

Chapter 5

Discretization schemes of a Brownian diffusion

In this chapter, (X_t) is a d -dimensional sol. of a SDE :

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$$

where:

$$\begin{cases} b: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \\ \sigma: [0, T] \times \mathbb{R}^d \rightarrow \mathcal{M}(d, q, \mathbb{R}) \\ W \text{ is a } q\text{-dim standard Brownian motion (BM)} \end{cases}$$

on $(\Omega, \mathcal{F}, \mathbb{P})$

$X_0: (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow \mathbb{R}^d$ is independent of W

We assume that b and σ are Lipschitz continuous in x uniformly with respect to $t \in [0, T]$, i.e.

$$\forall t \in [0, T], \forall x, y \in \mathbb{R}^d, \|b(t, x) - b(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq K \|x - y\| \quad (111)$$

$$\mathcal{F}_t = \sigma(X_0, W|_{[0, t]}, W_0, 0 \leq s \leq t)$$

$$\wedge K \|x - y\| \quad (\text{for some constant } K)$$

Th (strong solution of SDE)

There is a unique \mathcal{F}_t -adapted solution, its paths are \mathbb{P} -a.s. continuous

2. August

ammonium- und nitratstickstoffgehalt
des Bodens

Im Boden sind Stickstoffgehalte in Form von Ammonium- und Nitratstickstoff

$$NH_4^+ + NO_3^- = N$$

$$NH_4^+ = N$$

$$NH_4^+ = N$$

Stickstoffgehalt des Bodens

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7.1 Euler-Maruyama schemes

We want to compute quantities of the form :

$E(f(X_T))$ or $E(F((X_t)_{t \in [0, T]}))$. In general, we cannot simulate X_T or $(X_t)_{t \in [0, T]}$. We introduce various types of Euler schemes with step $\frac{T}{n}$ ($n \in \mathbb{N}^*$).

7.1.1 The discrete time and stepwise constant Euler schemes

Discrete time Euler scheme

$$t_k^n = \frac{kT}{n}, \quad k = 0, \dots, n$$

$(Z_k^n)_{1 \leq k \leq n}$ are iid, $\sim \mathcal{N}(0, T/n)$, $Z_k^n := \sqrt{\frac{n}{T}} (W_{t_k^n} - W_{t_{k-1}^n})$

$$\bar{X}_{t_{k+1}^n} = \bar{X}_{t_k^n} + \frac{T}{n} b(t_k^n, \bar{X}_{t_k^n}) + \sigma(t_k^n, \bar{X}_{t_k^n}) \sqrt{\frac{T}{n}} Z_{k+1}^n$$

Stepwise constant Euler scheme

notations : $\underline{t} := t_k^n$ if $t \in [t_k^n, t_{k+1}^n)$

$$\tilde{X}_t = \bar{X}_{\underline{t}}$$

7.1.2. The genuine (continuous) Euler scheme

We want to extend the definition of the Euler scheme at every instant $t \in [0, T]$:

$\forall k \in \{0, \dots, n-1\}, \forall t \in [t_k^n, t_{k+1}^n)$

$$\begin{cases} \bar{X}_t = \bar{X}_{\underline{t}} + (t - \underline{t}) b(\underline{t}, \bar{X}_{\underline{t}}) + \sigma(\underline{t}, \bar{X}_{\underline{t}}) (W_t - W_{\underline{t}}) \\ \bar{X}_0 = X_0 \end{cases}$$

$(\bar{X}_t)_{t \in [0, T]}$ is \mathcal{F}_t -adapted and has continuous paths

Proposition: Assume b and σ are continuous, the above (genuine) Euler scheme satisfies the following SDE (with frozen coefficients)

10/11/17

Chapter 10: Linear Algebra

Let V be a vector space over F . A linear transformation $T: V \rightarrow V$ is a map such that $T(av + bw) = aT(v) + bT(w)$ for all $v, w \in V$ and $a, b \in F$. The kernel of T is the set of all $v \in V$ such that $T(v) = 0$. The image of T is the set of all $w \in V$ such that $w = T(v)$ for some $v \in V$.

