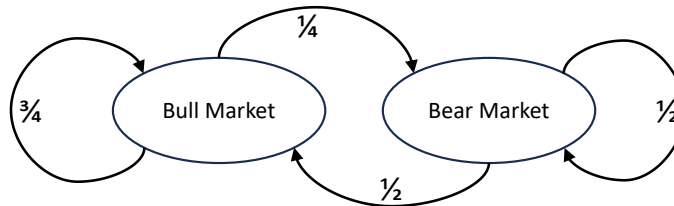


Solutions to Problem Sheet 2

1. A country's economy can be described as either a Bull market (where stock prices rise and things are going well), or a Bear market (where stock prices fall and things are not going so well). The economy is categorised as such each quarter. This process can be described as a discrete-time Markov chain, with probabilities of being in each state in the next quarter:



If the country is currently in a Bull market, what is the probability of being in either a Bull or a Bear market in three quarters times?

Solution 1 We have: $\pi_0 = (1, 0)$, and:

$$P = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/3 \end{pmatrix}$$

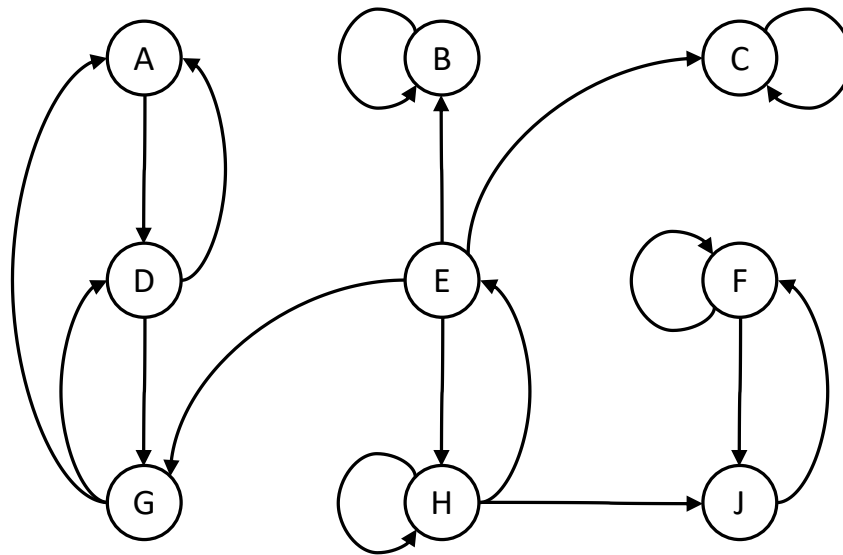
We want to find $\pi_3 = \pi_0 P^3$, so:

$$\begin{aligned}
 P^3 &= \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/3 \end{pmatrix} \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/3 \end{pmatrix} \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/3 \end{pmatrix} \\
 &= \begin{pmatrix} 9/16 + 1/8 & 3/16 + 1/8 \\ 3/8 + 1/4 & 1/8 + 1/4 \end{pmatrix} \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/3 \end{pmatrix} \\
 &= \begin{pmatrix} 11/16 & 5/16 \\ 5/8 & 3/8 \end{pmatrix} \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/3 \end{pmatrix} \\
 &= \begin{pmatrix} 33/64 + 5/32 & 11/64 + 5/32 \\ 15/32 + 3/16 & 5/32 + 3/16 \end{pmatrix} = \begin{pmatrix} 43/64 & 21/64 \\ 21/32 & 11/32 \end{pmatrix}
 \end{aligned}$$

And so:

$$\pi_3 = (1, 0) \begin{pmatrix} 43/64 & 21/64 \\ 21/32 & 11/32 \end{pmatrix} = (43/64, 21/64)$$

2. Consider the discrete-time Markov chain below with nine states. An arrow indicates that the probability of transitioning from one state to another is greater than zero.
- Identify all the irreducible classes and state whether they are closed or not.
 - Classify each state as either Recurrent, Transient, or Absorbing.



Solution 2 The irreducible classes are:

- $\{A, D, G\}$ which is closed,
- $\{B\}$ which is closed,
- $\{C\}$ which is closed,
- $\{E, H\}$ which is not closed,
- $\{F, J\}$ which is closed.

Therefore:

- the Recurrent states are: A, D, G, F, J ;
- the Transient states are: E, H ;
- the Absorbing states are: B, C .

