Sharing a Government? A Spatial Model of Centralisation Versus Decentralisation

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Abstract

I incorporate a spatial dimension to the analysis in Ventura (2019) to account for the fact that externalities will depend on how far apart regions are. Using this new framework, I perform a positive and a normative analysis on which parliaments will/should have decision powers upon projects. I find that regions no longer unanimously agree when voting the fiscal constitution, and the delegation of decision powers to the union/regional parliament(s) can be suboptimal. Voting the constitution under an hypothetical veil of ignorance could solve problems in a world where there are a large number of regions and where the representative of a region has rational expectations.

1 Introduction

When should a group of regions delegate decision powers to a union parliament? When will this happen? Will all regions agree to (do not) form a union? Space may be an important factor when seeking an answer to these questions. For instance, if China decides to enhance its connectivity to Europe by constructing a 21st-century silk road, economic spillovers on Kazakhstan will probably be larger than on New Zealand. On a smaller scale, if the Autonomous Community of Madrid lowers certain taxes, this will likely affect left-wing high-taxes regions and right-wing low-taxes regions asymmetrically.

In this paper, I enrich the framework of Ventura (2019)[3] by including the notion of space. More specifically, I add a previous stage to the three-stage model of Ventura (2019)[3], so that I have a four-stage model in which, first, nature randomly places regions onto some particular space; second, regions vote a constitution that assigns decision powers to either the union or the regional parliament; third, random payoffs for each project are realized; and, fourth, the union/regional parliaments (do not) implement a certain policy type depending on which chamber has decision powers. Introducing a previous stage in which nature places regions onto space allows me to separate regions by some notion of distance, which in turn allows me to express externalities as a function of distances.

I study three things by using a shortcut that makes the model more tractable and that (I believe) does not take us far from reality: that the externality that region j suffers from i is proportional to the benefits that that i enjoys from implementing its project. First, I study how the set of efficient projects depends on the isolation of a region. Secondly, I study the behavior of either chamber in the case they have decision powers and conditional on project-by-project voting. I show how the behavior of regional parliaments remains unaffected by the introduction of distances, whereas the distribution of distances is key to predict the behavior of the union parliament. Third, I study how regions will behave when voting the constitution, under the assumption that decision powers are assigned to either of the two type of chambers by majority voting. A result that arises from this simple framework is that regions can inefficiently assign decision powers if the preferences of the median voter do not coincide with those of the average voter. However, I also show how the veil of ignorance (i.e.

the hypothetical situation in which regions vote the constitution before being drawn onto the space) solves inefficiencies if the number of regions is large and the representative region has "rational expectations".

2 The model

There are $i = \{1, 2...N\}$ equally sized regions, each with one project of the same type[1]. To avoid situations in which majority voting leads to draws, assume $3 \leq N < \infty$ is an odd number. Each region is randomly placed onto a generic space of dimension d, $\mathbb{R}^d$; and any two regions i, j $(i \neq j)$ are separated by distance $d_{ij} > 0$. The space can intuitively be thought as the geographical space $(d = 2)$, although d can arbitrarily be larger to accommodate for (potentially) more complex spaces, like ideology.

Each region has one regional parliament, and there is one union parliament in addition. Regional parliaments are self-financed by each region, whereas the union parliament is financed according to a proportionality principle: economically larger regions contribute more to the union budget than smaller regions. To keep things simple, I will assume that regions are equally sized so that each region contributes the same to the union parliament.

The project in region i is a triplet composed by (i) an *ex-ante* known total cost $0 < \mathcal{C}_i < \infty$, (ii) a random benefit $\beta_i \in \mathbb{R}$ that accrues to region i, and (iii) a series of random externalities $\{\varepsilon_{ij}\}_{j \neq i}$ such that $\varepsilon_{ij} = f_{ij}(d_{ij}) \in \mathbb{R}$, i.e. the externality that neighbor j suffers from i's project is a function of the distance between i and j. Benefits and externalities for region i are drawn from a joint probability distribution with cdf $F_i \equiv F_i(a_1, a_2, ...a_N) = P(\beta_i \leq a_1, \varepsilon_{ij_1} \leq a_2, ...\varepsilon_{ij_{N-1}} \leq a_N)$. If the project is to be undertaken by the region, it will be the regional parliament the only one that decides upon implementation and the one assumes the cost. However, if the project is to be undertaken by the union parliament, each region will finance $\frac{\mathcal{C}_i}{N}$ part of the cost, and the (dis)approval of the project will be subject to a simple-majority vote in which there are N representatives – one for each region.

Regions decide, through majority voting, whether each region i will have the powers to undertake their own project or, on the contrary, whether the union parliament will have the

[1]The analysis that I will make is invariant to the number of projects that there is.

decision powers. More specifically, the model has the following four stages:

- $\underline{t=0}$. Nature randomly draws the position of each region $i = \{1, 2...N\}$ in the space.
- $\underline{t=1}$. Regions vote on which parliament will have decision powers.
- $\underline{t=2}$. Benefits and externalities are realized for each region $i = \{1, 2...N\}$.
- $\underline{t=3}$. Regional parliaments/union parliament decide whether to implement the project or not, depending on which one had been assigned decision powers.

3 Efficient implementation

Following a utilitarian approach, I will consider that the project of region i should be implemented if the total payoff of the project is positive, which leads to the following set of efficient projects for region i:

$$S_i^E = \{(\mathcal{C}_i, \beta_i, \{\varepsilon_{ij}(d_{ij})\}_{j \neq i}) \in \mathbb{R}^{N+1} : -\mathcal{C}_i + \beta_i + \sum_{j \neq i} \varepsilon_{ij}(d_{ij}) \geq 0\} \tag{1}$$

For the sake of visualisation, assume the following two cases aimed at reducing the dimensionality of S_i^E:

CASE A. Positive externalities. $\varepsilon_{ij}(d_{ij}) = \frac{\beta_i}{1+d_{ij}} \forall j = -i$. S_i^E becomes:

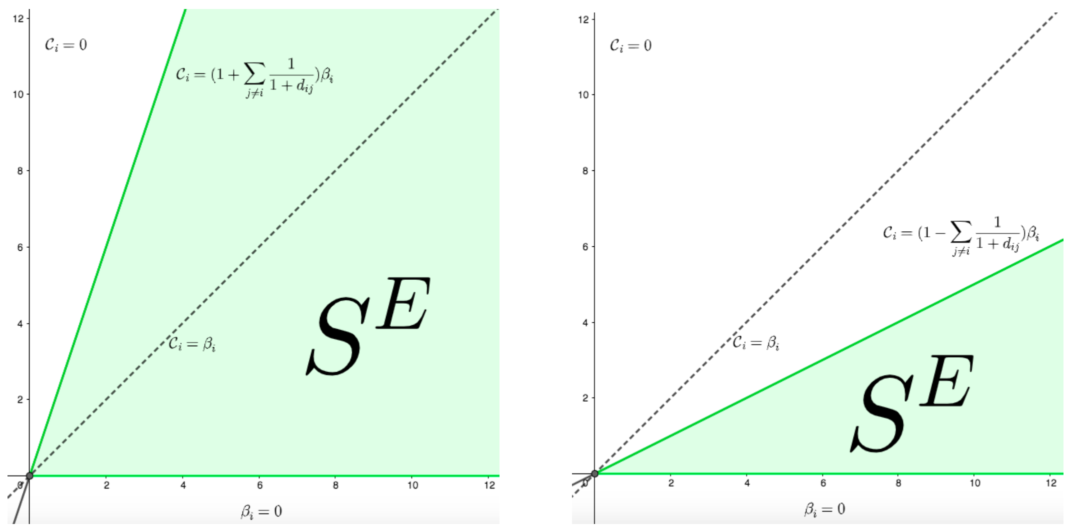
$$S_i^E = \{(\mathcal{C}_i, \beta_i) \in \mathbb{R}_+^2 : -\mathcal{C}_i + \beta_i + \sum_{j \neq i} \frac{\beta_i}{1 + d_{ij}} \geq 0\} \tag{2}$$

CASE B. Negative externalities. $\varepsilon_{ij}(d_{ij}) = \frac{\beta_i}{1+d_{ij}} \forall j = -i$. S_i^E becomes:

$$S_i^E = \{(\mathcal{C}_i, \beta_i) \in \mathbb{R}^2 : \mathcal{C}_i \geq 0 \text{ and } -\mathcal{C}_i + \beta_i - \sum_{j \neq i} \frac{\beta_i}{1 + d_{ij}} \geq 0\} \tag{3}$$

Graphically, these two sets can be represented as figure 1 depicts.

Figure 1 shows the effect of isolation on the set of efficient projects. First, S_i^E for isolated



(a) S_i^E when $\varepsilon_{ij}(d_{ij}) = \frac{\beta_i}{1+d_{ij}} \forall j = -i$, in green.

(b) S_i^E when $\varepsilon_{ij}(d_{ij}) = -\frac{\beta_i}{1+d_{ij}}$, in green. Note that S_i^E could also be in the fourth quadrant if $\sum_{j \neq i} \frac{1}{1+d_{ij}} > 1$.

Figure 1: Efficient sets

regions (in the sense that $min_{j \neq i}\{d_{ij}\} \to \infty$) will be invariant to the sign of externalities: externalities get diluted for regions whose neighbours are far apart, and as a result S_i^E is the set below the 45º line in either case. However, for connected regions, both sets will differ.

Having analysed the efficient benchmark, let me now proceed by asking a normative and a positive question. *Primo*, what parliaments should we assign decision powers at $t = 1$ to? *Secondo*, what decision powers will each parliament actually have? The next section analyses the ex-post behavior of each parliament, which confronted to the efficient benchmark just analysed helps answer the normative question. Section 5, instead, analyses how regions will vote at $t = 1$, thus answering the positive question.

4 Ex-post behavior of parliaments

In this section I analyse the normative question of which chamber(s) should have decision powers.

The following choice sets describe the regional parliaments choice of projects and union choice of projects, respectively:

$$S_i^P = \{(\mathcal{C}_i, \beta_i, \{\varepsilon_{ij}(d_{ij})\}_{j \neq i}) \in \mathbb{R}^{N+1} : -\mathcal{C}_i + \beta_i \geq 0\} \tag{4}$$

$$\begin{gathered} S_i^U = \{(\mathcal{C}_i, \beta_i, \{\varepsilon_{ij}(d_{ij})\}_{j \neq i}) \in \mathbb{R}^{N+1} : -\tfrac{\mathcal{C}_i}{N} + \beta_i < 0 \ and - \tfrac{\mathcal{C}_i}{N} + \varepsilon_{ij}(d_{ij}) \geq 0 \text{ for } j = \{1, 2, ...n\}, n > \tfrac{N}{2}, j \neq i\} \\ \cup \\ \{(\mathcal{C}_i, \beta_i, \{\varepsilon_{ij}(d_{ij})\}_{j \neq i}) \in \mathbb{R}^{N+1} : -\tfrac{\mathcal{C}_i}{N} + \beta_i \geq 0 \ and - \tfrac{\mathcal{C}_i}{N} + \varepsilon_{ij}(d_{ij}) \geq 0 \text{ for } j = \{1, 2, ...n\}, n \geq \tfrac{N-1}{2}, j \neq i\} \end{gathered} \tag{5}$$

Set 4 tells that, if region i has the powers to implement its own policy, it will implement the set of policies such that the benefit (net of cost) that accrues to the region is positive -i.e. $\mathcal{C}_i + \beta_i > 0$. A graphical representation of the set can be found below, in figure 2.

Set 5 describes all the combinations $(\mathcal{C}_i, \beta_i, \{\varepsilon_{ij}(d_{ij})\}_{j \neq i})$ that will be approved by the union parliament, if it is this parliament the one that decides upon implementation. Since regions vote by majority voting, this set is composed of two subsets: (i) the subset of policies that yield a negative payoff to region i but a non-negative payoff to at least $\frac{N}{2}$ neighbors of i, and (ii) the subset of policies that yield a non-negative payoff to both region i and, at least, $\frac{N-1}{2}$ neighbors. Also note that I am implicitly assuming that regions vote "approve" for projects that leave them indifferent, without loss of generality.

It is worth highlighting that the decisions of regional parliaments are invariant to the location of regions: S_i^P does not depend on d_{ij}. This result is a trivial consequence of the fact that regional parliaments disregard the externalities imposed upon other regions, a fact that has long been highlighted in the literature (Oates, 1972[2]; Besley and Coate, 2003[1]; Ventura, 2019[3]). The union parliament, nonetheless, cares about externalities, and the functional form of $\varepsilon(d_{ij})$ will thus matter. Again for the sake of tractability, let's consider set 5 in the case where externalities are a function of region i's benefits:

CASE A. Positive externalities. $\varepsilon_{ij}(d_{ij}) = \frac{\beta_i}{1+d_{ij}} \forall j = -i$. S_i^U becomes:

$$S_i^U = \{(\mathcal{C}_i, \beta_i) \in \mathbb{R}_+^2 : -\frac{\mathcal{C}_i}{N} + \frac{\beta_i}{1 + d_{ij}^{ME}} \geq 0\} \tag{6}$$

