MATH 256B - ALGEBRAIC GEOMETRY LECTURE NOTES

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Lecture 1

Why Cohomology?

Why study cohomology? Well, it is useful for determining $\mathscr{F}(X) = \Gamma(X, \mathscr{F})$ where X is a scheme and $\mathscr{F}$ is a sheaf of abelian groups on X.

Key Tool

Our key tool (so far) is Exercise II.1.8: If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves (of abelian groups), then

$$0 \to \Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'')$$

is exact. It is not always exact on the right. For example, Exercise II.1.21c: let k be a field, $X = \mathbb{P}^1_k$, $P = [1,0], Q = [0,1], Y = \{P, Q\}$ with reduced induced subscheme structure, and let $\mathscr{I}_Y$ be the sheaf of ideals of Y. Then

$$0 \to \mathscr{I}_Y \to \mathscr{O}_X \to \underbrace{\mathscr{O}_X/\mathscr{I}_Y}_{i_*\mathscr{O}_X, i: Y \hookrightarrow X} \to 0$$

is exact, but applying the $\Gamma(X, -)$ functor, we have

$$0 \to \underbrace{\Gamma(X, \mathscr{I}_Y)}_{0} \to \underbrace{\Gamma(X, \mathscr{O}_X)}_{k} \to \underbrace{\Gamma(X, \mathscr{O}_X/\mathscr{I}_Y)}_{k^2}$$

where the right arrow cannot be surjective.

Application 1

Given an exact sequence

$$0 \to \Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'')$$

we get a long exact sequence in cohomology

$$\begin{aligned} 0 \to\ & \Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'') \to \\ \to\ & H^1(X, \mathscr{F}') \to H^1(X, \mathscr{F}) \to H^1(X, \mathscr{F}'') \to \\ \to\ & H^2(X, \mathscr{F}') \to \cdots \end{aligned}$$

Sometimes you can show $H^1(X, \mathscr{F}') = 0$. Notation: $H^0(X, \mathscr{F}) = \Gamma(X, F)$.

Application 2

If X is a scheme over a field k, then $H^i(X, \mathscr{F})$ is a k-vector space (where $\mathscr{F}$ is a sheaf of $\mathscr{O}_X$-modules). Define

$$h^i(X, \mathscr{F}) = \dim_k H^i(X, \mathscr{F}).$$

Then the Riemann-Roch theorem gives a formula for $\displaystyle\sum_{i=0}^{\infty} (-1)^i h^i(X, \mathscr{F})$

How should it look?

The first part of this document is based on Tony's notes from Professor Vojta's lectures. Two (I think) of the lectures were taken from Dave Brown. Send comments and corrections to `geraschenko@gmail.com` .

1) We want it to have a long exact sequence (LES).
2) The LES should be functorial in the short exact sequence (SES). That is, given a morphism of short exact sequences, i.e. a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F}' \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{G}' \arrow[r] & \mathscr{G} \arrow[r] & \mathscr{G}'' \arrow[r] & 0 \end{tikzcd} \tag{1}$$

there should be an induced morphism of long exact sequences, i.e. a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^0(X,\mathscr{F}') \arrow[r] \arrow[d] & H^0(X,\mathscr{F}) \arrow[r] \arrow[d] & H^0(X,\mathscr{F}'') \arrow[r] \arrow[d] & H^1(X,\mathscr{F}') \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & H^0(X,\mathscr{G}') \arrow[r] & H^0(X,\mathscr{G}) \arrow[r] & H^0(X,\mathscr{G}'') \arrow[r] & H^1(X,\mathscr{G}') \arrow[r] & \cdots \end{tikzcd} \tag{2}$$

in a functorial way.

So lets require that $\mathscr{F} \mapsto H^i(X,\mathscr{F})$ be a functor $\mathcal{A}b(X) \xrightarrow{H^i} \mathcal{A}b$, where $\mathcal{A}b(X)$ is the category of sheaves of abelian groups on X and $\mathcal{A}b$ is the category of abelian groups.

Then we get

- 2/3 of the maps in the LES and all the vertical maps in (2)
- commutativity of 2/3 of the squares in (2)
- SES$\mapsto$ LES is functorial

So we need

- to find functors $\mathscr{F} \mapsto H^i(X,\mathscr{F})$ for each $i \in \mathbb{N}$
- for all short exact sequences, $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$, and for all $i \in \mathbb{N}$, maps
$$\delta^i : H^i(X,\mathscr{F}'') \to H^{i+1}(X,\mathscr{F}')$$
such that the LES is exact and so that for all diagrams (1), the diagram commutes:
$$\begin{tikzcd} H^i(X,\mathscr{F}'') \arrow[r, "\delta^i"] \arrow[d] & H^{i+1}(X,\mathscr{F}') \arrow[d] \\ H^i(X,\mathscr{G}'') \arrow[r, "\delta^i"] & H^{i+1}(X,\mathscr{G}') \end{tikzcd}$$
- an isomorphism of functors $H^0(X,\mathscr{F}) \cong \Gamma(X,\mathscr{F})$

Some Definitions

Definition. An *abelian category* is a category $\mathcal{A}$ together with

(i) a structure of an abelian group on $Hom(A,B$ for all objects $A, B \in \mathcal{A}$.
(ii) fore every morphism $A \to B$, a kernel $A' \to A$ and a cokernel $B \to B'$.

such that

(1) $Hom(A,B) \times Hom(B,C) \to Hom(A,C)$ is a bilinear map.

(2) Finite sums and products exist.
(3) Every monomorphism[1] is the kernel of its cokernel.
(4) Every epimorphism[2] is the cokernel of its kernel.
(5) Every morphism can be factored into an epimorphism followed by a monomorphism.

Some examples:

$\mathcal{A}b$
$\mathcal{A}b(X)$, where X is a topological space
$\mathcal{M}od(X)$, the category of $\mathscr{O}_X$-modules where X is a scheme

Remark. - All kernels are monomorphisms and all cokernels are epimorphisms.
- If $\mathcal{A}$ is an abelian category, the so is $\mathcal{A}^{\mathrm{op}}$.
- The empty product (resp. coproduct) is an initial (final) object; call it 0 (0'). There is a (doubly unique) morphism $0 \to 0'$, which is an isomorphism (exercise).

Definition. A covariant functor $F : \mathcal{A} \to \mathcal{B}$ of abelian categories is *additive* if for all $A, A' \in \mathcal{A}$, the induced map $Hom(A, A') \to Hom(FA, FA')$ is a homomorphism of abelian groups. Similarly for contravariant functors.

We will want $H^i(X, -)$ to be additive.

Definition. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B if $g \circ f = 0$ and the homology of the sequence (at B) is 0.

Definition. A *complex* $A^{\cdot}$ in an abelian category $\mathcal{A}$ is a set of objects A^i and morphisms $d^i : A^i \to A^{i+1}$ for all $i \in \mathbb{Z}$ such that $d^{i+1} \circ d^i = 0$. By convention, if A^i is only given for $i \in I \subset \mathbb{Z}$, we assume $A^i = 0$ for $i \in \mathbb{Z} \smallsetminus I$.

Theorem 1.1 (Freyd's Embedding Theorem)**.** *Every abelian can be embedded as a full subcategory of $\mathcal{A}b$*

Definition. An object in an abelian category $I \in \mathcal{A}$ is *injective* if the functor $Hom(-, I)$ is exact[3].

Definition. A *resolution* of an object A is a complex $I^{\cdot}$ and a morphism $A \to I^0$ such that

$$0 \to A \to I^0 \to I^1 \to \cdots$$

is exact. An *injective resolution* is a resolution where each I^i is injective.

Lecture 2

$$\begin{tikzcd} \ker f \arrow[d] & \mathrm{im} f \arrow[d] \\ A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ \mathrm{coim} f & \mathrm{coker} f \end{tikzcd}$$

Definition. Given a morphism $f : A \to B$, the *image* of f is the kernel of its cokernel, and the *coimage* of f is the cokernel of its kernel.

Remark. (1) If $f_i : A_i \to B$ for $i = 1, 2$ are monomorphisms, then there is at most one $g : A_1 \to A_2$ such that $f_2 \circ g = f_1$ ($\mathscr{H}om(A_1, A_2) \hookrightarrow \mathscr{H}om(A_1, B)$ since f_2 mono). Thus, there is a partial ordering on monomorphisms to B.

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[1] $A \to B$ monomorphism if for all C, $Hom(C, A) \to Hom(C, B)$ is injective.
[2] $A \to B$ epimorphism if for all C, $Hom(B, C) \to Hom(A, C)$ is injective.
[3] The contravariant functor $Hom(-, I)$ is always left exact.

(2) If $A \xrightarrow{f} B \xrightarrow{g} C$ with $g \circ f = 0$, then $\mathrm{im} f \subseteq \ker g$. We define the *homology* at B to be $\ker g/\mathrm{im} f = \ker(\mathrm{coker} f \to \mathrm{coim} g)$.

Definition. An abelian category has *enough injectives* if every object is a subobject of an injective object. That is, there is a monomorphism from any object to an injective object.

Lemma 2.1. *In an abelian category with enough injectives, every object has an injective resolution.*

Proof. Given $A \in \mathcal{A}$, there is a monomorphism to some injective object I^0. Assume inductively that we have $0 \to A \to I^0 \to \cdots \to I^{n-1} \xrightarrow{d^{n-1}} I^n$ exact. Then there must be a monomorphism $\mathrm{coker} d^{n-1} \to I^{n+1}$ with I^{n+1} injective. Then the sequence $0 \to A \to I^0 \to \cdots \to I^n \to I^{n+1}$ □

Remark.
(1) We haven't really used injectivity yet.
(2) In the following diagram, exactness of the horizontal sequence is equivalent to exactness of the diagonal sequences, where $K^0 = \mathrm{coker}(\epsilon)$ and $K^{i+1} = \mathrm{coker}(d^i)$

$$\begin{tikzcd}[column sep=tiny]
 & & & 0 \arrow[dr] & & & & 0 \arrow[dr] & & 0 & & \\
 & & & & A \arrow[dr] & & & & K^1 \arrow[ur] \arrow[dr] & & & \\
0 \arrow[rr] & & A \arrow[rrr, "\epsilon"'] & & & I^0 \arrow[rr, "d^0"] \arrow[dr] & & I^1 \arrow[rr, "d^1"'] \arrow[ur] & & I^2 \arrow[rr, "d^2"] \arrow[dr] & & \cdots \\
 & & & & & & K^0 \arrow[ur] \arrow[dr] & & & & K^2 \arrow[ur] \arrow[dr] & \\
 & & & & & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

Let X be a scheme (or a topological space), and supposed we have a cohomology theory for $\mathcal{A}b(X)$.

Definition. A sheaf $\mathscr{F} \in \mathcal{A}b(X)$ is *acyclic* if $H^i(X, \mathscr{F}) = 0$ for all $i > 0$.

Acyclic objects can be used to compute cohomology.

Theorem 2.2. *Suppose X as above with a cohomology theory. If a sheaf $\mathscr{F} \in \mathcal{A}b(X)$ has an acyclic resolution*

$$0 \to \mathscr{F} \to \mathscr{I}^0 \to \mathscr{I}^1 \to \cdots$$

then there is a natural isomorphism $H^i(X, \mathscr{F}) \cong h^i(H^0(X, \mathscr{I}^\cdot))$.

Proof. We have that

$$0 \to \mathscr{F} \to \mathscr{I}^0 \to K^0 \to 0$$

is exact, so we get an exact sequence in cohomology

$$0 \to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{I}^0) \to H^0(X, K^0) \to H^1(X, \mathscr{F}) \to \underbrace{H^1(X, \mathscr{I}^0)}_{0 \quad \mathscr{I}^0 \text{ acyclic}}$$

from which we can say that

$$\begin{aligned}
H^0(X, \mathscr{F}) &= \ker(H^0(X, \mathscr{I}^0) \to H^0(X, K^0)) \\
&= \ker(H^0(X, \mathscr{I}^0) \to H^0(X, \mathscr{I}^1)) \quad (K \hookrightarrow \mathscr{I}^1 \text{ and } H^0 \text{ left exact}) \\
&= h^0(H^0(X, \mathscr{I}^\cdot))
\end{aligned}$$

and

$$\begin{aligned} H^1(X,\mathscr{F}) &= H^0(X,K^0)/\mathrm{im}(H^0(X,\mathscr{I}^0) \to H^0(X,K^0)) \\ &= \frac{\ker(H^0(X,\mathscr{I}^0) \to H^0(X,\mathscr{I}^1))}{\mathrm{im}(H^0(X,\mathscr{I}^0) \to H^0(X,\mathscr{I}^1))} \qquad (H^0(X,-) \text{ left exact}) \\ &= h^1(H^0(X,\mathscr{I}^\cdot)) \end{aligned} \tag{3}$$

We also have that

$$\underbrace{H^i(X,\mathscr{I}^0)}_{0} \to H^i(X,K^0) \overset{\sim}{\to} H^{i+1}(X,\mathscr{F}) \to \underbrace{H^{i+1}(X,\mathscr{I}^0)}_{0}$$

for all $i > 0$. The exact sequences

$$0 \to K^i \to \mathscr{I}^{i+1} \to K^{i+1} \to 0$$

yield the isomorphisms $H^j(X,K^{i+1}) \cong H^{j+1}(X,K^i)$ for all $j > 0$ (since $\mathscr{I}^{i+1}$ is acyclic). Thus,

$$H^i(X,\mathscr{F}) \cong H^{i-1}(X,K^0) \cong \cdots \cong H^1(X,K^{i-2}).$$

But K^{i-2} has the acyclic resolution $K^{i-2} \to \mathscr{I}^{i-1} \to \mathscr{I}^i \to \cdots$, so we may apply formula (3) to get

$$H^1(X,K^{i-2}) \cong h^1(H^0(X,\mathscr{I}^{\cdot+(i-1)})) \cong h^i(H^0(X,\mathscr{I}^\cdot))).$$

as desired. □

Derived Functors

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Given $A \in \mathcal{A}$ with an injective resolution $0 \to A \to I^\cdot$, define

$$R^i_{I^\cdot} F(A) = h^i(F(I^\cdot))$$

for each $i \in \mathbb{N}$.

