CSE/AMS 547 Discrete Mathematics December 11, 2017

Homework 5

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Due by Monday, Dec. 11, 11:59pm.

Problem 6a

Solution :

$\left\{ {n+1 \atop m+1} \right\} = \sum_{k=0}^{k=n} \left\{ {k \atop m} \right\} (m+1)^{n-k}$

Using recurrence on the left side, we have:

$(m+1)\left\{ {n \atop m+1} \right\} + \left\{ {n \atop m} \right\}$

Applying recurrence again, we have:

$(m+1)((m+1)\left\{ {n-1 \atop m+1} \right\} + \left\{ {n-1 \atop m} \right\}) + \left\{ {n \atop m} \right\}$

$= (m+1)^2\left\{ {n-1 \atop m+1} \right\} + (m+1)\left\{ {n-1 \atop m} \right\} + \left\{ {n \atop m} \right\}$

$= (m+1)^3\left\{ {n-2 \atop m+1} \right\} + (m+1)^2\left\{ {n-2 \atop m} \right\} + (m+1)\left\{ {n-1 \atop m} \right\} + \left\{ {n \atop m} \right\}$

Applying recurrence consecutively, we have:

$(m+1)^n\left\{ {0 \atop m+1} \right\} + \sum_{k=0}^{k=n} \left\{ {k \atop m} \right\} (m+1)^{n-k}$

The first part always equates to 0 when m ¿ 0 which leads us to the fact that LHS = RHS.

Hence proved.

Problem 6b

Solution :

$\left\{ {m+n+1 \atop m} \right\} = \sum_{k=1}^{k=m} k\left\{ {n+k \atop k} \right\}$

We can prove this by induction on m

for m=1, we have LHS:

$\left\{ {n+2 \atop 1} \right\} = 1 - -ifn > 0$

and RHS:

$\sum_{k=0}^{k=m} k\left\{ {n+k \atop k} \right\} = (1)\left\{ {n+1 \atop 1} \right\} = 1 - -if n > 0$

Assume that it is true for m-1...Now for m, it becomes:

$\left\{ {m-1+n+1 \atop m-1} \right\} = \left\{ {m+n \atop m-1} \right\}$

$\left\{ {m+n+1 \atop m} \right\} = m\left\{ {m+n \atop m} \right\} + \left\{ {m+n \atop m-1} \right\}$

$= m\left\{ {m+n \atop m} \right\} + \sum_{k=0}^{k=m-1} k\left\{ {n+k \atop k} \right\}$

The RHS will turn out to be : $\sum_{k=0}^{k=m} k\left\{ {n+k \atop k} \right\}$

Thus, we have LHS = RHS...

which is the required result

Problem 6c

Solution :

We start with the recurrence relations for both the sides:

$\left\{ {n \atop k} \right\} = k\left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}$

$\left[{n \atop k} \right] = (n-1)\left[{n-1 \atop k} \right] + \left[{n-1 \atop k-1} \right]$

We see that:

$\left\{ {-k \atop -n} \right\} = (-n)\left\{ {-k-1 \atop -n} \right\} + \left\{ {k-1 \atop -n-1} \right\}$

$\left\{ {n \atop k} \right\} = (-n)\left\{ {n \atop k+1} \right\} + \left\{ {n+1 \atop k+1} \right\}$

We now substitute $\left\{ {-k-1 \atop -n} \right\}$ with $\left[{n \atop k+1} \right]$ and $\left\{ {-k-1 \atop -n-1} \right\}$ with $\left[{n+1 \atop k+1} \right]$

Now, from the recurrence for the 1st kind, we have:

$\left\{ {n+1 \atop k+1} \right\} = \left[{(n+1)-1 \atop (k+1)-1} \right] + (n+1-1)\left[{(n+1)-1 \atop k+1} \right]$

$= \left[{n+1 \atop k+1} \right] + n\left[{n \atop k+1} \right]$

Hence proved...

