Solutions to Problem Sheet 6

1. Assume that the costs $c_1, c_2, c_3, c_4, c_5, c_6 > 0$ and all distinct. Explain why the solution below cannot be optimal.

	X	Y	Z	
A	30 c_1	c_2	70 c_3	100
B	30 c_4	80 c_5	40 c_6	150
	60	80	110	250

Solution 1 *With 2 sources and 3 destinations, the optimal solution should have 2+3-1=4 basic variables. However this solution has 5, and so cannot be optimal.*

2. Consider the following transportation problem, with four sources A, B, C and D, and four destinations W, X, Y and Z:

	W	X	Y	Z	
A	6	3	10	2	5
B	1	14	7	4	19
C	5	9	8	12	6
D	2	11	15	13	5
	7	13	11	4	35

(a) How many basic and non-basic variables does this problem have?

(b) Find a feasible solution using the North West Corner method. What would be the cost of using this solution?

(c) Find a feasible solution using the Minimum Cost method. What would be the cost of using this solution?

Solution 2 (a) *There are 4 sources and 4 destinations. So we would expect 4+4-1=7 basic variables, and so 9 non basic variables.*

(b) *Using the North West Corner method:*

	W	X	Y	Z	
A	5 6	3	10	2	5
B	2 1	13 14	4 7	4	19
C	5	9	6 8	12	6
D	2	11	1 15	4 13	5
	7	13	11	4	35

This results in an overall cost of: $(5\times6)+(2\times1)+(13\times14)+(4\times7)+(6\times8)+(1\times15)+(5\times13) = 357$.

(c) *Using the Minimum Cost method:*

	W	X	Y	Z	
A	6	1 3	10	4 2	5
B	7 1	1 14	11 7	4	19
C	5	6 9	8	12	6
D	2	5 11	15	13	5
	7	13	11	4	35

This results in an overall cost of: $(1\times3)+(4\times2)+(7\times1)+(1\times14)+(6\times9)+(5\times11) = 218$.

3. ToyAutos is a small car manufacturer based in Wales, producing 100 vehicles per year. They have three factories, one in Barry producing 35 vehicles a year, one in Port Talbot producing 45 vehicles a year, and another in Wrexham producing 20 vehicles a year. They have three dealerships, one in Bangor selling 15 cars a year, one in Cardiff selling 70 cars a year, and another in Newport selling 15 cars a year.

   According to Google Maps, the distances between each of the factories and dealerships is as follows:

From	To	Miles
Barry	Bangor	188
Barry	Cardiff	10
Barry	Newport	25
Port Talbot	Bangor	165
Port Talbot	Cardiff	36
Port Talbot	Newport	42
Wrexham	Bangor	64
Wrexham	Cardiff	140
Wrexham	Newport	126

   ToyAutos would like to know how many cars to send to each dealership from each factory, to satisfy the supply and demand in each location, and to minimise the total number of miles travelled.

   (Hint: begin by producing an initial feasible solution with the North West Corner method, then use the Stepping-Stone algorithm)

Solution 3 *First set up the table and fill in using the North West Corner method:*

	Bangor (B)	*Cardiff (C)*	*Newport (N)*	
Barry (B)	*15* *188*	*20* *10*	*25*	*35*
Port Talbot (P)	*165*	*45* *36*	*42*	*45*
Wrexham (W)	*64*	*5* *140*	*15* *126*	*20*
	15	*70*	*15*	*100*

Solution 3 (continuing from p. 3) *The first round of the stepping stone algorithm:*

Path	*Cost*
BN → WN → WC → BC	+ 25 - 126 + 140 - 10 = 29
PB → BB → BC → PC	+ 165 - 188 + 10 - 36 = -49
PN → WN → WC → PC	+ 42 - 126 + 140 - 36 = 20
WB → BB → BC → WC	+ 64 - 188 + 10 - 140 = -254

So we choose to increase WB by 5:

	Bangor (B)	Cardiff (C)	Newport (N)	
Barry (B)	10 188	25 10	25	35
Port Talbot (P)	165	45 36	42	45
Wrexham (W)	5 64	140	15 126	20
	15	70	15	100

The second round of the stepping stone algorithm:

Path	*Cost*
BN → WN → WB → BB	+ 25 - 126 + 64 - 188 = -225
PB → BB → BC → PC	+ 165 - 188 + 10 - 36 = -49
PN → WN → WB → BB → BC → PC	+ 42 - 126 + 64 - 188 + 10 - 36 = -234
WC → WB → BB → BC	+ 140 - 64 + 188 - 10 = 254

So we choose to increase PN by 10:

	Bangor (B)	Cardiff (C)	Newport (N)	
Barry (B)	188	35 10	25	35
Port Talbot (P)	165	35 36	10 42	45
Wrexham (W)	15 64	140	5 126	20
	15	70	15	100

Solution 3 (continuing from p. 4) *The third round of the stepping stone algorithm:*

Path	*Cost*
BB → BC → PC → PN → WN → WB	*+ 188 - 10 + 36 - 10 + 126 - 64 = 266*
PN → PN → PC → BC	*+ 25 - 42 + 36 - 10 = 9*
PB → PN → WN → WB	*+ 165 - 42 + 126 - 64 = 185*
WC → PC → PN → WN	*+ 140 - 36 + 42 - 126 = 20*

And there is no more gains to be made. We have found the optimal allocation.

4. Consider the following transportation problem:

	X	Y	Z	
A	7	7	4	20
B	3	6	5	40
C	6	9	2	20
	30	10	40	80

(a) Find a feasible solution using the Minimum Cost method.

(b) The problem is degenerate. Identify a basic variable that is set to zero.

(c) Find an optimal solution using the stepping stone algorithm.

Solution 4 *(a) Using the Minimum Cost method, we get:*

	X	Y	Z	
A	7	7	20 4	20
B	30 3	10 6	5	40
C	6	9	20 2	20
	30	10	40	80

(b) Depending on whether the row or column was ticked off first, we can either choose x_{AY} or x_{BZ} as the basic variable that is set to zero.

(c) Choosing $x_{BZ} = 0$ as a basic variable, we get:

	X	Y	Z	
A	7	7	20 4	20
B	30 3	10 6	0 5	40
C	6	9	20 2	20
	30	10	40	80

The first round of the stepping stone algorithm:

Path	Cost
AX → AZ → BZ → BX	+ 7 - 4 + 5 - 3 = 5
AY → AZ → BZ → BY	+ 7 - 4 + 5 - 6 = 2
CX → BX → BZ → CZ	+ 6 - 3 + 5 - 2 = 6
CY → BY → BZ → CZ	+ 9 - 6 + 5 - 2 = 6

We have already reached the optimal solution.

5. A small fire department in rural Galicia runs five fire stations, F_1, F_2, F_3, F_4 and F_5. These fire stations house 3, 3, 1, 10, and 5 fire engines respectively. Two calls come in simultaneously: there have been two major fire disasters, D_1 and D_2. The department estimates that D_1 will require 14 fire engines, and D_2 will require 8 fire engines. The time (in minutes) from each fire station to the location of each disaster is given by:

	F_1	F_2	F_3	F_4	F_5
D_1	8	6	3	10	5
D_2	3	9	2	7	4

By first using the minimum cost method, and then the stepping stone method, devise a plan for which fire engines should be dispatched to which disaster.

