Problem Sheet 5

1. A toothpaste company can make two kinds of toothpaste:
   - 'Sure Smiles': a budget toothpaste that makes a profit of £1000 a tonne, and
   - 'Wicked Whites': a premium toothpaste that makes of profit of £8000 a tonne.

   Two of the ingredients need to be imported and so their daily use is limited: only 12 kilograms of Calcium Carbonate can be used each day, and only 24 kilograms of Sodium Fluoride can be used each day.
   - Each tonne of 'Sure Smiles' requires 3 kilograms of Sodium Fluoride and 1 kilogram of Calcium Carbonate.
   - Each tonne of 'Wicked Whites' requires 1 kilogram of Sodium Fluoride and 2 kilograms of Calcium Carbonate.

   Additionally, to ensure that there is enough budget toothpaste available to the population, the government has legislated that the company cannot produce more than 2 tonnes more of premium toothpaste than the budget toothpaste each day.

   (a) Using the graphical method, how many tonnes of each toothpaste should the company produce each day to maximise their daily profit?

   (b) If the government now legislates that the company can only make £1600 per tonne of 'Wicked Whites', how many tonnes of each toothpaste should the company produce each day to maximise their daily profit now?

2. Use the Simplex method to solve the following Linear programming problem:

Maximise:

$$3x_1 + 5x_2$$

subject to

$$\begin{aligned} -5x_1 + 17x_2 &\leq 425 \\ 5x_1 + 4x_2 &\leq 205 \\ x_1, x_2 &\geq 0 \end{aligned}$$

3. Consider the following linear programming problem:

Maximise:

$$3x_1 + x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 - x_2 + 4x_3 &\leq 17 \\ 2x_1 + x_3 &\leq 6 \\ 2x_2 + 3x_3 &\leq 14 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

   (a) Use the Simplex method to find one optimal solution.

   (b) Pivot one more time to find all optimal solutions.
   Give your answer in the form $\{(1-t)\underline{\mathbf{a}} + t\underline{\mathbf{b}} \text{ for all } t \in [0,1]\}$.

   (c) If we fix $x_3 = 1$, find the values that x_1 and x_2 must take for the solution to remain optimal.

4. Solve the following linear programming problem using the two-phase method:

Maximise:

$$2x_1 + 3x_2 + 4x_3$$

subject to

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &\leq 10 \\ 2x_1 + 3x_2 + 3x_3 &\leq 15 \\ x_1 + x_2 - x_3 &\geq 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

5. Cardiff University needs to create its exam timetable. It has a set M of exams (indexed by m) to schedule. For each pair of exams i, j, it has an indicator C_{ij} that is set to 1 if the modules cannot be scheduled at the same time (due to sharing students), and 0 if they can be scheduled at the same time. Let T be the set of time slots available, indexed by t. Formulate an linear programming problem that finds a feasible schedule using the least time slots.

   You are not asked to solve the linear programming problem!