Algebra, Calculus & Probability Refresher

MSc Maths Sills

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Algebra

Numbers

- Integers:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

- Rationals:

$$\mathbb{Q} = \left\{ a \mid \exists\ p, q \in \mathbb{Z} \text{ for which } a = \frac{p}{q} \right\}$$

- Real numbers:

$$\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Exponents

For $a, b \in \mathbb{R}^+$, $x, y \in \mathbb{R}$:

1. $a^x a^y = a^{x+y}$
2. $a^0 = 1$
3. $a^{-x} = \frac{1}{a^x}$
4. $(a^x)^y = a^{xy}$
5. $a^x b^x = (ab)^x$
6. $a^{\frac{1}{2}} = \sqrt{a}$

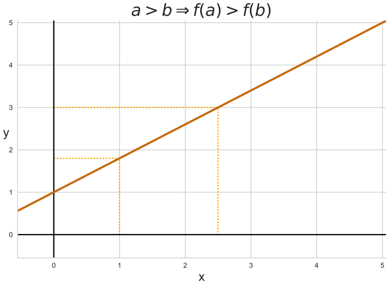
Logarithms

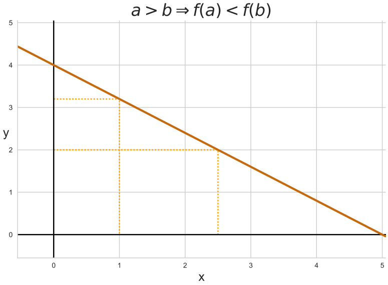
$$\log_a a^b = b$$

$$3^x = 81 \Leftrightarrow x = \log_3 81 = 4$$

Inequalities

Increasing & decreasing functions:





Inequalities

Solve:

$$5 - 2x \geq 13$$

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$$\begin{aligned} 5 - 2x - 5 &\geq 13 - 5 \\ -2x &\geq 8 \\ -2x \times \frac{1}{-2} &\leq 8 \times \frac{1}{-2} \quad \text{apply decreasing } f \\ x &\leq -4 \end{aligned}$$

Functions

Evaluate the function f when $a = 4$, $b = 2$, $c = -5$:

$$f(a, b, c) = \frac{a}{b} + 4c - a^2c + 10(a + b)$$

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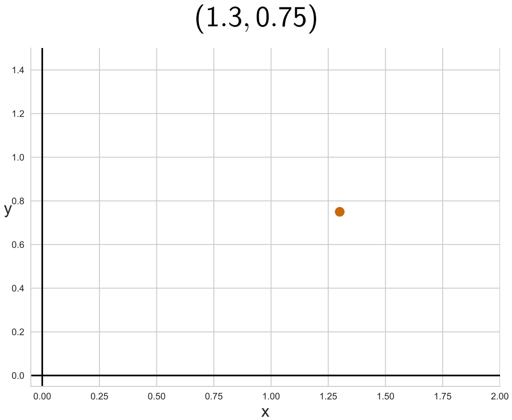
Solution:

$$\begin{aligned} f(4,2,-5) &= \frac{4}{2} + (4 \times -5) - \left(4^2 \times -5\right) + 10\,(4+2) \\ &= 2 - 20 - (16 \times -5) + 10\,(6) \\ &= 2 - 20 + 80 + 60 \\ &= 122 \end{aligned}$$

Coordinates in the plane

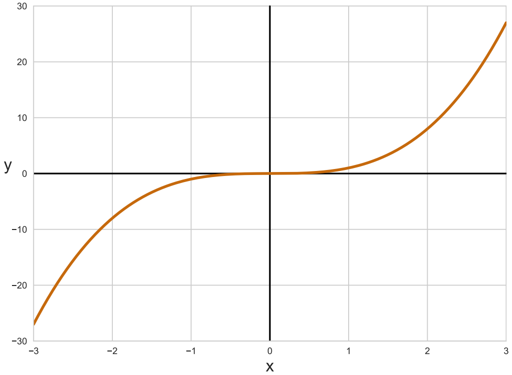
Right handed Cartesian axes:



For $P = (x, y)$, x/y is called the abscissa / ordinate of P.

