Algebra, Calculus & Probability Refresher

MSc Week 0

Geraint Palmer
Room: M/1.29
palmergi1@cardiff.ac.uk

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Algebra

Numbers

- Integers:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

- Rationals:

$$\mathbb{Q} = \left\{ a \mid \exists\ p, q \in \mathbb{Z} \text{ for which } a = \frac{p}{q} \right\}$$

- Real numbers:

$$\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

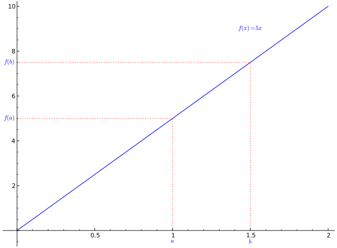
Exponents

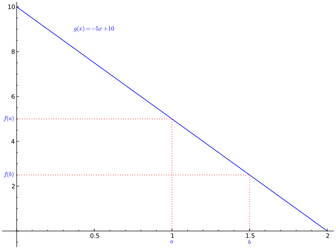
For $a, b \in \mathbb{R}^+$, $x, y \in \mathbb{R}$:

1. $a^x a^y = a^{x+y}$
2. $a^0 = 1$
3. $a^{-x} = \frac{1}{a^x}$
4. $(a^x)^y = a^{xy}$
5. $a^x b^x = (ab)^x$
6. $a^{\frac{1}{2}} = \sqrt{a}$

Inequalities

When solving inequalities it is important to keep in mind whether or not the operation we are using is *increasing* or *decreasing*.





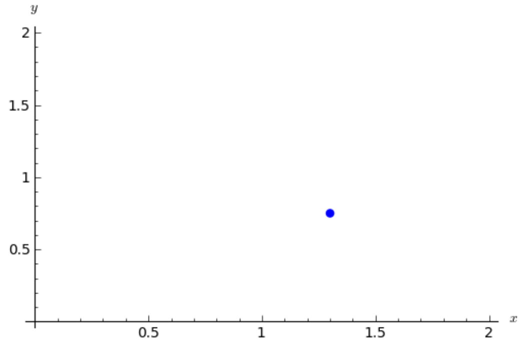
$a < b \Rightarrow f(a) < f(b)$

$a < b \Rightarrow g(a) > g(b)$

Coordinates in the plane

Right handed Cartesian axes:

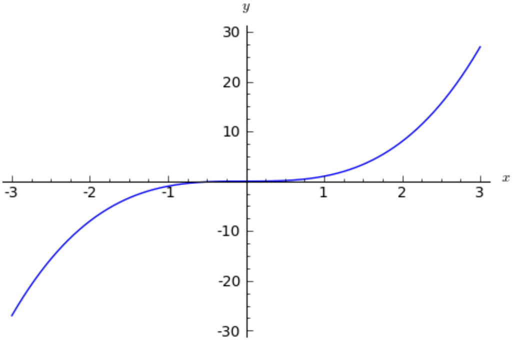
$$(1.3, 0.75)$$



For $P = (x, y)$, x/y is called the abscissa / ordinate of P.

Graphs

If x and y connected by an equation, then this relation can be represented by a curve or curves in the (x, y) plane which is known as the graph of the equation.



$$y = x^3$$

Graphs

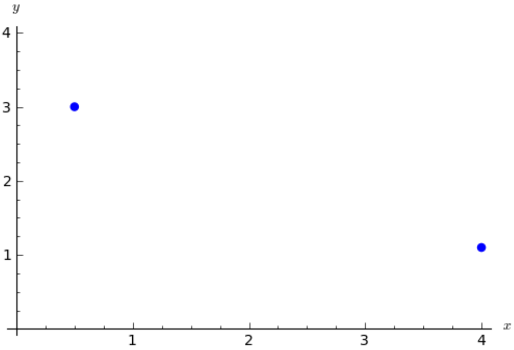
Graph of a straight line:

$$y = mx + c$$

- m is called the *gradient* of the line.
- c is called the *y-intercept* of the line.

