

7.2 Weak convergence

We want to compute $\mathbb{E}(f(X_T))$ for f Lipschitz (of constant k). We have

$$\begin{aligned}
 |\mathbb{E}(f(X_T)) - \mathbb{E}(f(\bar{X}_T^N))| &\stackrel{\text{Euler scheme}}{\leq} \mathbb{E}(|f(X_T) - f(\bar{X}_T^N)|) \\
 &\leq \mathbb{E}(|f(X_T) - f(\bar{X}_T^N)|^2)^{\frac{1}{2}} \\
 &\leq k \mathbb{E}(|X_T - \bar{X}_T^N|^2)^{\frac{1}{2}} \\
 &\stackrel{\uparrow}{\leq} \frac{k}{\sqrt{N}}
 \end{aligned}$$

Th. on strong error rate.

But we loose too much in  and we can do better.

Th. (Talay & Tubaro, 1990)

We suppose b and σ are C^∞ on $(0, T) \times \mathbb{R}^n$ with all derivatives bounded and that $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is C^∞ with derivatives at most polynomial, i.e.

$\forall a = (a_1, \dots, a_n) \in \mathbb{N}^n, \exists p, C > 0, \forall x \in \mathbb{R}^n$

$$\left| \frac{\partial^{a_1 + \dots + a_n} f}{\partial x_1^{a_1} \dots \partial x_n^{a_n}}(x) \right| \leq C(1 + |x|^p)$$

then \exists a sequence $(C_l)_{l \geq 1}$ of numbers $\forall s. t. : \forall l \in \mathbb{N}^*$,
we have : $\mathbb{E}(f(\bar{X}_T^N)) - \mathbb{E}(f(X_T)) = \frac{C_1}{N} + \frac{C_2}{N^2} + \dots + \frac{C_l}{N^l} + o\left(\frac{1}{N^{l+1}}\right)$

Remark: under these assumptions, $\beta = \frac{1}{2}$
so we have an improvement

EDO:

$$\left(\begin{aligned} \frac{\partial u}{\partial t}(t, x) + \frac{1}{2} \sum_{i,j=1}^m a_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j}(t, x) \\ + \sum_{i=1}^m b_i(t, x) \frac{\partial u}{\partial x_i}(t, x) = 0 \quad (t, x) \in [0, T] \times \mathbb{R}^n \\ u(T, x) = f(x), x \in \mathbb{R}^n \end{aligned} \right)$$

where $a(t, x) = \sigma(t, x) \sigma^*(t, x)$

$$(\forall i, j \in \{1, \dots, m\}, a_{ij}(t, x) = \sum_{l=1}^n \sigma_{il}(t, x) \sigma_{jl}(t, x))$$

Prop: Under assumptions of Th. [TT90], EDO (♥)has a unique solution. This sol. is C^∞ with derivatives at most polynomial. If (X_t) is an SDE sol.
 $(dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t)$ then $(u(t, X_t))_{t \in [0, T]}$ is a square integrable martingale with terminal value $f(X_T)$ and $u(0, X_0) = \mathbb{E}(f(X_T))$ (♥)
Proof: We do not prove existence, unicity, regularity of the EDO solution. We apply Itô's formula:

$$\begin{aligned} u(t, X_t) &= u(0, X_0) + \int_0^t \nabla_x u(s, X_s) \cdot \sigma(s, X_s) dW_s \\ &\quad + \int_0^t \underbrace{\left[\frac{\partial u}{\partial t} + \frac{1}{2} \sum_{i,j=1}^m a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + b \cdot \nabla_x u \right]}_{=0 \text{ by } (♥)}(s, X_s) ds \end{aligned}$$

The function $x \mapsto \nabla_x u(s, x) \cdot \sigma(s, x)$ is at most polynomial.