Solution 5 *Using the minimum cost method we get:*

	F_1	F_2	F_3	F_4	F_5	
D_1	8	3 6	3	10 10	1 5	14
D_2	3 3	9	1 2	7	4 4	8
	3	3	1	10	5	22

The first round of the stepping stone algorithm:

Path	*Cost*
$D_1F_1 \to D_1F_5 \to D_2F_5 \to D_2F_1$	*+ 8 - 5 + 4 - 3 = 4*
$D_1F_3 \to D_1F_5 \to D_2F_5 \to D_2F_3$	*+ 3 - 5 + 4 - 2 = 0*
$D_2F_2 \to D_1F_2 \to D_1F_5 \to D_2F_5$	*+ 9 - 6 + 5 - 4 = 4*
$D_2F_4 \to D_1F_4 \to D_1F_5 \to D_2F_5$	*+ 7 - 10 + 5 - 4 = -2*

So we choose to increase D_2F_4 by 4:

	F_1	F_2	F_3	F_4	F_5	
D_1	8	3 6	3	6 10	5 5	14
D_2	3 3	9	1 2	4 7	4	8
	3	3	1	10	5	22

Solution 5 (continuing from p. 7) *The second round of the stepping stone algorithm:*

Path	*Cost*
$D_1F_1 \to D_1F_4 \to D_2F_4 \to D_2F_1$	*+ 8 - 5 + 7 - 3 = 2*
$D_1F_3 \to D_1F_4 \to D_2F_4 \to D_2F_3$	*+ 3 - 10 + 7 - 2 = -2*
$D_2F_2 \to D_1F_2 \to D_1F_4 \to D_2F_4$	*+ 9 - 6 + 10 - 7 = 6*
$D_2F_5 \to D_2F_4 \to D_1F_4 \to D_1F_5$	*+ 4 - 7 + 10 - 5 = 2*

So we choose to increase D_1F_3 by 1:

	F_1	F_2	F_3	F_4	F_5	
D_1	 8	3 6	1 3	5 10	5 5	14
D_2	3 3	 9	 2	5 7	 4	8
	3	3	1	10	5	22

The third round of the stepping stone algorithm:

Path	*Cost*
$D_1F_1 \to D_1F_4 \to D_2F_4 \to D_2F_1$	*+ 8 - 5 + 7 - 3 = 2*
$D_2F_2 \to D_1F_2 \to D_1F_4 \to D_2F_4$	*+ 9 - 6 + 10 - 7 = 6*
$D_2F_3 \to D_1F_3 \to D_1F_4 \to D_2F_4$	*+ 2 - 3 + 10 - 7 = 2*
$D_2F_5 \to D_2F_4 \to D_1F_4 \to D_1F_5$	*+ 4 - 7 + 10 - 5 = 2*

And there is no more gains to be made. We have found the optimal allocation.

6. Consider the transportation problem below, with demand nodes A, B and C, and supply nodes X, Y and Z. First use the minimum cost method to find a basic assignment, then use the stepping stone algorithm to show that this is the optimal assignment. Which demand nodes are unsatisfied?

	X	Y	Z	
A	5	9	2	35
B	11	8	3	15
C	15	3	5	5
	5	10	20	

Solution 6 *Here the demand and supply are unequal, and so a dummy column needs to be introduced. Once this is introduced, we can use the minimum cost method (ignoring the dummy column) to find a basic feasible solution:*

	X	Y	Z	D	
A	5 5	9	20 2	10 0	35
B	11	5 8	3	10 0	15
C	15	5 3	5	0	5
	5	10	20	20	

The first round of the stepping stone algorithm:

Path	*Cost*
$AY \to AD \to BD \to BY$	*+ 9 - 0 + 0 - 8 = 1*
$BX \to AX \to AD \to BD$	*+ 11 - 5 + 0 - 0 = 6*
$BZ \to AZ \to AD \to BD$	*+ 3 - 2 + 0 - 0 = 1*
$CX \to AX \to AD \to BD \to BY \to CY$	*+ 15 - 5 + 0 - 0 + 8 - 3 = 15*
$CZ \to AZ \to AD \to BD \to BY \to CY$	*+ 5 - 2 + 0 - 0 + 8 - 3 = 8*
$CD \to CY \to BY \to BD$	*+ 0 - 3 + 8 - 0 = 5*

And there is no more gains to be made. We have found the optimal allocation. Demand nodes A and B are unsatisfied by 10 each.