CSE/AMS 547 Discrete Mathematics October 23, 2017

Homework 2

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Due by Monday, Oct. 23, 11:59pm.

Problem 2.22

Given :

Prove Lagrange's Identity (without induction)

Solution :

1. Splitting the original and using the distributive law, we get :

$$\sum_{1\leq j<k\leq n}(a_jb_k - a_kb_j)^2 = \sum_{1\leq j<k\leq n}(a_jb_k)^2 - 2\sum_{1\leq j<k\leq n}(a_jb_ka_kb_j) + \sum_{1\leq j<k\leq n}(a_kb_j)^2 \qquad (1)$$

2. The first and last terms are the sums of terms when $j \neq k$, ie. $\sum\limits_{1\leq j\neq k\leq n}(a_jb_k)^2$

3. Using a similar logic, we can split the 2^{nd} term and notice that two parts are of the same form as that of the previous step. There is only a change in the ordering of the terms. So, we can write it as $j \neq k$, ie. $\sum\limits_{1\leq j\neq k\leq n}(a_jb_ka_kb_j)^2$

4. To get the final equation as required, we need both j and k going from 1 to n (of the form $1 \leq j, k \leq n$), and to achieve the split we have to consider the cases where j=k.

5. We see that whenever j=k, the previous two terms are equal, ie. both are : $\sum\limits_{1\leq j\leq n}(a_j)^2(b_j)^2$. As these two cancel out, we can use them in the equation to change the limits.

6. The equation that we get is as follows :

$$\begin{aligned}\sum_{1\leq j\neq k\leq n}(a_jb_k)^2-\sum_{1\leq j\neq k\leq n}(a_jb_ka_kb_j)^2 &= \sum_{1\leq j,k\leq n}(a_jb_k)^2-\sum_{1\leq j,\leq n}(a_jb_ka_kb_j)^2\\ &= \sum_{1\leq j\leq n}(a_j)^2\sum_{1\leq j\leq n}(b_j)^2-\sum_{1\leq j\leq n}(a_jb_j)\sum_{1\leq k\leq n}(a_kb_k)\\ &= \sum_{1\leq k\leq n}(a_k)^2\sum_{1\leq k\leq n}(b_k)^2-(\sum_{1\leq k\leq n}(a_kb_k))^2\end{aligned} \tag{2}$$

which is the required result. For the second part of the question,

$\sum_{1\leq j<k\leq n}(a_jb_k-a_kb_j)(A_jB_k-A_kB_j)$

we can expand the given equation to get the four distinct terms : $a_jA_jb_kB_k$, $a_jB_jb_kA_k$, $b_jA_ja_kB_k$, $b_jB_ja_kA_k$

Before proceeding further, we can notice that the result remains unchanged when j and k are swapped (Since both terms just become their own negatives and result in the same positive sum).

The limit $1\leq j,k\leq n=(1\leq j<k\leq n)+(1\leq j=k\leq n)+(1\leq k<j\leq n)$

If we take the given sum as $S_{j,k}$, we can split the sum according to the previous step. When j = k, all the individual terms become 0 and the 1st and 3rd terms are equal to each other.

So we get : $\sum_{1\leq j,k\leq n}S_{j,k}=2\sum_{1\leq j<k\leq n}S_{j,k}$

7. The first term can be re-written in the following way :

$$\begin{aligned}\sum_{1\leq j<k\leq n}a_jA_jb_kB_k &= \sum_{j=1}^{n}\sum_{k=1}^{n}a_jA_jb_kB_k\\ &= (\sum_{j=1}^{n}a_jA_j)(\sum_{k=1}^{n}b_kB_k)\\ &= (\sum_{k=1}^{n}a_kA_k)(\sum_{k=1}^{n}b_kB_k)\end{aligned} \tag{3}$$

8. The second term can be re-written in the following way :

$$\sum_{1\le j<k\le n} a_jB_jb_kA_k = \sum_{j=1}^{n}\sum_{k=1}^{n} a_jB_jb_kA_k$$
$$= (\sum_{j=1}^{n} a_jB_j)(\sum_{k=1}^{n} b_kA_k) \tag{4}$$
$$= (\sum_{k=1}^{n} a_kB_k)(\sum_{k=1}^{n} b_kA_k)$$

