Ex 1. Schéma de Crank-Nicolson

On s'intéresse à la résolution du problème discret suivant :

Pour tout $n \in [0, N]$, trouver $(u_i^{n})_{i\in\{0,\dots,J+1\}}$ solution de

$$
\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_{i+1}^{n+\frac{1}{2}} - 2u_i^{n+\frac{1}{2}} + u_{i-1}^{n+\frac{1}{2}}}{h^2}, \quad \text{pour tout } i \in \{1, \dots, J\},
$$

- $u_0^n = 0, u_{J+1}^n = 0$, pour tout $n \in \{0, \dots, N\}$,
- $u_i^0 = u_0(x_i)$, pour tout $i \in \{0, \dots, J+1\}$.

où $u_i^{n+\frac{1}{2}} = \frac{u_i^n + u_i^{n+1}}{2}$, pour tout $i \in \{0, \dots, J + 1\}$.

1) Ce schéma est il explicite ou implicite ?
2) Étudier la consistance de ce schéma.
3) Étudier la stabilité au sens de Von Neumann.
4) Étudier la stabilité en norme L^∞ en espace.
5) Que pensez-vous de la convergence ?

Ex. 1.

$C_{h,\Delta t}$ → CN → Crank - Nickolson

Chaleur $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$

$\forall n \in [0, N]$, trouver $(u_i^n)_{i \in \{0,\dots,J+1\}}$

Sol. de

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_{i+1}^{n+\frac{1}{2}} - 2u_i^{n+\frac{1}{2}} + u_{i-1}^{n+\frac{1}{2}}}{h^2}, \forall i \in \{1, \dots, J\}$$

$$u_i^{n+\frac{1}{2}} = \frac{u_i^n + u_i^{n+1}}{2}, \forall i \in \{0, \dots, J+1\}$$

1) temps implicite ou explicite — car il apparait à droite

implicite

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\left(\frac{u_{i+1}^n + u_{i+1}^{n+1}}{2} - 2\,\frac{u_i^n + u_i^{n+1}}{2} + \frac{u_{i-1}^n + u_{i-1}^{n+1}}{2}\right)}{h^2}$$

2) consistance de schéma

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{2h^2}\left(\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) + \left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right)\right)$$

$u_i^n = u(x_i, t_n)$ $\qquad$ $u_i^{n+1} = u(x_i, t_n + \Delta t)$

$u_{i+1}^n = u(x_i + h, t_n)$ $\qquad$ $u_{i+1}^{n+1} = u(x_i + h, t_n + \Delta t)$

$u_{i-1}^n = u(x_i - h, t_m)$ $\qquad$ $u_{i-1}^{n+1} = u(x_i - h, t_n + \Delta t)$

$$\varphi(x_{i+1}) = \varphi(x_i) + h\varphi'(x_i) + \frac{h^2}{2}\varphi''(x_i) + \frac{h^3}{6}\varphi^{(3)}(x_i) + \frac{h^4}{24}\varphi^{(4)}(x_i + \theta_i^+ h),$$

$$\varphi(x_{i-1}) = \varphi(x_i) - h\varphi'(x_i) + \frac{h^2}{2}\varphi''(x_i) - \frac{h^3}{6}\varphi^{(3)}(x_i) + \frac{h^4}{24}\varphi^{(4)}(x_i + \theta_i^- h),$$

2 points (x_i, t_m) et $(x_i, t_m + \Delta t)$

autour (x_i, t_m) Fixe

$$u(x,t) = u(x_i,t_m) + (x-x_i)\, u_x(x_i,t_m) + \frac{1}{2}(x-x_i)^2 u_{xx}(x_i,t_m) + \frac{1}{6}(x-x_i)^3 u_{xxx}(x_i,t_m)$$
$$+ \frac{1}{24}(x-x_i)^4 u_{(4)x}(\xi, t_m) \quad \xi \in \{x_{i-1}, x_{i+1}\}$$

autour $(x_i, t_m + \Delta t) \to (x_i, t_m')$ Fixe

$$u(x,t') = u(x_i,t_m') + (x-x_i)\, u_x(x_i,t_m') + \frac{1}{2}(x-x_i)^2 u_{xx}(x_i,t_m') + \frac{1}{6}(x-x_i)^3 u_{xxx}(x_i,t_m')$$
$$+ \frac{1}{24}(x-x_i)^4 u_{(4)x}(\xi, t_m') \quad \xi \in \{x_{i-1}, x_{i+1}\}$$

$u_i^m = u(x_i, t_m)$

$u_{i+1}^m = u(x_i + h, t_m)$ On a donc.

