UNSW Mathematics Society Presents
MATH3611/5705 Seminar

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UNSW MathSoc Presents: MATH 3611/5705 Revision Seminar

Overview I

1. Sets and Cardinality

Countability

Countability

Let $\mathbb{N}$ be the natural numbers. A set S said is the be countable, if either S is finite or

$$|S| = |\mathbb{N}|,$$

and uncountable if

$$|\mathbb{N}| < |S|.$$

Some examples of countable sets are

- $\mathbb{Z} = 0, -1, 1, -2, 2 \ldots,$
- The set of rationals $\mathbb{Q}$.

Countability

Proposition: Union and Cartisian product of countable sets

Suppose I and S are countable sets, the finite union of S

$$\bigcup_{i \in I} S_i,$$

is countable.
The finite cartisian product of S

$$\prod_{i \in I} S^i$$

is countable.

This means that $\mathbb{N} \times \mathbb{N} \times \mathbb{N} = (0,0,0),(1,0,0),(0,1,0),\ldots$ is countable.

Countability

Definition

Suppose we have set A and B, the following statements are equivalent.

(i) If there exist a surjection $f : A \to B$ then $|B| \leq |A|$.

(ii) If there exist injection $f : B \to A$, then $|B| \leq |A|$.

Note that if $A = B$, then there exist bijection $f : A \to B$.

Theorem: Cantor-Bernstein

If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.

2020 Question 1

2020: Q1

Let S be the set of open disks in the complex plane which have rational radius. Is this set countable or uncountable?

Note that clearly there is a bijective function $f : \mathbb{C} \to \mathbb{R} \times \mathbb{R}$

$$f(a + bi) = (a, b).$$

Therefore $|\mathbb{C}| = |\mathbb{R} \times \mathbb{R}|$. Recall that $\mathbb{R}$ is uncountable, that is we have the surjection

$$g : \mathbb{R} \to \mathbb{N} : g(x) = \begin{cases} x, \text{if} x \in \mathbb{N} \\ 0, \text{if} x \notin \mathbb{N}. \end{cases}$$

So it follows that $\mathbb{R} \times \mathbb{R}$ is uncountable. Therefore we have that the set S is uncountable.

2022 Question 1

Q1: 2022

Determine whether the following sets are countable or uncountable.

(a) The rational numbers $\mathbb{Q}$

(b) The set of bit strings $B = \{(b_1, b_2, \dots) : b_i \in [0, 1]\}$.

(c) The set of clopen sets of $\mathbb{R}$ with the usual topology.

(d) The set of clopen sets of $\mathbb{R}$ with discrete topology .

(a) Countable. A bijection with $\mathbb{N}$

(b) Uncountable. Consider $P(\mathbb{N})$ (the power set of $\mathbb{N}$.

(c) Countable.

(d) Uncountable.

2021 Question 1

Q1: 2021

Is the set of functions from $\mathbb{Q}$ to $\mathbb{Q}$ countable or uncountable?

First note that $|\mathbb{N}| = |\mathbb{Q}|$, so instead of looking at the set of functions from $\mathbb{Q}$ to $\mathbb{Q}$, we can look at the set of functions from $\mathbb{N}$ to $\mathbb{N}$.
Also note that $\{0, 1\} \subset \mathbb{N}$, so we can look at the set of functions from $\mathbb{N}$ to $\mathbb{N}$.

Suppose we the set of function S is countable. Further suppose that $f_i : \mathbb{N} \to \{0, 1\}$, $i \in \mathbb{N}$, and let

$$\mathbf{b}_i = f_i(0)f_i(1)f_i(2)\dots,$$

where $f_i(\mathbb{N}) \in \{0, 1\}$.
Now construct $\mathbf{b} = (b_0, b_1, b_2, \dots)$ where $b_i = 1 - f_i(i)$.

Clearly $\mathbf{b} \neq \mathbf{b}_1$ as it differs in at least the 0th position: $b_0 \neq f_0(0)$.

2021 Question 1

Now $\mathbf{b}$ also differs to $\mathbf{b}_1$ in at least the 1st position: $b_1 \neq f_1(1)$ and so on. Therefore we can constructed a function $g : \mathbb{N} \to \{0, 1\}$ from $\mathbf{b}$ such that $g \notin S$.

It follows that there exist an uncountable amount of functions from $\mathbb{N}$ to $\{0, 1\} \implies$ an uncountable amount of functions from $\mathbb{N}$ to $\mathbb{N}$ and an uncountable amount of functions from $\mathbb{Q}$ to $\mathbb{Q}$.

2. Metric Spaces

Metric Spaces

Metric Space

Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is a **metric** if it satisfy the following conditions.

(1) $d(x, y) = 0 \iff x = y$.

(2) (Symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$

(3) (Triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The pair (X, d) is called a **metric space**.

Metric Spaces

Some examples are:

(i) $X = \mathbb{R}$, with $d(x, y) = |x - y|$, then (X, d) is the usual metric on $\mathbb{R}$.

(ii) $X =$ any non-empty set. Then function $d : \mathbb{X} \times \mathbb{X} \to \{0, 1\}$

$$d(x, y) = \begin{cases} 0x = y \\ 1x \neq y \end{cases}$$

is called the **discrete metric**.

(iii) The river metrics on $\mathbb{R}^2 : d((x, y), (x', y')) = |y - y'|$ if $x = x'$, but if $x \neq x'$ then $d((x, y), (x', y') = |x - x'| + |y| + |y'|$. The idea is that if $x \neq x'$ then$\{x\} \times \mathbb{R}$ is separated from $\{x' \times \mathbb{R}\}$ by mountains, so to travel from (x, y) to (x', y') you must travel to , and along the river $\mathbb{R} \times \{0\}$.

Convergence on metric space

Convergence

Let (X, d) be a metric space. Suppose that $\{x_k\}_{k=0}^{\infty}$ is a sequence of elements of X and that $x \in X$. Then $\{x_k\}_{k=0}^{\infty}$ converges to x in metrics d if

$$d(x_k, x) \to 0 \text{ as } k \to \infty.$$

The alternate delta-epislon definition is

> $x_k \to x$ as $k \to \infty$ in (X, d) if for all $\epsilon > 0$ there exist K for all $k \leq K, d(x_k, x) < \epsilon$.

Uniqueness of limits

Limits in a metric space are unique.

Topology of metrics spaces

Interior & boundary points

Suppose we have metric space (X, d) and $Y \subset X$

(i) x_0 is called an **interior point** of Y if there exist epsilon such that $B(x_0, \epsilon) \in Y$.

(ii) x_0 is called a **boundary point** of Y if for all $\epsilon > 0$, $B(x_0, \epsilon) \cap Y \neq \emptyset$ and $B(x_0, \epsilon) \cap X/Y \neq \emptyset$.

Open and closed sets

Suppose we have metric space (X, d) and $Y \subset X$, then

(i) Y is **open** if it contains only **interior points**.

(ii) Y is **close** if it contains all of its **boundary points**. Or if X/Y is **open**.

Continuity on Metric Space

Continuity at a point

Let (X, d_X) and (Y, d_Y) be metric spaces and suppose that $f : X \to Y$.

(1) f is continuous at $x_0 \in X$ if for all $\epsilon > 0$, there exist $\delta > 0$ such that if $d_X(x, x_0) < \delta$ then $d_Y(f(x), f(x_0)) < \epsilon$.

(2) f is continuous on X if f is continuous at every point of X.

Pre-image Theorem

Let (X, d_X) and (Y, d_Y) be metric spaces and suppose that $f : X \to Y$. Then f is continuous on X if and only if $f^{-1}(U)$ is an open set in (X, d_X) for every open set U in (Y, d_Y).

Boundedness & Compact

Definition

Suppose we have the metric space $\{X, d\}$, and $Y \subset X$. We say Y is bounded if there exist $x_0 \in X$ and $M \in \mathbb{R}$ such that for all $d(y, x_0) \leq M$ for all $y \in Y$.

If Y is closed and bounded, then Y is compact.

2020 Question 5

Question

Let $f : \mathbb{R} \to \mathbb{R}$ be the functions defined by

$$f(x) = \begin{cases} 2 + \frac{1}{x}, x > 1 \\ 1 + x, x \leq 1. \end{cases}$$

(i) Show that $d(x, y) = |f(x) - f(y)|$ defines a metric on $\mathbb{R}$.

(ii) What is $B(2, \frac{1}{2})$ in this metrics?

(iii) Find the closure of the interval $(2, \infty)$ in this metric.

2020 Question 5

(i) We will check by definition. Clearly, condition (1) and (2) holds, we will just check for triangle inequality.

$$\begin{aligned} d(x,y) &= |f(x) - f(y)| \\ &= |f(x) - f(z) + f(z) - f(y)| \\ &\leq |f(x) - f(z)| + |f(z) - f(y)| \\ &= d(x,z) + d(y,z). \end{aligned}$$

(ii) Now by definition $B(2, \frac{1}{2}) = \{x \in X : d(2,x) < \frac{1}{2}\}$ so $x \in \mathbb{R}$ such that $|f(2) - f(x)| < \frac{1}{2}$, $|\frac{5}{2} - f(x)| < \frac{1}{2}$ and so

$$3 > f(x) > 2.$$

Now notice that for all $x > 1$, $2 < f(x) < 3$. Therefore the open ball $B(2, \frac{1}{2})$ is the interval $(1, \infty)$ in this metrics.