9. The third term can be re-written in the following way :

$$\sum_{1\le j<k\le n} b_jA_ja_kB_k = \sum_{j=1}^{n}\sum_{k=1}^{n} b_jA_ja_kB_k$$
$$= (\sum_{j=1}^{n} b_jA_j)(\sum_{k=1}^{n} a_kB_k) \tag{5}$$
$$= (\sum_{k=1}^{n} b_kA_k)(\sum_{k=1}^{n} a_kB_k)$$

10. The fourth term can be re-written in the following way :

$$\sum_{1\le j<k\le n} b_jB_ja_kA_k = \sum_{j=1}^{n}\sum_{k=1}^{n} b_jB_ja_kA_k$$
$$= (\sum_{j=1}^{n} b_jB_j)(\sum_{k=1}^{n} a_kA_k) \tag{6}$$
$$= (\sum_{k=1}^{n} b_kB_k)(\sum_{k=1}^{n} a_kA_k)$$

11. The 1st term = last term and the middle two terms are equal, so the final combination of the 4 sums results in :

$2(\sum_{k=1}^{n} a_kA_k)(\sum_{k=1}^{n} b_kB_k) - 2(\sum_{k=1}^{n} a_kB_k)(\sum_{k=1}^{n} b_kA_k)$

This is the result for the limits : $1 \le j,k \le n$. To get the result for $1 \le j < k \le n$, we can just divide the result obtained above by 2. Thus the final required result is : $(\sum_{k=1}^{n} a_kA_k)(\sum_{k=1}^{n} b_kB_k) - (\sum_{k=1}^{n} a_kB_k)(\sum_{k=1}^{n} b_kA_k)$

Problem 2.27

Solution :

1. To begin, we see that $\Delta(c^{\underline{x}}) = (c^{\underline{x+1}}) - (c^{\underline{x}})$ (From the definition of δ in the book)

2. By the m^{th} power rule, we also get : $(c^{\underline{x}}) = c(c-1)(c-2)...(c-x+1)$

3. By the m^{th} power rule again, we get : $(c^{\underline{x+1}}) = c(c-1)(c-2)...(c-x+1)(c-x)$

4. By the m^{th} power rule again, we get : $(c^{\underline{x+2}}) = c(c-1)(c-2)...(c-x+1)(c-x)(c-x-1)$

5. Subtracting, we get : $c(c-1)(c-2)...(c-x+1)(c-x) - c(c-1)(c-2)...(c-x+1)$

6. Solving for this equation:

$$\begin{aligned} c(c-1)...(c-x+1)(c-x) - c(c-1)...(c-x+1) &= c(c-1)...(c-x+1)((c-x)-1) \\ &= c(c-1)...(c-x+1)(c-x-1) \\ &= c...(c-x-1)((c-x)/(c-x)) \\ &= (c^{\underline{x+2}})/(c-x) \end{aligned} \tag{7}$$

Which is the required result

7. To calculate $\sum_{k=1}^{n}((-2)^k/k)$, we substitute $c = -2$ and $x = x-2$

$$\begin{aligned} \Delta((-2)^k/k) &= \frac{(-2)^{\underline{(x-2)+2}}}{(-2-(x-2))} \\ &= \frac{(-2)^{\underline{x}}}{-x} \end{aligned} \tag{8}$$

8. We know that $\sum_{k=1}^{n}((-2)^k/k) = \sum_{k=1}^{n+1}((-2)^k/k)(\delta k)$

9. Solving for the previous equation :

$$\begin{aligned} \sum_{k=1}^{n+1}((-2)^k/k)(\delta k) &= (-(-2)^{k-2}) \mid_1^{n+1} \\ &= -(-2)^{\underline{n-1}} - -(-2)^{\underline{-1}} \\ &= (-2)^{\underline{-1}} - (-2)^{\underline{n-1}} \\ &= \frac{1}{-2+1} - (-2)^{\underline{n-1}} \\ &= -1 - ((-2)(-2-1)...(-2-(n-2))) \\ &= -1 + ((-1)(-2)(-3)...(-n)) \\ &= -1 + (-1)^n n! \end{aligned} \tag{9}$$

Which is the required result

Problem 4.16

Solution :

