# ch 3 _ VARIABLES ALEATOIRES CONTINUES

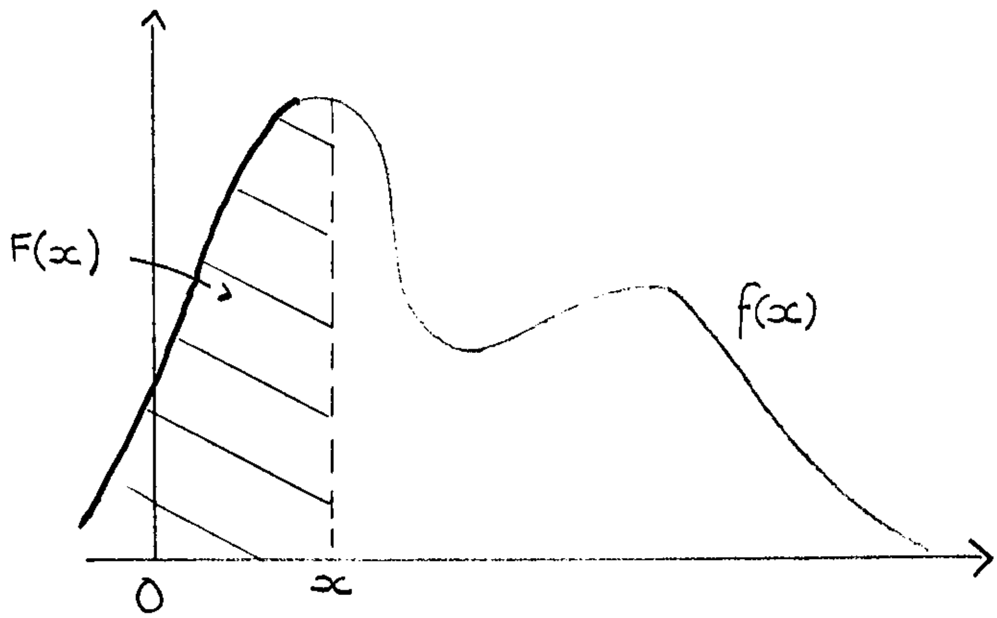
## (I) GENERALITÉS

Definition :

Une variable X est continue s'il existe une fonction f (appelée "Densité") telle que

$$F(x) = P(X \leqslant x) = \int_{-\infty}^{x} f(t)\,dt$$

F est 1 primitive de f, f est "la dérivée" de F



Propriétés

$$\boxed{\begin{array}{c} \displaystyle\int_{-\infty}^{+\infty} f(t)\,dt = 1 \\ \\ f(x) \geqslant 0. \end{array}}$$

$$\sum_{x=-\infty}^{+\infty} P(X=x) = 1$$

$$P(X=x) \geqslant 0$$

<u>Remarque</u> :
La densité $f$ pour 1 v.a. continue correspond à la distribution d'1 v.a. discrète.

$$P(a < X \leqslant b) = F(b) - F(a). = \int_a^b f(x)\,dx$$
$$= P(a \leq X \leq b)$$

<u>Remarque</u>
$P(X=b) = 0$, $\forall b \in \mathbb{R}$

Indicatrice :
$I \subset \mathbb{R}$, $\mathbb{1}_I(x) = \begin{cases} 1 & \text{si } x \in I \\ 0 & \text{si } x \notin I. \end{cases}$

Exemple
$$f(x) = C\, e^{-x}\,(x^3+1)\ \mathbb{1}_{(0,4)}(x)$$

Exercice 1.

$$f(x) = C\, x^{12}\, \mathbb{1}_{(0,1)}(x).$$

1) trouver C ?