So, by Prop [polynomial moments], we have

$$\mathbb{E} \left(\int_0^T |\sigma^*(s, X_s) \nabla_x u(s, X_s)|^2 ds \right) < \infty \quad \text{so}$$

 $u(t, X_t) = u(0, X_0) + \int_0^t \nabla_x u(s, X_s) \cdot \sigma(s, X_s) dW_s$ is a square

integrable martingale so: $E(u(T, X_T)) = u(0, X_0)$
 $E(f(X_T)) = u(0, X_0)$

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□

Proof of Th. [TT50]

We write the proof only for $d=1$. We are going to show only c. (first term of the development). By Equation

$$E(f(\bar{X}_T)) - E(f(X_T))$$

$$\stackrel{''}{=} E(u(T, \bar{X}_T)) - u(0, X_0)$$

$$\stackrel{''}{=} E(u(T, \bar{X}_T) - u(0, \bar{X}_0)) = \sum_{k=0}^{N-1} E_k$$

$$\text{where } E_k = E(u(t, \bar{X}_{t_{k+1}}) - u(t, \bar{X}_{t_k}))$$

On $[t_k, t_{k+1})$, $dX_t = r(t_k, \bar{X}_{t_k}) dW_t + b(t_k, \bar{X}_{t_k}) dt$, Itô's formula gives us:

$$\begin{aligned} u(t_{k+1}, \bar{X}_{t_{k+1}}) - u(t_k, \bar{X}_{t_k}) &= \int_{t_k}^{t_{k+1}} \sigma(t_k, \bar{X}_{t_k}) \frac{\partial u}{\partial x}(t, \bar{X}_t) dW_t \\ &+ \int_{t_k}^{t_{k+1}} \left(\frac{\partial u}{\partial t}(t, \bar{X}_t) + \frac{1}{2} \sigma^2(t_k, \bar{X}_{t_k}) \frac{\partial^2 u}{\partial x^2}(t, \bar{X}_t) \right. \\ &\quad \left. + b(t_k, \bar{X}_{t_k}) \frac{\partial u}{\partial x}(t, \bar{X}_t) \right) dt \end{aligned}$$

$\left(\dots \right) \frac{\partial u}{\partial x}(\dots)$ is at most polynomial } \rightarrow The integral in dW_t has null expectation
 + Prop (polynomial moments)

$$\text{So } E_k = \int_{t_k}^{t_{k+1}} E(v_k(t, \bar{X}_t)) dt$$

$$\text{with } v_k(t, x) = \frac{\partial u}{\partial t}(t, x) + \frac{1}{2} \sigma_k^2 \frac{\partial^2 u}{\partial x^2}(t, x) + b_k \frac{\partial u}{\partial x}(t, x)$$

$$\text{where } \sigma_k = \sigma(t_k, \bar{X}_{t_k}), b_k = b(t_k, \bar{X}_{t_k})$$

As u is solution of $\textcircled{1}$ then $u(t_k, \bar{X}_{t_k}) = 0$.

We apply Itô's formula: $(t_k \leq t \leq t_{k+1})$

$$\mathbb{E}(u(t, \bar{X}_t)) = \int_{t_k}^{t_{k+1}} \mathbb{E} \left(\frac{\partial u_k}{\partial t}(s, \bar{X}_s) + \frac{1}{2} \sigma_k^2 \frac{\partial^2 u_k}{\partial x^2}(s, \bar{X}_s) + b_k \frac{\partial u_k}{\partial x}(s, \bar{X}_s) \right) ds dt$$

We have, \forall integrable g :

$$\int_{t_k}^{t_{k+1}} \int_{t_k}^t g(s) ds dt \stackrel{\text{FUBINI}}{=} \int_{t_k}^{t_{k+1}} (t_{k+1} - s) g(s) ds$$

So: $\mathcal{E}_k = \int_{t_k}^{t_{k+1}} (t_{k+1} - s) \mathbb{E} \left(\frac{\partial u_k}{\partial t}(s, \bar{X}_s) + \frac{1}{2} \sigma_k^2 \frac{\partial^2 u_k}{\partial x^2}(s, \bar{X}_s) + b_k \frac{\partial u_k}{\partial x}(s, \bar{X}_s) \right) ds$