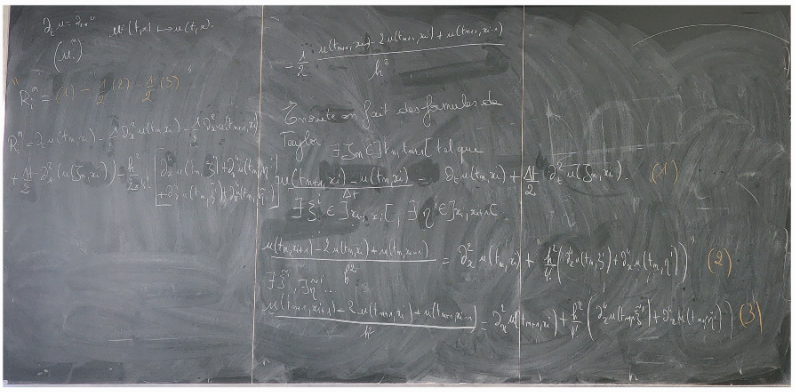
$u_{i-1}^m = u(x_i - h, t_m)$

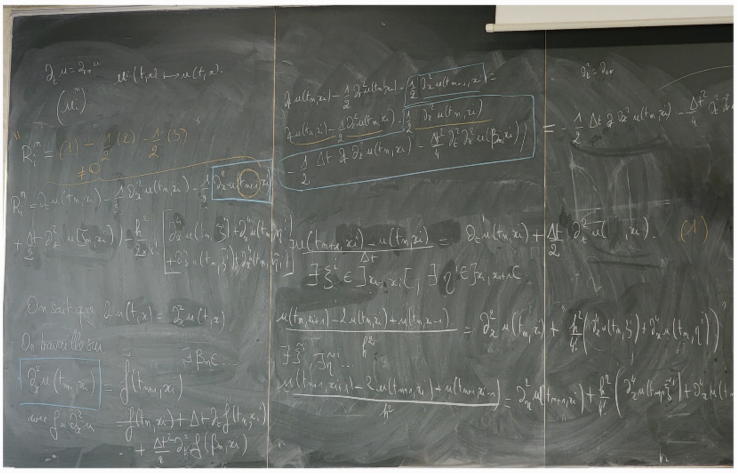
$$u(x_i+h,t_m) = u(x_i,t_m) + (h)\, u_x(x_i,t_m) + \frac{1}{2}(h)^2 u_{xx}(x_i,t_m) + \frac{1}{6}(h)^3 u_{xxx}(x_i,t_m)$$
$$+ \frac{1}{24}(h)^4 u_{(4)x}(\xi, t_m) \quad \xi \in \{x_{i-1}, x_{i+1}\}$$

