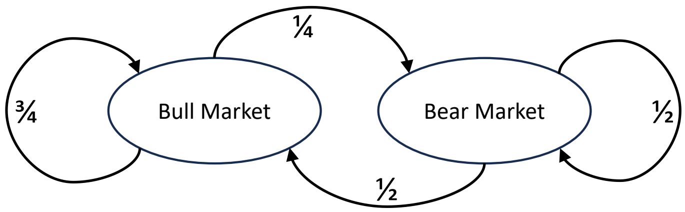
Solutions to Problem Sheet 2

1. A country's economy can be described as either a Bull market (where stock prices rise and things are going well), or a Bear market (where stock prices fall and things are not going so well). The economy is categorised as such each quarter. This process can be described as a discrete-time Markov chain, with probabilities of being in each state in the next quarter:



If the country is currently in a Bull market, what is the probability of being in either a Bull or a Bear marker in three quarters times?

Solution 1 *We have:* $\pi_0 = (1, 0)$, *and:*

$$P = \begin{pmatrix} {}^{3}\!/_{4} & {}^{1}\!/_{4} \\ {}^{1}\!/_{2} & {}^{1}\!/_{3} \end{pmatrix}$$

We want to find $\pi_3 = \pi_0 P^3$, *so:*

$$\begin{aligned} P^3 &= \begin{pmatrix} {}^{3}\!/_{4} & {}^{1}\!/_{4} \\ {}^{1}\!/_{2} & {}^{1}\!/_{3} \end{pmatrix} \begin{pmatrix} {}^{3}\!/_{4} & {}^{1}\!/_{4} \\ {}^{1}\!/_{2} & {}^{1}\!/_{3} \end{pmatrix} \begin{pmatrix} {}^{3}\!/_{4} & {}^{1}\!/_{4} \\ {}^{1}\!/_{2} & {}^{1}\!/_{3} \end{pmatrix} \\ &= \begin{pmatrix} {}^{9}\!/_{16} + {}^{1}\!/_{8} & {}^{3}\!/_{16} + {}^{1}\!/_{8} \\ {}^{3}\!/_{8} + {}^{1}\!/_{4} & {}^{1}\!/_{8} + {}^{1}\!/_{4} \end{pmatrix} \begin{pmatrix} {}^{3}\!/_{4} & {}^{1}\!/_{4} \\ {}^{1}\!/_{2} & {}^{1}\!/_{3} \end{pmatrix} \\ &= \begin{pmatrix} {}^{11}\!/_{16} & {}^{5}\!/_{16} \\ {}^{5}\!/_{8} & {}^{3}\!/_{8} \end{pmatrix} \begin{pmatrix} {}^{3}\!/_{4} & {}^{1}\!/_{4} \\ {}^{1}\!/_{2} & {}^{1}\!/_{3} \end{pmatrix} \\ &= \begin{pmatrix} {}^{33}\!/_{64} + {}^{5}\!/_{32} & {}^{11}\!/_{64} + {}^{5}\!/_{32} \\ {}^{15}\!/_{32} + {}^{3}\!/_{16} & {}^{5}\!/_{32} + {}^{3}\!/_{16} \end{pmatrix} = \begin{pmatrix} {}^{43}\!/_{64} & {}^{21}\!/_{64} \\ {}^{21}\!/_{32} & {}^{11}\!/_{32} \end{pmatrix} \end{aligned}$$

