Week 0

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Overview

Algebra

Wikipedia:

"Algebra is the branch of mathematics concerning the study of the rules of operations and relations, and the constructions and concepts arising from them, including terms, polynomials, equations and algebraic structures."

Numbers

- Integers:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

- Rationals:

$$\mathbb{Q} = \left\{ a \;\middle|\; \exists\ p, q \in \mathbb{Z} \text{ for which } a = \frac{p}{q} \right\}$$

- Real numbers:

$$\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

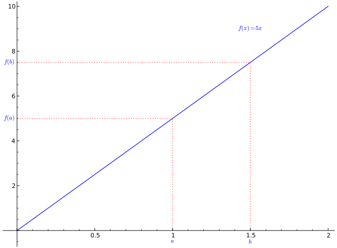
Exponents

If a and b are any positive real numbers and x and y are any real numbers then:

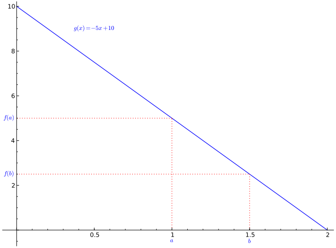
1. $a^x a^y = a^{x+y}$
2. $a^0 = 1$
3. $a^{-x} = \frac{1}{a^x}$
4. $(a^x)^y = a^{xy}$
5. $a^x b^x = (ab)^x$

Inequalities

When solving inequalities it is important to keep in mind whether or not the operation we are using is an *increasing* or a *decreasing* one.



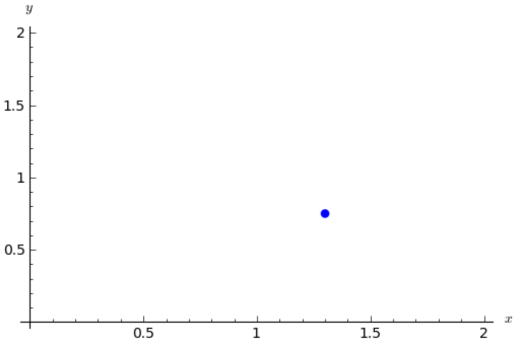
$$a < b \Rightarrow f(a) < f(b)$$



$$a < b \Rightarrow g(a) > g(b)$$

Coordinates in the plane

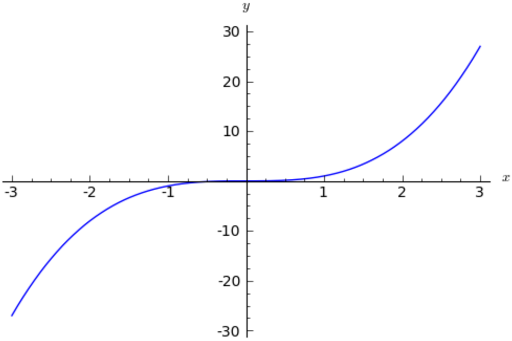
The location of a point in a plane can be specified in terms of right handed cartesian axes:



The point $(1.3, .75)$ is plotted above.
In general for a point $P = (x, y)$, x/y is called the abscissa/ordinate of P.

Graphs

If x and y connected by an equation, then this relation can be represented by a curve or curves in the (x, y) plance which is known as the graph of the equation.



The equation $y = x^3$ is plotted above.

Graphs

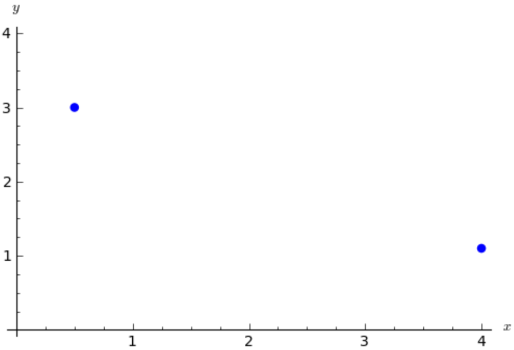
One particular type of graph is the graph of a line:

$$y = mx + b$$

- m is called the *gradient* of the line.
- b is called the *y-intercept* of the line.

