{if } j_1 < n_2+2 \text{ and } (i_1,j_1) \neq (n_1+2,n_2) \\ 0 & \text{otherwise} \end{array}\right\} & \text{if } \delta = (-1,1) \\ \left.\begin{array}{lr} r_{21}\mu_2 & \text{if } i_1 < n_1+2 \text{ and } (i_1,j_1) \neq (n_1,n_2+2) \\ 0 & \text{otherwise} \end{array}\right\} & \text{if } \delta = (1,-1) \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

$$q_{(i_1,j_1),(-1)} = \begin{cases} r_{11}\mu_1 & \text{if } i > n_1 \text{ and } j < n_2+2 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

$$q_{(i_1,j_1),(-2)} = \begin{cases} r_{22}\mu_2 & \text{if } j > n_2 \text{ and } i < n_1+2 \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

$$q_{(i_1,j_1),(-3)} = \begin{cases} r_{21}\mu_2 & \text{if } (i,j) = (n_1, n_2+2) \\ r_{12}\mu_1 & \text{if } (i,j) = (n_1+2, n_2) \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

and

$$q_{-1,s} = 0 \tag{12}$$

$$q_{-2,s} = 0 \tag{13}$$

$$q_{-3,s} = 0 \tag{14}$$

Note that the there are only two differences between this formulation and the formulation given in Subsection 5.2: the probabilities of leaving nodes 1 and 2 are now $(1-r_{11}-r12)\mu_1$ and $(1-r_{21}-r_{22})\mu_2$; and there are now two more ways to reach deadlock, Equation 9 and Equation 10.

For $n_1 = 1$ and $n_2 = 2$, the resulting Markov chain is shown in Figure 22.

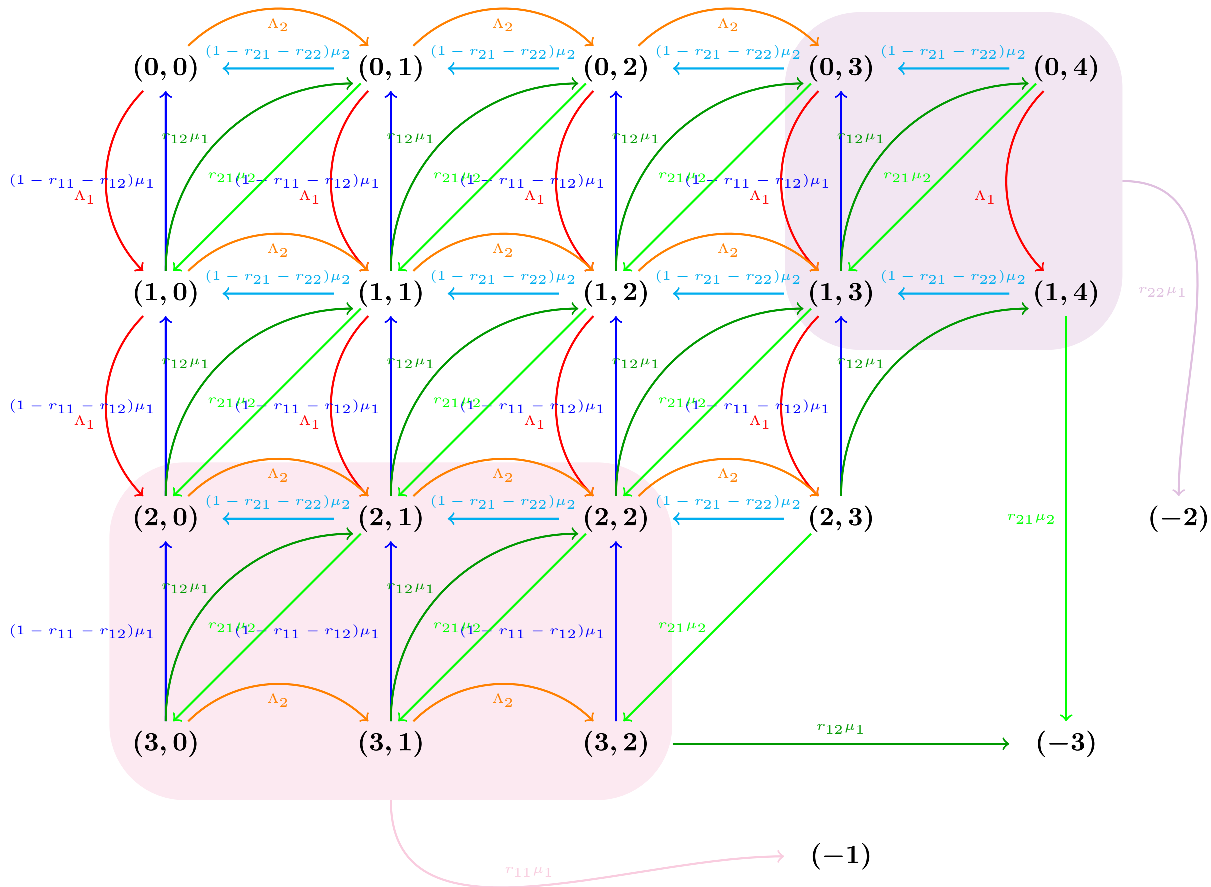
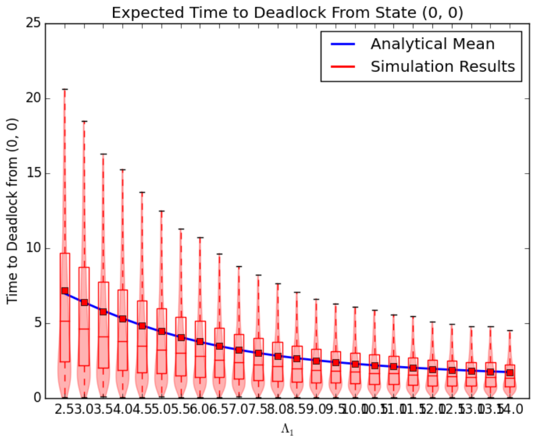
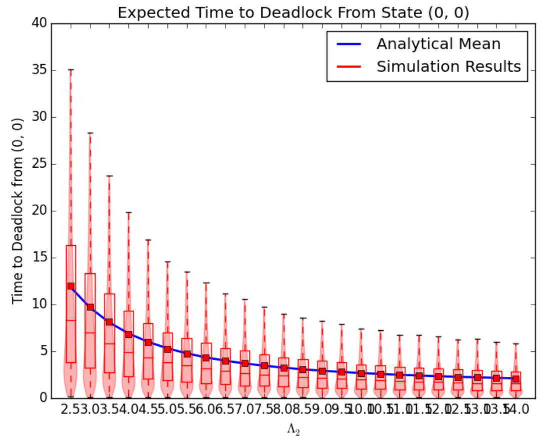