Using the regularity of u, σ, b , the $\mathbb{E}(\dots)$ is bounded and so:

$$|\mathcal{E}_k| \leq C \int_{t_k}^{t_{k+1}} (t_{k+1} - s) ds = \frac{C T^2}{2 N^2}$$

(for some C)

We sum the errors: $|\mathbb{E}(u(T, \bar{X}_T) - u(0, \bar{X}_0))| \leq \frac{C T^2}{N}$

To get an equivalent, we write, $\forall s \in (t_k, t_{k+1}]$:

$$\begin{cases} \frac{\partial u_k}{\partial t}(s, \bar{X}_s) = \frac{\partial u_k}{\partial t}(t_k, \bar{X}_{t_k}) + o\left(\frac{1}{N}\right) \\ \frac{\partial^2 u_k}{\partial x^2}(s, \bar{X}_s) = \dots \\ \dots \end{cases} \quad \text{(again by Itô's formula)}$$

We get $\mathcal{E}_k \approx \frac{T^2}{2N^2} \mathbb{E} \left[\left(\frac{\partial^2 u}{\partial t^2} + \sigma_k^2 \frac{\partial^3 u}{\partial t \partial x^2} + 2b_k \frac{\partial^3 u}{\partial t \partial x} + \frac{\sigma_k^4}{4} \frac{\partial^4 u}{\partial x^4} + \sigma_k^2 b_k \frac{\partial^3 u}{\partial x^3} + b_k^2 \frac{\partial^2 u}{\partial x^2} \right) (t_k, \bar{X}_{t_k}) \right]$

In the end:

$$E(f(X_T)) - E(f(X_0)) \sim \frac{T}{2N} \int_0^T E \left[\left(\frac{\partial^2 u}{\partial t^2} + \sigma^2 \frac{\partial^2 u}{\partial x^2} + 2b \frac{\partial^2 u}{\partial t \partial x} + \frac{\sigma^4 \partial^4 u}{4 \partial x^4} + \sigma^2 b \frac{\partial^3 u}{\partial x^3} + b^2 \frac{\partial^3 u}{\partial x^3} \right) (t, X_t) \right] dt$$

□

7.3 Milstein's scheme (dimension 1)

We always have the same equation:

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt, \quad X_0 = x_0$$

On $[t_k, t_{k+1}]$: $X_t = X_{t_k} + \int_{t_k}^t \sigma(X_s) dW_s + \int_{t_k}^t b(X_s) ds$

Itô's formula:

We want to improve the approximation of this term.

$$\sigma(X_t) = \sigma(X_{t_k}) + \int_{t_k}^t [\sigma'(X_s) b(X_s) + \frac{1}{2} \sigma''(X_s) \sigma^2(X_s)] ds + \int_{t_k}^t \sigma'(X_s) \sigma(X_s) dW_s$$

$$\int_{t_k}^t \sigma(X_s) dW_s = \underbrace{\int_{t_k}^t \sigma(X_{t_k}) dW_s}_{\textcircled{1}} + \underbrace{\int_{t_k}^t \int_{t_k}^s \sigma(X_u) \sigma'(X_u) dW_u dW_s}_{\textcircled{2}} + \int_{t_k}^t \int_{t_k}^s \underbrace{(\dots)}_{\substack{\text{bounded} \\ \text{or at most} \\ \text{linear growth}}} du ds$$

