$$f_i(H_i|\theta) = \frac{M_i(\theta)^{H_i}}{H_i!}e^{-M_i(\theta)} \Big/ \frac{M_i(\hat{\theta})^{H_i}}{H_i!}e^{-M_i(\hat{\theta})} = \left(\frac{M_i(\theta)}{M_i(\hat{\theta})}\right)^{H_i} e^{-(M_i(\theta)-M_i(\hat{\theta}))} \tag{1}$$

$$F(H|\theta) = \prod_i f_i \tag{2}$$

$$L(H|\theta) = -\mathrm{Log}(F(H|\theta)) = -\sum_i l_i(H_i|\theta) \tag{3}$$

$$l_i(H_i|\theta) = \mathrm{Log}(f_i(H_i|\theta)) = H_i\left[\mathrm{Log}(M_i(\theta)) - \mathrm{Log}(M_i(\hat{\theta}))\right] - M_i(\theta) + M_i(\hat{\theta}) \tag{4}$$

Fisher Information:

$$I = E\left[(\partial_\theta L(H|\theta))^2_{\theta_0}\right] = \sum_i E\left[(\partial_\theta l_i(H_i|\theta))^2_{\theta_0}\right] \tag{5}$$

$$\partial_\theta l_i(H_i|\theta_0) = \frac{\partial_\theta f_i(H_i|\theta_0)}{f_i(H_i|\theta_0)} \tag{6}$$

$$\partial_\theta f_i(H_i|\theta_0) = \left(\frac{M_i'(\theta_0)}{M_i(\theta_0)}H_i - M_i'(\theta_0)\right) f_i(H_i|\theta) \tag{7}$$

$$\begin{aligned} I &= \sum_i (M_i'(\theta_0))^2 E\left[\frac{H_i^2}{M_i^2(\theta_0)} - 2\frac{H_i}{M_i(\theta_0)} + 1\right] = \sum_i (M_i'(\theta_0))^2 \left(\frac{E[H_i^2]}{M_i^2(\theta_0)} - 2\frac{E[H_i]}{M_i(\theta_0)} + 1\right) \\ &= \sum_i (M_i'(\theta_0))^2 \left(\frac{Var[H_i] + E^2[H_i]}{M_i^2(\theta_0)} - 2\frac{E[H_i]}{M_i(\theta_0)} + 1\right) = \sum_i \frac{(M_i'(\theta_0))^2}{M_i(\theta_0)} \end{aligned} \tag{8}$$

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$$E[\partial^2_\theta L(H|\theta)] = \sum_i E\left[\frac{\partial^2_\theta f_i(H_i|\theta_0)}{f_i(H_i|\theta_0)} - \frac{(\partial_\theta f_i(H_i|\theta_0))^2}{f_i^2(H_i|\theta_0)}\right] = \sum_i E\left[\frac{\partial^2_\theta f_i(H_i|\theta_0)}{f_i(H_i|\theta_0)}\right] - I = -I \tag{9}$$

$$\partial^2_\theta f_i(H_i|\theta_0) = f_i(H_i|\theta_0)\left(\frac{M_i''}{M_i}H_i - \frac{(M_i')^2}{M_i^2}H_i - M_i'' + \frac{(M_i')^2}{M_i^2}H_i^2 - 2\frac{(M_i')^2}{M_i}H_i + (M_i')^2\right) \tag{10}$$

$$\sum_i E\left[\frac{\partial^2_\theta f_i(H_i|\theta_0)}{f_i(H_i|\theta_0)}\right] = \left(M_i'' - \frac{(M_i')^2}{M_i} - M_i'' + \frac{(M_i')^2}{M_i} + (M_i')^2 - 2(M_i')^2 + (M_i')^2\right) = 0 \tag{11}$$

We want to evaluate $L(H|\theta)$ near $\hat{\theta}$,

$$\begin{aligned} L(H|\Delta\theta) = -\sum_i H_i \mathrm{Log}\left(1 + \frac{M_i'(\hat{\theta})}{M_i(\hat{\theta})}\Delta\theta\right) - M_i'(\hat{\theta})\Delta\theta = \\ -\sum_i H_i\left(\frac{M_i'(\hat{\theta})}{M_i(\hat{\theta})}\Delta\theta - \frac{1}{2}\left(\frac{M_i'(\hat{\theta})}{M_i(\hat{\theta})}\right)^2\Delta\theta^2\right) - M_i'(\hat{\theta})\Delta\theta = \\ \frac{\Delta\theta^2}{2}\sum_i\left(\frac{M_i'(\hat{\theta})}{M_i(\hat{\theta})}\right)^2 H_i + \Delta\theta\partial_\theta F(H|\hat{\theta}) = \frac{\Delta\theta^2}{2}\sum_i\left(\frac{M_i'(\hat{\theta})}{M_i(\hat{\theta})}\right)^2 H_i \end{aligned} \tag{12}$$

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$$0 = L'(H|\hat{\theta}) = L'(H|\theta_0) + L''(H|\theta_0)\Delta\theta_0 \rightarrow \Delta\theta_0 = -\frac{L'(H|\theta_0)}{L''(H|\theta_0)} = -\frac{\sum_i l'_i(H_i|\theta_0)}{\sum_i l''_i(H_i|\theta_0)} \tag{13}$$

$$0 = E[L'(H|\theta_0)] \quad E[L''(H|\theta_0)] = -I \tag{14}$$

$$\Delta\theta_0 = -\frac{\sqrt{n}(1/n\sum_i l'_i - E[L'])}{\sqrt{n}(1/n\sum_i l''_i - E[L'']) - \sqrt{n}I} \tag{15}$$

$$\sqrt{n}I\Delta\theta_0 \sim \mathrm{Norm}(0, Var[L']) \rightarrow \Delta\theta_0 \sim \mathrm{Norm}(0, Var[L']/nI^2) \tag{16}$$

$$Var[L'] = Var[\sum_i \frac{M'_i}{M_i}H_i - M'_i] = \sum_i Var[\frac{M'_i}{M_i}H_i - M'_i] = \sum_i \left(\frac{M'_i}{M_i}\right)^2 Var[H_i] = I \tag{17}$$

$$\Delta\theta_0 \sim \mathrm{Norm(0,1/nI)} \tag{18}$$