CSE/AMS 547 Discrete Mathematics September 01, 2017

Academic Honesty Review

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Problem 1.10

Given:

Q_n - the minimum number of moves needed to transfer a tower of n disks from A to B if all moves must be clockwise

R_n - the minimum number of moves needed to transfer a tower of n disks from B back to A if all moves must be clockwise

The only moves allowed are $A \to B$, $B \to C$, and $C \to A$, where C is the temporary tower.

To Prove:

$$Q_n = \left\{ \begin{array}{cc} 0, & \text{if } n = 0 \\ 2R_{n-1} + 1, & \text{if } n > 0 \end{array} \right\}$$

$$R_n = \left\{ \begin{array}{cc} 0, & \text{if } n = 0 \\ Q_n + Q_{n-1} + 1, & \text{if } n > 0 \end{array} \right\}$$

Solution:

1. $Q_0 = 0$: This follows from the fact that if there are no disks, there won't be any moves required.

2. Similarly, $R_0 = 0$

3. $Q_1 = 1$: The number of moves required to transfer 1 disk from A to B (Since we can move the disks directly)

4. $R_1 = 1 + 1$: Moving the disk from B to A is done $B \to C \to A$

5. Transferring n disks from A to B: If the rules are the same as that of the Tower of Hanoi problem, we can proceed as follows:

5.1 - We first transfer the first $n - 1$ disks to a temporary tower, say 'C'. We know that $A \to C$ is not allowed directly, and neither is $B \to A$. So, transferring disks from $A \to C$

is similar to transferring disks from $B \to A$. From the given information, this is done in $\mathbf{R_{n-1}}$ moves.

5.2 - We then transfer the n^{th} disk from tower A to tower B in **1** move. We have seen earlier in step 1 that $Q_1 = 1$.

5.3 - We then transfer the initial $n-1$ disks from C to B. As shown in step 5.1, it is not a direct move and similar in nature to the move $B \to A$. As before, this takes another $\mathbf{R_{n-1}}$ moves.

5.4 - If we add the moves from the previous 3 steps, we get a total of $\mathbf{R_{n-1}+1+R_{n-1} = 2R_{n-1}+1 = Q_n}$ steps. This is the required result.

6. Transferring n disks from B to A: We proceed as follows

6.1 - similar to the previous approach, we now transfer $n-1$ disks from $B \to C$ in X steps.

6.2 - Now, we cannot directly transfer the n^{th} disk from $B \to A$. We must move it to C first, but that can only be done when C is empty, since the n^{th}disk is the largest. So, we move the $n-1$ disks from $C \to A$ in Y steps. Since we are transferring $n-1$ disks from B to A, we can use the given information to see that the minimum steps required is $\mathbf{R_{n-1}}$.

6.3 - We then transfer the n^{th} disk from B to C in **1** move.

6.4 - We then transfer the $n-1$ disks from A to B in $\mathbf{Q_{n-1}}$ moves.

6.5 - We then transfer the n^{th} disk from C to A in **1** move.

6.6 - Now we transfer the remaining $n-1$ disks from B to A in $\mathbf{R_{n-1}}$ moves.

6.7 - We add the above moves to get a total of $\mathbf{R_{n-1}+1+Q_{n-1}+1+R_{n-1} = 2R_{n-1}+1+1+Q_{n-1} = R_n}$.

6.8 - If we substitute the result $\mathbf{2R_{n-1}+1 = Q_n}$ from step 5.4 in the previous step, we get : $\mathbf{R_n = Q_n + Q_{n-1} + 1}$. This is the required result.

Problem 1.14

Solution:

1. Without any planes there is only 1 part, which is the initial piece itself. So, $P_0 = 1$

2. A single plane will always divide a 3D region into two parts. So, $P_1 = 2$.

3. Let us assume that the piece of cheese is a cube and add another plane to divide it. We can either add the new plane parallel to the existing plane, or add it in such a way that the two plane intersect. A parallel plane would create an extra region resulting in a total of 3 parts. But in the other case, when they intersect, a total of 4 parts is created.