Lemma 2.3. *Let A and B be objects with injective resolutions $I^\cdot$ and $J^\cdot$ together with a map $f : A \to B$*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\delta"] \arrow[d, "f"] & I^0 \arrow[r, "d^0"] & I^1 \arrow[r, "d^1"] & \cdots \\ 0 \arrow[r] & B \arrow[r, "\epsilon"] & J^0 \arrow[r, "e^0"] & J^1 \arrow[r, "e^1"] & \cdots \end{tikzcd} \tag{4}$$

, then there are functions $f^i : I^i \to J^i$ for each $i \in \mathbb{N}$ such that this diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\delta"] \arrow[d, "f"] & I^0 \arrow[r, "d^0"] \arrow[d, "f^0"] & I^1 \arrow[r, "d^1"] \arrow[d, "f^1"] & \cdots \\ 0 \arrow[r] & B \arrow[r, "\epsilon"] & J^0 \arrow[r, "e^0"] & J^1 \arrow[r, "e^1"] & \cdots \end{tikzcd}$$

Proof. Applying injectivity of J^0 to $\epsilon \circ f : A \to J^0$, we obtain f^0. Now assume inductively that we have $f^0, \ldots, f^n$. Let $K^0 = \mathrm{coker}(\epsilon)$ and $K^i =$

$\mathrm{coker}(d^{i-1})$ for $i > 0$, then by the universal property of cokernels, we have a map $K^n \to J^{n+1}$.

$$\begin{tikzcd} I^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "f^{n-1}"] & I^n \arrow[r] \arrow[d, "f^n"] & K^n \arrow[r] \arrow[d, dotted, "\exists!"] & 0 \\ J^{n-1} \arrow[r] & J^n \arrow[r] & J^{n+1} & \end{tikzcd}$$

And $K^n \hookrightarrow I^{n+1}$, so injectivity of J^{n+1} gives us f^{n+1}. □

Lemma 2.4. *Any two choices of $f^\cdot = \{f^0, f^1, \dots\}$ for the same data (4) are* homotopic*. That is, there is a morphism, $k^\cdot$, of chain complexes of degree -1 such that*

$$e^{n-1}k^n + k^{n+1}d^n = f^n - g^n$$

where $e^{-1} = \epsilon$.

Proof. Arrow chasing. □

Corollary 2.5. *If $f^\cdot$ and $g^\cdot$ are two maps of complexes as in Lemma (2.3), then the induced maps $R^iF(f^\cdot), R^iF(g^\cdot) : R^i_{I^\cdot}F(A) \to R^i_{J^\cdot}F(B)$ are the same, so we can call them $R^i_{I^\cdot,J^\cdot}F(f)$.*

Corollary 2.6. *If $B = A$ and $f = Id_A$, then these maps are isomorphisms.*

Proof. For two injective resolutions $I^\cdot$ and $J^\cdot$ of A, Lemma (2.3) tells us that the identity induces $f^\cdot$ and $g^\cdot$ so that $I^\cdot \xrightarrow{f^\cdot} J^\cdot \xrightarrow{g^\cdot} I^\cdot$. Commutativity of the diagrams and functoriality tell us that

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & I^\cdot \arrow[d, "f^\cdot"] \\ 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & J^\cdot \arrow[d, "g^\cdot"] \\ 0 \arrow[r] & A \arrow[r] & I^\cdot \end{tikzcd}$$

$$Id = R^i_{I^\cdot,I^\cdot}F(Id_A) = \underbrace{R^i_{I^\cdot,J^\cdot}F(Id_A)}_{g^\cdot} \circ \underbrace{R^i_{J^\cdot,I^\cdot}F(Id_A)}_{f^\cdot}$$

and likewise with $I^\cdot$ and $J^\cdot$ interchanged. □

Corollary 2.7. *$R^i_{I^\cdot}F(A)$ is independent of $I^\cdot$ up to unique isomorphism. These isomorphisms are compatible with $R^i_{I^\cdot,J^\cdot}F(f)$ at both ends. Thus, we have well defined functors $R^iF : \mathcal{A} \to \mathcal{B}$ for all $i \in \mathbb{N}$.*

Remark. If I is injective, then it is F-acyclic (i.e. $R^iF(I) = 0$ for all $i > 0$).

> *Proof.* $0 \to I \to I \to 0$ is an injective resolution, so $R^iF(I) = h^i(0 \to \underbrace{F(I)}_{\text{deg } 0} \to 0 \to 0 \to \cdots) = 0$ for $i > 0$. □

Theorem 2.8. *If $F : \mathcal{A} \to \mathcal{B}$ is as above, then $\{R^iF\}_{i>0}$ is a δ-functor with $R^0F \cong F$.*

Proof(sketch). Let $A \in \mathcal{A}$ and let $0 \to A \to I^\cdot$ be an injective resolution, then $0 \to F(A) \hookrightarrow F(I^0) \to F(I^1)$, so $F(A) \cong \ker(F(I^0) \to F(I^1)) =$

$h^0(F(I^\cdot)) = R^0F(A)$. To get the morphisms, chase diagrams until you catch one.

Given a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ with injective resolutions $I^\cdot$ and $J^\cdot$ for A and C, we have

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[rr] \arrow[d] & & B \arrow[rr] \arrow[dll, dotted, "I^0 \text{ inj}"'] \arrow[d, dotted] \arrow[drr] & & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^0 \arrow[rr] \arrow[d] & & I^0 \times J^0 \arrow[rr] & & J^0 \arrow[r] \arrow[d] & 0 \\
 & I^1 & & & & J^1 &
\end{tikzcd}$$

Then by the Snake Lemma, we have the exact sequence

$$0 \to \operatorname{coker}(A \to I^0) \to \operatorname{coker}(B \to I^0 \times J^0) \to \operatorname{coker}(C \to J^0) \to 0$$

and $\operatorname{coker}(A \to I^0) \hookrightarrow I^1$ and $\operatorname{coker}(C \to J^0) \hookrightarrow J^1$, so repeat. Then remove the top row and apply F. Note that the rows are still (split) short exact, and apply the Snake Lemma to get the long exact sequence[4] Verify commutativity of

$$\begin{tikzcd}
R^iF(C) \arrow[r, "\delta"] \arrow[d] & R^{i+1}FA \arrow[d] \\
R^iF(C) \arrow[r, "\delta"] & R^{i+1}FA'
\end{tikzcd}$$

□

The functors R^iF are called the *right derived functors* of F.

Lecture 3

For effaceable functors, see the book.

Recap of what we've done:

(1) Given a covariant left exact functor from an abelian category with enough injectives to an abelian category, there are right derived functors.
(2) A Cohomology should have some desirable properties.

Definition. An abelian group A is *divisible* if for all $a \in A$ and non-zero $n \in \mathbb{Z}$, there is some $a' \in A$ such that $na' = a$.

For example, 0, $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}, \mathbb{R}, \mathbb{R}/\mathbb{Z}, \ldots$ are divisible groups. Injective A-modules

Lemma 3.1. *An abelian group is injective if and only if it is divisible.*

[4] Anton doesn't see how to do this.

Proof. ($\Rightarrow$) Let A be an injective group, and let $a \in A, 0 \neq n \in \mathbb{Z}$, then

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot n"] \arrow[d, "\phi"] & \mathbb{Z} \arrow[dl, dotted, "\psi"] \\ & A & \end{tikzcd}$$

where $\phi(1) = a$. Then by injectivity, ψ exists, and $n \cdot \psi(1) = a$. Thus, A is divisible.

($\Leftarrow$) Let $M' \subseteq M$ be abelian groups, and let $\phi : M' \to A$ be a homomorphism, where A is divisible.

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "\phi"] & M \\ & A & \end{tikzcd}$$

Pick $x \in M \setminus M'$, and let d be a non-negative integer generator of the ideal $\{n \in \mathbb{Z} | nx \in M'\}$. If $d = 0$, then $\langle M', x\rangle \cong M' \oplus \mathbb{Z}$ and we can extend ϕ to $\langle M', x\rangle \to A$ by setting $\phi(x) = 0$. If $d \neq 0$, then pick $a \in A$ such that $da = \phi(dx)$. Then we can extend ϕ to $\langle M', x\rangle$ by setting $\phi(x) = a$. By the standard Zorn's Lemma argument, we can extend ϕ to a map $M \to A$, so A is injective. $\square$

Lemma 3.2. *The category $\mathcal{A}b$ of abelian groups has enough injectives.*

Proof. For an abelian group A, we define the *dual* $\hat{A} = \mathscr{H}om(A, \mathbb{Q}/\mathbb{Z})$. Then a homomorphism $f : A \to B$ has a dual $\hat{f} : \hat{B} \to \hat{A}$. We have a natural map $A \to \hat{\hat{A}} = \mathscr{H}om(\mathscr{H}om(A, \mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}), a \mapsto [\phi \mapsto \phi(a)]$

<u>Claim:</u> This map is one to one.

Indeed, if it were not, then there would be a non-zero $a \in A$ such that $\phi(a) = 0$ for all $\phi \in \hat{A}$. But since $\mathbb{Q}/\mathbb{Z}$ is injective, we have

$$\begin{tikzcd} 0 \arrow[r] & \langle a\rangle \arrow[r] \arrow[d, "\phi_0"'] & A \arrow[dl, dotted] \\ & \mathbb{Q}/\mathbb{Z} & \end{tikzcd}$$

For any $\phi_0 : \langle a\rangle \to \mathbb{Q}/\mathbb{Z}$ we have an extension $\phi : A \to \mathbb{Q}/\mathbb{Z}$ so it suffices to find $\phi_0 : \langle a\rangle \to \mathbb{Q}/\mathbb{Z}$ with $\phi_0(a) \neq 0$. But either $\langle a\rangle \cong \mathbb{Z}$, in which case we can send $a \mapsto 1/2 \in \mathbb{Q}/\mathbb{Z}$, or $\langle a\rangle \cong \mathbb{Z}/n\mathbb{Z}$, in which case we can send $a \mapsto 1/n \in \mathbb{Q}/\mathbb{Z}$. Thus, the claim is true.

So $A \hookrightarrow \hat{\hat{A}}$. Next, there is a surjection $\bigoplus_{i\in I} \mathbb{Z} \to \hat{A} \to 0$ for some index set I. By taking duals, we get a map

$$\begin{aligned} \hat{\hat{A}} \to \widehat{\bigoplus_{i\in I} \mathbb{Z}} &= \mathscr{H}om(\bigoplus \mathbb{Z}, \mathbb{Z}/\mathbb{Q}) \\ &= \prod \mathscr{H}om(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \\ &= \prod \mathbb{Q}/\mathbb{Z} \end{aligned}$$

which is divisible, and therefore injective. Also, the map $\hat{\hat{A}} \to \widehat{\bigoplus_{i\in I}\mathbb{Z}}$ is one to one because the dual of a surjective map is one to one. Thus, $A \hookrightarrow \hat{\hat{A}} \hookrightarrow \prod \mathbb{Q}/\mathbb{Z}$. $\square$

If A is a ring, T is an abelian group, and X is an A-module, then we can put an A-module structure on $\mathscr{H}om_{\mathbb{Z}}(X,T)$ by setting $a\cdot\phi = (x \mapsto \phi(a\cdot x))$ for each $a \in A$

Lemma 3.3. *If A is a ring, X is an A-module, and T is an abelian group, then*

$$\begin{aligned} \mathscr{H}om_{\mathbb{Z}}(X,T) &\xrightarrow{\sim} \mathscr{H}om_A(X, \mathscr{H}om(A,T)) \\ \phi &\mapsto \ (x \mapsto (a \mapsto \phi(ax))) \end{aligned}$$

as abelian groups (also as A-modules).

Proof. exercise. $\square$

Lemma 3.4. *If T is and injective abelian group, then $\mathscr{H}om_{\mathbb{Z}}(A,T)$ is an injective A-module.*

Proof. Given A-modules $0 \to X \to Y$, we have

Exercise: show that the diagram commutes

$$\begin{tikzcd} \mathscr{H}om_A(Y, \mathscr{H}om_{\mathbb{Z}}(A,T)) \arrow[r] \arrow[d, "\wr"] & \mathscr{H}om_A(X, \mathscr{H}om_{\mathbb{Z}}(A,T)) \arrow[r] \arrow[d, "\wr"] & 0 \\ \mathscr{H}om_{\mathbb{Z}}(Y,T) \arrow[r] & \mathscr{H}om_{\mathbb{Z}}(X,T) \arrow[r] & 0 \end{tikzcd}$$

with the top row exact. $\square$

Theorem 3.5. *The category of A-modules has enough injectives.*

Proof. Let M be an A-module. Embed it into an injective abelian group T, so $f : M \hookrightarrow T$. Define $g : M \to \mathscr{H}om_{\mathbb{Z}}(A,T)$ by $g(m) = (a \mapsto f(am))$. If $m \neq 0$, then $g(m)(1) = f(1\cdot m) \neq 0$, so g is one to one. $\square$

Theorem 3.6. *If $(X, \mathscr{O}_X)$ is a ringed space, then $\mathcal{M}od(X)$ has enough injectives.*

Proof. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. For each $x \in X$, $\mathscr{F}_x$ is an $\mathscr{O}_{X,x}$-module, and can therefore be embedded into an injective module I_x. Define $\mathscr{J} = \prod_{x\in X} j_*\mathscr{I}_x$, where $\mathscr{I}_x = I_x$ as a sheaf at x and $j : \{x\} \hookrightarrow X$ is the inclusion, so $j_*\mathscr{I}_x$ is a skyscraper sheaf.

Then we have that

$$\mathscr{H}om(\mathscr{F}, \mathscr{J}) = \prod_{x\in X} \mathscr{H}om(\mathscr{F}, j_*\mathscr{I}_x) = \prod_{x\in X} \mathscr{H}om_{\mathscr{O}_{X,x}}(\mathscr{F}_x, I_x).$$

By taking an injection from each factor, we get a map $\mathscr{F} \to \mathscr{J}$ which must be an injection because it is an injection on all stalks.

Finally, we must show injectivity of $\mathscr{J}$. Given any $\mathscr{G}_1 \to \mathscr{G}_2$, we have that

$$\begin{aligned} \mathscr{H}om_{\mathscr{O}_{X,x}}(\mathscr{G}_{2,x}, I_x) &\twoheadrightarrow \mathscr{H}om_{\mathscr{O}_{X,x}}(\mathscr{G}_{1,x}, I_x) \\ \mathscr{H}om(\mathscr{G}_{2,x}, j_*\mathscr{I}_x) &\twoheadrightarrow \mathscr{H}om(\mathscr{G}_{1,x}, j_*\mathscr{I}_x) \\ \mathscr{H}om(\mathscr{G}_{2,x}, \mathscr{J}) &\twoheadrightarrow \mathscr{H}om(\mathscr{G}_{1,x}, \mathscr{J}) \qquad \text{(prod of surj is surj)} \end{aligned}$$

□

Corollary 3.7. *If X is a topological space, then $\mathcal{A}b(X)$ has enough injectives.*

Proof. Make X into a ringed space by setting $\mathscr{O}_X$ to the constant sheaf $\mathbb{Z}$. Then $\mathcal{M}od((X, \mathscr{O}_X)) = \mathcal{A}b(X)$. □

$H^\cdot(X, -)$ defined

Definition. Let X be a topological space, then the cohomology functors $H^\cdot(X, -)$ are the right derived functors of $\Gamma(X, -) : \mathcal{A}b(X) \to \mathcal{A}b$.