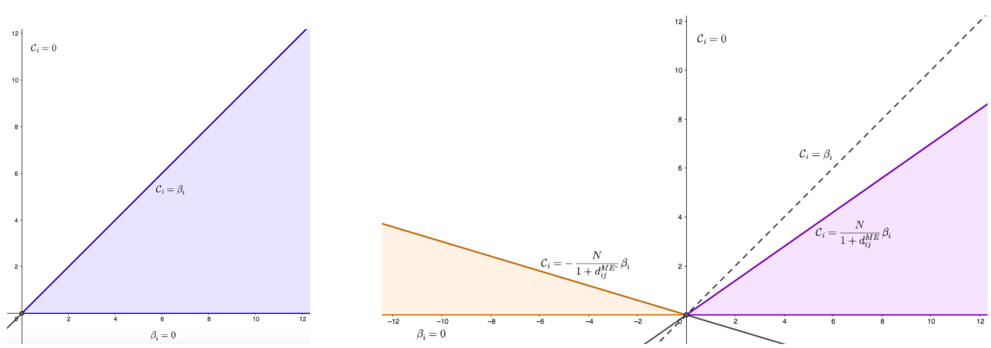
where d_{ij}^{ME} denotes the median of the set of distances $\{d_{ij}\}_{j=\{1,...,N\}}$. From now on, I will refer to the region j such that $d_{ij} = d_{ij}^{ME}$ as the "median neighbor" (of i). The purple area in figure 2 graphically represents this set.

CASE B. Negative externalities. $\varepsilon_{ij}(d_{ij}) = -\frac{\beta_i}{1+d_{ij}} \forall j = -i$. S_i^U becomes:

$$S_i^U = \{(\mathcal{C}_i, \beta_i) \in \mathbb{R}^2 : \mathcal{C}_i \geq 0 \text{ and } -\frac{\mathcal{C}_i}{N} - \frac{\beta_i}{1 + d_{ij}^{ME^*}} \geq 0\} \tag{7}$$

where $d_{ij}^{ME^*}$ denotes the $(\frac{N+3}{2N})^{th}$ quantile of the set of distances $\{d_{ij}\}_{j=\{1,...,N\}}$. In other words, ME^* is the distance between i and the region after the median neighbor, in the order of distances (I will refer to this region as the "median+1 neighbor"). The orange area in figure 2 graphically represents this set.

Note that equations 6 and 7 are nothing else but an application of the median voter theorem: given that projects are voted separately and that preferences are single peaked, the median (in a distance sense) neighbor's preferences decide whether the project of region i will be approved or not.



(a) S_i^P.

(b) S_i^U when $\varepsilon_{ij}(d_{ij}) = \frac{\beta_i}{1+d_{ij}} \forall j = -i$, in purple; and S_i^U when $\varepsilon_{ij}(d_{ij}) = -\frac{\beta_i}{1+d_{ij}} \forall j = -i$, in orange.

Figure 2: Parliaments' choices

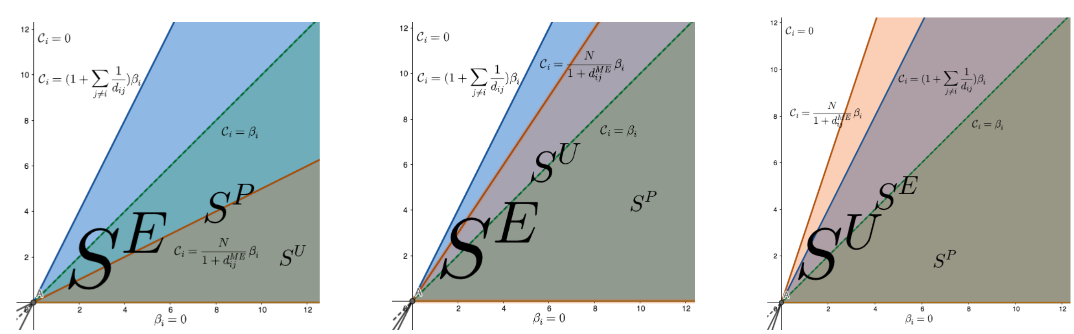
Before analysing the positive question of what decision powers each parliament will have, let me state some results that follow from this simplified framework.

Proposition 1. *In the presence of positive externalities,* $\exists\ d_{ij}^1$, d_{ij}^2 *such that for* $d_{ij}^{ME} \geq d_{ij}^2$ *the regional parliament performs at least as efficient a job as the union parliament, for*

$d_{ij}^1 < d_{ij}^{ME} \leq d_{ij}^2$ *the union parliament performs a better job than the regional parliament, and for* $d_{ij}^{ME} < d_{ij}^1$*, the relative performance of either parliament depends on* $F_i(\cdot)$*.*

A formal proof of this proposition can be found in the Appendix.

Proposition 1 tells that it suffices to know d_{ij}^{ME} to know whether the union parliament or the regional parliament will perform a more efficient job. For d_{ij}^{ME} large enough, the median neighbor will inefficiently block any project that regional parliament i would have efficiently approved, because the positive externality that the median neighbor receives is insufficient to cover for the shared cost $\frac{\mathcal{C}_i}{N}$. Therefore, the regional parliaments of such regions should retain the power to approve their own projects. For $d_{ij}^1 < d_{ij}^{ME} < d_{ij}^2$, the regional parliament of region i performs worse than the union parliament because it disregards that the median neighbor is close enough to enjoy the positive externalities arising from their own project. In this case, the union parliament should have the powers to decide upon the project of region i. Finally, for d_{ij}^{ME} small enough, both the union parliament will "over-approve" inefficient projects (because the median neighbor is close enough to enjoy a positive benefit net of costs), and the regional parliament will tend to under-approve because it disregards positive externalities. Therefore, similar to Ventura (2019)[3], decision powers should be assigned to either of the two parliaments depending on which parliament will make more mistakes on average – something that will depend on where the largest mass of β_i is located, i.e. $F_i(\cdot)$. Figure 3 below summarizes the different possible cases.



(a) $d_{ij}^{ME} \geq N-1$ and, consequently, $S_i^U \subseteq S_i^P \subset S_i^E$.

(b) $(\overline{\frac{1}{1+d_{ij}}})^{-1} - 1 \leq d_{ij}^{ME} < N-1$ and, consequently, $S_i^P \subset S_i^U \subseteq S_i^E$.

(c) $d_{ij}^{ME} < (\overline{\frac{1}{1+d_{ij}}})^{-1} - 1$ and, consequently, $S_i^P \subset S_i^E \subset S_i^U$.

Figure 3: Efficiency of regional/union parliament's choices

Another noteworthy result that arises from proposition 1 (and, more specifically, from lemma 3 – see appendix) is that isolated regions in the sense that $min\{d_{ij}\}_{j \neq i} \to \infty$ should never form a union. This is because externalities are close to zero when regions are isolated[2]. Consequently, the regional parliament does not make mistakes from disregarding *de facto* inexistent externalities. Note that this accounts for a trivial pattern that can be observed in the world: trade/political unions are more easily formed between regions that are geographically close to each other.

