Modelling Queues Where Customers Randomly Change Priority Classes While Waiting

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1 Introduction

There are a number of situations in which a customer's priority in a queue might change during their time queueing, or equivalently where their priority depends on the amount of time they have already spent in the queue. Classic examples arise in healthcare systems, for example when a patient's medical urgency might increase the longer they spend waiting due to health degeneration. Another example would be a prioritisation scheme that attempts a trade-off between medical need and waiting times. These are both examples where a patient's priority has the chance to upgrade over time while in the queue. However there also might be situations in which a patient's priority can downgrade over time: consider a medical intervention that can improve a patient's outcome if caught early, if a patient has been waiting a long time already then they might be passed over for a newly referred patient who will gain more benefit from the intervention. In this case a patient's priority is downgraded the longer they wait.

In this paper a single $M/M/c$ queue is modelled, with multiple classes of customer of different priorities. While waiting in the queue customers change their class to any other class at specific rates. Thus upgrades, downgrades, and 'skip-grades' (moving to a priority class not immediately above or below the current class) are modelled.

This is first modelled using simulation, where we describe generalisable logic. This is implemented in version v2.3.0 of the Ciw library in Python [17]. Then two Markov chain models are defined for the system, which are used to find steady state distributions and expected sojourn times for each customer class. These Markov chains give some insights into the behaviour of the systems under different combinations of parameters; and numerical experiments give further behaviours.

This paper is structured as follows: Section 2 defines the system under consideration in detail. Section 3 highlights some previous and related work. Section 4 discusses the contribution to the Ciw library and the logic required to simulate this system. Section 5 defines two Markov chain models of the system, one useful for considering system-wide statistics such as state probabilities, and one useful for considering customers' statistics such as average sojourn time. Section 6 explores a bounded approximation for numerically tractable analysis, and gives guidelines on choosing a large enough bound so as to sufficiently approximate an unbounded system. Section 7 presents results from numerical experiments that give insight into the behaviour of the system under various parameters. Section 8 experimentally justifies the use of these models to model scenarios where prioritisation rules are unknown.

2 System Under Consideration

Consider an $M/M/c$ queue with K classes of customer labelled $0, 1, 2, \ldots, K-1$. Let:

- λ_k be the arrival rate of customers of class k,
- μ_k be the service rate of customers of class k,

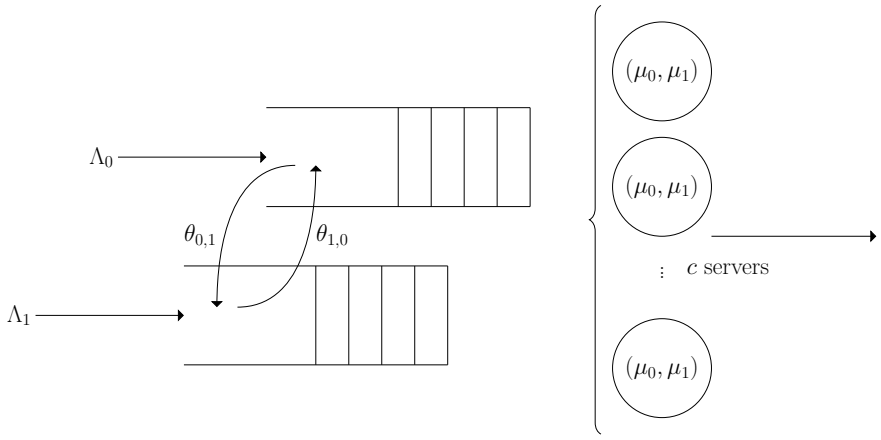


Figure 1: An example of a two-class priority queue.

$$\Theta = \begin{pmatrix} - & \theta_{01} & \theta_{02} & \theta_{03} & \theta_{04} \\ \theta_{10} & - & \theta_{12} & \theta_{13} & \theta_{14} \\ \theta_{20} & \theta_{21} & - & \theta_{23} & \theta_{24} \\ \theta_{30} & \theta_{31} & \theta_{32} & - & \theta_{34} \\ \theta_{40} & \theta_{41} & \theta_{42} & \theta_{43} & - \end{pmatrix}$$

—— downgrades
- - - upgrades
····· 'skip-grades'

Figure 2: Representations of parts of the matrix Θ. Example when $K = 5$.

- $\theta_{i,j}$ be the rate at which customers of class i change to customers of class j while they are waiting in line.

Customers of class i have priority over customers of class j if $i < j$. Customers of the same class are served in the order they arrived to that class.

Figure 1 shows an example with two classes of customer.

The key feature here is the $K \times K$ class change matrix $\Theta = (\theta_{i,j})$. All elements $\theta_{i,j}$ where $i \neq j$ are rates, and so are non-negative real numbers, if customers of class i cannot change to customers of class j directly, then $\theta_{i,j} = 0$. The diagonal values $\theta_{i,i}$ are unused as customers cannot change to their own class. All elements $\theta_{i,i-1}$ represent the direct upgrade rates; all elements $\theta_{i,i+1}$ represent the direct downgrade rates, while all other elements can be thought of as 'skip-grades'. This is shown in Figure 2.