Exercise

Find the equation for the line going through the points $\{(0.5, 3), (4, 1.1)\}$:



Solution

General form of $y = mx + c$ through $\{(x_1, y_1), (x_2, y_2)\}$:

$$\left.\begin{aligned} y_1 &= mx_1 + c \\ y_2 &= mx_2 + c \end{aligned}\right\} \Rightarrow m(x_1 - x_2) = y_1 - y_2$$

which gives:

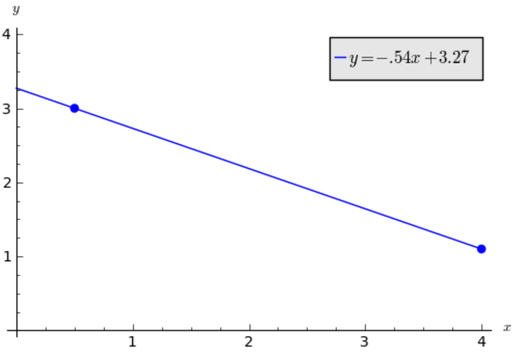
$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

$$c = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$$

Solution

So for $(x_1, y_1) = (0.5, 3)$ and $(x_2, y_2) = (4, 1.1)$ we have:

$$m = \frac{1.9}{-3.5} \approx -0.54$$
$$c = \frac{11.45}{3.5} \approx 3.27$$



Exercise

Where does the line $y = -0.54x + 3.27$ intersect the y-axis and the x-axis?

Exercise

Where does the line $y = -0.54x + 3.27$ intersect the y-axis and the x-axis?
This is equivalent to solving:

$$y = -0.54 \times 0 + 3.27$$

and

$$0 = -0.54x + 3.27$$

Solving Linear Equations

In general equations of the form:

$$y = mx + c$$

are solved by multiplying or adding various constants.

$$\begin{aligned}
0 = -0.54x + 3.27 \quad &\Leftrightarrow 0 - 3.27 &&= (-0.54x + 3.27) - 3.27 \\
&\Leftrightarrow -3.27 &&= -0.54x \\
&\Leftrightarrow -3.27 \times \frac{1}{-0.54} &&= 0.54x \times \frac{1}{-0.54} \\
&\Leftrightarrow 6.06 &&\approx x
\end{aligned}$$

Solving Linear Equations

In general equations of the form:

$$y = mx + c$$

are solved by multiplying or adding various constants.

$$\begin{aligned} y = mx + c \Leftrightarrow y - c &= (mx + c) - c \\ \Leftrightarrow y - c &= mx \\ \Leftrightarrow (y - c) \times \frac{1}{m} &= mx \times \frac{1}{m} \\ \Leftrightarrow \frac{y - c}{m} &= x \end{aligned}$$

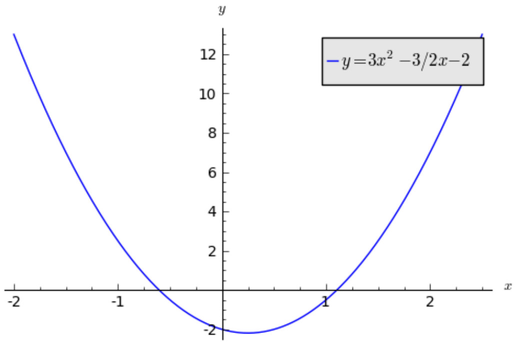
Quadratic

A "quadratic" is an expression of the form:

$$ax^2 + bx + c$$

- a is called the quadratic coefficient,
- b is called the linear coefficient,
- c is called the constant term or free term.

Quadratic



Solving a Quadratic Equation

General solution of the equation:

$$ax^2 + bx + c = 0$$

is given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exercise

Solve the equation:

$$3x^2 - \frac{3}{2}x - 2 = 0$$

Solution

From the previous formula we have:

$$
\begin{aligned}
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &\Leftrightarrow x = \frac{\frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 4 \times 3 \times (-2)}}{2 \times 3} \\
&\Leftrightarrow x = \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4} + 24}}{6} \\
&\Leftrightarrow x = \frac{3}{12} \pm \frac{\frac{1}{2}\sqrt{9 + 96}}{6} \\
&\Leftrightarrow x = \frac{1}{4} \pm \frac{\sqrt{105}}{12}
\end{aligned}
$$

Exercise

Solve the equation:

$$4x^2 - 2x + 10 = 3$$

Solution

From the previous formula we have:

$$
\begin{aligned}
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &\Leftrightarrow x = \frac{2 \pm \sqrt{2^2 - 4 \times 4 \times 7}}{2 \times 4} \\
&\Leftrightarrow x = \frac{2 \pm \sqrt{-108}}{8} \\
&\Leftrightarrow x = \frac{2 \pm \sqrt{i^2 108}}{8} \\
&\Leftrightarrow x = \frac{2 \pm i\sqrt{3 \times 36}}{8} \\
&\Leftrightarrow x = \frac{2 \pm 6i\sqrt{3}}{8} = \frac{1}{4} \pm \frac{3}{4} i\sqrt{3}
\end{aligned}
$$

Complex Numbers

$$i^2 = -1$$

Complex numbers:

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

If $z = a + ib$:

- a is the real part of z.
- b is the imaginary part of z.