Exercise

Find the equation for the line going through the points $\{(.5, 3), (4, 1.1)\}$:



Solution

General form of line $y = mx + b$ through $\{(x_1, y_1), (x_2, y_2)\}$ can be obtained:

$$\left.\begin{array}{l} y_1 = mx_1 + b \\ y_2 = mx_2 + b \end{array}\right\} \Rightarrow \left.\begin{array}{r} m(x_1 - x_2) = y_1 - y_2 \\ c(a_2 - a_1) = a_2 b_1 - a_1 b_1 \end{array}\right\}$$

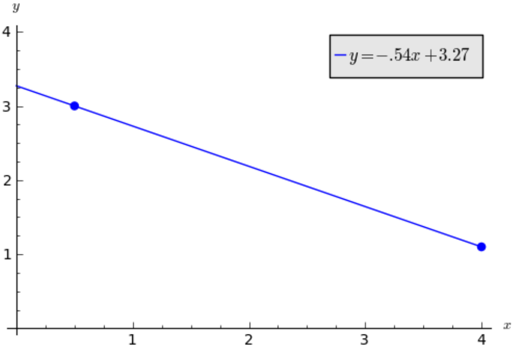
which gives:

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$
$$c = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$$

Solution

So for $(x_1, y_1) = (.5, 3)$ and $(x_2, y_2) = (4, 1.1)$ we have:

$$m = \frac{1.9}{-3.5} \approx -.54$$
$$c = \frac{11.45}{3.5} \approx 3.27$$



Exercise

Where does the line $y = -.54x + 3.27$ intersect the y-axis and the x-axis?

Exercise

Where does the line $y = -.54x + 3.27$ intersect the y-axis and the x-axis?
This is equivalent to solving:

$$y = -.54 \times 0 + 3.27$$

and

$$0 = -.54x + 3.27$$

Solving Linear Equations

In general equations of the form:

$$y = mx + b$$

are solved by muliplying or adding various constants.

$$0 = -.54x + 3.27 \Leftrightarrow 0 - 3.27 = (-.54x + 3.27) - 3.27$$
$$-3.27 = -.54x \Leftrightarrow -3.27 \times \frac{1}{-.54} = .54x \times \frac{1}{-.54}$$
$$x \approx 6.06$$

Solving Linear Equations

In general equations of the form:

$$y = mx + b$$

are solved by muliplying or adding various constants.

$$y = mx + b \Leftrightarrow y - b = (mx + b) - b$$
$$y - b = mx \Leftrightarrow (y - b) \times \frac{1}{m} = mx \times \frac{1}{m}$$
$$x = \frac{y - b}{m}$$

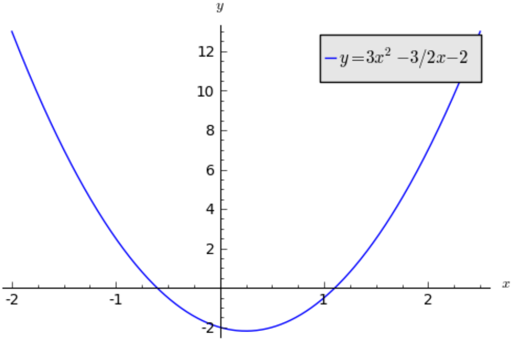
Quadratic

A “quadratic” is an expression of the form:

$$ax^2 + bx + c$$

- a is called the quadratic coefficient,
- b is called the linear coefficient,
- c is called the constant term or free term.

Quadratic



Solving a Quadratic Equation

General solution of the equation:

$$ax^2 + bx + c = 0$$

is given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exercise

Solve the equation:

$$3x^2 - \frac{3}{2}x - 2 = 0$$

Solution

From the previous formula we have:

$$x = \frac{\frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 4 \times 3 \times (-2)}}{2 \times 3} \Leftrightarrow x = \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4} + 24}}{6}$$

$$x = \frac{3}{12} \pm \frac{\frac{1}{2}\sqrt{9 + 96}}{6} \Leftrightarrow x = \frac{1}{4} \pm \frac{\sqrt{105}}{12}$$

Exercise

Solve the equation:

$$4x^2 - 2x + 10 = 3$$

Solution

From the previous formula we have:

$$x = \frac{2 \pm \sqrt{2^2 - 4 \times 4 \times 7}}{2 \times 4} \Leftrightarrow x = \frac{2 \pm \sqrt{-108}}{8}$$

$$x = \frac{2 \pm \sqrt{i^2 108}}{8} \Leftrightarrow x = \frac{2 \pm i\sqrt{3 \times 36}}{8}$$

$$x = \frac{2 \pm 6i\sqrt{3}}{8} = \frac{1}{4} \pm \frac{3}{4} i\sqrt{3}$$

Very brief description of Complex Numbers

$$i^2 = -1$$

Complex numbers:

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

If $z = a + ib$:

- a is the real part of z.
- b is the imaginary part of z.