Note: We use injective filtrations in $\mathcal{A}b(X)$ rather than $\mathcal{M}od(X)$ because sometimes we will want to consider sheaves of abelian groups which are not $\mathscr{O}_X$-modules. The following theorem says that we get the same cohomology as we would have if we used $\mathcal{M}od(X)$ injectives.

Theorem 3.8 (*)**.** *If $(X, \mathscr{O}_X)$ is a ringed space (e.g. a scheme), then the right derived functors $\mathcal{M}od(X) \to \mathcal{A}b$ associated to $\Gamma(X, -)$ coincide with $H^\cdot(X, -)$, restricted to $\mathcal{M}od(X)$.*

Recall that if X is a topological space and $\mathscr{F}$ is a sheaf (of abelian groups) on X, then we say $\mathscr{F}$ is flasque if $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$ is surjective for all open sets $V \subseteq U$.

Injective sheaves are flasque

Lemma 3.9. *Let $(X, \mathscr{O}_X)$ be a ringed space. Then any injective element of $\mathcal{M}od(X)$ is flasque.*

Proof. Let $V \subseteq U$ be open. Regard $\mathscr{O}_U$ and $\mathscr{O}_V$ as sheaves on X by extending by zero (see exercise II.1.19b), so $W \mapsto \mathscr{O}_U(W)$ if $W \subseteq U$ and $W \mapsto 0$ otherwise.

We have a map $\mathscr{O}_V \to \mathscr{O}_U$ as $\mathscr{O}_X$-modules, which is injective on stalks, and therefore injective. Now look at

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_U, \mathscr{F}) \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_V, \mathscr{F}) \to 0.$$

The sequence is exact by injectivity of $\mathscr{F}$. But we have a natural isomorphism $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_U, \mathscr{F}) \cong \mathscr{F}(U)$ (since a morphism is determined by the image of $1 \in \mathscr{O}_U(U)$), so $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective. □

Proposition 3.10. *Let X be a topological space and $\mathscr{F}$ a flasque sheaf on X, the $\mathscr{F}$ is acyclic.*

Proof. Embed $\mathscr{F}$ into an injective sheaf and let $\mathscr{G}$ be the quotient.

$$0 \to \mathscr{F} \to \mathscr{I} \to G \to 0.$$

Then $\mathscr{F}$ and $\mathscr{I}$ are flasque, so $\mathscr{G}$ must also be flasque (Exercise in chapter II), and we get

$$0 \to \Gamma(X,\mathscr{F}) \to \Gamma(X,\mathscr{I}) \to \Gamma(X,\mathscr{G}) \to 0$$

from the same exercise. Therefore, we get the long exact sequence

$$\begin{array}{lllll} 0 & \to \Gamma(X,\mathscr{F}) & \to \Gamma(X,\mathscr{I}) & \twoheadrightarrow \Gamma(X,\mathscr{G}) & \xrightarrow{0} \\ & \xrightarrow{0} H^1(X,\mathscr{F}) & \to 0 & \to H^1(X,\mathscr{G}) & \to \\ & \xrightarrow{0} H^2(X,\mathscr{F}) & \to 0 & \to H^2(X,\mathscr{G}) & \to \cdots \end{array}$$

where the middle column is 0 because $\mathscr{I}$ is injective. So we have that $H^1(X,F)=0$ and $H^i(X,\mathscr{F})=H^{i-1}(X,\mathscr{G})$. But since $\mathscr{G}$ is also flasque, we have that $H^{i-1}(X,\mathscr{G})=0$ by induction. □

So flasque sheaves can be used to compute cohomology.

Proof of (*). Let $\mathscr{F}\in\mathcal{M}od(X)$ and let $0\to\mathscr{F}\to\mathscr{I}^{\cdot}$ be an injective resolution (in $\mathcal{M}od(X)$). Then $h^i(\Gamma(X,\mathscr{I}^{\cdot}))$ computes the right derived functors of $\Gamma(X,-):\mathcal{M}od(X)\to\mathcal{A}b$. But the $\mathscr{I}^i$ are flasque and therefore acyclic, and so $h^i(\Gamma(X,\mathscr{I}^{\cdot}))=H^i(X,\mathscr{F})$. □

For next time, read page 209.

Lecture 4

Last time, we showed that injective objects in $\mathcal{M}od(X)$ are acyclic in $H^{\cdot}(X,-)$. In particular, the right derived functors of $\Gamma(X,-):\mathcal{M}od(X)\to\mathcal{A}b$ coincide with $H^{\cdot}(X,-)|_{\mathcal{M}od(X)}$.

Corollary 4.1. *Let $(X,\mathscr{O}_X)$ be a ringed space with $B=\Gamma(X,\mathscr{O}_X)$. Then for all $\mathscr{F}\in\mathcal{M}od(X)$ and $i\in\mathbb{N}$, $H^i(X,\mathscr{F})$ has a natural B-module structure. Thus, if X is a scheme over* Spec A*, then the groups $H^i(X,\mathscr{F})$ are A-modules.*

Remark. Let $\mathcal{A}$ be an abelian category with enough injectives and $\mathcal{B}$ and $\mathcal{C}$ abelian categories. If $F:\mathcal{A}\to\mathcal{B}$ is left exact and $G:\mathcal{B}\to\mathcal{C}$ is exact, then the δ-functor $\{R^i(G\circ F)\}$ is isomorphic to $\{G\circ R^iF\}$.

Theorem 4.2. *Let X be a noetherian topological space of dimension n. Then $H^i(X,\mathscr{F})=0$ for all $i>n$ and for all $\mathscr{F}\in\mathcal{A}b(X)$.*

Grothendieck Vanishing Theorem

Note: For the rest of this lecture, X is a noetherian topological space.

Lemma 4.3. *The direct limit of flasque sheaves is flasque.*

Proof. Let $\{\mathscr{F}_\alpha\}_{\alpha\in A}$ be a directed system of flasque sheaves and let $V\subseteq U\subseteq X$ be open sets. Then we have that

$$\begin{tikzcd} (\varinjlim \mathscr{F}_\alpha)(U) \arrow[r] \arrow[d, "\wr"'] & (\varinjlim \mathscr{F}_\alpha)(V) \arrow[d, "\wr"'] \\ \varinjlim \mathscr{F}_\alpha(U) \arrow[r, two heads] & \varinjlim \mathscr{F}_\alpha(V) \end{tikzcd}$$

Where the bottom arrow is surjective because the $\mathscr{F}_\alpha$ are flasque and $\varinjlim$ is an exact functor on abelian groups, and the vertical arrows are isomorphisms by exercises II.1.10 and II.1.11. $\square$

Lemma 4.4. *Let $\{\mathscr{F}_\alpha\}$ be a directed system in $\mathcal{A}b(X)$. Then there are natural isomorphisms for all $i \in \mathbb{N}$*

$$\varinjlim H^i(X, \mathscr{F}_\alpha) \xrightarrow{\sim} H^i(X, \varinjlim \mathscr{F}_\alpha).$$

Proof. Let $\mathcal{C}$ be the category of A-directed systems in $\mathcal{A}b(X)$. Then $(\mathscr{F}_\alpha) \in \mathcal{C}$. For each α, inject $\mathscr{F}_\alpha$ into its sheaf of discontinuous sections $\mathscr{G}^0_\alpha$. Then inject the cokernel of this map into *its* sheaf of discontinuous sections, $\mathscr{G}^1_\alpha$, etc. This yields a flasque resolution $0 \to \mathscr{F}_\alpha \to \mathscr{G}^\cdot_\alpha$ which is functorial in $\mathscr{F}_\alpha$, so we get an A-directed system of complexes. In particular, $(\mathscr{G}^i_\alpha)_\alpha \in \mathcal{C}$.

Then we have that

$$\begin{aligned}
\varinjlim_\alpha H^i(X, \mathscr{F}_\alpha) &\cong \varinjlim_\alpha h^i(\Gamma(X, \mathscr{G}^\cdot_\alpha)) && \text{(definition)}\\
&\cong h^i(\varinjlim_\alpha \Gamma(X, \mathscr{G}^\cdot_\alpha)) && (\varinjlim \text{ exact in } \mathcal{A}b)\\
&\cong h^i(\Gamma(X, \underbrace{\varinjlim_\alpha \mathscr{G}^\cdot_\alpha}_{\text{flasque}})) && \text{(nontrivial, need } X \text{ noetherian)}
\end{aligned}$$

If $0 \to \varinjlim \mathscr{F}_\alpha \to \varinjlim \mathscr{G}^0_\alpha \to \cdots$ is exact, then we are done. To see that it is exact, observe that $\varinjlim$ commutes with taking stalks:

$$(\varinjlim \mathscr{F}_\alpha)_P = \varinjlim_{P \in U} \varinjlim_\alpha \mathscr{F}_\alpha(U) = \varinjlim_\alpha \varinjlim_{P \in U} \mathscr{F}_\alpha(U) = \varinjlim (\mathscr{F}_\alpha)_P$$

$\square$

Lemma 4.5. *Let $j : Y \hookrightarrow X$ be a closed immersion of topological spaces and let $\mathscr{F} \in \mathcal{A}b(Y)$. Then*

$$H^i(X, j_*\mathscr{F}) \cong H^i(Y, \mathscr{F})$$

for all i.

Proof. H^0 coincides for all j by definition of j_*. If $0 \to \mathscr{F} \to \mathscr{I}^\cdot$ is a flasque resolution of $\mathscr{F}$ on Y, then $0 \to j_*\mathscr{F} \to j_*\mathscr{I}^\cdot$ is exact (look at the stalks), so the computation is the same $\square$

If $Y \subseteq X$ is a closed subset, $U = X \smallsetminus Y$ and $\mathscr{F} \in \mathcal{A}b(X)$, then define

$$\mathscr{F}_Y = j_*(\mathscr{F}|_Y)$$

where $j : Y \hookrightarrow X$ and $\mathscr{F}|_Y = j^{-1}\mathscr{F}$. We also define

$$\mathscr{F}_U = i_!(\mathscr{F}|_U)$$

where $i : U \hookrightarrow X$ is the inclusion. Then by exercise II.1.19, we have that

$$0 \to \mathscr{F}_U \to \mathscr{F} \to \mathscr{F}_Y \to 0.$$

Proof of Theorem. Use induction on $n = \dim X$.

Step 1: Reduce to the case where X is irreducible by induction on the number of components. If the number of components is 0, then we are done (since vacuously). If it is 1, then X is irreducible. If there is more than one irreducible component, then let Y be one component and let $U = X \smallsetminus Y$. Then

$$0 \to \mathscr{F}_U \to \mathscr{F} \to \mathscr{F}_Y \to 0$$

is exact, so by the long exact sequence in cohomology, we have the exact sequence

$$H^i(X, \mathscr{F}_U) \to H^i(X, \mathscr{F}) \to H^i(X, \mathscr{F}_Y)$$

for each $i > n$, but $H^i(X, \mathscr{F}_Y) = H^i(Y, \mathscr{F}|_Y) = 0$ by the inductive hypothesis (and Lemma (4.5)). Then

$$0 \to \underbrace{(\mathscr{F}_U)_{X \smallsetminus \overline{U}}}_{0} \to \mathscr{F}_U \xrightarrow{\sim} (\mathscr{F}_U)_{\overline{U}} \to 0$$

Hence $H^i(X, \mathscr{F}_U) \cong H^i(X, (\mathscr{F}_U)_{\overline{U}} = H^i(\overline{U}, (\mathscr{F}_U)_{\overline{U}}) = 0$ by induction. Thus, $H^i(X, \mathscr{F}) = 0$ for all $i > n$.

Step 2: If X is irreducible of dimension 0, then the open subsets are $\varnothing$ and X, so $\mathscr{F}$ is flasque and therefore acyclic. So $H^i(X, \mathscr{F}) = 0$ for all $i > 0$, as desired.

Step 3: Assume X is irreducible (with $n = \dim X > 0$). Let B be a generating set for $\mathscr{F}$ (e.g. $\coprod_{U \subseteq X} \mathscr{F}(U)$). Let $A = \{\text{finite subsets of } B\}$, then A is a directed system. For $\alpha \in A$, let $\mathscr{F}_\alpha$ be the subsheaf of $\mathscr{F}$ generated by α. That is, $\mathscr{F}_\alpha$ is the set of sections that can be obtained from α by restrictions and gluing.

Since X is noetherian, all open subsets are quasi-compact, so all gluings are finite. Thus, any section of $\mathscr{F}$ comes from a finite subset of B, so $\mathscr{F} = \varinjlim \mathscr{F}_\alpha$.

By Lemma (4.4), it is enough to show the theorem for the $\mathscr{F}_\alpha$.

Finally, we do induction on the number of elements in α. If the number of elements in α is greater than 1, choose a proper subset α'. Then

$$0 \to \mathscr{F}_{\alpha'} \to \mathscr{F}_\alpha \to \underbrace{\text{quotient}}_{\text{gen by images of } \alpha \smallsetminus \alpha'} \to 0.$$

By the long exact sequence, we have

$$H^i(X, \mathscr{F}_{\alpha'}) \to H^i(X, \mathscr{F}_\alpha) \to H^i(X, \text{quotient}).$$

By induction on $|\alpha|$, we have that the side terms are zero, and so the middle is zero. Thus, we may assume that $\mathscr{F}$ is generated by a single element (over some open set U). Then we have the exact sequence

$$0 \to \mathscr{R} \to \mathbb{Z}_U \to \mathscr{F} \to 0$$

which gives us the exact sequence

$$H^i(X, \mathbb{Z}_U) \to H^i(X, \mathscr{F}) \to H^{i+1}(X, \mathscr{R}).$$

So it is enough to show the result for $\mathbb{Z}_U$ and its subsheaves.

