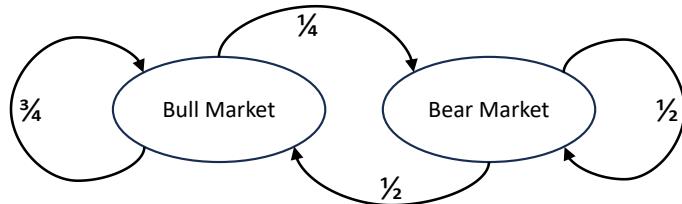


## Solutions to Problem Sheet 2

1. A country's economy can be described as either a Bull market (where stock prices rise and things are going well), or a Bear market (where stock prices fall and things are not going so well). The economy is categorised as such each quarter. This process can be described as a discrete-time Markov chain, with probabilities of being in each state in the next quarter:



If the country is currently in a Bull market, what is the probability of being in either a Bull or a Bear market in three quarters times?

**Solution 1** We have:  $\pi_0 = (1, 0)$ , and:

$$P = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/3 \end{pmatrix}$$

We want to find  $\pi_3 = \pi_0 P^3$ , so:

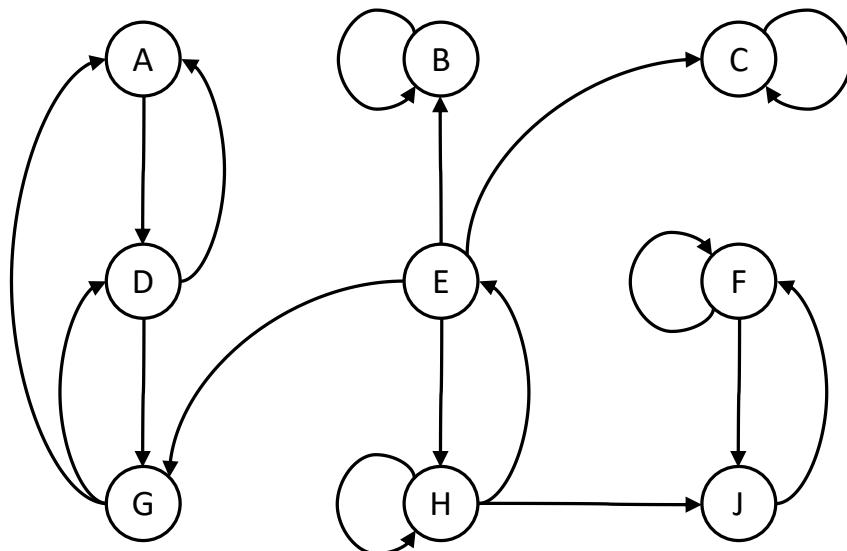
$$\begin{aligned} P^3 &= \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/3 \end{pmatrix} \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/3 \end{pmatrix} \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/3 \end{pmatrix} \\ &= \begin{pmatrix} 9/16 + 1/8 & 3/16 + 1/8 \\ 3/8 + 1/4 & 1/8 + 1/4 \end{pmatrix} \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/3 \end{pmatrix} \\ &= \begin{pmatrix} 11/16 & 5/16 \\ 5/8 & 3/8 \end{pmatrix} \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/3 \end{pmatrix} \\ &= \begin{pmatrix} 33/64 + 5/32 & 11/64 + 5/32 \\ 15/32 + 3/16 & 5/32 + 3/16 \end{pmatrix} = \begin{pmatrix} 43/64 & 21/64 \\ 21/32 & 11/32 \end{pmatrix} \end{aligned}$$

And so:

$$\pi_3 = (1, 0) \begin{pmatrix} 43/64 & 21/64 \\ 21/32 & 11/32 \end{pmatrix} = (43/64, 21/64)$$

2. Consider the discrete-time Markov chain below with nine states. An arrow indicates that the probability of transitioning from one state to another is greater than zero.

- (a) Identify all the irreducible classes and state whether they are closed or not.
- (b) Classify each state as either Recurrent, Transient, or Absorbing.



**Solution 2** The irreducible classes are:

- $\{A, D, G\}$  which is closed,
- $\{B\}$  which is closed,
- $\{C\}$  which is closed,
- $\{E, H\}$  which is not closed,
- $\{F, J\}$  which is closed.

Therefore:

- the Recurrent states are: A, D, G, F, J;
- the Transient states are: E, H;
- the Absorbing states are: B, C.