Solving Systems of Equations

A system of equations is a collection of equations involving the same set of variables. For example:

$$
\begin{aligned}
3x + 2y =& 1 \\
2x - 2y =& -2
\end{aligned}
$$

Various techniques can be used to solve such a problem.

Elimination of Variables

- Use first equation to obtain expression for first variable as a function of other variables.
- Substitute and use second equation to obtain expression for second variable as a function of other variables.
- etc...

Exercise

Solve:

$$
\begin{aligned}
3x + 2y &= 1 \\
2x - 2y &= -2
\end{aligned}
$$

Solution

First equation gives:

$$3x + 2y = 1 \Rightarrow x = \frac{1 - 2y}{3}$$

Substituting in to second equation gives:

$$2\left(\frac{1 - 2y}{3}\right) - 2y = -2$$

which implies:

$$y = \frac{4}{5}$$

Substituting in to our expression for x we get:

$$x = -\frac{1}{5}$$

Shorthand notation

- Summation:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

- Multiplication:

$$\prod_{i=1}^{n} a_i = a_1 \times a_2 \times a_3 \times \cdots \times a_n$$

Examples

- Summation:

$$\sum_{i=1}^{4} i \times 2^i = 1 \times 2 + 2 \times 2^2 + 3^3 + 4 \times 2^4 = 2 + 8 + 3 \times 8 + 4 \times 16 = 98$$

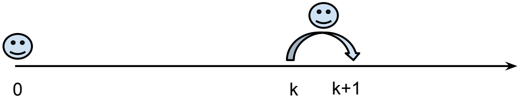
- Multiplication:

$$\prod_{k=1}^{3} k^2 = 1 \times 2^2 \times 3^2 = 36$$

Proof by Induction

Technique often used to prove algebraic relationships. Basic idea:

- Prove that something is true at the start.
- Prove that if something is true at point k then it is true at point $k + 1$.



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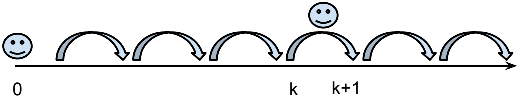

Exercise

Prove that:

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Solution

- True for $n = 0$?:

$$\sum_{i=0}^{0} i = 0 \text{ and } \frac{n(n+1)}{2} = 0$$

- If true for $n = k$, true for $n = k + 1$?:

$$\sum_{i=0}^{k+1} i = \sum_{i=0}^{k} i + k + 1 = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$$

Calculus

Wikipedia:

"Calculus is the study of change, in the same way that geometry is the study of shape and algebra is the study of operations and their application to solving equations"

Functions

A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

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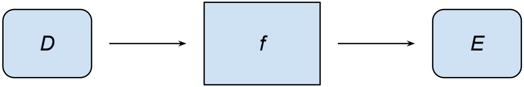

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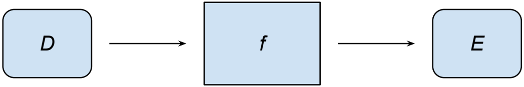
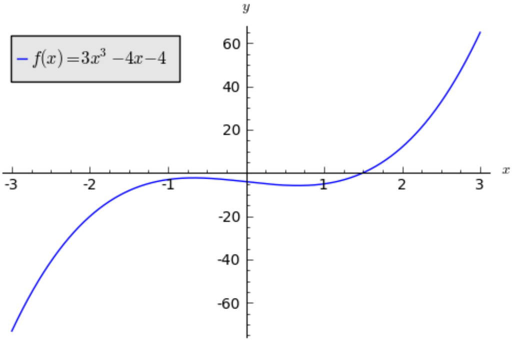


- We usually consider functions for which the sets D and E are sets of real numbers.
- The set D is called the domain of the function.
- The range of f is the set of all possible values of $f(x)$ as x varies throughout the domain.
- A symbol that represents an arbitrary number in the domain of a function f is call an independent variable.
- A symbol that represents a number in the range of f is called a dependent variable.

