CS 224: Advanced Algorithms	Spring 2017

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1 Overview

In the last lecture, we analyzed Ford-Fulkerson. In this lecture, we consider two ways to find max flows more quickly than Ford-Fulkerson.

2 Scaling Capacities

Recall that the Ford-Fulkerson algorithm has runtime $O(mf^*)$, where f^* is the maximum flow. We want to improve on the factor of f^*. The idea is to think of each capacity $c(e) \in \{1, 2 \dots U\}$ as a bit integer $u(e) \in \{0,1\}^{\lceil \log_2 U \rceil}$. Then, we will "process" each $u(e)$ one bit at a time (from left to right), maintaining capacities $u'(e) = u(e)_{n \dots (n-k)}$ (th k most significant bits of u) and a flow f' such that f' is feasible on $u'(e)$. We will actually ensure that f' is optimal at the beginning of each iteration. Then, when we finish, $u'(e) = u(e)$ and we will have $f' = f^*$. The complete scaling algorithm is given below. This algorithm is due to Gabow [2].

2.1 Scaling Max-Flow Algorithm

- initialize $u'(e) = 0, f'_e = 0$ for all $e \in E$.
- for $k = (\lceil \log_2 U \rceil - 1) \dots 0$
  - for each $e \in E$
    - set $u'(e) = 2u'(e)$ and $f'_e = 2f'_e$ (add trailing 0 to bit vectors of u', f')
  - for each $e \in E$
    - set $u'(e) = u'(e) + u(e)_k$ (add k'th bit of u to u')
  - augment f' such that it is a max flow for G with capacities $u'(e)$

2.2 Analysis

Claim. *At the beginning of every iteration of the scaling algorithm, $f' = (f')^*$, the maximum flow on G with capacities $u'(e)$.*

Proof. At the beginning of the first iteration $u'(e) = 0$ $\forall e \in E$ so $(f')^* = 0 = f'$. Now, suppose that $f' = (f')^*$ before we update $u'(e)$. Then, there must exist some saturated $S-T$ cut in (G, u'). After we double $u'(e)$ and f'_e, the cut is still saturated. After we add $u(e)_k$ to u', f' may not be maximal, but it is still feasible. But then we augment f' so that it must equal $(f')^*$ on the new capacities. □

At the k'th iteration of the algorithm $u'(e)$ is the k most significant bits of $u(e)$, so after $\lceil \log_2 U \rceil$ iterations $u'(e) = u(e)$. Hence, by the claim, the output f' of the scaling algorithm will be the maximal flow for G with capacities $u'(e) = u(e)$, as desired.

The scaling algorithm runs for $O(\log_2 U)$ iterations, each iteration dominated by the time to augment f' into a maximal flow. If we use Ford-Fulkerson, we know that this time will be $O(mf^*)$. What can we say about f^*?

Claim. *In each iteration of the scaling algorithm, $f' = f^* \leq m$.*

Proof. At the beginning to the current iteration $f' = (f')^*$, so there must exist some saturated $S-T$ cut of (G, u'). Now, as before, after we double $u'(e), f'_e$, this cut will still have 0 capacity. After we set $u'(e) = u(e)_k$, we can have increased $u'(e)$ by at most 1. Since the cut cannot contain more than m edges, its residual capacity is $\leq m$ now, so the minimum cut in (G, u') must have capacity $\leq m$ which implies that $f^* \leq m$ (by the max-flow min-cut theorem). □

Hence, if we use Ford-Fulkerson to augment f', each iteration takes $O(m + mf^*) = O(m^2)$ steps, so we can achieve a runtime of $O(m^2 \log U)$ for maximum flow using the scaling algorithm. Hence, the scaling algorithm is weakly polynomial.

3 Blocking Flow

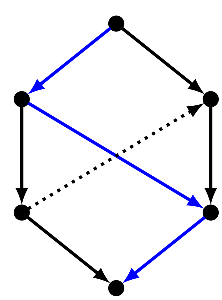
Suppose we have a residual graph G_f. For any $v \in V$, let its **level** ℓ_v denote the length of the shortest path distance from s to v in G_f.

Definition. *An edge $(u, v) \in E$ is **admissible** if $\ell_v = \ell_u + 1$. An s-t flow on G_f is **admissible** if it traverses only admissible edges.*

Definition. *The **level graph** L associated with G_f is the subgraph of G_f comprising only its admissible edges.*

Definition. *A **blocking flow** f is an admissible s-t flow that saturates at least one edge in every other admissible flow f'.*

Example. *Consider the following unit-capacity residual graph.*



The blue flow is a blocking flow but clearly not a max flow. On the other hand, by definition, every admissible max flow is a blocking flow. Moreover, all edges here except for the dotted one are in the associated level graph.

3.1 A new algorithm for max flow

Recall that one of the shortcomings of Ford-Fulkerson is that the algorithm can run up to f^* iterations. We will see how blocking flows remedy this problem.

Lemma 3.1. *Augmenting along a blocking flow strictly increases $d(s, t)$ in the new level graph.*

Proof. Let L' denote the level graph after augmenting along a blocking flow. Since we augmented along a flow in L, it is clear that

$$L' \subseteq L \cup \mathrm{rev}(L) \cup (G_f \backslash L),$$

where $\mathrm{rev}(L)$ denotes the set comprising the edges in L reversed. But this implies that $d_{L'}(s, t) \geq d_L(s, t)$. However, if $d_{L'}(s, t) = d_L(s, t)$, then L and L' must share some s-t path, which contradicts the fact that we constructed L' from L by augmenting along a blocking flow. $\square$

Why is this useful? Consider the following algorithm, given by Dinitz in [1].