Solving Systems of Equations

A system of equations is a collection of equations involving the same set of variables. For example:

$$\begin{aligned} 3x + 2y &= 1 \\ 2x - 2y &= -2 \end{aligned}$$

Various techniques can be used to solve such a problem.

Solution

First equation gives:

$$3x + 2y = 1 \Rightarrow x = \frac{1-2y}{3}$$

Substituting in to second equation gives:

$$2\left(\frac{1-2y}{3}\right) - 2y = -2$$

which implies:

$$y = \frac{4}{5}$$

Substituting in to our expression for x we get:

$$x = -\frac{1}{5}$$

Shorthand notation

- Summation:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

- Multiplication:

$$\prod_{i=1}^{n} a_i = a_1 \times a_2 \times a_3 \times \cdots \times a_n$$

Examples

- Summation:

$$\sum_{i=1}^{4} i \times 2^i = 1 \times 2 + 2 \times 2^2 + 3^3 + 4 \times 2^4$$
$$= 2 + 8 + 3 \times 8 + 4 \times 16 = 98$$

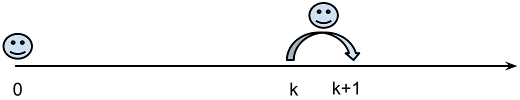
- Multiplication:

$$\prod_{k=1}^{3} k^2 = 1 \times 2^2 \times 3^2 = 36$$

Proof by Induction

Technique often used to prove algebraic relationships. Basic idea:

- Prove that something is true at the start.
- Prove that if something is true at point k then it is true at point $k + 1$.



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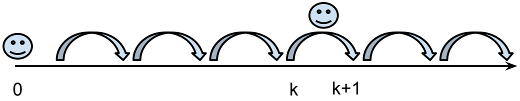


Exercise

Prove that:

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Solution

- True for $n = 0$?:

$$\sum_{i=0}^{0} i = 0 \text{ and } \frac{n(n+1)}{2} = 0$$

Solution

- True for $n = 0$?:

$$\sum_{i=0}^{0} i = 0 \quad \text{and} \quad \frac{n(n+1)}{2} = 0$$

- If true for $n = k$, true for $n = k + 1$?:

$$\begin{aligned}
\sum_{i=0}^{k+1} i &= \sum_{i=0}^{k} i + k + 1 \\
&= \frac{k(k+1)}{2} + k + 1 \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

Infinite Sums

$$\sum_{k=0}^{\infty} a^k = \frac{a}{1-a}$$

$$\sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2}$$

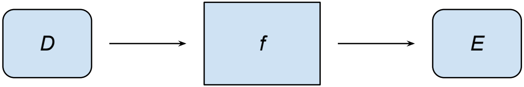
$$\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$$

https://en.wikipedia.org/wiki/List_of_mathematical_series

Calculus

Functions

A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.



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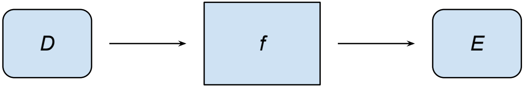
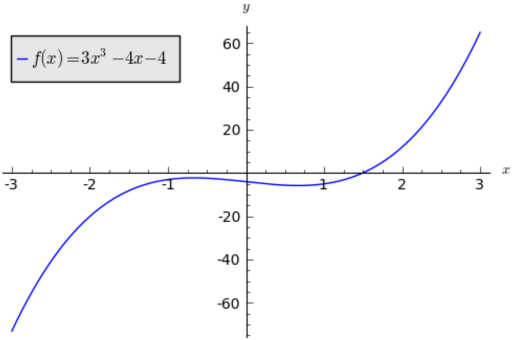


- We usually consider functions for which the sets D and E are sets of real numbers.
- The set D is called the domain of the function.
- The range of f is the set of all possible values of $f(x)$ as x varies throughout the domain.
- A symbol that represents an arbitrary number in the domain of a function f is call an independent variable.
- A symbol that represents a number in the range of f is called a dependent variable.