Proposition 2. *In the presence of positive externalities and for a given* $(\overline{\frac{1}{1+d_{ij}}}(N))^{-1} - 1 < d_{ij}^{ME}(N) < 1 + d_{ij}^{ME}(N-1)$, $\exists\ \tilde{N} > 0$ *such that the union parliament performs a more efficient job than the regional parliament in deciding upon the implementation of i's project.*

The proof of this result lies on the fact that $S_i^P \subset S_i^U$ when $1 < \frac{N}{1+d_{ij}^{ME}}$, so that for N large enough this condition will eventually be true as long as (i) $d_{ij}^{ME}(N) - d_{ij}^{ME}(N-1) < 1$ (i.e., for each member that it is added to the union, d_{ij}^{ME} grows by less than one), and (ii) $d_{ij}^{ME}(N) \geq (\overline{\frac{1}{1+d_{ij}}}(N))^{-1} - 1$ (i.e. for each member that it is added to the union, the condition $d_{ij}^{ME} \geq (\overline{\frac{1}{1+d_{ij}}})^{-1} - 1$ still holds). Intuitively, the relative efficiency of the union parliament (with respect to the regional parliament) in approving the project of region i grows if the union expands in such a way that d_{ij}^{ME} remains relatively unchanged. This is unsurprising if we bear in mind that such growth will bring the median neighbor closer to the average neighbor, so that the efficiency of the union parliament will increase.

Note as well that, unless we consider an N-dymensional simplex in which all regions are equally far apart from each other, it is impossible to improve the overall efficiency of the union parliament only by making the union grow in such a way that $(\overline{\frac{1}{1+d_{ij}}}(N))^{-1} - 1 < d_{ij}^{ME}(N) < 1 + d_{ij}^{ME}(N-1)$. This is because, outside an N-dimensional simplex world, adding regions to the union will asymmetrically affect d_{ij}^{ME} for each $i = \{1, 2...N\}$.

Proposition 3. *In the presence of negative externalities,* $\exists\ \tilde{d}_{ij}$ *such that for* $d_{ij}^{ME^*} \geq \tilde{d}_{ij}$ *the union parliament performs a more efficient job than the regional parliament, and for*

[2]Note that it is straightforward to show that the same argument is true for negative externalities here.

$d_{ij}^{ME^*} < \tilde{d}_{ij}$ *the relative performance of either parliament depends on* $F_i(\cdot)$.

The proof of this result can be found in the Appendix. There, I also proof that $\tilde{d}_{ij} = -\frac{1}{\widehat{\frac{1}{1+d_{ij}}}} - 1$, where $\widehat{\frac{1}{1+d_{ij}}} \equiv \frac{1}{N}(1 - \sum_{j=-i}^{N} \frac{1}{1-d_{ij}})$.

The intuition goes as follows. In the presence of negative externalities, the regional parliament will always implement inefficient projects whose β_i is too large. This is because the average neighbor suffers a negative externality that the parliament of region i completely disregards. The relative performance of region i's parliament with respect to the union parliament will therefore depend on whether the union parliament reduces this mistake that i's parliament is doing or not.

Two cases arise here. First, as long as the externality on the median+1 neighbor is smaller than the externality on the average neighbor (which happens when the median+1 neighbor is far enough so that $d_{ij}^{ME^*} > -\frac{1}{\widehat{\frac{1}{1+d_{ij}}}} - 1$), the union parliament will unambiguously improve the job of the regional parliament: the median+1 neighbor is closer to the average neighbor than region i, since the median+1 neighbor at least takes into account some level of externality – which makes the median+1 neighbor veto inefficient projects with $\beta_i > 0$ that the regional parliament would have approved. In fact, note that when the externality on the median+1 neighbor equals the externality on the average neighbor, the union parliament implements exactly the efficient set of projects. The second case, in which $d_{ij}^{ME^*} < -\frac{1}{\widehat{\frac{1}{1+d_{ij}}}} - 1$, is however ambiguous, and the relative performance of the regional parliament with respect to the union parliament again depends on $F_i(\cdot)$. This is because, if the externality on the median+1 neighbor is larger than the externality on the average neighbor, the union parliament now makes two class of mistakes (depending on the sign of $\widehat{\frac{1}{1+d_{ij}}}$, which captures whether the average neighbor is close to or far away from i) that the regional parliament does not make. More specifically, the median+1 neighbor will be vetoing efficient projects that the regional parliament would have implemented (only when $\widehat{\frac{1}{1+d_{ij}}} > 0$ -i.e. the average neighbor is far away from i) and accepting inefficient projects that the regional parliament would have vetoed (regardless of the sign of $\widehat{\frac{1}{1+d_{ij}}}$).

5 Constitutional outcome

Let me now switch to the positive question. What parliament(s) will have the decision powers upon the implementation of project type t? In Ventura (2019)[3], all regions unanimously agreed, in t=1, to assign decision powers to the chamber(s) that maximised the ex-ante value of projects. This resulted from two implicit assumptions: (i) payoffs were valued in monetary terms and regions were "payoff maximisers", i.e. regions based their choice on regional vs. union parliament depending on which one maximised their expected payoff, and (ii) the symmetrical structure of the model: all regions received, on average, the same expected payoff.

Now however, when voting the constitution regions are not identical. Nature has already placed regions onto space, which means that regions no longer live under a Rawlsian veil of ignorance. More specifically, when voting which parliament will decide upon implementation, region i will vote for the union parliament when:

$$E_i\{-\sum_{i=1}^{N}\frac{\mathcal{C}_i}{N}+\beta_i+\sum_{j=-i}\varepsilon_{ji}(d_{ij})|U\}\geq E_i\{-\mathcal{C}_i+\beta_i+\sum_{j=-i}\varepsilon_{ji}(d_{ij})|P\}, \tag{8}$$

whereas it will vote for the regional parliament otherwise[3].

Some straightforward conclusions that arise from this framework are:

1. Regions whose $\mathcal{C}_i$ is above average will tend to vote for the union parliament since this dilutes the large cost, *ceteris paribus* benefits and externalities.

2. *Ceteris paribus* costs and externalities, regions will tend to prefer the union parliament when the union "over-approves" projects with $\beta_i > 0$ with respect to the regional parliament (i.e. $E_i\{\beta_i|U\} > E_i\{\beta_i|P\}$).

3. *Ceteris paribus* costs and benefits, regions will tend to vote for the union parliament when the union parliament "over-approves" ("under-approves") projects with positive (negative) externalities with respect to regional parliaments (i.e. $E_i\{\sum_{j=-i}\varepsilon_{ji}(d_{ij})|U\} > E_i\{\sum_{j=-i}\varepsilon_{ji}(d_{ij})|P\}$).

[3] I am assuming regions vote for the union parliament in the case of indifference without loss of generality.