$$T(v) = 0 \implies v \in \ker(T)$$

$$T(v) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

The rank of T is the dimension of the image of T .

$$\text{rank}(T) + \dim(\ker(T)) = \dim(V)$$

Let $T: V \rightarrow V$ be a linear transformation. The characteristic polynomial of T is defined by

$$p_T(\lambda) = \det(T - \lambda I)$$

$$p_T(\lambda) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}$$

$$p_T(\lambda) = (a - \lambda)(d - \lambda) - bc$$

The eigenvalues of T are the roots of the characteristic polynomial $p_T(\lambda)$.

Let λ be an eigenvalue of T and v be an eigenvector of T corresponding to λ . Then $T(v) = \lambda v$. The eigenspace of T corresponding to λ is the set of all eigenvectors of T corresponding to λ , together with the zero vector.

$$\bar{X}_t = X_0 + \int_0^t b(\underline{s}, \bar{X}_s) ds + \int_0^t \sigma(\underline{s}, \bar{X}_s) dW_s$$

$t \in [0, T]$

Proof: $\forall t \in (t_1^*, t_{n+1}^*)$

$$\bar{X}_t = \bar{X}_{t_1^*} + \int_{t_1^*}^t b(\underline{s}, \bar{X}_s) ds + \int_{t_1^*}^t \sigma(\underline{s}, \bar{X}_s) dW_s \quad \square$$

7.2 Strong error rate and polynomial moments

7.2.1 Properties of the solution (X_t)

We start by proving a polynomial moment control.

Prop (polynomial moments) We suppose that b and σ are Borel functions such that

$$\forall t \in [0, T], \forall x \in \mathbb{R}^d, \|b(t, x)\| + \|\sigma(t, x)\| \leq c(1 + |x|) \quad (H2)$$

(c a constant, T fixed) then $\exists K_{p,d,T} \geq 0$ s.t.

for (X_t) a strong solution:

$$\sup_{t \leq T} \mathbb{E}(|X_t|^{2p}) \leq K_{p,d,T} (1 + |y|^{2p})$$

Proof: $f: \mathbb{R}^n \rightarrow |x|^{2p}$ has the following gradient: $\nabla f(x) = 2p|x|^{2p-2}x$ and the following Hessian: $\nabla^2 f(x) = 2p|x|^{2p-4}(|x|^2 I_n + (2p-2)x \otimes x)$

where $x \otimes x$ is the matrix $(x_i x_j)_{1 \leq i, j \leq n}$.

We set $a(t, x) := \sigma \sigma^*(t, x) \in \mathbb{R}^{n \times n}$ (matrix $n \times n$).

The Itô Formula gives us

$$|X_t|^{2p} = |y|^{2p} + \int_0^t 2p|X_0|^{2p-2} X_0 \cdot b(s, X_0) ds + p|X_0|^{2p-4} \text{tr}[(|X_0|^2 I_n + (2p-2)X_0 \otimes X_0) a(s, X_0)] ds + \int_0^t 2p|X_0|^{2p-2} X_0 \cdot \sigma(s, X_0) dW_s$$

scalar product

scalar product

We will use a localization procedure:

$$V_m = \inf \{ t \geq 0 : |X_t| \geq m \}$$

$$\mathbb{E} \left(\int_0^{V_m \wedge t} 2p |X_s|^{2p-2} X_s \cdot \sigma(s, X_s) dW_s \right) = 0 \quad (\forall m)$$

