Chapter 5
Discretization schemes of a Brownian diffusion

In this chapter, (X_t) is a d-dimensional sol. of a SDE : $dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$

where: $\begin{cases} b : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d \\ \sigma : [0,T] \times \mathbb{R}^d \longrightarrow \mathcal{M}(d, q, \mathbb{R}) \\ W \text{ is a } q\text{-dim standard Brownian motion (BM) on } (\Omega, \mathcal{A}, \mathbb{P}) \end{cases}$

$X_0 : (\Omega, \mathcal{A}, \mathbb{P}) \longrightarrow \mathbb{R}^d$ is independant of W

We assume that b and σ are Lipschitz continuous in x uniformly with respect to $t \in [0,T]$, i.e. (H1)

$$\forall t \in [0,T], \forall x, y \in \mathbb{R}^d, |b(t,x) - b(t,y)| + \|\sigma(t,x) - \sigma(t,y)\| \leq K|x-y| \text{ (for some constant } K)$$

$\mathcal{F}_t = \sigma(X_0, \mathcal{N}_{\mathbb{P}}, W_s, 0 \leq s \leq t)$

Th (strong solution of SDE)
There is a unique $\mathcal{F}_t$-adapted solution, its paths are $\mathbb{P}$-a.s. continuous

7.1 Euler-Maruyama schemes

We want to compute quantities of the form: $\mathbb{E}(f(X_T))$ or $\mathbb{E}(F((X_t)_{t\in[0,T]}))$. In general, we cannot simulate X_T (or $(X_t)_{t\in[0,T]}$). We introduce various types of Euler schemes with step $\frac{T}{n}$ ($n\in\mathbb{N}^*$).

7.1.1 The discrete time and stepwise constant Euler schemes

Discrete time Euler scheme

$t_k^n = \frac{kT}{n}$, $k=0,\dots,n$

$(Z_k^n)_{1\le k\le}$ are iid, $\sim\mathcal{N}(0,I_q)$, $Z_k^n := \sqrt{\frac{n}{T}}(W_{t_k^n}-W_{t_{k-1}^n})$

$$\bar{X}_{t_{k+1}^n} = \bar{X}_{t_k^n} + \frac{T}{n} b(t_k^n,\bar{X}_{t_k^n}) + \sigma(t_k^n,\bar{X}_{t_k^n})\sqrt{\frac{T}{N}}\, Z_{k+1}^n$$

Stepwise constant Euler scheme

notations: $\underline{t} := t_k^n$ if $t\in[t_k^n, t_{k+1}^n)$

$\tilde{X}_t = \bar{X}_{\underline{t}}$

7.1.2. The genuine (continuous) Euler scheme

We want to extend the definition of the Euler scheme at every instant $t\in[0,T]$:

$\forall k\in\{0,\dots,n-1\}$, $\forall t\in[t_k^n, t_{k+1}^n)$

$$\begin{cases} \bar{X}_t = \bar{X}_{\underline{t}} + (t-\underline{t})\, b(\underline{t},\bar{X}_{\underline{t}}) + (W_t - W_{\underline{t}})\times\sigma(\underline{t},\bar{X}_{\underline{t}}) \\ \bar{X}_0 = X_0 \end{cases}$$

$(\bar{X}_t)_{t\in[0,T]}$ is $\mathcal{F}_t$-adapted and has continuous paths

Proposition: Assume b and σ are continuous, the above (genuine) Euler scheme satisfies the following SDE (with frozen coefficients)

$$\boxed{\bar{X}_t = X_0 + \int_0^t b(\underline{s}, \bar{X}_{\underline{s}})\,ds + \int_0^t \sigma(\underline{s}, \bar{X}_{\underline{s}})\,dW_s \qquad t \in [0,T]}$$

<u>Proof</u>: $\forall t \in [t_i^n, t_{i+1}^n]$
$$\bar{X}_t = \bar{X}_{t_i^n} + \int_{t_i^n}^t b(\underline{s}, \bar{X}_{\underline{s}})\,ds + \int_{t_i^n}^t \sigma(\underline{s}, \bar{X}_{\underline{s}})\,dW_s$$
$\square$

7.2 Strong error rate and polynomial moments

7.2.1 Properties of the solution (X_t)

We start by proving a polynomial moment control.

<u>Prop (polynomial moments)</u> We suppose that b and σ are Borel functions such that

(H2)

$$\forall t \in [0,T],\ \forall x \in \mathbb{R}^d,\ |b(t,x)| + \|\sigma(t,x)\| \leq c(1+|x|)$$

(c a constant, T fixed) then $\exists K_{p,d,T} > 0$ s.t. for (X_t) a strong solution:

$$\sup_{t \leq T} \mathbb{E}\left(|X_t|^{2p}\right) \leq K_{p,d,T}\left(1+|y|^{2p}\right)$$

<u>Proof</u>: $f : \mathbb{R}^n \longrightarrow |x|^{2p}$ has the following gradient: $\nabla f(x) = 2p|x|^{2p-2}x$ and the following Hessian: $D^2 f(x) = 2p|x|^{2p-4}\left(|x|^2 I_n + (2p-2)\, x \otimes x\right)$

where $x \otimes x$ is the matrix $(x_i x_j)_{1 \leq i,j \leq n}$.