A natural result that arises from this framework is that the fiscal constitution that will arise in $t = 1$ is not necessarily the one that coincides with social preferences: the preferences of the median voter in $t = 1$ do not necessarily coincide with the preferences of the average voter in $t = 1$. Hence, all sorts of conflicts can arise in this framework. For instance, in the simplified case where $\varepsilon_{ij}(d_{ij}) = \frac{\beta_i}{1+d_{ij}} \forall j = -i$ and costs and benefits are symmetrical ex-ante[4], the union parliament is more likely to have decision powers when (i) the median region in $t = 1$ is close to its median neighbor and (ii) the probability of disagreement between regional parliaments and the union parliament is large, given projects of positive value for the median region (see analysis under equation 13 in the appendix). If conditions (i) and (ii) are satisfied, but at the same time the average region is isolated (which I prove, under equation 10 in the appendix, to favor regional parliaments), then the model generates an (inefficient) tyranny of the majority: the union parliament will tend to approve the (possibly inefficient) projects of the close majority (case in figure 3 c), whereas it will tend to veto the (efficient) projects of the isolated minority (case in figure 3 a) since the shared cost does not compensate for the low externalities that the core receives. In this case, isolated regions pay for the projects of the center and barely get anything in return. Another sort of conflict would arise if the median region was isolated and preferred regional parliaments, whereas the average region was connected and preferred a union. In that case, regional parliaments would keep decision powers and would be vetoing efficient projects that the union parliament would have undertaken.

To summarize, conflicts arise when the interests of the average region in $t = 1$ do not coincide with those of the median region in $t = 1$ [5]. Hence, let me end by asking the following question: could the possibility of voting the constitution under the veil of ignorance (i.e., at $t = 0$) bring us closer to the preferences of the average region in $t = 1$? Proposition 4 answers the question.

Proposition 4. *Voting the constitution under the veil of ignorance solves conflicts in* $t = 1$

[4] In the appendix I provide specific details on how the median and the average regions behave in the simplified cases where $\varepsilon_{ij}(d_{ij}) = \pm\frac{\beta_i}{1+d_{ij}} \forall j = -i$ and costs and benefits are symmetrical ex-ante.

[5] Note, however, the fact that the median and the average regions are placed onto different points in the space is necessary but not sufficient to generate disagreement. Both regions may be far away from each other and still have the same interests depending on the distribution of benefits and externalities.

when (i) N is large, (ii) nature places regions onto space in an i.i.d fashion, (iii) $E|E_i\{-\mathcal{C}+\beta_i+\sum_{j=-i}\varepsilon_{ij}(d_{ij})|U\}-E_i\{-\mathcal{C}+\beta_i+\sum_{j=-i}\varepsilon_{ij}(d_{ij})|P\}|<\infty$ *and (iv) the representative region under the veil of ignorance has "rational expectations" and knows how nature places regions onto space.*

The proof of this result is an application Kinchine's weak law of large numbers, for which i.i.d. draws (condition ii) and finite expectation of the random variable (condition iii) are necessary and sufficient conditions. This is, by Kinchine's WLLN:

$$\frac{1}{N}\sum_{i=1}^{N}E_i\{-\mathcal{C}_i+\beta_i+\sum_{j=-i}\varepsilon_{ij}(d_{ij})|U\}-\frac{1}{N}\sum_{i=1}^{N}E_i\{-\mathcal{C}_i+\beta_i+\sum_{j=-i}\varepsilon_{ij}(d_{ij})|P\}\rightarrow$$
$$E[E_i\{-\mathcal{C}_i+\beta_i+\sum_{j=-i}\varepsilon_{ij}(d_{ij})|U\}-E_i\{-\mathcal{C}_i+\beta_i+\sum_{j=-i}\varepsilon_{ij}(d_{ij})|P\}] \text{ as } N\rightarrow\infty,$$

i.e. the average voter in $t=1$ becomes more and more similar to the representative region in the veil of ignorance as N grows, provided that the representative region in the veil of ignorance forms its expectations according to the rule that nature uses to draw regions onto space (condition iv).

Although the result is unsurprising, let me highlight two implications. First, the theoretical appeal of the veil of ignorance as an ideal benchmark in which actors decide best is stronger when we talk about individuals (what will the social position of an individual be?) and not regions (in which position will a region be placed?). This is because the veil of ignorance works for N large, but the number of regions in a country/union generally is not an N-large world. Second, the Rawlsian veil of ignorance is, by definition, unfeasible in the real world. Still, modelling it has an appeal if somehow regions recognize that abstracting from their (already drawn) position in the real world is desirable before voting a constitution. With this in mind, I believe that a more realistic way to model a situation that resembled the unfeasible veil of ignorance would have been to (i) impose that regions behave *as if* they were in the veil of ignorance (although they already know their position in the real world) and (ii) assume regions imperfectly form their expectations of how nature works depending on the position in which a region has been placed[6].

[6] I do not develop this case for space and time constraints, though.

6 Conclusion

In this paper, I have extended the framework in Ventura (2019)[3] to incorporate a spatial dimension to the analysis of when regions should/do form union or not. When externalities depend on the distance between regions, isolated regions will prefer regional parliaments whereas connected regions will tend to prefer the union parliament if this chamber increases exposure to large positive externalities or effectively shields against negative externalities from other regions' projects. In my framework, the delegation of decision powers to either of the two chambers will thus depend on the preferences of the median voter when voting the fiscal constitution, which will in turn depend on the whole distribution of regions onto space. As a result, if the median voter preferences differ from the average voter preferences, the decision made in this voting stage may be inefficient; an issue that could be fixed if a large number of regions with rational expectations could vote the constitution under the veil of ignorance.

Finally, let me suggest some paths for further investigation. First, almost all the results in this framework depend on the distribution of distances between regions. Although I have shown how it all boils down to finding average/median voters and its respective average/median neighbors, this individuals will change depending on the shape of the space and how nature distributes regions onto it. Further investigation can thus inquire in which spaces favor delegation to regional parliaments and which favor delegation to the union parliament. Another interesting path for further investigation is to think deeper about the conflicts that may arise after the fiscal constitution has been voted. In the case where a central core repeatedly exerts a tyranny of the majority upon the periphery through the union parliament, how would the possibility of secession change the picture? What if the voting in $t = 1$ was done under a fake veil of ignorance in which regions do not have rational expectations but recognize that some regions win and some others lose? All this matters, apart from strengthening the paper with various extensions (heterogeneous sizes, risk-averse regions, empirical testing of the predictions of the model), are exciting paths to keep investigating.

References

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A Appendix

Proof of Proposition 1. To prove this result, let me begin by stating three lemmas.

Lemma 1. *$S_i^U = S_i^P$ iff $d_{ij}^{ME} = N-1$. Furthermore, $S_i^U \subset S_i^P$ iff $d_{ij}^{ME} > N-1$ and vice-versa.*

I prove this lemma in three steps. (i) I first start by proving that $S_i^U = S_i^P$ iff $d_{ij}^{ME} = N-1$:

Direction 1. $S_i^U = S_i^P \Rightarrow d_{ij}^{ME} = N-1$.

