Dynamic Priority Classes (?)

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1 The Model

Here we consider an $M/M/c$ queue with K classes of customer. Order and label the customer classes $0, 1, 2, \dots, K-1$, with customer classes with lower labels having priority over customer classes of higher labels. The index k will be used to represent customer classes. Let:

- λ_k be the arrival rate of customers of class k,
- μ_k be the service rate of customers of class k,
- θ_{k_i,k_j} be the rate at which customers of class k_i change to customers of class k_j.

Figure 1 shows an example with two classes of customer.

2 State Markov Chain Formulation

Let $\underline{\mathbf{s}}_t = (s_{0,t}, s_{1,t}, \dots, s_{K-1,t}) \in \mathbb{R}^K$ represent the state of the system at time step t, where $s_{k,t}$ represents the number of customers of class k present at time step t.

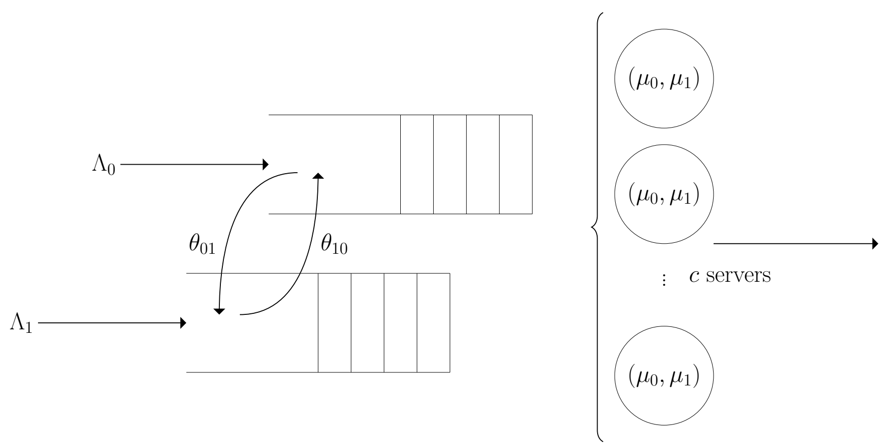

Figure 1: An example of a two-class priority queue.

Then the rates of change between $\underline{\mathbf{s}}_t$ and $\underline{\mathbf{s}}_{t+1}$ are given by Equation 1, where $\underline{\delta} = \underline{\mathbf{s}}_t - \underline{\mathbf{s}}_{t+1}$,

$$q_{\underline{\mathbf{s}}_t,\underline{\mathbf{s}}_{t+1}} = \begin{cases} \lambda_k & \text{if } \delta_k = 1 \text{ and } \delta_i = 0 \; \forall \; i \neq k \\ B_{k,t}\mu_k & \text{if } \delta_k = 1 \text{ and } \delta_i = 0 \; \forall \; i \neq k \text{ and } \sum_{i<k} s_{i,t} < c \\ (s_{k,t} - B_{k,t})\theta_{k_0,k_1} & \text{if } \delta_{k_0} = -1 \text{ and } \delta_{k_1} = 1 \text{ and } \delta_i = 0 \; \forall \; i \neq k_0, k_1 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

and $B_{k,t}$, representing the number of customers of class k currently in service at time step t, is given by Equation 2.

$$B_{k,t} = \min\left(c - \min\left(\sum_{i<k} s_{i,t}, c\right), s_{k,t}\right) \tag{2}$$

3 Sojourn Time Markov Chain Formulation

Let $\underline{\mathbf{z}}_t = (z_{0,t}, z_{1,t}, \dots, z_{n,t} \dots, z_{K-1,t}, b_t, n_t) \in \mathbb{R}^{K+2}$ represent the state of a particular customer time step t, where n_t represents that customer's class at time t; $z_{k,t} \; \forall \; k < n$ represents the number of customers of class k in front of the customer in the queue at time t; $z_{k,t} \; \forall \; n < k < K$ represents the number of customers of class k behind the customer in the queue at time t; and b_t represent the number of customers of class n_t behind the customer in the queue at time t. Also let $\star$ represent an absorbing state, representing the state where that customer has finished service and left the system.

Then the rates of change between $\underline{\mathbf{z}}_t$ and $\underline{\mathbf{z}}_{t+1}$ are given by Equation 3, where $\underline{\delta} = \underline{\mathbf{z}}_t - \underline{\mathbf{z}}_{t+1}$,

$$q_{\underline{\mathbf{z}}_t,\underline{\mathbf{z}}_{t+1}} = \begin{cases} \mu_n & \text{if } z_{t+1} = \star \text{ and } \sum_{k \leq n} z_{k,t} < c \\ \lambda_n & \text{if } \delta_K = 1 \text{ and } \delta_i = 0 \; \forall \; i \neq K \\ \lambda_k & \text{if } \delta_k = 1 \text{ and } \delta_i = 0 \; \forall \; i \neq k \text{ and } k \neq n \\ A_{k,n,t}\mu_k & \text{if } \delta_k = -1 \text{ and } \delta_i = 0 \; \forall \; i \neq k \text{ and } k < K \\ \tilde{A}_{n,t}\mu_n & \text{if } \delta_K = -1 \text{ and } \delta_i = 0 \; \forall \; i \neq K \\ (z_{k_0,t} - A_{k_0,n,t})\theta_{k_0,k_1} & \text{if } \delta_{k_0} = -1 \text{ and } \delta_{k_1} = 1 \text{ and } \delta_i = 0 \; \forall \; i \neq k_0, k_1 \text{ and } k_0 < K \text{ and } k_1 \neq n, K, K+1 \\ (z_{K,t} - \tilde{A}_{n,t})\theta_{n,k} & \text{if } \delta_K = -1 \text{ and } \delta_k = 1 \text{ and } \delta_i = 0 \; \forall \; i \neq k, n \text{ and } k < K \\ (z_{k,t} - A_{k,n,t})\theta_{k,n} & \text{if } \delta_k = -1 \text{ and } \delta_K = 1 \text{ and } \delta_i = 0 \; \forall \; i \neq k, K \\ \theta_{n,k} & \text{if } \delta_n = z_{K,t} \text{ and } \delta_K = -z_{K,t} \text{ and } \delta_{K+1} = n - k \text{ and } \delta_i = 0 \text{ otherwise, and } \sum_{k \leq n} z_{k,t} < c \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

and $A_{k,n,t}$ and $\tilde{A}_{n,t}$ are given by Equations 4 and 5.

$$A_{k,n,t} = \begin{cases} \min\left(c - \min\left(\sum_{i<k} z_{i,t}, c\right), z_{k,t}\right) & \text{if } k \leq n \\ \min\left(c - \min\left(1 + \sum_{i<k} z_{i,t}, c\right), z_{k,t}\right) & \text{if } n < k < K \end{cases} \tag{4}$$

$$\tilde{A}_{n,t} = \min\left(c - \min\left(1 + \sum_{i \leq n} z_{i,t}, c\right), z_{K,t}\right) \tag{5}$$

The expected time to absorption can be calculated from each state. Customers arrive in all states where $z_{K,t} = 0$, and their class can be determined by n. Combining these times to absorption with the state probabilities found in the previous section, the sojourn times for each customer class can be found.