Graphs

If x and y connected by an equation, then this relation can be represented by a curve or curves in the (x, y) plane which is known as the graph of the equation.



$$y = x^3$$

Graphs

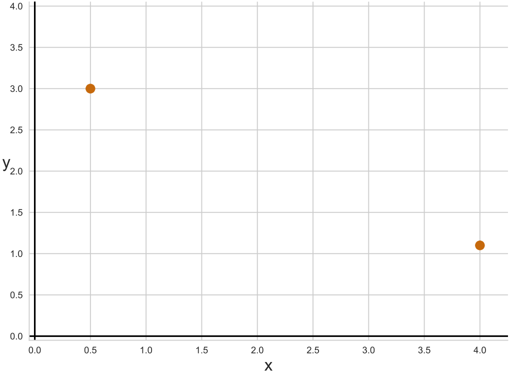
Graph of a straight line:

$$y = mx + c$$

- m is called the *gradient* of the line.
- c is called the *y-intercept* of the line.

Exercise

Find the equation for the line going through the points $\{(0.5, 3), (4, 1.1)\}$:



Solution

General form of $y = mx + c$ through $\{(x_1, y_1), (x_2, y_2)\}$:

$$\left.\begin{aligned} y_1 &= mx_1 + c \\ y_2 &= mx_2 + c \end{aligned}\right\} \Rightarrow m(x_1 - x_2) = y_1 - y_2$$

which gives:

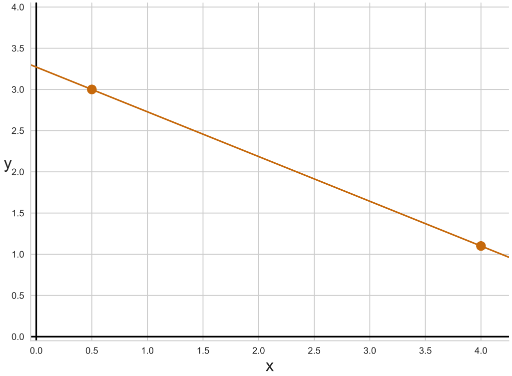
$$m = \tfrac{y_1 - y_2}{x_1 - x_2}$$

$$c = \tfrac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$$

Solution

So for $(x_1, y_1) = (0.5, 3)$ and $(x_2, y_2) = (4, 1.1)$ we have:

$$m = \frac{1.9}{-3.5} \approx -0.54$$
$$c = \frac{11.45}{3.5} \approx 3.27$$



Exercise

Where does the line $y = -0.54x + 3.27$ intersect the y-axis and the x-axis?

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Where does the line $y = -0.54x + 3.27$ intersect the y-axis and the x-axis?
This is equivalent to solving:

$$y = -0.54 \times 0 + 3.27$$

and

$$0 = -0.54x + 3.27$$

Solving Linear Equations

In linear equations are solved by multiplying or adding various constants.

$$
\begin{aligned}
0 = -0.54x + 3.27 \quad \Leftrightarrow 0 - 3.27 &= (-0.54x + 3.27) - 3.27 \\
\Leftrightarrow -3.27 &= -0.54x \\
\Leftrightarrow -3.27 \times \frac{1}{-0.54} &= 0.54x \times \frac{1}{-0.54} \\
\Leftrightarrow 6.06 &\approx x
\end{aligned}
$$

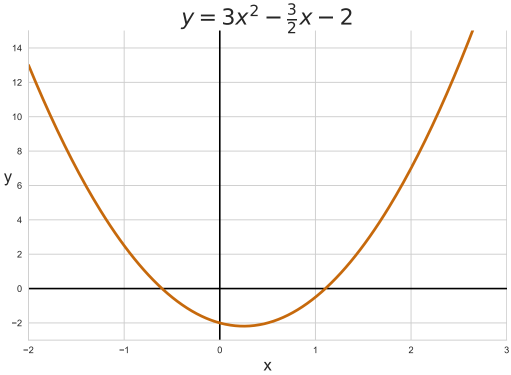
Quadratic

A "quadratic" is an expression of the form:

$$ax^2 + bx + c$$

- a is called the quadratic coefficient,
- b is called the linear coefficient,
- c is called the constant term or free term.