(iii) By definition, let $Y = (2, \infty)$, then the closure of Y, $\mathrm{cl}(Y)$ consists of Y and all of its limit points.

2020 Question 5

Now looking at our end points. Suppose we have a sequence $\{x_k\}_{k=1}^{\infty} \in Y, x_k \to \infty$ as $k \to \infty$. Then clearly $f(x_k) \to 2$. Now notice

$$\begin{aligned} d(x_k, 1) &= |f(x_k) - f(1)| \\ &= |f(x_k) - 2| \to 0. \end{aligned}$$

Therefore 1 is a limit point of Y. Similarly, suppose we have a sequence $\{y_k\}_{k=1}^{\infty} \in Y, x_k \to 2$ as $k \to \infty$. Then clearly $f(y_k) \to 2.5$. Now notice

$$\begin{aligned} d(y_k, 2) &= |f(y_k) - f(2)| \\ &= |f(y_k) - 2.5| \to 0. \end{aligned}$$

Therefore 2 is a limit point. It follows that the the closure of Y is $[1] \cup [2, \infty)$.

2021 Question 6

Question 6

Let (X, d) be a metric space.

(i) Show that for a given point x, the function $d_x : X \to \mathbb{R}$ defined by $d_x(y) = d(x, y)$ is continuous (with respect to the usual metric on $\mathbb{R}$).

(ii) Show that if Y is a non-empty compact subset of X, then for each point $x \in X$, there is a closest point to x in Y , i.e. a point $y \in Y$ such that

$$d(x, y) \leq d(x, y'), \quad \forall y' \in Y.$$

2021 Question 6

(i) We wish to show that d_x is continuous. Then by definition we need to show that for all $y, y_0 \in X$ and suppose that $|y - y_0| < \delta$ then $|d_x(y) - d_x(y_0)| < \epsilon$. Now

$$\begin{aligned}
|d_x(y) - d_x(y_0)| &= ||x - y| - |x - y_0|| \\
&\leq |x - y - x + y_0| \quad \text{by reverse triangle ineq} \\
&= |y_0 - y| = |y - y_0| < \epsilon.
\end{aligned}$$

Therefore by picking $\delta = \epsilon$, we have that d_x is continuous.

(ii) Since Y is a non-empty compact set of X then by definition Y is closed and bounded.

(Cont.)

Now recall that $d_x(y) = d(x, y)$ is continuous. Since Y is compact, it follows that $d_x(Y)$ is also compact (continuity preserve compactness). Therefore there exist y' such that $d_x(y') = \min_{y \in Y} d_x(y)$ and so

$$d(x, y') \leq d(x, y)$$

for all $y \in Y$.

Normed Spaces

Normed Space

Let V be a real vector space a function $\|\cdot\| : V \to \mathbb{R}$ is a **norm** if

(i) $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0$ iff $v = 0$

(ii) $\|\lambda v\| = |\lambda|\|v\|$ for all $\lambda \in \mathbb{R}$, $v \in V$.

(iii) $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$.

Note that if $\|\cdot\|$ is a norm on a vector V, then setting $d(u, v) = \|u - v\|$ gives us a metric on V.

Some examples of normed spaces are

(1) $V = \mathbb{R}^n, \|\mathbf{x}\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$

(2) $V = \ell^1 = \{\mathbf{x} = (x_1, x_2, \dots) : \sum_{k=1}^{\infty} |x_k| < \infty\}$ with $\|\mathbf{x}\| = \sum_{k=1}^{\infty} |x_k|$

(3) $V = C[0, 1]$, $\|f\|_\infty = \sup_{t \in [0,1]} |f(t)|$.

(4) $V = C[0, 1]$, $\|f\|_2 = \left(\int_0^1 |f(t)|^2 dt\right)^{\frac{1}{2}}$.

Inner Product Space

Inner Product Space

Let V be a real vector space. An inner product on V is a map $\langle \cdot, \cdot \rangle$ such that

(i) $\langle v, v \rangle \geq 0$ for all $v \in V$ and $\langle v, v \rangle = 0$ iff $v = 0$.

(ii) $\langle \lambda v, u \rangle = \lambda \langle v, u \rangle$ for all $\lambda \in \mathbb{R}$, $u, v \in \mathbb{R}$.

(iii) $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$.

(iv) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.

Note the the same definition could also be applied to complex vector space.

Some examples of inner product spaces are

(1) $\mathbb{R}^n$ and $\mathbb{C}^n$ with dot products.

(2) $V = C[0, 1]$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$

Completeness

Cauchy Sequences

Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in a metric space (X, d). We say that $\{x_n\}$ is a *Cauchy sequence* if for all $\varepsilon > 0$ there exists K such that for all $k, l \geq K$, $d(x_k, x_l) < \varepsilon$.

A nice result is that every convergent sequence is Cauchy.

Completeness

A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges to an element in X.

Completeness (Cont.)

Hilbert Spaces

A *Hilbert space* is a complete inner product space.

Banach Spaces

A *Banach space* is a complete normed vector space.

Contraction Mappings

Contraction Map

Let (X, d) be a metric space and suppose that $f : (X, d) \to (X, d)$. We say that f is a *contraction* if there exists a constant $c \in [0, 1)$ such that for all $x, y \in X$,

$$d(f(x), f(y)) \leq cd(x, y).$$

Contraction Mapping Theorem

Let (X, d) be a complete metric space. If $f : (X, d) \to (X, d)$ is a contraction then there exists some unique point $x_F \in X$ such that $f(x_F) = x_F$.

2016 Exam Question 3b

Question

Let (X, d) be $[1, \infty)$ with the usual metric $d(x, y) = |x - y|$. Define $f : X \to X$ by $f(x) = x + 1/x$.

i) Show that $d(f(x), f(y)) < d(x, y)$ for all $x \neq y \in X$.

ii) Is (X, d) a complete metric space? Give brief reasons for your answer.

iii) Show that f does not have a fixed point in X. Explain why this doesn't contradict the Contraction Mapping Theorem.

2016 Exam Question 3b (Cont.)

Question

$$f : [1, \infty) \to [1, \infty);\ f(x) = x + 1/x.$$

i) Show that $d(f(x), f(y)) < d(x, y)$ for all $x \neq y \in X$.

Fix $x > y$ by the symmetry of $|\cdot|$. One can see that

$$\begin{aligned} |f(x) - f(y)| &= \left| x - y - \frac{1}{x} + \frac{1}{y} \right| \\ &= (x - y)\left(1 - \frac{1}{xy}\right) \\ &< d(x, y) \end{aligned}$$

since $xy > 1$.

2016 Exam Question 3b (Cont.)

Question

$$f : [1, \infty) \to [1, \infty);\ f(x) = x + 1/x.$$

ii) Is (X, d) a complete metric space? Give brief reasons for your answer.

Proposition 2.10.12

Let (X, d) be a complete metric space and suppose that $Y \subseteq X$. (Y, d) is complete if and only if Y is closed in X.

The result follows from the facts that $[1, \infty)$ is closed in and $(, |\cdot|)$ is a complete metric space.

2016 Exam Question 3b (Cont.)

Question

$$f : [1, \infty) \to [1, \infty);\; f(x) = x + 1/x.$$

iii) Show that f does not have a fixed point in X. Explain why this doesn't contradict the Contraction Mapping Theorem.

Considering the fixed point problem $f(x) = x$ gives $1/x = 0$, which cannot happen for $x \in [1, \infty)$. This doesn't violate the contraction mapping theorem however, since there is no $c \in [0, 1)$ such that $d(f(x) - f(y)) \leq cd(x, y)$ for all $x, y \in X$. f is thus not a contraction, and the Contraction Mapping Theorem does not apply.

3. Sequences and Series of Functions

Types of Convergence

Pointwise Convergence

Let A be a set. For $k \in_{+}$, let $f_k : A \to$ and let $f : A \to$. We say that the sequence $\{f_k\}$ converges *pointwise* to f if for all $a \in A$
$f_k(a) \to f(a)$ in as $k \to \infty$.

Ian Doust Lecture Notes Example 3.2.2

Let $f_k : [-1, 1] \to$ be defined as $f_k(t) = \tan^{-1}(kt)$. Then

$$f_k(t) \to f(t) = \begin{cases} -\pi/2 & -1 \leq t < 0 \\ 0 & t = 0 \\ \pi/2 & 0 < t \leq 1. \end{cases}$$

Types of Convergence (Cont.)

Question

$$f : [1, \infty) \to [1, \infty);\ f(x) = x + 1/x.$$

i) Show that $d(f(x), f(y)) < d(x, y)$ for all $x \neq y \in X$.