3 Related Work

Systems of this kind have been investigated previously:

- [9] (1960): Non preemptive M/M/1 where customers are served in order of the difference between their waiting time and urgency number (that is priorities increasing linearly over time). Solved by considering event probabilities at clock ticks.
- [10] (1964): Another formulation giving the same behaviour as [9], but now for both non preemptive and preemptive priorities, and multiple customer classes. Called 'delay dependent' or 'time dependant' priorities, and recently by 'accumulating' priorities.

- [8] (1971): Similar to [9], but treat each urgency number as a separate customer class, and not considering clock ticks. Upper and lower bounds on the waiting times, based on FIFO and static priorities.
- [15] (1979): Now considers the case where priorities increase non-linearly but concavely over time.
- [6] (1990): Non preemptive M/G/1 queue with two classes of customers, where priorities switch if the number from one class exceeds a given threshold. Lower priority customers have a finite waiting capacity, higher have infinite capacity.
- [22] (1995): Introduces the generalised $c\mu$-rule (first conceived in [20]), which applies a class- and waiting time-dependant cost to each customer. This acts as a scheduling rule, but can also model dynamic priorities amongst customers.
- [12] (2003): Similar to [6] but with Markovian services and infinite waiting capacities for both customers.
- [23] (2008): Preemptive n-priority-classes M/M/c with exponential upgrades. Customers only upgrade to the priority immediately higher than themselves. Stability considered.
- [4] (2010): Preemptive two-priority-classes M/M/c with exponential upgrades. Customers cannot upgrade if the number of lower priority customers is below a given threshold. Holding costs considered.
- [7] (2012): Extension of [4], allows batch arrivals, multiple classes, phase-type upgrades and services. Customers only upgrade to the priority immediately higher than themselves.
- [21] (2014): Furthers the work of [10] to look at the maximum priority of the waiting customers in a single server queue as a stochastic process. This is extended in [19] to multi-server queues.
- [11] (2020): Upgrades and downgrades after a random, exponentially distributed, amount of time. Models two priority classes in a single server, finite buffer system with batch arrivals. Extended in [5] to include unreliable services and impatient customers.
- [1] (2022): Analytical (truncated) expressions for a two class delayed accumulating priority M/G/1 queue. Customer priorities increase linearly over time, at different rates according to class, after an initial fixed delay.
- [3] (2019): Evidence that in Canada that decision makers often use their own discretion in deciding which patients to be seen, rather than FIFO within each triage category. Fits prioritisation rules to this discretionary behaviour.

4 Simulation Model Logic

The Ciw library [17] is an open-source Python library for discrete-event simulation of open queueing networks. A key contribution of this work is the adaptation of the library's logic to facilitate the type of dynamically changing priority classes described in Section 2. This adaptation is released in version Ciw v2.3.0, with usage documentation at `https://ciw.readthedocs.io/en/latest/Guides/change-class-while-queueing.html`.

The core of Ciw's logic is the event scheduling approach, described in [18]. This is a variant of the three-phase approach, with an **A**-phase which advances the clock to the next scheduled event, a **B**-phase where scheduled events are carried out, and a **C**-phase where conditional events are carried out. Figure 3 shows a flow diagram of the logic of the event scheduling approach.

The primary scheduled, or **B**-events are customers arriving to a queue, and customers finishing service. The conditional, or **C**-events are those that happen immediately after, and because of, these **B**-events. The primary ones are customers beginning service, and customers leaving the queue.

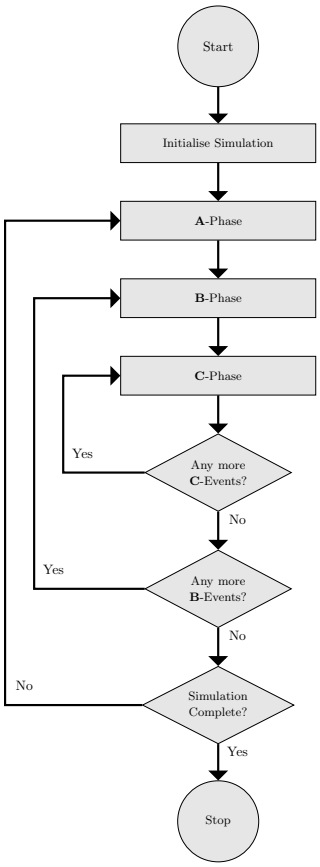


Figure 3: Flow diagram of the event scheduling approach used by Ciw, taken from [16].

All other features of queueing systems that can be simulated with the Ciw library involve increasing the range of **B**- and **C**-events that can happen during the simulation run. In the case of customers randomly changing priority classes while waiting, one additional **B**-event and one additional **C**-event are included:

- Upon arrival to the queue customers are assigned a date in which they will change customer class, determined by randomly sampling from a distribution. Therefore each customer's event of changing customer class is scheduled for the future, and are therefore **B**-events. If those customers begin service (which might not be scheduled yet) before that event has occurred, then their changing customer class event is cancelled.
- Upon changing class, they immediately schedule another changing class event for the future, again sampling a date from a given distribution. This happens immediately after the above, and so is a **C**-event.