We set $a(t,x) := \sigma\sigma^*(t,x) \in \mathbb{R}^{n \otimes n}$ (matrix $d \times d$).

The Itô Formula gives us

scalar product

$$|X_t|^{2p} = |y|^{2p} + \int_0^t 2p|X_s|^{2p-2} X_s . b(s,X_s) + p|X_s|^{2p-4}\, \mathrm{tr}\left[\left(|X_s|^2 I_n + (2p-2) X_s \otimes X_s\right) a(s,X_s)\right] ds$$
$$+ \int_0^t 2p|X_s|^{2p-2} X_s . \sigma(s,X_s)\,dW_s$$

scalar product

We will use a localization procedure:

$$\nu_m = \inf\{t \geq 0 : |X_t| \geq m\}$$

$$\mathbb{E}\left(\int_0^{\nu_m \wedge t} 2p|X_s|^{2p-2} X_s \cdot \sigma(s, X_s)\, dW_s\right) = 0 \quad (\forall m)$$

So:

$$\mathbb{E}(|X_{\nu_m \wedge t}|^{2p}) = |y|^{2p} + \mathbb{E}\Big(\int_0^{\nu_m \wedge t} 2p|X_s|^{2p-2} X_s \cdot b(s, X_s)$$
$$+ p|X_s|^{2p-4}\, \mathrm{tr}\left[(|X_s|^2 I_n + (2p-2) X_s \otimes X_s)\, a(s, X_s)\right] ds$$

$$\leq |y|^{2p} + \int_0^t \mathbb{E}\left(\text{same with } X_{\nu_m \wedge s}\right) ds$$

(some at C) H2

$$\leq |y|^{2p} + C\int_0^t \mathbb{E}\Big(|X_{\nu_m \wedge s}|^{2p-1}(1 + |X_{\nu_m \wedge s}|)$$
$$+ |X_{\nu_m \wedge s}|^{2p-2}(1 + |X_{\nu_m \wedge s}|)^2\, ds$$

$$\leq |y|^{2p} + C\int_0^t 1 + \mathbb{E}(|X_{\nu_m \wedge s}|^{2p})\, ds$$

because $|x|^{2p-2} + |x|^{2p-1} + |x|^{2p} \leq 3(1 + |x|^{2p})$

$$\leq |y|^{2p} + Ct + C\int_0^t \mathbb{E}(|X_{\nu_m \wedge s}|^{2p})\, ds$$

As $|X_{\nu_m \wedge s}| \leq \mathbb{1}_{\{\nu_m = 0\}}|y| + \mathbb{1}_{\{\nu_m > 0\}} m \leq |y| \vee m$, then the function $s \mapsto \mathbb{E}(|X_{\nu_m \wedge s}|^{2p})$ is locally integrable.

By Gronwall's Lemma, we get:

$$\forall t \leq T,\ \mathbb{E}(|X_{\nu_m \wedge t}|^{2p}) \leq (|y|^{2p} + CT)e^{CT}$$ dim. du dos

As $m \to +\infty$, $\nu_m \to T$, $X_{\nu_m \wedge t} \to X_t$ (a.s.), so by Fatou's Lemma: $\mathbb{E}(|X_t|^{2p}) \leq \liminf_{m \to \infty} \mathbb{E}(|X_{\nu_m \wedge t}|^{2p})$

so, $\forall t \leq T$, $\mathbb{E}(|X_t|^{2p}) \leq (|y|^{2p} + CT)e^{CT}$ $\square$

Posons $f(t) = \mathbb{E}(|X_{t\wedge\tau_m}|^{2p})$.