Problem 6.12

Solution :

Let's say that:

if we assume that :

$g(n) = \sum_k \left[{n \atop k} \right](-1)^k f(k)$

we can see that :

$\sum_k \left[{n \atop k}\right](-1)^k g(k) = \sum_k \left[{n \atop k}\right](-1)^k \sum_m \left[{n \atop m}\right](-1)^m f(m)$

$= \sum_{k,m} (-1)^{k+m} f(m) \left[{n \atop k}\right]\left\{{k \atop m}\right\}$

$= \sum_{k,m} (-1)^{2n-k-m} f(m) \left[{n \atop k}\right]\left\{{k \atop m}\right\}$

$= \sum_m (-1)^{n-m} f(m) \sum_k (-1)^{n-k} \left[{n \atop k}\right]\left\{{k \atop m}\right\}$

$= \sum_m (-1)^{n-m} f(m)[m = n]$

$= f(n)$

Now, if we perform the exact procedure with the parts of f and g reversed, we prove the equation as required.

That is, we assume: $f(n) = \sum_k \left[{n \atop k}\right](-1)^k g(k)$

$\sum_k \left[{n \atop k}\right](-1)^k f(k) = \sum_k \left[{n \atop k}\right](-1)^k \sum_m \left[{n \atop m}\right](-1)^m g(m)$

$= \sum_{k,m} (-1)^{k+m} g(m) \left[{n \atop k}\right]\left\{{k \atop m}\right\}$

$= \sum_{k,m} (-1)^{2n-k-m} g(m) \left[{n \atop k}\right]\left\{{k \atop m}\right\}$

$= \sum_m (-1)^{n-m} g(m) \sum_k (-1)^{n-k} \left[{n \atop k}\right]\left\{{k \atop m}\right\}$

$= \sum_m (-1)^{n-m} g(m)[m = n]$

$= g(n)$

This is the required result.

Problem 6.15

Solution :

we have : $x^k = \sum_k \left\langle{n \atop k}\right\rangle \binom{x+k}{n}$

We know that : $\Delta \binom{x+k}{n} = \binom{x+k}{n-1}$

From that we have: $\delta^m (x)^n = \sum_k \left\langle{n \atop k}\right\rangle \binom{x+k}{n-m}$...

when x=0,

RHS $= \sum_k \left\langle{n \atop k}\right\rangle \binom{k}{n-m}$...(1)

Now,

$x^n = \sum_k \left\{{n \atop k}\right\} x^{\underline{k}}$

The previous LHS now becomes: $\sum_k \left\{{n \atop k}\right\} (k)(k-1)(k-2)...(k-m+1)x^{\underline{k-m}}$

When x=0, all terms in the summation are 0 except when $k - m = 0$, Thus $k = m$

Now the RHS = $\left\{{n \atop m}\right\} m(m-1)...1$

$= \left\{{n \atop m}\right\} m!...(2)$

Thus from the previous result , we combine to get:

$m!\left\{{n \atop m}\right\} = \sum_k \left\langle{n \atop k}\right\rangle \left\langle{k \atop n-m}\right\rangle$

Which is the required result.

Problem 6.26

Solution :

We know that: $H_k = H_{k-1} + 1/k$

Now, $H_0 = 0$

From these two, we have :

$S_n = \sum_{k=1}^{k=n} H_{k-1}/k + \sum_{k=1}^{k=n} 1/k^2 = T_n + {H_n}^{(2)}$

Now if we can compute T_n, we can compute S_n.