↳ in L^2 norm
This is a $O((t_{k+1} - t_k)^2)$

so we denote it by $O_{L^2}((t_{k+1} - t_k)^2)$
dérivée de σ'

$$\textcircled{1} = \sigma(X_{t_k}) \times (W_t - W_{t_k})$$

$$\textcircled{2} = \int_{t_k}^t \int_{t_k}^s [\sigma(X_{t_k}) \sigma'(X_{t_k}) + \sigma(X_{t_k}) (u - t_k)] dW_u dW_s$$

$$= \int_{t_k}^t \sigma(X_{t_k}) \sigma'(X_{t_k}) (W_s - W_{t_k}) dW_s + \sigma(X_{t_k}) \int_{t_k}^t \int_{t_k}^s (u - t_k) dW_u dW_s$$

Ito's formula:

$$\frac{(W_t - W_{t_k})^2 - (t - t_k)}{2} = 0 + \int_{t_k}^t (W_s - W_{t_k}) dW_s - \int_{t_k}^t \frac{1}{2} ds + \frac{1}{2} \int_{t_k}^t d\langle W \rangle_s = 0$$

So: $\int_{t_k}^t W_s - W_{t_k} dW_s$
 \parallel
 $\frac{(W_t - W_{t_k})^2 - (t - t_k)}{2}$

And we have: $\int_{t_k}^t \int_{t_k}^s (u - t_k) dW_u dW_s$
 \downarrow

$$\begin{aligned} E(\dots)^2 &= \int_{t_k}^t E\left(\int_{t_k}^s (u - t_k) dW_u\right)^2 ds \\ &= \int_{t_k}^t \int_{t_k}^s |u - t_k|^2 du ds \\ &= \int_{t_k}^t \frac{(s - t_k)^3}{3} ds \\ &= \frac{(s - t_k)^4}{12} \end{aligned}$$

All in all: ② = $\frac{(W_t - W_{t_k})^2 - (t - t_k)}{2} + O_P((t_{k+1} - t_k)^2)$

$$\begin{aligned} \int_{t_k}^t h(X_s) ds &= \int_{t_k}^t h(X_{t_k}) + h'(X_{t_k})(s - t_k) + O(|s - t_k|^2) ds \\ &= h(X_{t_k})(t - t_k) + h'(X_{t_k})\frac{(t - t_k)^2}{2} + O((t - t_k)^3) \end{aligned}$$

SCHEME:

$$\begin{aligned} \bar{X}_t^{\text{mil},n} = & \bar{X}_t^{\text{mil}} + \left[b(\bar{X}_t^{\text{mil}}) - \frac{1}{2} \sigma \sigma'(\bar{X}_t^{\text{mil}}) \right] (t - t) \\ & + \sigma(\bar{X}_t^{\text{mil}}) (W_t - W_t) + \frac{1}{2} \sigma \sigma'(\bar{X}_t^{\text{mil}}) (W_t - W_t)^2 \\ & + \frac{1}{2} \sigma \sigma'(\bar{X}_t^{\text{mil}}) (W_t - W_t)^2 \end{aligned} \quad \left[\frac{45}{2} \right]$$

Th (Strong L^p rate for the Milstein scheme)

Assume b and σ are bounded, α_b and α_σ -Hölder continuous first derivatives ($\alpha := \min(\alpha_b, \alpha_\sigma)$). Then

$$\| \max_{0 \leq k \leq n} |X_{t_k} - \bar{X}_{t_k}^{\text{mil},n}| \|_p$$

$$\begin{aligned} & \Rightarrow \\ \| \sup_{t \in [0, T]} |X_t - \bar{X}_t^{\text{mil},n}| \|_p \\ & \Rightarrow \end{aligned}$$

$$C_{b, \sigma, T, p} (1 + \|X_0\|_p) \left(\frac{T}{n} \right)^{\frac{1+\alpha}{2}}$$

↓

For b and σ 1-Lip, we get $\left(\frac{1}{n} \right)^{\frac{1}{2}}$. This has to be compared with the: $\frac{1}{N^{1/2}}$ obtained for the regular Euler scheme under the same assumptions.

