Université Nice-Sophia Antipolis
SMEMP302 - ECUE Probabilitic computational methods, 2022-2023
Sylvain Rubenthaler (`https://math.unice.fr/~rubentha/`)

Exercices (6)

We consider the Euler scheme of the Black-Scholes SDE

$$dX_t = X_t(rdt + \sigma dW_t)$$

with the following values for the parameters

$$X_0 = 100\,,\ r = 0.15\,,\ \sigma = 1.0\,,\ T = 1\,.$$

We consider the Euler scheme of this SDE with step $h = T/n$, namely

$$\overline{X}_{t_{k+1}} = \overline{X}_{t_k}(1 + rh + \sigma\sqrt{h}Z_{k+1})\,,\ \overline{X}_0 = X_0\,,$$

where $t_k = kh$, $k = 0, \dots, n$ and $(Z_k)_{1\le k\le n}$ is a sequence of i.i.d. variables of law $\mathcal{N}(0,1)$. We choose a coarse discretization step $n = 10$ so that $h = 1/10$.

We want to price a vanilla Call option with strike $K = 100$, i.e. to compute

$$C_0 = e^{-rt}\mathbb{E}((X_T - K)_+)$$

using a crude Monte Carlo simulation and a Richardson-Romber (RR) extrapolation with consistent Brownian increments. The Black-Scholes reference premium is $C_0^{BS} = 42.9571$. To equalize the complexity of the crude simulation and its RR extrapolated counterpart, we use M sample paths, $M = 2^k$, $k = 10, \dots, 19$ for the RR-extrapolated simulationand $3M$ for the crude Monte-Carlo simulation.

(1) Compute the crude Monte-Carlo approximation and RR-extrapolated simulation for $M = 2^k$, $k = 14, \dots, 2$. Show the results on a graph. The simulation is large enough so that, at its end, the observed error is approximatively representative of the residual bias. The result should look like Figure 0.1.

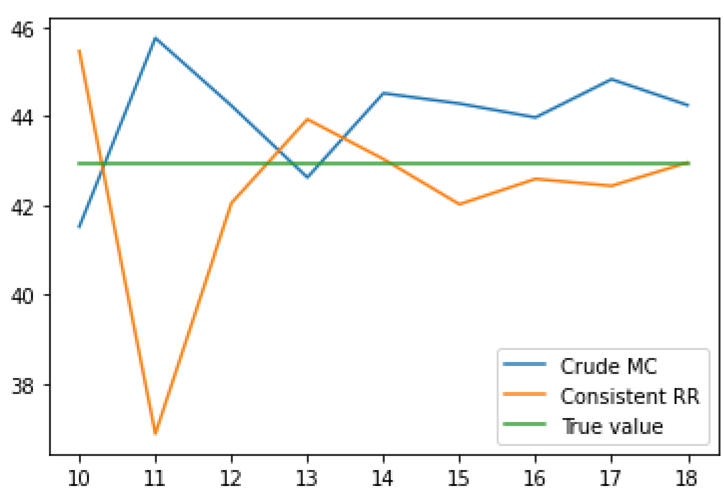

Figure 0.1. Crude MC vs consistent RR

(2) Let X, Y in $L^2(\Omega, \mathcal{A}, \mathbb{P})$.
   (a) Show that

$$|\mathrm{Cov}(X,Y)| \le \sigma(X)\sigma(Y) \text{ and } \sigma(X+Y) \le \sigma(X) + \sigma(Y)\,,$$

where $\sigma(X) = \sqrt{\mathrm{Var}(X)} = \|X - \mathbb{E}(X)\|_2$ (this is the standard deviation of X).

(b) Show that $|\sigma(X) - \sigma(Y)| \leq \sigma(X - Y)$. Deduce that, for every $\alpha \in (-\infty, 0] \cup [1, +\infty)$, in the case where $\sigma(X) = \sigma(Y)$:

$$\mathrm{Var}(\alpha X + (1-\alpha)Y) \geq \mathrm{Var}(X)\,.$$

(c) Deduce that consistent Brownian increments produce the RR-extrapolated simulation with the lowest asymptotic variance as n goes to infinity.

(3) In the above numerical illustration, carry on testing the RR extrapolation based on Euler scheme vs crude Monte-Carlo simulation with steps T/n and $T/(2n)$, $n = 5, 10, 20, 50$, respectively with
- independant Brownian increments,
- consistent Brownian increments.

Compute the estimator of the variance of the estimators in both settings and compare the obtained results. The result could look like Figure

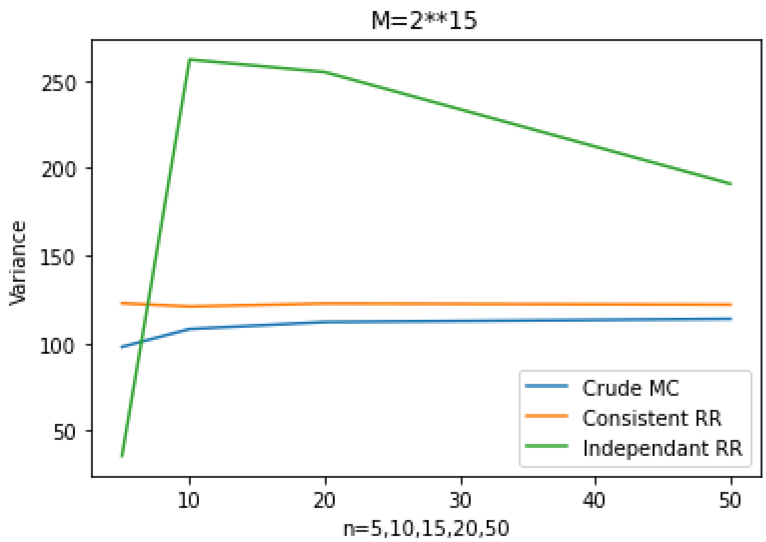

FIGURE 0.2. Variances