$$\int_{-\infty}^{+\infty} f(x)dx = 1 = C\int_0^1 x^{12}dx = \frac{C}{13} \Rightarrow C = 13.$$

$$f(x) = 13\, x^{12}\, \mathbb{1}_{(0,1)}(x).$$

2) Fonction de repartition

$$F(x) = \begin{cases} 0 & \text{si } x \leqslant 0 \\ \int_0^x 13\, t^{12}\, dt = x^{13} & \text{si } x \in (0\ 1) \\ 1 & \text{si } x \geqslant 1 \end{cases}$$

3) Faire 1 calcul de probabilité

$$P\left(\frac{1}{4} < X < \frac{3}{4}\right) = \left(\frac{3}{4}\right)^{13} - \left(\frac{1}{4}\right)^{13} = P\left(\frac{1}{4} \leq X \leq \frac{3}{4}\right)$$

$$P\left(\frac{1}{4} < X < 5\right) = 1 - \left(\frac{1}{4}\right)^{13}$$

$$P(-1 < X < 8) = 1 - 0 = 1$$

## Exercice 2.

$$f(x) = C \times \frac{1}{\sqrt{x}} \ \mathbb{1}_{(0,1)}(x)$$

1) $C$ ?

$C = \frac{1}{2}$ $\left( \int_0^1 f(x)\,dx = 1 \right)$

$$f(x) = \frac{1}{2} \times \frac{1}{\sqrt{x}} \ \mathbb{1}_{(0\,1)}(x).$$

2) $F(x)$ ?

$$F(x) = \begin{cases} 0 \text{ si } x \leqslant 0 \\ \displaystyle\int_0^x \frac{dt}{2\sqrt{t}} = \sqrt{x} \quad \text{si } x \in (0,1). \\ 1 \text{ si } x \geqslant 1 \end{cases}$$

3) $P(\frac{1}{4} < X < 4) = 1 - \sqrt{\frac{1}{4}} = \frac{1}{2}$

## Exercice 3.

$$f(x) = \frac{c}{1+x^2}$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1 = c\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = c\,[\tan^{-1} x]_{-\infty}^{+\infty}$$

$$= c\,[\arctan]_{-\infty}^{+\infty}$$

$\Leftrightarrow c\pi = 1$

$C = \frac{1}{\pi}$

$$f(x) = \frac{1}{\pi(1+x^2)}.$$

2) $F(x)$ ?

$$F(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dt}{(1+x^2)} = \frac{1}{\pi} \left[ \tan^{-1} t \right]_{-\infty}^{x}$$

$$= \frac{1}{\pi} \left[ \tan^{-1} x + \frac{\pi}{2} \right].$$

3) $(-1 < x < 1) = \frac{1}{\pi} \left[ \tan^{-1}(1) - \tan^{-1}(-1) \right].$

$$= \frac{1}{\pi} \left[ \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) \right] = \frac{1}{2}$$

Notation.

- $f(x) = C\, e^{-x} \cos x \quad \mathbb{1}_{(0, \frac{\pi}{2})}(x).$

Le support de la variable est $(0, \frac{\pi}{2})$.

$(0 ; \frac{\pi}{2}) = \text{supp}\,(X).$

$$\int_a^b f(x)\,dx = \int_{(ab)} f(x)\,dx = \int_{a<x<b} f(x)\,dx = P(a<X<b)$$

ex:

$$P(|X|>3) = \int_{|x|>3} f(x)\,dx$$

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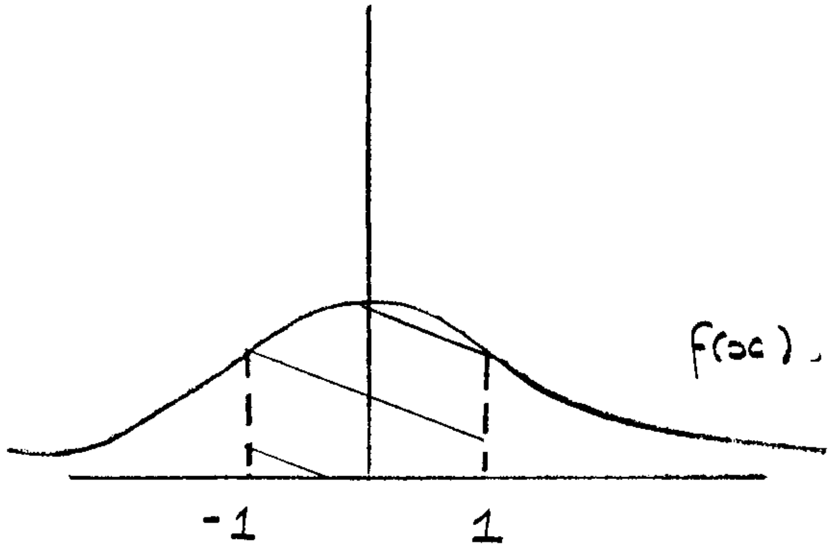



$$P(|X|<1) = \int_{-1}^{1} f(x)\,dx = \int_{|x|<1} f(x)\,dx$$

$$P(|X|>1) = \int_{-\infty}^{-1} f(x)\,dx + \int_{1}^{+\infty} f(x)\,dx$$

$$= \int_{|x|>1} f(x)\,dx$$

## Exercice.

$X$ de densite $f(x)$

$$E(X^2) = \int_{\mathbb{R}} x^2 f(x)\,dx \geqslant \int_{|x|>t} x^2 f(x)\,dx \geqslant t^2 \int_{|x|>t} f(x)\,dx$$

$$\geqslant t^2 \, P(|X|>t)$$

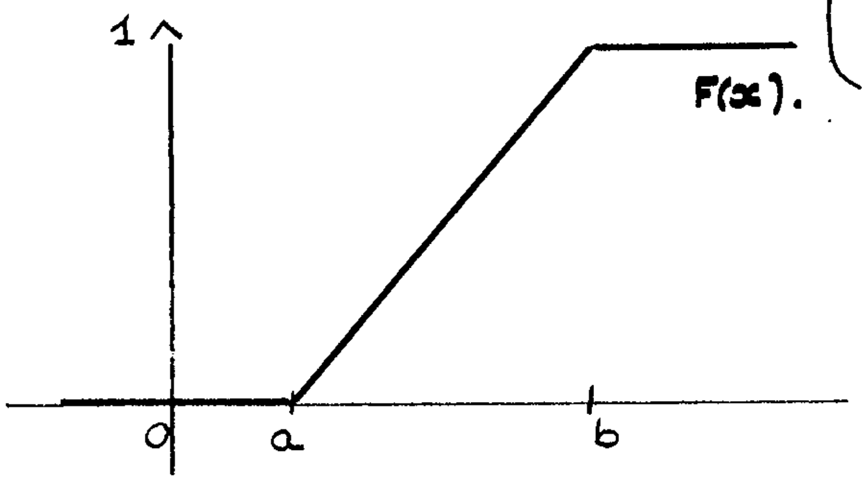
$$\Rightarrow \boxed{P(|X|>t) \leqslant \frac{E(x^2)}{t^2}.}$$

## (II) V. A. Continues fondamentales.

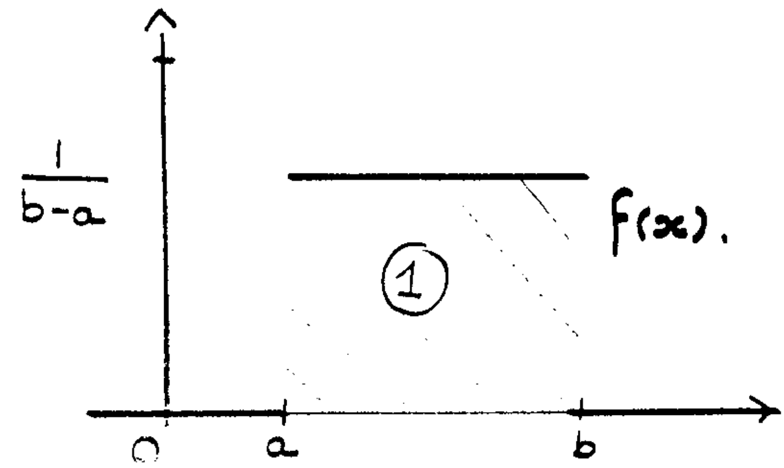
### (1) Les lois uniformes. $X \sim \mathcal{U}(a,b)$.