This is the maximum number of parts that can be created by 2 planes, so, $P_2 = 4$. Thus, the maximum output value is obtained when the new plane intersects each of the existing planes.

4. In order to visualize the addition and intersection of more planes, we can, for now, look at a face of the cube. This is in a shape of a square. And say, if we only added planes that are perpendicular to that face, we would see an intersection of lines on a $2d$ plane. In essence, this problem boils down to a line intersection problem with the added component of the new number of areas created by the introduction of a new plane. For example, in case of two planes, when the two planes intersect, we can take a look at a single plane from a 2d point of view to see one line (which is the 2nd plane) intersecting the plane into two.

5. We know that the number of regions created by lines intersecting each other is given by $L_n = S_n + 1$, where $S_n = n(n+1)/2$.

6. Thus, a new plane intersecting the old planes will create $P_n = P_{n-1} + L_{n-1}$ new regions.

7. Now to calculate P_5,

7.1 - $P_1 = 2, P_2 = 4$

7.2 - $P_3 = P_2 + L_2 = 4 + 4 = 8$

7.3 - $P_4 = P_3 + L_3 = 8 + 7 = 15$

7.4 - $P_5 = P_4 + L_4 = 15 + 11 = \mathbf{26}$, which is the required answer

Problem 1.16

Given:

$g(1) = \alpha$

$g(2n + j) = 3g(n) + \gamma n + \beta_j$, for $j = 0, 1$ and $n \geq 1$

Solution:

General form of the equation $g(n) = \alpha A(n) + \beta_0 B(n) + \beta_1 C(n) + \gamma D(n) +$

1. Say, $g(n) = 1$

1.1 $g(1) = 1 = \alpha$

1.2 $g(2) = g(2 * 1 + 0) = 3 + \gamma + \beta_0 = 1$. So, $\gamma + \beta_0 = -2$

1.3 $g(3) = g(2 * 1 + 1) = 3 + \gamma + \beta_1 = 1$. So, $\gamma + \beta_1 = -2$

1.4 $g(4) = g(2 * 2 + 0) = 3 + 2\gamma + \beta_0 = 1$. So, $2\gamma + \beta_0 = -2$

1.5 Solving the 4 equations, we get the values of $\alpha = 1$, $\gamma = 0$, $\beta_0 = -2$, and $\beta_1 = -2$

1.6 We get the equation of the form : $A(n) - 2B(n) - 2C(n) = 1$

2. Say, $g(n) = n$

2.1 $g(1) = 1 = \alpha$

2.2 $g(2) = g(2*1+0) = 3+\gamma+\beta_0 = 2$. So, $\gamma+\beta_0 = -1$

2.3 $g(3) = g(2*1+1) = 3+\gamma+\beta_1 = 3$. So, $\gamma+\beta_1 = 0$

2.4 $g(4) = g(2*2+0) = 6+2\gamma+\beta_0 = 4$. So, $2\gamma+\beta_0 = -2$

2.5 Solving the 4 equations, we get the values of $\alpha = 1$, $\gamma = -1$, $\beta_0 = 0$, and $\beta_1 = 1$

2.6 We get the equation of the form : $A(n) + C(n) - D(n) = n$

3. Say, $g(n) = n^2$

3.1 $g(1) = 1 = \alpha$

3.2 $g(2) = g(2*1+0) = 3+\gamma+\beta_0 = 4$. So, $\gamma+\beta_0 = 1$

3.3 $g(3) = g(2*1+1) = 3+\gamma+\beta_1 = 9$. So, $\gamma+\beta_1 = 6$

3.4 $g(4) = g(2*2+0) = 12+2\gamma+\beta_0 = 16$. So, $2\gamma+\beta_0 = 4$

3.5 Solving the 4 equations, we get the values of $\alpha = 1$, $\gamma = 3$, $\beta_0 = -2$, and $\beta_1 = 3$

3.6 We get the equation of the form : $A(n) - 2B(n) + 3C(n) + 3D(n) = n^2$

....Unable to proceed further....