Quadratic



Solving a Quadratic Equation

General solution of the equation:

$$ax^2 + bx + c = 0$$

is given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exercise

Solve the equation:

$$3x^2 - \frac{3}{2}x - 2 = 0$$

Solution

From the previous formula we have:

$$
\begin{aligned}
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Leftrightarrow x &= \frac{\frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 4 \times 3 \times (-2)}}{2 \times 3} \\
\Leftrightarrow x &= \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4} + 24}}{6} \\
\Leftrightarrow x &= \frac{3}{12} \pm \frac{\frac{1}{2}\sqrt{9 + 96}}{6} \\
\Leftrightarrow x &= \frac{1}{4} \pm \frac{\sqrt{105}}{12}
\end{aligned}
$$

Exercise

Solve the equation:

$$4x^2 - 2x + 10 = 3$$

Solution

From the previous formula we have:

$$\begin{aligned} x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &\Leftrightarrow x = \frac{2 \pm \sqrt{2^2 - 4 \times 4 \times 7}}{2 \times 4} \\ &\Leftrightarrow x = \frac{2 \pm \sqrt{-108}}{8} \\ &\Leftrightarrow x = \frac{2 \pm \sqrt{i^2 108}}{8} \\ &\Leftrightarrow x = \frac{2 \pm i\sqrt{3 \times 36}}{8} \\ &\Leftrightarrow x = \frac{2 \pm 6i\sqrt{3}}{8} = \frac{1}{4} \pm \frac{3}{4} i\sqrt{3} \end{aligned}$$

Complex Numbers

$$i^2 = -1$$

Complex numbers:

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

If $z = a + ib$:

- a is the real part of z.
- b is the imaginary part of z.

Solving Systems of Equations

A system of equations is a collection of equations involving the same set of variables. For example:

$$3x + 2y = 1$$
$$2x - 2y = -2$$

Various techniques can be used to solve such a problem.

Solution

First equation gives:

$$3x + 2y = 1 \Rightarrow x = \frac{1-2y}{3}$$

Substituting in to second equation gives:

$$2\left(\frac{1-2y}{3}\right) - 2y = -2$$

which implies:

$$y = \frac{4}{5}$$

Substituting in to our expression for x we get:

$$x = -\frac{1}{5}$$

Shorthand notation

- Summation:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

- Multiplication:

$$\prod_{i=1}^{n} a_i = a_1 \times a_2 \times a_3 \times \cdots \times a_n$$

Examples

- Summation:

$$\sum_{i=1}^{4} i \times 2^i = 1 \times 2 + 2 \times 2^2 + 3^3 + 4 \times 2^4$$
$$= 2 + 8 + 3 \times 8 + 4 \times 16 = 98$$

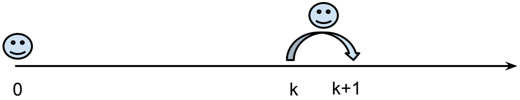
- Multiplication:

$$\prod_{k=1}^{3} k^2 = 1 \times 2^2 \times 3^2 = 36$$

Proof by Induction

Technique often used to prove algebraic relationships. Basic idea:

- Prove that something is true at the start.
- Prove that if something is true at point k then it is true at point $k + 1$.