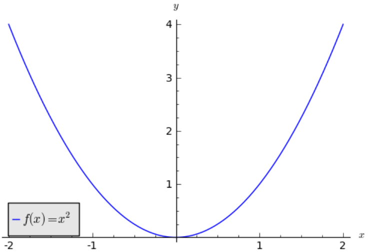
Example

$$f(x) = 3x^3 - 4x - 4$$

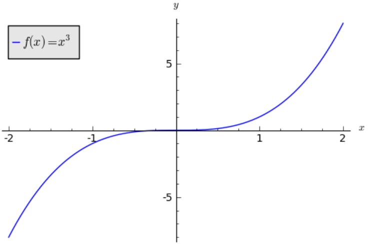


Even and Odd Functions

- If $f(-x) = f(x)$ for all x in its domain then f is *even*:



- If $f(-x) = -f(x)$ for all x in its domain then f is *odd*:

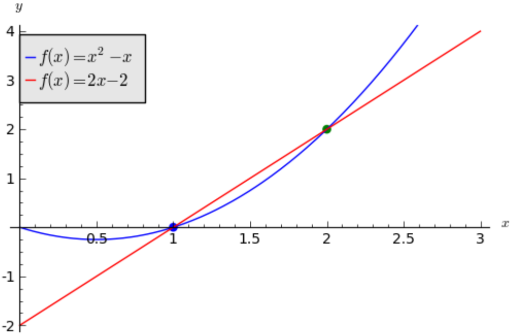


Tangent Curves

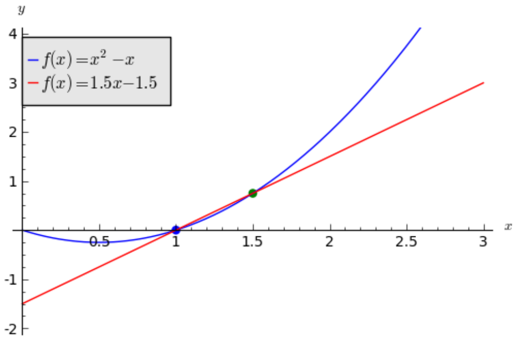
The tangent line to the curve $y = f(x)$ at the point $P = (a, f(a))$ is the line through P with gradient:

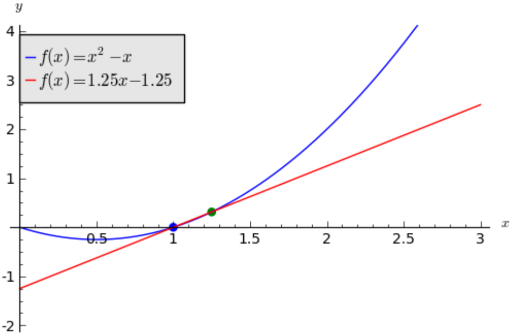
$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Tangent Curves

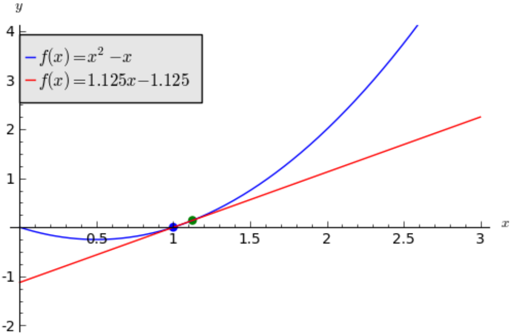


Tangent Curves

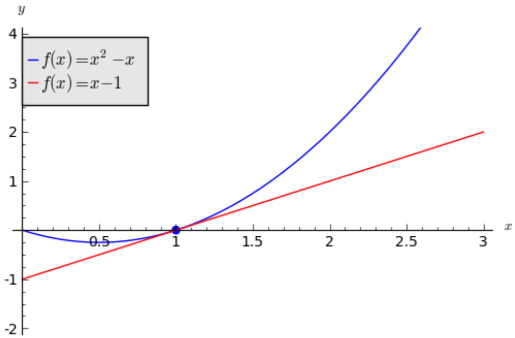



y
4
3
2
1
-1
-2
0.5
1
1.5
2
2.5
3
x
$f(x) = x^2 - x$
$f(x) = 1.25x - 1.25$

Tangent Curves



Tangent Curves



Derivative

The derivative of a function f at a number a, denoted by $f'(a)$ is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Exercise

Find the derivative of

$$f(x) = x^2 - 3x + 2$$

at the number a.