Proof for $\mathbb{Z}_U$: Let $Y = X \smallsetminus U$. We have that

$$0 \to \mathbb{Z}_U \to \mathbb{Z} \to \mathbb{Z}_Y \to 0,$$

and $\mathbb{Z}$ is flasque since X is irreducible, so for all $i > n$,

$$H^{i-1}(X, \mathbb{Z}_Y) \to H^i(X, \mathbb{Z}_U) \to \underbrace{H^i(X, \mathbb{Z})}_{0}.$$

The leftmost term is $H^{i-1}(X, \mathbb{Z}_Y) \cong H^{i-1}(Y, \mathbb{Z}|_Y) = 0$ by induction on dimension ($\dim Y < n$). Thus, $H^i(X, \mathbb{Z}_U) = 0$ for all $i > n$.

Proof for $\mathscr{R}$: For all $x \in U$, $\mathscr{R}_x$ is a subgroup of $(\mathbb{Z}_U)_x = \mathbb{Z}$. Let d be a minimal positive element in $\mathbb{Z}$ such that $d \in \mathscr{R}_x$ for some $x \in U$. Then $d \in \mathscr{R}(V)$ for some open set $x \in V \subseteq U$. Since $\mathbb{Z}$ is a PID, we have that $\mathscr{R}|_V \subseteq d \cdot \mathbb{Z}_V$. But we also have that $d \cdot \mathbb{Z}_V \subseteq \mathscr{R}|_V$, so $\mathscr{R}|_V \cong \mathbb{Z}$ on V. Thus, we have

$$0 \to \mathbb{Z}_V \to \mathscr{R} \to \text{quotient} \to 0.$$

We have already shown that $H^i(X, \mathbb{Z}_V) = 0$ for $i > n$, and the quotient is a sheaf on the lower dimensional $X \smallsetminus V$, so $H^i(X, \text{quotient}) = 0$ for $i > n$ by induction on dimension. Therefore, $H^i(X, \mathscr{R}) = 0$ for all $i > n$.

$\square$

Example: Exercise III.2.1a. Let $X = \mathbb{A}^1_k$ for an infinite field k. Let P, Q be distinct closed points. Let $Y = \{P, Q\}$, and let $U = X \smallsetminus Y$. Then $H^1(X, \mathbb{Z}_U) \neq 0$ (i.e. the bound given by the theorem cannot be improved).

Proof. From

$$0 \to \mathbb{Z}_U \to \mathbb{Z} \to \mathbb{Z}_Y \to 0$$

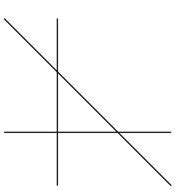
we get

$$\underbrace{H^0(X, \mathbb{Z})}_{\mathbb{Z}} \xrightarrow{\text{not onto}} \underbrace{H^0(X, \mathbb{Z}_Y)}_{\mathbb{Z}\times\mathbb{Z}} \to H^1(X, \mathbb{Z}_U).$$

Therefore, $H^1(X, \mathbb{Z}_U) \neq 0$. $\square$

Part (b)*: Show for all $n > 0$ that if $X = \mathbb{A}^n_k$, $Y =$ union of $n+1$ hyperplanes with empty intersection (in $\mathbb{P}^n_k$) and $U = X \smallsetminus Y$, then $H^n(X, \mathbb{Z}_U) \neq 0$.

In the case $n = 2$, we have



$$0 \to \mathbb{Z}_U \to \underbrace{\mathbb{Z}}_{\text{flasque}} \to \mathbb{Z}_Y \to 0$$

so

$$\underbrace{H^{n-1}(X, \mathbb{Z})}_{0} \to H^{n-1}(X, \mathbb{Z}_Y) \xrightarrow{\sim} H^n(X, \mathbb{Z}_U) \to \underbrace{H^n(X, \mathbb{Z})}_{0}$$

so it suffices to show that $H^{n-1}(Y, \mathbb{Z}_Y) \neq 0$. Let $Y' =$ the three points of intersection and let $U' = Y \smallsetminus Y'$. Then

$$0 \to \mathbb{Z}_{U'} \to \mathbb{Z} \to \mathbb{Z}_{Y'} \to 0$$

so

$$\underbrace{H^0(Y,\mathbb{Z})}_{\mathbb{Z}} \to \underbrace{H^0(Y,\mathbb{Z}_{Y'})}_{\mathbb{Z}^3} \xrightarrow{\text{not onto}} \underbrace{H^1(Y,\mathbb{Z}_{U'})}_{\mathbb{Z}^3} \to H^1(Y,\mathbb{Z}).$$

To see that $H^1(Y,\mathbb{Z}_{U'}) = \mathbb{Z}^3$, observe that $U' = U_1' \cup U_2' \cup U_3'$, where each $U_i' \cong \mathbb{A}^1_k \smallsetminus 2$ points. So $H^1(Y,\mathbb{Z}_{U'}) = \bigoplus_{i=1}^3 H^1(Y,\mathbb{Z}_{U_i'}) = \mathbb{Z}^3$ by the Mayer-Vietoris sequence (exercise III.2.4). Therefore, $H^1(Y,\mathbb{Z}) \neq 0$, as desired.

Lecture 5

Note on Freyd's Theorem: every abelian category can be embedded as a full subcategory of $\mathcal{A}b$. This does *not* imply that $\varinjlim$ is always exact (just because it is exact in $\mathcal{A}b$). For example, $\varprojlim$ (which is the $\varinjlim$ in $\mathcal{A}b^{\mathrm{op}}$) is not exact.

III §3

Theorem 5.1 (Goal)**.** *Let $\mathscr{F}$ be a quasi-coherent sheaf on a noetherian affine scheme $X = \operatorname{Spec} A$, then $\mathscr{F}$ is acyclic.*

Qco sheaves on noetherian affine schemes are acyclic

Note: A is noetherian (Proposition II.3.2) and $\mathscr{F} = \tilde{M}$ for some A-module M.

Lemma 5.2 (Key Lemma)**.** *If I is an injective module over a noetherian ring A, then the sheaf $\tilde{I}$ on $\operatorname{Spec} A$ is flasque.*

Proof assuming Key Lemma. Let $0 \to M \to I^{\cdot}$ be an injective resolution of M (in $\mathcal{M}od A$. Then $0 \to \tilde{M} \to \tilde{I}^{\cdot}$ is exact ($\tilde{\ }$ is exact). By the Key Lemma, this is an acyclic resolution, so we have $H^i(X,\mathscr{F}) = h^i(\Gamma(X, I^{\cdot})) = h^i(I^{\cdot}) = 0$ for all $i > 0$. □

Lemma 5.3. *Let A be a ring and J an A-module. Then J is injective if and only if for all ideals $\mathfrak{a} \subseteq A$, all maps $\phi : \mathfrak{a} \to J$ extend to $A \to J$.*

Proof. ($\Rightarrow$) Trivial from definition of injectivity:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a} \arrow[r] \arrow[d, "\phi"'] & A \arrow[dl, dotted, "J \text{ inj}"] \\ & J & \end{tikzcd}$$

($\Leftarrow$) Let $M' \subseteq M$ be A-modules and let $\phi : M' \to J$. By the usual Zorn's Lemma argument, it suffices to extend ϕ to a map $\langle M', x\rangle \to J$ for some $x \in M \smallsetminus M'$. We have the obvious surjection $M' \oplus A \to \langle M', x\rangle$. Let $\mathfrak{a} = \{a \in A | ax \in M'\}$ be the kernel

$$\begin{aligned} 0 \to \mathfrak{a} &\to M' \oplus A \to \langle M', x\rangle \to 0 \\ a &\mapsto (ax, -a) \end{aligned}$$

To extend ϕ, we need a map $\psi : A \to J$ such that $\phi + \psi : M' \oplus A \to J$ vanishes on $\mathfrak{a}$. This is exactly what we get from the assumption. □

Lemma 5.4. *Let I be an injective module over a noetherian ring A and let $f \in A$. Then the localization map $\theta : I \to I_f$ is surjective.*

Proof. For all $i \in \mathbb{N}$, let $\mathfrak{b}_i = \operatorname{Ann}(f^i)$. Then $0 = \mathfrak{b}_0 \subseteq \mathfrak{b}_1 \subseteq \cdots$. By the noetherian hypothesis, there is some r such that $\mathfrak{b}_r = \mathfrak{b}_{r+1} = \cdots$. Pick $x \in I_f$ and write $x = y/f^n$ with $y \in I$ and $n \in \mathbb{N}$. Define $\phi : (f^{n+r}) \to I$ by $f^{n+r} \mapsto f^r y$ [if $af^{n+r} = 0$, then $a \in \mathfrak{b}_{n+r} = \mathfrak{b}_r$, so $af^r = 0$ and so the map is well defined]. By injectivity, there is some $\psi : A \to I$ such that $\psi|_{(f^{n+r})} = \phi$.

$$\begin{tikzcd} 0 \arrow[r] & (f^{n+r}) \arrow[r] \arrow[d, "\phi"'] & A \arrow[dl, dotted, "\psi"] \\ & I & \end{tikzcd}$$

Let $z = \psi(1)$, then $\theta(z) = f^{-n}y = x$. $\square$

Definition (Exercise II.5.6)**.** Let A be a ring, let $\mathfrak{a}$ be a principal ideal in A, and let M be an A-module. Then $J = \Gamma_{\mathfrak{a}}(M) = \{x \in M | \mathfrak{a}^n x = 0 \text{ for some } n \in \mathbb{N}\}$.

Lemma 5.5. *Let $X = \operatorname{Spec} A$, I be an A-module, $\mathfrak{a} = (f) \subseteq A$ a principal ideal, $J = \Gamma_{\mathfrak{a}}(I)$, $U \subseteq X$ open, and $Z = Z(f) = V_f$. Then*

$$J = \Gamma_Z(X, \tilde{I}) = \{s \in \Gamma(X, \tilde{I}) | s_P = 0 \textit{ for all } P \notin Z\}$$

and

$$\Gamma(U, \tilde{J}) = \Gamma_Z(U, \tilde{I}).$$

Proof.

$$\begin{aligned}
\Gamma_Z(X, \tilde{I}) &= \{x \in I : x \mapsto 0 \in I_P \text{ for all } P \notin Z\} \\
&= \{x \in I | x \mapsto 0 \in I_f\} \\
&= \ker(I \to I_f) \\
&= \{x \in I | f^n x = 0 \text{ for some } n \in \mathbb{N}\} \\
&= \{x \in I | \mathfrak{a}^n x = 0 \text{ for some } n \in \mathbb{N}\} \\
&= \Gamma_{\mathfrak{a}}(I) = J.
\end{aligned}$$

For the other claim, start with the special case $U = D(g)$, so $\Gamma(U, \tilde{J}) = J_g = (\ker(I \to I_f))_g$. So $x \in \Gamma(U, \tilde{J})$ if and only if there are $n, m \in \mathbb{N}$ such that $x = g^{-n}y$, $y \in I$ and $f^m y = 0$. On the other hand, $\Gamma_Z(U, \tilde{I}) = \ker(I_g \to (I_g)_f)$, So $x \in \Gamma_Z(U, \tilde{I})$ if and only if there are $n, n', m \in \mathbb{N}$ such that $x = g^{-n}y$, $y \in I$ and $g^{n'} f^m y = 0$. Thus, $\Gamma_Z(U, \tilde{I}) = \Gamma(U, \tilde{J})$ in this case. For the general case, cover U with open affine sets an glue. $\square$

Lemma 5.6. *Let A be a noetherian ring, $\mathfrak{a} \in A$ an ideal, I an A-module, and $J = \Gamma_{\mathfrak{a}}(I)$. If I is injective, then so is J.*

Proof. By Lemma (5.3), it suffices to complete the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{b} \arrow[r] \arrow[d, "\phi"'] & A \arrow[dl, dotted] \\
& J &
\end{tikzcd}$$

for any ideal $\mathfrak{b}$ and any $\phi : \mathfrak{b} \to J$. For all $b \in \mathfrak{b}$, there is some n such that $\phi(\mathfrak{a}^n b) = \mathfrak{a}^n \phi(b) = 0$. Since $\mathfrak{b}$ is finitely generated, we may choose one n that works for all b. Thus, $\mathfrak{a}^n \mathfrak{b} \subseteq \ker \phi$.

Recall Krull's Theorem: Let $\mathfrak{a} \subseteq A$ be an ideal in a noetherian ring, and let $M \subseteq N$ be finitely generated A-modules. Then the $\mathfrak{a}$-adic topology on M is induced by the $\mathfrak{a}$-adic topology on N. That is, for all n, there is some n' such that $\mathfrak{a}^n M \supseteq M \cap \mathfrak{a}^{n'} N$. Thus, taking $M = \mathfrak{b}$ and $N = A$, $\mathfrak{a}^n \mathfrak{b} \supseteq \mathfrak{b} \cap \mathfrak{a}^{n'}$.

So we have

$$\begin{tikzcd}
A \arrow[r, two heads] & A/\mathfrak{a}^{n'} \arrow[rr, dotted, bend left, "\psi"] \arrow[r, dotted] & J \arrow[r, hook] & I \\
\mathfrak{b} \arrow[u, hook] \arrow[r] \arrow[rr, bend right, "\phi"'] & \mathfrak{b}/\mathfrak{b}\cap\mathfrak{a}^{n'} \arrow[u, hook] \arrow[r, "\text{Krull}"] & J &
\end{tikzcd}$$

By injectivity of I, ψ exists such that the diagram commutes. If we can show $\mathrm{im}\psi \subseteq J$, then we're done. This is true because the image is killed by $\mathfrak{a}^{n'}$. $\square$

Proof of Key Lemma. Statement:if $X = \operatorname{Spec} A$ is noetherian, $I \in \mathcal{M}od(A)$ injective, then $\tilde{I}$ is flasque.

