Operators

➜ An operator is a mapping between functions in its domain (input) and functions (output) in its range
   ◆ Examples:
   ● Multiplication operator
   ● Differentiation operator
   ● Integral operator

   ◆ Classes:
   ◆ Linear operators: If A and B are linear operators, f and g functions, and k a constant, then (A+B)f=Af+Bf, A(f+g)=Af+Ag, Ak=kA
   ◆ Differential Operators: include differentiation of the functions to which they are applied

$$\hat{A}\,(\psi) = \varphi$$
;

Figure 1. Â is an operator, acting on a function (ψ). Therefore, Â will map one state vector into another one.
Source: Quantum Mechanical Operators - Quantum Chemistry - PSIBERG.

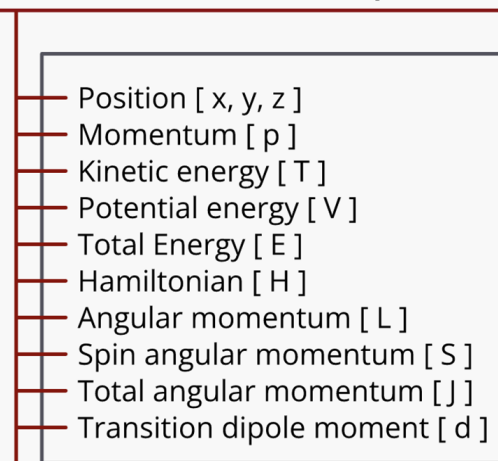For every parameter that we can measure in physical systems, a quantum mechanical operator exists.

Figure. Examples of quantum mechanical operators. Source: Quantum Mechanical Operators - Quantum Chemistry - PSIBERG

Commutation of Operators

Differential operators act on the function(s) to their right, they do not necessarily commute with other operators containing the same independent variable. The commutator of operators A and B is therefore useful: $[A, B] = AB - BA$. Let's do an example: We will evaluate the commutator $[x, p]$ where $p = -id/dx$. We apply $[x, p]$ to an arbitrary function $f(x)$.

$[x, p]f(x) = (xp - px)f(x)$
$= - ixdf(x)/dx - (-id/dx)(xf(x))$
$= - ixf'(x) + i(f(x) + xf'(x)) = if(x)$

Therefore, we see that $[x, p] = i$. This also indicates that $[x, p]f(x) = if(x)$ for all f.

In general, if A, B, and C are operators and k is a constant, we have the following properties:

$$[A, B] =- [B, A]$$
$$[A, B + C] = [A, B] + [A, C]$$
$$k[A, B] = [kA, B] = [A, kB]$$

Identity, Inverse, Adjoint

The identity is an operator that leaves a function unchanged. It is generally denoted by I.

Some operators have an inverse, which is an operator that will "undo" its effect. Let A denote an operator, then A^{-1} will donte the inverse of A. Furthermore, the inverse will have the following property: $A^{-1}A = AA^{-1} = I$.

Associated with many operators will be another one called its adjoint. This operator is denoted: A^+. For all functions f and g in Hilbert space, we have the following: $< f|Ag >=< A^+f|g >$. We can see that if we apply A to the right of any scalar product produces the same result as if we apply A^+ to the left of any scalar product. If we have the case that $A = A^+$, then the operator A is defined as self-adjoint or Hermitian. If we have that $A^+ =- A$ then A is called anti-Hermitian.

In addition, if the adjoint of an operator is equal to its inverse, $A^{-1} = A^+$, then the operator A is called unitary. [275-277]

Sources:

Mathematical Methods for Physicists by Arfken, George B. & Webber. Pages 275-277.
Quantum Mechanical Operators - Quantum Chemistry - PSIBERG