Types of Convergence (Cont.)

Pointwise Convergence (Generalised For Metric Spaces)

Let A be a set and let (X, d) be a metric space. For $k \in_{+}$, let $f_k : A \to X$ and let $f : A \to X$. We say that the sequence $\{f_k\}$ converges *pointwise* to f in (X, d) if for all $a \in A$, $f_k(a) \xrightarrow{d} f(a)$ in X as $k \to \infty$.

We won't cover this here in depth. This is covered in metric spaces.

Types of Convergence (Cont.)

Uniform Convergence

Let A be a set and $\mathcal{B}(A)$ be the set of bounded functions with domain A. For a sequence of functions $\{f_k\}$ in $\mathcal{B}(A)$ and f in $\mathcal{B}(A)$, we say that f_k converges *uniformly* (on A) to f if it converges to f in the d_∞ metric. That is, if $d_\infty(f, f_k) = \|f - f_k\|_\infty \to 0$.

2016 Exam Question 7a (Modified)

Question

For $k \in_+$, define $f_k : [0, 1] \to$ by

$$f_k(x) = \begin{cases} 1 & x = 0 \\ \frac{\sin(kx)}{kx} & 0 < x \leq 1. \end{cases}$$

Discuss the pointwise and uniform convergence of $\{f_k\}$.

2016 Exam Question 7a (Modified) (Cont.)

Question

$$f_k : [0,1] \to ;\, x \mapsto \begin{cases} 1 & x = 0 \\ \frac{\sin(kx)}{kx} & 0 < x \leq 1. \end{cases}$$

Discuss the pointwise and uniform convergence.

Let's first identify the pointwise limit function f. Obviously, $f(0) = 1$. Consider $x \in (0,1]$. Since $\sin(t) \in [-1,1]$ for all t, a careful pinching argument gives that $f_k(x) \to 0$ for all $x \in (0,1]$. f may then be expressed as

$$f : [0,1] \to ;\, x \mapsto \begin{cases} 1 & x = 0 \\ 0 & 0 < x \leq 1. \end{cases}$$

2016 Exam Question 7a (Modified) (Cont.)

Question

$$f_k : [0,1] \to; x \mapsto \begin{cases} 1 & x = 0 \\ \frac{\sin(kx)}{kx} & 0 < x \leq 1. \end{cases}$$

Discuss the ~~pointwise~~ and uniform convergence.

A Helpful Result (2019 Lecture Notes Example 3.3.6)

If $\{g_k\}_{k=1}^{\infty}$ converges uniformly then its limit is continuous.

The contrapositive of this statement gives us that since the pointwise limit f of f_k is discontinuous at 0, then f_k does not converge uniformly to f.

2016 Exam Question 7a (Modified) (Cont.)

Question

$$f_k : [0,1] \to ; x \mapsto \begin{cases} 1 & x = 0 \\ \frac{\sin(kx)}{kx} & 0 < x \leq 1. \end{cases}$$

Discuss the ~~pointwise~~ and ~~uniform~~ convergence.

The sequence $\{f_k\}$ thus converges to

$$f : [0,1] \to ; x \mapsto \begin{cases} 1 & x = 0 \\ 0 & 0 < x \leq 1. \end{cases}$$

pointwise but not uniformly.

Series of Functions

Weierstrass M-test (Ian Doust Corollary 3.4.5 Generalised)

Suppose that $\{f_k\}$ is a sequence of functions in $C_b(X)$ and that there is a sequence of constants $\{M_k\}$ such that $\|f_k\|_\infty \leq M_k$ for $k \in_+$. If $\sum_{k=1}^\infty M_k$ converges then the series $\sum_{k=1}^\infty f_k$ converges uniformly to a function $f \in C_b(X)$.

To use Weierstrass M-tests, it helps to know ways of proving convergence of series.

Convergence of Series

The series $\sum w_k$ converges absolutely if

- (Ratio Test) $\limsup_k \left|\frac{w_{k+1}}{w_k}\right| < 1$ or
- (Root Test) $\limsup_k \sqrt[k]{|w_k|} < 1$.

2022 Exam Question 5

Question

For $k \in_+$, define $f_k : [0,1] \to$ by

$$f_k(t) = \frac{t^k e^{-kt}}{k}.$$

Prove that the series

$$S(t) = \sum_{k=1}^{\infty} f_k(t)$$

converges uniformly to a continuous function on $[0,1]$.

2022 Exam Question 5 (Cont.)

Question

$$f_k : [0,1] \to; x \mapsto \frac{t^k e^{-kt}}{k}.$$

Prove that $S(t) = \sum_{k=1}^{\infty} f_k(t)$ converges uniformly on $[0,1]$.

Let's construct our sequence $\{M_k\}$. Fix k. $M_k \geq \|f_k\|_\infty$ gives

$$M_k \geq \sup_{t \in [0,1]} \left| \frac{t^k e^{-kt}}{k} \right|.$$

Since f_k is continuous and positive on $[0,1]$, it suffices to consider local maximums via derivatives and boundary points in finding this supremum. Differentiating gives that there's a maximum at $t = 1$. Boundary points give $f_k(0) = 0$ and $f_k(1) = (ke^k)^{-1}$. One may thus consider $M_k = (ke^k)^{-1}$.

2022 Exam Question 5 (Cont.)

Question

$$f_k : [0,1] \to ; x \mapsto \frac{t^k e^{-kt}}{k}.$$

Prove that $S(t) = \sum_{k=1}^{\infty} f_k(t)$ converges uniformly on $[0,1]$.

Since we have found $M_k = (ke^k)^{-1}$ such that $M_k \geq \|f_k\|_\infty$ for $k \in_+$, then $S(t)$ converges if $\sum_{k=1}^{\infty} M_k$ converges. We can see that

$$\frac{M_{k+1}}{M_k} = \frac{\left((k+1)e^{k+1}\right)^{-1}}{(ke^k)^{-1}} = \frac{1}{e}\frac{k}{k+1},$$

and that $\limsup_k |M_{k+1}/M_k| \leq 1/e < 1$. So $\sum_{k=1}^{\infty} M_k$ converges by the ratio test. S thus converges uniformly to some $f \in C[0,1]$.

2022 Exam Question 5 (Cont.)

Question

$$f_k : [0,1] \to ;\ x \mapsto \frac{t^k e^{-kt}}{k}.$$

Prove that $S(t) = \sum_{k=1}^{\infty} f_k(t)$ converges uniformly on $[0,1]$.

Since we have found $M_k = (ke^k)^{-1}$ such that $M_k \geq \|f_k\|_\infty$ for $k \in_+$, then $S(t)$ converges if $\sum_{k=1}^{\infty} M_k$ converges. We can see that $\sqrt[k]{|M_k|} = \sqrt[k]{(ke^k)^{-1}} = (ek^{1/k})^{-1}$, and that

$$\limsup_k \sqrt[k]{|M_k|} \leq \frac{1}{e} \sup_{x \in_+} x^{1/x} = \frac{e^{-1/e}}{e} < 1$$

So $\sum_{k=1}^{\infty} M_k$ converges by the ratio test. S thus converges uniformly to some $f \in C[0,1]$.

Differentiating and Integrating Limits of Sequences

Suppose that $\{f_k\}_{k=1}^{\infty} \subseteq C[0,1]$.

Differentiating Limits of Sequences

If the f_k are differentiable on $(0,1)$, $f_k \to f$ uniformly, $f_k' \in C_b(0,1)$ and $f_k' \to \phi$ uniformly, then f is differentiable on $(0,1)$ and $f' = \phi$.

Integrating Limits of Sequences

If the sequence $\{f_k\}$ converges uniformly to $f : [0,1] \to$ then

$$\int_0^1 f_k(t)\,dt \to \int_0^1 f(t)\,dt.$$

Differentiating and Integrating Limits of Sequences (Cont.)

Differentiation of Power Series

Suppose that $S(z) = \sum_{k=0}^{\infty} a_k z^k$ is a power series which converges pointwise on the disk $\{z \in : |z| < R\}$. Then S is differentiable on this disk with

$$S'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}.$$

Differentiating and Integrating Limits of Sequences (Cont.)

These topics don't come up too much on their own in the exams. They're usually used to aid in questions more steeped in other topics (covered in other parts of the seminar). We'll focus here now on questions of the convergence of series. They're the most common kind of question from this topic in the exams.

2021 Exam Question 5 (Modified)

Question

Consider the sequence of piecewise linear functions of $\{f_n\}_{n=1}^{\infty} \subset C[0,1]$ where f_n is zero outside the interval $[\frac{n-1}{n}, \frac{n}{n+1}]$, and has a symmetric spike of height $1/n$ within that interval.

i) Does the sequence $\{f_n\}$ converge pointwise and/or uniformly on $[0,1]$? Explain your answer.

ii) Does the series

$$\sum_{k=1}^{\infty} f_k(x)$$

converge pointwise and/or uniformly on $[0,1]$? Explain your answer.