We want to show that for all $U \subseteq V$, $\Gamma(V, \tilde{I}) \to \Gamma(U, \tilde{I})$ is surjective. It suffices to check the case where $V = X$. Let $Y = \operatorname{Supp}\tilde{I} = \{P \in X | I_P \neq 0\}$. Now we apply noetherian induction on Y. If $Y \cap U = \varnothing$, then $\Gamma(U, \tilde{I}) = 0$ and there is nothing to show. So we may assume that Y meets U. Then there is some $f \in A$ such that $D(f) \subseteq U$ and $D(f) \cap Y \neq \varnothing$. Let $Z = X \smallsetminus D(f) = Z(f)$ and consider the commutative diagram

$$\begin{tikzcd}
\Gamma(V,\tilde{I}) \arrow[r] & \Gamma(U,\tilde{I}) \\
\Gamma(X,\tilde{I}) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
I = \Gamma(X,\tilde{I}) \arrow[r] & \Gamma(U,\tilde{I}) \arrow[r] & \Gamma(D(f),\tilde{I}) = I_f & \text{(row \textit{not} exact)} \\
\Gamma_Z(X,\tilde{I}) \arrow[u, hook] \arrow[r] \arrow[d, equal] & \Gamma_Z(U,\tilde{I}) \arrow[u, hook] \arrow[d, equal] & & \\
J = \Gamma(X,\tilde{J}) \arrow[r] & \Gamma(U,\tilde{J}) & &
\end{tikzcd}$$

Let $s \in \Gamma(U, \tilde{I})$, and let $s' \in \Gamma(D(f), \tilde{I})$ be its image. By Lemma (5.4) there is some $t \in \Gamma(X, \tilde{I})$ mapping to s'. Let $t' = t|_U \in \Gamma(U, \tilde{I})$, so that $s - t' \in \Gamma_Z(U, \tilde{I})$ [s and t' agree on $D(f) = Z^c$]. So it suffices to check that the bottom map is surjective.

Lem 5.4: $I \to I_f$ surjective

By Lemma (5.6), J is injective . Also, $\operatorname{Supp}J \subseteq Y \cap Z \subsetneq Y$, so we can apply induction on the dimension of $Y = \operatorname{Supp}I$. $\square$

Lem 5.6: I inj $\Rightarrow$ $J = \Gamma_{\mathfrak{a}}(I)$ inj

Corollary 5.7 (to the Theorem)**.** *Let $\mathscr{F}$ be a quasi-coherent sheaf on a noetherian scheme X. Then $\mathscr{F}$ can be embedded into a flasque, quasi-coherent sheaf.*

Proof. Cover X with finitely many open affine sets $U_i = \operatorname{Spec} A_i$, $i = 1 \dots n$. For each i, $\mathscr{F}|_{U_i} \cong \tilde{M}_i$ for some A_i-module M_i. Embed M_i into an injective A_i-module I_i, and let

$$\mathscr{G} = \bigoplus_{i=1}^{n} f_{i*}(\tilde{I}_i)$$

where $f_i : U_i \to X$ is the inclusion map. We have $\mathscr{F}|_{U_i} \to \tilde{I}_i$, which is injective for each i, and we have

$$\mathscr{F} \to \underbrace{f_{i*}(\mathscr{F}|_{U_i})}_{V \mapsto \mathscr{F}(V \cap U_i)} \to f_{i*}(\tilde{I}_i).$$

Composing and adding these maps, we get an injection $\mathscr{F} \to \mathscr{G}$ [check on stalks that $\mathscr{F} \to f_{i*}(\tilde{I}_i)$ is injective]. By the Key Lemma (5.2), $\tilde{I}_i$ is flasque for each i, so $f_{i*}(\tilde{I}_i)$ is flasque by exercise II.1.16d, so $\mathscr{G}$ is flasque. Similarly, $\mathscr{G}$ is quasi-coherent. □

Lecture 6

Application to Corollary (5.7): Every such sheaf has a quasi-coherent flasque resolution which can therefore be used to compute cohomology.

Also (Exercise III.3.6) $\mathfrak{Qco}(X)$ has enough injectives, and the resulting derived functor cohomology theory agrees with the usual one.

Theorem 6.1 (Serre)**.** *Let X be a noetherian scheme. Then the following are equivalent:*

1. *X is affine*
2. *All quasi-coherent sheaves on X are acyclic*
3. *For all (quasi-)coherent sheaves of ideals $\mathscr{I}$ on X, $H^1(X, \mathscr{I}) = 0$.*

Proof. (1) ⇒ (2) is Theorem (5.1).

(2) ⇒ (3) is trivial.

(3) ⇒ (1): Assume (3) and let $A = \Gamma(X, \mathscr{O}_X)$.

> **Claim.** *Every closed point $P \in X$ has an open affine neighborhood of the form X_f for some $f \in A$.*
>
> *Proof of Claim.* Let U be any open affine neighborhood of P, and let $Y = X \smallsetminus U$. Consider
>
> $$0 \to \mathscr{I}_{Y \cup \{P\}} \to \mathscr{I}_Y \to k(P) \to 0$$
>
> where $k(P)$ is the residue field skyscraper sheaf at P, and Y and $Y \cup \{P\}$ are taken as reduced subschemes of X. By the long exact sequence, we have
>
> $$H^0(X, \mathscr{I}_Y) \to \underbrace{H^0(X, k(P))}_{k(P)} \to \underbrace{H^1(X, \mathscr{I}_{Y \cup \{P\}})}_{0 \text{ by } (3)}.$$
>
> Therefore, there is some $f \in \Gamma(X, \mathscr{I}_Y)$ mapping to $1 \in k(P)$. In particular, $f \notin \mathfrak{m}_P$, so $P \in X_f$ [$f \in H^0(X, \mathscr{I}_Y) \subseteq H^0(X, \mathscr{O}_X) = A$]. Also, for all $Q \in Y$, $f \in \mathscr{I}_Y \subseteq \mathfrak{m}_Q$, so $Q \notin X_f$. Thus, $X_f \subseteq U$. Let $B = \Gamma(U, \mathscr{O}_X)$ and $\bar{f} = f|_U \in B$. Then $X_f = X_f \cap U = D(\bar{f}) = \operatorname{Spec} B_{\bar{f}}$ (in U). So X_f is affine. $\square_{\text{Claim}}$

Since X is noetherian, the sets X_f cover X. Take a finite subcover, $X_{f_1}, \dots, X_{f_r}$. By the handout (Exercises II.2.16 and II.2.17), it will suffice to check that the ideal $(f_1, \dots, f_n) \subseteq A$ is all of A. Define a map of sheaves $\mathscr{O}_X^r \to \mathscr{O}_X$ by $(a_1, \dots, a_r) \mapsto \sum a_i f_i$. We may assume it is onto, because if $P \in X$, then $P \in X_{f_i}$ for some i and $e_i = (0, \dots, 1, \dots, 0) \mapsto f_i \in \mathscr{O}_X$, and $f_i \notin \mathfrak{m}_P$.

Now blah blah □ Start:Fast forward

Application: Exercise III.3.1 (neat filtration trick)

Exercise III.3.8: Lemma (5.4) (Hartshorne Lemma III.3.3) and Lemma (5.2) (Hartshorne Proposition III.3.4) are false without the noetherian hypothesis.

Chapter II, §4: Separated and Proper morphisms

Definition. Let $f : X \to Y$ be a morphism of schemes. Then the diagonal map is the morphism $\Delta : X \to X \times_Y X$ defined by the diagram

$$\begin{tikzcd} X \arrow[dr, dotted, "\Delta"] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\ & X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

We say that f is *separated* if Δ is a closed immersion. In this case, we say that X is *separated over* Y. A *separated scheme* is a scheme X which is separated over $\operatorname{Spec}\mathbb{Z}$.

Notation:

1) In EGA, prescheme is the same as scheme in Hartshorne, and scheme means separated scheme.
2) If $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ are morphisms in $\mathfrak{Sch}(S)$, then (f_1, f_2) is the S-morphism defined by the diagram

$$\begin{tikzcd} X \arrow[dr, dotted, "(f_1,f_2)"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\ & Y_1 \times_S Y_2 \arrow[r] \arrow[d] & Y_2 \arrow[d] \\ & Y_1 \arrow[r] & S \end{tikzcd}$$

Thus, $\Delta = (\mathrm{id}_X, \mathrm{id}_X)$.

On the other hand, if $f_i : X_i \to Y_i$ are morphisms for $i = 1, 2$, then $f_1 \times f_2 = (f_1 \circ pr_1, f_2 \circ pr_2)$.

Example: The affine line with two origins is not separated over k because $\Delta \subseteq X \times_k X = \mathbb{A}^2$ with double axes and quadruple origin contains only two of the origins, but $\overline{\Delta}$ contains all four. Thus, Δ is not a closed immersion.

Lecture 7

Recall that $f : X \to Y$ is separated if $\Delta : X \to X \times_Y X$ is a closed immersion.

Proposition 7.1. *If $f : X \to Y$ is a morphism of affine schemes, then it is separated.*

Proof. Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Then $X \times_Y X = \operatorname{Spec}(A \otimes_B A)$ and Δ corresponds to the morphism $A \otimes_B A \to A$, $a \otimes a' \mapsto aa'$, which is onto. Thus, Δ is a closed immersion. ∎

Corollary 7.2. *All affine schemes are separated.*

Corollary 7.3. *A morphism $f : X \to Y$ is separated if and only if $\Delta(X)$ is a closed subset of $X \times_Y X$*

Proof. ($\Rightarrow$) obvious.

($\Leftarrow$) We need to show that

(1) Δ is a homeomorphism onto $\Delta(X)$
(2) $\mathscr{O}_{X \times_Y X} \to \Delta_* \mathscr{O}_X$ is onto

(1) We have

$$\begin{tikzcd}
X \arrow[dr, dotted, "\Delta"] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\
& X \times_Y X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

Since $\pi_1 \circ \Delta : X \to \Delta(X) \to X$ is the identity, Δ must be one to one, and π_1 is a continuous inverse, so Δ is a homeomorphism $X \to \Delta(X)$.

(2) It is enough to look at stalks. Let $P = \Delta(P') \in \Delta(X)$ for some $P' \in X$. Restrict to an open affine neighborhood V of $f(P')$ in Y, and an open affine neighborhood U of P' in $f^{-1}(V)$. Then we've reduced to the affine case, where we know the result holds. ∎

Valuative Criterion for Separatedness

Theorem 7.4 (Valuative Criterion of Separatedness)**.** *A morphism $f : X \to Y$ of noetherian schemes is separated if and only if the following criterion holds: for any field K and any valuation ring R of K (Frac(R) $= K$), and for any diagram*

$$\begin{tikzcd}
U := \operatorname{Spec} K \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\
T := \operatorname{Spec} R \arrow[r] \arrow[ur, dotted, "\exists\leq 1"] & Y
\end{tikzcd}$$

there exists at most one $h : T \to X$ such that the diagram commutes.

Remark. (1) You really only need X noetherian.
(2) Criterion fails for the affine line with the doubled origin.
(3) Have to use valuation rings rather than curves.

Lemma 7.5. *Let R, K, U, T be as above, and let X be a scheme, then*

(a) *To give a map $U \to X$ is equivalent to giving a point $x \in X$ and a field extension $k(x) \hookrightarrow K$, where $k(x) = \mathscr{O}_x/\mathfrak{m}_x$.*
(b) *Giving a map $T = \operatorname{Spec} R \to X$ is equivalent to giving points $x_0, x_1 \in X$ with $x_1 \rightsquigarrow x_0$ and*[5] *an inclusion $k(x_1) \hookrightarrow K$ such that if you let $z = \overline{\{x_1\}} \subseteq X$ (with reduced induced subscheme structure), then R dominates the local ring $\mathscr{O}_{X,z}$*

[5] $x_1 \rightsquigarrow x_0$ means that x_0 is a specialization of x_1. i.e. x_0 is in the closure of $\{x_1\}$

Proof. in the works. □

Lemma 7.6. *Let $f : X \to Y$ be a quasi-compact morphism of schemes, and let $Z = f(X) \subseteq Y$. Then Z is closed if and only if it is stable under specialization.*

Proof. in the works. □

Remark. Since X is noetherian, $\Delta : X \to X \times_Y X$ is quasi-compact.

Proof of Valuative Criterion. in the works

proof continued in next lecture

Lecture 8

Valuative Criterion proof continued. still in the works. □

Separatedness on noetherian schemes

Corollary 8.1. *When working with noetherian schemes:*

(a) *Open and closed immersions are separated.*
(b) *Compositions of separated morphisms are separated.*
(c) *Separatedness is stable under base extension. i.e. if $f : X \to Y$ is separated, then $f' : X \times_Y Y' \to Y'$ is separated.*

$$\begin{array}{ccc} X \times_Y Y' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

(d) *If $f : X \to Y$ and $f' : X' \to Y'$ are separated S-morphisms, then so is $f \times f' : X \times_S X' \to Y \times_S Y$.*
(e) *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms and $g \circ f$ is separated, then so is f.*
(f) *Separatedness is local on the base. i.e. $f : X \to Y$ is separated if and only if there is an open cover $\{U_i\}$ of Y such that $f^{-1}(U_i) \to U_i$ is separated for all i.*
(g) *$f : X \to Y$ is separated if and only if $f_{red} : X_{red} \to Y_{red}$ is separated.*

Proof of (e). in the works. □

Proof of (g). in the works. □

Remark. All of these things are true without the noetherian hypothesis.

Corollary 8.2 (of the Corollary)**.** *A morphism of separated schemes is separated. More precisely, if $f : X \to Y$ with X separated (over $\mathbb{Z}$), then f is separated.*

Proof. Use (e) of the Corollary above. □

Corollary 8.3 (of the Corollary)**.** *Most schemes you work with will be separated. An exception: gluing.*

Properness

In topology, a map is proper if the inverse image of a compact set is compact. We want $\mathbb{P}^1_k \to \operatorname{Spec} k$ to be proper, but $\mathbb{A}^1_k \to \operatorname{Spec} k$ not to be proper.

Definition. A morphism $f : X \to Y$ is *closed* if for all closed subschemes $Z \subseteq X$, $f(Z)$ is closed in Y. We say that f is *universally closed* if for all morphisms $Y' \to Y$, the pullback of f is closed.

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

Example: $\mathbb{A}^1_k \to \operatorname{Spec} k$ is closed, but not universally closed. Take $Y' = \mathbb{A}^1_k$, so $X' = \mathbb{A}^2_k$, and f' is a projection. Then look at $Z = \{xy = 1\}$.

Definition. A morphism is *proper* if it is separated, of finite type, and universally closed.

Valuative Criterion for Properness

Theorem 8.4 (Valuative Criterion for Properness)**.** *Let $f : X \to Y$ be a finite type morphism of noetherian schemes, then f is proper if and only if*

$$\begin{tikzcd} U := \operatorname{Spec} K \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ T := \operatorname{Spec} R \arrow[r] \arrow[ur, dotted, "\exists!"] & Y \end{tikzcd}$$

Proof. in the works. $\square$

Lecture 9

A question from last time: If $f : \operatorname{Spec} K \to X \times_Y X$ sends the point to $\xi \in \Delta$, then f factors though Δ as a morphism of schemes.

$$\begin{tikzcd} X \arrow[dr, "\Delta"] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\ & X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} k(x_1) & & \\ & k(\xi) \arrow[ul] & k(x_1) \arrow[l] \arrow[ull, bend right, "\mathrm{id}"'] \\ & k(x_1) \arrow[u] \arrow[uul, bend left, "\mathrm{id}"] & \end{tikzcd}$$

Since $\xi \in \Delta$, there is some $x_1 \in X$ such that $\Delta(x_1) = \xi$. So we get the diagram on the right, in which every arrow must be an isomorphism (since all the morphisms are between fields). Now $\operatorname{Spec} K \to X \times_Y X$ gives $k(\xi) \cong k(x_1) \to K$, so f factors through Δ.