Solution

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0} \frac{\left((a+h)^2-3(a+h)+2\right)-\left(a^2-3a+2\right)-f(a)}{h} \\
&= \lim_{h\to 0} \frac{a^2+2ah+h^2-3a-3h+2-a^2+3a-2}{h} \\
&= \lim_{h\to 0} \frac{2ah+h^2-3h}{h} \\
&= \lim_{h\to 0} 2a+h-3 \\
&= 2a-3
\end{aligned}
$$

Rules of Differentiation

- The Power Rule:

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

- The Constant Multiple Rule:

$$\frac{d}{dx}\left(cf(x)\right) = c\frac{d}{dx}\left(f(x)\right)$$

- The Sum Rule:

$$(f+g)' = f' + g'$$

Rules of Differentiation

- The Product Rule:

$$(fg)' = f'g + fg'$$

- The Quotient Rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

The Chain Rule

If $f\left(g(x)\right) = f \circ g$:

$$(f \circ g)' = \left(f' \circ g\right) g'$$

Exercise

Differentiate $F(x) = \sqrt{x^2 + 1}$.

Solution

If we let $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ and $g(x) = x^2 + 1$ then we have:

$$f' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

and

$$g' = 2x$$

Using the Chain Rule we have:

$$\begin{aligned} F'(x) &= \left(\frac{1}{2\sqrt{x^2+1}}\right)(2x) \\ &= \frac{x}{\sqrt{x^2+1}} \end{aligned}$$

Table of Derivatives

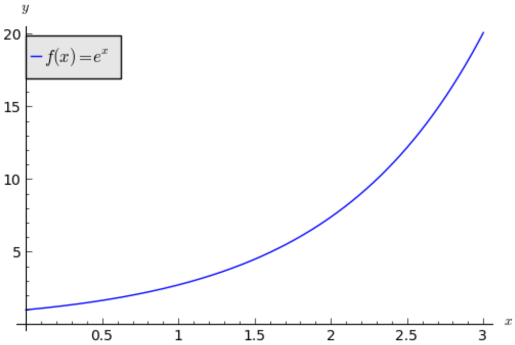
$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x)$$
$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

http://www.math.wustl.edu/~freiwald/131derivativetable.pdf

Natural Exponential Function

The mathematical constant e can be defined as the real number such that:

$$\frac{d}{dx}e^x = e^x$$



Logarithms

$$\log_a a^b = b$$

$$3^x = 81 \Leftrightarrow x = \log_3 81 = 4$$

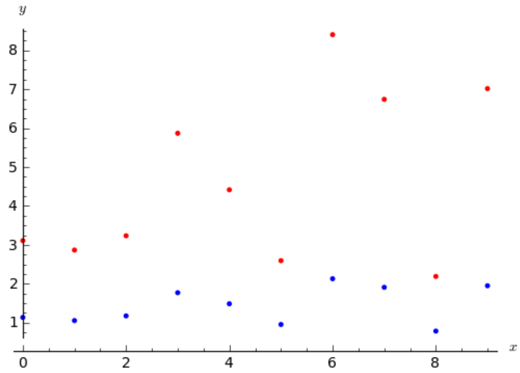
Naural Logarithm

$$\ln x = \log_e x$$

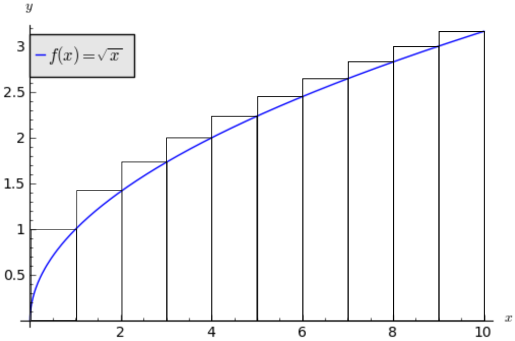
$$y = e^x \Leftrightarrow x = \ln y$$

Logarithms in Statistics

In the following graph the red dots are contracted by log. This is a standard procedure sometimes used when analysing data.