Figure 22: Markov chain of the two node system with self loops, $n_1 = 1$ and $n_2 = 2$.

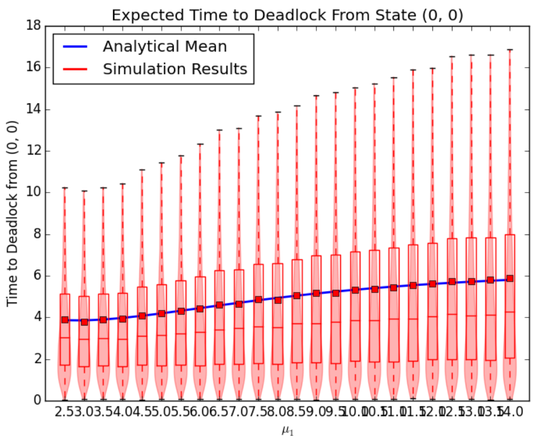
Figure 23 shows the effect of varying the parameters of the above Markov model. Base parameters of $\Lambda_1 = 4$, $\Lambda_2 = 5$, $n_1 = 3$, $n_2 = 2$, $\mu_1 = 10$, $\mu_2 = 8$, $r_{11} = 0.1$, $r_{12} = 0.25$, $r_{21} = 0.15$ and $r_{22} = 0.1$ were used.



