then there are two equilibria in the domain of the differential equation.

39. In this problem we will determine the stability of equilibria in an SIRS model that includes mortality. Consider a population of size $N = 100$. The SIRS model with mortality for this population is:

$$\begin{aligned}\frac{dS}{dt} &= -\frac{1}{200}SI + \frac{1}{10}R + \frac{1}{3}I\\ \frac{dI}{dt} &= \frac{1}{200}SI - \frac{2}{3}I\\ \frac{dR}{dt} &= \frac{1}{3}I - \frac{1}{10}R\end{aligned}$$

(a) What is the mortality rate for this disease?

(b) Rewrite the system of differential equations as a part of differential equations with S and I as dependent variables.

(c) Find all equilibria lying within the domain of the system.

(d) By linearizing the differential equations around the equilibria that you discovered in (c), classify each of the equilibria (e.g., as stable node, spiral, or saddle).

40. In this problem we will determine the stability of equilibria in an SIRS model that includes mortality. Consider a population of size $N = 250$. Assuming a mortality rate $m = 1/4$, our SIRS model becomes:

$$\begin{aligned}\frac{dS}{dt} &= -\frac{1}{500}SI + \frac{1}{5}R + \frac{1}{4}I\\ \frac{dI}{dt} &= \frac{1}{500}SI - \frac{1}{3}I\\ \frac{dR}{dt} &= \frac{1}{12}I - \frac{1}{5}R\end{aligned}$$

(a) Write the system of differential equations as a part of differential equations with S and I as dependent variables.

(b) Find all equilibria lying within the domain for this model.

(c) By linearizing the differential equations around the equilibria that you discovered in (c), classify each of the equilibria (e.g., as stable node, spiral, or saddle).

41. Assume that all individuals in the population are equally likely to be parents to the mI offspring added to the population in each unit of time. If an offspring is born to an infected parent it will be born infected (i.e., into the infectious class). Similarly, offspring born to susceptible parents are susceptible, and offspring born to recovered parents are recovered. Derive an SIRS model to describe the spread of this disease. There is no need to analyze your model.

42. Lifelong Immunity A particular infectious disease confers lifelong immunity to any individual who recovers from the disease. The population size is $N = 200$. Assume that the spread of the disease can be described by an SIR model:

$$\begin{aligned}\frac{dS}{dt} &= -\frac{1}{100}SI\\ \frac{dI}{dt} &= \frac{1}{100}SI - 6I\\ \frac{dR}{dt} &= 6I\end{aligned}$$

Assuming that $R(0) = 0$ initially and $I(0) = 5$, calculate a bound on the maximum number of individuals who will catch the disease.

43. Effect of Vaccination A particular infectious disease confers lifelong immunity to any individual who recovers from the disease. The population size is $N = 100$. Assume that the spread of the disease can be described by an SIR model:

$$\begin{aligned}\frac{dS}{dt} &= -\frac{1}{300}SI\\ \frac{dI}{dt} &= \frac{1}{300}SI - \frac{1}{9}I\\ \frac{dR}{dt} &= \frac{1}{9}I\end{aligned}$$

(a) Assuming that $R(0) = 0$ initially and $I(0) = 5$, calculate a bound on the maximum number of individuals who will catch the disease.

(b) Assume that a vaccination program means that half of the population start out immune to the disease, i.e., $R(0) = 50$. Assume also that there are initially 5 infected individuals (i.e., $I(0) = 5$). Recalculate the maximum bound on the number of individuals who will eventually catch the disease.

Relapsing Infections

In some diseases (such as herpes simplex), an individual may apparently recover from the disease, and in fact gain immunity to it, but the disease continues to be harbored in the person's body, breaking out some time after they recover from the initial infection. To model this process we will modify our SIRS model as follows: Since there is no loss of immunity, $a = 0$. However, in each unit of time a fraction r (r is a constant called the rate of relapse) of the individuals from the recovered class become infected with the disease. In Problems 44–47 you will analyze models for relapsing infections.

44. (a) Write down a system of differential equations to describe the spread of this disease through the population.

(b) Find all of the equilibria that lie in the domain of the system of differential equations you derived in part (a).

45. Assume that the following model can be used to represent the spread of a relapsing infection in a population of size $N = 100$ and with relapse rate $r = \frac{1}{100}$:

$$\begin{aligned}\frac{dS}{dt} &= -\frac{1}{50}SI\\ \frac{dI}{dt} &= \frac{1}{50}SI - \frac{1}{10}I + \frac{1}{100}R\\ \frac{dR}{dt} &= \frac{1}{10}I - \frac{1}{100}R\end{aligned}$$

(a) What is the domain for this differential equation system?