$X$ suit 1 loi uniforme sur l intervale $(a,b$

$$F(x) = \begin{cases} 0 \text{ si } x < a \\ \dfrac{x-a}{b-a} \text{ si } x \in (a,b \\ 1 \text{ si } x > b \end{cases}$$

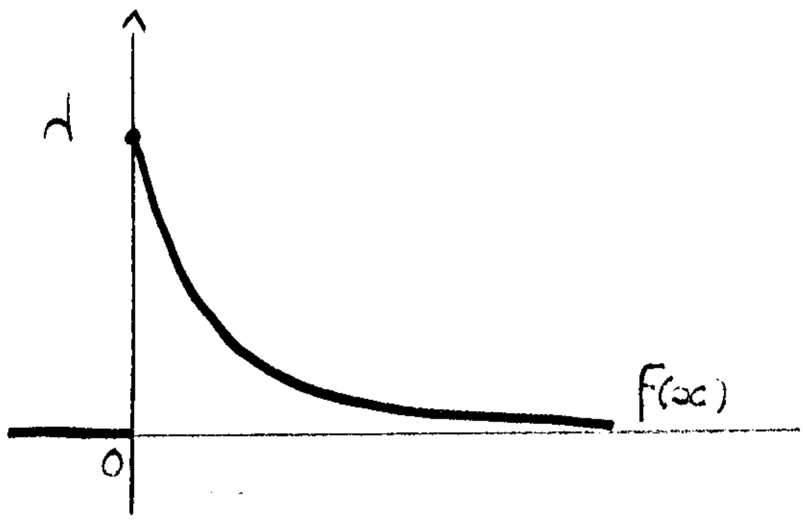


$$f(x) = \frac{1}{b-a} \ \mathbb{1}_{(ab)}(x).$$

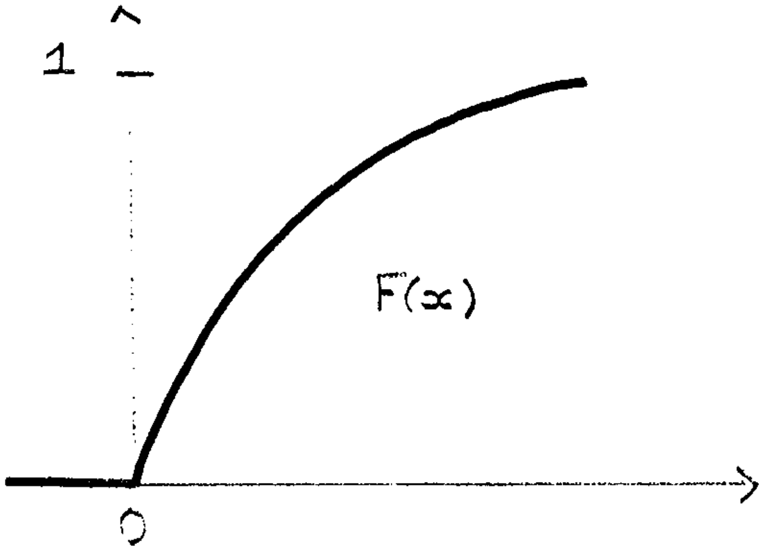



② <u>loi exponentielle</u> $X \sim Exp(\lambda) \quad \lambda > 0$

PAR EXEMPLE: $f(x) = \lambda \ e^{-\lambda x} \quad \mathbb{1}_{\mathbb{R}^+}(x)$
DURÉE de
VIE d'1 AMPOULE



$$F(x) = P(X \leq x) = \begin{cases} 0 & \text{si } x < 0 \\ \displaystyle\int_0^x \lambda e^{-\lambda t}\, dt = 1 - e^{-\lambda x} & \text{si } x \geq 0 \end{cases}$$




$P(X > x) = e^{-\lambda x}$ (pour $x \geq 0$).

