Université Nice-Sophia Antipolis
SMEMP302 - ECUE Probabilitic computational methods, 2022-2023
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Exercices (3)

Exercise 1. We want to compute the price of the *Best of Call* option with strike K. This example is developped at the end of chapter 2 in the course. We recall the following formulas.

$$\begin{aligned}&e^{-rT}h_T \overset{\text{(law)}}{=} \phi(Z^1,Z^2)\\ &:= \left(\max\left(x_0^1\exp\left(-\frac{\sigma_1^2}{2}T+\sigma_1\sqrt{T}Z^1\right), x_0^2\exp\left(-\frac{\sigma_2^2}{2}T+\sigma_2\sqrt{T}(\rho Z^1+\sqrt{1-\rho^2}Z^2)\right)\right)-Ke^{-rT}\right)_+,\end{aligned}$$

where $Z=(Z^1,Z^2)\sim\mathcal{N}(0,I_2)$. We set

$$X_T^1 = x_0^1\exp\left(rT-\frac{\sigma_1^2}{2}T+\sigma_1\sqrt{T}Z^1\right),\ X_T^2 = x_0^2\exp\left(rT-\frac{\sigma_2^2}{2}T+\sigma_2\sqrt{T}(\rho Z^1+\sqrt{1-\rho^2}Z^2)\right).$$

We would like to compute

$$\mathbb{E}(h_T) = \mathbb{E}((\max(X_T^1,X_T^2)-K)_+).$$

(1) Using the convexity inequality

$$\sqrt{ab}\le\max(a,b)\ (a,b>0),$$

show that

$$k_T := \left(\sqrt{X_T^1X_T^2}-K\right)_+$$

is a natural control variate for h_T.

(2) Show that $\mathbb{E}(k_T)$ has a closed form (this is known as the *geometric mean option*). Show that this closed form can be written as Black-Scholes formula with appropriate parameters.

(3) Check on a simulation that this procedure reduces the variance (use the parameters of Exercices (2)).

Exercise 2. We are interested in the following processes (Black-Scholes model):

$$X_t^0 = x_0^0e^{rt}\ \text{(riskless)},$$

$$X_t^1 = x_0^1\exp\left(\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t\right)\ \text{(risky asset)},$$

where (W_t) is a standard Brownian motion. We set $\varphi : x\mapsto (x-K)_+$. The parameters are: $r=0$, $\sigma=0.2$, $x_0^1=70$, $T=1$, $K=100$. The price of a call option is

$$\mathrm{Call}_0(x_0^1,K,r,\sigma,T)=\mathbb{E}(\varphi(Z)),\ Z\sim\mathcal{N}(0,1)$$

where

$$\varphi(z) = (x_0^1e^{(r-\frac{\sigma^2}{2})T+\sigma z}-K)_+$$

As seen in the course $(\forall\theta)$:

$$\begin{aligned}\mathbb{E}(\varphi(Z)) &= \int_{\mathbb{R}}\varphi(z)\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz\\ (z=u+\theta)\quad &= \int_{\mathbb{R}}\varphi(u+\theta)\frac{e^{-u^2/2}e^{-\theta^2/2}e^{-u\theta}}{\sqrt{2\pi}}du\\ &= e^{-\frac{\theta^2}{2}}\mathbb{E}(\varphi(Z+\theta)e^{-\theta Z})\\ &= e^{\theta^2/2}\mathbb{E}(\varphi(Z+\theta)e^{-\theta(Z+\theta)})\end{aligned}$$

Compare the performances of

(1) a "crude" Monte-Carlo simulation,

(2) an importance sampling method with

$$\theta = -\frac{\log(x_0^1/K)}{\sigma\sqrt{T}}\,,$$

(3) an importance sampling method with

$$\theta = \frac{-\log(x/K) + \mu T}{\sigma\sqrt{T}}$$

($\mu = r - \sigma^2/2$).

Exercice 3. Exercise inspired by exercise 1.7 of [Pardoux-2008]. We want to compute

$$I = \mathbb{E}(\mathbb{1}_{X>0}e^{\beta X})\,,$$

where $X \sim \mathcal{N}(0,1)$ and $\beta = 5$. You have to estimate the variance at each step of the exercise

(1) Compute (by Monte-Carlo) the variance of the naive method (when we draw X_1, X_2,... of law $\mathcal{N}(0,1)$ and we approximate I bu $\frac{1}{M}\sum_{i=1}^{M}\mathbf{1}_{X_i>0}e^{\beta X_i}$).
(2) Propose an importance sampling method.
(3) Propose a control variate method.
(4) Propose an even better method using antithetic variables.

PYTHON TIPS

Print real numbers (x, y) with a formatting and inside a text:

```
print('Crude variance : {0:.3f} and reduced variance
                : {1:.3f}'.format(x,y))
```

Add a constant `theta` to a data frame `df` with k columns (numbered from 0 to $k-1$)

```
df[k]=theta
```