If $S_i^U = S_i^P$, then (i) $S_i^U \subseteq S_i^P$, which means that $1 \geq \frac{N}{1+d_{ij}^{ME}}$ or, in other words, $d_{ij}^{ME} \geq N-1$; and (ii) $S_i^P \subseteq S_i^U$, which means that $1 \leq \frac{N}{1+d_{ij}^{ME}}$ or, in other words, $d_{ij}^{ME} \leq N-1$. Then it must be that $d_{ij}^{ME} = N-1$.

Direction 2. $d_{ij}^{ME} = N-1 \Rightarrow S_i^U = S_i^P$.

If $d_{ij}^{ME} = N-1$, then $S_i^U = \{(\mathcal{C}_i, \beta_i) \in \mathbb{R}^2 : -\frac{\mathcal{C}_i}{N} + \frac{\beta_i}{1+N-1} \geq 0\}$ or, in other words, $S_i^U = \{(\mathcal{C}_i, \beta_i) \in \mathbb{R}^2 : -\frac{\mathcal{C}_i}{N} + \frac{\beta_i}{N} \geq 0\}$, which is precisely S_i^P.

(ii) Now I prove that $S_i^U \subset S_i^P$ iff $d_{ij}^{ME} > N-1$:

$S_i^U \subset S_i^P \iff 1 > \frac{N}{1+d_{ij}^{ME}} \iff d_{ij}^{ME} > N-1$.

(iii) Finally, the proof of $S_i^P \subset S_i^U$ iff $d_{ij}^{ME} < N-1$ is identical to the last proof.

Lemma 2. *Let $\overline{\frac{1}{1+d_{ij}}} \equiv \frac{1}{N}\sum_{i=1}^{N} \frac{1}{1+d_{ij}}$. $S_i^E = S_i^U$ iff $d_{ij}^{ME} = (\overline{\frac{1}{1+d_{ij}}})^{-1} - 1$. Furthermore, $S_i^E \subset S_i^U$ iff $d_{ij}^{ME} < (\overline{\frac{1}{1+d_{ij}}})^{-1} - 1$ and vice-versa.*

The proof of this lemma is isomorphic to the proof of lemma 1, but changing S_i^P for S_i^E and $d_{ij}^{ME} = N-1$ for $d_{ij}^{ME} = (\overline{\frac{1}{1+d_{ij}}})^{-1} - 1$.

Lemma 3. *$S_i^P \subset S_i^E$. Furthermore, for $N < \infty$, $\lim_{min\{d_{ij}\}, j\neq i \to \infty} S_i^E = S_i^P$.*

The proof of $S_i^P \subset S_i^E$ goes by noting that, for $d_{ij} > 0$ with $j \neq i$, $1 < N\overline{\frac{1}{1+d_{ij}}}$ which means that $\exists$ a subset $\mathcal{S}$ such that $S_i^E = S_i^P \cup \mathcal{S}$ and $S_i^P \cap \mathcal{S} = \emptyset$.

The proof that, for $N < \infty$, $\lim_{min\{d_{ij}\}, j\neq i \to \infty} S_i^E = S_i^P$ goes by noting that we can bound terms as follows: $\sum_{j\neq i} \frac{1}{1+d_{ij}} \leq \sum_{j\neq i} \frac{1}{1+min\{d_{ij}\}} = (N-1)\frac{1}{1+min\{d_{ij}\}} \to 0$ as $min\{d_{ij}\} \to \infty$.

Now, by lemmas 1 and 3, $\exists d_{ij}^1$ $(= N-1)$ such that for $d_{ij}^{ME} \geq d_{ij}^1$, $S_i^U \subseteq S_i^P \subset S_i^E$ and, consequently, the regional parliament performs at least as efficient a job as the union parliament. Similarly, by lemmas 1, 2, and 3 $\exists d_{ij}^2$ $(= (\overline{\frac{1}{1+d_{ij}}})^{-1} - 1)$ such that for $d_{ij}^2 \leq d_{ij}^{ME} < N-1$, $S_i^P \subset S_i^U \subseteq S_i^E$ and, consequently, the regional parliament performs a more efficient job than the union parliament. Furthermore, for $d_{ij}^{ME} < d_{ij}^2$, $S_i^P \subset S_i^E \subset S_i^U$, so that, similarly to Ventura (2019)[3], the relative performance of either parliament depends on $F_i(\cdot)$. $\square$

Proof of Proposition 3*.* Let me begin proving this propostition by stating two lemmas.

Lemma 4. *In the presence of negative externalities,* $\widehat{\frac{1}{1+d_{ij}}} \geq 0 \iff S_i^E \subset S_i^P$ *and* $\widehat{\frac{1}{1+d_{ij}}} < 0 \iff S_i^E \cap S_i^P$.

The proof of the first result goes by noting that, for $d_{ij} > 0$ with $j \neq i$, $1 > N\widehat{\frac{1}{1+d_{ij}}}$. Thus $1 > N\widehat{\frac{1}{1+d_{ij}}} \geq 0 \iff S_i^E \subset S_i^P$.

The proof of the second result goes by noting that $\widehat{\frac{1}{1+d_{ij}}} < 0 \iff S_i^E \in \{(\beta_i, \mathcal{C}_i) \in \mathbb{R}^2 : \beta_i < 0 \text{ and } \mathcal{C}_i > 0\}$, whereas $S_i^P \in \{(\beta_i, \mathcal{C}_i) \in \mathbb{R}^2 : \beta_i > 0 \text{ and } \mathcal{C}_i > 0\}$.

Lemma 5. *In the presence of negative externalities,* $S_i^P \cap S_i^U = \emptyset$

The proof goes by noting that $S_i^P \in \{(\beta_i, \mathcal{C}_i) \in \mathbb{R}^2 : \beta_i > 0 \text{ and } \mathcal{C}_i > 0\}$ (since $1 > 0$), whereas $S_i^U \in \{(\beta_i, \mathcal{C}_i) \in \mathbb{R}^2 : \beta_i < 0 \text{ and } \mathcal{C}_i > 0\}$ (since $-\frac{N}{1+d_{ij}^{ME^*}} < 0$ for $d_{ij}^{ME^*} > 0$, which holds since $d_{ij} > 0$ by assumption).

Now I proceed by considering two mutually exclusive and exhaustive cases.

(i) Case $\widehat{\frac{1}{1+d_{ij}}} \geq 0$. Note that, in this case, $d_{ij}^{ME^*} > -\frac{1}{\widehat{\frac{1}{1+d_{ij}}}} - 1$ $\forall d_{ij}^{ME^*} > 0$. By lemma 4, $S_i^E \subset S_i^P$ and it must then be, by lemma 5, that $S_i^E \cap S_i^U = \emptyset$, in which case the relative performance of either parliament depends on $F_i(\cdot)$.