So:

$$\begin{aligned} \mathbb{E}(|X_{V_m \wedge t}|^{2p}) &= |y|^{2p} + \mathbb{E} \left(\int_0^{V_m \wedge t} 2p |X_s|^{2p-2} X_s \cdot b(s, X_s) \right. \\ &\quad \left. + p |X_s|^{2p-4} b[(|X_s|^2 I_n + (2p-2) X_s \otimes X_s) a(s, X_s)] ds \right) \end{aligned}$$

$$\leq |y|^{2p} + \int_0^t \mathbb{E}(\text{same with } X_{V_m \wedge s}) ds$$

(some at C) H2 $\leq |y|^{2p} + C \int_0^t \mathbb{E}(|X_{V_m \wedge s}|^{2p-1} (1 + |X_{V_m \wedge s}|) + |X_{V_m \wedge s}|^{2p-2} (1 + |X_{V_m \wedge s}|)^2) ds$

$$\leq |y|^{2p} + C \int_0^t 1 + \mathbb{E}(|X_{V_m \wedge s}|^{2p}) ds$$

because $|x|^{2p-2} + |x|^{2p-1} + |x|^{2p} \leq 3(1 + |x|^{2p})$

$$\leq |y|^{2p} + Ct + C \int_0^t \mathbb{E}(|X_{V_m \wedge s}|^{2p}) ds$$

As $|X_{V_m \wedge s}| \leq \mathbb{1}_{\{V_m = \infty\}} |y| + \mathbb{1}_{\{V_m < \infty\}} m \leq |y| \vee m$, then the function $s \mapsto \mathbb{E}(|X_{V_m \wedge s}|^{2p})$ is locally integrable.

By Gronwall's Lemma, we get:

$$\forall t \leq T, \mathbb{E}(|X_{V_m \wedge t}|^{2p}) \leq (|y|^{2p} + CT) e^{Ct}$$

dém. au das

As $m \rightarrow +\infty, V_m \rightarrow T, X_{V_m \wedge t} \rightarrow X_t$ (a.s.), so by

Fatou's Lemma: $\mathbb{E}(|X_t|^{2p}) \leq \liminf_{m \rightarrow \infty} \mathbb{E}(|X_{V_m \wedge t}|^{2p})$

$$\text{so, } \forall t \leq T, \mathbb{E}(|X_t|^{2p}) \leq (|y|^{2p} + CT) e^{Ct}$$

□

Parsons $f(t) = E(|X_{t+T_w}|^{2p})$

Now we have

$$f(t) \leq |y|^{2p} + ct + \int_0^t c f(s) ds$$

Parsons $g(t) = e^{-ct} \int_0^t c f(s) ds$. Now we have:

$$g'(t) = e^{-ct} \left(-c \int_0^t c f(s) ds + c f(t) \right)$$

$$g'(t) \leq e^{-ct} c (|y|^{2p} + n) \quad (c \text{ put changer})$$

$$g(t) \leq \frac{g(0)}{=0} + \int_0^t e^{-cs} c (|y|^{2p} + n) ds$$

$$f(t) \leq |y|^{2p} + ct + \int_0^t c f(s) ds$$

$$\leq |y|^{2p} + ct + e^{ct} g(t)$$

$$\leq |y|^{2p} + ct + \int_0^t e^{c(t-s)} c (|y|^{2p} + n) ds$$

$$\leq |y|^{2p} \times C e^t + c(t + t e^{ct})$$

Burkholder - Davis - Gundy's inequality
(as seen in Rémi Catellier's course?)

you have
seen Doob's
(i.e. $p=1$)

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Lemma: $p \geq 1$, \exists constant $C_p > 0$ such as for all process $(H_s)_{s \geq 0}$ (taking values in $\mathbb{R}^{n \times d}$) which is \mathcal{F}_0 -adapted and all $t \geq 0$ such that $\mathbb{P} \left(\int_0^t |H_s|^2 ds < +\infty \right) = 1$, we have

$$\mathbb{E} \left(\sup_{0 \leq s \leq t} \left| \int_0^s H_s dW_s \right|^{2p} \right) \leq C_p \mathbb{E} \left(\left(\int_0^t |H_s|^2 ds \right)^p \right)$$

(technical) Lemma $\forall q \geq 1$

$$\left| \sum_{k=1}^K a_k \right|^q \leq K^{q-1} \sum_{k=1}^K |a_k|^q$$

and for all $f: \mathbb{R}^1 \rightarrow \mathbb{R}^n$ (measurable)

$$\left| \int_A f(x) dx \right|^q \leq |A|^{q-1} \int_A |f(x)|^q dx$$

($|A|$ = Lebesgue measure of A)