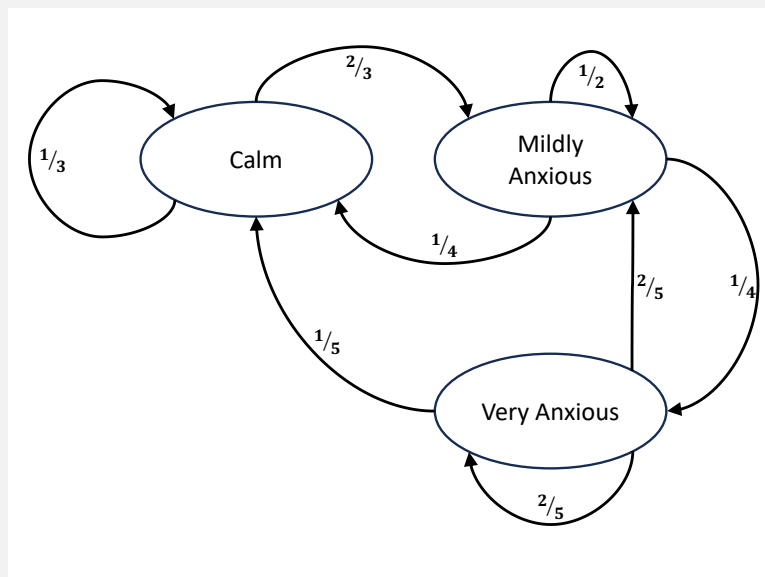
3. A mental health doctor is trying to understand a patient's mental state. They ask the patient to record daily whether they feel Calm, Mildly Anxious, or Very Anxious. Crunching the data the doctor finds:
- On a calm day, $\frac{1}{3}$ of the time they will remain calm tomorrow, and $\frac{2}{3}$ of the time they will become mildly anxious tomorrow;
 - On a mildly anxious day, $\frac{1}{4}$ of the time they will become calm tomorrow, $\frac{1}{2}$ the time they remain mildly anxious tomorrow, while $\frac{1}{4}$ of the time they become very anxious tomorrow;

- On a very anxious day, only $\frac{1}{5}$ of the time will they become calm tomorrow, $\frac{2}{5}$ of the time they will become mildly anxious, however $\frac{2}{5}$ of the time they remain very anxious tomorrow.
- (a) Draw the discrete-time Markov chain describing the patient's mental state.
- (b) Find the steady-state probabilities.
- (c) The doctor devises a medication plan: on calm days the patient should not take any medication; on mildly anxious days they should take a pill of type A, costing $1p$ per pill; and on very anxious days they should take a pill of type B, costing $23p$ per pill. What is the expected yearly cost for this medication plan?

Solution 3 Ordering the states 1-Calm, 2-Mildly Anxious, then 3-Very Anxious, we have:

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}$$

(a) Visualising the Markov chain:



(b) To find steady state we solve $\underline{\pi} = \underline{\pi}P$ and $\sum \underline{\pi} = 1$:

$$\pi_1 = \frac{1}{3}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{5}\pi_3 \quad (1)$$

$$\pi_2 = \frac{2}{3}\pi_1 + \frac{1}{2}\pi_2 + \frac{2}{5}\pi_3 \quad (2)$$

$$\pi_3 = \frac{1}{4}\pi_2 + \frac{2}{5}\pi_3 \quad (3)$$

$$1 = \pi_1 + \pi_2 + \pi_3 \quad (4)$$

Solution 3 (continuing from p. 3) We can solve for π_3 in Equation 3:

$$\begin{aligned}\pi_3 &= \frac{1}{4}\pi_2 + \frac{2}{5}\pi_3 \\ \frac{3}{5}\pi_3 &= \frac{1}{4}\pi_2 \\ \pi_3 &= \frac{5}{12}\pi_2\end{aligned}$$

And solve for π_2 in Equation 2:

$$\begin{aligned}\pi_2 &= \frac{2}{3}\pi_1 + \frac{1}{2}\pi_2 + \frac{2}{5}\pi_3 \\ \pi_2 &= \frac{2}{3}\pi_1 + \frac{1}{2}\pi_2 + \frac{2}{5}\left(\frac{5}{12}\pi_2\right) \\ \pi_2 &= \frac{2}{3}\pi_1 + \pi_2\left(\frac{1}{2} + \frac{1}{6}\right) \\ \frac{1}{3}\pi_2 &= \frac{2}{3}\pi_1 \\ \pi_2 &= 2\pi_1\end{aligned}$$

And finally solve for π_1 in Equation 4:

$$\begin{aligned}\pi_1 + \pi_2 + \pi_3 &= 1 \\ \pi_1\left(1 + 2 + \left(2 \times \frac{5}{12}\right)\right) &= 1 \\ \frac{23}{6}\pi_1 &= 1 \\ \pi_1 &= \frac{6}{23}\end{aligned}$$

Implying that $\underline{\pi} = (6/23, 12/23, 5/23)$.

