

# Algebra, Calculus & Probability Refresher

## MSc Week 0

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# *Algebra*

# Numbers

- Integers:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

- Rationals:

$$\mathbb{Q} = \left\{ a \mid \exists p, q \in \mathbb{Z} \text{ for which } a = \frac{p}{q} \right\}$$

- Real numbers:

$$\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

# Exponents

For  $a, b \in \mathbb{R}^+$ ,  $x, y \in \mathbb{R}$ :

$$1. a^x a^y = a^{x+y}$$

$$4. (a^x)^y = a^{xy}$$

$$2. a^0 = 1$$

$$5. a^x b^x = (ab)^x$$

$$3. a^{-x} = \frac{1}{a^x}$$

$$6. a^{\frac{1}{2}} = \sqrt{a}$$

# Logarithms

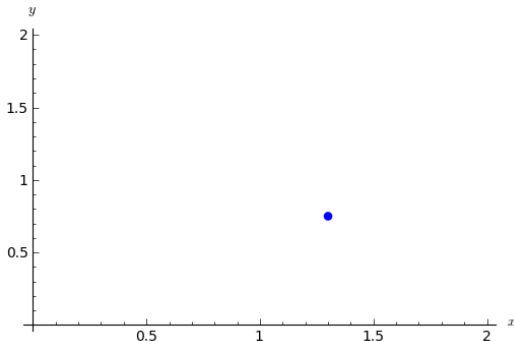
$$\log_a a^b = b$$

$$3^x = 81 \Leftrightarrow x = \log_3 81 = 4$$

# Coordinates in the plane

Right handed Cartesian axes:

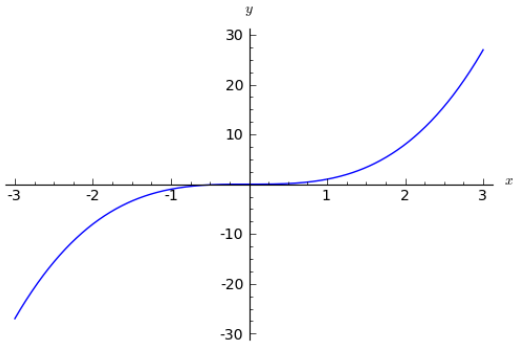
$(1.3, 0.75)$



For  $P = (x, y)$ ,  $x/y$  is called the abscissa / ordinate of  $P$ .

# Graphs

If  $x$  and  $y$  connected by an equation, then this relation can be represented by a curve or curves in the  $(x, y)$  plane which is known as the graph of the equation.



$$y = x^3$$

# Graphs

Graph of a straight line:

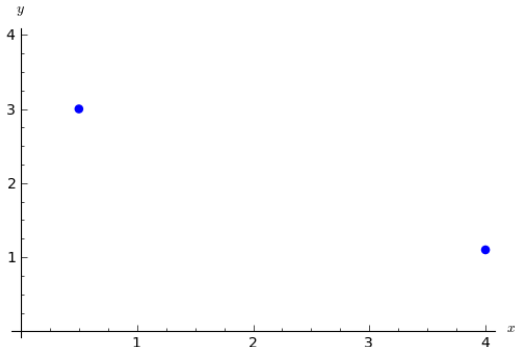
$$y = mx + c$$

- $m$  is called the *gradient* of the line.
- $c$  is called the *y-intercept* of the line.



# Exercise

Find the equation for the line going through the points  $\{(0.5, 3), (4, 1.1)\}$ :



# Solution

General form of  $y = mx + c$  through  $\{(x_1, y_1), (x_2, y_2)\}$ :

$$\left. \begin{array}{l} y_1 = mx_1 + c \\ y_2 = mx_2 + c \end{array} \right\} \Rightarrow m(x_1 - x_2) = y_1 - y_2$$

which gives:

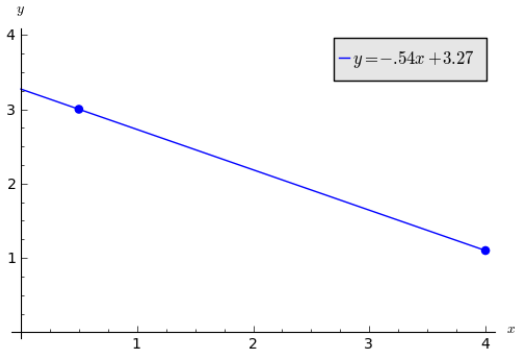
$$m = \frac{y_1 - y_2}{x_1 - x_2}$$
$$c = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$$

# Solution

So for  $(x_1, y_1) = (0.5, 3)$  and  $(x_2, y_2) = (4, 1.1)$  we have:

$$m = \frac{1.9}{-3.5} \approx -0.54$$

$$c = \frac{11.45}{3.5} \approx 3.27$$



## Exercise

Where does the line  $y = -0.54x + 3.27$  intersect the  $y$ -axis and the  $x$ -axis?

## Exercise

Where does the line  $y = -0.54x + 3.27$  intersect the  $y$ -axis and the  $x$ -axis?

This is equivalent to solving:

$$y = -0.54 \times 0 + 3.27$$

and

$$0 = -0.54x + 3.27$$

# Solving Linear Equations

In general equations of the form:

$$y = mx + c$$

are solved by multiplying or adding various constants.

$$\begin{aligned}0 = -0.54x + 3.27 &\Leftrightarrow 0 - 3.27 &&= (-0.54x + 3.27) - 3.27 \\&\Leftrightarrow -3.27 &&= -0.54x \\&\Leftrightarrow -3.27 \times \frac{1}{-0.54} = 0.54x \times \frac{1}{-0.54} \\&\Leftrightarrow 6.06 &&\approx x\end{aligned}$$

# Solving Linear Equations

In general equations of the form:

$$y = mx + c$$

are solved by multiplying or adding various constants.

$$\begin{aligned}y = mx + c &\Leftrightarrow y - c &&= (mx + c) - c \\&\Leftrightarrow y - c &&= mx \\&\Leftrightarrow (y - c) \times \frac{1}{m} &&= mx \times \frac{1}{m} \\&\Leftrightarrow \frac{y - c}{m} &&= x\end{aligned}$$

# Quadratic

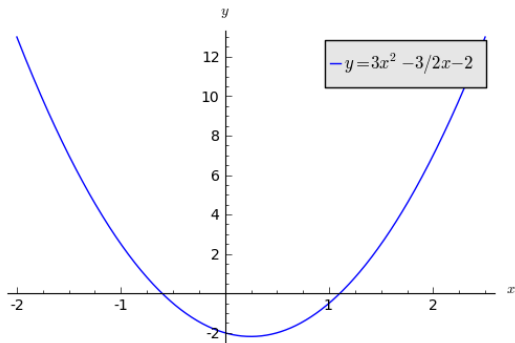
A “quadratic” is an expression of the form:

$$ax^2 + bx + c$$

- $a$  is called the quadratic coefficient,
- $b$  is called the linear coefficient,
- $c$  is called the constant term or free term.



# Quadratic



# Solving a Quadratic Equation

General solution of the equation:

$$ax^2 + bx + c = 0$$

is given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

## Exercise

Solve the equation:

$$3x^2 - \frac{3}{2}x - 2 = 0$$

# Solution

From the previous formula we have:

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Leftrightarrow x = \frac{\frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 4 \times 3 \times (-2)}}{2 \times 3} \\&\Leftrightarrow x = \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4} + 24}}{6} \\&\Leftrightarrow x = \frac{3}{12} \pm \frac{\frac{1}{2}\sqrt{9 + 96}}{6} \\&\Leftrightarrow x = \frac{1}{4} \pm \frac{\sqrt{105}}{12}\end{aligned}$$

# Exercise

Solve the equation:

$$4x^2 - 2x + 10 = 3$$

# Solution

From the previous formula we have:

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Leftrightarrow x = \frac{2 \pm \sqrt{2^2 - 4 \times 4 \times 7}}{2 \times 4} \\&\Leftrightarrow x = \frac{2 \pm \sqrt{-108}}{8} \\&\Leftrightarrow x = \frac{2 \pm \sqrt{i^2 108}}{8} \\&\Leftrightarrow x = \frac{2 \pm i\sqrt{3 \times 36}}{8} \\&\Leftrightarrow x = \frac{2 \pm 6i\sqrt{3}}{8} = \frac{1}{4} \pm \frac{3}{4}i\sqrt{3}\end{aligned}$$

# Complex Numbers

$$i^2 = -1$$

Complex numbers:

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

If  $z = a + ib$ :

- $a$  is the real part of  $z$ .
- $b$  is the imaginary part of  $z$ .