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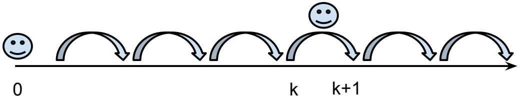


Exercise

Prove that:

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Solution

- True for $n = 0$?:

$$\sum_{i=0}^{0} i = 0 \text{ and } \frac{n(n+1)}{2} = 0$$

Solution

- True for $n = 0$?:

$$\sum_{i=0}^{0} i = 0 \quad \text{and} \quad \frac{n(n+1)}{2} = 0$$

- If true for $n = k$, true for $n = k + 1$?:

$$\begin{aligned} \sum_{i=0}^{k+1} i &= \sum_{i=0}^{k} i + k + 1 \\ &= \frac{k(k+1)}{2} + k + 1 \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

Infinite Sums

$$\sum_{k=0}^{\infty} a^k = \frac{a}{1-a}$$

$$\sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2}$$

$$\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$$

`https://en.wikipedia.org/wiki/List_of_mathematical_series`

Infinite Sums

$$S = \sum_{k=0}^{\infty} a^k$$
$$S = a^0 + a^1 + a^2 + a^3 + a^4 + \dots$$
$$aS = a^1 + a^2 + a^3 + a^4 + a^5 + \dots$$

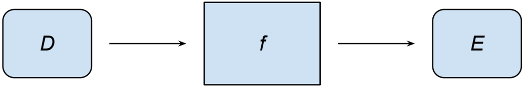
Consider $S - aS$:

$$S - aS = a^0$$
$$S - aS = 1$$
$$S(1 - a) = 1$$
$$S = \frac{1}{(1 - a)}$$

Calculus

Functions

A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.



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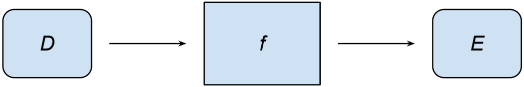
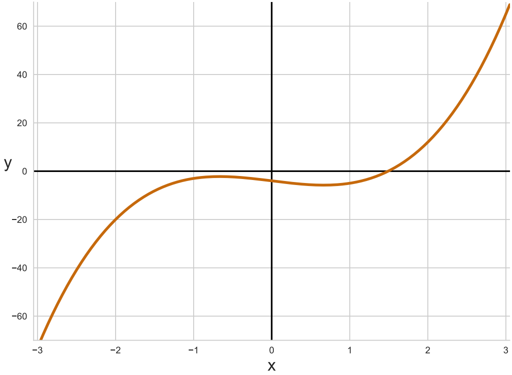


- We usually consider functions for which the sets D and E are sets of real numbers.
- The set D is called the domain of the function.
- The range of f is the set of all possible values of $f(x)$ as x varies throughout the domain.
- A symbol that represents an arbitrary number in the domain of a function f is call an independent variable.
- A symbol that represents a number in the range of f is called a dependent variable.

Example

$$f(x) = 3x^3 - 4x - 4$$

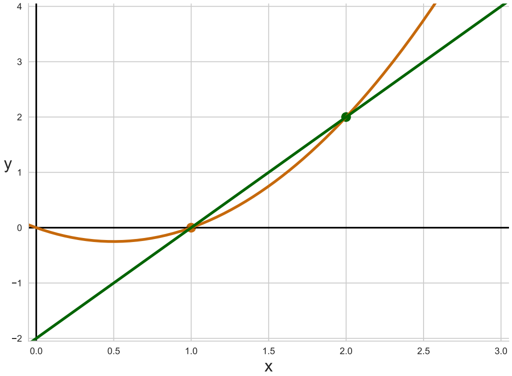


Tangent Curves

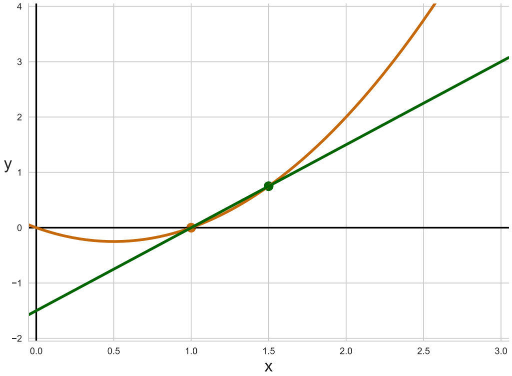
The tangent line to the curve $y = f(x)$ at the point $P = (a, f(a))$ is the line through P with gradient:

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Tangent Curves



Tangent Curves



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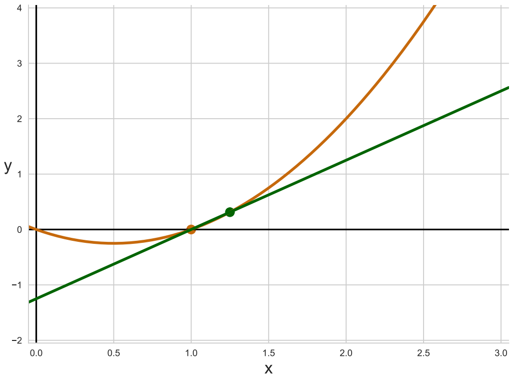

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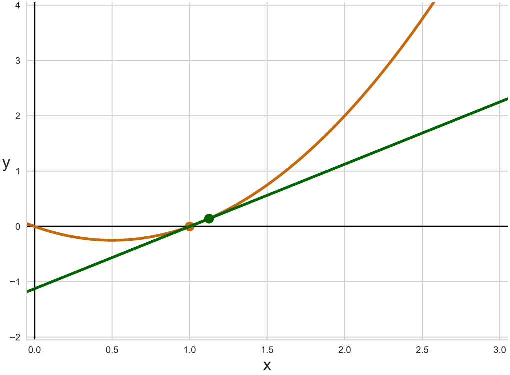

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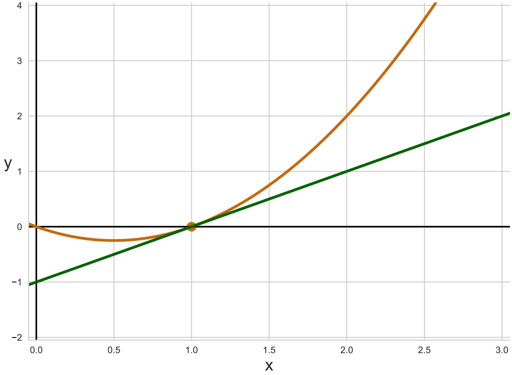

Derivative

The derivative of a function f at a number a, denoted by $f'(a)$ is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

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For polynomials:

$$f = x^n$$
$$\implies f' = nx^{n-1}$$

Exercise

Find the derivative of

$$f(x) = x^2 - 3x + 2$$

at $x = 6$.

Solution

$$
\begin{aligned}
f(x) &= x^2 && - 3x && + 2 \\
f(x) &= 1 \times x^2 && - 3 \times x^1 && + 2 \times x^0 \\
\\
f'(x) &= 1 \times 2 \times x^{2-1} && - 3 \times 1 \times x^{1-1} && + 2 \times 0 \times x^{0-1} \\
f'(x) &= 2x^1 && - 3x^0 && + 0 \\
f'(x) &= 2x && - 3 \\
\\
f'(6) &= 2 \times 6 && - 3 \\
f'(6) &= 9
\end{aligned}
$$

Rules of Differentiation

- The Power Rule:

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

- The Constant Multiple Rule:

$$\frac{d}{dx}\left(cf(x)\right) = c\frac{d}{dx}\left(f(x)\right)$$

- The Sum Rule:

$$(f+g)' = f' + g'$$

Rules of Differentiation

- The Product Rule:

$$(fg)' = f'g + fg'$$

- The Quotient Rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

The Chain Rule

If $f\left(g(x)\right) = f \circ g$:

$$(f \circ g)' = \left(f' \circ g\right) g'$$

Exercise

Differentiate $F(x) = \sqrt{x^2 + 1}$.