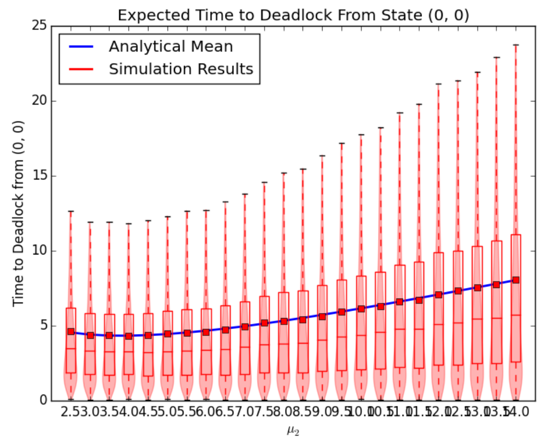
(a) Varying Λ_1



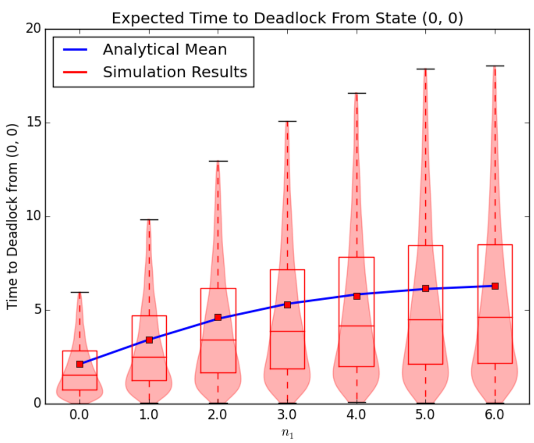
(b) Varying Λ_2



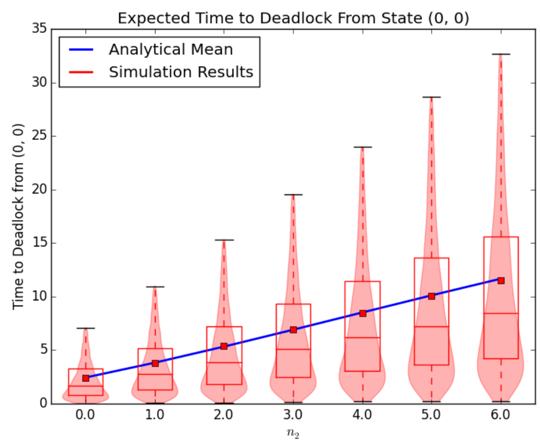
(c) Varying μ_1



(d) Varying μ_2



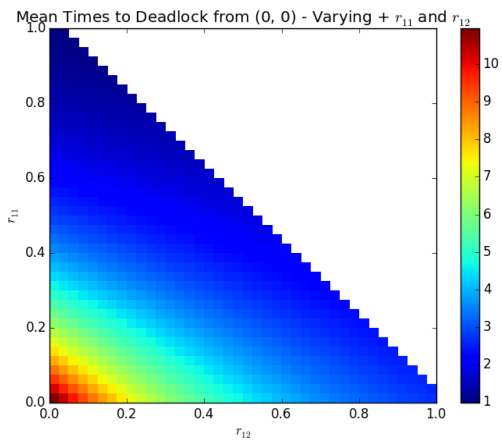
(e) Varying n_1



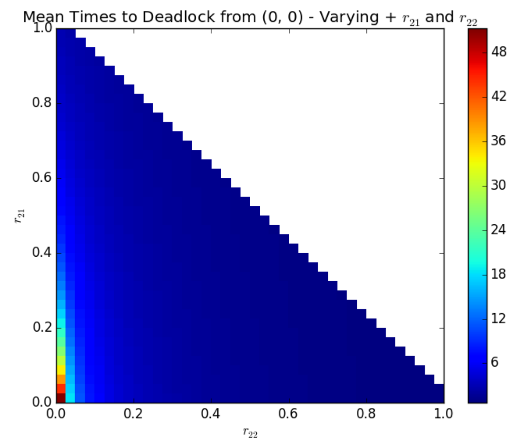
(f) Varying n_2

Figure 23: Analytical & Simulation Results of Times to Deadlock (10,000 iterations)

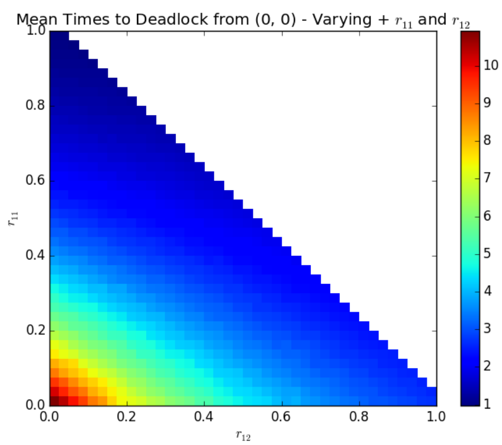
The heatmaps in Figure 24 illustrate how varying the two transition probabilities out of each node affects the time to deadlock. Note the shape of the heatmap, this is due to the restriction than $r_{11} + r12 \leq 1$ and $r_{21} + r_{22} \leq 1$. We can see for both nodes it is the rejoining probability (r_{11} and r_{22}) that has the most drastic effect on time to deadlock. This effect is greater for Node 2, the node that has the smaller queueing capacity.