# Solving Systems of Equations

A system of equations is a collection of equations involving the same set of variables. For example:

$$3x + 2y = 1$$

$$2x - 2y = -2$$

Various techniques can be used to solve such a problem.



# Solution

First equation gives:

$$3x + 2y = 1 \Rightarrow x = \frac{1 - 2y}{3}$$

Substituting in to second equation gives:

$$2\left(\frac{1 - 2y}{3}\right) - 2y = -2$$

which implies:

$$y = \frac{4}{5}$$

Substituting in to our expression for  $x$  we get:

$$x = -\frac{1}{5}$$

# Shorthand notation

- Summation:

$$\sum_{i=1}^n a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

- Multiplication:

$$\prod_{i=1}^n a_i = a_1 \times a_2 \times a_3 \times \cdots \times a_n$$

# Examples

- Summation:

$$\begin{aligned}\sum_{i=1}^4 i \times 2^i &= 1 \times 2 + 2 \times 2^2 + 3^3 + 4 \times 2^4 \\ &= 2 + 8 + 3 \times 8 + 4 \times 16 = 98\end{aligned}$$

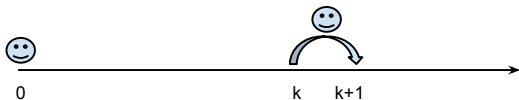
- Multiplication:

$$\prod_{k=1}^3 k^2 = 1 \times 2^2 \times 3^2 = 36$$

# Proof by Induction

Technique often used to prove algebraic relationships. Basic idea:

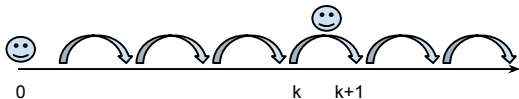
- Prove that something is true at the start.
- Prove that if something is true at point  $k$  then it is true at point  $k + 1$ .



# Proof by Induction

Technique often used to prove algebraic relationships. Basic idea:

- Prove that something is true at the start.
- Prove that if something is true at point  $k$  then it is true at point  $k + 1$ .



# Exercise

Prove that:

$$\sum_{i=0}^n i = \frac{n(n+1)}{2}$$

# Solution

- True for  $n = 0$ ?:

$$\sum_{i=0}^0 i = 0 \quad \text{and} \quad \frac{n(n+1)}{2} = 0$$

# Solution

- True for  $n = 0$ ?:

$$\sum_{i=0}^0 i = 0 \quad \text{and} \quad \frac{n(n+1)}{2} = 0$$

- If true for  $n = k$ , true for  $n = k + 1$ ?:

$$\begin{aligned}\sum_{i=0}^{k+1} i &= \sum_{i=0}^k i && + k + 1 \\ &= \frac{k(k+1)}{2} && + k + 1 \\ &= \frac{(k+1)(k+2)}{2}\end{aligned}$$



# Infinite Sums

$$\sum_{k=0}^{\infty} a^k = \frac{a}{1-a}$$

$$\sum_{k=0}^{\infty} ka^k = \frac{a}{(1-a)^2}$$

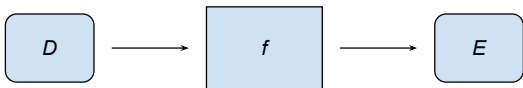
$$\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$$

[https://en.wikipedia.org/wiki/List\\_of\\_mathematical\\_series](https://en.wikipedia.org/wiki/List_of_mathematical_series)

# *Calculus*

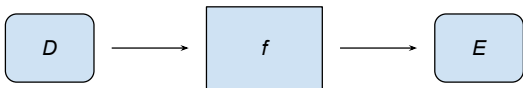
# Functions

A function  $f$  is a rule that assigns to each element  $x$  in a set  $D$  exactly one element, called  $f(x)$ , in a set  $E$ .