Solution

If we let $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ and $g(x) = x^2 + 1$ then we have:

$$f' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

and

$$g' = 2x$$

Using the Chain Rule we have:

$$\begin{aligned} F'(x) &= \left(\frac{1}{2\sqrt{x^2+1}}\right)(2x) \\ &= \frac{x}{\sqrt{x^2+1}} \end{aligned}$$

Table of Derivatives

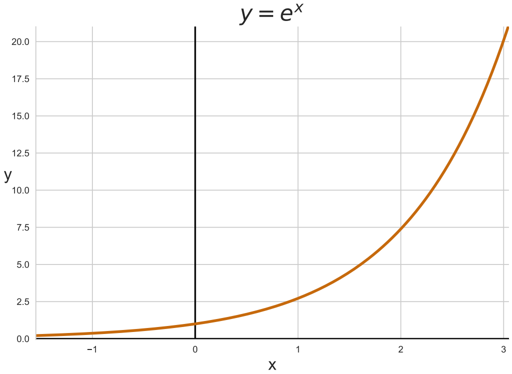
$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x)$$
$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

http://www.math.wustl.edu/~freiwald/131derivativetable.pdf

Natural Exponential Function

The mathematical constant e can be defined as the real number such that:

$$\frac{d}{dx}e^x = e^x$$

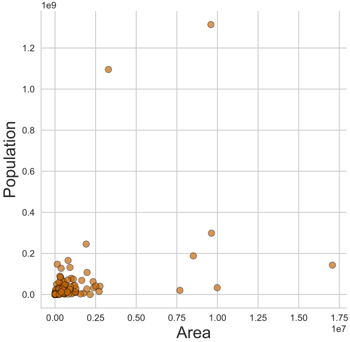


Natural Logarithm

$$\ln x = \log_e x$$

$$y = e^x \Leftrightarrow x = \ln y$$

A standard procedure used when analysing data is to transform with log. Here we compare populations and areas of the countries of the world:



Logarithms in Statistics

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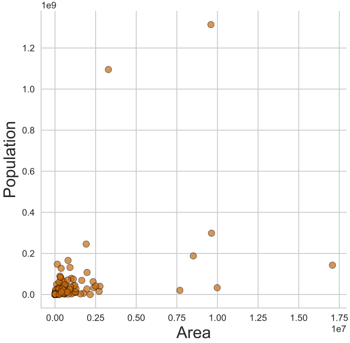
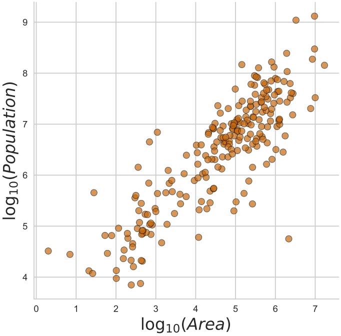
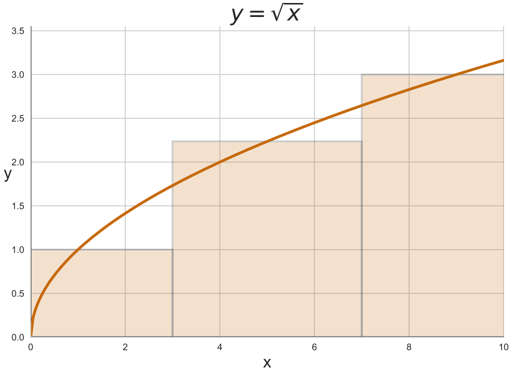




Area under a graph



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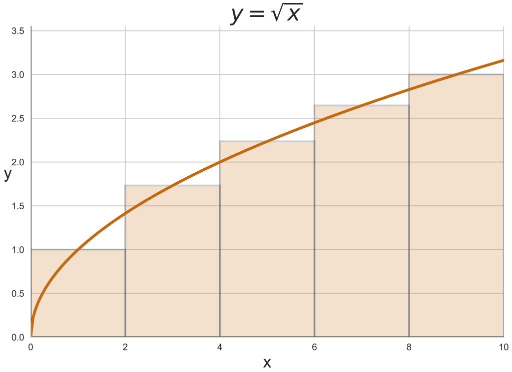