Corollary 9.1 (of the Valuative Criterion for Properness)**.**

(a) *Closed immersions are proper (but not open immersions).*
(b) *Compositions of proper morphisms are proper.*
(c) *Properness is stable under base extension. i.e. if $f : X \to Y$ is proper, then $f' : X \times_Y Y' \to Y'$ is proper.*

$$\begin{tikzcd} X \times_Y Y' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

(d) *If $f : X \to Y$ and $f' : X' \to Y'$ are proper S-morphisms, then so is $f \times f' : X \times_S X' \to Y \times_S Y$.*

(e) *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms and $g \circ f$ is proper and g is separated, then f is proper.*

(f) *Properness is local on the base. i.e. $f : X \to Y$ is proper if and only if there is an open cover $\{U_i\}$ of Y such that $f^{-1}(U_i) \to U_i$ is proper for all i.*

(g) *If $f : X \to Y$ is proper, then so is $f_{red} : X_{red} \to Y_{red}$. If f is of finite type, the the converse also holds.*

Partial Proof. In each case, we need to check finite-type-ness first. □

Again, this result actually holds without the noetherian assumption.

The Valuative criterion for properness is *important*, as illustrated by the following fact.

An important application of the Valuative Criterion

Let Y be proper over a noetherian scheme S. Let X be a noetherian regular S-scheme of dimension 1. Assume f is a rational map such that the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$$

commutes. Then f extends (uniquely) to a morphism $X \to Y$.

Proof. We have a dense open $U \subseteq X$ and an S-morphism $f : U \to Y$. Since $X \smallsetminus U$ is zero dimensional and X is noetherian, $X \smallsetminus U$ has a finite number of points.

For $x \in X \smallsetminus U$, let $R = \mathscr{O}_{X,x}$. Then R is a regular noetherian local ring of dimension 1. By remark II.6.11.2A on page 142[6], it is entire (an integral domain), so by theorem I.6.2A on page 40[7], it is a DVR. Let $K = \operatorname{Frac} R$. Let $T = \operatorname{Spec} R$ with $t =$ the closed point, and $\tau =$ the generic point. Let $g : T \to X$ be the canonical map. Let $\xi = g(\tau) \in X$, then $\xi \in U$ since it is not a closed point of X, so we get the diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[rr] \arrow[d] & & Y \arrow[d] \\ T = \operatorname{Spec} R \arrow[rr] \arrow[urr, dotted, "\exists! h"] \arrow[dr, hook] & & S \\ & X \arrow[ur] \arrow[uur, dashed, "f"] & \end{tikzcd}$$

By the valuative criterion, we get the map h. Then we can extend f by saying $f(x) = h(t)$.

We may assume S is affine, equal to $\operatorname{Spec} C$. Let $\operatorname{Spec} B$ be an open affine neighborhood of $h(t) \in Y$, and let $X' = \operatorname{Spec} A$ be an open affine

[6]A regular local ring is a UFD.

[7]For a noetherian local domain of dimension 1, TFAE: i) is a DVR ii) is integrally closed iii) is regular iv) max'l ideal is principal.

neighborhood of $x \in X$. Let $\mathfrak{p} \subseteq A$ correspond to x, then we get the diagram

$$\begin{tikzcd}
K & & B \arrow[ll] \arrow[dll, "h^*"'] & = & C[y_1, \dots, y_r] \\
A_{\mathfrak{p}} \arrow[u] & & C \arrow[u] \arrow[dl] & & \\
& A \arrow[ul] & & &
\end{tikzcd}$$

There is some $a \notin \mathfrak{p}$ such that $h^*(y_i) \in A_a$ for all i. So replace A with A_a. Then we get a map $B \to A$ and still have $x \in \operatorname{Spec} A$. From an exercise, we have that

$$\begin{tikzcd}
\mathscr{H}om_S(U \cap X', \operatorname{Spec} B) \arrow[r, "\sim"] & \mathscr{H}om_C(B, \Gamma(U \cap X', \mathscr{O}_{X'})) \\
\mathscr{H}om_S(X', \operatorname{Spec} B) \arrow[r] \arrow[u] \arrow[ur, phantom, "\circlearrowleft"] & \mathscr{H}om_C(B, \Gamma(X', \mathscr{O}_{X'})) \arrow[u]
\end{tikzcd}$$

So we have some $f' : X' \to \operatorname{Spec} B$ restricting to f. Therefore, f extends to X. Uniqueness follows from Exercise II.4.2, which says that if two morphisms from a reduced scheme to a separated scheme agree on a dense open subset, then they agree everywhere. $\square$

Example 1: The assumption that X is regular is necessary. Take $X = \{y^2 = x^2 + x^3\} \subseteq \mathbb{A}^2_k$ for k an algebraically closed field of characteristic $\neq 2$. Take $U = X \smallsetminus \{(0,0)\}$ and $f : U \to \mathbb{P}^1_k$, $(x, y) \mapsto [x, y]$ the projection from the point $(0,0)$. Then the map cannot extend to the point $(0,0)$.

Example 2: The assumption $\dim X = 1$ is necessary. Let $X = \mathbb{P}^2_k$, and let Y be the blow-up of $\mathbb{P}^2_k$ at the origin, $[0,0,1]$. Then there is an obvious birational equivalence between X and Y, but they are not isomorphic.

Key Application: (Number Theory) If R is a Dedekind ring and $Y = \operatorname{Spec} R$, $K = \operatorname{Frac}(R)$, and if X is proper over Y (i.e. $\operatorname{Spec} R$), then any K-morphism $\operatorname{Spec} K \to X$ gives a rational map from Y to X by stuff from chapter I §4. Thus, we get a unique R-morphism $Y \to X$ (i.e. a section of $X \to Y$). Therefore, $X(K) = X(R)$.[8]

Recall that if A is a ring and $n \in \mathbb{N}$, then $\mathbb{P}^n_Y = \operatorname{Proj} A[x_0, \dots, x_n]$. If $A \to A'$ is a homomorphism, then $\mathbb{P}^n_{A'} = \mathbb{P}^n_A \times_{\operatorname{Spec} A} \operatorname{Spec} A'$.

Definition. If Y is an arbitrary scheme, then $\mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times Y$.

[8] Notation: For X and Y S-schemes, $Y(X)$ means $\mathscr{H}om_S(X, Y)$. Also, by $X(K)$ we mean $X(\operatorname{Spec} K)$.

Definition. A morphism $f : X \to Y$ is *projective* if there is a closed immersion $i : X \to \mathbb{P}^n_Y$ for some $n \in \mathbb{N}$ such that the diagram commutes:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{P}^n_Y \arrow[d] \\ & Y \end{tikzcd}$$

Caution: EGA uses a more general definition.

Note: Neither definition is local on the base. Why not? For an open cover $\{U_i\}$, we could have $f^{-1}(U_i) \to \mathbb{P}^{n_i}_{U_i}$ with n_i unbounded. Or, more to the point, we need an $\mathscr{O}(1)$ on X that works globally.

Example: If S is a graded ring, generated over $S_0 = A$ by finitely many elements of degree 1. Then $\operatorname{Proj} S \to \operatorname{Spec} A$ is projective.

Proof. Let $s_0, \ldots, s_n \in S_1$ be a generating set. Then $T = A[t_0, \ldots, t_n] \twoheadrightarrow S$. So we have

$$\begin{tikzcd} \operatorname{Proj} S \arrow[r, hook, "\text{closed}", "\text{imm}"'] \arrow[dr] & \operatorname{Proj} T = \mathbb{P}^n_A \arrow[d] \\ & \operatorname{Spec} A \end{tikzcd}$$

□

Lemma 9.2. *Let $n \in \mathbb{N}$. Then $\mathbb{P}^n_{\mathbb{Z}}$ is proper over $\mathbb{Z}$.*

Proof. in the works.

proof continued in next lecture

Lecture 10

continued proof. in the works. □

end fast forward

It follows that $\mathbb{P}^n_Y \to Y$ is proper (it is a base extension). So given a projective morphism $f : X \to Y$ we have that $f = \pi \circ i$ is a composition of proper morphisms, so it is proper.

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbb{P}^n_Y \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

Projective morphisms are proper

The converse is false. There are proper morphisms (even over $\operatorname{Spec} k$ with k algebraically closed) which are not projective.

Lemma 10.1 (Chow's Lemma, ex II.4.10)**.** *Given a proper morphism $f : X \to S$, with S noetherian, there is a birational morphism*[9] *$g : X' \to X$ such that $f \circ g$ is projective.*

Definition. A morphism $f : X \to Y$ is *quasi-projective* if there is an open immersion $i : X \hookrightarrow X'$ and a projective morphism $X' \to Y$ such that $f = g \circ i$:

$$\begin{tikzcd} X \arrow[r, hook, "\text{open}"] \arrow[dr, "f"'] & X' \arrow[r, hook, "\text{closed}"] \arrow[d, "\text{proj}"] & \mathbb{P}^n_Y \arrow[dl, bend left] \\ & Y & \end{tikzcd}$$

(therefore, X is isomorphic to a subscheme of $\mathbb{P}^n_Y$ for some n)

[9] A *birational morphism* is a morphism which, as a rational map, is birational.

Definition. A *subscheme* of a scheme Y is an immersion $X \hookrightarrow Y$. An *immersion* is a map that can be written as an open immersion followed by a closed immersion (or vice versa (exercise)).

Theorem 10.2. *If $f : X \to Y$ is quasi-projective and Y is noetherian, then f is separated and of finite type.*

Proof. Let $X \to X'$ be an open immersion with X' projective. Then $X \to X' \to Y$ is separated (it is a composition of separated morphisms) and of finite type (it is a composition of finite type morphisms - Ex II.3.3c; X' noeth $\Rightarrow X$ noeth $\Rightarrow X$ quasi-compact). □

Near Converse(Nagata): If $f : X \to Y$ is separated and of finite type, then there is an open immersion $X \hookrightarrow X'$ with X' proper over Y.

Definition. Let k be an algebraically closed field. A *variety* over k is an integral scheme, separated and of finite type over $\operatorname{Spec} k$. (i.e. it is a separated, finite type morphism $X \to \operatorname{Spec} k$ with X integral).

It is *projective* (resp. *quasi-projective)* if the morphism is, and it is *complete* if the morphism is proper.

Note: not everybody uses this definition. Some say X is reduced instead of integral. Many allow arbitrary k (in which integral may be replaced by geometrically integral[10]).

§III.4 Čech Cohomology

In this section:

- X is a topological space
- $\mathcal{U} = (U_i)_{i \in I}$ is an open cover of X with a *well ordered* index set I
- $\mathscr{F}$ is a sheaf of abelian groups on X
- $C^p(\mathcal{U}, \mathscr{F}) := \prod_{i_0 < i_1 < \cdots < i_p} \mathscr{F}(U_{i_0 i_1 \cdots i_p})$, with $U_{i_0 i_1 \cdots i_p} := U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_p}$ for any $p \in \mathbb{N}$

If $\alpha \in C^p(\mathcal{U}, \mathscr{F})$, we write its components as $\alpha_{i_0 i_1 \cdots i_p} \in \mathscr{F}(U_{i_0 i_1 \cdots i_p})$. For arbitrary $(p+1)$-tuples $i_0 i_1 \cdots i_p$ we write

$$\alpha_{i_0 i_1 \cdots i_p} = \begin{cases} 0 & \text{if the tuple contains a repeat} \\ (-1)^{|\sigma|} \alpha_{\sigma(i_0)\sigma(i_1)\cdots\sigma(i_p)} & \text{where } \sigma(i_0) < \cdots < \sigma(i_n) \end{cases}$$

Define $d : C^p(\mathcal{U}, \mathscr{F}) \to C^{p+1}(\mathcal{U}, \mathscr{F})$ by

$$(d\alpha)_{i_0 i_1 \cdots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0 \cdots \hat{i}_j \cdots i_{p+1}} |_{U_{i_0 \cdots i_{p+1}}}.$$

Observe that the definition is compatible with the convention for arbitrary $i_0 i_1 \cdots i_{p+1}$. Also, we have that $d^2 = 0$.