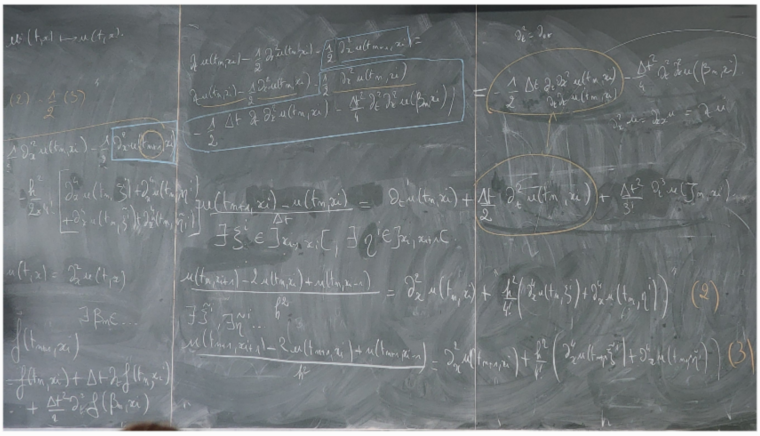
$+$

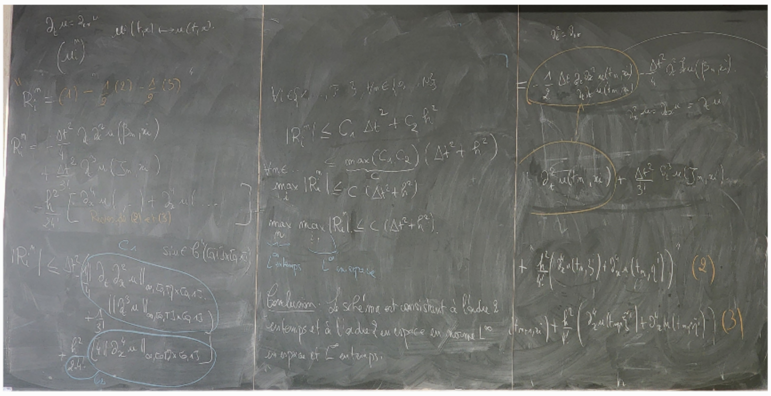
$$u(x_i-h,t_m) = u(x_i,t_m) - (h)\, u_x(x_i,t_m) + \frac{1}{2}(h)^2 u_{xx}(x_i,t_m) - \frac{1}{6}(h)^3 u_{xxx}(x_i,t_m)$$
$$+ \frac{1}{24}(h)^4 u_{(4)x}(\xi, t_m) \quad \xi \in \{x_{i-1}, x_{i+1}\}$$

$$\begin{matrix} u(x_i+h,t_m) \\ + \\ u(x_i-h,t_m) \end{matrix} =$$









$$u(x_i+h,t_m) = u(x_i,t_m) + \cancel{(h)\,u_x(x_i,t_m)} + \frac{1}{2}(h)^2 u_{xx}(x_i,t_n) + \cancel{\frac{1}{6}(h)^3 u_{xxx}(x_i,t_n)}$$

$$+ \frac{1}{24}(h)^4 u_{(4)x}(\xi,t_m) \qquad \xi \in \{x_{i-1}, x_{i+1}\}$$

$+$

$$u(x_i-h,t_m) = u(x_i,t_m) - \cancel{(h)\,u_x(x_i,t_m)} + \frac{1}{2}(h)^2 u_{xx}(x_i,t_n) - \cancel{\frac{1}{6}(h)^3 u_{xxx}(x_i,t_n)}$$

$$+ \frac{1}{24}(h)^4 u_{(4)x}(\xi,t_m) \qquad \xi \in \{x_{i-1}, x_{i+1}\}$$

$$u(x_i+h,t_m) + u(x_i-h,t_m) = 2u(x_i,t_m) + h^2 u_{xx}(x_i,t_m) + \frac{1}{24}(h)^4\left(u_{(4)x}(\xi',t_m) + u_{(4)x}(\eta',t_m)\right)$$

$\xi \in \{x_{i-1}, x_{i+1}\}$
$\eta \in \{x_{i-1}, x_{i+1}\}$

même chose autour $(x_i, t_m+\Delta t) \to (x_i, t'_m)$

$$u(x_i+h,t'_m) + u(x_i-h,t'_m) = 2u(x_i,t'_m) + h^2 u_{xx}(x_i,t'_m) + \frac{1}{12}(h)^4\left(u_{(4)x}(\check{\xi},t_m) + u_{(4)x}(\check{\eta},t_m)\right)$$

donc

$$\begin{matrix} u(x_i+h,t_m) \\ + \\ u(x_i-h,t_m) \end{matrix} - 2u(x_i,t_m) = h^2 u_{xx}(x_i,t_m) + \frac{1}{24}(h)^4\left(u_{(4)x}(\xi',t_m)+u_{(4)x}(\eta',t_m)\right)$$

$$\begin{matrix} u(x_i+h,t'_m) \\ + \\ u(x_i-h,t'_m) \end{matrix} - 2u(x_i,t'_m) = h^2 u_{xx}(x_i,t'_m) + \frac{1}{24}(h)^4\left(u_{(4)x}(\breve{\xi},t_m)+u_{(4)x}(\breve{\eta},t_m)\right)$$