1. The reciprocal of a Euclid number is : $\frac{1}{e_n}$

2. From the recurrence relation given in the book, we know that $e_n = e_{n-1}(e_{n-1}-1)+1$

3. Following the previous statement, we also have $e_{n+1} = e_n(e_n - 1) + 1$

4. Solving for the previous equation we have,

$$\begin{aligned}
e_{n+1} &= e_n(e_n - 1) + 1 \\
e_{n+1} - 1 &= e_n(e_n - 1), Now\, taking\, the\, reciprocals \\
\frac{1}{e_{n+1} - 1} &= \frac{1}{e_n(e_n - 1)} - 1 \\
&= \frac{1}{e_n - 1} - \frac{1}{e_n} \\
\frac{1}{e_n} &= \frac{1}{e_n - 1} - \frac{1}{e_{n+1} - 1}
\end{aligned} \tag{10}$$

We see that this form for the term will give us an oscillating series when we sum it

5. Summing it we get $\sum_{k=1}^{k=n} \frac{1}{e_k - 1} - \frac{1}{e_{k+1}-1}$

6. Except for the first and last terms all cancel each other out, and we are left with

$\frac{1}{e_1-1} - \frac{1}{e_{n+1}-1}$

$= \frac{1}{2-1} - \frac{1}{e_{n+1}-1}$

$= 1 - \frac{1}{e_{n+1}-1}$

Which is the required result.

Problem 4.24

Solution :

1. We know that $\epsilon_p(n!) = \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + ... = \sum_{k\geq 1} \lfloor \frac{n}{p^k} \rfloor$

2. Also, $\nu_p(n)$ is the sum of digits in the 'p-base' representation of n.

3. We can assume the p-base representation of n to be of the form $a_x a_{x-1}...a_0$, which is similar to a binary representation of the number.

4. Therefore, $n = a_x p^x + ... + a_1 p + a_0$ in base p

5. Dividing a term by p^i, we get : $\lfloor \frac{n}{p^i} \rfloor = a_x p^{x-i} + ... + a_{i+1}p + a_i$ (from its definition)

6. The sum mentioned in step 1 becomes :

$$\begin{aligned}
\sum_{k=1}^{x}(a_x p^{x-k} + ... + a_{i+1}p + a_i) &= \sum_{k=1}^{x}\sum_{j=i}^{x} a_j p^{j-i} \\
&= \sum_{j=i}^{x}\sum_{k=1}^{x} a_j p^{j-i} \\
&= \sum_{j=i}^{x} n_j \frac{p^j - 1}{p-1} \\
&= \frac{1}{p-1}\sum_{j=1}^{x}(n_j p^j - n_j) \\
&= \frac{1}{p-1}\sum_{j=0}^{x}(n_j p^j - n_j) \\
&= \frac{1}{p-1}(n - \nu_p(n))
\end{aligned} \tag{11}$$

In the previous set of equations, the 2nd last step is done because adding j=0 will result in no extra addition to the final sum but helps us get the term a_0.

This is the required result.

Problem 4.26

Solution :

If we notice the terms given, we have terms which are obtained by adding the two adjacent terms, eg. 0/1 and 1/9 result in 1/10. This is basically the construction of a Stern-Brocot tree. The given function representing the terms is just a subtree of the Stern-Brocot tree and within that tree for m/n preceding m'/n', m'n-mn' = 1.

Problem 4.30

Was unable to solve this problem after trying.

Problem 4.38

Was unable to solve this problem after trying.

Problem 4.42

Solution :

1. We know that $k \perp m$ and $k \perp n \iff k \perp mn$

2. Also, if $k \perp m$, then $k \perp m + xk$. This follows from the fact that gcd(a,b) = gcd(a,b+ax)

3. So $m \perp n$ and $n' \perp n \iff n \perp mn'$

4. Using the relation in step two we also get $n \perp mn' + nm'$ (where nm' is a multiple of n)

5. Also, $m' \perp n'$ and $n' \perp n \iff n' \perp m'n$

6. Using the relation in step two we also get $n' \perp mn' + nm'$ (where nm' is a multiple of n)

7. Combining steps 4 and 6, we get $nn' \perp mn' + nm'$

8. This shows us that the result that we get from adding the two fractions, ie. $\frac{mn'+nm'}{nn'}$ is in its reduced form as the numerator and denominator are relatively prime as show above. Therefore the denominator is nn'.

This is the required result.