Note that the particular distributions used to sample class change dates in these cases are generic, and any of Ciw's currently pre-programmed distributions can be chosen, or custom distributions can also be input. For the systems described in this paper, we choose Exponential distributions with rates determined by the class change matrix Θ.

5 Markov Chain Models

The situation described in words in Section 2 can be described precisely as two different Markov chains. The first, described in Section 5.1, describes the overall changes in state, where a state records the number of customers of each class at the node. This is useful for analysing system-wide statistics such as average queue size. The second, described in Section 5.2, describes how an individual arriving customer experiences the system until their exit. This is useful for analysing individual customers' statistics such as average sojourn time.

5.1 Discrete State Markov Chain Formulation

Let $\underline{\mathbf{s}}_t = (s_{0,t}, s_{1,t}, \ldots, s_{K-1,t}) \in \mathbb{N}^K$ represent the state of the system at time step t, where $s_{k,t}$ represents the number of customers of class k present at time step t. Let S denote set of all states $\underline{\mathbf{s}}_t$.

The rates of change between $\underline{\mathbf{s}}_t$ and $\underline{\mathbf{s}}_{t+1}$ are given by Equation 1, where $\underline{\delta} = \underline{\mathbf{s}}_{t+1} - \underline{\mathbf{s}}_t$,

$$q_{\underline{\mathbf{s}}_t, \underline{\mathbf{s}}_{t+1}} = \begin{cases} \lambda_k & \text{if } \delta_k = 1 \text{ and } \delta_i = 0 \; \forall \; i \neq k \\ B_{k,t}\mu_k & \text{if } \delta_k = 1 \text{ and } \delta_i = 0 \; \forall \; i \neq k \text{ and } \sum_{i<k} s_{i,t} < c \\ (s_{k,t} - B_{k,t})\theta_{k_0,k_1} & \text{if } \delta_{k_0} = -1 \text{ and } \delta_{k_1} = 1 \text{ and } \delta_i = 0 \; \forall \; i \neq k_0, k_1 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

and $B_{k,t}$, representing the number of customers of class k currently in service at time step t, is given by Equation 2, where c is the number of servers.

$$B_{k,t} = \min\left(c - \min\left(\sum_{i<k} s_{i,t}, c\right), s_{k,t}\right) \tag{2}$$

Let $\pi_{\underline{\mathbf{s}}}$ denote the steady state probability of being in state $\underline{\mathbf{s}} \in S$.

5.2 Sojourn Time Markov Chain Formulation

Let $\underline{\mathbf{z}}_t = (z_{0,t}, z_{1,t}, \ldots, z_{n,t} \ldots, z_{K-1,t}, m_t, n_t) \in \mathbb{N}^{K+2} \times (1, \ldots, K-1)$ represent the state of a particular customer at time step t, where n_t represents that customer's class at time t; $z_{k,t}\ \forall\ k < n$ represents the number of customers of class k in front of the customer in the queue at time t; $z_{k,t}\ \forall\ n < k < K$ represents the number of customers of class k behind the customer in the queue at time t; and m_t represent the number of customers of class n_t behind the customer in the queue at time t. Also let $\star$ represent an absorbing state, representing the state where that customer has finished service and left the system. Let Z denote set of all states $\underline{\mathbf{z}}_t$ and $\star$.

Then the rates of change between $\underline{\mathbf{z}}_t$ and $\underline{\mathbf{z}}_{t+1}$ are given by Equation 3, where $\underline{\delta} = \underline{\mathbf{z}}_{t+1} - \underline{\mathbf{z}}_t$,

$$q_{\underline{\mathbf{z}}_t, \underline{\mathbf{z}}_{t+1}} = \begin{cases} \mu_n & \text{if } z_{t+1} = \star \text{ and } \sum_{k \le n} z_{k,t} < c \\ \lambda_n & \text{if } \delta_K = 1 \text{ and } \delta_i = 0\ \forall\ i \neq K \\ \lambda_k & \text{if } \delta_k = 1 \text{ and } \delta_i = 0\ \forall\ i \neq k \text{ and } k \neq n \\ A_{k,n,t}\mu_k & \text{if } \delta_k = -1 \text{ and } \delta_i = 0\ \forall\ i \neq k \text{ and } k < K \\ \tilde{A}_{n,t}\mu_n & \text{if } \delta_K = -1 \text{ and } \delta_i = 0\ \forall\ i \neq K \\ (z_{k_0,t} - A_{k_0,n,t})\theta_{k_0,k_1} & \text{if } \delta_{k_0} = -1 \text{ and } \delta_{k_1} = 1 \text{ and } \delta_i = 0\ \forall\ i \neq k_0, k_1 \text{ and } k_0 < K \text{ and } k_1 \neq n, K, K+1 \\ (z_{K,t} - \tilde{A}_{n,t})\theta_{n,k} & \text{if } \delta_K = -1 \text{ and } \delta_k = 1 \text{ and } \delta_i = 0\ \forall\ i \neq k, n \text{ and } k < K \\ (z_{k,t} - A_{k,n,t})\theta_{k,n} & \text{if } \delta_k = -1 \text{ and } \delta_K = 1 \text{ and } \delta_i = 0\ \forall\ i \neq k, K \\ \theta_{n,k} & \text{if } \delta_n = z_{K,t} \text{ and } \delta_K = -z_{K,t} \text{ and } \delta_{K+1} = n - k \text{ and } \delta_i = 0 \text{ otherwise, and } \sum_{k \le n} z_{k,t} < c \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

and $A_{k,n,t}$ and $\tilde{A}_{n,t}$, representing the number of customers of class k currently in service, are given by Equations 4 and 5.