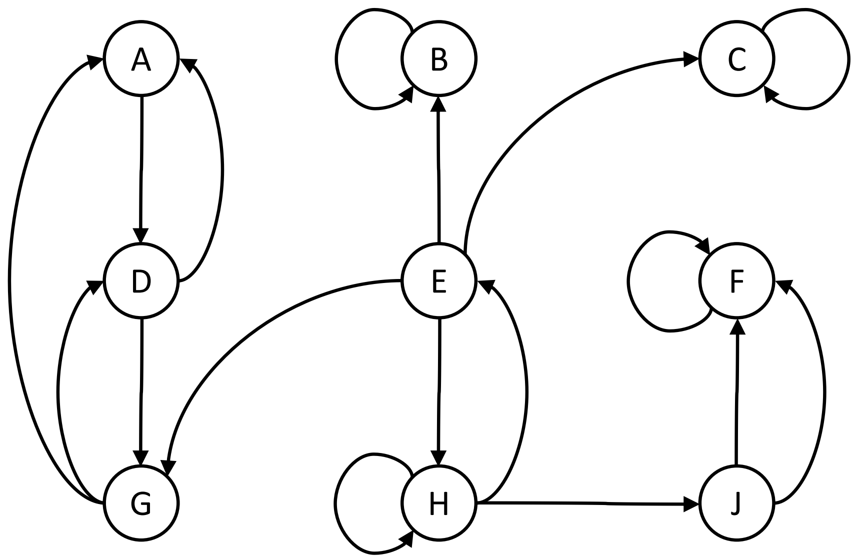
And so:

$$\pi_3 = (1, 0) \begin{pmatrix} {}^{43}\!/_{64} & {}^{21}\!/_{64} \\ {}^{21}\!/_{32} & {}^{11}\!/_{32} \end{pmatrix} = ({}^{43}\!/_{64}, {}^{21}\!/_{64})$$

2. Consider the discrete-time Markov chain below with nine states. An arrow indicates that the probability of transitioning from one state to another is greater than zero.

   (a) Identify all the irreducible classes and state whether they are closed or not.

   (b) Classify each state as either Recurrent, Transient, or Absorbing.



Solution 2 *The irreducible classes are:*

- $\{A, D, G\}$ *which is closed,*
- $\{B\}$ *which is closed,*
- $\{C\}$ *which is closed,*
- $\{E, H\}$ *which is not closed,*
- $\{F, J\}$ *which is closed.*

Therefore:

- *the Recurrent states are:* A, D, G, F, J;
- *the Transient states are:* E, H;
- *the Absorbing states are:* B, C.

3. A mental health doctor is trying to understand a patient's mental state. They ask the patient to record daily whether they feel Calm, Mildly Anxious, or Very Anxious. Crunching the data the doctor finds:

   - On a calm day, $1/3$ of the time they will remain calm tomorrow, and $2/3$ of the time they will become mildly anxious tomorrow;
   - On a mildly anxious day, $1/4$ of the time they will become calm tomorrow, $1/2$ the time they remain mildly anxious tomorrow, while $1/4$ of the time they become very anxious tomorrow;

- On a very anxious day, only $1/5$ of the time will they become calm tomorrow, $2/5$ of the time they will become mildly anxious, however $2/5$ of the time they remain very anxious tomorrow.

(a) Draw the discrete-time Markov chain describing the patient's mental state.

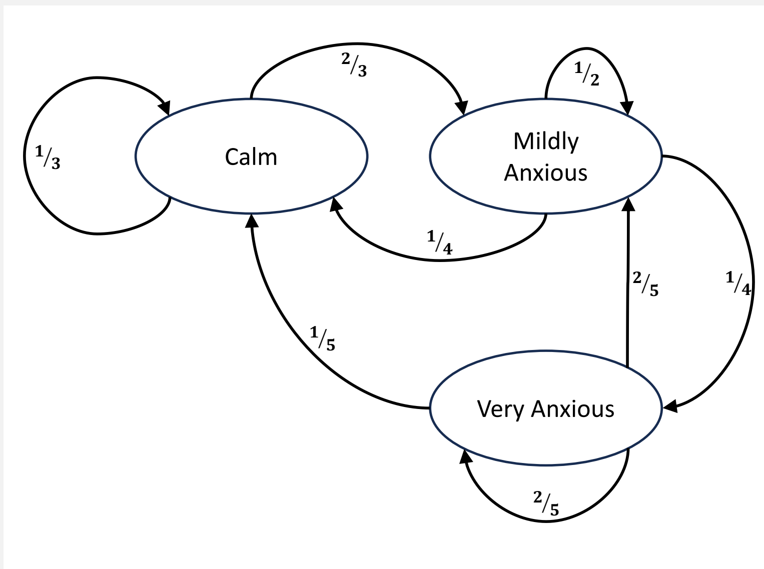
(b) Find the steady-state probabilities.