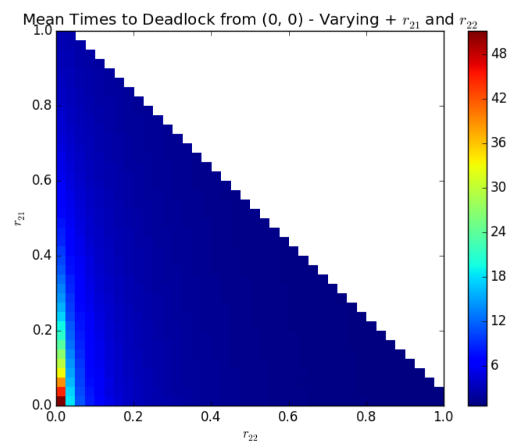
(a) Analytical: Varying transition probabilities at Node 1



(b) Analytical: Varying transition probabilities at Node 2



(c) Simulation: Varying transition probabilities at Node 1



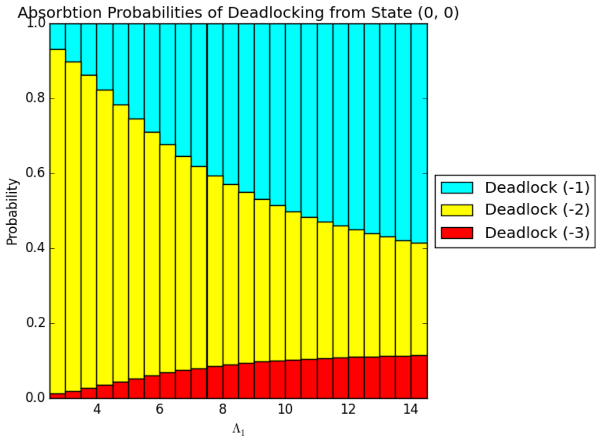
(d) Simulation: Varying transition probabilities at Node 2

Figure 24: Analytical & Simulation Results of Times to Deadlock, varying Transition Probabilities (10,000 iterations)

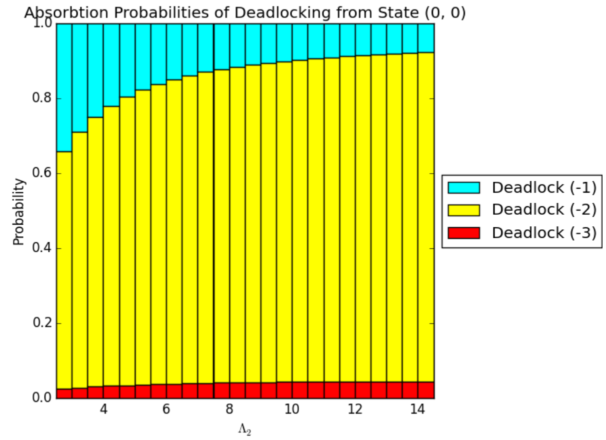
Times to deadlock in this case means the time to the first instance of deadlock. There is however three different deadlocked states which the queueing network can fall into, -1, -2 and -3.

A method is shown in [5] to find the probabilities of which absorbing state a Markov chain will reach. The discrete transition matrix must be in the form $P = \left(\begin{smallmatrix} T & U \\ 0 & I \end{smallmatrix}\right)$. Then $A = (I - T)^{-1}U$, and the $(i, j)^{\text{th}}$ element of A corresponds to the probability of reaching absorbing state j from transient state i.