$$A_{k,n,t} = \begin{cases} \min\left(c, \sum_{i \le k} z_{i,t}\right) - \min\left(c \sum_{i<k} z_{i,t}\right) & \text{if } k \le n \\ \min\left(c, \sum_{i \le k} z_{i,t} + 1 + z_{K,t}\right) - \min\left(c \sum_{i<k} z_{i,t} + 1 + z_{K,t}\right) & \text{if } n < k < K \end{cases} \tag{4}$$

$$\tilde{A}_{n,t} = \min\left(c, \sum_{i \le n} z_{i,t} + 1 + z_{K,t}\right) - \min\left(c, \sum_{i \le n} z_{i,t} + 1\right) \tag{5}$$

Let $a_{\underline{\mathbf{z}}}$ denote the expected time to absorption from state $\underline{\mathbf{z}} \in Z$.

5.2.1 Mean sojourn time calculation

The expected time to absorption can be calculated from each state. Customers arrive in all states where $z_K = 0$, and their class can be determined by n. First define $\tilde{Z} = \{\underline{\mathbf{z}} \in Z \setminus \{\star\} \mid z_K = m = 0\} \subset Z$ as the set of all states where the newly arriving customer can can arrive. Let $c : \tilde{Z} \to S$ be a map between states in $\tilde{Z}$ and S, given in Equation 6.

$$c\left(\underline{\mathbf{z}} = (z_0, z_1, \ldots, z_{K-1}, m, n)\right) = (z_0, z_1, \ldots, z_{K-1}) \tag{6}$$

Note that c is a surjective map, but not injective. In fact, for every element $\underline{\mathbf{s}} \in S$ exactly K states in $\tilde{Z}$ map to it. These correspond to states at which each of the K classes of customer can arrive. In each of

these states, the probability of a customer from class k arriving is $\frac{\lambda_k}{\sum_{i=0}^{K-1}\lambda_i}$. Therefore, we can combine this to get the overall mean sojourn time in Equation 7:

$$\overline{\Psi} = \sum_{\underline{\mathbf{z}}\in\tilde{Z}} \sum_{k=0}^{K-1} \frac{\lambda_k}{\sum_{i=0}^{K-1}\lambda_i} \pi_{c(\underline{\mathbf{z}})} a_{\underline{\mathbf{z}}} \tag{7}$$

Defining $\tilde{Z}_k = \{\underline{\mathbf{z}} \in \tilde{Z} \mid z_{K-1} = n = k\} \subset Z$ as the states that customers of class k arrive to, the mean sojourn times for customers who arrive as a given customer class k is given by Equation 8.

$$\Psi_k = \sum_{\underline{\mathbf{z}}\in\tilde{Z}_k} \pi_{c(\underline{\mathbf{z}})} a_{\underline{\mathbf{z}}} \tag{8}$$

6 Bounded Approximation

In order to analyse the above Markov chain models numerically, finite approximations can be made. Let $b \in \mathbb{N}$, and define the b-bounded version of the infinite queueing system described in Section 2, such that the maximum allowed number of customers of each priority class is b, and customer losses when that number is exceeded. The equivalent b-bounded Markov chains associated with this system are identical to those described in Sections 5.1 and 5.2 except with bounded state spaces $\underline{\mathbf{s}}_t = (s_{0,t}, s_{1,t}, \ldots, s_{K-1,t}) \in (0, 1, \ldots, b)^K$ and $\underline{\mathbf{z}}_t = (z_{0,t}, z_{1,t}, \ldots, z_{n,t} \ldots, z_{K-1,t}, m_t, n_t) \in (0, 1, \ldots, b)^{K+2}$ respectively. These Markov chains are finite and so stationary.

If the unbounded system is stationary, that is the system reaches steady state and has steady state probabilities $\underline{\pi}$, then the steady states of the b-bounded system, $\underline{\tilde{\pi}}$ is an approximation of $\underline{\pi}$. As b increases the probability of the number of customers of a particular customer class in the unbounded system exceeding b approaches zero as b increases. Therefore as b increases the b-bounded system becomes a better and better approximation of the unbounded system.

Choosing an appropriate value for b is a trade off between accuracy and model size, and so computational time. An inefficient way to choose b would be to sequentially build bounded models, increasing b each time, calculating the statistics of interest, and observing when the relationship between b and that statistic levels off. It would be more efficient to choose a b and be able to immediately measure if the accuracy is sufficient. We propose two measures, one for ergodic Markov chains (the model in Section 5.1), and one for absorbing Markov chains (the model in Section 5.2).