7.4 Richardson-Romberg Extrapolation with consistent Brownian increments

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We set $W^{(1)} = W$, $X^{(1)} = X$, $X_0^{(1)} = X_0$. A regular Monte-Carlo simulation based on M independent copies $(\bar{X}_T^{(1)})^m$ ($m=1, \dots, M$) of the Euler scheme $\bar{X}_T^{(1)}$ with step $\frac{T}{m}$ induces the following global (squared) quadratic error

$$\left\| \mathbb{E}(f(X_T)) - \frac{1}{M} \sum_{m=1}^M f(\bar{X}_T^{(1)})^m \right\|_2^2$$

=

$$\left(\mathbb{E}(f(X_T)) - \mathbb{E}(f(\bar{X}_T^{(1)})) \right)^2 + \left\| \mathbb{E}(f(\bar{X}_T^{(1)})) - \frac{1}{M} \sum_{m=1}^M f(\bar{X}_T^{(1)})^m \right\|_2^2$$

+ 0 (because Euler Scheme error and MC error are independent)

$$= \left(\frac{C_1}{m} \right)^2 + \frac{\text{Var}(f(\bar{X}_T^{(1)}))}{M} + o\left(\frac{1}{m^3}\right)$$

This is a bias-variance decomposition of the approximation error of the Monte-Carlo estimator.

Richardson-Romberg extrapolation

Strong solution $X^{(2)}$ of a "copy" of $X^{(1)}$, driven by a second Brownian motion $W^{(2)}$ and starting from $X_0^{(2)}$ (ind. of everybody, with same distribution as $X_0^{(1)}$).

Then we consider the Euler scheme with a twice smaller step $\frac{T}{2m}$, denoted by $\bar{X}^{(2)}$ associated to $X^{(2)}$ (i.e. starting from $X_0^{(2)}$, with Brownian increments built from $W^{(2)}$).

Combining the two time discretization errors related to $X^{(1)}$

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$$\begin{aligned} & \mathbb{E} (2f(\bar{X}_T^{(2)}) - f(\bar{X}_T^{(1)})) - \mathbb{E} (f(X_T)) = \\ & 2 [\mathbb{E} (f(\bar{X}_T^{(1)})) - \mathbb{E} (f(X_T))] - [\mathbb{E} (f(\bar{X}_T^{(1)})) - \mathbb{E} (f(X_T))] = \\ & 2 \left(\frac{C_1}{2n} + \frac{C_2}{4n^2} + o\left(\frac{1}{n^3}\right) \right) - \left(\frac{C_1}{n} + \frac{C_2}{n^2} + o\left(\frac{1}{n^3}\right) \right) = \\ & -\frac{C_2}{2n^2} + o\left(\frac{1}{n^3}\right) \end{aligned}$$

Then the new global (squared) quadratic error becomes

$$\left\| E(f(X_T)) - \frac{1}{M} \sum_{m=1}^M 2f(\bar{X}_T^{(m)}) - f(\bar{X}_T^{(1)}) \right\|_2^2$$

$$\approx \left(\frac{c_2}{2m^2} \right)^2 + \frac{\text{Var}(2f(\bar{X}_T^{(1)}) - f(\bar{X}_T^{(1)}))}{M} + o\left(\frac{1}{m^5}\right)$$

Can we control the variance term $\text{Var}(2f(\bar{X}_T^{(2)}) - f(\bar{X}_T^{(1)}))$?

Consistent simulation (of the Brownian increments)

If $w^{(i)} = w$ ($i=1,2$) and $x_0^{(i)} = x_0$ ($i=1,2$) then

$$\text{Var}(2f(\bar{X}_T^{(1)}) - f(\bar{X}_T^{(2)})) \xrightarrow{(n \rightarrow \infty)} \text{Var}(2f(X_T) - f(X_T))$$

" "

$$\text{Var}(f(X_T))$$

since the Euler schemes $\bar{X}^{(i)}$ ($i=1, 2$) both converge in $\mathbb{E}(\mathbb{P})$ to X .