Calculer ?

$$P(X > x+y \,/\, X > y) = \frac{P(X > x+y)}{P(X > y)}$$

$$= \frac{e^{-\lambda(x+y)}}{e^{-\lambda y}}$$

$$= e^{-\lambda x} = P(X > x).$$

la loi exponentielle est (1 temps d'attente) sans memoire

une V.A.

③ <u>loi gamma</u> $X \sim \gamma_\alpha \quad \alpha > 0.$

$$\boxed{\alpha > 0 \qquad \Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx.}$$

$$\Gamma(\alpha+1) = \int_0^{\infty} e^{-x} x^{\alpha} dx$$

$$= \begin{bmatrix} u = x^{\alpha} & v' = e^{-x} \\ u' = \alpha x^{\alpha-1} & v = -e^{-x} \end{bmatrix}$$

$$= \cancel{\left[-x^{\alpha} e^{-x}\right]_0^{\infty}} + \alpha \int_0^{\infty} e^{-x} x^{\alpha-1} dx$$

$$\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha).$$

$$\Gamma(1) = \int_0^{+\infty} e^{-x} x^0 \, dx = \left[-e^{-x}\right]_0^{\infty} = 1$$

$$\Gamma(1) = 1$$

- $n \in \mathbb{N}^*$ : $\Gamma(n+1) = n\,\Gamma(n).$

$$\Gamma(n) = n-1\,\Gamma(n-1)$$

$$\Gamma(n-1) = n-2\,\Gamma(n-2).$$

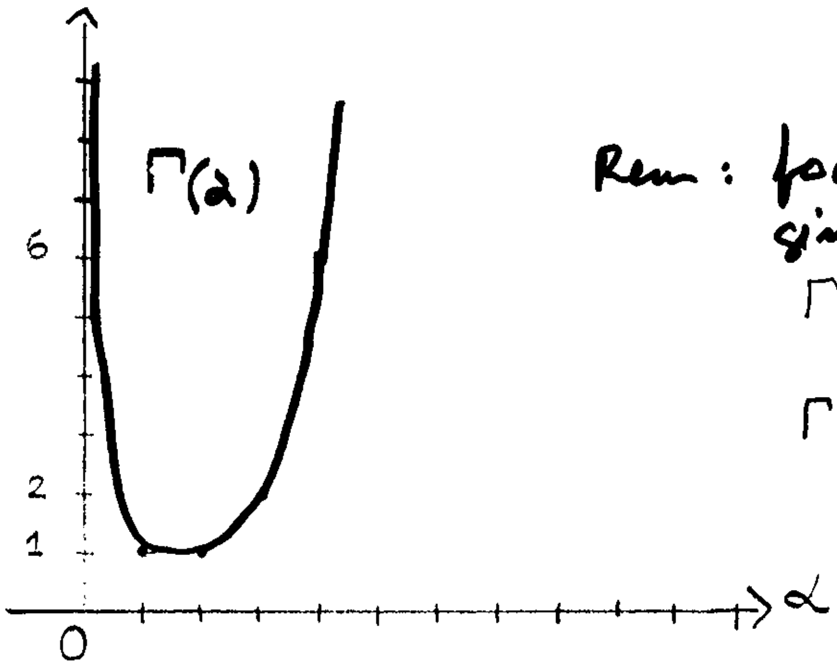
$$\Gamma_2 = 1\,\Gamma(1)$$

- $n \in \mathbb{N}$ : $\Gamma(n+1) = n!$

$$\boxed{\Leftrightarrow \int_0^{\infty} e^{-x} x^n dx = n!}$$

$$\int_0^{\infty} e^{-x}\,(3x^2)+1)\,dx = 7$$

$$\int_0^{+\infty} e^{-y}\,(x^2 y + x\,y^2)\,dy = x^2 + 2x$$




Rem : factorielle généralisée :

$\Gamma(\alpha) = (\alpha - 1)!$

$\Gamma\left(\frac{1}{2}\right) = \left(-\frac{1}{2}\right)! = \sqrt{\pi}$

$\Rightarrow$ e.g.

