The Sunflower Lemma and its Modifications

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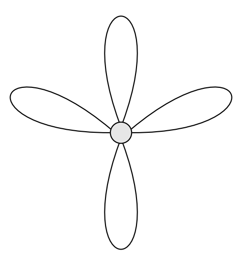
Chapter 1

Introduction

The *Sunflower Lemma* is one the earliest and most beautiful results in modern extremal combinatorics, which was discovered by Erdös and Rao in 1960. Very crudely speaking, the lemma asserts that if we begin with a collection of large and uniform sets, then we must obtain some configuration that has some very regular and ordered property regardless of the size of the ground set of our universe set. This exposition aims to give insight into the lemma, as well as discuss a few applications and modifications to the lemma.

Consider a collection $\mathcal{S}$ of sets $S_1, \ldots, S_k$. A natural restriction we might like to impose on the sets of $\mathcal{S}$ is that each pair of distinct sets S_i, S_j with $i \neq j$ have common intersection; that is, for each $i \neq j$, $S_i \cap S_j = Y$ for some set Y. Pictorially, one can imagine this as a sunflower where each set $S_i \setminus Y$ forms a petal and Y forms the *core* of the sunflower.



A sunflower with four petals. Each petal is denoted by the set $S_i \setminus Y$, whilst the core is denoted by the set Y.

This leads naturally to our first definition.

Definition. A *sunflower with* k *petals* and *core* Y is a family $\mathcal{S}$ of k sets $S_1, \ldots, S_k$ such that for each $i \neq j$, $S_i \cap S_j = Y$.

We do not require Y to be non-empty; a sunflower with k petals such that $S_i \cap S_j = \emptyset$ is perfectly fine as well; the core is just empty. The *sunflower lemma* shows the existence of a small sunflower when the family of sets we start with is very large. Even when the family looks seemingly random, the lemma ensures that a sunflower with k petals exist. Finding such a sunflower is quite difficult though! We now state the *Sunflower Lemma*.

Theorem (Sunflower Lemma)**.** *Let* $\mathcal{F}$ *be a family of sets, each of cardinality at most* ℓ*. If* $|\mathcal{F}| > \ell! \cdot (k-1)^\ell$*, then* $\mathcal{F}$ *contains a sunflower with* k *petals.*

Proof. We induct on ℓ.

For the base case $\ell = 1$, each set contains one element and so, each set must be disjoint from any other set. Therefore, if $|\mathcal{F}| > \ell! \cdot (k-1)^\ell = k-1$, then we can choose any k sets in $\mathcal{F}$ to form our sunflower with k petals and empty core.

Now, let $\ell \geq 2$ and consider any maximal family $\mathcal{F}' = \{\mathcal{F}_1, \ldots, \mathcal{F}_t\}$ of pairwise disjoint members of $\mathcal{F}$. If $t \geq k$, we are done since $\mathcal{F}'$ forms the sunflower with k petals.

We may, therefore, assume that $t \leq k-1$. Let $\mathcal{A} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_t$. Since each set contains at most ℓ elements, we first see that $|\mathcal{A}| \leq \ell \cdot (k-1)$. Since $\mathcal{F}'$ is a maximal family of pairwise disjoint, it follows that $\mathcal{A}$ must intersect every set in $\mathcal{F}'$; otherwise, we could add $\mathcal{A}$ into $\mathcal{F}'$.

Since there are at most $\ell \cdot (k-1)$ elements in $\mathcal{A}$ and there are more than $\ell! \cdot (k-1)^\ell$ elements in $\mathcal{F}$, then there must exist some element $x \in \mathcal{A}$ that is contained in at least

$$\frac{|\mathcal{F}|}{|\mathcal{A}|} \geq \frac{\ell! \cdot (k-1)^\ell}{\ell \cdot (k-1)} = (\ell-1)! \cdot (k-1)^{\ell-1}$$

sets of $\mathcal{F}$. Consider these collection of sets without x; that is, consider the following collection of sets

$$\mathcal{F}'_x = \{S \setminus \{x\} : S \in \mathcal{F},\, x \in S\}.$$

By our inductive hypothesis, $\mathcal{F}'_x$ contains a sunflower with k petals. Adding x back into each of these sets still gives us a sunflower with k petals but these sets are now also in $\mathcal{F}$. Therefore, there is a sunflower with k petals in $\mathcal{F}$, as required. $\square$

What this theorem tells us is that even if the universe set is large, we still obtain highly regular configurations within our family. A natural question to ask now is whether the bound proposed in the sunflower lemma is tight; that is, can we reduce the inequality and still ensure the existence of a sunflower?

1.1 Bounds on the Sunflower Lemma

To make the exposition cleaner, we shall define the following function.

Definition. Let $f(s,k)$ be the least integer such that any family of $f(s,k)$ non-empty sets, each of cardinality s, contains a sunflower with k petals.

The sunflower lemma asserts that $f(s,k) \leq s! \cdot (k-1)^s + 1$. Is there any other bound that we can make? Certainly!

Firstly, consider the following family: take s disjoint sets $A_1, \dots, A_s$ with $|A_1| = \dots |A_s| = k-1$. Define the family $\mathcal{F}$ to be the family of sets obtained by choosing one element from each set A_i; that is, consider the family $\mathcal{F} = A_1 \times \cdots \times A_s$. It is easy to see that $|\mathcal{F}| = (k-1)^s$, and each set of $\mathcal{F}$ consists of s elements.

On the other hand, suppose that $\mathcal{F}$ contains a sunflower with k petals, say $\{S_1, \dots, S_k\} \subseteq \mathcal{F}$. Then for each $i \in \{1, \dots, s\}$, the family $\{S_1 \cap A_i, \dots, S_k \cap A_i\}$ also forms a sunflower. However, since A_i consists of $k-1$ elements, this is only true when $S_1 \cap A_i = \cdots = S_k \cap A_i = \{a_i\}$, which contradicts the existence of a sunflower with k petals. This suggests that a lower bound of $f(s,k)$ is $(k-1)^s$. In other words, we know that

$$(k-1)^s < f(s,k) \leq s! \cdot (k-1)^s + 1.$$

Chapter 2

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2.1 Relaxation of the core

2.2 Relaxation of the petals

Chapter 3

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