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = \frac{1}{2h^2}\left(\left(u_{i+1}^n-2u_i^n+u_{i-1}^n\right)+\left(u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}\right)\right)$$

après

Fixe

$$u_i^{n+1} = u(x_i,t_n+\Delta t) = u(x_i,t'_m)$$

$$u_i^n = u(x_i,t_n)$$

autour $\longrightarrow$

$$u(x_i,t_m+\Delta t) = u(x_i,t_m) + (\Delta t)\,u_t(x_i,t_m) + \frac{1}{2}(\Delta t)^2 u_{tt}(x_i,t_n) + \frac{1}{6}(\Delta t)^3 u_{ttt}(x_i,t_n) + \frac{1}{24}(\Delta t)^4 u_{(4)t}(x_i,\zeta)$$

$$u(x_i,t_m+\Delta t) - u(x_i,t_m) = (\Delta t)\,u_t(x_i,t_m) + \frac{1}{2}(\Delta t)^2 u_{tt}(x_i,t_n) + \frac{1}{6}(\Delta t)^3 u_{ttt}(x_i,t_n) + \frac{1}{24}(\Delta t)^4 u_{(4)t}(x_i,\zeta)$$

donc on a

$$\frac{u(x_i+h,t_m)-2u(x_i,t_m)+u(x_i-h,t_m)}{h^2} = u_{xx}(x_i,t_m) + \frac{1}{24}(h)^4\left(u_{(4)x}(\xi',t_m)+u_{(4)x}(\eta',t_m)\right) \qquad \xi\in\{x_{i-1},x_{i+1}\}$$

$$\frac{u(x_i+h,t'_m)-2u(x_i,t'_m)+u(x_i-h,t'_m)}{h^2} = u_{xx}(x_i,t'_m) + \frac{1}{24}(h)^4\left(u_{(4)x}(\breve{\xi},t_m)+u_{(4)x}(\breve{\eta},t_m)\right) \qquad \eta\in\{x_{i-1},x_{i+1}\}$$

$$\frac{u(x_i,t_m+\Delta t)-u(x_i,t_m)}{\Delta t} = u_t(x_i,t_m) + \frac{1}{2}(\Delta t)\,u_{tt}(x_i,t_n) + \frac{1}{6}(\Delta t)^2 u_{ttt}(x_i,t_n) + \frac{1}{24}(\Delta t)^3 u_{(4)t}(x_i,\zeta) \qquad \zeta\in\{t_n,t_{n+1}\}$$

$$R_j^m = \frac{u(x_i, t_m+\Delta t) - u(x_i,t_m)}{\Delta t} - \frac{1}{2}\left(\frac{u(x_i+h,t'_m) - 2u(x_i,t'_m) + u(x_i-h,t'_m)}{h^2} + \frac{u(x_i+h,t'_m) - 2u(x_i,t'_m) + u(x_i-h,t'_m)}{h^2}\right)$$

$$= u_t(x_i,t_m) + \frac{1}{2}(\Delta t)\, u_{tt}(x_i,t_m) + \frac{1}{6}(\Delta t)^2 u_{ttt}(x_i,t_m) + \frac{1}{24}(\Delta t)^3 u_{(4)t}(x_i,\xi)$$

$$\xi \in \{t_m, t_{m+1}\}$$

$$- \frac{1}{2}\left(u_{xx}(x_i,t_m) + \frac{1}{24}(h)^4\left(u_{(4)x}(\xi',t_m) + u_{(4)x}(\eta',t_m)\right) + u_{xx}(x_i,t'_m) + \frac{1}{24}(h)^4\left(u_{(4)x}(\xi',t_m) + u_{(4)x}(\eta',t_m)\right)\right)$$

E.C.

$$u_t = u_{xx}$$

$$u_{xx}(x_i,t'_m) = f(x_i,t'_m) \quad \text{autour } (x_i,t_m)$$

$$u_{xx}(x_i,t_m+\Delta t) = u_{xx}(x_i,t_m) + (\Delta t)\, u_t\, u_{xx}(x_i,t_m) + \frac{1}{2}(\Delta t)^2 u_{tt}\, u_{xx}(x_i,\tilde{\xi})$$

$$= u_{xx}(x_i,t_m) + \Delta t\, u_{tt}(x_i,t_m) + \frac{\Delta t^2}{2} u_{ttt}(x_i,\tilde{\xi})$$