For use of summation by parts, we have: $u(k) = H_{k-1}$ and $v(k) = 1/k$, then, $\Delta u(k) = 1/k$

We can also set $v(k) = h_{k-1}$, so that: $Ev(k) = H_k$

$$\sum_1^{n+1} u(x)\Delta v(x)\delta x = u(x)v(x)\Bigg|_{x=n+1}^{x=1} - \sum_1^{n+1} Ev(x)\Delta u(x)\delta x$$

$$= (H_{x-1})^2\Bigg|_{x=n+1}^{x=1} - \sum_{k=1}^{k=n} H_k/k$$

$= {H_n}^2 - S_n$

Now, $S_n = T_n + {H_n}^(2) = {H_n}^2 - S_n + {H_n}^(2)$

From this, we have:

$S_n = ({H_n}^2 + {H_n}^{(2)})/2$

Problem 6.34

Solution :

Using the recurrence relation:

$\left\langle{n \atop k}\right\rangle = (k+1)\left\langle{n-1 \atop k}\right\rangle + (n-k)\left\langle{n-1 \atop k-1}\right\rangle$

if $k = 0$, then $\left\langle{-1 \atop k}\right\rangle = 1$

For $k > 0$,

$\left\langle {-1 \atop k} \right\rangle = \frac{k}{k+1} \left\langle {-1 \atop k-1} \right\rangle$

$= \frac{k}{k+1} \frac{k-1}{k} \left\langle {-1 \atop k-2} \right\rangle$

$= \frac{k}{k+1} \frac{k-1}{k} ... \frac{1}{2} \left\langle {-1 \atop 0} \right\rangle$

All the terms cancel out to give:

$= \frac{1}{k+1}$

Now, in the original recurrence, we substitute -1 for n, to get:

$\left\langle {-1 \atop k} \right\rangle = (k+1) \left\langle {-2 \atop k} \right\rangle + (-1-k) \left\langle {-2 \atop k-1} \right\rangle$

Using our previous answer ,we have:

$\frac{1}{k+1} = (k+1)(\left\langle {-2 \atop k} \right\rangle - \left\langle {-2 \atop k-1} \right\rangle)$

$\frac{1}{(k+1)^2} = (\left\langle {-2 \atop k} \right\rangle - \left\langle {-2 \atop k-1} \right\rangle)$

similar to the previous way, we have:

$\left\langle {-2 \atop k} \right\rangle = \frac{1}{(k+1)^2} + \left\langle {-2 \atop k-1} \right\rangle$

$\left\langle {-2 \atop k} \right\rangle = \frac{1}{(k+1)^2} + \frac{1}{k^2} + \left\langle {-2 \atop k-2} \right\rangle$

$\left\langle {-2 \atop k} \right\rangle = \frac{1}{(k+1)^2} + \frac{1}{k^2} + ... + \left\langle {-2 \atop 0} \right\rangle$

$\left\langle {-2 \atop k} \right\rangle = {H_{k+1}}^{(2)}$

Which is the required result...

Problem 6.39

Solution :

$\sum_{k=1}^{k=n} {H^2}_k$

$= \sum_{k=1}^{k=n} H_k H_k$

Now, $H_k = \sum_{j=1}^{j=k} \frac{1}{j}$

Thus, $= \sum_{k=1}^{k=n} (\sum_{j=1}^{j=k} \frac{1}{j}) H_k$

Switching the sum variables, we have

$= \sum_{j=1}^{k=n} (1/j)(\sum_{k=j}^{k=n} H_k)$

$= \sum_{j=1}^{k=n} (1/j)((\sum_{k=1}^{k=n} H_k) - (\sum_{k=1}^{k=j-1} H_k))$

Expanding it using the given formula in 6.67, we have:

$= nH^2{}_n - nH_n - \sum\left((j-1)H_{j-1} - (j-1)\right)/j$

$= nH^2{}_n - nH_n - \sum(H_{j-1} - H_{j-1}/j - 1 + 1/j)$

Now, $H_{n-1} = H - n - 1/n$

$= nH^2{}_n - nH_n - \sum(H_j - 1/j - (H_j - 1/j)/j - 1 + 1/j)$

$= nH^2{}_n - nH_n - \sum(H_j - H_j/j + 1/j^2 - 1)$

Simplifying and solving we get the final answer as:

$(n+1)H^(2)_n - (2n+1)H_n + 2n$

Which is the required result

Problem 6.21 Unable to solve after trying