(ii) Case $\widehat{\frac{1}{1+d_{ij}}} < 0$. In this case, by lemmas 4 and 5 both S_i^U and S_i^P are located in the fourth quadrant of the cartesian plain $(\beta_i, \mathcal{C}_i)$. Two subcases arise here: either $d_{ij}^{ME^*} \geq -\frac{1}{\widehat{\frac{1}{1+d_{ij}}}} - 1$, in which case it is straightforward to show that $S_i^U \subseteq S_i^E$ (i.e. the union is strictly improving the performance of the regional parliament); or $d_{ij}^{ME^*} < -\frac{1}{\frac{1}{1+d_{ij}}} - 1$, in which case it is straightforward to show that $S_i^E \subset S_i^P$ (i.e. the relative performance of either parliament depends on $F_i(\cdot)$). $\square$

A.1 ***Social vs individual ex-ante preferences in t=1***

A.1.1 *Social ex-ante preferences*

The rule that maximises ex-ante welfare should assign decision powers to the union parliament if:

$$\sum_{i=1}^{N} E_i\{-\sum_{i=1}^{N} \frac{\mathcal{C}_i}{N} + \beta_i + \sum_{j=-i} \varepsilon_{ij}(d_{ij})|U\} \geq \sum_{i=1}^{N} E_i\{-\mathcal{C}_i + \beta_i + \sum_{j=-i} \varepsilon_{ij}(d_{ij})|P\} \tag{9}$$

otherwise, regional parliaments should keep decision powers.

Now, let me consider the simplified cases where $\varepsilon_{ij}(d_{ij}) = \pm\frac{\beta_i}{1+d_{ij}}$. I will also assume ex-ante symmetry of benefits and costs to simplify algebra, i.e. $F_i(\cdot) = F(\cdot)$ and $\mathcal{C}_i = \mathcal{C}$ $\forall i = \{1, 2...N\}$.

CASE A. Positive externalities. Some straightforward algebra reveals that equation 9 can be written as:

$$\sum_{i=1}^{N} E_i\{-\mathcal{C} + \beta_i + \sum_{j=-i} \frac{\beta_i}{1+d_{ij}}|U\} \geq \sum_{i=1}^{N} E_i\{-\mathcal{C} + \beta_i + \sum_{j=-i} \frac{\beta_i}{1+d_{ij}}|P\}$$

$$\Longleftrightarrow$$

$$\sum_{i=1}^{N}[E_i\{-\mathcal{C} + \beta_i + \sum_{j=-i} \frac{\beta_i}{1+d_{ij}}|U\} - E_i\{-\mathcal{C} + \beta_i + \sum_{j=-i} \frac{\beta_i}{1+d_{ij}}|P\}] \geq 0$$

$$\Longleftrightarrow$$

$$E_{\tilde{i}}\{-\mathcal{C} + \beta_{\tilde{i}} + \sum_{j=-\tilde{i}} \frac{\beta_{\tilde{i}}}{1+d_{\tilde{i}j}}|U=1\}P_{\tilde{i}}(U=1) - E_{\tilde{i}}\{-\mathcal{C} + \beta_{\tilde{i}} + \sum_{j=-\tilde{i}} \frac{\beta_{\tilde{i}}}{1+d_{\tilde{i}j}}|P=1\}P_{\tilde{i}}(P=1) \geq 0$$

$$\Longleftrightarrow$$

$$E\{-\mathcal{C} + N\overline{\frac{1}{1+d_{\tilde{i}j}}}\beta|U=1\}P_{\tilde{i}}(U=1) - E\{-\mathcal{C} + N\overline{\frac{1}{1+d_{\tilde{i}j}}}\beta|P=1\}P_{\tilde{i}}(P=1) \geq 0$$

$$\Longleftrightarrow$$

$$E\{-\mathcal{C}+N\overline{\frac{1}{1+d_{\tilde{i}j}}}\beta|\beta \geq \frac{C}{N}(1+d_{\tilde{i}j}^{ME})\}P(\beta \geq \frac{C}{N}(1+d_{\tilde{i}j}^{ME}))-E\{-\mathcal{C}+N\overline{\frac{1}{1+d_{\tilde{i}j}}}\beta|\beta \geq \mathcal{C}\}P(\beta \geq \mathcal{C}) \geq 0 \tag{10}$$

where $\tilde{i}$ is the hypothetical average region such that $N[E_{\tilde{i}}\{-\mathcal{C} + \beta_{\tilde{i}} + \sum_{j=-\tilde{i}} \frac{\beta_{\tilde{i}}}{1+d_{\tilde{i}j}}|U\} - E_{\tilde{i}}\{-\mathcal{C} + \beta_{\tilde{i}} + \sum_{j=-\tilde{i}} \frac{\beta_{\tilde{i}}}{1+d_{\tilde{i}j}}|P\}] = \sum_{i=1}^{N} E_i\{-\mathcal{C} + \beta_i + \sum_{j=-i} \frac{\beta_i}{1+d_{ij}}|U\} - \sum_{i=1}^{N} E_i\{-\mathcal{C} + \beta_i + \sum_{j=-i} \frac{\beta_i}{1+d_{ij}}|P\}$ and $\overline{\frac{1}{1+d_{\tilde{i}j}}} \equiv \frac{1}{N}\sum_{j=1}^{N} \frac{\beta_{\tilde{i}}}{1+d_{\tilde{i}j}}$. Note that the second line uses the fact that $E_{\tilde{i}}\{-\mathcal{C}+\beta+\sum_{j=-\tilde{i}} \frac{\beta}{1+d_{\tilde{i}j}}|U=0\}Pr_{\tilde{i}}(U=0) = 0$ (*idem for P=0*), the third line uses the fact that $F_i(\cdot) = F(\cdot)$, and the fourth line uses the fact that $P_{\tilde{i}}(U=1) = P(-\frac{C}{N} + \frac{\beta}{1+d_{\tilde{i}jME}} \geq 0)$ and $P_{\tilde{i}}(U=1) = P(-\mathcal{C}+\beta \geq 0)$.

Equation 10 tells that, in this simplified case, it suffices to know the relative position (with respect to other regions) of the average region on the space in order to determine which are social preferences. Some limit cases:

- The average region is close to its median neighbor, i.e $d_{\tilde{i}j}^{ME} \to 0$.