Question

$$f_n : [0,1] \to; x \mapsto \begin{cases} \text{symm. spike height } 1/n & x \in \left[\frac{n-1}{n}, \frac{n}{n+1}\right] \\ 0 & \text{otherwise.} \end{cases}$$

i) Does the sequence $\{f_n\}$ converge pointwise and/or uniformly on $[0,1]$? Explain your answer.

As before, let's find try to find the pointwise limit function f of f_n. One can see that $0 \leq f_n(x) \leq 1/n$ for all $x \in [0,1]$ and for all $n \in_+$. So by a $\limsup$ pinching argument, the limit function f of f_n may be characterised as the zero function on $[0,1]$.

2021 Exam Question 5 (Modified) (Cont.)

Question

$$f_n : [0,1] \to; x \mapsto \begin{cases} \text{symm. spike height } 1/n & x \in \left[\frac{n-1}{n}, \frac{n}{n+1}\right] \\ 0 & \text{otherwise.} \end{cases}$$

i) Does the sequence $\{f_n\}$ converge ~~pointwise~~ and/or uniformly on $[0,1]$? Explain your answer.

The sequence $\{f_n\}$ converges uniformly to the zero function f on $[0,1]$ if $\|f - f_n\|_\infty \to 0$, or $\sup_{t\in[0,1]} |f_n(t)| \to 0$. It's easy to see that $\sup_{t\in[0,1]} |f_n(t)| = 1/n \to 0$. So the convergence of the sequence $\{f_n\}$ is uniform.

2021 Exam Question 5 (Modified) (Cont.)

Question

$$f_n : [0,1] \to; x \mapsto \begin{cases} \text{symm. spike height } 1/n & x \in \left[\frac{n-1}{n}, \frac{n}{n+1}\right] \\ 0 & \text{otherwise.} \end{cases}$$

ii) Does the series $\sum_{k=1}^{\infty} f_k(x)$ converge pointwise and/or uniformly on $[0,1]$? Explain your answer.

This is a case where the Weierstrass M-test would fail. To discuss the convergence of the series $\sum_{k=1}^{\infty} f_k(x)$, let's introduce a helpful term. For $n \in_+$ let's say that f_n is *active* for any t satisfying $f_n(t) \neq 0$.

2021 Exam Question 5 (Modified) (Cont.)

Question

$$f_n : [0,1] \to ; x \mapsto \begin{cases} \text{symm. spike height } 1/n & x \in \left[\frac{n-1}{n}, \frac{n}{n+1}\right] \\ 0 & \text{otherwise.} \end{cases}$$

ii) Does the series $\sum_{k=1}^{\infty} f_k(x)$ converge pointwise and/or uniformly on $[0,1]$? Explain your answer.

Let $n \in_+$. Consider the largest set that f_n is active for,

$$\left(\frac{n-1}{n}, \frac{n}{n+1}\right).$$

This set is disjoint from the largest set that f_{n+1} is active for,

$$\left(\frac{n}{n+1}, \frac{n+1}{n+2}\right).$$

So no two functions in the sequence $\{f_n\}$ are ever both active.

2021 Exam Question 5 (Modified) (Cont.)

Question

$$f_n : [0,1] \to;\ x \mapsto \begin{cases} \text{symm. spike height } 1/n & x \in \left[\frac{n-1}{n}, \frac{n}{n+1}\right] \\ 0 & \text{otherwise.} \end{cases}$$

ii) Does the series $\sum_{k=1}^{\infty} f_k(x)$ converge pointwise and/or uniformly on $[0,1]$? Explain your answer.

The series $\sum_{k=1}^{\infty} f_k(x)$ may then compared pointwise to S defined on $[0,1]$ by

$$S(x) = \begin{cases} f_n(x) & f_n \text{ is active for } x \\ 0 & \text{otherwise.} \end{cases}$$

This is well-defined since no two functions of the sequence $\{f_n\}$ are ever similtaneously active. The 'otherwise' portion is accurate because every f_k is zero where inactive, so the sum is zero where none are active.

Question

$$f_n : [0,1] \to ;\, x \mapsto \begin{cases} \text{symm. spike height } 1/n & x \in \left[\frac{n-1}{n}, \frac{n}{n+1}\right] \\ 0 & \text{otherwise.} \end{cases}$$

ii) Does the series $\sum_{k=1}^{\infty} f_k(x)$ converge ~~pointwise~~ and/or uniformly on $[0,1]$? Explain your answer.

Consider the sequence $\{S_n\}_{n=1}^{\infty}$ of partial sums S_n defined on $[0,1]$ such that

$$S_n(x) = \sum_{k=1}^{n} f_n(x).$$

The sum S_n is the collection of all spikes up to and including the nth spike.

Question

$$f_n : [0,1] \to; x \mapsto \begin{cases} \text{symm. spike height } 1/n & x \in \left[\frac{n-1}{n}, \frac{n}{n+1}\right] \\ 0 & \text{otherwise.} \end{cases}$$

ii) Does the series $\sum_{k=1}^{\infty} f_k(x)$ converge ~~pointwise~~ and/or uniformly on $[0,1]$? Explain your answer.

This observation can be used to analyse $\|S - S_n\|_\infty$. One may note that when $n \in_+$, $S(x) - S_n(x)$ is the curve with every spike from the $(n+1)$th spike onwards. So, $\sup_{k \geq n} |S(x) - S_k(x)| = \frac{1}{n+1}$ and $\|S - S_n\|_\infty \to 0$.

Question

$$f_n : [0,1] \to; x \mapsto \begin{cases} \text{symm. spike height } 1/n & x \in \left[\frac{n-1}{n}, \frac{n}{n+1}\right] \\ 0 & \text{otherwise.} \end{cases}$$

ii) Does the series $\sum_{k=1}^{\infty} f_k(x)$ converge ~~pointwise~~ and/or ~~uniformly~~ on $[0,1]$? Explain your answer.

The series $\sum_{k=1}^{\infty} f_k(x)$ thus converges both pointwise and uniformly to S defined on $[0,1]$ by

$$S(x) = \begin{cases} f_n(x) & f_n \text{ is active for } x \\ 0 & \text{otherwise.} \end{cases}$$

4. Topological Spaces

Topological Spaces

We developed the theory of *metric spaces* by defining suitable distance functions called *metrics*. Unfortunately, the concept of convergence cannot be completely described in metric spaces. We need a more general way of describing these concepts of convergence.

> To do this, we extract properties of *open sets* and in doing so, it turns out this is all that's needed for the concepts of convergence to make sense!

Topological Spaces

Topology

Let X be a non-empty set. A family τ of subsets of X is called a *topology* if

1. $\emptyset, X \in \tau$;
2. $\{V_i\}_{i\in I} \subseteq \tau$ implies that $\bigcup_{i\in I} V_i \in \tau$;
3. $V_1, \ldots, V_n \in \tau$ implies that $\bigcap_{i=1}^n V_i \in \tau$.

The pair (X, τ) forms a *topological space* and the elements of τ form the *open sets* of X.

Topological Spaces – Important Topologies

Some important topologies:

- *Discrete topology* – $\tau = \mathcal{P}(X)$.
- *Coarse topology* – $\tau = \{\emptyset, X\}$. Also known as the *indiscrete topology*.
- *Metric topology* – Let (X, d) be a metric space. Then (X, d) induces a topology τ_d.
- *Cofinite topology* – $\tau = \{\emptyset\} \cup \{U \subseteq X : X \setminus U \text{ is finite}\}$.
- *Cocountable topology* – $\{\emptyset\} \cup \{U \subseteq X : X \setminus U \text{ is countable}\}$.
- *Subspace topology* – Let (X, τ) be a topological space and consider a subset $S \subseteq X$. Then

  $$\tau_S = \{S \cap U : U \in \tau\}$$

  forms a topology on S.

Basic Results in Topology

Theorem. Let (X, τ) be a topological space. A subset $Y \subseteq X$ is *closed* if Y^c is open.

We can define a similar notion for the set of closed sets of a topological space. Namely,

Let (X, τ) be a topological space, and denote $\mathcal{C}(X)$ by the closed sets of X. Then,

- $\emptyset, X \in \mathcal{C}(X)$;
- If $\{V_i\}_{i \in I} \subseteq \mathcal{C}(X)$, then $\bigcap_{i \in I} V_i \in \mathcal{C}(X)$;
- If $V_1, \ldots, V_n \in \mathcal{C}(X)$, then $\bigcup_{i=1}^n V_i \in \mathcal{C}(X)$.

The union and intersection finiteness property is flipped!

Let (X, τ) be a topological space.

Let $x \in X$ be some point in X. We say that an *open neighbourhood* of x is an open set $V \in \tau$ such that $x \in V$. A *neighbourhood* of x is any set that contains an open neighbourhood of x.
We denote the *collection of neighbourhoods* by $\mathrm{Nbhd}(x)$.

Let $Y \subseteq X$ be a subset of X; the *interior* of Y is denoted by

$$\mathrm{Int}(Y) = \{y \in Y : \exists V_y \in \mathrm{Nbhd}(y) \text{ such that } V_y \subseteq Y\}.$$

It turns out that there is a much more useful characterisation of the interior of Y.