# Functions

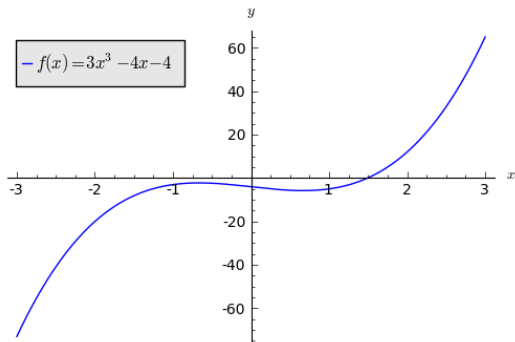
A function  $f$  is a rule that assigns to each element  $x$  in a set  $D$  exactly one element, called  $f(x)$ , in a set  $E$ .



- We usually consider functions for which the sets  $D$  and  $E$  are sets of real numbers.
- The set  $D$  is called the domain of the function.
- The range of  $f$  is the set of all possible values of  $f(x)$  as  $x$  varies throughout the domain.
- A symbol that represents an arbitrary number in the domain of a function  $f$  is called an independent variable.
- A symbol that represents a number in the range of  $f$  is called a dependent variable.

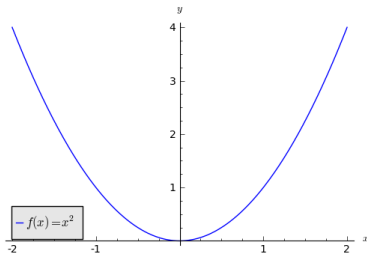
# Example

$$f(x) = 3x^3 - 4x - 4$$

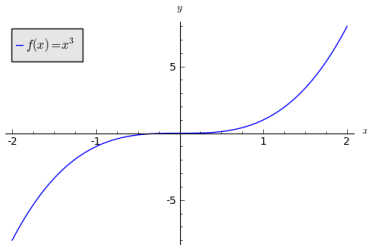


# Even and Odd Functions

- If  $f(-x) = f(x)$  for all  $x$  in its domain then  $f$  is *even*:



- If  $f(-x) = -f(x)$  for all  $x$  in its domain then  $f$  is *odd*:

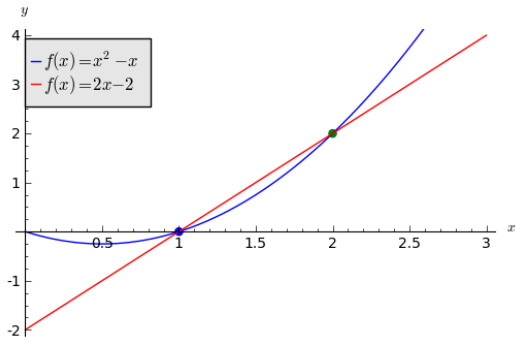


# Tangent Curves

The tangent line to the curve  $y = f(x)$  at the point  $P = (a, f(a))$  is the line through  $P$  with gradient:

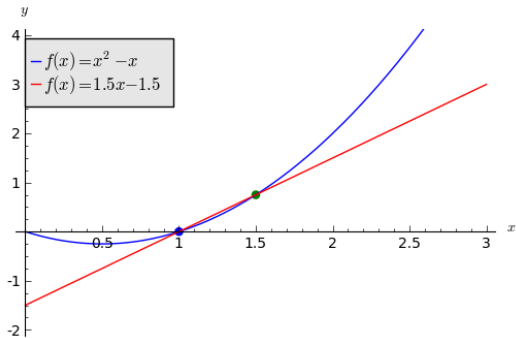
$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