6.1 Accuracy measure for the ergodic Markov chain

Let $S_b = \{\underline{\mathbf{s}} \in S \mid b \in \underline{\mathbf{s}}\}$, the set of states that lie on the Markov chain boundary. We wish to choose b large enough that the boundary is irrelevant, that is that the Markov chain hardly ever reaches the boundary. Therefore we propose the *relative* probability of being at the boundary, $\mathcal{Q}(b)$, to be a measure of accuracy; if this is sufficiently small, then the bound b is large enough. This is given in Equation 9, it is the ratio of the probability of being at the boundary in the b-bounded system, and the probability of being at the boundary if every state was equally likely. This is necessary as the larger b is, the larger the state space is, meaning that the steady state probabilities are spread over more states and so are not comparable alone, whereas the relative probability of being at the boundary is comparable over different sizes of b.

$$\mathcal{Q}(b) = \frac{|S|}{|S_b|} \sum_{s\in S_b} \tilde{\pi}_s \tag{9}$$

To demonstrate the effect of b on $\mathcal{Q}(b)$ under different systems, consider the dynamic classes system with two customer classes, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{2}$, $c = 1$, $\mu_1 = \frac{1}{\rho}$, $\mu_2 = \frac{1}{\rho}$, $\theta_{12} = 1$, and $\theta_{21} = 1$; where $0 < \rho < 1$ is some given traffic intensity. Figure 4 shows the effect of b on $\mathcal{Q}(b)$ for this system, for different values of ρ. In all cases as b increases, $\mathcal{Q}(b)$ decreases, indicating greater accuracy of the bounded system. As expected, as ρ increases, we expect more customers in the queue, and so the boundary b needs to be much larger before it can be considered irrelevant.

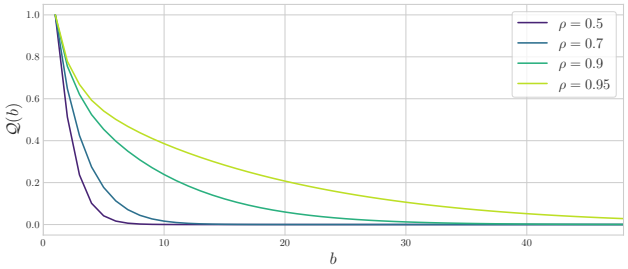


Figure 4: Demonstration of the effect of b on $\mathcal{Q}(b)$

6.2 Accuracy measure for the absorbing Markov chain

The above check isn't possible for absorbing Markov chains as they will not reach steady state, so another check is required. Define $h_{i,J}$ as the hitting probabilities of a set of states J from state i, that is, what is the probability of ever reaching any state in J when starting from state i. These are defined recursively by Equation 10.

$$h_{iJ} = \begin{cases} \sum_k p_{ik} h_{kJ} & \text{if } i \notin J \\ 1 & \text{if } i \in J \end{cases} \tag{10}$$

Relating this to the absorbing Markov chain described in Section 5.2, and letting $Z_b \subset Z$ be the set of boundary states such that $Z_b = \{\underline{\mathbf{z}} \in Z \mid b \in \underline{\mathbf{z}}\}$, then if a customer arrives to state i, the probability of that customer's state reaching the boundary is h_{iS_b}. Therefore we propose the probability of an arriving customer hitting the boundary, $\mathcal{P}(b)$, to be a measure of accuracy; if this is sufficiently small, then the bound b is large enough. This is calculated in a similar way to the mean sojourn time in Section 5.2.1, and given in Equation 11.

$$\mathcal{P}(b) = \sum_{\underline{\mathbf{z}} \in \tilde{Z}} \sum_{k=0}^{K-1} \frac{\lambda_k}{\sum_{i=0}^{K-1} \lambda_i} \pi_{c(\underline{\mathbf{z}})} h_{\underline{\mathbf{z}}, Z_b} \tag{11}$$

To demonstrate the effect of b on $\mathcal{P}(b)$ under different system, consider the same dynamic classes system with two customer classes used in the previous demonstration. Figure 5 shows the effect of b on $\mathcal{P}(b)$ for this system, for different values of ρ. Again, in all cases as b increases, $\mathcal{P}(b)$ decreases, indicating greater accuracy of the bounded system; and similarly as ρ increases the boundary b needs to be larger before it can be considered irrelevant.

7 Effect of Parameters on System Behaviour

In order to explore the effect that the parameters have on system behaviour, numerical experiments we run. Table 1 gives the ranges of value for which simulation experiments were run. These experiments will be

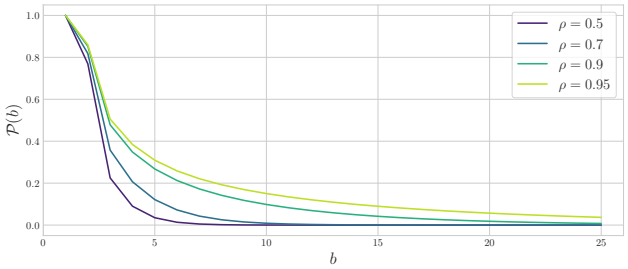


Figure 5: Demonstration of the effect of b on $\mathcal{P}(b)$

Parameters	Values
Λ_1	1
Λ_2	$1/2$, $2/3$, 1, $3/2$, 2
μ_1	$1/2$, 1, 2, 3, 4
μ_2	$1/2$, 1, 2, 3, 4
θ_{12}	$1/2$, $2/3$, 1, $3/2$, 2
θ_{21}	$1/2$, $2/3$, 1, $3/2$, 2
c	1, 2, 3

Table 1: Table of parameters explored, for two classes of customer. Simulations were run for 5000 time units, with a warm-up time of 200 and cool down time of 200 time units.

used to explore the models in the next subsections. For these simulations, upgraded individuals pre-empt individuals of a less prioritised class, and these pre-empted individuals have their service time re-sampled.