(c) The doctor devises a medication plan: on calm days the patient should not take any medication; on mildly anxious days they should take a pill of type A, costing $1p$ per pill; and on very anxious days they should take a pill of type B, costing 23p per pill. What is the expected yearly cost for this medication plan?

Solution 3 *Ordering the states 1-Calm, 2-Mildly Anxious, then 3-Very Anxious, we have:*

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/4 & 1/2 & 1/4 \\ 1/5 & 2/5 & 2/5 \end{pmatrix}$$

(a) Visualising the Markov chain:



(b) To find steady state we solve $\underline{\pi} = \underline{\pi}P$ and $\sum \underline{\pi} = 1$:

$$\pi_1 = \frac{1}{3}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{5}\pi_3 \tag{1}$$
$$\pi_2 = \frac{2}{3}\pi_1 + \frac{1}{2}\pi_2 + \frac{2}{5}\pi_3 \tag{2}$$
$$\pi_3 = \frac{1}{4}\pi_2 + \frac{2}{5}\pi_3 \tag{3}$$
$$1 = \pi_1 + \pi_2 + \pi_3 \tag{4}$$

Solution 3 (continuing from p. 3) *We can solve for π_3 in Equation 3:*

$$\begin{aligned}\pi_3 &= \frac{1}{4}\pi_2 + \frac{2}{5}\pi_3 \\ \frac{3}{5}\pi_3 &= \frac{1}{4}\pi_2 \\ \pi_3 &= \frac{5}{12}\pi_2\end{aligned}$$

And solve for π_2 in Equation 2:

$$\begin{aligned}\pi_2 &= \frac{2}{3}\pi_1 + \frac{1}{2}\pi_2 + \frac{2}{5}\pi_3 \\ \pi_2 &= \frac{2}{3}\pi_1 + \frac{1}{2}\pi_2 + \frac{2}{5}\left(\frac{5}{12}\pi_2\right) \\ \pi_2 &= \frac{2}{3}\pi_1 + \pi_2\left(\frac{1}{2} + \frac{1}{6}\right) \\ \frac{1}{3}\pi_2 &= \frac{2}{3}\pi_1 \\ \pi_2 &= 2\pi_1\end{aligned}$$

And finally solve for π_1 in Equation 4:

$$\begin{aligned}\pi_1 + \pi_2 + \pi_3 &= 1 \\ \pi_1\left(1 + 2 + \left(2 \times \frac{5}{12}\right)\right) &= 1 \\ \frac{23}{6}\pi_1 &= 1 \\ \pi_1 &= \frac{6}{23}\end{aligned}$$

Implying that $\underline{\pi} = (^6/_{23}, ^{12}/_{23}, ^5/_{23})$.

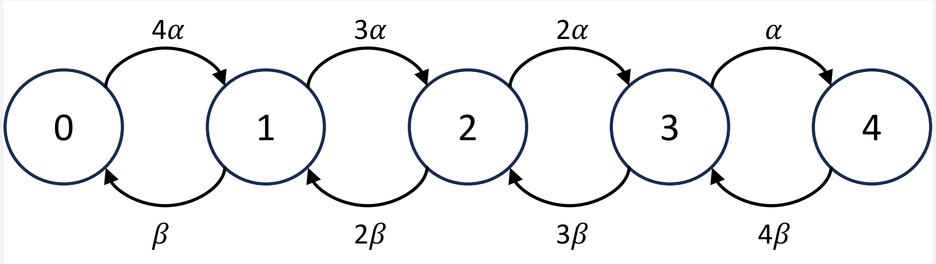
(c) It will cost $1p$ per pill each day they are in state 2, and $23p$ per pill each day they are in state 3. That is the yearly cost C is:

$$\begin{aligned}C &= 365\left(1\pi_2 + 23\pi_3\right)p \\ &= 365\left(^{12}/_{23} + 23^5/_{23}\right)p \\ &= 365\left(^{127}/_{23}\right)p \\ &= 2015.43p \\ &= £20.15\end{aligned}$$

4. A printing shop owns four printers. Each printer breaks down at a rate of β. Once broken down, it is sent for repair. The rate at which printers get repaired is α.