Nous avons
$$f(t) \leq |y|^{2p} + Ct + \int_0^t C f(s)\,ds .$$

Posons $g(t) = e^{-Ct}\int_0^t C f(s)\,ds$. Nous avons :

$$g'(t) = e^{-Ct}\left(-C\int_0^t C f(s)\,ds + C f(t)\right)$$

$$g'(t) \leq e^{-Ct}\, C(|y|^{2p} + t) \quad (C \text{ peut changer})$$

$$g(t) \leq \underbrace{g(0)}_{=0} + \int_0^t e^{-Cs}\, C(|y|^{2p} + s)\,ds$$

$$\begin{aligned}
f(t) &\leq |y|^{2p} + Ct + \int_0^t C f(s)\,ds \\
&\leq |y|^{2p} + Ct + e^{Ct} g(t) \\
&\leq |y|^{2p} + Ct + \int_0^t e^{C(t-s)}\, C(|y|^{2p} + s)\,ds \\
&\leq |y|^{2p} \times Ce^{t} + C(t + te^{Ct})
\end{aligned}$$

Burkholder - Davis - Gundy's inequality (as seen in Rémi Catellier's course ?) → you have seen Doob (i.e. $p=1$)

Lemma: $p \geq 1$, $\exists$ constant $C_p > 0$ such as for all process $(H_s)_{s\geq 0}$ (taking values in $\mathbb{R}^{n\times d}$) which is $\mathcal{F}_s$-adapted and all $t \geq 0$ such that $\mathbb{P}\left(\int_0^t |H_s|^2 ds < +\infty\right) = 1$, we have

$$\mathbb{E}\left(\sup_{s\leq t}\left|\int_0^s H_s dW_s\right|^{2p}\right) \leq C_p\, \mathbb{E}\left(\left(\int_0^t |H_s|^2 ds\right)^p\right)$$

(technical) Lemma $\forall q \geq 1$

$$\left|\sum_{k=1}^K a_k\right|^q \leq K^{q-1} \sum_{k=1}^K |a_k|^q$$

and for all $f : \mathbb{R}^d \longrightarrow \mathbb{R}^n$ (measurable)

$$\left|\int_A f(x)\, dx\right|^q \leq |A|^{q-1} \int_A |f(x)|^q dx$$

($|A|$ = Lebesgue measure of A)

Proof: Jensen's Inequality

$$\left(\frac{1}{K}\sum_{k=1}^K |a_k|\right)^q \leq \frac{1}{K}\sum_{k=1}^K |a_k|^q$$

and $\left(\frac{1}{|A|}\int_A |f(x)| dx\right)^q \leq \frac{1}{|A|}\int_A |f(x)|^q dx$ $\square$

Proposition (A): Let $p \geq 1$, $\exists\, C$ ($C(p,\sigma,b,T)$) such that $\forall\, 0 \leq s \leq t \leq T$, $\mathbb{E}(|X_t - X_s|^{2p}) \leq C(1+|y|^{2p})(t-s)^{p-1}$

Proof: We start with the triangular inequality

$$|X_t - X_s| \leq \left|\int_s^t \sigma(r, X_r)\, dW_r\right| + \left|\int_s^t b(r, X_r)\, dr\right|$$

Using BDG and the above technical Lemma, we get

$$\mathbb{E}(|X_t - X_s|^{2p}) \leq 2^{2p-1}\left[\mathbb{E}\left(\left|\int_s^t \sigma(r, X_r)\, dW_r\right|^{2p}\right) + \mathbb{E}\left(\left|\int_s^t b(r, X_r)\, dr\right|^{2p}\right)\right]$$

$$\leq C\left[\mathbb{E}\left(\left(\int_s^t |\sigma(r, X_r)|^2\, dr\right)^p\right) + (t-s)^{2p-1}\mathbb{E}\left(\int_s^t |b(r, X_r)|^{2p}\, dr\right)\right]$$

$$\leq C\left[(t-s)^{p-1}\int_s^t \mathbb{E}(|\sigma(r, X_r)|^{2p})\, dr + T^p (t-s)^{p-1}\int_s^t \mathbb{E}(|b(r, X_r)|^{2p})\, dr\right]$$

By our assumptions, we have:

$$\mathbb{E}(|\sigma(r, X_r)|^{2p} + |b(r, X_r)|^{2p}) \overset{\text{H2}}{\leq} C\,\mathbb{E}((1 + |X_r|)^{2p})$$

$$\leq C\,\mathbb{E}(1 + |X_r|^{2p})$$

Prop. on polynomial moments

$$\leq C(1 + |y|^{2p})$$

$\square$

7.2.2. Strong error rate

Th: Under H1 and H2 and under

$\exists \alpha, K > 0, \forall x \in \mathbb{R}^n, \forall (s, t) \in [0, T]$ (H3)

$$|\sigma(t, x) - \sigma(s, x)| + |b(t, x) - b(s, x)| \leq K(1 + |x|)(t-s)^\alpha$$

Then, for $\beta = \min(\alpha, \frac{1}{2})$

$$\forall p \geq 1, \exists C_p > 0, \forall y \in \mathbb{R}^n, \forall N \in \mathbb{N}^*, \mathbb{E}\left(\sup_{t \leq T} |X_t - \bar{X}_t^N|^{2p}\right) \leq C_p \frac{(1 + |y|^{2p})}{N^{2\beta p}}$$

$$\boxed{\text{If, } \gamma < \beta,\ N^{\gamma} \sup_{t \le T} |X_t - \bar{X}_t^N| \xrightarrow[N \to \infty]{a.s.} 0 .}$$