# Tangent Curves

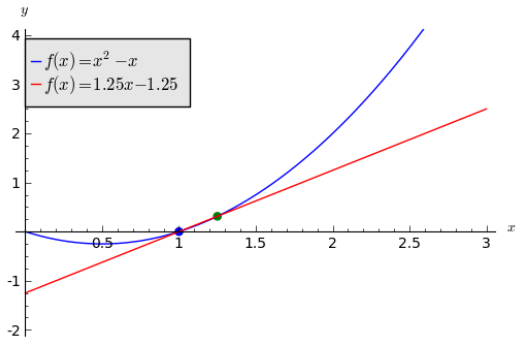




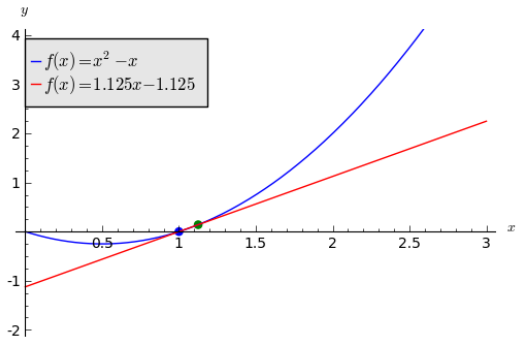
# Tangent Curves



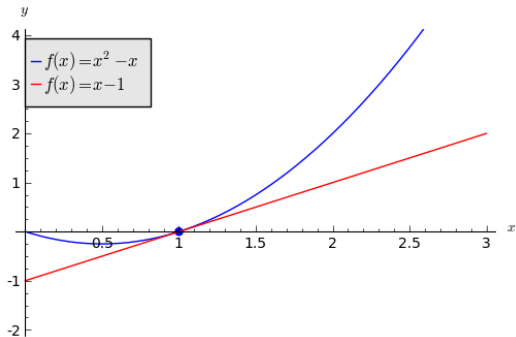
# Tangent Curves



# Tangent Curves



# Tangent Curves



# Derivative

The derivative of a function  $f$  at a number  $a$ , denoted by  $f'(a)$  is:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

# Exercise

Find the derivative of

$$f(x) = x^2 - 3x + 2$$

at the number  $a$ .

# Solution

$$\begin{aligned}f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\&= \lim_{h \rightarrow 0} \frac{((a+h)^2 - 3(a+h) + 2) - (a^2 - 3a + 2) - f(a)}{h} \\&= \lim_{h \rightarrow 0} \frac{a^2 + 2ah + h^2 - 3a - 3h + 2 - a^2 + 3a - 2}{h} \\&= \lim_{h \rightarrow 0} \frac{2ah + h^2 - 3h}{h} \\&= \lim_{h \rightarrow 0} 2a + h - 3 \\&= 2a - 3\end{aligned}$$

# Rules of Differentiation

- The Power Rule:

$$\frac{d}{dx} (x^n) = nx^{n-1}$$

- The Constant Multiple Rule:

$$\frac{d}{dx} (cf(x)) = c \frac{d}{dx} (f(x))$$

- The Sum Rule:

$$(f + g)' = f' + g'$$



# Rules of Differentiation

- The Product Rule:

$$(fg)' = f'g + fg'$$

- The Quotient Rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

# The Chain Rule

If  $f(g(x)) = f \circ g$ :

$$(f \circ g)' = (f' \circ g) g'$$

# Exercise

Differentiate  $F(x) = \sqrt{x^2 + 1}$ .

## Solution

If we let  $f(x) = \sqrt{x} = x^{\frac{1}{2}}$  and  $g(x) = x^2 + 1$  then we have:

$$f' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

and

$$g' = 2x$$

Using the Chain Rule we have:

$$\begin{aligned} F'(x) &= \left( \frac{1}{2\sqrt{x^2 + 1}} \right) (2x) \\ &= \frac{x}{\sqrt{x^2 + 1}} \end{aligned}$$