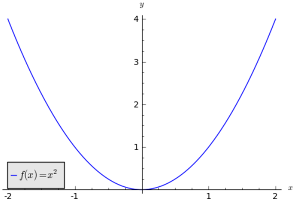
Example

The function $f(x) = 3x^3 - 4x - 4$ is plotted below:

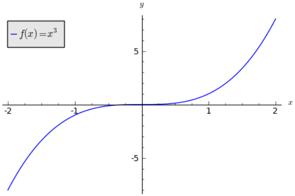


Even and Odd Functions

- If a function f satisfies $f(-x) = f(x)$ for all x in its domain then f is called an even function:



- If a function f satisfies $f(-x) = -f(x)$ for all x in its domain then f is called an odd function:

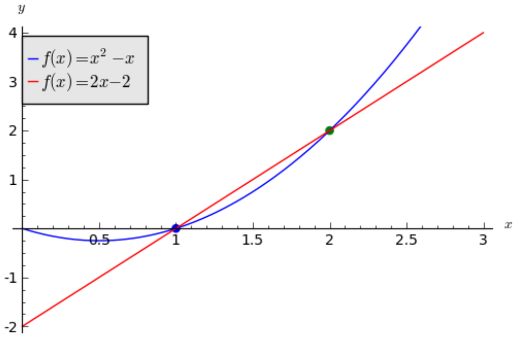


Tangent Curves

The tangent line to the curve $y = f(x)$ at the point $P = (a, f(a))$ is the line through P with gradient:

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Tangent Curves



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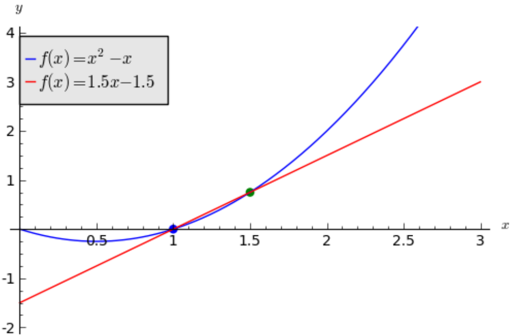

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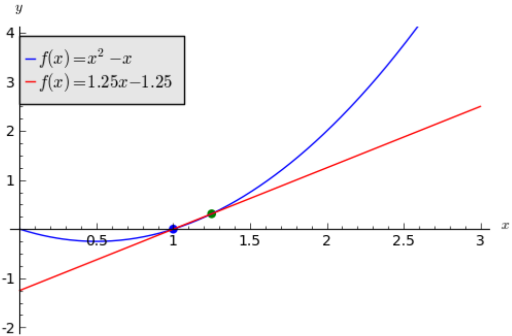

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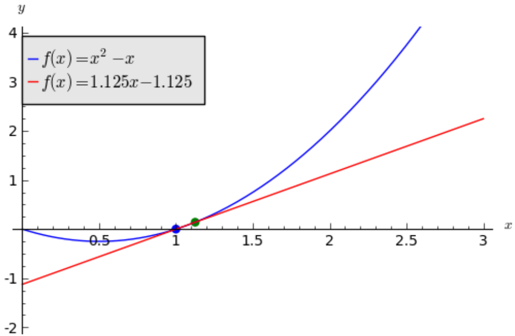

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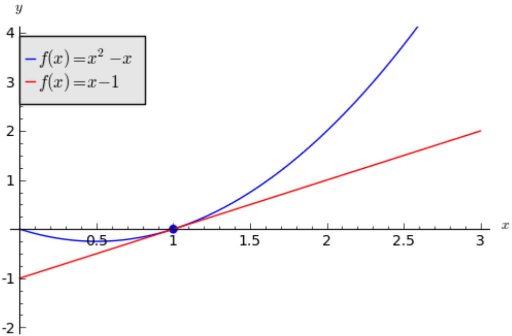


Derivative

The derivative of a function f at a number a, denoted by $f'(a)$ is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Exercise

Find the derivative of the function $f(x) = x^2 - 3x + 2$ at the number a.

Solution

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{\big((a+h)^2 - 3(a+h) + 2\big) - \big(a^2 - 3a + 2\big) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - 3a - 3h + 2 - a^2 + 3a - 2}{h} \\
&= \lim_{h\to 0} \frac{2ah + h^2 - 3h}{h} = \lim_{h\to 0} 2a + h - 3 \\
&= 2a - 3
\end{aligned}
$$

Rules of Differentiation

- The Power Rule:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

- The Constant Multiple Rule:

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x))$$

- The Sum Rule:

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

Rules of Differentiation

- The Product Rule:

$$\frac{d}{dx}\left(f(x)g(x)\right) = g(x)\frac{d}{dx}\left(f(x)\right) + f(x)\frac{d}{dx}\left(g(x)\right)$$

- The Quotient Rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}\left(f(x)\right) - f(x)\frac{d}{dx}\left(g(x)\right)}{g(x)^2}$$

The Chain Rule

If g and f are two functions with derivatives g' and f' respectively then:

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))g'(x)$$

Exercise

Differentiate $F(x) = \sqrt{x^2 + 1}$.