$$= u_t(x_i,t_m) + \frac{1}{2}(\Delta t)\, u_{tt}(x_i,t_m) + \frac{1}{6}(\Delta t)^2 u_{ttt}(x_i,t_m) + \frac{1}{24}(\Delta t)^3 u_{(4)t}(x_i,\xi)$$

$$- \frac{1}{2} u_{xx}(x_i,t_m) - \frac{1}{2}\Delta t\, u_{tt}(x_i,t_m) - \frac{\Delta t^2}{4} u_{ttt}(x_i,\tilde{\xi}) - \frac{1}{2} u_{xx}(x_i,t_m)$$

$$- \frac{1}{24}\left((h)^4\left(u_{(4)x}(\xi',t_m) + u_{(4)x}(\eta',t_m)\right) + (h)^4\left(u_{(4)x}(\xi',t_m) + u_{(4)x}(\eta',t_m)\right)\right)$$

$$= \cancel{u_t(x_i,t_m)} + \cancel{\frac{1}{2}(\Delta t)\, u_{tt}(x_i,t_m)} + \frac{1}{6}(\Delta t)^2 u_{ttt}(x_i,t_m) + \frac{1}{24}(\Delta t)^3 u_{(4)t}(x_i,\xi)$$

$$- \cancel{\frac{1}{2} u_t(x_i,t_m)} - \cancel{\frac{1}{2}\Delta t\, u_{tt}(x_i,t_m)} - \frac{\Delta t^2}{4} u_{ttt}(x_i,\tilde{\xi}) - \cancel{\frac{1}{2} u_t(x_i,t_m)}$$

$$- \frac{1}{24}\left((h)^4\left(u_{(4)x}(\xi',t_m) + u_{(4)x}(\eta',t_m)\right) + (h)^4\left(u_{(4)x}(\xi',t_m) + u_{(4)x}(\eta',t_m)\right)\right)$$

$$= \frac{1}{6}(\Delta t)^2 u_{ttt}(x_i, t_n) + \boxed{\frac{1}{24}(\Delta t)^3 u_{(4)t}(x_i, \varsigma)} - \frac{\Delta t^2}{4} u_{ttt}(x_i, \tilde{\varsigma})$$

$$- \frac{1}{24}\left((h)^4 \left(u_{(4)x}(\xi_i', t_n) + u_{(4)x}(\eta_i', t_n)\right) + (h)^4 \left(u_{(4)x}(\tilde{\xi}_i, t_n) + u_{(4)x}(\tilde{\eta}_i, t_n)\right) \right)$$

$$= \frac{1}{6}(\Delta t)^2 u_{ttt}(x_i, \beta) - \frac{1}{4}(\Delta t)^2 u_{ttt}(x_i, \hat{\varsigma})$$

$$- \frac{h^4}{24}\left(\left(u_{(4)x}(\xi_i', t_n) + u_{(4)x}(\eta_i', t_n)\right) + \left(u_{(4)x}(\tilde{\xi}_i, t_n) + u_{(4)x}(\tilde{\eta}_i, t_n)\right) \right)$$

il n'y pas dependances de Δt et h^4 pour C_1 C_2

$$= \left[\frac{1}{6} u_{ttt}(x_i, \beta) - \frac{1}{4} u_{ttt}(x_i, \hat{\varsigma})\right] \cdot \Delta t^2$$

$$- \frac{1}{24}\left[\left(u_{(4)x}(\xi_i', t_n) + u_{(4)x}(\eta_i', t_n)\right) + \left(u_{(4)x}(\tilde{\xi}_i, t_n) + u_{(4)x}(\tilde{\eta}_i, t_n)\right) \right] h^2$$

$\|\cdot\|_\infty \to \|\cdot\|_{\infty\, [0,1]\times[0,T]}$

$$\leq \left[\frac{1}{6}\|u_{ttt}\|_\infty - \frac{1}{4}\|u_{ttt}\|_\infty\right] \Delta t^2 - \frac{1}{24} 4 \cdot \|u_{(4)x}\| \cdot h^2$$

$$\frac{1}{6} - \frac{1}{4} = \frac{6-4}{24} = \frac{2}{24} = \frac{1}{12}$$

$$\leq \frac{1}{12}\|u_{ttt}\|_\infty \cdot \Delta t^2 - \frac{1}{6}\|u_{(4)x}\| \cdot h^2$$