(b) Find all of the possible equilibria for this system of differential equations.

(c) Use the fact that $S + I + R = 100$ to eliminate S from the system, and to write it as a pair of differential equations with I and R as dependent variables.

(d) By linearizing the differential equation system near each of the equilibria that you discovered in part (b), classify these equilibria (e.g., as a stable node, spiral, or saddle).

46. Assume that the following model can be used to represent the spread of the infection in a population of size $N = 250$ and with relapse rate $r = \frac{1}{50}$:

$$\frac{dS}{dt} = -\frac{1}{100}SI$$
$$\frac{dI}{dt} = \frac{1}{100}SI - \frac{1}{5}I + \frac{1}{50}R$$
$$\frac{dR}{dt} = \frac{1}{5}I - \frac{1}{50}R$$

(a) What is the domain for this differential equation system?

(b) Find all of the possible equilibria for this system of differential equations.

(c) Use the fact that $S + I + R = 250$ to eliminate S from the system and to write it as a pair of differential equations with I and R as dependent variables.

(d) By linearizing the differential equation system near each of the equilibria that you discovered in part (b), classify these equilibria (e.g., as a stable node, spiral, or saddle).

(e) We will now sketch the direction of flow for the system in the IR-plane.

(i) First sketch the $\frac{dI}{dt} = 0$ and $\frac{dR}{dt} = 0$ isoclines.

(ii) Add arrows to show the directions of the vector field on the isoclines that you drew in part (i). Then add arrows showing the direction of the vector field in the regions between isoclines.

Chapter 11 Review

Key Terms

Discuss the following definitions and concepts:

1. Linear first-order equation
2. Homogeneous
3. Vector field, direction vector
4. Solution of a system of linear differential equations
5. Eigenvalue, eigenvector
6. Superposition principle
7. General solution
8. Stability
9. Sink, or stable node
10. Saddle point
11. Source, or unstable node
12. Spiral
13. Euler's formula
14. Compartment model
15. Conserved quantity
16. Harmonic oscillator
17. Models for love
18. Nonlinear autonomous system of differential equations
19. Critical point
20. Zero isoclines, or null clines
21. Graphical approach to stability
22. Lotka–Volterra model of competition
23. Monoculture
24. Founder control, coexistence
25. Lotka–Volterra predator–prey model
26. Community matrix
27. Fitzhugh–Nagumo model
28. Bistable
29. Excitable
30. Michaelis–Menten law
31. Chemostat model
32. SIRS model
33. Endemic disease
34. Basic reproductive number

Review Problems

In Problems 1–4 classify the equilibrium point at $(x, y) = (0, 0)$.

1. $\frac{dx}{dt} = 2x + 3y, \quad \frac{dy}{dt} = -2y$

2. $\frac{dx}{dt} = 2y, \quad \frac{dy}{dt} = 3x$

3. $\frac{dx}{dt} = 2y, \quad \frac{dy}{dt} = -3x$

4. $\frac{dx}{dt} = x + y, \quad \frac{dy}{dt} = y - x$

5. Two-Compartment Model Matter flows into and between two compartments, as shown in Figure 11.86.

(a) Based on the flows shown in the figure, write down differential equations for the amounts of material $x_1(t)$ and $x_2(t)$ in the two compartments.

(b) Find the equilibrium values for $\hat{x}_1$ and $\hat{x}_2$.

(c) By linearizing your equations from part (a), determine whether the equilibrium you found in part (b) is stable or unstable.

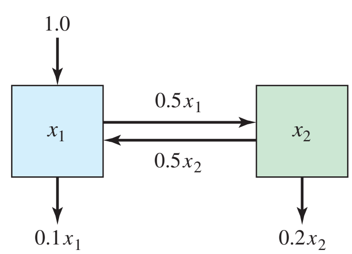

Figure 11.86 Problem 5

6. Carbohydrate Flow in a Forest As we saw in Section 11.3, plants receive nitrogen from microorganism partners, fungi and bacteria, in their roots. In return, the plant shares the carbohydrates that it makes by photosynthesis with those partners. We will build a two-compartment model for this flow of carbohydrates, with one compartment representing the free carbohydrates stored in a plant, and the other representing the free carbohydrates in the plant's fungal partners.

Suppose that each day the plant produces 1 g of free carbohydrate. Now assume that each day 50% of the free