

A Brief Introduction to Markov Chains

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1 Discrete Time Markov Chains

- Steady-State Probabilities
- Higher Order Markov Chains

2 Absorbing Markov Chains

- Snakes & Ladders

3 Continuous Time Markov Chains

- A Simple Queue



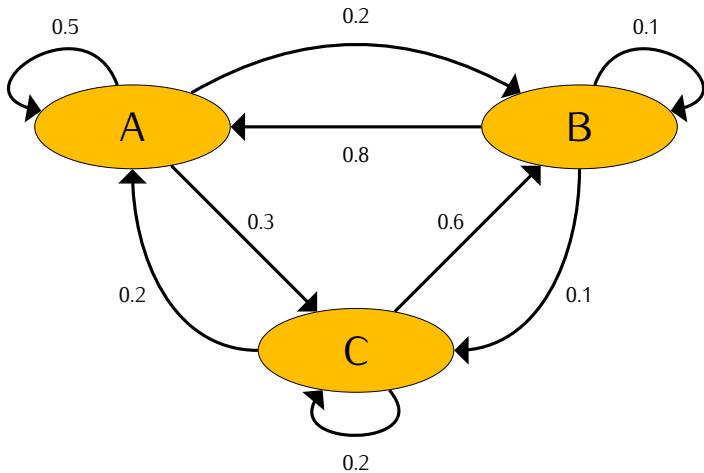
Figure: from Flickr [brickset](#)

Andrei Andreyevich Markov



Figure: Markov chain pioneer.

What is a Markov Chain?



$$P = \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.8 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.2 \end{pmatrix}$$

Initial state $\pi_0 = (1, 0, 0)$:



$$\pi_{t+1} = \pi_t P$$

$$\pi_t = \pi_0 P^t$$



$$\pi_3 = (0.521, 0.262, 0.217)$$

Initial state $\pi_0 = (103, 147, 82)$:



$$\pi_{t+1} = \pi_t P$$

$$\pi_t = \pi_0 P^t$$



$$\pi_3 = (167.875, 88.725, 75.400)$$

Initial state $\pi_0 = (0.1, 0.3, 0.6)$:



$$\pi_{t+1} = \pi_t P$$

$$\pi_t = \pi_0 P^t$$



$$\pi_3 = (0.5093, 0.2569, 0.2338)$$

Steady-State Probabilities

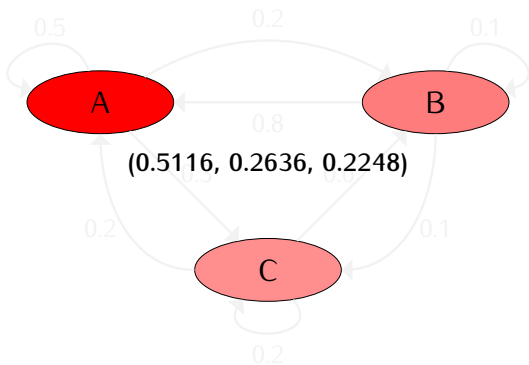
$$\pi = \pi P$$

$$\sum \pi = 1$$

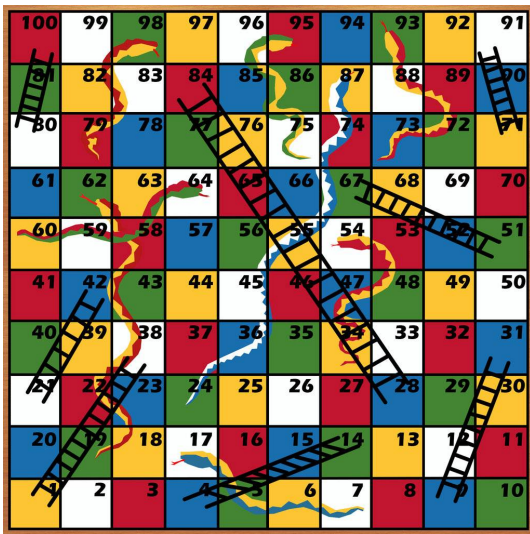
Steady-State Probabilities

$$\pi = \pi P$$

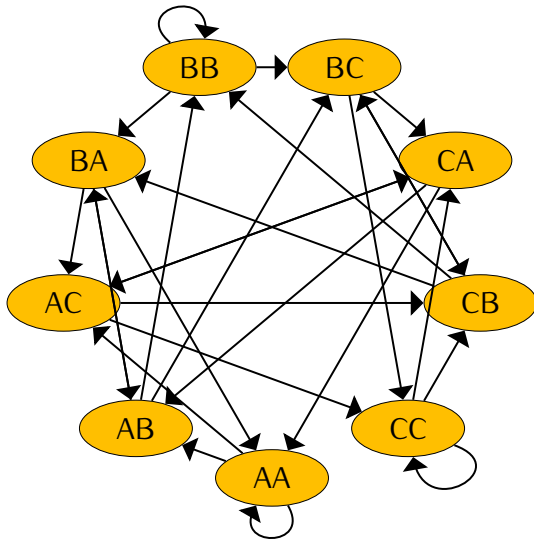
$$\sum \pi = 1$$



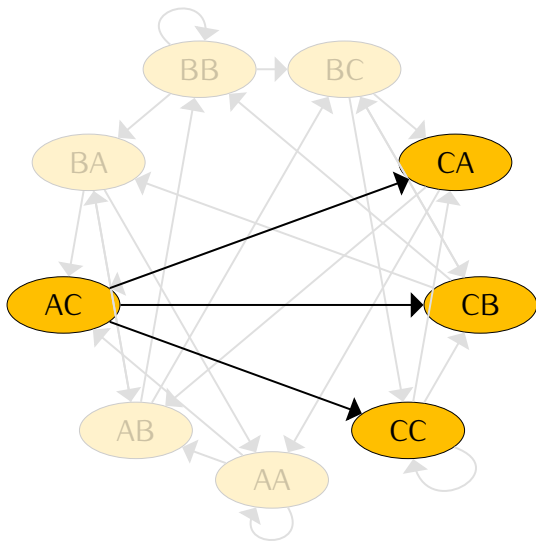
The Markov Property



Higher Order Markov Chains



Higher Order Markov Chains



the

GRAIN

MOTHER

WALL

SUCCESS

ZONE

SHADOW

JOB

T.A.R.D.I.S.

CLOCK

TIDE

DISAPPOINTMENT

against the

GRAIN

MOTHER

WALL

ZONE

SUCCESS

SHADOW

JOB

T.A.R.D.I.S.

CLOCK

TIDE

DISAPPOINTMENT

race against the

GRAIN

MOTHER

WALL

ZONE

SUCCESS

SHADOW

JOB

T.A.R.D.I.S.

CLOCK

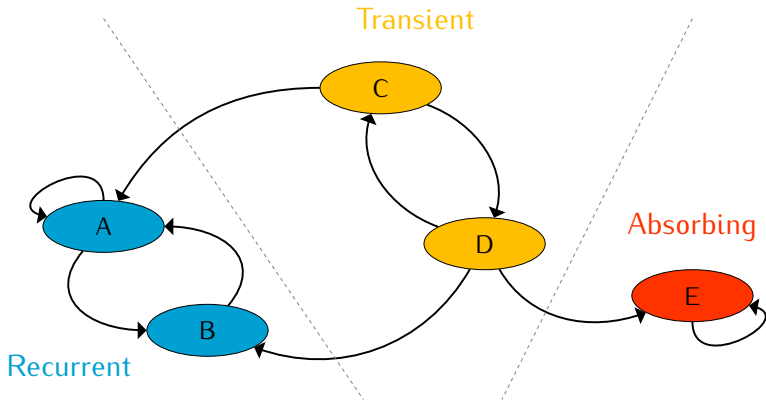
TIDE

DISAPPOINTMENT

Generating Music with Markov Chains

<https://www.youtube.com/watch?v=q0Z2Q-Ls48U>

Classification of States



Absorbing Markov Chains

Probability of Absorption

$$\mathbb{P}(\text{absorption in } t \text{ steps from } s) = P_{(s,a)}^t$$

$$\lim_{t \rightarrow \infty} P_{(s,a)}^t \rightarrow 1$$

Absorbing Markov Chains

Probability of Absorption

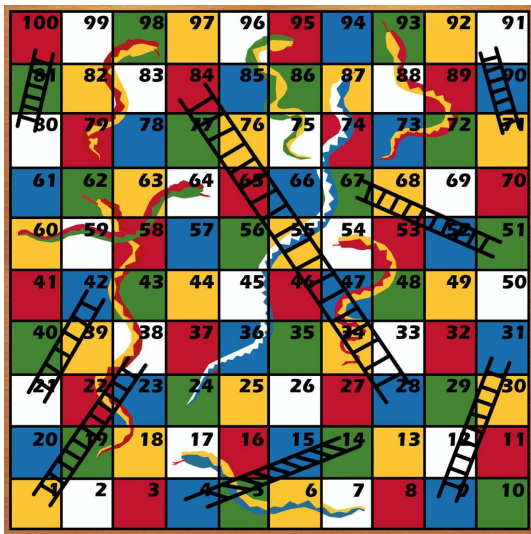
$$\mathbb{P}(\text{absorption in } t \text{ steps from } s) = P_{(s,a)}^t$$

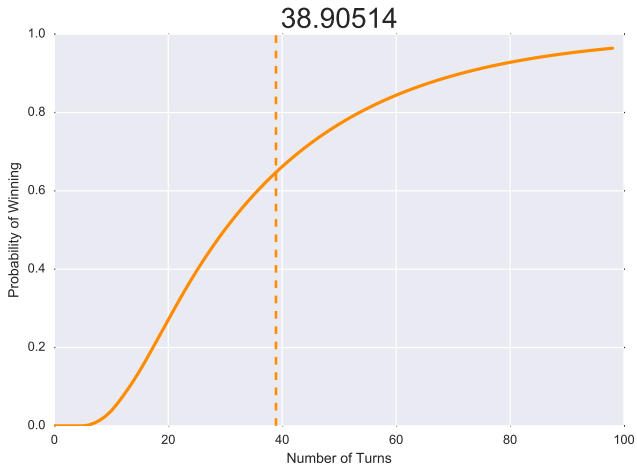
$$\lim_{t \rightarrow \infty} P_{(s,a)}^t \rightarrow 1$$