Proof: Jensen's Inequality

$$\left(\frac{1}{K} \sum_{k=1}^K |a_k| \right)^q \leq \frac{1}{K} \sum_{k=1}^K |a_k|^q$$

$$\text{and } \left(\frac{1}{|A|} \int_A |f(x)| dx \right)^q \leq \frac{1}{|A|} \int_A |f(x)|^q dx \quad \square$$

Proposition: Let $p \geq 1$, $\exists C(p, \sigma, b, T)$ such
 that $\forall 0 \leq s \leq t \leq T$, $\mathbb{E}(|X_t - X_s|^{2p}) \leq C(1 + |y|^{2p})(t-s)^{p-1}$

1. The first part of the book is devoted to a general discussion of the theory of the atom.

2. The second part of the book is devoted to a detailed discussion of the theory of the atom, and the third part is devoted to a detailed discussion of the theory of the atom.

3. The fourth part of the book is devoted to a detailed discussion of the theory of the atom, and the fifth part is devoted to a detailed discussion of the theory of the atom.

4. The sixth part of the book is devoted to a detailed discussion of the theory of the atom, and the seventh part is devoted to a detailed discussion of the theory of the atom.

5. The eighth part of the book is devoted to a detailed discussion of the theory of the atom, and the ninth part is devoted to a detailed discussion of the theory of the atom.

Proof: We start with the triangular inequality

$$|X_t - X_s| \leq \left| \int_s^t \sigma(r, X_r) dW_r \right| + \left| \int_s^t b(r, X_r) dr \right|$$

Using BDG and the above technical Lemma, we get

$$\begin{aligned} \mathbb{E}(|X_t - X_s|^{2p}) &\leq 2^{2p-1} \left[\mathbb{E} \left(\left| \int_s^t \sigma(r, X_r) dW_r \right|^{2p} \right) + \mathbb{E} \left(\left| \int_s^t b(r, X_r) dr \right|^{2p} \right) \right] \\ &\leq C \left[\mathbb{E} \left(\left| \int_s^t |\sigma(r, X_r)|^2 dr \right|^p \right) + (t-s)^{2p-1} \mathbb{E} \left(\int_s^t |b(r, X_r)|^{2p} dr \right) \right] \\ &\leq C \left[(t-s)^{p-1} \int_s^t \mathbb{E}(|\sigma(r, X_r)|^{2p}) dr + T^p (t-s)^{p-1} \int_s^t \mathbb{E}(|b(r, X_r)|^{2p}) dr \right] \end{aligned}$$

By our assumptions, we have:

$$\begin{aligned} \mathbb{E}(|\sigma(r, X_r)|^{2p} + |b(r, X_r)|^{2p}) &\stackrel{(H2)}{\leq} C \mathbb{E}(|1 + |X_r||^{2p}) \\ &\leq C \mathbb{E}(|1 + |X_r|^{2p}|) \stackrel{\text{Prop. on polynomial moments}}{\leq} C(1 + |y|^{2p}) \end{aligned}$$

□

7.2.2. Strong error rate

Th: Under (H1) and (H2) and under

$\exists \alpha, k > 0, \forall x \in \mathbb{R}^n, \forall (s, t) \in [0, T]$

$$|\sigma(t, x) - \sigma(s, x)| + |b(t, x) - b(s, x)| \stackrel{(H3)}{\leq} K(1 + |x|)(t-s)^\alpha$$

Then, for $\beta = \min(\alpha, \frac{1}{2})$

$$\forall p > 1, \exists C_p > 0, \forall y \in \mathbb{R}^n, \forall N \in \mathbb{N}^*, \mathbb{E} \left(\sup_{t \leq T} |X_t - \bar{X}_t^N|^{2p} \right)$$

$$C_p \frac{(1 + |y|^{2p})}{N^{2\beta p}}$$

$$\text{If } \gamma < \beta, N^\gamma \sup_{t \leq T} |X_t - \bar{X}_t| \xrightarrow[N \rightarrow \infty]{a.s.} 0.$$