# Table of Derivatives

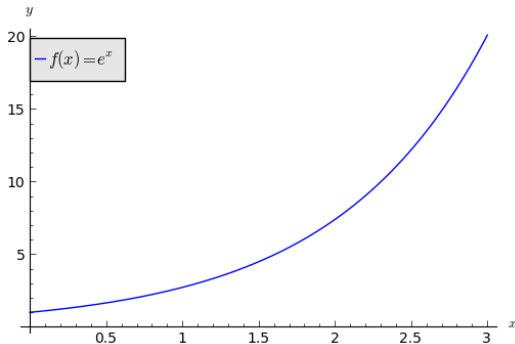
$$\begin{aligned}\frac{d}{dx} \sin(x) &= \cos(x) \\ \frac{d}{dx} \cos(x) &= -\sin(x) \\ \frac{d}{dx} \tan(x) &= \sec^2(x)\end{aligned}$$

<http://www.math.wustl.edu/~freiwald/131derivativetable.pdf>

# Natural Exponential Function

The mathematical constant  $e$  can be defined as the real number such that:

$$\frac{d}{dx}e^x = e^x$$



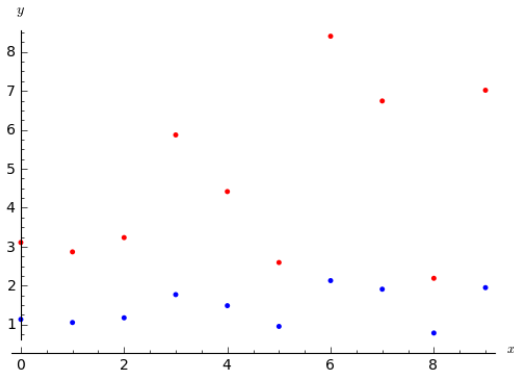
# Natural Logarithm

$$\ln x = \log_e x$$

$$y = e^x \Leftrightarrow x = \ln y$$

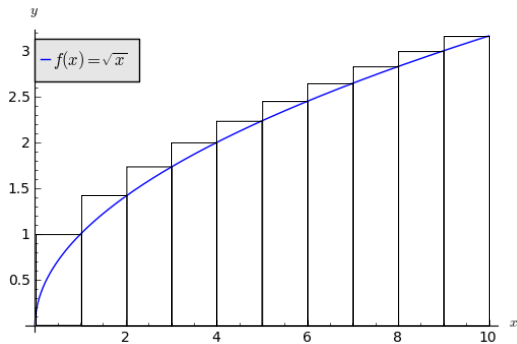
# Logarithms in Statistics

In the following graph the red dots are contracted by log. This is a standard procedure sometimes used when analysing data.





# Area under a graph



# Fundamental Theorem of Calculus

Let  $f, F$  be continuous on  $[a, b]$  then:

1. If  $F(x) = \int_a^x f(t)dt$  then  $F' = f$ .

2.  $\int_a^b f(x)dx = F(b) - F(a)$

# Fundamental Theorem of Calculus

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2.  $\int_a^b f(x)dx = F(b) - F(a)$

$$\int f(x)dx = F(x) \text{ means } \frac{d}{dx}F = f$$

# Tables of Indefinite Integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int cf(x) dx = c \int f(x) dx$$

$$\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$$

$\vdots$

<http://integral-table.com/downloads/single-page-integral-table.pdf>

# Exercise

Calculate:

$$\int x^2 + \sin(x) dx$$

## Exercise

Calculate:

$$\int x^2 + \sin(x) dx$$

$$\int x^2 + \sin(x) dx = \frac{x^3}{3} - \cos(x) + C$$

# Integration by Parts

If  $u = f(x)$  and  $v = g(x)$ :

$$\int u dv = uv - \int v du$$

# Exercise

Calculate:

$$\int x \cos(x) dx$$



# Solution

Letting  $u = x$  and  $dv = \cos(x)dx$  we have  $du = dx$  and  $v = \sin(x)$ , thus:

$$\begin{aligned}\int x \cos(x) dx &= \int u dv = uv - v du \\ &= x \sin(x) - \int \sin(x) dx \\ &= x \sin(x) + \cos(x) + C\end{aligned}$$

# The Substitution Rule

If  $u = g(x)$  then:

$$\int f(g(x))g'(x)dx = \int f(u)du$$

# Exercise

Calculate:

$$\int x^3 \cos(x^4 + 2) dx$$

## Solution

Letting  $u = x^4 + 2$ , we have  $du = 4x^3 dx$ , thus:

$$\begin{aligned}\int x^3 \cos(x^4 + 2) dx &= \int \cos(u) \frac{1}{4} du \\&= \frac{1}{4} \int \cos(u) du \\&= \frac{1}{4} \sin(u) + C \\&= \frac{1}{4} \sin(x^4 + 2) + C\end{aligned}$$

# *Probability*

# Random Variables

In trials where the outcome is numerical, the outcomes are values of random variables.

