Université Nice-Sophia Antipolis
SMEMP302 - ECUE Probabilitic computational methods, 2023-2024
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Answers for final exam (2h30)

(1) We compute

$$\begin{aligned} X_t - \overline{X}_t &= \int_0^t b(X_s) - b(\overline{X}_{\underline{s}})ds \\ &= \int_0^t b(X_s) - b(\overline{X}_s)ds + \int_0^t b(\overline{X}_s)ds - \int_0^t b(\overline{X}_{\underline{s}})ds \,, \end{aligned}$$

then, by basic triangular inequality and mean value's inequality

$$\left|X_t - \overline{X}_t\right| \leq \left|\int_0^t b(X_s) - b(\overline{X}_s)ds\right| + \sup_{x\in\mathbb{R}} |b'(x)| \int_0^t |\overline{X}_s - \overline{X}_{\underline{s}}|ds \,.$$

(2)

(a) We use the integration by parts equation (for the processes $s \mapsto t_k - s$, $s \mapsto \int_{t_{k-1}}^s b'(X_u)dW_u$)

$$\begin{aligned} (t_k - t_k) \times \int_{t_{k-1}}^{t_k} b'(X_u)dW_u - (t_k - t_{k-1}) \times \int_{t_{k-1}}^{t_{k-1}} b'(X_u)dW_u = \\ - \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s b'(X_u)dW_u ds + \int_{t_{k-1}}^{t_k} (t_k - u)b'(X_u)dW_u + 0 \end{aligned}$$

(quadratic variation is zero).

(b) We compute, using Itô's formula, for $t_{k-1} \leq s \leq t_k$,

$$b(X_s) - b(X_{t_{k-1}}) = \int_{t_{k-1}}^s b'(X_u)(b(X_u)du + dW_u) + \frac{1}{2}\int_{t_{k-1}}^s b''(X_u)du \,.$$

So

$$\begin{aligned} \int_{t_{k-1}}^{t_k} b(X_s) - b(X_{t_{k-1}})ds = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s b'(X_u)b(X_u) + \frac{1}{2}b''(X_u)duds \\ + \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s b'(X_u)dW_u ds \,. \end{aligned}$$

We can use Fubini's Theorem on the first part (as b' and b'' are bounded):

$$\begin{aligned} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s b'(X_u)b(X_u) + \frac{1}{2}b''(X_u)duds &= \int_{t_{k-1}}^{t_k} \left(b'(X_u)b(X_u) + \frac{1}{2}b''(X_u)\right) \times \int_u^{t_k} dsdu \\ &= \int_{t_{k-1}}^{t_k} \left(b'(X_u)b(X_u) + \frac{1}{2}b''(X_u)\right)(t_k - u)du \,. \end{aligned}$$

For the second part, we use the previous question. So

$$\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s b'(X_u)dW_u ds = \int_{t_{k-1}}^{t_k} (t_k - u)b'(X_u)dW_u \,.$$

So we get

$$\begin{aligned} \int_{t_{k-1}}^{t_k} b(X_s) - b(X_{t_{k-1}})ds = \int_{t_{k-1}}^{t_k} \left(b'(X_u)b(X_u) + \frac{1}{2}b''(X_u)\right)(t_k - u)du \\ + \int_{t_{k-1}}^{t_k} (t_k - u)b'(X_u)dW_u \,. \end{aligned}$$

(3) We have

$$\begin{aligned}\left|\int_0^t b(X_s)-b(X_{\underline{s}})ds-\int_0^{\underline{t}} b(X_s)-b(X_{\underline{s}})ds\right| &= \left|\int_{\underline{t}}^t b(X_s)-b(X_{\underline{s}})ds\right| \\ \text{(triangular inequality)} &\leq \int_{\underline{t}}^t |b(X_s)|+|b(X_{\underline{s}})|ds \\ &\leq 2\sup_{x\in\mathbb{R}}|b(x)|\times\Delta t\,.\end{aligned}$$

(4)

