III Fonction d'1 v.a.

A Calcul de la densité $g(u)$ de $U = u(X)$.

X : densité $f(x)$, supp(X).

$U = u(x)$: on cherche d'abord les supp (U)

L'application u est-elle 1 bijection de Supp (X) dans Supp(U)

1 Si u n'est pas bijection

1) Exprimer la f.r de U, G, à l'aide de la f.r. de X, F

2) $G(u) = P(U \leq u) = P(u(X) \leq u)$.

(on transforme l'év. complexe en événement simple)

$= P(a(u) \leq X \leq b(u)) = F[b(u)] - F(a(u))$

$$g(u) = G'(u) = f[b(u)] \times b'(u) - f(a(u)) \times a'(u).$$

3) Ne pas oublier l'indicatrice du support de U.

2 Si u est 1 bijection : (dérivable dans les 2 sens)

$u = u(x) \Longleftrightarrow x = x(u)$	$\lvert x'(u) \rvert = \left\lvert \dfrac{dx}{du} \right\rvert = \dots$
$g(u) = f(x(u)) \times \left\lvert \dfrac{dx}{du} \right\rvert$	$\mathbb{1}_{\text{Supp}(U)}(u)$

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B) Exemple.

1) $X \sim \mathcal{U}(0,1)$
$U = \ln X$

- $f(x) = \mathbb{1}_{(0,1)}(x)$ $\quad$ $supp(X) = (0,1)$.

$u = \ln x$ $\qquad$ $supp(U) = \mathbb{R}^{*-}$

- $u = \ln x \iff x = e^{u}$ $\qquad$ $\frac{dx}{dx} = e^{u}$

$\left|\frac{dx}{du}\right| = e^{u}$

- $g(u) = f[e^{u}] \times e^{u}$ $\qquad$ $\mathbb{1}_{\mathbb{R}^{*,-}}(u) = e^{u} \mathbb{1}_{\mathbb{R}^{*,-}}(u)$

2) $f(x) = \frac{3x^2}{2}$ $\qquad$ $supp(X) = (-1,1)$

$U = X^2 = u(X)$

- $U = X^2$ $\qquad$ $supp(U) = (0,1)$

- $G(U) = P(U \leqslant u) = P(u(X) \leqslant u) = P(X^2 < u)$
$P[-\sqrt{u} < X < \sqrt{u}] = F(\sqrt{u}) - F(-\sqrt{u})$.

- $g(u) = f(\sqrt{u}) \times \frac{1}{2\sqrt{u}} + f(-\sqrt{u}) \times \frac{1}{2\sqrt{u}}$

$= \frac{1}{2\sqrt{u}}\left(\frac{3u}{2} + \frac{3u}{2}\right) = \frac{1}{\sqrt{u}}\left(\frac{6u}{4}\right) = \frac{3}{2}\sqrt{u}$

$\mathbb{1}_{(0,1)}(u)$

③ $f(x) = \frac{1}{125} e^{-\frac{x^2}{5}} \times x^5 \mathbb{1}_{(0;+\infty)}(x)$, $\text{supp}(X) = \mathbb{R}^{*,+}$

$U = \frac{X^2}{5}$ $\qquad$ $\text{supp}(U) = \mathbb{R}^{*,+}$

– $u = \frac{x^2}{5} \iff x = \sqrt{5}\sqrt{u}$ $\qquad$ $\left|\frac{dx}{du}\right| = \frac{1}{2\sqrt{u}} \times \sqrt{5}$

– $g(u) = f(\sqrt{5}\sqrt{u}) \times \frac{\sqrt{5}}{2\sqrt{u}} \mathbb{1}_{\mathbb{R}^{*,+}}(u)$

$= \frac{1}{125} e^{-u} \times 5^{5/2} \times u^{5/2} \times \frac{\sqrt{5}}{2\sqrt{u}} \mathbb{1}_{\mathbb{R}^{*,+}}(u)$

$= \frac{1}{125} e^{-u} \times 5^3 u^2 \mathbb{1}_{\mathbb{R}^{*,+}}(u)$

$= e^{-u} \frac{u^2}{2} \mathbb{1}_{\mathbb{R}^+}(u) \sim \gamma(3)$.

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Ⓒ Exemple fondamentaux.

1) loi uniformes

$X \sim \mathcal{U}(a,b)$ $\qquad$ $f(x) = \frac{1}{b-a}$ $\qquad$ $\text{Supp}(X) : (a;b)$

$U = cX + d$ $\qquad$ $\text{Supp}(U) : (ca+d ; cb+c)$
(si $c > 0$)

$u = cx + d \iff \frac{u-d}{c}$ $\qquad$ $\left|\frac{dx}{du}\right| = \left|\frac{1}{c}\right|$

$$g(u) = f\left(\frac{u-d}{c}\right) \times \frac{1}{|c|} = \frac{1}{b-a} \times \frac{1}{|c|} \quad \mathbb{1}_{(ca+d,\, cb+d)}(u)$$

Theoreme :

les lois uniformes st stables par transformation affine.

exemple

$X \sim U(-1,1)$

$U = 2X-1 \sim U(-3;1)$.