3. A mental health doctor is trying to understand a patient's mental state. They ask the patient to record daily whether they feel Calm, Mildly Anxious, or Very Anxious. Crunching the data the doctor finds:

- On a calm day,  $\frac{1}{3}$  of the time they will remain calm tomorrow, and  $\frac{2}{3}$  of the time they will become mildly anxious tomorrow;
- On a mildly anxious day,  $\frac{1}{4}$  of the time they will become calm tomorrow,  $\frac{1}{2}$  the time they remain mildly anxious tomorrow, while  $\frac{1}{4}$  of the time they become very anxious tomorrow;

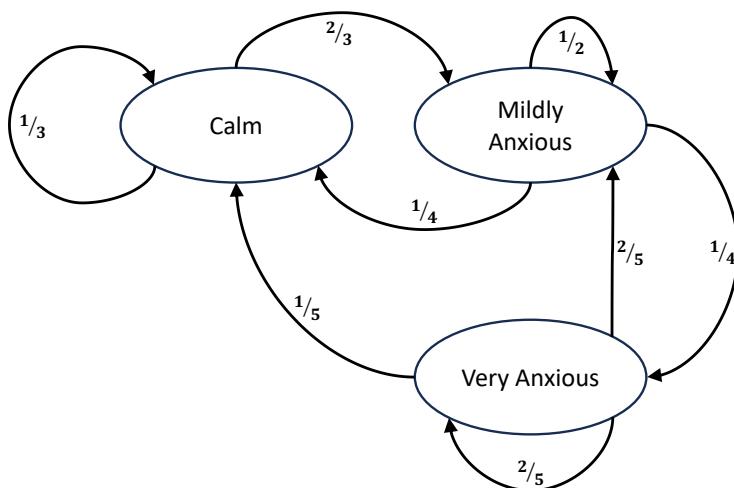
- On a very anxious day, only  $\frac{1}{5}$  of the time will they become calm tomorrow,  $\frac{2}{5}$  of the time they will become mildly anxious, however  $\frac{2}{5}$  of the time they remain very anxious tomorrow.

- Draw the discrete-time Markov chain describing the patient's mental state.
- Find the steady-state probabilities.
- The doctor devises a medication plan: on calm days the patient should not take any medication; on mildly anxious days they should take a pill of type A, costing 1p per pill; and on very anxious days they should take a pill of type B, costing 23p per pill. What is the expected yearly cost for this medication plan?

**Solution 3** Ordering the states 1-Calm, 2-Mildly Anxious, then 3-Very Anxious, we have:

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/4 & 1/2 & 1/4 \\ 1/5 & 2/5 & 2/5 \end{pmatrix}$$

(a) Visualising the Markov chain:



(b) To find steady state we solve  $\underline{\pi} = \underline{\pi}P$  and  $\sum \underline{\pi} = 1$ :

$$\pi_1 = \frac{1}{3}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{5}\pi_3 \quad (1)$$

$$\pi_2 = \frac{2}{3}\pi_1 + \frac{1}{2}\pi_2 + \frac{2}{5}\pi_3 \quad (2)$$

$$\pi_3 = \frac{1}{4}\pi_2 + \frac{2}{5}\pi_3 \quad (3)$$

$$1 = \pi_1 + \pi_2 + \pi_3 \quad (4)$$

**Solution 3 (continuing from p. 3)** We can solve for  $\pi_3$  in Equation 3:

$$\begin{aligned}\pi_3 &= \frac{1}{4}\pi_2 + \frac{2}{5}\pi_3 \\ \frac{3}{5}\pi_3 &= \frac{1}{4}\pi_2 \\ \pi_3 &= \frac{5}{12}\pi_2\end{aligned}$$

And solve for  $\pi_2$  in Equation 2:

$$\begin{aligned}\pi_2 &= \frac{2}{3}\pi_1 + \frac{1}{2}\pi_2 + \frac{2}{5}\pi_3 \\ \pi_2 &= \frac{2}{3}\pi_1 + \frac{1}{2}\pi_2 + \frac{2}{5}\left(\frac{5}{12}\pi_2\right) \\ \pi_2 &= \frac{2}{3}\pi_1 + \pi_2\left(\frac{1}{2} + \frac{1}{6}\right) \\ \frac{1}{3}\pi_2 &= \frac{2}{3}\pi_1 \\ \pi_2 &= 2\pi_1\end{aligned}$$

And finally solve for  $\pi_1$  in Equation 4:

$$\begin{aligned}\pi_1 + \pi_2 + \pi_3 &= 1 \\ \pi_1 \left(1 + 2 + \left(2 \times \frac{5}{12}\right)\right) &= 1 \\ \frac{23}{6}\pi_1 &= 1 \\ \pi_1 &= \frac{6}{23}\end{aligned}$$

Implying that  $\underline{\pi} = (6/23, 12/23, 5/23)$ .