7.1 Existence of Stationary Distributions

A key difference between using a bounded approximation to analysing infinite Markov chains is the existence of stationary distributions. All bounded approximations have steady state distributions, even if the corresponding infinite Markov chain does not, leading to spurious results in these cases. Theorem 1 gives a naive check for the existence or non-existence of steady states, but does not cover all possibilities.

Theorem 1. *For an $M/M/c$ work conserving queue with K classes of customer, with arrival rate and service rate λ_k and μ_k for customers of class k, respectively; then*

1. *it will reach steady state if* $\rho_{\max} = \frac{\sum_i \lambda_i}{c \min_i \mu_i} < 1$,
2. *it will never reach steady state if* $\rho_{\min} = \frac{\sum_i \lambda_i}{c \max_i \mu_i} \geq 1$.

Note that this result does not assume any particular service discipline such as first-in-first-out or prioritised classes, but holds for any work conserving discipline.

Proof. The queue will reach steady state if the rate at which customers are added to the queue is less than the rate at which customers leave the queue. As arrivals are not state dependent, customers are added to the queue at a rate $\sum_i \lambda_i$ when in any state. The rate at which customers leave the queue is state dependent, depending on the service discipline.

We do not need to consider cases when there are less than c customers present, as here any new arrival will increase the rate at which customers leave the queue, as that arrival would enter service immediately. Considering the cases where there are c or more customers in the queue, there are two extreme cases, either:

1. all customers in service are of the class with the slowest service rate. In this case the rate at which customers leave the queue is $c \min_i \mu_i$, which is the slowest possible rate at which customers can leave the queue. If $\sum_i \lambda_i < c \min_i \mu_i$ then the rate at which customers enter the queue is smaller than the smallest possible rate at which customers leave the queue, and so will always be smaller than the rate at which customers leave the queue in all states. Therefore the system will reach steady state. Or:
2. all customers in service are of the class with the fastest service rate. In this case the rate at which customers leave the queue is $c \max_i \mu_i$, which is the fastest possible rate at which customers can leave the queue. If $\sum_i \lambda_i \geq c \max_i \mu_i$ then the rate at which customers enter the queue is greater than or equal to the largest possible rate at which customers leave the queue, and so will always be greater or equal to than the rate at which customers leave the queue in all states. Therefore the system cannot reach steady state.

□

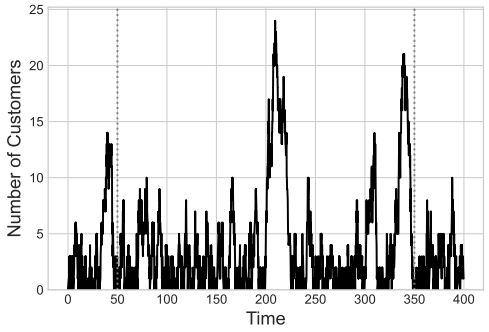
If $c \min_i \mu_i \leq \sum_i \lambda_i < c \max_i \mu_i$ then more investigation is needed. In the case of dynamic priority classes the class change matrix Θ may be significant. For example the service rate of customers of one class may be very slow, however if the rate at which customers leave that class is sufficiently large then that service rate may not have an effect. Alternatively if the rate at which customers of the other classes change to that class is large, then that slow service rate could be a bottleneck for the system.

We can however approximately test if a system reaches steady state or not using simulation. Consider the time series $x(t)$, representing the total number of customers in the system at time t. In Ciw, this can be empirically recorded using a state tracker object. If the system reaches steady state, then the $x(t)$ will be stochastic with non-increasing trend, therefore it would be a stationary time series. Conversely, if the system does not reach steady state, then $x(t)$ will be stochastic with increasing trend, therefore it would be a non-stationary time series.

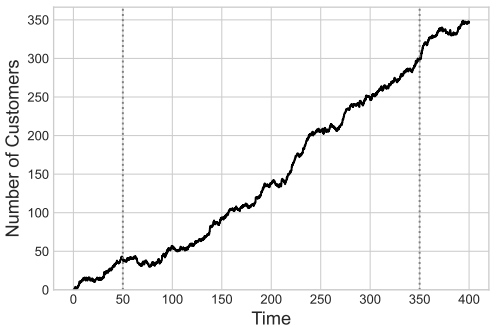
The Augmented Dicky-Fuller (ADF) test [2] tests for the non-stationarity of a stochastic time series, and so can be utilised here to test if a simulation has reached steady state or not. Note here that the time series $x(t)$ recorded by Ciw has irregular gaps (time stamps are the discrete time points where a customer arrives or leaves the system), and the ADF test requires regularly spaced time stamps; therefore the Traces library [13] is used to take regularly-spaced moving averages before the hypothesis test is undertaken.