   Letting i be the state that there are i printers in operation, draw the Markov chain for this system, and find the steady-state probabilities for general α and β, and for when $\alpha = \beta$.

Solution 4 *When there are i printers in operation, there are $4 - i$ printers being repaired, so:*



with:

$$Q = \begin{pmatrix} -4\alpha & 4\alpha & 0 & 0 & 0 \\ \beta & -(\beta+3\alpha) & 3\alpha & 0 & 0 \\ 0 & 2\beta & -(2\beta+2\alpha) & 2\alpha & 0 \\ 0 & 0 & 3\beta & -(3\beta+\alpha) & \alpha \\ 0 & 0 & 0 & 4\beta & -4\beta \end{pmatrix}$$

To find steady-state probabilities, solve $Q\underline{\pi} = 0$ and $\sum \underline{\pi} = 1$:

$$\begin{aligned} 4\alpha\pi_0 &= \beta\pi_1 \\ (\beta+3\alpha)\pi_1 &= 4\alpha\pi_0 + 2\beta\pi_2 \\ (2\beta+2\alpha)\pi_2 &= 3\alpha\pi_1 + 3\beta\pi_3 \\ (3\beta+\alpha)\pi_3 &= 2\alpha\pi_3 + 4\beta\pi_4 \\ 4\beta\pi_4 &= \alpha\pi_3 \end{aligned}$$

Putting everything in terms of π_0 gives, for π_1:

$$\pi_1 = \frac{4\alpha}{\beta}\pi_0$$

For π_2:

$$\begin{aligned} (\beta+3\alpha)\pi_1 &= 4\alpha\pi_0 + 2\beta\pi_2 \\ (\beta+3\alpha)\left(\frac{4\alpha}{\beta}\right)\pi_0 &= 4\alpha\pi_0 + 2\beta\pi_2 \\ \frac{12\alpha^2}{\beta}\pi_0 &= 2\beta\pi_2 \\ \pi_2 &= \frac{6\alpha^2}{\beta^2}\pi_0 \end{aligned}$$

Solution 4 (continuing from p. 5) *For π_3:*

$$
\begin{aligned}
(2\beta + 2\alpha)\pi_2 &= 3\alpha\pi_1 + 3\beta\pi_3 \\
(2\beta + 2\alpha)\left(\frac{6\alpha^2}{\beta^2}\right)\pi_0 &= 3\alpha\left(\frac{4\alpha}{\beta}\right)\pi_0 + 3\beta\pi_3 \\
\frac{12\alpha^3}{\beta_2}\pi_0 &= 3\beta\pi_3 \\
\pi_3 &= \frac{4\alpha^3}{\beta_3}\pi_0
\end{aligned}
$$

And finally for π_4:

$$
\begin{aligned}
4\beta\pi_4 &= \alpha\pi_3 \\
\pi_4 &= \frac{\alpha}{4\beta}\pi_3 \\
\pi_4 &= \frac{\alpha^4}{\beta^4}\pi_0
\end{aligned}
$$

And we get the value of π_0 with:

$$
\begin{aligned}
\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1 \\
\pi_0\left(1 + \frac{4\alpha}{\beta} + \frac{6\alpha^2}{\beta^2} + \frac{4\alpha^3}{\beta^3} + \frac{\alpha^4}{\beta^4}\right) &= 1 \\
\pi_0 &= \frac{1}{\left(1 + \frac{4\alpha}{\beta} + \frac{6\alpha^2}{\beta^2} + \frac{4\alpha^3}{\beta^3} + \frac{\alpha^4}{\beta^4}\right)}
\end{aligned}
$$

Now when $\alpha = \beta$:

$$\underline{\pi} = ({}^1/_{16}, {}^1/_4, {}^3/_8, {}^1/_4, {}^1/_{16})$$