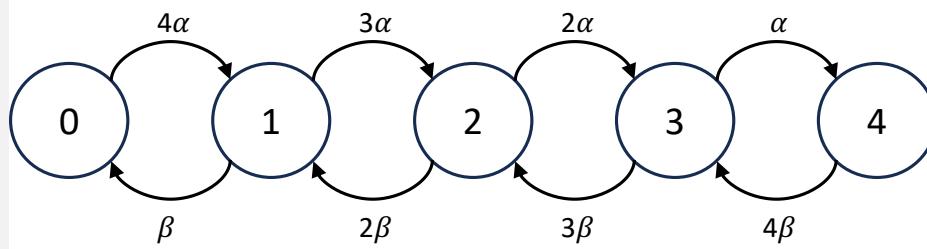
- (c) It will cost 1p per pill each day they are in state 2, and 23p per pill each day they are in state 3. That is the yearly cost  $C$  is:

$$\begin{aligned}C &= 365(1\pi_2 + 23\pi_3)p \\ &= 365(12/23 + 23^5/23)p \\ &= 365(127/23)p \\ &= 2015.43p \\ &= £20.15\end{aligned}$$

4. A printing shop owns four printers. Each printer breaks down at a rate of  $\beta$ . Once broken down, it is sent for repair. The rate at which printers get repaired is  $\alpha$ .

Letting  $i$  be the state that there are  $i$  printers in operation, draw the Markov chain for this system, and find the steady-state probabilities for general  $\alpha$  and  $\beta$ , and for when  $\alpha = \beta$ .

**Solution 4** When there are  $i$  printers in operation, there are  $4 - i$  printers being repaired, so:



with:

$$Q = \begin{pmatrix} -4\alpha & 4\alpha & 0 & 0 & 0 \\ \beta & -(\beta + 3\alpha) & 3\alpha & 0 & 0 \\ 0 & 2\beta & -(2\beta + 2\alpha) & 2\alpha & 0 \\ 0 & 0 & 3\beta & -(3\beta + \alpha) & \alpha \\ 0 & 0 & 0 & 4\beta & -4\beta \end{pmatrix}$$

To find steady-state probabilities, solve  $Q\pi = 0$  and  $\sum \pi = 1$ :

$$\begin{aligned} 4\alpha\pi_0 &= \beta\pi_1 \\ (\beta + 3\alpha)\pi_1 &= 4\alpha\pi_0 + 2\beta\pi_2 \\ (2\beta + 2\alpha)\pi_2 &= 3\alpha\pi_1 + 3\beta\pi_3 \\ (3\beta + \alpha)\pi_3 &= 2\alpha\pi_2 + 4\beta\pi_4 \\ 4\beta\pi_4 &= \alpha\pi_3 \end{aligned}$$

Putting everything in terms of  $\pi_0$  gives, for  $\pi_1$ :

$$\pi_1 = \frac{4\alpha}{\beta}\pi_0$$

For  $\pi_2$ :

$$\begin{aligned} (\beta + 3\alpha)\pi_1 &= 4\alpha\pi_0 + 2\beta\pi_2 \\ (\beta + 3\alpha)\left(\frac{4\alpha}{\beta}\right)\pi_0 &= 4\alpha\pi_0 + 2\beta\pi_2 \\ \frac{12\alpha^2}{\beta}\pi_0 &= 2\beta\pi_2 \\ \pi_2 &= \frac{6\alpha^2}{\beta^2}\pi_0 \end{aligned}$$

**Solution 4 (continuing from p. 5)** For  $\pi_3$ :

$$\begin{aligned} (2\beta + 2\alpha)\pi_2 &= 3\alpha\pi_1 + 3\beta\pi_3 \\ (2\beta + 2\alpha) \left( \frac{6\alpha^2}{\beta^2} \right) \pi_0 &= 3\alpha \left( \frac{4\alpha}{\beta} \right) \pi_0 + 3\beta\pi_3 \\ \frac{12\alpha^3}{\beta^2} \pi_0 &= 3\beta\pi_3 \\ \pi_3 &= \frac{4\alpha^3}{\beta^3} \pi_0 \end{aligned}$$

And finally for  $\pi_4$ :

$$\begin{aligned} 4\beta\pi_4 &= \alpha\pi_3 \\ \pi_4 &= \frac{\alpha}{4\beta}\pi_3 \\ \pi_4 &= \frac{\alpha^4}{\beta^4}\pi_0 \end{aligned}$$

And we get the value of  $\pi_0$  with:

$$\begin{aligned} \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1 \\ \pi_0 \left( 1 + \frac{4\alpha}{\beta} + \frac{6\alpha^2}{\beta^2} + \frac{4\alpha^3}{\beta^3} + \frac{\alpha^4}{\beta^4} \right) &= 1 \\ \pi_0 &= \frac{1}{\left( 1 + \frac{4\alpha}{\beta} + \frac{6\alpha^2}{\beta^2} + \frac{4\alpha^3}{\beta^3} + \frac{\alpha^4}{\beta^4} \right)} \end{aligned}$$

Now when  $\alpha = \beta$ :

$$\underline{\pi} = (1/16, 1/4, 3/8, 1/4, 1/16)$$