Consider two examples:

- Example 1, that is guaranteed to reach steady state by Theorem 1: $\lambda_1 = 2$, $\lambda_2 = 1$, $\mu_1 = 4$, $\mu_2 = 4$, $\theta_{12} = 1$, $\theta_{21} = 1$, $c = 1$;
- Example 2, that is guaranteed not to reach steady state by Theorem 1: $\lambda_1 = 2$, $\lambda_2 = 1$, $\mu_1 = 1$, $\mu_2 = 1$, $\theta_{12} = 1$, $\theta_{21} = 1$, $c = 2$.



(a) State time series for Example 1.



(b) State time series for Example 2.

p-Value Threshold	Does Not Reach Steady state	Reaches Steady State
0.001	0.75%	100%
0.005	1.25%	100%
0.01	1.44%	100%
0.05	3.38%	100%
0.1	5.25%	100%

Table 2: ADF test p-Values from numerical experiments for parameters sets that do and do not reach steady state according to Theorem 1.

Figures 6a and 6b shows their state time series' $x(t)$ respectively. It is clear that the state time series for Example 1 is stationary, and the state time series for Example 2 is non-stationary and increasing. When performing the ADF test on these, Example 1 gives a p-value of 0.0004, rejecting the null hypothesis that the time series is non-stationary, while Example 2 gives a p-value of 0.9961, and the null hypothesis cannot be rejected.

This test is evidenced by the parameter sweep: Table 2 gives the percentage of experiments with an ADF p-value less than a given value, for all those experiments with parameters that do or do not reach steady state according to Theorem 1. We see that all those that reach steady state produce low p-values, while very small proportions of those that do not reach steady state have low p-values, as expected from hypothesis testing.

This evidences that the p-value of the ADF test can be confidently used to determine if a parameter set is able to reach steady state or not. Figure 7 shows the mean and maximum ADF p-values obtained across the parameter sweep for pairs of $\rho_{\min}$ or $\rho_{\max}$, and $\Delta\theta = \theta_{12} - \theta_{21}$. This gives a strong indication that $\Delta\theta$ has no effect on whether the system reaches steady state, and thus the rate at which customers change class is of no consequence. It also heavily indicates that the condition for reaching steady state is simply $\rho_{\max} < 1$.

7.2 Effect of Θ on Customer Experience

Figure 8 shows the effect of Θ on the number of customers and sojourn times, broken down by customer class. For each θ_{12}, θ_{21} pair, the plots show the average number of customers and average sojourn time obtained over all the experiments in the parameter sweep where $\rho_{\max} < 1$, that is those that reach steady state. We see that as θ_{21} increases, and θ_{12} decreases, that is more customers are being upgraded than downgraded, then:

E1. the number of Class 1 customers present increases,

E2. the number of Class 2 customers present decreases,

E3. this effect is more pronounced for customers of Class 1 than Class 2;

E4. the sojourn time of Class 1 customers increases,

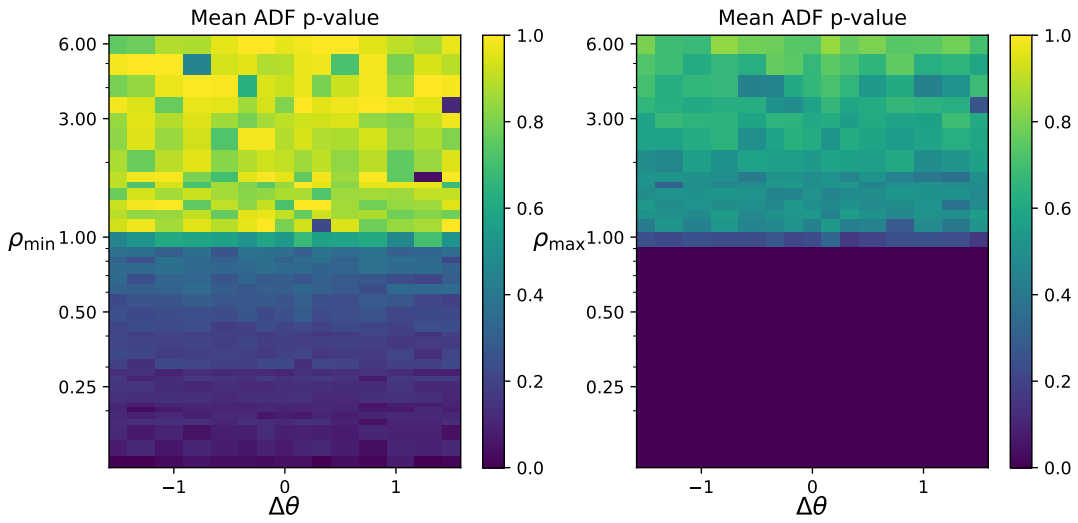
E5. the sojourn time of Class 2 customers decreases,

E6. this effect is more pronounced for customers of Class 2 than Class 1;