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Proof: $\forall u \in [0, T]$

$$X_u - \bar{X}_u = \int_0^u \sigma(s, X_s) - \sigma(s, \bar{X}_s) dW_s + \int_0^u b(s, X_s) - b(s, \bar{X}_s) ds$$

Using B-D-G's Inequality, we get

$$\mathbb{E} \left(\sup_{u \leq t} |X_u - \bar{X}_u|^{2p} \right)$$

$$\leq 2^{2p-1} \mathbb{E} \left(\sup_{u \leq t} \left| \int_0^u \sigma(s, X_s) - \sigma(s, \bar{X}_s) dW_s \right|^{2p} \right)$$

$$+ 2^{2p-1} \mathbb{E} \left(\sup_{u \leq t} \left| \int_0^u b(s, X_s) - b(s, \bar{X}_s) ds \right|^{2p} \right)$$

$$\stackrel{\text{B-D-G}}{\leq} t^{p-1} \int_0^t \mathbb{E} (|\sigma(s, X_s) - \sigma(s, \bar{X}_s)|^{2p}) ds$$

$$+ 2^{2p-1} t^{2p-1} \int_0^t \mathbb{E} (|b(s, X_s) - b(s, \bar{X}_s)|^{2p}) ds$$

With Technical Lemma + (H3)

$$|\sigma(s, X_s) - \sigma(s, \bar{X}_s)|^{2p} \leq 3^{2p-1} (|\sigma(s, X_s) - \sigma(s, X_2)|^{2p} + |\sigma(s, X_s) - \sigma(s, X_2)|^{2p} + |\sigma(s, X_2) - \sigma(s, \bar{X}_2)|^{2p})$$

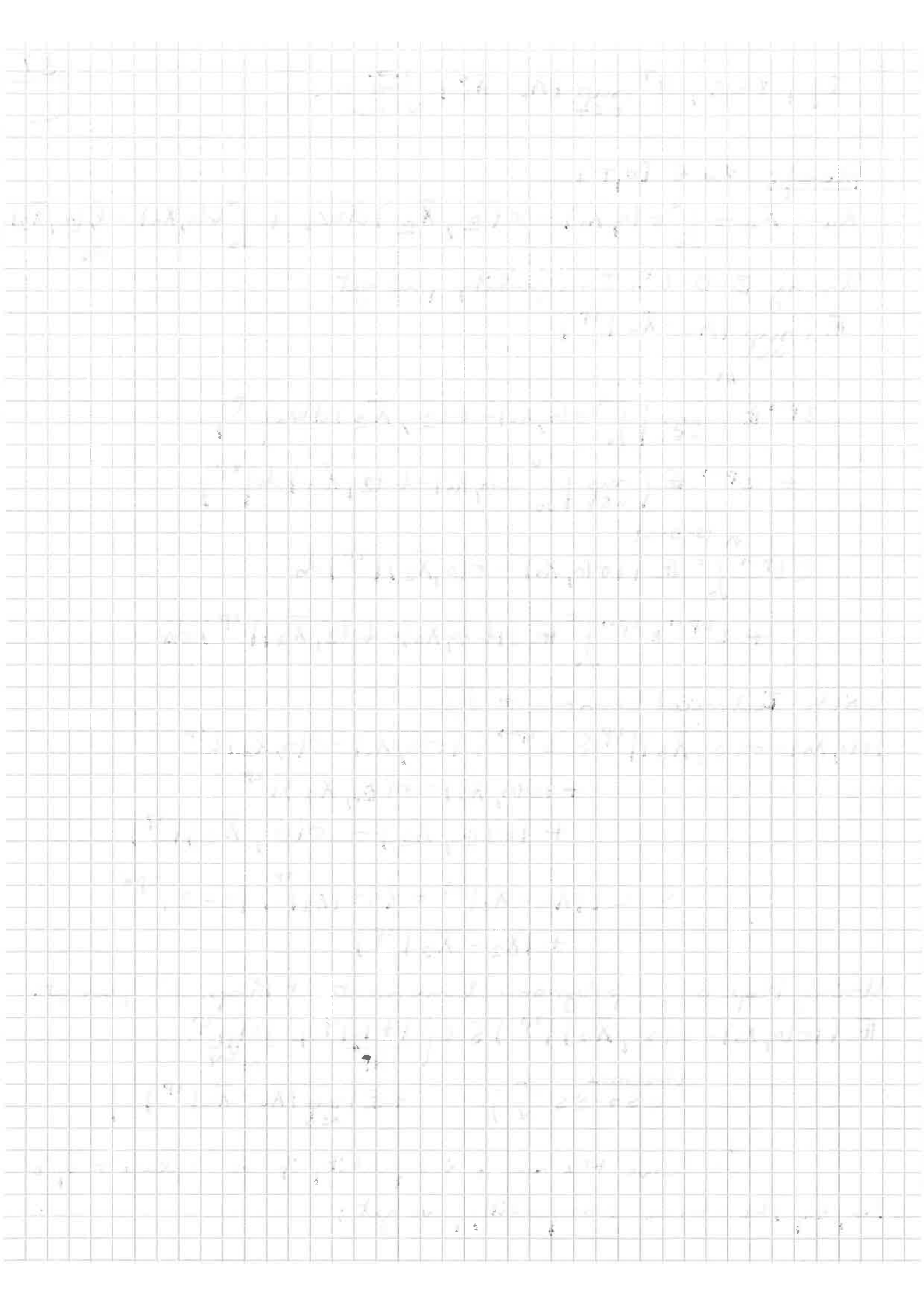
$$\leq C (|X_s - X_2|^{2p} + (1 + |X_2|^{2p}) (s - 2)^{2p} + |X_2 - \bar{X}_2|^{2p})$$