$$\leq C_1 \Delta t^2 + C_2 h^2$$

$$|R_j^n| \leq \max(C_1, C_2)(\Delta t^2 + h^2)$$

$\forall i \in \{0, \dots, 1\}$ $\quad \forall n \in \{0, \dots T\}$

$$\max_n \max_j |R_j^n| \leq C \, (\Delta t^2 + h^2)$$

donc

$$\lim_{\Delta t, h \to 0} \max_n \max_j |R_j^n| = 0$$

+ donc le schéma est consistante en L^∞ et d'ordre 2 en temps et espace

3) Étudier la stabilité au sens von Neuman

$$\left(\sum_{J=1}^{N} h \left|U_J^n\right|^2\right)^{1/2} \leq K \left(\sum_{J=1}^{N} h \left(U_0(x_J)\right)^2\right)^{1/2}, \quad n \in \{0, \dots, M\}$$

$$\frac{u(x_i, t_m+\Delta t) - u(x_i, t_m)}{\Delta t} = \frac{1}{2}\left(\frac{u(x_i+h, t_m) - 2u(x_i, t_m) + u(x_i-h, t_m)}{h^2} + \frac{u(x_i+h, t_m') - 2u(x_i, t_m') + u(x_i-h, t_m')}{h^2}\right)$$

Rappelle $C_m(x \longmapsto f(x \pm \delta x)) = e^{\pm 2\pi i m \delta x} C_m(f(x))$

$$C_m(U_i^n) = C_m(u(x_i, t_n)) = C_m(u(x_i, t_n))$$

$$C_m(U_{i+1}^n) = C_m(u(x_i+h, t_n)) = e^{2\pi i m h} C_m(u(x_i, t_n))$$

$$C_m(U_{i-1}^n) = C_m(u(x_i-h, t_m)) = e^{-2\pi i m h} C_m(u(x_i, t_n))$$

$$C_m(U_i^{n+1}) = C_m(u(x_i, t_m+\Delta t)) = C_m(u(x_i, t_m+\Delta t))$$

$$C_m(U_{i+1}^{n+1}) = C_m(u(x_i+h, t_m+\Delta t)) = e^{2\pi i m h} C_m(u(x_i, t_m+\Delta t))$$

$$C_m(U_{i-1}^{n+1}) = C_m(u(x_i-h, t_m+\Delta t)) = e^{-2\pi i m h} C_m(u(x_i, t_m+\Delta t))$$

$$\frac{u(x_i, t_m+\Delta t) - u(x_i, t_m)}{\Delta t} = \frac{1}{2}\left(\frac{u(x_i+h, t_m) - 2u(x_i, t_m) + u(x_i-h, t_m)}{h^2} + \frac{u(x_i+h, t_m') - 2u(x_i, t_m') + u(x_i-h, t_m')}{h^2}\right)$$

$$\frac{u(x_i, t_m+\Delta t) - u(x_i, t_m)}{\Delta t} = \frac{C_m(u(x_i, t_m+\Delta t)) - C_m(u(x_i, t_n))}{\Delta t}$$

$$\frac{u(x_i+h, t_m) - 2u(x_i, t_m) + u(x_i-h, t_m)}{h^2} = \frac{\left(e^{2\pi i m h} + e^{-2\pi i m h} - 2\right) C_m(u(x_i, t_n))}{h^2}$$

$$\frac{u(x_i+h, t_m') - 2u(x_i, t_m') + u(x_i-h, t_m')}{h^2} = \frac{\left(e^{2\pi i m h} + e^{-2\pi i m h} - 2\right) C_m(u(x_i, t_m+\Delta t))}{h^2}$$

$$\frac{1}{2}\left(\frac{u(x_i+h,t_m)-2u(x_i,t_m)+u(x_i-h,t_m)}{h^2}+\frac{u(x_i+h,t_m')-2u(x_i,t_m')+u(x_i-h,t_m')}{h^2}\right)=$$

$$=\frac{\left(\frac{e^{2\pi i m h}+e^{-2\pi i m h}}{2}-1\right)C_m(u(x_i,t_n))}{h^2}+\frac{\left(\frac{e^{2\pi i m h}+e^{-2\pi i m h}}{2}-1\right)C_m(u(x_i,t_m+\Delta t))}{h^2}$$