$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\,\frac{1}{2}\,\sqrt{\pi}$

<u>Definition</u>

$X \sim \gamma(\alpha)$ si sa densité

$$\boxed{f(x) = \frac{1}{\Gamma(\alpha)}\, e^{-x}\, x^{(\alpha-1)}\, \mathbb{1}_{\mathbb{R}^{*+}}(x) \qquad \mathbb{R}^{*+} = (0;+\infty)}$$

$(\Leftrightarrow X \sim \gamma_\alpha)$

<u>Remarque</u> : exemple

$$\gamma_3 \Rightarrow \frac{1}{2}\, e^{-x}\, x^2\, \mathbb{1}_{(0,+\infty)}(x)$$

$C\, e^{-x} x^7 \; \mathbb{1}_{[0;+\infty)}(x) \Rightarrow \gamma(8)$

$C\, e^{-x} \sqrt{x} \; \mathbb{1}_{[0;+\infty)}(x) \Rightarrow \gamma\left(\frac{3}{2}\right)$

$C\, \frac{e^{-x}}{\sqrt{x}} \; \mathbb{1}_{[0;+\infty)}(x) \Rightarrow \gamma\left(\frac{1}{2}\right)$

$$\boxed{\gamma_1 = Exp(1).}$$

Plus généralement on montrera que $Exp(\lambda) = \frac{\gamma_1}{\lambda}$

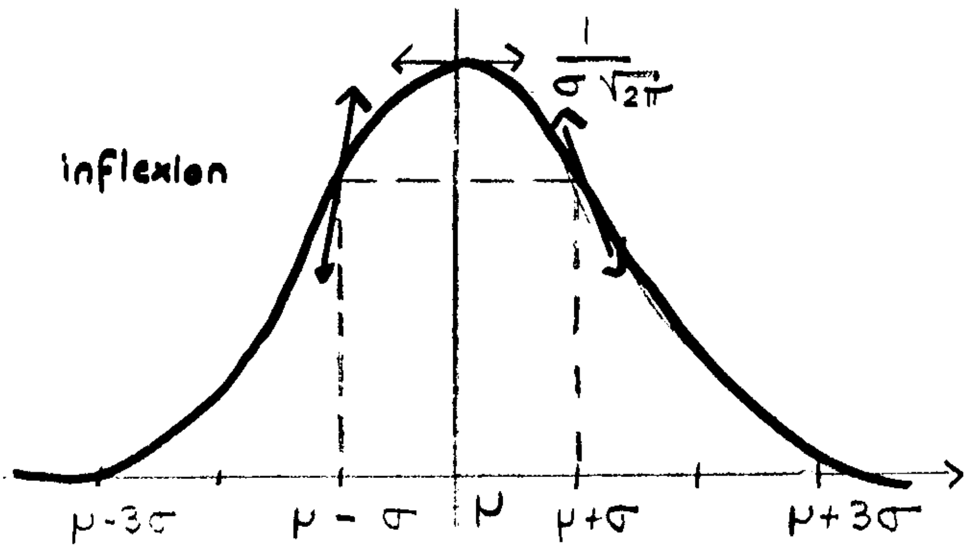
$\Leftrightarrow$ ( Si $X \sim \gamma_1$ : $\frac{X}{\lambda} \sim Exp(\lambda)$ ; $\lambda > 0$ )

(4) loi normale. $X \sim N(\mu; \sigma)$ $\sigma > 0$

moyenne espérance — écart type

Elle représente tout ce qui est "compliqué"

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}}$$



$N(0,1)$ = loi normale standard (ou centrée réduite)
densite $\varphi(x) = \frac{1}{\sqrt{2\pi}} \operatorname{Exp}\left\{-\frac{x^2}{2}\right\}$.

et sa fonction de repartition.