Problem 1.21

Given:

Variation of the Josephus problem with 2n people in a circle and the first n people as good guys and the rest bad.

To Show:

There is always an integer M depending on N such that if we execute the M^{th} person in the circle, the bad guys are the first to go.

Solution:

1. We can note that in order to remove the bad guys first, the m^{th} person cannot be $<= n$, since all the good guys are $<= n$. Thus $m > n$ is quite obvious.

2. If we take an example of $n = 2$, which leaves us 4 people to deal with, we see that in the first round, either the 3^{rd} or 4^{th} person must be eliminated. Say, if $m = 3$ or $m = 4$, we will end up eliminating a good guy before all the bad guys are removed. But still, even if $m > n$, it must still remove all bad guys before a good guy. If x rounds are made,

m must still point to a member of the 2^{nd} half of people.

3. One way the above should hold true is when m is a multiple of one of $[n+1, n+2....2n]$. But this is not true always, as can be seen when $n = 2$ and $m = 6$. Person 2 gets eliminated before 4.

4. A stronger rule would be to have m be a multiple of all of $[n+1, n+2....2n]$. This would always work because during every round, the last guy (who is bad) will be removed until there are no more bad guys left (which should be after n rounds). After every round, when one guy gets removed, the m^{th} guy would still be the last guy since m is also a multiple of $2n-1$.

5. Thus, $m = LCM(n+1, n+2, ..., 2n)$, would be a value that results in the elimination of all the bad guys first.

Problem 2.14

Solution:

1. In order to separate the given form $\sum_{k=1}^{n} k2^k$, we must find out a representation of k.

2. k can be represented as $\sum_{j=1}^{k} 1$.

3. We can rewrite it as :

$$\begin{aligned}\sum_{1\le k\le n} k2^k &= \sum_{1\le k\le n} 2^k * \sum_{1\le j\le k} 1 \\ &= \sum_{1\le j\le k\le n} 2^k \\ &= \sum_{1\le j\le n}\sum_{j\le k\le n} 2^k\end{aligned} \tag{1}$$

4. The double sum is now solved first over k, ie., $j \le k \le n$. We also split the total sum over $[1...n]$ as the sum of the first j values and the next $n-j$ values. The value being the result of a geometric progression, we can make use of the knowledge that $\sum_{j=1}^{n} 2^j = \frac{2^{n+1}-2}{2-1}$

$$\begin{aligned}\sum_{j\le k\le n} 2^k &= \sum_{1\le k\le n} 2^k - \sum_{1\le k<j} 2^k \\ &= 2^{n+1} - 2 - (2^j - 2) \\ &= 2^{n+1} - 2^j\end{aligned} \tag{2}$$

5. We now sum this result over j to get the final result:

$$\begin{aligned}\sum_{1\le j\le n} 2^{n+1} - 2^j &= n2^{n+1} - (2^{n+1} - 2) \\ &= n2^{n+1} - 2^{n+1} + 2\end{aligned} \tag{3}$$

Problem 2.21

Solution: - We will use the perturbation method as required.

1. $S_n = \sum_{k=0}^{n} (-1)^{n-k}$

1.1 It follows that $S_{n+1} = \sum_{k=0}^{n+1} (-1)^{n+1-k}$, when we replace n with n+1.

1.2 Evaluation of S_{n+1} by splitting it for the first term :

$$\begin{aligned}
\sum_{k=0}^{n+1} (-1)^{n+1-k} &= (-1)^{n+1-0} + \sum_{1 \le k \le n+1} (-1)^{n+1-k} \\
&= (-1)^{n+1} + \sum_{1 \le k+1 \le n+1} (-1)^{n+1-(k+1)} \\
&= (-1)^{n+1} + \sum_{0 \le k \le n} (-1)^{n-k} \\
&= (-1)^{n+1} + S_n
\end{aligned} \tag{4}$$