Let (X, τ) be a topological space, and let $Y \subseteq X$. Then

$$\operatorname{Int}(Y) = \bigcup_{\{V \in \tau : V \subseteq Y\}} V.$$

Therefore, instead of looking at *neighbourhoods* at a point, one can look at open sets in the topology that is contained *completely* inside of Y.

Think about how this generalises the concept of *interior* points on metric spaces!

It turns out that the interior set is an open set.

Basic Results in Topology

Once we define the notion of an *interior point*, we can start defining the boundary and the closure.

Let (X, τ) be a topological space, and consider a subset $Y \subseteq X$.

- The *boundary* is
$$\mathrm{Bd}(Y) = X \setminus (\mathrm{Int}(Y) \cup \mathrm{Int}(Y^c)).$$
- The *closure* is
$$\mathrm{cl}(Y) = \mathrm{Int}(Y) \cup \mathrm{Bd}(Y).$$

Much like the *interior*, the closure has an alternative and equivalent characterisation.

Let (X, τ) be a topological space, and let $Y \subseteq X$. Then

$$\mathrm{cl}(Y) = \bigcap_{\{V \in \mathcal{C}(X) : V \supseteq Y\}} V.$$

In other words, we can characterise the *closure* of Y in a few different ways:

- The *closure* of Y is the intersection of all closed sets containing Y.
- The *closure* of Y is the smallest closed set that contains Y.
- The *closure* of Y is the union of the interior points of Y and its boundary points.
- The *closure* of Y is Y together with its limit points.

Basic Results in Topology

We are now at a point where we can meaningfully talk about *convergence* of a sequence $\{x_k\}_{k=1}^{\infty}$. When we discussed convergence of a sequence on a metric space, we ensured that:

For every $\epsilon > 0$, we could always find some *large enough* N such that, for all $n > N$, we have that $d(x, x_n) < \epsilon$.

On a topological space, we replace the metric with *open sets*. Each distance can be meaningfully replaced with some neighbourhood of x and we need to ensure that the neighbourhood captures all *large enough* points in the sequence.

For every neighbourhood of x, we could always find some *large enough* N such that, for all $n > N$, we have that x_n is encapsulated inside of the neighbourhood.

We now make this concrete.

Let (X, τ) be a topological space. A sequence $\{x_k\}_{k=1}^{\infty} \subseteq X$ *converges* to $x \in X$ if, for every *neighbourhood* $V \in \mathrm{Nbhd}(x)$, there exists a $K(V) \in \mathbb{N}$ such that, for all $n > K(V)$, we have that $x_n \in V$.

- Note that the natural number K *can* depend on V, so one can think of $K(V)$ simply as a function of V.
- There is good reason why *uniform convergence* cannot really be defined on a topological space – it is not invariant under homeomorphisms. Simply put, it is not an interesting concept to talk about on topological spaces.

(2020, Problem 2)

Determine which of the following sequences (if any) converge, and what points they converge to.

i) The sequence $(1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, \dots)$ in $X = \{0, 1\}$ with the topology $\{\emptyset, \{0\}, X\}$.

ii) The sequence $\left(1, \frac{1}{2}, \frac{1}{3}, \dots\right)$ in $\mathbb{R}$ with the cofinite topology.

(2020, Problem 2)

Determine which of the following sequences (if any) converge, and what points they converge to.

i) The sequence $(1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, \dots)$ in $X = \{0, 1\}$ with the topology $\{\emptyset, \{0\}, X\}$.

i) Consider every neighbourhood of 1. Clearly, that is just X itself and it is easy to see that $x_n \in X$ for every $n \geq 1$. Therefore, the sequence converges to 1. Does it converge to any other point in X?

The only other point in X is 0. Does the sequence converge to 0? The neighbourhoods of 0 consist of $\{0\}$ and X. However, there are infinitely many 1's, so no matter how far you go, you'll always find a 1 in the sequence, which means that the sequence cannot be contained inside the neighbourhood $\{0\}$. Therefore, the sequence simply converges to 1.

(2020, Problem 2)

Determine which of the following sequences (if any) converge, and what points they converge to.

ii) The sequence $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$ in $\mathbb{R}$ with the cofinite topology.

ii) Consider some point $x \in \mathbb{R}$. What do the neighbourhoods of x look like under the cofinite topology? Recall that the cofinite topology is the set of open sets whose complement is finite.

If $V \in \text{Nbhd}(x)$, then $\mathbb{R} \setminus V$ is finite, so we can enumerate all of the missing points – say $\{a_1, a_2, \ldots, a_k\}$. Consider the smallest missing point, say a_i, and choose N such that $\frac{1}{N} < a_i$. Then, for all $n > N$, we have that $x_n = \frac{1}{n} < \frac{1}{N} < a_i$ and so, we have that $x_n \in V$ for all such $n > N$. We can make the same argument for all neighbourhoods of a.

Therefore, the sequence converges to x. But there's nothing special about x, we can make the same argument for all $x \in \mathbb{R}$. Therefore, the sequence converges to every point in $\mathbb{R}$.

(2021, Problem 2)

Determine which of the following sequences (if any) converge, and what points they converge to.

i) The sequence $\left(1, \frac{1}{2}, \frac{1}{3}, \dots\right)$ in $X = \mathbb{R}$ with the discrete topology.

ii) The sequence $(a, b, a, b, \dots)$ in $X = \{a, b, c\}$ with the topology

$$\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}.$$

(2021, Problem 2)

Determine which of the following sequences (if any) converge, and what points they converge to.

i) The sequence $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$ in $X = \mathbb{R}$ with the discrete topology.

i) Consider any point $x \in \mathbb{R}$. What do the neighbourhoods of x look like under the discrete topology? Since the discrete topology is $\tau = \mathcal{P}(X)$, any neighbourhood must include the set $\{x\}$.

Therefore, a sequence converges to x under the discrete topology if and only if the sequence eventually becomes constant at x. We see that the sequence does not *eventually* become constant; therefore, the sequence does not converge to any point in $\mathbb{R}$.

(2021, Problem 2)

Determine which of the following sequences (if any) converge, and what points they converge to.

ii) The sequence $(a, b, a, b, \dots)$ in $X = \{a, b, c\}$ with the topology

$$\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}.$$

ii) Consider the point $a \in X$. The neighbourhoods of a under the topology τ are the sets $\{a\}$, $\{a, b\}$, and $\{a, b, c\}$. We see that a and b alternate in the sequence; therefore, no matter what N we propose, the sequence is not *eventually* contained inside $\{a\}$ and so, the sequence does not converge to a.

The sequence converges to b and c.

Continuity on Topological Spaces

Our next piece of machinery is to make sense of what continuity of functions look like on topological spaces.

> On metric spaces (X, d_X) and (Y, d_Y), we said that a function $f : X \to Y$ was continuous if the preimage of each open set $U \in (Y, d_Y)$ is also an open set; that is, for each open set $U \in (Y, d_Y)$, $f^{-1}(U)$ is also an open set in (X, d_X).

We can generalise this by observing that each open set of X, where (X, τ) is now a topological space, belongs to τ. Therefore, the natural generalisation goes as follows:

Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is continuous if, for every $U \in \tau_Y$, we have that $f^{-1}(U) \in \tau_X$.

Notice that we didn't need to specify that U was an open set – because each element of the topology is in fact an open set!

Continuity on Topological Spaces

(2022, Problem 8)

Suppose that (X_1, τ_1) and (X_2, τ_2) are topological spaces and that $f : X_1 \to X_2$ is continuous. For each of the following statements, say whether it must be true. If that is not the case, give a counterexample.

i) $f(U) \in \tau_2$ for all $U \in \tau_2$.

ii) $f^{-1}(V) \in \tau_1$ for all $V \in \tau_2$.

Hint. Continuity does not tell you information about the *image* of an open set unless the subset is compact.

Hausdorff Topological Spaces

Let (X, τ) be a topological space. In the previous slides, we explored the concept of "closeness" of two points in the space X under the topology τ.

We still need a notion of "farness".

What sort of definition would allow us to satisfactorily say that two points are "far away"?

On a metric space, if x and y are not the same point, then a metric should ensure that $d(x, y) > 0$.

How do we define this on a topology?

If x, y are not the same point, then we can always find a pair of open sets in τ that do not intersect!

Hausdorff Topological Spaces

Definition. Let (X, τ) be a topological space. We say that (X, τ) is *Hausdorff* if, for all $x, y \in X$ such that $x \neq y$, there exist $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

If a topological space is Hausdorff, limits of sequences on the space behave nicely! We obtain the uniqueness of limits property that we come to expect.

Hausdorff Topological Spaces

(2018, Problem 8)

Let (X, τ) be a topological space. Let (X', τ') be a Hausdorff topological space, and suppose that

$$f : X \to X'$$

is a continuous, injective function. Prove that X is a Hausdorff space as well.