1. Initialize f to 0.
2. While there exists an s-t path in G_f, augment f along a blocking flow.

It is clear that when we terminate the process, we have found a max flow.

Observation 1. *Dinitz's runtime is given by* (*# iterations*) $\times$ (*time to find blocking flow*)*.*

Lemma 3.1 implies that the first term is at most $n-1$. We will consider the second term below.

3.2 Finding a blocking flow

We do so by running a modified version of DFS on L from s to build a blocking flow $f \in \mathbb{R}^m$.

We initialize f to 0 and H to L. From our current vertex v, we may run three kinds of moves.

- advance(v, w): move from v to w for $(v, w) \in L$,
- retreat(u, v): if $\nexists\ (v, w) \in L$, return its ancestor u in the call stack and delete (u, v) from H,
- augment: if $v = t$, augment f along the minimum residual flow on the found s-t path and delete saturated edges.

3.2.1 Runtime analysis

Claim. *In unit capacity graphs, Section 3.2's algorithm considers each edge $O(1)$ times, which implies $O(m)$ time to find a blocking flow.*

Proof. Observe that the algorithm

- advances at most once along any edge because L is a DAG by construction,
- retreats at most once along any edge because we delete it afterwards,
- augments at most once along any edge because one augmentation immediately saturates it.

It follows that the algorithm processes each edge only a constant number of times. $\square$

Then Observation 1 implies the following.

Corollary. *In unit-capacity graphs, Dinitz's algorithm runs in $O(mn)$ time.*

Note that in unit-capacity graphs, the cut crossing the edges from s encounters at most $n-1$ units of flow. Then min-cut implies that $f^* \leq n-1$, so Ford-Fulkerson runs in $O(mn)$ time.

It seems like our new algorithm exhibits no improvement. But it turns out that it guarantees better performance on unit-capacity graphs.

Claim. *In unit-capacity graphs, Dinitz's algorithm runs* $O(\min\{\sqrt{m},\, n^{2/3}\})$ *iterations of finding blocking flows.*

Proof. $\sqrt{\mathbf{m}}$**-bound.** Lemma 3.1 implies that $d(s,\, t) \geq k$ after k iterations of finding blocking flows. Unit capacity implies that any s-t flow can be decomposed into an edge-disjoint collection of s-t paths, so at most $\frac{m}{k}$ s-t path remain. Therefore, the residual max flow (equivalently, the number of iterations) is at most $\frac{m}{k}$. This means that the total number of iterations is at most $k + \frac{m}{k}$, which equals $2\sqrt{m}$ when $k = \sqrt{m}$.

$\mathbf{n^{2/3}}$**-bound.** In L, define $D_i := \{v \in V : \ell_v = i\}$. Suppose that we have already run $k \geq 2n^{2/3}$ iterations, so $0 \leq i \leq k$. Since at most half of the D_i contain at least $n^{1/3}$ vertices, it follows that there must exist consecutive sets D_i, D_{i+1} such that both contain at most $n^{1/3}$ vertices. Therefore, the number of edges from D_i to D_{i+1} is at most $|D_i|\cdot|D_{i+1}| \leq n^{2/3}$. This means that the cut crossing these edges encounters at most $n^{2/3}$ units of flow. Then min-cut implies that the remaining flow is at most $n^{2/3}$, so we run at most $2n^{2/3} + n^{2/3} = O(n^{2/3})$ iterations in total. □

Corollary. *In unit-capacity graphs, Dinitz's algorithm runs in* $O(m \min\{\sqrt{m},\, n^{2/3}\})$ *time.*

3.3 General capacity graphs

We run the same algorithm, although the runtime now differs.

- We retreat at most once along each edge, so retreats contribute $O(m)$ time.
- We saturate at least one edge each time we augment, so we augment at most m times. The length of an s-t path is at most n, so augmentations contribute $O(mn)$ time.
- We can advance at most n times before retreating, so advances contribute $O(mn)$ time.

This implies the following.

Claim. *In general-capacity graphs, Dinitz's runs* $O(mn)$ *iterations of finding blocking flows.*

Corollary. *In general-capacity graphs, Dinitz's runs in* $O(mn^2)$ *time.*

It is also possible to show the in general capacity graphs, Dinitz' algorithm has runtime $O(mn + nf^*)$. Then since f^* in each round of scaling is at most m (since the previously saturated minimum cut now has at most m units of added capacity), each round of scaling when using Dinitz' algorithm would take $O(mn)$ time. Thus using the capacity-scaling approach with Dinitz' algorithm used in each iteration would yield an overall runtime of $O(mn \log U)$.

References

[1] Yefim Dinitz. Algorithm for solution of a problem of maximum flow in a network with power estimation. *Dokl. Akad. Nauk SSSR*, 11(5):1277–1280, 1970.

[2] Harold N. Gabow. Scaling Algorithms for Network Problems. *J. Comput. Syst. Sci.* 31(2), pages 148–168, 1985.