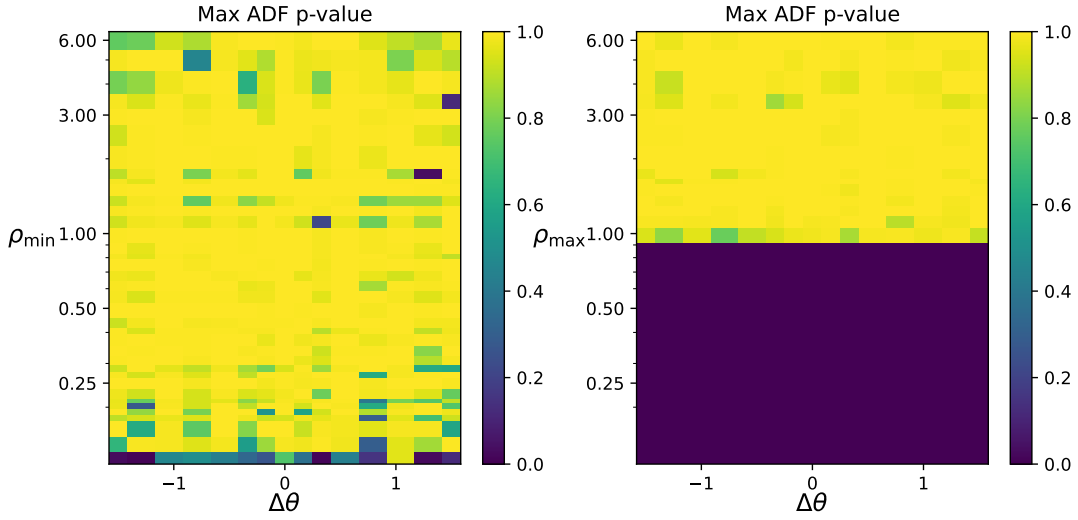
also, as θ_{21} increases, independent of θ_{12}

E7. the overall number of customers present increases.

Effects E1 and E2 come directly from the fact that increasing θ_{21} and decreasing θ_{12} upgrades more customers than are downgraded. Effect E7 is explained by the pre-emption: when an interrupted individual restarts service their service time is re-sampled, causing more work for the system and causing a build-up of the queue. Effect E4 is a consequence of effects E1 and E2, as there are more customers of Class 1 in the queue

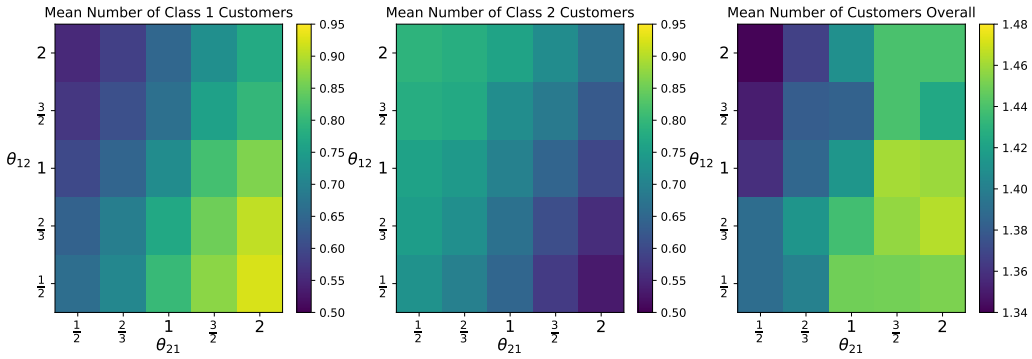


(a) Mean ADF.

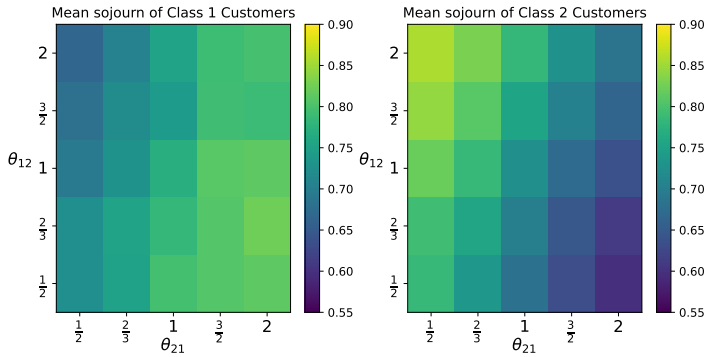


(b) Maximum ADF.

Figure 7: ADF p-values obtained across the parameter sweep, by $\rho_{\min}$ or $\rho_{\max}$, and $\Delta\theta = \theta_{12} - \theta_{21}$ pairs.



(a) Effect of Θ on customer numbers.



(b) Effect of Θ on sojourn times.

Figure 8: Effect of Θ on customer experience.

on average, their average sojourn time will increase, as stated by Little's Theorem [14]. Effect E5 however is due to upgrading customers at a higher rate, which effectively stops lower priority customers waiting a long time, as they will be upgraded before they end their service. This is more pronounced (effect E6), as it is accounting for effect E4 pushing the waiting times of customers of Class 2 up, allowing more time to upgrade them before they reach service.

8 Experimental Validation

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