Area under a graph



Fundamental Theorem of Calculus

Let f, F be continuous on $[a, b]$ then:

1. If $F(x) = \int_a^x f(t)dt$ then $F' = f$.
2. $\int_a^b f(x)dx = F(b) - F(a)$

Fundamental Theorem of Calculus

Let f, F be continuous on $[a, b]$ then:

1. If $F(x) = \int_a^x f(t)dt$ then $F' = f$.
2. $\int_a^b f(x)dx = F(b) - F(a)$

$$\int f(x)dx = F(x) \;\text{ means }\; \frac{d}{dx}F = f$$

Tables of Indefinite Integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

$$\int cf(x)dx = c\int f(x)dx$$

$$\int (f(x) + g(x))\,dx = \int f(x)dx + \int g(x)dx$$

$$\vdots$$

`http://integral-table.com/downloads/single-page-integral-table.pdf`

Exercise

Calculate:

$$\int x^2 + \sin(x) dx$$

Exercise

Calculate:

$$\int x^2 + \sin(x)dx$$

$$\int x^2 + \sin(x)dx = \frac{x^3}{3} - \cos(x) + C$$

Integration by Parts

If $u = f(x)$ and $v = g(x)$:

$$\int u dv = uv - \int v du$$

Exercise

Calculate:

$$\int x \cos(x) dx$$

Solution

Letting $u = x$ and $dv = \cos(x)dx$ we have $du = dx$ and $v = \sin(x)$, thus:

$$\begin{aligned} \int x\cos(x)dx &= \int udv = uv - vdu \\ &= x\sin(x) - \int \sin(x)dx \\ &= x\sin(x) + \cos(x) + C \end{aligned}$$

The Substitution Rule

If $u = g(x)$ then:

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Exercise

Calculate:

$$\int x^3 \cos(x^4 + 2) dx$$

Solution

Letting $u = x^4 + 2$, we have $du = 4x^3 dx$, thus:

$$\begin{aligned}
\int x^3 \cos(x^4 + 2) dx &= \int \cos(u) \frac{1}{4} du \\
&= \frac{1}{4} \int \cos(u) du \\
&= \frac{1}{4} \sin(u) + C \\
&= \frac{1}{4} \sin(x^4 + 2) + C
\end{aligned}$$

Probability

Random Variables

In experiments or trials in which the outcome is numerical, the outcomes are values of what is known as a random variable.

For example, suppose that a coin is spun 3 times and we record the outcomes and ask: how many heads appear? If we denote the random variable associated with the number of heads by X and denote the sample space by S_X then we have:

$$S_X = \{0, 1, 2, 3\}$$

Probability Distributions for Discrete Random Variables

The probability distribution $P(X = x_i) = p_i$ has the following properties:

- $0 \leq p_i \leq 1$
- $\sum_{i=1}^{n} p_i = 1$, if X has n possible outcomes, or $\sum_{i=1}^{\infty} p_i = 1$ if X has a countably infinite set of outcomes

Exercise

Write down the probability distribution for the random variable X associated with the rolling of a six sided dice.

Solution

We have $S_X = \{1, 2, 3, 4, 5, 6\}$ and $P(X = x_i) = \frac{1}{6}$ for all i:

x_i	1	2	3	4	5	6
$P(X = x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Cumulative Distribution for Discrete Random Variables

For a given probability distribution $P(X = x_i) = p_i$ we have the cumulative distribution $F(x) = P(X \leq x)$:

$$F(x) = \sum_{i=1}^{x} P(X = x_i)$$

Exercise

Write down the cumulative probability distribution for the random variable X associated with the rolling of a six sided dice.

Solution

We have $S_X = \{1, 2, 3, 4, 5, 6\}$ and $P(X = x_i) = \frac{1}{6}$ for all i:

x_i	1	2	3	4	5	6
$P(X = x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$F(x_i)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1

Probability Distributions for Continuous Random Variables

In many applications the discrete random variable which takes its values from a countable list is inappropriate. For example, the random variable X could be the time from, say, $t = 0$, until a light bulb fails. In these cases we use a continuous random variable, which is defined for the continuous variable $t \geq 0$, and is no longer a countable list of values.
Instead of the sequence of probabilities $\{P(X = x_i)\}$, we define a probability density function $f(x)$ over $\mathbb{R}$ which has the properties:

- $f(x) \geq 0$
- $\int_{-\infty}^{\infty} f(x)dx = 1$
- for any $x_1 < x_2$:

$$P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(x)dx$$

Cumulative Distribution for Continuous Random Variables

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du$$

Mean and Variance of Continuous Random Variables

Mean:

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

Variance:

$$Var(X) = \int_{-\infty}^{\infty} (x - E(X))f(x)dx$$

Exercise

Find the mean of the negative exponential distribution:

$$f(x) = \lambda e^{-\lambda x} \text{ defined for } 0 < x < \infty$$

Support Material

https://intranet.cardiff.ac.uk/students/your-study/study-skills/maths-support

https://github.com/drvinceknight/MSc_week_0/wiki