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt.$$

19/11 $X \sim N(\mu, \sigma)$

$$F(x) = P(X \leqslant x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-\frac{(t-\mu)^2}{2\sigma^2}\right\} dt$$

$$\frac{t-\mu}{\sigma} = u$$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\mu}{\sigma}} ex - \frac{u^2}{2} \,\cancel{\sigma}\, du.$$

$$F(x) = P(X \leqslant x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$X \sim N(\mu, \sigma)$ : $P(a \leqslant x \leqslant b)$

$$\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

$P(X \leqslant b) = \Phi\left(\frac{b-\mu}{\sigma}\right)$

$P(X > a) = 1 - \Phi\left(\frac{a-\mu}{\sigma}\right).$

<u>Exercice</u> : Table de $\mathcal{N}(0;1)$

$X \sim N(-3,2)$

$P(X<1) = \Phi\left(\frac{1-(-3)}{2}\right) = \Phi(2) = 0,977$

$P(X > -1) = 1 - \Phi(1) = 0,159$

$P(0<X<4) = \Phi(3,5) - \Phi(1,5) = 1 - 0,933 = 0,067$

$P(\ln |X| < 0) = \Phi$ (on transforme "L'évènement complexe" $\ln |x|$ en 1 évènement simple : $a < X < b$)

$P(\ln |X| < 0) = P(|X| < 1) = P(-1 < x < 1).$

$\Phi(2) - \Phi(1) = \ldots$

$P\left(\frac{|X|}{1+|X|} < 0,5\right) = P(|X| < 1) = P(-1 < X < 1).$

Trouver a tq $P(X > a) = 0{,}8 = 1 - \Phi\left(\frac{a+3}{2}\right)$

$$= \Phi\left(\frac{-a-3}{2}\right)$$

$$= \Phi(0{,}85) = 0$$

$$\frac{-a-3}{2} = 0{,}85$$

$$-a-3 = 1{,}7$$

$$a = -4{,}7.$$

<u>Exercice</u>

$X \sim N(\mu, \sigma)$

$$P(\mu - \ell < X < \mu + \ell) = \Phi\left(\frac{\mu+\ell-\mu}{\sigma}\right) - \Phi\left(\frac{\mu-\ell-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{\ell}{\sigma}\right) - \Phi\left(-\frac{\ell}{\sigma}\right)$$

$$= 2\,\Phi\left(\frac{\ell}{\sigma}\right) - 1$$

Règles des $3\sigma$ : $P(\mu - 3\sigma < X < \mu + 3\sigma)$

$$= \Phi\left(\frac{3\sigma - \mu + \mu}{\sigma}\right) - \Phi\left(\frac{\mu - 3\sigma - \mu}{\sigma}\right)$$

$$= \Phi(3) - \Phi(-3)$$

$$= 2\,\Phi(3) - 1 = 100\%.$$

"Ecart probable" = E

$P(|X-\mu| < E) = 0,5$.

$P(\mu - E < X < \mu + E) = 2\,\Phi\left(\frac{E}{\sigma}\right) - 1 = 0,5$

$2\,\Phi\left(\frac{E}{\sigma}\right) = 1,5$.

$\Phi\left(\frac{E}{\sigma}\right) = 0,75$

$\Phi(0,67) = \Phi\left(\frac{2}{3}\right)$

$E = \frac{2}{3}\sigma$

les notes à 1 examen suivent 1 loi normale
$N(10, 3)$

On peut affirmer que toutes les notes sont comprises entre
$1 < x < 19$.

Entre quelles notes se trouveront la moitié des notes ?
$8 < X < 12$