Thus, we have a complex $C^{\cdot}(\mathcal{U}, \mathscr{F})$

Definition. The Čech Cohomology of $\mathscr{F}$ with respect to $\mathcal{U}$ is

$$\check{H}^p(\mathcal{U}, \mathscr{F}) = h^p(C^{\cdot}(\mathcal{U}, \mathscr{F})).$$

[10] X is *geometrically integral* if $X \times_{\operatorname{Spec} k} \bar{k}$ is integral. See exercise II.3.15

Example: $X, \mathscr{F}$ as above, with $\mathcal{U} = \{X\}$. Then $C^p(\mathcal{U}, \mathscr{F}) = 0$ for all $p \neq 0$ and $C^0(\mathcal{U}, \mathscr{F}) = \Gamma(X, \mathscr{F})$, so

$$\check{H}^p(\mathcal{U}, \mathscr{F}) = \begin{cases} \Gamma(X, \mathscr{F}) & p = 0 \\ 0 & p \neq 0. \end{cases}$$

In particular, Čech Cohomology might not have a long exact sequence. We will eventually show that if all the $U_{i_1 \cdots i_p}$ are acyclic for $\mathscr{F}$, then

$$\check{H}^{\cdot}(\mathcal{U}, \mathscr{F}) \cong H^{\cdot}(X, \mathscr{F}).$$

Example: Lets find $\check{H}(\mathcal{U}, \mathscr{O}(1))$ where

$X = \mathbb{P}^1_k = \operatorname{Proj} k[x, y]$ for k a field
$\mathcal{U} = \{U, V\}$ with
$U = D_+(x) = \operatorname{Spec} k[1/t]$ where $t = x/y$
$V = D_+(y) = \operatorname{Spec} k[t]$

Then

$$\begin{aligned} C^0 &= \Gamma(U, \mathscr{O}(1)) \times \Gamma(V, \mathscr{O}(1)) \\ &\cong \Gamma(U, \mathscr{O}_U) \times \Gamma(V, \mathscr{O}_V) \\ &= k[1/t] \times k[t] \\ C^1 &= \Gamma(U \cap V, \mathscr{O}(1)) \\ &\cong \Gamma(U \cap V, \mathscr{O}_{U \cap V}) \\ &= k[t]_t = k[t, 1/t] \end{aligned}$$

What is the map d? Recall that $\mathscr{O}(1) = \widetilde{S(1)}$, so

$$\begin{aligned} \mathscr{O}(1)|_{D_+(x)} &= \widetilde{S(1)_{(x)}} = \widetilde{xk[1/t]} \\ \mathscr{O}(1)|_{D_+(y)} &= \widetilde{S(1)_{(y)}} = \widetilde{yk[t]} \end{aligned}$$

So

$$\begin{array}{rlll} \Gamma(U, \mathscr{O}(1)) = xk[1/t] \cong k[1/t] & & \ni 1 \mapsto & \\ \Gamma(V, \mathscr{O}(1)) = yk[t] \cong k[t] & \ni 1 \mapsto & & \\ \Gamma(U \cap V, \mathscr{O}(1)) = xk[t]_t \cong k[t, 1/t] & & 1/t & 1 \end{array}$$

Then we have $\check{H}^1(\mathcal{U}, \mathscr{O}(1)) = C^1 / \operatorname{im}(C^0 \to C^1)$. But

$$\operatorname{im}(\Gamma(U, \mathscr{O}(1)) \to C^1) = \bigoplus_{n \leq 0} kt^n$$

$$\operatorname{im}(\Gamma(V, \mathscr{O}(1)) \to C^1) = \bigoplus_{n \geq -1} kt^n$$

so $\mathrm{im}(C^0 \to C^1)$ is all of C^1, and therefore $\check{H}^1(\mathcal{U}, \mathscr{O}(1)) = 0$.

$$\begin{aligned}
\check{H}^0(\mathcal{U}, \mathscr{O}(1)) &= \ker(C^0 \to C^1) \\
&= \mathrm{im}(\Gamma(U, \mathscr{O}(1)) \to \Gamma(U \cap V, \mathscr{O}(1))) \qquad \text{(since these maps} \\
&\quad \cap \mathrm{im}(\Gamma(V, \mathscr{O}(1)) \to \Gamma(U \cap V, \mathscr{O}(1))) \qquad \text{are one to one)} \\
&= \bigoplus_{n \le 0} kt^n \cap \bigoplus_{n \ge -1} kt^n \\
&= kt^{-1} \oplus kt^0 = k^2 = \Gamma(X, \mathscr{O}(1))
\end{aligned}$$

More generally, for all $X, \mathscr{F}, \mathcal{U}$, $\check{H}^0(\mathcal{U}, \mathscr{F}) = \Gamma(X, \mathscr{F})$.

Proof.

$$\begin{tikzcd}
0 \arrow[r] & \check{H}^0(\mathcal{U}, \mathscr{F}) \arrow[r] & C^0(\mathcal{U}, \mathscr{F}) \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & C^1(\mathcal{U}, \mathscr{F}) \arrow[d, hook] \\
0 \arrow[r] & \Gamma X, \mathscr{F} \arrow[r] & \prod \mathscr{F}(U_i) \arrow[r, "g"'] & \prod_{i,j} \mathscr{F}(U_i \cap U_j)
\end{tikzcd}$$

The top row is exact, and the bottom row is exact by the sheaf axioms. Commutativity of the square tells us that $\ker f = \ker g$, as desired. □

Lecture 11

We will show that $\check{H}^p(\mathcal{U}, \mathscr{F}) \xrightarrow{\sim} H^p(X, \mathscr{F})$ whenever $\mathcal{U}$ is such that $\mathscr{F}|_{U_{i_0 \cdots i_p}}$ is acyclic with respect to derived functor cohomology.

Define a sheaf version of $C^{\cdot}(\mathcal{U}, \mathscr{F})$ in the following way:

$$\mathscr{C}^p(\mathcal{U}, \mathscr{F}) = \prod_{i_0 < \cdots < i_p} f_*(\mathscr{F}|_{U_{i_0 \cdots i_p}})$$

where f has components $f_{i_0 \ldots i_p} : U_{i_0 \ldots i_p} \hookrightarrow X$. Then we have that $\Gamma(X, \mathscr{C}^p(\mathcal{U}, \mathscr{F})) = C^p(\mathcal{U}, \mathscr{F})$. Also, define $d : \mathscr{C}^p(\mathcal{U}, \mathscr{F}) \to \mathscr{C}^{p+1}(\mathcal{U}, \mathscr{F})$ in the obvious way.

Lemma 11.1. *For all $X, \mathcal{U}, \mathscr{F}$, the complex $\mathscr{C}^{\cdot}(\mathcal{U}, \mathscr{F})$ is a resolution of $\mathscr{F}$.*

Proof. Define $\epsilon : \mathscr{F} \to \mathscr{C}^0(\mathcal{U}, \mathscr{F}) = \prod_i f_*(\mathscr{F}|_{U_i})$ by $\mathscr{F}(V) \ni s \mapsto (s|_{U_i \cap V})_i$. Then we need to show that the sequence

$$0 \to \mathscr{F} \xrightarrow{\epsilon} \mathscr{C}^0(\mathcal{U}, \mathscr{F}) \xrightarrow{d} C^1(\mathcal{U}, \mathscr{F}) \xrightarrow{d} \cdots$$

is exact.

First, ϵ is injective because it is injective on $\mathscr{F}(V)$ for all V (since $\mathscr{F}$ is a sheaf). So view $\mathscr{F}$ as a subsheaf of $\mathscr{C}^0(\mathcal{U}, \mathscr{F})$. Then exactness at $p = 0$ is equivalent to

$$\ker(\mathscr{C}^0(\mathcal{U}, \mathscr{F}) \to \mathscr{C}^1(\mathcal{U}, \mathscr{F})) = \mathscr{F}.$$

But that follows from

$$\begin{aligned}
\ker(\mathscr{C}^0(\mathcal{U}, \mathscr{F})(V) \to \mathscr{C}^1(\mathcal{U}, \mathscr{F})(V)) &= \ker(C^0(\mathcal{U}|_V, \mathscr{F}|_V) \to C^1(\mathcal{U}|_V, \mathscr{F}|_V)) \\
&= \check{H}^0(\mathcal{U}|_V, \mathscr{F}|_V) = \Gamma(V, \mathscr{F}|_V) \\
&= \mathscr{F}(V)
\end{aligned}$$

To show exactness everywhere else, fix $x \in X$ and j such that $x \in U_j$. For each $p \geq 1$, define

$$k : \mathscr{C}^p(\mathcal{U}, \mathscr{F})_x \to \mathscr{C}^{p-1}(\mathcal{U}, \mathscr{F})_x$$

in the following way: given $\alpha_x \in \mathscr{C}^p(\mathcal{U}, \mathscr{F})_x$, lift it to $\mathscr{C}^p(\mathcal{U}, \mathscr{F})(V)$ for some open neighborhood $x \in V \subseteq U_j$. Then for any $i_0 < \cdots < i_{p-1}$, let

$$(k\alpha)_{i_0 \ldots i_{p-1}} = \alpha_{j, i_0 \ldots i_{p-1}}.$$

Then $k(\alpha_x) = (k\alpha)_x$. This is well defined (independent of V).

$$\begin{tikzcd}
\mathscr{C}^0(\mathcal{U},\mathscr{F}) \arrow[r, "d"] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "0"] & \mathscr{C}^1(\mathcal{U},\mathscr{F}) \arrow[r, "d"] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "0"] \arrow[dl, "k"'] & \mathscr{C}^1(\mathcal{U},\mathscr{F}) \arrow[r, "d"] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "0"] \arrow[dl, "k"'] & \cdots \arrow[dl, "k"'] \\
\mathscr{C}^0(\mathcal{U},\mathscr{F}) \arrow[r, "d"] & \mathscr{C}^1(\mathcal{U},\mathscr{F}) \arrow[r, "d"] & \mathscr{C}^1(\mathcal{U},\mathscr{F}) \arrow[r, "d"] & \cdots
\end{tikzcd}$$

Claim. *$(kd + dk)(\alpha_x) = \alpha_x$ for all $\alpha_x \in \mathscr{C}^p(\mathcal{U}, \mathscr{F})_x$. That is, k is a homotopy between the identity map on the complex $\mathscr{C}^\cdot(\mathcal{U}, \mathscr{F})$ and the zero map.*

Proof of Claim. Compute away:

$$\begin{aligned}
(d(k(\alpha_x)))_{i_0 \ldots i_p} &= \sum_{l=0}^{p} (-1)^l (k(\alpha_x))_{i_0 \ldots \hat{i}_l \ldots i_p}|_{U_{i_0 \ldots i_p}} \\
&= \sum_{l=0}^{p} (-1)^l \alpha_{j, i_0 \ldots \hat{i}_l \ldots i_p}|_{U_{i_0 \ldots i_p}} \\
(k(d(\alpha_x)))_{i_0 \ldots i_p} &= (d\alpha)_{j, i_0 \ldots i_p} \\
&= \alpha_{i_0 \ldots i_p} + \sum_{l=0}^{p} (-1)^{l+1} \alpha_{j, i_0 \ldots \hat{i}_l \ldots i_p}|_{U_{i_0 \ldots i_p}}
\end{aligned}$$

Now add the two and you get $\alpha_{i_0 \ldots i_p}$. $\square_{\text{Claim}}$

Since the identity map on the complex $\mathscr{C}^\cdot(\mathcal{U}, \mathscr{F})$ is homotopic to the identity, the induced maps on homologies are the same, so the homologies are zero[11]. That is, the sequence is exact for $p \geq 1$, as desired. $\square$

Lemma 11.2. *Let X,$\mathcal{U}$,$\mathscr{F}$ be as usual. Then there is a map*

$$\check{H}^p(\mathcal{U}, \mathscr{F}) \to H^p(X, \mathscr{F})$$

for all $p \in \mathbb{N}$ which is natural in $\mathscr{F}$.

Proof. Let $0 \to \mathscr{F} \to \mathscr{I}^\cdot$ be an injective resolution of $\mathscr{F}$. Then by Lemma (2.3), there are maps f^p for all $p \in \mathbb{N}$ such that

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \arrow[r, "\epsilon"] \arrow[d, equal] & \mathscr{C}^0(\mathcal{U},\mathscr{F}) \arrow[r, "d"] \arrow[d, "f^0"] & \mathscr{C}^1(\mathcal{U},\mathscr{F}) \arrow[r, "d"] \arrow[d, "f^1"] & \cdots \\
0 \arrow[r] & \mathscr{F} \arrow[r, "\epsilon"'] & \mathscr{I}^0 \arrow[r] & \mathscr{I}^1 \arrow[r] & \cdots
\end{tikzcd}$$

[11]Another way to say this: if α is a cocycle, then we have that $\alpha = (kd+dk)\alpha = d(k\alpha)$, so it is a coboundary.

and the system of maps $f^\cdot$ is unique up to homotopy. Eliminating the first column and taking global sections, we have

$$\begin{tikzcd}
0 \arrow[r] & C^0(\mathcal{U},\mathscr{F}) \arrow[r, "d"] \arrow[d] & C^1(\mathcal{U},\mathscr{F}) \arrow[r, "d"] \arrow[d] & \cdots \\
0 \arrow[r] & \Gamma(X,\mathscr{I}^0) \arrow[r] & \Gamma(X,\mathscr{I}^1) \arrow[r] & \cdots
\end{tikzcd}$$

Taking homologies, we get well defined maps

$$\check{H}^p(\mathcal{U},\mathscr{F}) \to H^p(X,\mathscr{F})$$

for all $p \in \mathbb{N}$.

Finally, we need to show naturality. That is, we need to show that for all maps $\mathscr{F} \to \mathscr{G}$, the box

$$\begin{tikzcd}
\check{H}^p(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d] & H^p(X,\mathscr{F}) \arrow[d] \\
\check{H}^p(\mathcal{U},\mathscr{G}) \arrow[r] & H^p(X,\mathscr{G})
\end{tikzcd}$$

commutes. To see this, let $\mathscr{J}^\cdot$ be an injective resolution of $\mathscr{G}$, then observe that the diagram of complexes

$$\begin{tikzcd}
C^\cdot(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d] & \mathscr{I}^\cdot \arrow[d] \\
C^\cdot(\mathcal{U},\mathscr{G}) \arrow[r] & \mathscr{J}^\cdot
\end{tikzcd}$$

commutes up to homotopy[12]. Thus, when we take homologies, we get a commutative diagram. $\square$

Lemma 11.3. *Let $X, \mathcal{U}, \mathscr{F}$ be as usual. If $\mathscr{F}$ is flasque, then*

$$\check{H}^p(\mathcal{U},\mathscr{F}) = 0$$

for all $p > 0$.

Proof. For all $V \subseteq X$ open and for all p, we have

$$\begin{tikzcd}
\mathscr{C}^p(\mathcal{U},\mathscr{F})(X) \arrow[r] \arrow[d, equal] & \mathscr{C}^p(\mathcal{U},\mathscr{F})(V) \arrow[d, equal] \\
\prod \mathscr{F}(U_{i_0\ldots i_p}) \arrow[r] & \prod \mathscr{F}(U_{i_0\ldots i_p} \cap V)
\end{tikzcd}$$

where the bottom arrow is surjective because it is surjective componentwise. Thus, the top arrow is surjective. This shows that $\mathscr{C}^p(\mathcal{U},\mathscr{F})$ is flasque. Thus, $\mathscr{C}^\cdot(\mathcal{U},\mathscr{F})$ is a flasque resolution of $\mathscr{F}$, so we can use it to compute

[12] I didn't actually check this.

derived functor cohomology.

$$\begin{aligned} H^p(X,\mathscr{F}) &= h^p(\Gamma(X,\mathscr{C}^{\cdot}(\mathcal{U},\mathscr{F}))) \\ &= h^p(C^{\cdot}(\mathcal{U},\mathscr{F})) \\ &= \check{H}^p(\mathcal{U},\mathscr{F}) \end{aligned}$$

But $H^p(X,\mathscr{F})=0$ for all $p>0$ because $\mathscr{F}$ is flasque. $\square$

Čech Cohomology agrees with Derived Functor Cohomology for the right $\mathcal{U}$

Theorem 11.4 (Exercise III.4.11). *Let X,$\mathcal{U}$,$\mathscr{F}$ be as usual. Assume that $H^l(U_{i_0\dots i_p},\mathscr{F}|_{U_{i_0\dots i_p}})=0$ for all $p\in\mathbb{N}$, $i_0<\cdots i_p$, and $l>0$. Then the maps from Lemma (11.2) are isomorphisms:*

$$\check{H}^p(\mathcal{U},\mathscr{F}) \xrightarrow{\sim} H^p(X,\mathscr{F}).$$

Proof. We apply induction on p. If $p=0$, then both cohomologies are isomorphic to $\Gamma(X,\mathscr{F}$.