Proof: $\forall u \in [0,T]$

$$X_u - \bar{X}_u = \int_0^u \sigma(s, X_s) - \sigma(\underline{s}, \bar{X}_{\underline{s}})\, dW_s + \int_0^u b(s, X_s) - b(\underline{s}, \bar{X}_{\underline{s}})\, ds$$

Using B-D-G's Inequality, we get

$$\mathbb{E}\left(\sup_{u \le t} |X_u - \bar{X}_u|^{2p}\right)$$

$$\leq$$

$$2^{2p-1} \mathbb{E}\left(\sup_{u \le t} \left| \int_0^u \sigma(s, X_s) - \sigma(\underline{s}, \bar{X}_{\underline{s}})\, dW_s \right|^{2p}\right)$$

$$+ 2^{2p-1} \mathbb{E}\left(\sup_{u \le t} \left| \int_0^u b(s, X_s) - b(\underline{s}, \bar{X}_{\underline{s}})\, ds \right|^{2p}\right)$$

$$\overset{(B-D-G)}{\leq}$$

$$C t^{p-1} \int_0^t \mathbb{E}\left(|\sigma(s, X_s) - \sigma(\underline{s}, \bar{X}_{\underline{s}})|^{2p}\right) ds$$

$$+ 2^{2p-1} t^{2p-1} \int_0^t \mathbb{E}\left(|b(s, X_s) - b(\underline{s}, \bar{X}_{\underline{s}})|^{2p}\right) ds$$

With Technical Lemma + (H3)

$$|\sigma(s, X_s) - \sigma(\underline{s}, \bar{X}_{\underline{s}})|^{2p} \le 3^{2p-1} \Big(|\sigma(s, X_s) - \sigma(s, X_{\underline{s}})|^{2p}$$
$$+ |\sigma(s, X_{\underline{s}}) - \sigma(\underline{s}, X_{\underline{s}})|^{2p}$$
$$+ |\sigma(\underline{s}, X_{\underline{s}}) - \sigma(\underline{s}, \bar{X}_{\underline{s}})|^{2p} \Big)$$

$$\le C \Big(|X_s - X_{\underline{s}}|^{2p} + (1 + |X_{\underline{s}}|^{2p})(s - \underline{s})^{2p\alpha}$$
$$+ |X_{\underline{s}} - \bar{X}_{\underline{s}}|^{2p} \Big)$$

Using Prop. on polynomial moments + Prop. ☆, we get:

$$\mathbb{E}\left(|\sigma(s, X_s) - \sigma(\underline{s}, \bar{X}_{\underline{s}})|^{2p}\right) \le C \left(\frac{1 + |y|^{2p}}{N^p} + \frac{1 + |y|^{2p}}{N^{2\alpha p}} + \mathbb{E}\left(\sup_{u \le s} |X_u - \bar{X}_u|^{2p}\right) \right)$$

(because $0 \le s - \underline{s} \le \frac{T}{N}$)

we have the same inequality if we replace σ by b

So, by the above inequality, we get:

$$\mathbb{E}\left(\sup_{u\leq t}|X_u-\overline{X}_u|^{2p}\right)\leq C\left(\frac{1+|y|^{2p}}{N^{2(\alpha\wedge\frac{1}{2})p}}+\int_0^t\mathbb{E}\left(\sup_{u\leq s}|X_u-\overline{X}_u|^{2p}\right)ds\right)$$

By Gronwall's Lemma (and using a localization technique: $\nu_m := \inf\{t\geq 0, |X_t-\overline{X}_t|\geq m\}$ such that $\mathbb{E}\left(\sup_{u\leq t}|X_{u\wedge\nu_m}-\overline{X}_{u\wedge\nu_m}|^{2p}\right)<+\infty$)

(for $\gamma\leq\beta$) $\mathbb{E}\left(\left(N^{\gamma}\sup_{u\leq t}|X_u-\overline{X}_u|\right)^{2p}\right)\leq\frac{C}{N^{2(\beta-\gamma)p}}$.

With $p>\frac{1}{2(\beta-\gamma)}$, we get:

$$\mathbb{E}\left(\sum_{N\geq 1}\left(N^{\gamma}\sup_{u\leq T}|X_u-\overline{X}_u|^{2p}\right)\right)<+\infty$$

and $\mathbb{P}\left(\sum_{N\geq 1}\sup_{u\leq T}|X_u-\overline{X}_u|)^{2p}<+\infty\right)=1$ (Borel-Cantelli)

and so $N^{\gamma}\sup_{u\leq t}|X_u-\overline{X}_u|\xrightarrow[N\to\infty]{a.s.}0$

$\square$