Mean Steps to Absorption

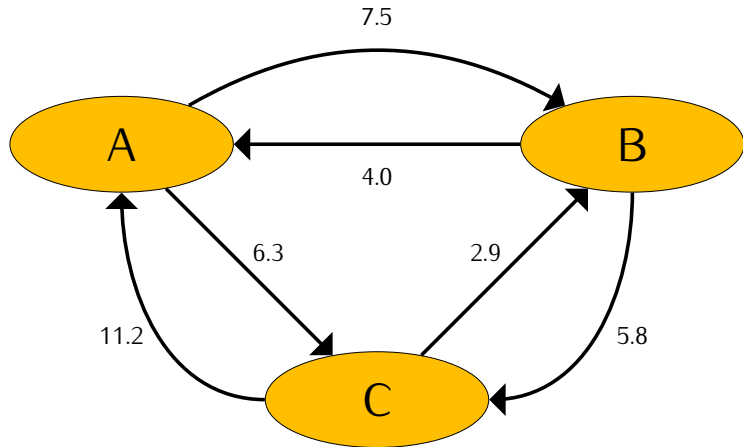
$$P = \begin{pmatrix} Q & R \\ 0 & 1 \end{pmatrix}$$

$$\mathbb{E}[\text{steps to absorption from } s] = (\mathbb{I} - Q)^{-1}_{(s)}$$





Continuous-Time Markov Chains



$$Q = \begin{pmatrix} -13.8 & 7.5 & 6.3 \\ 4.0 & -9.8 & 5.8 \\ 11.2 & 2.9 & -14.1 \end{pmatrix}$$

Discrete

$$\pi_t = \pi_0 P^t$$

$$\pi = \pi P$$

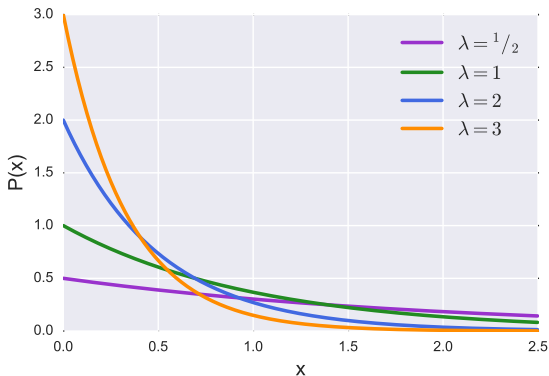
Continuous

$$\pi_t = \pi_0 \left(\mathbb{I} + \sum_{k=1}^{\infty} \frac{Q^k t^k}{k!} \right)$$

$$0 = \pi Q$$

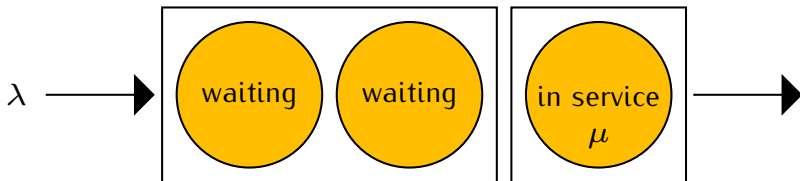
The Exponential Distribution

$$f(x) = \lambda e^{-\lambda x}$$

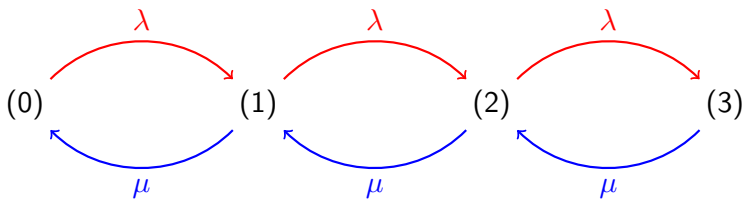


$$\mathbb{P}(T > s + t \mid T > s) = \mathbb{P}(T > s)$$

Modelling a Queue



Arrivals $\sim \text{Poisson}(\lambda)$
Service time $\sim \text{Exponential}(\mu)$



$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ \mu & -(\lambda + \mu) & \lambda & 0 \\ 0 & \mu & -(\lambda + \mu) & \lambda \\ 0 & 0 & \mu & -\mu \end{pmatrix}$$

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ \mu & -(\lambda + \mu) & \lambda & 0 \\ 0 & \mu & -(\lambda + \mu) & \lambda \\ 0 & 0 & \mu & -\mu \end{pmatrix}$$

$$\pi_0 = \frac{\mu^3}{\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3}$$

$$\pi_1 = \frac{\lambda\mu^2}{\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3}$$

$$\pi_2 = \frac{\lambda^2\mu}{\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3}$$

$$\pi_3 = \frac{\lambda^3}{\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3}$$

Thank You!