Solution

If we let $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$ then we have:

$$\frac{d}{dx}f(x) = \frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}} \text{ and } \frac{d}{dx}g(x) = 2x$$

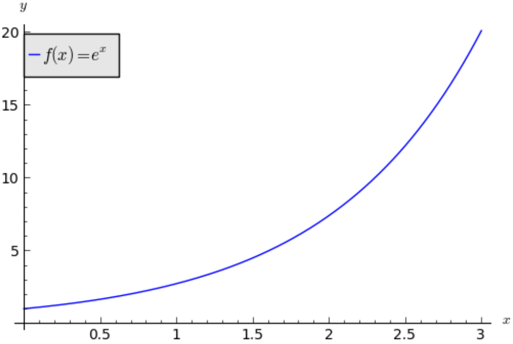
Using the Chain Rule we have:

$$\frac{d}{dx}F(x) = \frac{1}{2\sqrt{x^2+1}}2x = \frac{x}{\sqrt{x^2+1}}$$

Natural Exponential Function

The mathematical constant e can be defined as the real number such that:

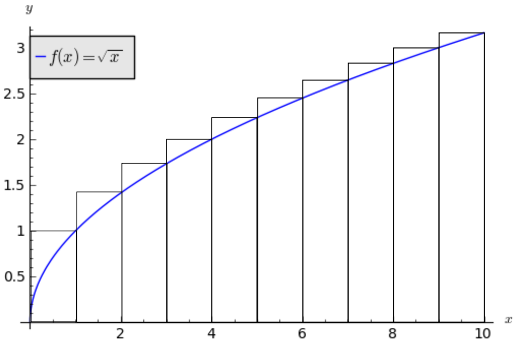
$$\frac{d}{dx}e^x = e^x$$



Trigonometric Functions

$$
\begin{aligned}
\frac{d}{dx}\sin(x) &= \cos(x) & \frac{d}{dx}\csc(x) &= -\csc(x)\cot(x) \\
\frac{d}{dx}\cos(x) &= -\sin(x) & \frac{d}{dx}\sec(x) &= \sec(x)\tan(x) \\
\frac{d}{dx}\tan(x) &= \sec^2(x) & \frac{d}{dx}\cot(x) &= -\csc^2(x)
\end{aligned}
$$

Area under a graph



Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ then:

1. If $g(x) = \int_a^x f(t)dt$ then $\frac{d}{dx}g = f$.
2. $\int_a^b f(x)dx = F(b) - F(a)$ where F is any function such that $\frac{d}{dx}F = f$.

Indefinite Integrals

$$\int f(x)dx = F(x) \text{ means } \frac{d}{dx}F = f$$

Exercise

Calculate:

$$\int x^2 + \sin(x) dx$$

Solution

Since $\frac{d}{dx}x^3 = 3x^2$ we have $\int x^2 dx = \frac{x^3}{3}$. Similarly, since $\frac{d}{dx}\cos(x) = -\sin(x)$ we have:

$$\int x^2 + \sin(x) dx = \frac{x^3}{3} - \cos(x) + C$$

where C is any constant.

The Substitution Rule

If $u = g(x)$ then:

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Exercise

Calculate:

$$\int x^3 \cos(x^4 + 2) dx$$

Solution

Letting $u = x^4 + 2$, we have $du = 4x^3 dx$, thus:

$$\begin{aligned} \int x^3 \cos(x^4 + 2) dx &= \int \cos(u) \frac{1}{4} du = \frac{1}{4} \int \cos(u) du \\ &= \frac{1}{4} \sin(u) + C \\ &= \frac{1}{4} \sin(x^4 + 2) + C \end{aligned}$$

Integration by Parts

If $u = f(x)$ and $v = g(x)$:

$$\int u dv = uv - \int v du$$

Exercise

Calculate:

$$\int x \cos(x) dx$$

Solution

Letting $u = x$ and $dv = \cos(x)dx$ we have $du = dx$ and $v = \sin(x)$, thus:

$$\begin{aligned}\int x\cos(x)dx &= \int udv = uv - vdu \\ &= x\sin(x) - \int \sin(x)dx \\ &= x\sin(x) + \cos(x) + C\end{aligned}$$

Tables of Indefinite Integrals

$$\int cf(x)dx = c\int f(x)dx$$

$$\int (f(x) + g(x))\, dx = \int f(x)dx + \int g(x)dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$$

$$\int \frac{1}{x} dx = ln(|x|) + C$$

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\vdots$$

Integration is . . . ...