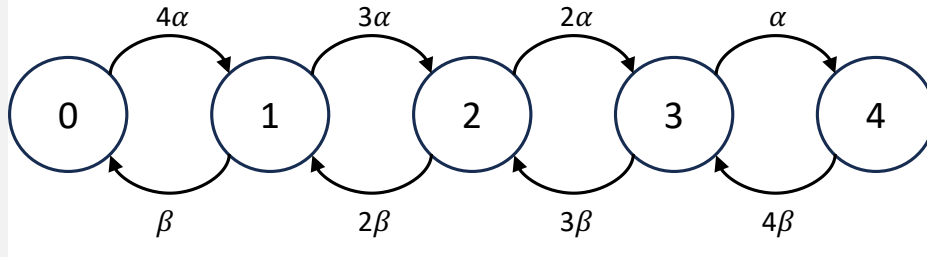
(c) It will cost $1p$ per pill each day they are in state 2, and $23p$ per pill each day they are in state 3. That is the yearly cost C is:

$$\begin{aligned}C &= 365(1\pi_2 + 23\pi_3)p \\ &= 365\left(\frac{12}{23} + 23\frac{5}{23}\right)p \\ &= 365\left(\frac{127}{23}\right)p \\ &= 2015.43p \\ &= \text{£}20.15\end{aligned}$$

4. A printing shop owns four printers. Each printer breaks down at a rate of β . Once broken down, it is sent for repair. The rate at which printers get repaired is α .

Letting i be the state that there are i printers in operation, draw the Markov chain for this system, and find the steady-state probabilities for general α and β , and for when $\alpha = \beta$.

Solution 4 When there are i printers in operation, there are $4 - i$ printers being repaired, so:



with:

$$Q = \begin{pmatrix} -4\alpha & 4\alpha & 0 & 0 & 0 \\ \beta & -(\beta + 3\alpha) & 3\alpha & 0 & 0 \\ 0 & 2\beta & -(2\beta + 2\alpha) & 2\alpha & 0 \\ 0 & 0 & 3\beta & -(3\beta + \alpha) & \alpha \\ 0 & 0 & 0 & 4\beta & -4\beta \end{pmatrix}$$

To find steady-state probabilities, solve $Q\pi = 0$ and $\sum \pi = 1$:

$$\begin{aligned} 4\alpha\pi_0 &= \beta\pi_1 \\ (\beta + 3\alpha)\pi_1 &= 4\alpha\pi_0 + 2\beta\pi_2 \\ (2\beta + 2\alpha)\pi_2 &= 3\alpha\pi_1 + 3\beta\pi_3 \\ (3\beta + \alpha)\pi_3 &= 2\alpha\pi_3 + 4\beta\pi_4 \\ 4\beta\pi_4 &= \alpha\pi_3 \end{aligned}$$

Putting everything in terms of π_0 gives, for π_1 :

$$\pi_1 = \frac{4\alpha}{\beta}\pi_0$$

For π_2 :

$$\begin{aligned} (\beta + 3\alpha)\pi_1 &= 4\alpha\pi_0 + 2\beta\pi_2 \\ (\beta + 3\alpha)\left(\frac{4\alpha}{\beta}\right)\pi_0 &= 4\alpha\pi_0 + 2\beta\pi_2 \\ \frac{12\alpha^2}{\beta}\pi_0 &= 2\beta\pi_2 \\ \pi_2 &= \frac{6\alpha^2}{\beta^2}\pi_0 \end{aligned}$$

Solution 4 (continuing from p. 5) For π_3 :

$$\begin{aligned}(2\beta + 2\alpha)\pi_2 &= 3\alpha\pi_1 + 3\beta\pi_3 \\ (2\beta + 2\alpha) \left(\frac{6\alpha^2}{\beta^2} \right) \pi_0 &= 3\alpha \left(\frac{4\alpha}{\beta} \right) \pi_0 + 3\beta\pi_3 \\ \frac{12\alpha^3}{\beta^2} \pi_0 &= 3\beta\pi_3 \\ \pi_3 &= \frac{4\alpha^3}{\beta^3} \pi_0\end{aligned}$$

And finally for π_4 :

$$\begin{aligned}4\beta\pi_4 &= \alpha\pi_3 \\ \pi_4 &= \frac{\alpha}{4\beta} \pi_3 \\ \pi_4 &= \frac{\alpha^4}{\beta^4} \pi_0\end{aligned}$$

And we get the value of π_0 with:

$$\begin{aligned}\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1 \\ \pi_0 \left(1 + \frac{4\alpha}{\beta} + \frac{6\alpha^2}{\beta^2} + \frac{4\alpha^3}{\beta^3} + \frac{\alpha^4}{\beta^4} \right) &= 1 \\ \pi_0 &= \frac{1}{\left(1 + \frac{4\alpha}{\beta} + \frac{6\alpha^2}{\beta^2} + \frac{4\alpha^3}{\beta^3} + \frac{\alpha^4}{\beta^4} \right)}\end{aligned}$$

Now when $\alpha = \beta$:

$$\underline{\pi} = (1/16, 1/4, 3/8, 1/4, 1/16)$$