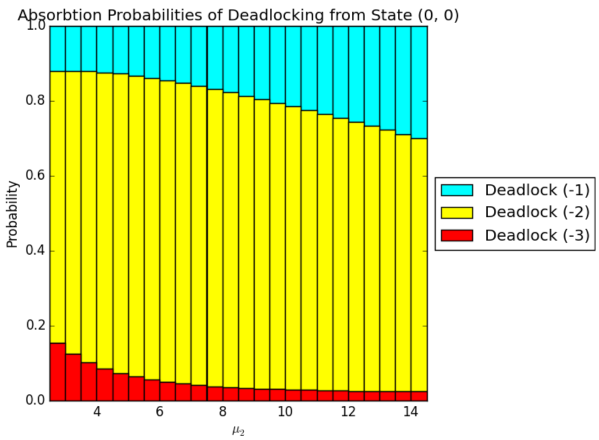
Figure 25 shows how varying the parameters of the queueing network affects the absorption probabilities.



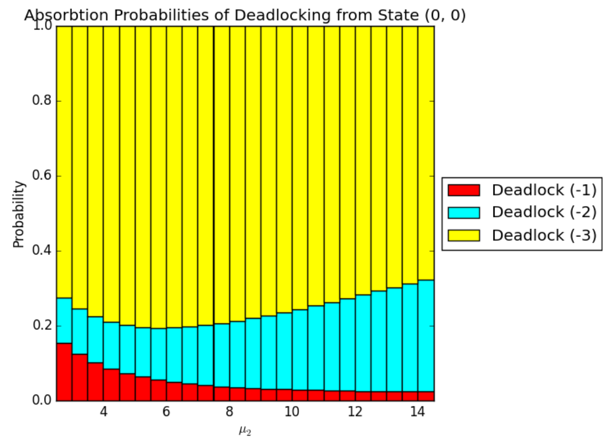
(a) Varying Λ_1



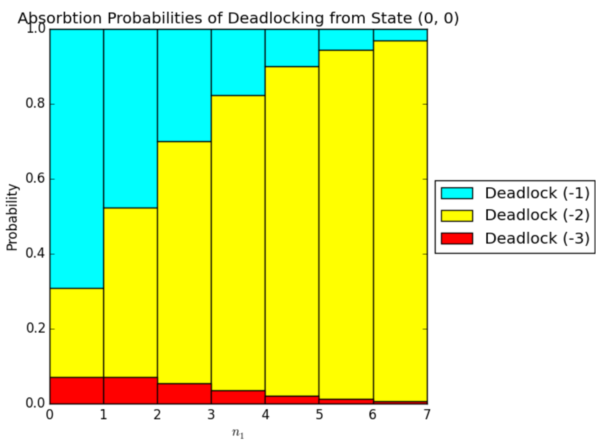
(b) Varying Λ_2



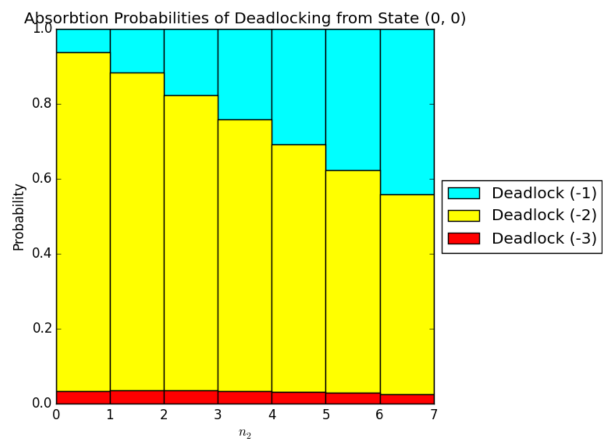
(c) Varying μ_1



(d) Varying μ_2



(e) Varying n_1



(f) Varying n_2

Figure 25: Probabilities of reaching each deadlocked state.

5.4 One Node Network, Multi-Server

Consider the one node network with feedback loop discussed in Subsection 5.1, now with c parallel servers.

State space:

$$S = \{i \in \mathbb{N} \mid 0 \leq i \leq n + 2c\}$$

where i denotes number of individuals at the node plus the number of individuals blocked at that node. For example, $i = n + c + 2$ denotes a full system, $n + c$ individuals in the node, and 2 of those individuals are also blocked. The state $i = n + 2c$ denotes the deadlocked state.