Let $x_1, x_2 \in X$ be two distinct points in X; we want to show that we can find disjoint open neighbourhoods $U_1, U_2 \in \tau$ such that $x_1 \in U_1$ and $x_2 \in U_2$ such that $U_1 \cap U_2 = \emptyset$.

- Since f is *injective*, then x_1, x_2 being distinct points in X imply that $f(x_1)$ and $f(x_2)$ are distinct points in X'.
- Since X' is Hausdorff with respect to the topology τ', we have open neighbourhoods $V_1, V_2 \in \tau'$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$, and $V_1 \cap V_2 = \emptyset$. Note that this required that $f(x_1)$ and $f(x_2)$ are distinct.
- Consider the preimages $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$. Since f is continuous, U_1 and U_2 are open neighbourhoods.

    Firstly, since $f(x_1) \in V_1$, it follows that $x_1 \in f^{-1}(V_1) = U_1$ and similarly, $x_2 \in U_2$.
    Since $V_1 \cap V_2 = \emptyset$, it follows that $U_1 \cap U_2 = \emptyset$. Therefore, U_1 and U_2 form disjoint open neighbourhoods around x_1 and x_2 respectively.

Since x_1, x_2 were arbitrary points in X, the argument holds for all pairs of distinct points in X; thus, (X, τ) is Hausdorff.

Bases for Topologies

There are many ways to define a topology on the same space. Oftentimes, we want a simple description of the topology. For example, if $\{x\} \in \tau$ and $\{y\} \in \tau$, then by the union closure property, we have that $\{x, y\} \in \tau$.

Therefore, if the topology is $\tau = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$, then in some sense, the subset $\tau' = \{\{x\}, \{y\}\}$ describes the *same* topology. Why is that so?

A topology must be closed by taking unions. Therefore, if τ' describes a topology, then the topology that it generates must include $\{x\} \cup \{y\} = \{x, y\}$.

Similarly, a topology must be closed by taking finite intersections. Therefore, if τ' describes a topology, then the topology it generates must include $\{x\} \cap \{y\} = \emptyset$.

This generates exactly the open sets in τ. We now make this notion more rigorous.

Bases for Topologies

Definition. Let (X, τ) be a topological space.
A *base* for τ is a subset $\mathcal{B} \subseteq \tau$ such that every open set $V \in \tau$ can be expressed as a union of elements in $\mathcal{B}$; that is, for every $V \in \tau$, one can express V as

$$V = \bigcup_{i \in I} V_i, \text{ where } V_i \in \mathcal{B} \ \ \forall i \in I.$$

We also have a notion of a base at a point $x \in X$. One can think of this as a localisation of the notion of a base of a topology; instead of looking at the entire space, one can define a topology at a point.

Definition. Let (X, τ) be a topological space, and consider a point $x \in X$. A *local base* for τ at x is a collection $\mathcal{LB}_x \subseteq \tau$ of open neighbourhoods of x such that if U is any neighbourhood of x, then there exist some $V \in \mathcal{LB}_x$ such that $V \subseteq U$.

Bases for Topologies

(2022, Problem 6)

a) Give the definition of a **base for a topology** τ on a set X.

b) Let $\mathcal{G}$ consist of the empty set and the collection of all open rectangles in the plane. That is,

$$\mathcal{G} = \{\emptyset\} \cup \{(a,b) \times (c,d) \subseteq \mathbb{R}^2 : a < b,\ c < d\}.$$

Prove that $\mathcal{G}$ is a base for a topology on $\mathbb{R}^2$.

Bases for Topologies

As always, there are different ways to characterise a *base* $\mathcal{B}$. Recall that a base for a topology τ can be thought of as the *smallest* set of open sets such that the union encapsulates all of the open sets in X.
We must, therefore, require a few properties to hold:

(1) $\mathcal{B}$ must cover all of the elements in X.

  This is because X is an open set in X, so we must be able to cover X.

(2) $\mathcal{B}$ should be as "small" as possible, in the sense that, if $x \in V_1 \cap V_2$ for $V_1, V_2 \in \mathcal{B}$, then we can always find some other open set $V \in \mathcal{B}$ such that $x \in V$.

It turns out that these are enough to define a base of a topology! We state this in more formal language.

Bases for Topologies

Theorem. Let X be a set, and let $\mathcal{B} \subseteq \mathcal{P}(X)$ be a collection of subsets. Then

$$\tau = \{V \subseteq X : V \text{ is a union of sets in } \mathcal{B}\}$$

is a topology if and only if the following properties are satisfied:

(1) $\bigcup_{V \in \mathcal{B}} V = X$; in other words, $\mathcal{B}$ *covers* X.

(2) For every V_1 and V_2 in $\mathcal{B}$ and every $x \in V_1 \cap V_2$, there exist a $V \in \mathcal{B}$ such that
$$x \in V \subseteq V_1 \cap V_2.$$

The collection union of sets in $\mathcal{B}$ forms a topology with base $\mathcal{B}$.

What about the topology generated by the collection of all finite intersections?

Bases for Topologies

It turns out that the collection of open sets generated by taking all finite intersections is also a topology! Specifically, if $S \subseteq \mathcal{P}(X)$ is *any* collection of subsets of X, then the collection

$$\mathcal{B} = \{V_1 \cap \cdots \cap V_n : V_k \in S, \quad k = 1, \ldots, n\}$$

forms a base of the same topology

$$\tau = \{V \subseteq X : V \text{ is a union of sets in } \mathcal{B}\}.$$

In particular, S *generates* τ. We say that S is a *subbase* and we often write τ as $\tau(S)$ to mean that S generates the topology τ; that is, we also have that

$$\tau(S) = \{V \subseteq X : V \text{ is a union of sets in } \mathcal{B}\}.$$

We make this definition formal.

Bases for Topologies

Definition. Let X be a set, and let $S \subseteq \mathcal{P}(X)$ be *any* collection of subsets. Define $\mathcal{B}$ to be the set of all **finite** intersections of sets in S; that is,

$$\mathcal{B} = \{V_1 \cap \cdots \cap V_n : V_k \in S, \quad k = 1, \ldots, n\}.$$

Then $\mathcal{B}$ forms a base of the topology

$$\tau(S) = \{V \subseteq X : V \text{ is a union of sets in } \mathcal{B}\}.$$

We call S a *subbase* for $\tau(S)$ and say that τ is generated by S.

We remark that we also allow the empty intersection; by convention, this gives the entire space X.

Convergence in topological spaces

- We talked about convergence on a topological space previously with a sequence of points. What about a sequence of functions?
- We can mimic the same ideas.

Remember that a sequence of points $\{x_n\}_{n=1}^{\infty}$ converge to a point $x \in X$ if, for each open neighbourhood of x, all *sufficiently large* points in the sequence belong to the neighbourhood.

We can mimic this idea but we need to be a little careful with what we mean by "*sufficiently large points*" because functions are not points.

Convergence in topological spaces

To make sense of this, consider a family F of functions from a set X to some space Y.

- You can think of each function $f : X \to Y$ as a point in the Cartesian product $Y^{|X|}$.
- The Cartesian product $Y^{|X|}$ is a topological space endowed with the *product topology*. It follows that $F \subseteq Y^{|X|}$ inherits this topology.

How do we then make sense of convergence?

Pointwise convergence

Definition. Let X be a set and $Y = F(X, \mathbb{R})$. For each $x \in X, y \in \mathbb{R}$, and $\epsilon > 0$, define

$$V_{x,y,\epsilon} = \{g \in Y : |g(x) - y| < \epsilon\}.$$

Then let

$$S = \{V_{x,y,\epsilon}\}_{x \in X, y \in \mathbb{R}, \epsilon > 0}$$

forms a topology $\tau_{pt} = \tau(S)$ generated by S as a subbase; we call this the *topology of pointwise convergence*.

The topology of pointwise convergence on a set X forms a Hausdorff topological space.

A sequence $\{f_n\}_{n=1}^{\infty}$, where $f_n : X \to \mathbb{R}$ converges *pointwise* to f if and only if $f_n \to f$ in the topology τ_{pt}.

Weak convergence

Definition. Let H be a Hilbert space. We say that a sequence of vectors $\{\mathbf{x}_n\}_{n=1}^{\infty}$ converges *weakly* to a vector $\mathbf{x} \in \mathsf{H}$ if, for each vector $y \in \mathsf{H}$, we have that

$$\langle \mathbf{x}_n, \mathbf{y} \rangle \to \langle \mathbf{x}, \mathbf{y} \rangle.$$

This allows us to define a notion of a *weak topology* in a similar light. Notice that this requirement is equivalent to ensuring that

$$\langle \mathbf{x}_n - \mathbf{x}, y \rangle \to 0.$$

To make sense of this on a topological space, we construct the topology

$$V_{\mathbf{x},\mathbf{y},\epsilon} = \{\mathbf{z} : |\langle \mathbf{z} - \mathbf{x}, \mathbf{y} \rangle| < \epsilon\}.$$

This gives the *weak topology* $\tau_{weak} = \tau(S)$ generated by the subbase

$$S = \{V_{\mathbf{x},\mathbf{y},\epsilon}\}_{\mathbf{x} \in \mathsf{H}, \mathbf{y} \in \mathbb{R}, \epsilon > 0}.$$

Nets

Until now, we've treated sequences distinctly from functions. However, another perspective is to connect these ideas together in the following way:

> A sequence is a function $f : \mathbb{N} \to X$ where X can either be a set or a space.
>
> In particular, each $n \in \mathbb{N}$ maps to the n-th coordinate x_n.