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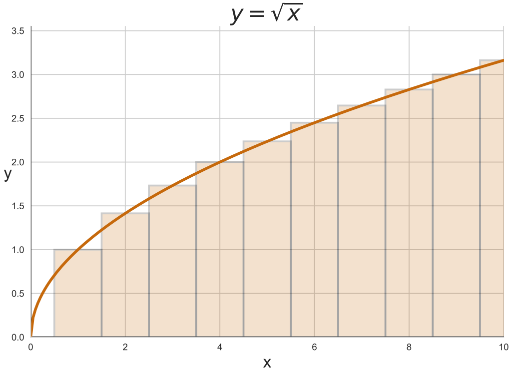

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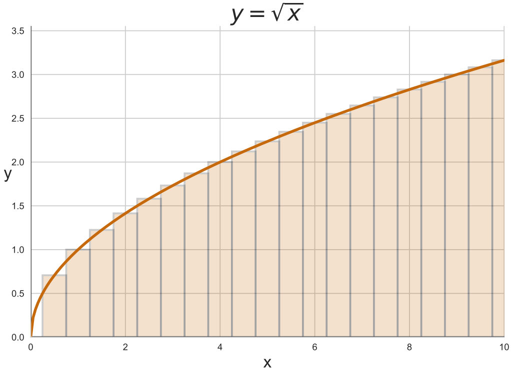

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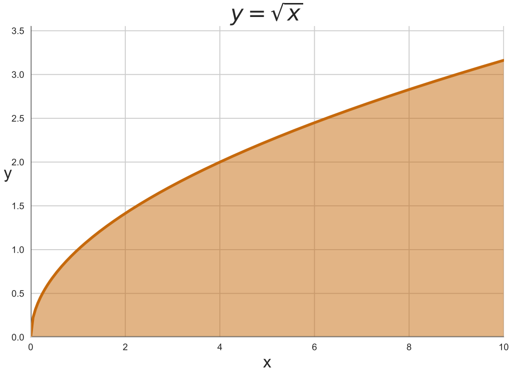

Fundamental Theorem of Calculus

Let f, F be continuous on $[a, b]$ then:

1. If $F(x) = \int_a^x f(t)dt$ then $F' = f$.
2. $\int_a^b f(x)dx = F(b) - F(a)$

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2. $\int_a^b f(x)dx = F(b) - F(a)$

$$\int f(x)dx = F(x) \text{ means } \frac{d}{dx}F = f$$

Tables of Indefinite Integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

$$\int cf(x)dx = c\int f(x)dx$$

$$\int (f(x) + g(x))\,dx = \int f(x)dx + \int g(x)dx$$

$$\vdots$$

http://integral-table.com/downloads/single-page-integral-table.pdf

Exercise

Calculate:

$$\int x^2 + \sin(x) dx$$

Exercise

Calculate:

$$\int x^2 + \sin(x)dx$$

$$\int x^2 + \sin(x)dx = \frac{x^3}{3} - \cos(x) + C$$

Integration by Parts

If $u = f(x)$ and $v = g(x)$:

$$\int u dv = uv - \int v du$$

Exercise

Calculate:

$$\int x \cos(x) dx$$

Solution

Letting $u = x$ and $dv = \cos(x)dx$ we have $du = dx$ and $v = \sin(x)$, thus:

$$\begin{aligned}\int x\cos(x)dx &= \int udv = uv - vdu \\ &= x\sin(x) - \int \sin(x)dx \\ &= x\sin(x) + \cos(x) + C\end{aligned}$$

The Substitution Rule

If $u = g(x)$ then:

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Exercise

Calculate:

$$\int x^3 \cos(x^4 + 2)dx$$

Solution

Letting $u = x^4 + 2$, we have $du = 4x^3dx$, thus:

$$\begin{aligned}
\int x^3 \cos(x^4+2)dx &= \int \cos(u)\frac{1}{4}du \\
&= \frac{1}{4}\int \cos(u)du \\
&= \frac{1}{4}\sin(u) + C \\
&= \frac{1}{4}\sin(x^4+2) + C
\end{aligned}$$

Probability

Random Variables

In trials where the outcome is numerical, the outcomes are values of random variables.