If we define $\delta = i_2 - i_1$ for all $i_k \geq 0$, then the transitions are given by:

$$q_{i_1,i_2} = \left\{ \begin{array}{rr} \Lambda & \text{if } \delta = 1 \\ (1 - r_{11})\mu\text{min}(i, c) & \text{if } \delta = -1 \\ 0 & \text{otherwise} \end{array} \right\} \quad \text{if } i_i < n + c \tag{15}$$

$$q_{i_1,i_2} = \left\{ \begin{array}{rr} (c - k)r_{11}\mu & \text{if } \delta = 1 \\ (1 - r_{11})(c - k)\mu & \text{if } \delta = -j - 1 \\ 0 & \text{otherwise} \end{array} \right\} \quad \text{if } i_i = n + c + j \quad \forall \quad 0 \leq j \leq c \tag{16}$$

Increasing the amount of servers has a similar effect to increasing the queueing capacity, there are now more transient spaces to go through before reaching the deadlocked state. Varying the amount of servers has a greater effect on the time to deadlock however, as any states in which customers are blocked ($i = n + c + 1$ to $i = n + 2c$) can jump back to state $i = n + c - 1$ simply with a service and an exit. Increasing the amount of servers also increases the rate at which i are reduced for most states, but not the rates at which i is increased.

Figure 26 shows the effect of varying the parameters of the above Markov model. Base parameters of $\Lambda = 6$, $n = 3$, $\mu = 2$ and $r_{11} = 0.5$ were used.

5.5 A Bound on the Mean Time to Deadlock

In this section we shall define four deadlocking queueing networks as follows:

- Define Ω_{1_1} as the 1 node queueing network described in Subsection 5.1 with the parameter set $\{\Lambda_1$, μ_1, n_1, $r_{11}\}$. Let its mean time to deadlock be denoted by ω_{1_1}.
- Define Ω_{1_2} as the 1 node queueing network described in Subsection 5.1 with the parameter set $\{\Lambda_2$, μ_2, n_2, $r_{22}\}$. Let its mean time to deadlock be denoted by ω_{1_2}.
- Define Ω_2 as the 2 node queueing network described in Subsection 5.2 with the parameter set $\{\Lambda_1$, Λ_2, μ_1, μ_2, n_1, n_2, r_{12}, $r_{21}\}$. Let its mean time to deadlock be denoted by ω_2.
- Define Ω as the 2 node queueing network described in Subsection 5.3 with the parameter set $\{\Lambda_1$, Λ_2, μ_1, μ_2, n_1, n_2, r_{11}, r_{12}, r_{21}, $r_{22}\}$. Let its mean time to deadlock be denoted by ω.

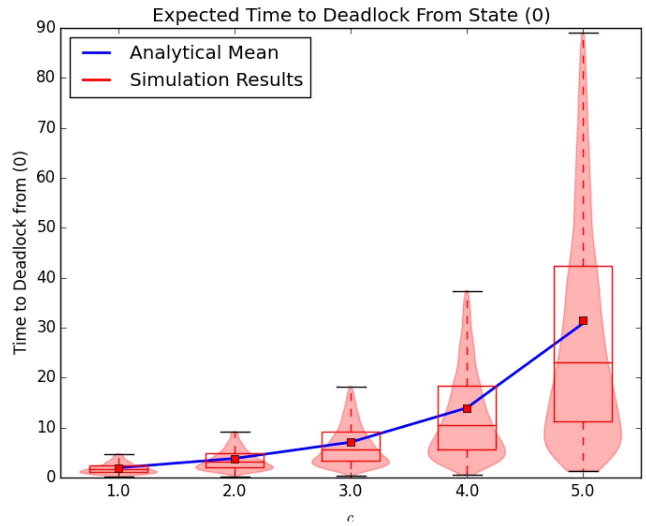
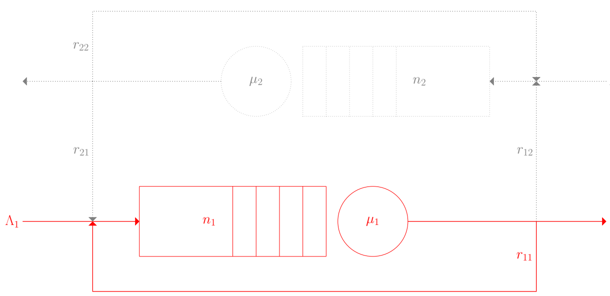
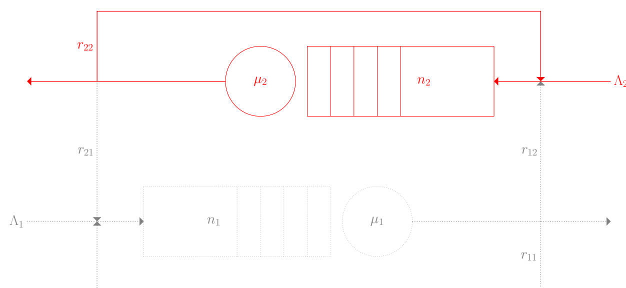


Figure 26: The effect of c on times to deadlock.