$$=\frac{\left(\cos(2\pi m h)-1\right)C_m(u(x_i,t_n))}{h^2}+\frac{\left(\cos(2\pi m h)-1\right)C_m(u(x_i,t_m+\Delta t))}{h^2}$$

$$\frac{C_m(u(x_i,t_n+\Delta t))-C_m(u(x_i,t_n))}{\Delta t}=\frac{\left(\cos(2\pi m h)-1\right)C_m(u(x_i,t_n))}{h^2}+\frac{\left(\cos(2\pi m h)-1\right)C_m(u(x_i,t_n+\Delta t))}{h^2}$$

$$\left(\frac{1}{\Delta t}-\frac{1}{h^2}(\cos 2\pi m h-1)\right)C_m(u(x_i,t_n+\Delta t))=\left(\frac{1}{h^2}(\cos 2\pi m h-1)+\frac{1}{\Delta t}\right)C_m(u(x_i,t_n))$$

$$C_m(u(x_i,t_n+\Delta t))=\frac{\left(\frac{1}{h^2}(\cos 2\pi m h-1)+\frac{1}{\Delta t}\right)}{\left(\frac{1}{\Delta t}-\frac{1}{h^2}(\cos 2\pi m h-1)\right)}C_m(u(x_i,t_n))$$

$$\frac{e^{-x}+e^{x}}{2}=\cos x$$

$$C_m(u(x_i,t_n+\Delta t))=\frac{\Delta t\,(\cos 2\pi m h-1)+h^2}{h^2-\Delta t\,(\cos 2\pi m h-1)}C_m(u(x_i,t_n))$$

$$\theta=\pi m h$$

$$\left|\frac{\Delta t\,(\cos 2\pi m h-1)+h^2}{h^2-\Delta t\,(\cos 2\pi m h-1)}\right|\le 1$$

$$\cos 2\theta = 1-2\sin^2\theta$$

$$\cos 2\theta - 1 = -2\sin^2\theta$$

$$\left|\frac{h^2+\Delta t(-2\sin^2\pi m h)}{h^2-\Delta t(-2\sin^2\pi m h)}\right|\le 1$$

$$\left|\frac{1-\frac{2\Delta t}{h^2}\sin^2\pi m h}{1+\frac{2\Delta t}{h^2}\sin^2\pi m h}\right|\le 1$$

$$\left|\frac{h^2-2\Delta t\sin^2\pi m h}{h^2+2\Delta t\sin^2\pi m h}\right|\le 1$$

$$\Leftrightarrow \left|1-\frac{2\Delta t}{h^2}\sin^2\right|\le \underbrace{1+\frac{2\Delta t}{h^2}\sin^2}_{\geq 0}$$

$$\Leftrightarrow -1-\cancel{\frac{2\Delta t}{h^2}}\sin^2 \underset{\text{ok}}{\le} 1-2\cancel{\frac{\Delta t}{h^2}}\sin^2 \underset{\text{ok}}{\le} 1+\frac{2\Delta t}{h^2}\sin^2$$

$$|a|\le b$$

$$\Leftrightarrow -b\le a\le b$$

④ Estabilité en norme L^∞ en espace.

$$\underbrace{\max_{1\le j\le N} |U_j^n|}_{\|U^n\|_\infty} \le K \, \|U_0\|_\infty$$

$$|u_i^{n+1}| = |u_i^n + \frac{\Delta t}{h^2} \quad u_i^{n+1} \quad |$$

$$\le \underbrace{|\quad| + |\quad| + |\quad|}$$

$$|u_i^n| \le \|u^n\|_\infty$$

$$|u_i^{n+1}| \le \|u^{n+1}\|_\infty$$

$$U^n = \begin{pmatrix} u_1^n \\ \vdots \\ u_J^n \end{pmatrix}$$

$$\forall i, \forall n, \; |\alpha| \, |u_i^{n+1}| \le C_1 \|u^n\|_\infty + C_2 \|u^{n+1}\|_\infty$$

$$|\alpha| \, \|u^{n+1}\|_\infty \le$$

$$(\underbrace{|\alpha| - C_2}_{=1 \; \ge 1}) \|u^{n+1}\|_\infty \le \underbrace{C_1}_{\ge 1} \|u^n\|_\infty$$