Example: A coin is spun 3 times, how many heads appear? Denote the random variable associated with the number of heads by X. Denote the sample space by S_X then:

$$S_X = \{0, 1, 2, 3\}$$

Discrete Probability Distributions

For a probability distribution $P(X = x_i) = p_i$:

- $0 \leq p_i \leq 1$
- All probabilities sum to 1:
  - $\sum_{i=1}^{n} p_i = 1$ if X has n possible outcomes
  - $\sum_{i=1}^{\infty} p_i = 1$ if X has a countably infinite set of outcomes

Exercise

Write down the state space and probability distribution for the random variable X associated with the rolling of a six sided die.

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$$S_X = \{1, 2, 3, 4, 5, 6\}$$

x_i	1	2	3	4	5	6
$P(X = x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Cumulative Distribution for Discrete Random Variables

For $P(X = x_i) = p_i$ the cumulative distribution $F(x) = P(X \leq x)$:

$$F(x) = \sum_{i=1}^{x} P(X = x_i)$$

Exercise

Write down the cumulative probability distribution for the random variable X associated with the rolling of a six sided dice.

Exercise

Write down the cumulative probability distribution for the random variable X associated with the rolling of a six sided dice.

x_i	1	2	3	4	5	6
$P(X = x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$F(x_i)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1

Mean and Variance

Mean / Average / Expected Value:

$$E(X) = \sum_{i=1}^{n} x_i p_i$$

Variance:

$$Var(X) = \sum_{i=1}^{n} (x_i - E(X))^2 \, p_i$$

Exercise

Calculate the mean and variance for the random variable X associated with the rolling of a six sided dice.

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Calculate the mean and variance for the random variable X associated with the rolling of a six sided dice.

Mean:

$$
\begin{aligned}
E(X) &= \frac{0}{6} + \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} \\
&= 3.5
\end{aligned}
$$

Variance:

$$
\begin{aligned}
Var(X) &= \frac{(0-3.5)^2}{6} + \frac{(1-3.5)^2}{6} + \frac{(2-3.5)^2}{6} + \frac{(3-3.5)^2}{6} \\
&+ \frac{(4-3.5)^2}{6} + \frac{(5-3.5)^2}{6} + \frac{(6-3.5)^2}{6} \\
&\approx 2.9
\end{aligned}
$$

Continuous Random Variables

The random variable X is the time from $t = 0$ until a light bulb fails. X is a continuous random variable, defined for the continuous variable $t \geq 0$, and is not a countable list of values.

Define a probability density function $f(x)$ over $\mathbb{R}$:

- $f(x) \geq 0$
- $\int_{-\infty}^{\infty} f(x)dx = 1$
- for any $x_1 < x_2$:

$$P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(x)dx$$

Continuous CDF

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du$$

Mean and Variance of Continuous Random Variables

Mean:

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

Variance:

$$Var(X) = \int_{-\infty}^{\infty} (x - E(X))^2 f(x) dx$$

Exercise

Find the mean of the negative exponential distribution:

$$f(x) = \lambda e^{-\lambda x} \text{ defined for } 0 < x < \infty$$

Solution

$$
\begin{aligned}
E(X) &= \int_0^\infty x f(x) dx \\
&= \lambda \int_0^\infty x e^{-\lambda x} dx \\
&= \lambda \left(uv - \int v du \right)_0^\infty \\
&= \lambda \left[\frac{x}{\lambda} e^{-\lambda x} + \int \frac{1}{\lambda} e^{-\lambda x} \right]_0^\infty \\
&= \lambda \left[\frac{x}{\lambda} e^{-\lambda x} + \frac{1}{\lambda^2} e^{-\lambda x} \right]_0^\infty \\
&= \left[x e^{-\lambda x} + \frac{1}{\lambda} e^{-\lambda x} \right]_0^\infty \\
&= 0 - 0 - 0 + \frac{1}{\lambda} = \frac{1}{\lambda}
\end{aligned}
$$

Support Material

- https://intranet.cardiff.ac.uk/students/your-study/study-skills/maths-support
- https://github.com/drvinceknight/MSc_week_0/wiki
- http://www.geraintianpalmer.org.uk/teaching/