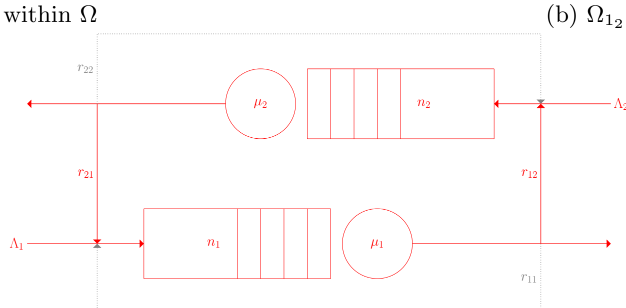
Figure 27 shows how Ω contains, and is made up by, Ω_{1_1}, Ω_{1_2} and Ω_2.



(a) Ω_{1_1} within Ω



(b) Ω_{1_2} within Ω



(c) Ω_2 within Ω

Figure 27: Decomposition of Ω into Ω_{1_1}, Ω_{1_2} and Ω_2

Theorem 2. *For any parameter sets the following inequality holds:* $\omega \leq \min(\omega_{1_1}, \omega_{1_2}, \omega_2)$

Proof. The proof will be here. $\square$

6 Resolving Deadlock

Once the system falls into a deadlocked state for the first time, that is the first transient deadlocked state since the last resolution, the simulated needs to automatically resolve the deadlock and allow services to resume again. This is not necassarily as simple as moving a blocked customer to his next node, as we need to conserve the numbers of customers at each service station. Closer inspection of the state digraph is required in order to find a way to resolve deadlock whilst conserving this property.

At deadlock, the service stations can be classified into the following three mutually exclusive categories:

- Nodes that are not deadlocked: These are nodes that do not contain any blocked individuals.
- Causation nodes, nodes causing deadlock: These are nodes where every server is occupied by a blocked individual, and there is at least one blocked individual waiting to enter that node who is directly or indirectly being blocked by an individual in this node.
- Affected nodes, nodes affected by deadlock: These are nodes containing at least one blocked individual who is directly or indirectly being blocked by an individual at a node that is causing deadlock, but is not classified as causing deadlock itself.

At the first instance of deadlock, there will only be one knot in $D(t)$. Let us denote the knot as K. The vertices of K correspond to servers. As there is no sink node, all vertices of K have an out-edge, and so all vertices in K contain a blocked individual. Therefore, there are no vertices in K belonging to nodes that are not deadlocked. All vertices of K correspond to servers of causation nodes, and a causation node has all its servers belonging to K. An affected node has servers belonging to the same weakly connected component as K, but does ot have servers in K.

When choosing which customer to move in order to resolve deadlock, we must be careful to conserve the number of customers at each service station. Causation nodes have full queues, and a customer may only be moved into a full queue if this causes another customer to simultaneously move from this node. Another complication arises due to the blocking mechanism used, in which those customers who have been blocked longer must be moved first. This property may have to be broken in order to ensure the conservation property is not. Assume that we have weighted the edges of the digraph with the time that they were created.

The following algorithm is proposed in order to resolve deadlock:

Find the knot K in $D(t)$
Find the cycle $C \in K$ whose average edge weight is minimum
Start at V_0
for V_i *in* C **do**
    Move the individual who is waiting to get to V_{i+1}
end
Redraw $D(t)$

References

[1] H. Cho, T. Kumaran, and R. Wysk. Graph-theoretic deadlock detection and resolution for flexible manufacturing systems. *IEEE transactions on robotics and automation*, 11(3):413–421, 1995.

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[4] J. Liebeherr and I. Akyildiz. Deadlock properties of queueing networks with finite capacities and multiple routing chains. *Queueing systems*, 20(3-4):409–431, 1995.

[5] W. Stewart. *Probability, markov chains, queues, and simulation.* Princeton university press, 2009.