  The social rule assigns decision powers depending on $F(\cdot)$. For instance, if $E\{-\mathcal{C} + N\overline{\frac{1}{1+d_{\tilde{i}j}}}\beta|\beta \geq \frac{C}{N}(1+d_{\tilde{i}j}^{ME})\} < 0$ and $E\{-\mathcal{C}+N\overline{\frac{1}{1+d_{\tilde{i}j}}}\beta|\beta \geq \mathcal{C}\} > 0$, regional parliaments should keep decision powers; but for $E\{-\mathcal{C} + N\overline{\frac{1}{1+d_{\tilde{i}j}}}\beta|\beta \geq \frac{C}{N}(1 + d_{\tilde{i}j}^{ME})\} > 0$ and $E\{-\mathcal{C} + N\overline{\frac{1}{1+d_{\tilde{i}j}}}\beta|\beta \geq \mathcal{C}\} > 0$, the union parliament may keep decision powers if $\frac{P(\beta \geq \frac{C}{N}(1+d_{\tilde{i}j}^{ME}))}{P(\beta \geq \mathcal{C})}$ is large enough.

- The average region is far away from its median neighbor, i.e $d_{\tilde{i}j}^{ME} \to \infty$.

  In this case (and as long as β is uniformly integrable, i.e. $E\{\beta|\beta \geq k\}P(\beta \geq k) \to 0$ as $k \to \infty$), regional parliaments should keep decision powers.

CASE B. Negative externalities. The union parliament should retain decision powers if:

$$E\{-\mathcal{C}+N\widehat{\frac{1}{1+d_{\tilde{i}j}}}\beta|\beta \leq -\frac{C}{N}(1+d_{\tilde{i}j}^{ME^*})\}P(\beta \leq -\frac{C}{N}(1+d_{\tilde{i}j}^{ME^*}))-E\{-\mathcal{C}+N\widehat{\frac{1}{1+d_{\tilde{i}j}}}\beta|\beta \geq \mathcal{C}\}P(\beta \geq \mathcal{C}) \geq 0 \tag{11}$$

where $\tilde{i}$ is the hypothetical average region such that $N[E_{\tilde{i}}\{-\mathcal{C}+\beta_{\tilde{i}}-\sum_{j=-\tilde{i}} \frac{\beta_{\tilde{i}}}{1+d_{\tilde{i}j}}|U\}-E_{\tilde{i}}\{-\mathcal{C}+\beta_{\tilde{i}}-\sum_{j=-\tilde{i}} \frac{\beta_{\tilde{i}}}{1+d_{\tilde{i}j}}|P\}] = \sum_{i=1}^{N} E_i\{-\mathcal{C}+\beta_i-\sum_{j=-i} \frac{\beta_i}{1+d_{ij}}|U\}-\sum_{i=1}^{N} E_i\{-\mathcal{C}+\beta_i-\sum_{j=-i} \frac{\beta_i}{1+d_{ij}}|P\}$ and $\widehat{\frac{1}{1+d_{\tilde{i}j}}} \equiv \frac{1}{N}(1-\sum_{j=-\tilde{i}} \frac{1}{1+d_{\tilde{i}j}})$. Otherwise, the regional parliaments would fare better.

Note that, again, social preferences will prefer regional parliaments if the average region is isolated (i.e., $d_{\tilde{i}j}^{ME^*} \to \infty$ and β is uniformly integrable) (recall footnote 3!), whereas results will depend on $F(\cdot)$ when $d_{\tilde{i}j}^{ME^*} \to 0$.

A.1.2 Individual ex-ante preferences

Equation 8 summarizes when region i will vote for the union parliament in $t = 1$. Note that, since the constitution is approved by majority voting, the union will have the decision powers when:

$$Me[\{E_i\{-\sum_{i=1}^{N}\frac{\mathcal{C}_i}{N}+\beta_i+\sum_{j=-i}\varepsilon_{ji}(d_{ij})|U\}-E_i\{-\mathcal{C}_i+\beta_i+\sum_{j=-i}\varepsilon_{ji}(d_{ij})|P\}\}_{i=1}^{N}]\geq 0 \tag{12}$$

i.e., the median voter prefers the union parliament to the regional parliament.

In the simplified cases where $\varepsilon_{ji}(d_{ij}) = \pm\frac{\beta_j}{1+d_{ij}}$, $F_i(\cdot) = F(\cdot)$ and $\mathcal{C}_i = \mathcal{C}\ \forall i = \{1, 2...N\}$, this is:

CASE A. Positive externalities. The union parliament will have decision powers when:

$$E\{-\mathcal{C}+N\overline{\frac{1}{1+d_{\tilde{i}j}}}\beta|\beta\geq\frac{C}{N}(1+d_{\tilde{i}j}^{ME})\}P(\beta\geq\frac{C}{N}(1+d_{\tilde{i}j}^{ME}))-E\{-\mathcal{C}+N\overline{\frac{1}{1+d_{\tilde{i}j}}}\beta|\beta\geq\mathcal{C}\}P(\beta\geq\mathcal{C})\geq 0, \tag{13}$$

where $\tilde{i}$ is the median of the set $\{E\{-\mathcal{C}+N\overline{\frac{1}{1+d_{ij}}}\beta|\beta\geq\frac{C}{N}(1+d_{ij}^{ME})\}P(\beta\geq\frac{C}{N}(1+d_{ij}^{ME}))-E\{-\mathcal{C}+N\overline{\frac{1}{1+d_{ij}}}\beta|\beta\geq\mathcal{C}\}P(\beta\geq\mathcal{C})\}_{i=1}^{N}$ and $\overline{\frac{1}{1+d_{ij}}}\equiv\frac{1}{N}\sum_{j=1}^{N}\frac{\beta_i}{1+d_{ij}}$. Otherwise, regional parliaments will have decision powers.

Limit cases are the same as the ones derived from equation 10.

CASE B. Negative externalities. The union parliament will have decision powers when:

$$E\{-\mathcal{C}+N\widehat{\frac{1}{1+d_{\tilde{i}j}}}\beta|\beta\leq-\frac{C}{N}(1+d_{\tilde{i}j}^{ME^*})\}P(\beta\leq-\frac{C}{N}(1+d_{\tilde{i}j}^{ME^*}))-E\{-\mathcal{C}+N\widehat{\frac{1}{1+d_{\tilde{i}j}}}\beta|\beta\geq\mathcal{C}\}P(\beta\geq\mathcal{C})\geq 0, \tag{14}$$

where $\tilde{i}$ is the median of the set $\{E\{-\mathcal{C}+N\widehat{\frac{1}{1+d_{ij}}}\beta|\beta\leq-\frac{C}{N}(1+d_{ij}^{ME^*})\}P(\beta\leq-\frac{C}{N}(1+d_{ij}^{ME^*}))-E\{-\mathcal{C}+N\widehat{\frac{1}{1+d_{ij}}}\beta|\beta\geq\mathcal{C}\}P(\beta\geq\mathcal{C})\}_{i=1}^{N}$ and $\widehat{\frac{1}{1+d_{ij}}}\equiv\frac{1}{N}(1-\sum_{j=-i}\frac{1}{1+d_{ij}})$. Otherwise, regional parliaments will have decision powers.

Limit cases are the same as the ones derived from equation 11.