$X \sim U(0\ 2)$

$U = \frac{X}{2} \sim U(0;1)$

2) lois normales

$X \sim N(\mu, \sigma)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \operatorname{Exp}\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \qquad \operatorname{supp}(X) = \mathbb{R}.$$

$$U = aX+b. \qquad \operatorname{Supp}(U) = \mathbb{R}$$

$$u = ax+b \iff x = \frac{u-b}{a} \qquad \left|\frac{dx}{du}\right| = \frac{1}{|a|}$$

$$g(u) = f\left(\frac{u-b}{a}\right) \times \frac{1}{|a|}$$

$$= \frac{1}{|a|\sigma\sqrt{2\pi}} \operatorname{Exp} - \left\{\frac{\left(\frac{u-b}{a}-\mu\right)^2}{2\sigma^2}\right\} = \frac{1}{|a|\sigma\sqrt{2\pi}} \exp\left\{-\frac{[u-(a\mu+b)]^2}{2|a|^2\sigma^2}\right\}$$

$\sim \mathcal{N}(a\mu+b;\ |a|\sigma)$

Theoreme
Les lois normales st stables par transformation affine.

Exercice

$$X \sim \mathcal{N}(\mu,\sigma) \Rightarrow \frac{X-\mu}{\sigma} \sim \mathcal{N}(0;1)$$

$$X \sim \mathcal{N}(0;1) \Rightarrow \sigma X+\mu \sim \mathcal{N}(\mu,\sigma).$$

Rappel :
$E(aX+b) = a\,E(X)+b.$
$V(aX+b) = a^2\,V(X).$

ε écart type $\sqrt{\sigma^2}$

* $X \sim \mathcal{N}(-2;3)$

$U = 5X-1 \sim \mathcal{N}(-11,\ 15)$

* $X \sim \mathcal{N}(-1,1)$
$U = X+1 \sim \mathcal{N}(0,1)$

3) Lois gammas

$X \sim \gamma(\alpha)$

$$f(x) = \frac{1}{\Gamma(\alpha)} \exp(-x)\, x^{(\alpha-1)} \qquad \mathrm{Supp}(X) = \mathbb{R}^{+,*} = (0;+\infty)$$

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$U = \frac{X}{\lambda}$ $(\lambda > 0)$ $\qquad$ $Supp(U) = \mathbb{R}^{*,+}$

$u = \frac{x}{\lambda} \iff \lambda x = \lambda u$ $\qquad$ $\left|\frac{dx}{du}\right| = \lambda$

$g(u) = f(\lambda u) \times \lambda$

$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \cdot e^{-\lambda u} u^{\alpha - 1} \quad \mathbb{1}_{\mathbb{R}^+}(u)$

$\frac{\gamma_\alpha}{\lambda} : \frac{\lambda^{\alpha}}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha - 1} \quad \mathbb{1}_{\mathbb{R}^+}(x)$

Application

$$\boxed{\frac{\gamma_1}{\lambda} = Exp(\lambda)}$$

$X \sim Exp(2)$

$U = \frac{X}{3} \sim Exp(6)$

④ loi du khi-deux à 1 degré de liberté (ddl)
$\chi^2(1)$.

Definition
$X \sim \mathcal{N}(0, 1) \Rightarrow X^2 \sim \chi^2(1)$

$$\boxed{\chi^2(1) = N^2(0,1)}$$

$X \sim N(0\ 1)$

$\varphi(x) = \frac{1}{\sqrt{2\pi}} \exp^{-\frac{x^2}{2}}$ $\quad$ $\mathrm{Supp}(X) = \mathbb{R}$

$U = X^2$ $\quad$ $\mathrm{Supp}(U) = \mathbb{R}^+$

$u = x^2 \iff$

$$G(u) = P(U \leqslant u) = P(X^2 \leqslant u) = P(-\sqrt{u} \leqslant X \leqslant \sqrt{u})$$
$$\Phi(\sqrt{u}) - \Phi(-\sqrt{u}).$$
$$\Phi(\sqrt{u}) - (1 - \Phi(\sqrt{u}))$$
$$2\,\Phi(\sqrt{u}) - 1$$

$$g(u) = 2\,\varphi(\sqrt{u})\,\frac{1}{2\sqrt{u}}$$

$$= \frac{1}{\sqrt{u}}\,\varphi(\sqrt{u})$$
$$= \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}u}\,u^{\frac{1}{2}-1} \qquad \mathbb{1}_{\mathbb{R}^+}(u)$$

$$\boxed{\chi^2(1) = 2\,\gamma(\tfrac{1}{2}).}$$

$$\frac{1}{\sqrt{2\pi}} = \frac{(\frac{1}{2})^{\frac{1}{2}}}{\Gamma(\frac{1}{2})}.$$