- Differentiation requires the application of rules (following a recipe).
- Integration is an art.

Logarithms

By definition:

$$\log_a a^b = b$$

"Natural Log":

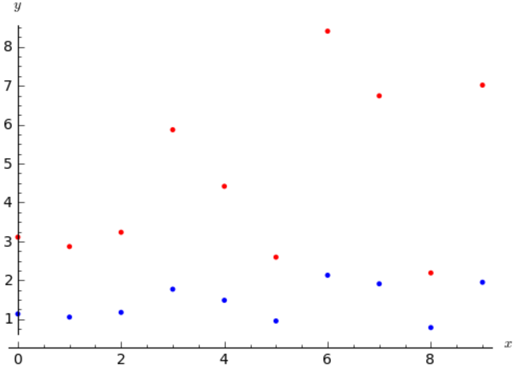
$$\ln x = \log_e x$$

(recall:)

$$\int \frac{1}{x} dx = ln(|x|) + C$$

Use of Logarithms in Statistics

In the following graph the red dots are contracted by log. This is a standard procedure sometimes used when analysing data.



Probability

Wikipedia:

“Probability theory is the branch of mathematics concerned with analysis of random phenomena.”

Random Variables

In experiments or trials in which the outcome is numerical, the outcomes are values of what is known as a random variable.

For example, suppose that a coin is spun 3 times and we record the outcomes and ask: how many heads appear? If we denote the random variable associated with the number of heads by X and denote the sample space by S_X then we have:

$$S_X = \{0, 1, 2, 3\}$$

Probability Distributions for Discrete Random Variables

The probability distribution $P(X = x_i) = p_i$ has the following properties:

- $0 \leq p_i \leq 1$
- $\sum_{i=1}^{n} p_i = 1$, if X has n possible outcomes, or $\sum_{i=1}^{\infty} p_i = 1$ if X has a countably infinite set of outcomes

Exercise

Write down the probability distribution for the random variable X associated with the rolling of a six sided dice.

Solution

We have $S_X = \{1, 2, 3, 4, 5, 6\}$ and $P(X = x_i) = \frac{1}{6}$ for all i:

x_i	1	2	3	4	5	6
$P(X = x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Cumulative Distribution for Discrete Random Variables

For a given probability distribution $P(X = x_i) = p_i$ we have the cumulative distribution $F(x) = P(X \leq x)$:

$$F(x) = \sum_{i=1}^{x} P(X = x_i)$$

Exercise

Write down the cumulative probability distribution for the random variable X associated with the rolling of a six sided dice.

Solution

We have $S_X = \{1, 2, 3, 4, 5, 6\}$ and $P(X = x_i) = \frac{1}{6}$ for all i:

x_i	1	2	3	4	5	6
$P(X = x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$F(x_i)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1

Probability Distributions for Continuous Random Variables

In many applications the discrete random variable which takes its values from a countable list is inappropriate. For example, the random variable X could be the time from, say, $t = 0$, until a light bulb fails. In these cases we use a continuous random variable, which is defined for the continuous variable $t \geq 0$, and is no longer a countable list of values.
Instead of the sequence of probabilities $\{P(X = x_i)\}$, we define a probability density function $f(x)$ over $\mathbb{R}$ which has the properties:

- $f(x) \geq 0$
- $\int_{-\infty}^{\infty} f(x)dx = 1$
- for any $x_1 < x_2$:

$$P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(x)dx$$

Cumulative Distribution for Continuous Random Variables

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du$$

Mean and Variance of Continuous Random Variables

Mean:

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

Variance:

$$Var(X) = \int_{-\infty}^{\infty} (x - E(X))f(x)dx$$

Exercise

Find the mean of the negative exponential distribution:

$$f(x) = \lambda e^{-\lambda x} \text{ defined for } 0 < x < \infty$$

Support Material

https://intranet.cardiff.ac.uk/students/your-study/study-skills/maths-support

https://github.com/drvinceknight/MSc_week_0/wiki