Example: A coin is spun 3 times, how many heads appear?  
Denote the random variable associated with the number of heads by  $X$ . Denote the sample space by  $S_X$  then:

$$S_X = \{0, 1, 2, 3\}$$

# Discrete Probability Distributions

For a probability distribution  $P(X = x_i) = p_i$ :

- $0 \leq p_i \leq 1$
- All probabilities sum to 1:
  - $\sum_{i=1}^n p_i = 1$  if  $X$  has  $n$  possible outcomes
  - $\sum_{i=1}^{\infty} p_i = 1$  if  $X$  has a countably infinite set of outcomes

## Exercise

Write down the state space and probability distribution for the random variable  $X$  associated with the rolling of a six sided die.





# Cumulative Distribution for Discrete Random Variables

For  $P(X = x_i) = p_i$  the cumulative distribution  $F(x) = P(X \leq x)$ :

$$F(x) = \sum_{i=1}^x P(X = x_i)$$

## Exercise

Write down the cumulative probability distribution for the random variable  $X$  associated with the rolling of a six sided dice.

## Exercise

Write down the cumulative probability distribution for the random variable  $X$  associated with the rolling of a six sided dice.

$x_i$	1	2	3	4	5	6
$P(X = x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$F(x_i)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1

# Mean and Variance

Mean / Average / Expected Value:

$$E(X) = \sum_{i=1}^n x_i p_i$$

Variance:

$$Var(X) = \sum_{i=1}^n (x_i - E(X))^2 p_i$$

## Exercise

Calculate the mean and variance for the random variable  $X$  associated with the rolling of a six sided dice.

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Calculate the mean and variance for the random variable  $X$  associated with the rolling of a six sided dice.

Mean:

$$\begin{aligned} E(X) &= \frac{0}{6} + \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} \\ &= 3.5 \end{aligned}$$

Variance:

$$\begin{aligned} \text{Var}(X) &= \frac{(0 - 3.5)^2}{6} + \frac{(1 - 3.5)^2}{6} + \frac{(2 - 3.5)^2}{6} + \frac{(3 - 3.5)^2}{6} \\ &\quad + \frac{(4 - 3.5)^2}{6} + \frac{(5 - 3.5)^2}{6} + \frac{(6 - 3.5)^2}{6} \\ &\approx 2.9 \end{aligned}$$

# Continuous Random Variables

The random variable  $X$  is the time from  $t = 0$  until a light bulb fails.  $X$  is a continuous random variable, defined for the continuous variable  $t \geq 0$ , and is not a countable list of values.

Define a probability density function  $f(x)$  over  $\mathbb{R}$ :

- $f(x) \geq 0$
- $\int_{-\infty}^{\infty} f(x)dx = 1$
- for any  $x_1 < x_2$ :

$$P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(x)dx$$



## Continuous CDF

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(u) du$$

# Mean and Variance of Continuous Random Variables

Mean:

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

Variance:

$$Var(X) = \int_{-\infty}^{\infty} (x - E(X))^2 f(x)dx$$

## Exercise

Find the mean of the negative exponential distribution:

$$f(x) = \lambda e^{-\lambda x} \text{ defined for } 0 < x < \infty$$

# Support Material

[https://intranet.cardiff.ac.uk/students/your-study/  
study-skills/maths-support](https://intranet.cardiff.ac.uk/students/your-study/study-skills/maths-support)  
[https://github.com/drvinceknight/MSc\\_week\\_0/wiki](https://github.com/drvinceknight/MSc_week_0/wiki)