CSE/AMS 547 Discrete Mathematics December 4, 2017

Homework 4

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Due by Monday, Dec. 4, 11:59pm.

Problem 5.18

Solution :

- 5.35 can be written as :

$\binom{r}{k}\binom{r-\frac{1}{2}}{k} = \frac{\binom{2r}{2k}\binom{2k}{k}}{2^{2k}}$

Also, the expansion of $\binom{r}{k}$ can be written as:

$\binom{r}{k} = \frac{r!}{(r-k)!(k!)} = \frac{(r)(r-1)(r-2)...(r-k+1)}{k!}$

Applying this expansion to the given equation:

$\binom{r}{k}\binom{r-1/3}{k}\binom{r-2/3}{k}$

$= (\frac{(r)(r-1)(r-2)...(r-k+1)}{k!})(\frac{(r-1/3)(r-1/3-1)...(r-1/3-k+1)}{k!})(\frac{(r-2/3)(r-2/3-1)(r-2/3-2)...(r-2/3-k+1)}{k!})$

$= (\frac{(r)(r-1)(r-2)...(r-k+1)}{k!})(\frac{(r-1/3)(r-4/3)...(r-k+2/3)}{k!})(\frac{(r-2/3)(r-5/3)(r-8/3)...(r-k+1/3)}{k!})$

$= (\frac{(r)(r-1/3)(r-2/3)...(r-k+1)(r-k+2/3)(r-k+1/3)}{k!k!k!})$

We can multiply this throughout by 3^{3k}

$= \frac{(3r)(3r-1)(3r-2)(3r-3)...(3r-3k+3)(3r-3k+2)(3r-3k+1)}{k!k!k!} \frac{1}{k!k!k!3^{3k}}$

$= (3r)(3r-1)(3r-2)(3r-3)...(3r-3k+3)(3r-3k+2)(3r-3k+1)(\frac{1}{k!k!k!3^{3k}})$

$= (\frac{(3r)!}{3r-3k})(\frac{1}{k!k!k!3^{3k}})$

Multiplying and dividing throughout by $(3k)!$ and $(2k)!$, we get

$= (\frac{(3r)!}{3r-3k})(\frac{1}{k!k!k!3^{3k}})(\frac{(3k)!}{(3k)!})(\frac{(2k)!}{(2k)!})$

$= (\frac{(3r)!}{(3r-3k)!(3k)!})(\frac{(3k)!}{(2k!)k!})(\frac{(2k)!}{k!k!})(\frac{1}{3^{3k}})$

$= \frac{(\binom{3r}{3k})(\binom{3k}{2k})(\binom{2k}{k})}{3^{3k}}$

which is the required result

Problem 5.19

Solution :

5.58 is : $B_t(z) = \sum_{k\geq 0}(tk)^{\underline{k-1}}(\frac{z^k}{k!})$

Expanding this, we get:

$= \sum_{k\geq 0} \frac{(tk)(tk-1)(tk-2)...(tk-k+2)(z^k)}{k!}$

Multiplying and dividing throughout by $(tk - k + 1)$, we get:

$= \sum_{k\geq 0} \frac{(tk)(tk-1)(tk-2)...(tk-k+2)(tk-k+1)(z^k)}{(tk-k+1)k!}$

$= \sum_{k\geq 0} \frac{(\binom{tk}{k})(z^k)}{(tk-k+1)}$

We have equation 5.60 as:

$B_t(z)^r = \sum_{k\geq 0} \binom{tk+r}{k} \frac{r}{tk+r} z^k$

Let r=-1 and substitute t with 1-t and z with -z :

$B_{1-t}(-z)^-1 = \sum_{k\geq 0} \binom{k-tk-1}{k} \frac{-1}{k-tk-1} -z^k$

Applying upper negation to the form above, we have:

$= \sum_{k\geq 0} \binom{k-tk-1}{k} \frac{-1}{k-tk-1} -1^k z^k$

$= \sum_{k\geq 0} \binom{k-tk-1}{k} \frac{-1}{k-tk-1} (-1^k) z^k$

$= \sum_{k\geq 0} \binom{k-(k-tk-1)-1}{k} \frac{-1}{k-tk-1} z^k$

$= \sum_{k\geq 0} \binom{tk}{k} \frac{1}{tk-k+1} z^k$

$= \sum_{k\geq 0} \frac{(\binom{tk}{k})(z^k)}{(tk-k+1)}$

Which is equal to the previous result. Hence, proved...

