1) $T(N) = 2T(N-1)+1$

$\Rightarrow 2\left(2T(N-2)+1\right)+1 - 2^2T(N-2)+2+1$

$\Rightarrow 2^3T(N-3)+2^2+2+1$

$\Rightarrow 2^kT(N-k)+2^0+2^1+\dots+2^{k-1}$

$\Rightarrow 2^kT(N-k)+\left(2^k-1\right)$

when $N-k=1 \rightarrow k=N-1$ hence,

$T(N) = 2^{N-1}T(1)+\left(2^{N-1}-1\right)$

since $T(1)$ is a constant $= O(1)$

$\Rightarrow O(2^N)$

2) $T(N) = 3T(N-1)+N$

$\Rightarrow 3\left(3T(N-2)+(N-1)\right)+N$

$\Rightarrow 3^2T(N-2)+3(N-1)+N$

$\Rightarrow 3^3T(N-3)+3^2(N-2)+3(N-1)+N$

$\Rightarrow 3^kT(N-k)+N\left(1+3+3^2+\dots+3^{k-1}\right)-\left(1+2\cdot 1\cdot 3+\dots k-1)\right)$

$\Rightarrow$ using geometric formula and the given geometric series

hence $(3^k-1)/(3-1)$ and $(k-1)k/2$

$a=3$ $k=1$
$b=1$
$f(n)=n$

$T(n) = O(n^1\cdot\frac{3n}{3})$

when $N-k=1 \Rightarrow k=N-1$ hence $\Rightarrow O(n^1\cdot n) = 2$

$T(N) \Rightarrow O(n^2)$

3) $9T(N/2) + N^2$

using master theorem $= T(N) = aT(\frac{N}{b}) + f(N)$

where $a = 9$, $b = 2$, $f(N) = N^2$

Hence, $n^{\log_b a} \Rightarrow N^{\log_2 9} \Rightarrow N^{\log_2 32} \Rightarrow N^{(3.17)}$

Hence, $T(N) = \Theta(N^{\log_2 (9)}) = \Theta(n^{3.17})$

recursilvely getting 9 children which have a size of $N/2$ and each node contributes n^2

so $T = n^{2(\frac{4}{9})^{\log N}}$ → geometric series

using property of logarithms, $\left(\frac{9}{4}\right)^{\log N} = N^{\log 9 - \log 4}$

$\Rightarrow N^{\log 9 - 2}$

simplify $\Rightarrow \Theta(n^{3.17})$

4) $T(N) = 100T(\frac{N}{2}) + n^{\log 2cn+1}$

$\log 2(100) = 6.64$

$6.64 > \log 2cn+1$, $F(n) = \Theta(n^{6.64})$

recursivly 100 children have size of $\frac{N}{2}$ where each node contributes $n^{\log 2cn+1}$ using this approximation $T = n^{\log 2cn+1} \cdot \frac{4}{100}^{\log N}$ → geometric series

$\log 2(100) = 6.69$ simplify $N^{\log 100-2} = \Theta n^{6.64}$

5) $4T(N/2) + n^2 \log n$

$a = 4$, $b = 2$, $F(N) = n^2 \log n$

$\log_2 4 = 2$

$F(N) = \Omega(N^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $aF(N/b) \leq k F(N)$ for $k < 1$ and large N

Hence $T(N) = \theta(F(N))$

$4\left(\frac{N}{2}\right)^2 \log\left(\frac{N}{2}\right) \leq k N^2 \log N$

$\Rightarrow N^2 \log\left(\frac{N}{2}\right) \leq k N^2 \log N$

$\Rightarrow \log N - \log 2 \leq k \log N$

$\Rightarrow \log N - 1 \leq k \log N$

$\Rightarrow \log N$

$\Rightarrow -1 \leq (k-1) \log N$

Hence $\theta(n^2 \log n)$

6) $a = 5 \quad b = 2 \quad F(N) = \frac{n^2}{\log n}$

$\Rightarrow 5T(N/2) + \frac{n^2}{\log 2 \cdot \log n}$

$\Rightarrow 5T\left(\frac{N}{2}\right) + \frac{c \cdot n^2}{\log n}, \quad c = \frac{1}{\log 2}$

by case 3 of master theorem

$\Rightarrow T(N) = \Theta F(n) = \Theta\left(\frac{n^2}{\log n}\right)$