(a) Let $0\leq k\leq N$. We deduce from the previous question that

$$\begin{aligned}\int_0^{t_k} b(X_s)-b(X_{\underline{s}})ds &= \sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} b(X_s)-b(X_{t_j})ds \\ &= \sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}(\underline{u}+\Delta t-u)\left(b'(X_u)b(X_u)+\frac{1}{2}b''(X_u)\right)du \\ &\qquad +\int_{t_j}^{t_{j+1}}(\underline{u}+\Delta t-u)b'(X_u)dW_u \\ &= \int_0^{t_k}(\underline{u}+\Delta t-u)\left(b'(X_u)b(X_u)+\frac{1}{2}b''(X_u)\right)du \\ &\qquad +\int_0^{t_k}(\underline{u}+\Delta t-u)b'(X_u)dW_u\,.\end{aligned}$$

(b) This is a consequence of question 3.

(5) Using the triangular inequality and the above question, we get

$$\begin{aligned}\sup_{u\in[0,T]}\left|\int_0^u b(X_s)-b(X_{\underline{s}})ds\right| &\leq 2\sup_{x\in R}|b(x)|\times\Delta t+\sup_{0\leq k\leq N}\left|\int_0^{t_k}(\underline{u}+\Delta t-u)\left(b'(X_u)b(X_u)+\frac{1}{2}b''(X_u)\right)du\right| \\ &\qquad +\sup_{0\leq k\leq N}\left|\int_0^{t_k}(\underline{u}+\Delta t-u)b'(X_u)dW_u\right|\end{aligned}$$

We compute the following bounds

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\leq k\leq N}\left|\int_0^{t_k}(\underline{u}+\Delta t-u)b'(X_u)dW_u\right|\right)^2 \\ &\qquad\leq\mathbb{E}\left(\sup_{u\in[0,T]}\left|\int_0^u(\underline{r}+\Delta t-r)b'(X_r)dW_r\right|\right)^2 \\ &\text{(Cauchy-Schwarz)}\leq\mathbb{E}\left(\sup_{u\in[0,T]}\left(\int_0^u(\underline{r}+\Delta t-r)b'(X_r)dW_r\right)^2\right) \\ &\text{(Burckholder-Davis-Gundy)}\leq\mathbb{E}\left(\int_0^T\Delta t^2\times|b'(X_r)|^2dr\right)\,.\end{aligned}$$

So we get

$$\begin{aligned}\mathbb{E}\left(\sup_{u\in[0,T]}\left|\int_0^u b(X_s)-b(X_{\underline{s}})ds\right|\right) &\leq 2\sup_{x\in R}|b(x)|\times\Delta t+\Delta t\int_0^T\mathbb{E}\left(\left|b(X_s)b'(X_s)+\frac{1}{2}b''(X_s)\right|\right)ds \\ &\qquad +\Delta t\sqrt{\int_0^T\mathbb{E}((b'(X_s))^2)ds}\,.\end{aligned}$$

(6) By question 1, we have for all $0 \le t \le T$,

$$\left|X_t - \overline{X}_t\right| \le \left|\int_0^t b(X_s) - b(\overline{X}_s)ds\right| + \sup_{x\in\mathbb{R}} |b'(x)| \int_0^t |\overline{X}_s - \overline{X}_{\underline{s}}|ds$$
$$\le \left|\int_0^t b(X_s) - b(\overline{X}_s)ds\right| + \sup_{x\in\mathbb{R}} |b'(x)| \int_0^t \sup_{u\in[0,s]} |\overline{X}_u - \overline{X}_{\underline{u}}|ds \,.$$

So there exists a constant C such that

$$z(t) \le C\left(\Delta t + \int_0^t z(s)ds\right) \,.$$

(7) From Gronwall's Lemma, we get, for all $0 \le t \le T$,

$$z(t) \le C\Delta t e^{CT} \,.$$

This proves the desired result.

(8)
```
import numpy as np
import scipy.stats as sps
T=1; x0=1
def b(x):
        return(np.sin(x))
def f(x):
        return(x)
def euler(N):
       N=int(N)
       z=sps.norm.rvs(size=(N))
       x=x0
       for k in range(N):
               x=x+b(x)*T/N+np.sqrt(T/N)*z[k]
       return(f(x))
```

(9)
```
 def mc(N,M):
        tab=np.ones(M)*N
return(np.mean(list(map(euler,tab))))
```