Problem 5.40

Solution :

$\sum_{j=1}^{m} -1^{k+1} \binom{r}{j} \sum_{k=1}^{n} \binom{-j+rk+s}{m-j}$

Swapping the summations of j and k, we have:

$= \sum_{j=1}^{m} -1^{j+1} \binom{r}{j} \sum_{k=1}^{n} (-1)^{m-j} \binom{(m-j)-j+rk+s-1}{m-j}$

$= \sum_{j=1}^{m} -1^{j+1} \binom{r}{j} \sum_{k=1}^{n} (-1)^{m-j} \binom{m-rk-s-1}{m-j}$

Combining the two summations:

$= \sum_{k=1}^{n} \sum_{j=1}^{m} (-1)^{j+1}(-1)^{m-j} \binom{r}{j} \binom{m-rk-s-1}{m-j}$

$= \sum_{k=1}^{n} \sum_{j=1}^{m} (-1)^{m+1} \binom{r}{j} \binom{m-rk-s-1}{m-j}$

$= \sum_{k=1}^{n} \sum_{j=1}^{m} \binom{r}{j} \binom{m-r(k-1)-s-1}{m} - \binom{m-r(k)-s-1}{m}$

$= (-1)^{m+1} \sum_{k=1}^{m} \binom{r}{j} \binom{m-r(k-1)-s-1}{m} - \binom{m-r(k)-s-1}{m}$

$= (-1)^{m} \sum_{k=0}^{m} \binom{m-r(k)-s-1}{m} - \binom{m-s-1}{m}$

$= (-1)^{m} \binom{m-rn-s-1}{m} - \binom{m-s-1}{m}$

$= \binom{rn+s}{m} - \binom{s}{m}$

Which is the required result

Problem 5.41

Solution :

$\sum_{k} \binom{n}{k} \frac{k!}{(n+1+k)!}$

expanding the binomial, we have:

$= \sum_{k} \frac{n!}{(n-k)!k!} \frac{k!}{(n+1+k)!}$

Multiplying and dividing by $(2n+1)!$

$= \sum_{k} \frac{n!}{(n-k)!k!} \frac{k!}{(n+1+k)!} \frac{(2n+1)!}{(2n+1)!}$

$= \sum_{k} \frac{n!}{(2n+1)!} \frac{(2n+1)!}{(n-k)!(n+1+k)!}$

$= \frac{n!}{(2n+1)!} \sum_{k} \frac{(2n+1)!}{(n-k)!(n+1+k)!}$

$= \frac{n!}{(2n+1)!} \sum_{k=0}^{n} \binom{2n+1}{n+k+1}$

Now, if replace (n+k+1) with k

$= \frac{n!}{(2n+1)!} \sum_{k=n+1}^{2n+1} \binom{2n+1}{k}$

$= \frac{n!}{(2n+1)!} \sum_{k=0}^{n} \binom{2n+1}{k}$

Adding these two, we get:

$= \frac{n!}{(2n+1)!} (2)^{2n+1}$

The required result is half the previous result, which gives us:

$= \frac{n!}{(2n+1)!} (\frac{1}{2}) (2)^{2n+1}$

$= \frac{n!}{(2n+1)!} (2)^{2n}$

Problem 5.60

Solution :

$\binom{m+n}{n}$

Expanding this, we have:

$= \frac{(m+n)!}{n!m!}$ - This is the form when $m! = n$

$= \frac{(2n)!}{n!n!}$ - This is the form when $m = n$

Using Stirling's approximation:

$\simeq \frac{\sqrt{2\pi(2n)}\frac{2n}{e}^{2n}}{(\sqrt{2\pi n}(\frac{n}{e})^n)(\sqrt{2\pi n}(\frac{n}{e})^n)}$

We can simplify this to get :

$\frac{4^n}{\sqrt{\pi n}}$

Which is the required result

Now, when $m! = n$:

$\simeq \frac{\sqrt{2\pi(m+n)}\frac{m+n}{e}^{m+n}}{(\sqrt{2\pi m}(\frac{m}{e})^m)(\sqrt{2\pi n}(\frac{n}{e})^n)}$

$\simeq \frac{\sqrt{m+n}(m+n)^{m+n}}{\sqrt{2\pi mn}m^m n^n}$

$\simeq \sqrt{\frac{m+n}{mn}\frac{1}{2\pi}}(\frac{m+n}{m})^m(\frac{m+n}{m})^n$

$\simeq \sqrt{\frac{1}{2\pi}(\frac{1}{m}+\frac{1}{n})}(1+\frac{m}{n})^n(1+\frac{n}{m})^m$

Which is the required result

Problem 5.80

Solution :

To prove:

$\binom{n}{k} \leq (\frac{en}{k})^k$

We can do this via induction:

We can see that this is true when k=1 :

$n \leq en$

Assume this is true for all k. Now, we try and prove it for k+1.

Taking ratios and comparing the rate of growth:

$\frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{n-k}{k+1}$

$\frac{(\frac{en}{k+1})^{k+1}}{\frac{en}{k}} = (\frac{n}{k+1})(\frac{e}{(1+1/k)^k})$

This is $\geq \frac{n}{k+1}$

Now, clearly,

$\frac{n}{k+1} \geq \frac{n-k}{k+1}$

Thus, we see that $(\frac{en}{k})^k$ is increasing at a faster rate than $\binom{n}{k}$

Hence, proved...