1.3 Evaluation of S_{n+1} by splitting it for the last term :

$$\begin{aligned}
\sum_{k=0}^{n+1} (-1)^{n+1-k} &= \sum_{0 \le k \le n} (-1)^{n+1-k} + (-1)^{(n+1)-(n+1)} \\
&= \sum_{0 \le k \le n} (-1)^{n+1-k} + (-1)^{0} \\
&= (-1) * \sum_{0 \le k \le n} (-1)^{n-k} + 1 \\
&= 1 - S_n
\end{aligned} \tag{5}$$

1.4 Solving the two equations, we get $2S_n = 1 - (-1)^{n+1}$

1.5 Thus, $S_n = \frac{1-(-1)^{n+1}}{2}$

2. $T_n = \sum_{k=0}^{n} (-1)^{n-k} k$

2.1 It follows that $T_{n+1} = \sum_{k=0}^{n+1} (-1)^{n+1-k} k$, when we replace n with n+1.

2.2 Evaluation of T_{n+1} by splitting it for the first term :

$$\begin{aligned}\sum_{k=0}^{n+1}(-1)^{n+1-k}k &= (-1)^{n+1-0} * 0 + \sum_{1\le k\le n+1}(-1)^{n+1-k}k \\ &= 0 + \sum_{1\le k+1\le n+1}(-1)^{n+1-(k+1)}(k+1) \\ &= \sum_{0\le k\le n}(-1)^{n-k}k + \sum_{0\le k\le n}(-1)^{n-k} \\ &= T_n + S_n\end{aligned} \tag{6}$$

2.3 Evaluation of T_{n+1} by splitting it for the last term :

$$\begin{aligned}\sum_{k=0}^{n+1}(-1)^{n+1-k} &= \sum_{k=0}^{n+1}(-1)^{n+1-k}k + (-1)^{n+1-(n+1)}(n+1) \\ &= (-1)\sum_{k=0}^{n}(-1)^{n-k}k + n + 1 \\ &= n + 1 - T_n\end{aligned} \tag{7}$$

2.4 Solving the two equations, we get $2T_n = n + 1 - S_n$

2.5 Thus, $T_n = \frac{n+1-S_n}{2}$

3. $U_n = \sum_{k=0}^{n}(-1)^{n-k}k^2$

3.1 It follows that $U_{n+1} = \sum_{k=0}^{n+1}(-1)^{n+1-k}k^2$, when we replace n with n+1.

3.2 Evaluation of U_{n+1} by splitting it for the first term :

$$\begin{aligned}\sum_{k=0}^{n+1}(-1)^{n+1-k}k^2 &= (-1)^{n+1-0} * 0 + \sum_{1\le k\le n+1}(-1)^{n+1-k}k^2 \\ &= 0 + \sum_{1\le k+1\le n+1}(-1)^{n+1-(k+1)}(k+1)^2 \\ &= 0 + \sum_{1\le k+1\le n+1}(-1)^{n-k}(k^2+2k+1) \\ &= \sum_{0\le k\le n}(-1)^{n-k}k^2 + 2\sum_{0\le k\le n}(-1)^{n-k}k + \sum_{0\le k\le n}(-1)^{n-k} \\ &= U_n + 2T_n + S_n\end{aligned} \tag{8}$$

3.3 Evaluation of U_{n+1} on by splitting it for the last term :

$$\begin{aligned}\sum_{k=0}^{n+1}(-1)^{n+1-k}k^2 &= \sum_{k=0}^{n+1}(-1)^{n+1-k}k^2 + (-1)^{n+1-(n+1)}(n+1)^2 \\ &= (-1)\sum_{k=0}^{n}(-1)^{n-k}k^2 + (n+1)^2 \\ &= -U_n + (n+1)^2\end{aligned} \tag{9}$$

3.4 Solving the two equations, we get $2U_n = (n+1)^2 - S_n - 2T_n = (n+1)^2 - (n+1-S_n) - S_n$

3.5 Thus, $U_n = \frac{n^2+n}{2}$

Problem 2.28

Solution:

We can see that in the third step, due to the interchanging of j and k in the sum limits, the term of the sum actually approaches ∞, in the event that $k > j$. The series is no longer convergent due to this step.