But then to talk about the *limit* of a sequence (as we take $n \to \infty$), we have a notion of *direction*.

> As we walk along the sequence, we walk along the direction of the limit point.

Nets

A *directed set* is a set Λ, together with a binary relation $\leq$ satisfying the following properties, for all $i, j, k \in \Lambda$.

(1) $i \leq i$ (i.e. $(\Lambda, \leq)$ is reflexive).

(2) $i \leq j$, $j \leq k$ implies that $i \leq k$ (i.e. $(\Lambda, \leq)$ is transitive).

(3) There exists some $m \in \Lambda$ such that, for all pairs $i, j \in \Lambda$, we have that $i, j \leq m$.

This is the *upper bound property* for pairs of elements in Λ.

Like how a sequence $\{x_k\}_{k \in \mathbb{N}}$ can be viewed as a function from $\mathbb{N}$ to some set X, we can view sequences $\{x_\lambda\}_{\lambda \in \Lambda}$ as a function from the directed set Λ to some set X.

The way we map an element $\lambda \in \Lambda$ from Λ to the set X is called a *net*.

Nets

Definition. A *net* is a function from a directed set $(\Lambda, \leq)$ to a set X.

With this in mind, how can we meaningfully talk about convergence of a net?

> Remember that a net comes from a *directed* set, so the same idea to how we normally think of convergence applies.

Definition. Let (X, τ) be a topological space. A net $\{x_\lambda\}_{\lambda \in \Lambda}$ converges to a point $x \in X$ if, for every neighbourhood V of x, there exist an $\alpha(V) \in \Lambda$ such that, for all $\lambda \geq \alpha$, we have that $x_\lambda \in V$.

Homeomorphisms

One big question of interest is when two spaces are more-or-less *the same*.

> The classic example is that we can always deform a coffee mug into a donut and vice versa without tearing or cutting the surface.
>
> We can ask the same sort of question on topological spaces.

That is, given topological spaces (X, τ_X) and (Y, τ_Y), when are these *the same*?

> When we deform a coffee mug into a donut, we are applying a transformation from a point on a coffee mug to a point on a donut.
>
> For the transformation to be smooth, it makes sense for the transformation to be continuous.
>
> However, if the two spaces are *the same*, then we should also be able to reverse the process. Therefore, we should also require the inverse map to be continuous.

This is called a homeomorphism!

Homeomorphisms

Definition. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A bijection $f : X \to Y$ is called a *homeomorphism* if f and f^{-1} are continuous. We say that X and Y are *homeomorphic* if there exists a homeomorphism $f : X \to Y$.

(2020, Problem 7)

Let (X, τ_X) be a compact topological space and let (Y, τ_Y) be a Hausdorff topological space.

i) Show that if $f : X \to Y$ is a continuous map and $Z \subseteq X$ is closed, then $f(Z) \subseteq Y$ is closed.

ii) Show that if $f : X \to Y$ is a continuous bijection, then it is a homeomorphism.

i) We firstly observe that X is compact and that $Z \subseteq X$ is closed. Therefore, Z is compact. Since f is continuous, it follows that $f(Z) \subseteq Y$ is compact. But Y is Hausdorff; therefore, $f(Z)$ is closed.

ii) Let $g = f^{-1}$. We argue that g is continuous. It suffices to show that, if Z is closed in (X, τ_X), then $g^{-1}(Z)$ is closed in (Y, τ_Y). Suppose that $Z \subseteq X$ is closed in (X, τ_X). By part (i), this implies that $f(Z)$ is closed in (Y, τ_Y). But $f(Z) = g^{-1}(Z)$; therefore, $g^{-1}(Z)$ is closed which implies that g is continuous.
We have shown that f and f^{-1} are continuous and so, f is a homeomorphism.

(2022, Problem 10)

a) Let (X_1, τ_1) and (X_2, τ_2) be topological spaces. Define carefully what it means to say that these spaces are **homeomorphic**.

b) Let $X_1 = \{z \in \mathbb{C} : |z| = 1\}$ and $X_2 = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$, both with the usual metric topology inherited from the plane. Explain why X_1 is not homeomorphic to X_2.

c) Let $\Omega = C[0, 1]$ with the usual $\|\cdot\|_\infty$ topology. For $t \in [0, 1]$, define $\phi_t : \Omega \to \mathbb{R}$, $\phi_t(f) = f(t)$. Let

$$Y_1 = \{\phi_t : t \in [0, 1]\}$$

and let τ_1 be the topology on Y_1 of pointwise convergence on Ω. Let $Y_2 = [0, 1]$. Prove that (Y_1, τ_1) is homeomorphic to Y_2 equipped with its usual metric topology.

Homeomorphic invariants

Definition. A topological space (X, τ) is *connected* if it is not the union of two disjoint nonempty open subsets.

> On an intuitive level, this definition is saying that, if a space X is connected, then it can't be cut up into two parts that are disconnected.

The nice thing about connected spaces is that it is invariant under continuous functions; that is, if (X, τ_X) and (Y, τ_Y) are topological spaces where (X, τ_X) is connected, then $f(X) \subseteq Y$ is also connected.

Theorem. Let (X, τ_X) and (Y, τ_Y) be topological spaces, and suppose that $f : X \to Y$ is a continuous function. If X is connected, then so if $f(X)$.

Homeomorphic invariants

Definition. A topological space (X, τ) is *path-connected* if, for every pair of points $x, y \in X$, there is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

Path-connectedness implies connectedness.

(2018, Problem 7)

(a) Skipped.

(b) Prove that $X = \mathbb{R} \setminus \{0\}$ and $Y = \mathbb{R}^2 \setminus \{(0,0)\}$ are not homeomorphic.

(c) Prove that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.

Hints.

Y is connected (because it is path-connected), while X is not.

Let $f : \mathbb{R} \to \mathbb{R}^2$ be a homeomorphism and consider the restriction $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}^2 \setminus \{f(0)\}$, which is homeomorphic to $\mathbb{R}^2 \setminus \{(0,0)\}$.

5. Compactness

Compactness

In some sense, compactness is a way to deal with the notion of *finiteness*.

On a topological space (X, τ), we can look at open sets that cover the entire space X. Ordinarily, we do not concern ourselves with the number of open sets required to cover X, just that it covers X.

> But if we want to deal with *finiteness*, then one satisfying solution is that anytime we cover the entire space, we only want to do so with a finite number of open sets.

This gives us a concrete way to define *finiteness* on a topological space.

Compactness

Definition. A topological space (X, τ) is *compact* if, for every $\{V_i\}_{i \in I} \subseteq \tau$ such that
$$\bigcup_{i \in I} V_i = X,$$
there is a *finite* subset $\{i_1, \ldots, i_n\} \subseteq I$ such that
$$\bigcup_{k=1}^{n} V_{i_k} = X.$$

In other words, every cover has a *finite subcover*.

For example, the interval $[0, 1]$ is compact.

It turns out there are some interesting properties that arise out of compact spaces!

Compactness

- Let (X, τ) be a compact topological space. If $Y \subseteq X$ is a closed subset of X, then Y is compact.
- Let (X, τ) be a Hausdorff topological space. If $Y \subseteq X$ is a compact subset of X, then Y is closed.

Compactness

Theorem (Heine-Borel)**.** A subset $X \subseteq \mathbb{R}^n$ is compact if and only if X is closed and bounded.

Theorem (Bolzano-Weierstrass)**.** Every bounded sequence of real numbers has a convergent subsequence.

Therefore, every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Basic Compactness Results

Definition. A topological space (X, τ) is *sequentially compact* if every sequence in X has a convergent subsequence.

- It turns out that, if $X \subseteq \mathbb{R}^n$, then the notion of compactness, closed and boundedness, and sequentially compactness are equivalent.

Let $X \subseteq \mathbb{R}^n$. The following are equivalent.

- X is compact.
- X is sequentially compact.
- X is closed and bounded.

Continuity over compact spaces

Theorem. Let (X, τ_X) and (Y, τ_Y) be compact topological spaces. If $f : X \to Y$ is continuous and X is compact, then $f(X) \subseteq Y$ is compact.

We saw that $[0, 1]$ was compact.

More generally, $[a, b] \subseteq \mathbb{R}$ is also compact.

Therefore, if we take a continuous function $f : X \to \mathbb{R}$ with $X = [a, b]$, then $f(X)$ is compact (and thus, closed and bounded on $\mathbb{R}$). This implies the max-min theorem.

Theorem (Max-Min Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f attains maximum and minimum values.