Using Prop. on polynomial moments + Prop. (★), we get:

$$\mathbb{E} (|\sigma(s, X_s) - \sigma(s, \bar{X}_s)|^{2p}) \leq C \left(\frac{1 + |y|^{2p}}{N^{2p}} + \frac{1 + |y|^{2p}}{N^{2p}} \right. \\ \left. \begin{matrix} \text{(because } 0 \leq s - 2 \leq \frac{T}{N} \text{)} \\ + \mathbb{E} \left(\sup_{u \leq s} |X_u - \bar{X}_u|^{2p} \right) \end{matrix} \right)$$

↓ we have the same inequality if we replace σ by b

So, by the above inequality, we get:



$$\mathbb{E} \left(\sup_{u \leq t} |X_u - \bar{X}_u|^{2p} \right) \leq C \left(\frac{1 + |y|^{2p}}{N^{2(\alpha + \frac{1}{2})p}} + \int_0^t \mathbb{E} \left(\sup_{u \leq s} |X_u - \bar{X}_u|^{2p} \right) ds \right) \quad (38)$$

By Gronwall's Lemma (and using a localization technique: $V_m := \inf \{ t \geq 0, |X_t - \bar{X}_t| \geq m \}$ such that

$$\mathbb{E} \left(\sup_{u \leq t} |X_{u \wedge V_m} - \bar{X}_{u \wedge V_m}|^{2p} \right) < +\infty$$

$$(\text{for } \gamma < \beta) \quad \mathbb{E} \left(\left(N^\gamma \sup_{u \leq t} |X_u - \bar{X}_u| \right)^{2p} \right) \leq \frac{C}{N^{2(\beta - \gamma)p}}.$$

With $p > \frac{1}{2(\beta - \gamma)}$, we get:

$$\mathbb{E} \left(\sum_{N \geq 1} \left(N^\gamma \sup_{u \leq t} |X_u - \bar{X}_u| \right)^{2p} \right) < +\infty$$

$$\text{and } \mathbb{P} \left(\sum_{N \geq 1} \sup_{u \leq t} |X_u - \bar{X}_u|^{2p} < +\infty \right) = 1 \quad (\text{Borel - Cantelli})$$

$$\text{and so } N^\gamma \sup_{u \leq t} |X_u - \bar{X}_u| \xrightarrow[N \rightarrow \infty]{a.s.} 0$$

□