Continuity over compact spaces

Recall that a function $f : X \to Y$ over metric spaces (X, d_X) and (Y, d_Y) is *uniformly continuous* if, for all $\epsilon > 0$, there exists some $\delta(\epsilon)$ such that

$$d_Y(f(x'), f(x)) < \epsilon$$

as long as $d_X(x, x') < \delta$.
We consider the case where $Y = \mathbb{R}$.

> If X is compact and f is continuous, then f is *uniformly continuous.*

Let (X, d) be a compact metric space, and let $f : X \to \mathbb{R}$ be a continuous function. Then f is *uniformly continuous.*

Continuity over compact spaces

(2018, Problem 3)

Let (X, τ) and (X', τ') be topological spaces.

(a) Skipped.

(b) Prove that if $Y \subseteq X$ is compact and $f : X \to X'$ is continuous, then $f(Y)$ is a compact subset of X'.

(c) Suppose that Y' is a compact subset of X' and $f : X \to X'$ is continuous. Is $f^{-1}(Y')$ necessarily compact?

(a) Skipped.

(b) Consider any open cover $\{V_i\}_{i \in I}$ of $f(Y)$. For each $V_i \in \tau'$, consider $U_i = f^{-1}(V_i) \in \tau$. Since V_i is open, so is U_i. Therefore, $\{U_i\}_{i \in I}$ form an open cover of Y, where $U_i = f^{-1}(V_i)$. However, since Y is compact, each open cover of Y has a finite subcover. The images of the finite subcover form a finite subcover of $f(Y)$; therefore, $f(Y)$ is also compact.

(c) False; this statement is true when X is Hausdorff.

Compactness on metric spaces

We consider a metric space (X, d). We can talk about compactness on metric spaces by discussing concepts of boundedness.

Definition. A metric space (X, d) is said to be *totally bounded* if, for every $\epsilon > 0$, there is a *finite* set $\{x_1, \ldots, x_n\} \subseteq X$ such that

$$X = \bigcup_{k=1}^{n} B(x_k, \epsilon).$$

- If $Y \subseteq X$ is totally bounded, then *every* subset of Y is also totally bounded.
- Similarly, the closure of Y is also totally bounded.

Compactness on a metric space

Let (X, d) be a metric space. The following are equivalent.

- X is compact.
- X is sequentially compact.
- X is complete and totally bounded.

In other words, every compact metric space is *complete*.

However, not all complete metric spaces are *compact*.

$\mathbb{R}$ with its usual metric is complete but not compact.
Complete metric spaces require total boundedness for it to be compact.

Equicontinuity

Definition. Let (X, d_X) and (Y, d_Y) be metric spaces. A subset S of $C(X, Y)$ is said to be

(1) Pointwise equicontinuous if, for every point $x \in X$ and $\epsilon > 0$, there exists $\delta(x, \epsilon)$ such that, for every $f \in S$, we have that

$$d_Y(f(x'), f(x)) < \epsilon \quad \text{whenever } d_X(x', x) < \delta.$$

(2) Uniformly equicontinuous if, for every $\epsilon > 0$, there exists $\delta(\epsilon)$ such that, for every $f \in S$, we have that

$$d_Y(f(x'), f(x)) < \epsilon \quad \text{whenever } d_X(x', x) < \delta.$$

Note that *pointwise* depends on the point and threshold ϵ, whereas *uniform* must hold for every point in X.

Clearly, we see that uniform equicontinuity implies pointwise equicontinuity. When does pointwise equicontinuity imply uniform equicontinuity?

When X is compact!

Theorem. Let (X, d_X) and (Y, d_Y) be metric spaces, with X compact. Then $S \subseteq C(X, Y)$ is pointwise equicontinuous if and only if S is uniform continuous.

This gives rise to an important theorem, the *Arzela-Ascoli Theorem.*

Theorem (Arzela-Ascoli)**.** A bounded subset of $(C[0, 1], \|\cdot\|_\infty)$ is totally bounded if and only if it is equicontinuous.

More generally, $[0, 1]$ can be replaced with any compact metric space.

The Arzela-Ascoli Theorem implies that any subset of $(C[0, 1], \|\cdot\|_\infty)$ is compact if and only if it is closed, bounded, and equicontinuous.

The Weierstrass Approximation Theorem

We continue to look at the metric space $C[a, b]$ endowed with the $\|\cdot\|_\infty$ metric.

Theorem (Weierstrass Approximation Theorem)**.** Let f be a continuous function on a *closed* and *bounded* interval $[a, b]$. For any $\epsilon > 0$, there exists a polynomial $p(x)$ such that

$$\|f - p\|_\infty < \epsilon.$$

(2022, Problem 9)

Let

$$V = \left\{ f \in C[0,1] : \int_0^1 f(t)\,dt = 0 \right\}$$

and let P denote the set of polynomials which lie in V.

(a) Give an example of a nonzero element of P.

(b) What is the closure of P in $(V, \|\cdot\|_\infty)$?

(c) Is $(V, \|\cdot\|_\infty)$ complete?

(a) Any polynomial whose antiderivative has roots 0 and/or 1 suffices. An easy example is $f(x) = 2x - 1$ since its antiderivative is $g(x) = x^2 - x$ and $\int_0^1 g(x)\,dx = 0$.

(b) We note that $C[0,1]$ is the set of continuous functions on the closed and bounded interval $[0,1]$. Therefore, any function $f \in V$ has a polynomial $p(x)$ such that

$$\|f - p\|_\infty < \epsilon/2.$$

Additionally, we see that

$$\begin{aligned}\left|\int_0^1 p(x)\,dx\right| &= \left|\int_0^1 p(x) - f(x) + f(x)\,dx\right| \\ &\leq \left|\int_0^1 p(x) - f(x)\,dx\right| + \left|\int_0^1 f(x)\,dx\right| \\ &\leq \int_0^1 |p(x) - f(x)|\,dx \leq \|f - p\|_\infty < \frac{\epsilon}{2}.\end{aligned}$$

We now consider $q(x) = p(x) - \int_0^1 p(x)\,dx$. Firstly, since p is a polynomial, so is q. Secondly, we see that

$$\begin{aligned}
\int_0^1 q(x)\,dx &= \int_0^1 \left(p(x) - \int_0^1 p(x)\,dx\right)\,dx \\
&= \int_0^1 p(x)\,dx - \int_0^1 \int_0^1 p(x)\,dx \\
&= \int_0^1 p(x)\,dx - \left(\int_0^1 p(x)\,dx\right) \cdot \int_0^1 1\,dx \\
&= \int_0^1 p(x)\,dx - \int_0^1 p(x)\,dx = 0.
\end{aligned}$$

Therefore, $q \in P$. We also see that

$$\|f - q\|_\infty = \left\|f - p + \int_0^1 p\right\|_\infty \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, P is dense in V and so, $\overline{P} = V$.

(c) Recall that $(C[0,1], \|\cdot\|_\infty)$ is complete. Therefore, it suffices to show that $(V, \|\cdot\|_\infty)$ is closed. Here's a sketch of the proof.

> Consider the map $f : C[0,1] \to \mathbb{R}$ where $f(g) = \int_0^1 g(t)\, dt$.
> Show that f is a continuous map.
> Then notice that $V = f^{-1}(\{0\})$ and $\{0\}$ is closed.

Therefore, V is a closed subset of $C[0,1]$. Since $C[0,1]$ is complete and V is closed, it follows that V is complete.

Separation of points

Definition. Let X and Y be two sets. A set S of functions between X and Y is said to be *separate points* if, for all distinct $x, y \in X$, there is a function $f \in S$ such that $f(x) \neq f(y)$.

Theorem (Urysohn's Lemma)**.** Let X be a Hausdorff and compact space. Then $C(X, \mathbb{R})$ separates points.

Theorem (Stone-Weierstrass)**.** Let X be a Hausdorff and compact space. Let $A \subseteq C(X, \mathbb{R})$ be a unital subalgebra. Then A is dense with respect to $\|\cdot\|_\infty$ if and only if A separates points.

Recall that an algebra of functions is a vector space of functions that is closed under pointwise multiplication of functions.

A *unital* algebra is an algebra that also contains the unit constant function 1.

(2019, Problem 9)

(a) Skipped.

(b) Let S be the span of the set of functions $\{e^{kt}\}_{k=0,1,\ldots} \subset C[0,1]$. What are the interior, boundary, and closure of S in the uniform norm of $C[0,1]$?

Recall that the uniform norm is the *supremum norm* $\|f\|_\infty$.

- $C[0,1]$ with the supremum norm is Hausdorff and compact.
- S is a unital subalgebra.
- Note that $f_k(t) = e^{kt}$ is injective, so $f_k(x) \neq f_k(y)$ for any pair of distinct points $x, y \in [0,1]$. Therefore, S *separates* points.

Use Stone-Weierstrass to show that the closure of S is $C[0,1]$.

- The boundary is $C[0,1] \setminus S$.
- The interior is empty because every smooth function can be approximated arbitrarily closely by non-smooth functions.