For $p>0$, assume the result up to $p-1$. Embed $\mathscr{F}$ into a flasque sheaf $\mathscr{G}$, and let $\mathscr{R}$ be the quotient:

$$0\to\mathscr{F}\to\mathscr{G}\to\mathscr{R}\to 0$$

Then we get the exact sequence

$$0\to\mathscr{F}(U_{i_0\dots i_p})\to\mathscr{G}(U_{i_0\dots i_p})\to\mathscr{R}(U_{i_0\dots i_p})\to\underbrace{H^1(U_{i_0\dots i_p},\mathscr{F}|_{U_{i_0\dots i_p}})}_{\text{0 by assumption}}$$

That is,

$$0\to C^{\cdot}(\mathcal{U},\mathscr{F})\to C^{\cdot}(\mathcal{U},\mathscr{G})\to C^{\cdot}(\mathcal{U},\mathscr{R})\to 0 \tag{5}$$

is exact, so we get a long exact sequence in Čech cohomology:

$$0\to\check{H}^0(\mathcal{U},\mathscr{F})\to\check{H}^0(\mathcal{U},\mathscr{G})\to\check{H}^0(\mathcal{U},\mathscr{R})\to\check{H}^1(\mathcal{U},\mathscr{F})\to\underbrace{\check{H}^1(\mathcal{U},\mathscr{G})}_{\text{0 since }\mathscr{G}\text{ flasque}}$$

Let $0\to\mathscr{F}\to\mathscr{I}^{\cdot}$ and $0\to\mathscr{R}\to\mathscr{J}^{\cdot}$ be injective resolutions. Then, as in Theorem (2.8), we see that $\mathscr{I}^{\cdot}\oplus\mathscr{J}^{\cdot}$ is an injective resolution for $\mathscr{G}$. such that

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{G} \arrow[r] \arrow[d] & \mathscr{R} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{I}^0 \arrow[r] \arrow[d] & \mathscr{I}^0\times\mathscr{J}^0 \arrow[r] \arrow[d] & \mathscr{J}^0 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

commutes. Then we will get a diagram

$$\begin{tikzcd}
0 \arrow[r] & C^{\cdot}(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d] & C^{\cdot}(\mathcal{U},\mathscr{G}) \arrow[r] \arrow[d] & C^{\cdot}(\mathcal{U},\mathscr{R}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(X,\mathscr{I}^{\cdot}) \arrow[r] & \Gamma(X,\mathscr{I}^{\cdot}\oplus\mathscr{J}^{\cdot}) \arrow[r] & \Gamma(X,\mathscr{J}^{\cdot}) \arrow[r] & 0
\end{tikzcd}$$

(loose end 1)

such that the rows are exact and it commutes.

Taking long exact sequences, we get

$$\begin{tikzcd}
0 \arrow[r] & \check{H}^0(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d, "\wr"] & \check{H}^0(\mathcal{U},\mathscr{G}) \arrow[r] \arrow[d, "\wr"] & \check{H}^0(\mathcal{U},\mathscr{R}) \arrow[r] \arrow[d, "\wr"] & \check{H}^1(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(\mathcal{U},\mathscr{F}) \arrow[r] & H^0(\mathcal{U},\mathscr{G}) \arrow[r] & H^0(\mathcal{U},\mathscr{R}) \arrow[r] & H^1(\mathcal{U},\mathscr{F}) \arrow[r] & 0
\end{tikzcd}$$

So the map on the right must also be an isomorphism (though we still need to show it is the same as the map obtained in Lemma (11.2) ... this is loose end 2).

For all $p > 1$, the long exact sequence obtained from (5) give us that

$$\begin{tikzcd}
0 = \check{H}^{p-1}(\mathcal{U},\mathscr{G}) \arrow[r] & \check{H}^{p-1}(\mathcal{U},\mathscr{R}) \arrow[r, "\sim"] \arrow[d, "\wr \text{ induction}"] & \check{H}^p(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d] & \check{H}^p(\mathcal{U},\mathscr{G}) = 0 \\
0 = H^{p-1}(X,\mathscr{G}) \arrow[r] & H^{p-1}(X,\mathscr{R}) \arrow[r, "\sim"] & H^p(X,\mathscr{F}) \arrow[r] & H^p(X,\mathscr{G}) = 0
\end{tikzcd}$$

To use induction, we need $\mathscr{R}$ to satisfy the hypothesis. That is, we need to check that $H^l(U_{i_0\dots i_p},\mathscr{R}|_{U_{i_0\dots i_p}}) = 0$ for all $l > 0$, $p \in \mathbb{N}$, and $i_0 < \cdots < i_p$. To see this, note that we have

$$0 \to \mathscr{F}|_{U_{i_0\dots i_p}} \to \mathscr{G}|_{U_{i_0\dots i_p}} \to \mathscr{R}|_{U_{i_0\dots i_p}} \to 0$$

exact, so the long exact sequence in cohomology tells us that

$$\underbrace{H^l(U_{i_0\dots i_p},\mathscr{G}|_{U_{i_0\dots i_p}})}_{0 \text{ since } \mathscr{G} \text{ flasque}} \to H^l(U_{i_0\dots i_p},\mathscr{R}|_{U_{i_0\dots i_p}}) \to \underbrace{H^{l+1}(U_{i_0\dots i_p},\mathscr{F}|_{U_{i_0\dots i_p}})}_{0 \text{ by assumption}}$$

for all $l > 0$, so $H^l(U_{i_0\dots i_p},\mathscr{R}|_{U_{i_0\dots i_p}}) = 0$, as required.

Now only two loose ends remain. proof continued in next lecture

Lecture 12

Corollary 12.1 (of unfinished theorem)**.** *Let $\mathscr{F}$ be a quasi-coherent sheaf on a noetherian separated scheme X, and let $\mathcal{U}$ be an open affine cover of X. Then the conclusion of the theorem holds. The maps from Lemma (11.2) are isomorphisms:*

$$\check{H}^p(\mathcal{U},\mathscr{F}) \xrightarrow{\sim} H^p(X,\mathscr{F}).$$

Theorem 5.1:qco sheaves are acyclic on affine schemes

Proof. All the $U_{i_0\dots i_p}$ are affine by exercise II.4.3, so use Theorem (5.1). $\square$

Continued proof of 11.4. Loose end 1: we need to construct the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C^\cdot(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d] & C^\cdot(\mathcal{U},\mathscr{G}) \arrow[r] \arrow[d] & C^\cdot(\mathcal{U},\mathscr{R}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(X,\mathscr{I}^\cdot) \arrow[r] & \Gamma(X,\mathscr{I}^\cdot \oplus \mathscr{J}^\cdot) \arrow[r] & \Gamma(X,\mathscr{J}^\cdot) \arrow[r] & 0
\end{tikzcd} \tag{6}$$

To do this, we switch to sheaves:

$$
\begin{tikzcd}
0 \arrow[r] & \mathscr{C}^{\cdot}(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d, dotted, "1"] & \mathscr{C}^{\cdot}(\mathcal{U},\mathscr{G}) \arrow[r] \arrow[d, dotted, "2"] & \mathscr{C}^{\cdot}(\mathcal{U},\mathscr{R}) \arrow[d, dotted, "3"] & \text{(not surjective)} \\
0 \arrow[r] & \mathscr{I}^{\cdot} \arrow[r] & \mathscr{I}^{\cdot} \oplus \mathscr{J}^{\cdot} \arrow[r] & \mathscr{J}^{\cdot} \arrow[r] & 0
\end{tikzcd}
$$

Note that the sequence of sheaves is not exact on the right. We wish to construct arrows 1,2, and 3. Assume we have constructed them up to $p-1$. Then we have the commutative diagram

$$
\begin{tikzcd}
0 \arrow[rr] & & \mathscr{K}' \arrow[rr] \arrow[dr, hook] \arrow[dd] & & \mathscr{K} \arrow[rr] \arrow[dr, hook] \arrow[dd] & & \mathscr{K}'' \arrow[dr, hook] \arrow[dd] & & \text{(not surjective)} & \\
& 0 \arrow[rr] & & \mathscr{C}^p(\mathcal{U},\mathscr{F}) \arrow[rr] \arrow[dd, dotted, "1"] & & \mathscr{C}^p(\mathcal{U},\mathscr{G}) \arrow[rr, "\psi"] \arrow[dd, dotted, "2"] & & \mathscr{C}^p(\mathcal{U},\mathscr{R}) \arrow[dd, dotted, "3"] & & \text{(not surjective)} \\
0 \arrow[rr] & & \mathscr{M}' \arrow[rr] \arrow[dr, hook] & & \mathscr{M} \arrow[rr] \arrow[dr, hook] & & \mathscr{M}'' \arrow[rr] \arrow[dr, hook] & & 0 & \\
& 0 \arrow[rr] & & \mathscr{I}^p \arrow[rr] & & \mathscr{I}^p \oplus \mathscr{J}^p \arrow[rr] & & \mathscr{J}^p \arrow[rr] & & 0
\end{tikzcd}
$$

where the $\mathscr{K}$'s and $\mathscr{M}$'s are the cokernels of the map from $(p-2)$-th to the $(p-1)$-th terms in the complexes. Since each complex is exact (Lemma 11.1), these cokernels inject into the p-th terms. If $p=0$, then $\mathscr{K}' = \mathscr{M}' = \mathscr{F}$, $\mathscr{K} = \mathscr{M} = \mathscr{G}$, and $\mathscr{K}'' = \mathscr{M}'' = \mathscr{R}$, with the downward morphisms identity maps. Note that the rows of $\mathscr{K}$'s and $\mathscr{M}$'s are exact (to the degree shown). The maps from the $\mathscr{K}$'s to the $\mathscr{M}$'s are the induced cokernel maps.

To construct arrow 1, note that $\mathscr{K}'$ maps to $\mathscr{I}^p$, then injectivity of $\mathscr{I}^p$ produces the arrow.

To construct arrow 2, consider the map $\varphi : \mathscr{C}^p(\mathcal{U},\mathscr{F}) \oplus \mathscr{K} \to \mathscr{C}^p(\mathcal{U},\mathscr{G})$ given by addition of the images of the coordinates. Then

$$
\begin{aligned}
\ker\varphi &= \{(x,-x) | x \in \mathscr{C}^p(\mathcal{U},\mathscr{F}) \cap \mathscr{K}\} \\
&= \mathscr{C}^p(\mathcal{U},\mathscr{F}) \cap \mathscr{K} \\
&= (\ker\psi) \cap \mathscr{K} \\
&= \ker(\psi|_{\mathscr{K}}) \\
&= \ker(\mathscr{K} \to \mathscr{K}'') = \mathscr{K}'
\end{aligned}
$$

Also, both $\mathscr{C}^p(\mathcal{U},\mathscr{F})$ and $\mathscr{K}$ map to $\mathscr{I}^p \oplus \mathscr{J}^p$, so we get a map $\mathscr{C}^p(\mathcal{U},\mathscr{F}) \oplus \mathscr{K} \to \mathscr{I}^p \oplus \mathscr{J}^p$ by adding the images, and the kernel of this map contains $\mathscr{C}^p(\mathcal{U},\mathscr{F}) \cap \mathscr{K} = \mathscr{K}' = \ker\varphi$, so we get an induced map from the image of φ to $\mathscr{I}^p \oplus \mathscr{J}^p$:

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{im}\varphi \arrow[r] \arrow[d] & \mathscr{C}^p(\mathcal{U},\mathscr{G}) \arrow[dl, dotted, "2"] \\
& \mathscr{I}^p \oplus J^p &
\end{tikzcd}
$$

Then we get arrow 2 from injectivity of $\mathscr{I}^p \oplus \mathscr{J}^p$.

To construct arrow 3, note that the existence of arrows 1 and 2 imply that there is a map from the image of ψ to $\mathscr{J}^p$ (since ψ^{-1} followed by 2 followed by projection is well defined). By injectivity of $\mathscr{J}^p$, we get arrow 3.

Taking global sections of the front face of the diagram, we get the diagram (6) and tie up our loose end (note that we get surjectivity of the top row).

Loose end 2: When we take long exact sequences of the rows in diagram (6), we get induced maps from Čech cohomology to derived functor cohomology, and we need to know that these maps are the same as those obtained in Lemma (11.2). In the lemma, we constructed a map of resolutions $\mathscr{C}^{\cdot}(\mathcal{U},\mathscr{F}) \xrightarrow{f^{\cdot}} \mathscr{I}^{\cdot}$, took global sections, and looked at the induced maps in homology. The way we tied up loose end 1 makes it clear that the maps obtained in the theorem are the same. $\square$

Exercise III.4.4

Exercise III.4.4: Let X be a topological space, and let $\mathscr{F} \in \mathcal{A}b(X)$, then we will show that

$$\varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U},\mathscr{F}) \longrightarrow H^1(X,\mathscr{F})$$

is an isomorphism.

(a) Let $\mathcal{U} = (U_i)_{i\in I}$ and $\mathcal{V} = (V_j)_{j\in J}$ be open covers of X. Suppose we're also given a function $\lambda : J \to I$ such that $V_j \subseteq U_{\lambda(j)}$ for all j (that is, $\mathcal{V}$ is a refinement of $\mathcal{U}$). Then for all p, there is an induced map $\lambda^p : \check{H}^p(\mathcal{U},\mathscr{F}) \to \check{H}^p(\mathcal{V},\mathscr{F})$. To see this, define

$$\begin{aligned} C^p(\lambda) : C^p(\mathcal{U},\mathscr{F}) &\to C^p(\mathcal{V},\mathscr{F}) \\ \alpha &\mapsto \beta \end{aligned}$$

where $\beta_{j_0\dots j_p} = \alpha_{\lambda(j_0)\dots\lambda(j_p)}|_{V_{j_0\cdots j_p}}$ (with the usual sign convention). As p varies, these maps commute with the coboundary maps of $C^{\cdot}(\mathcal{U},\mathscr{F})$ and $C^{\cdot}(\mathcal{V},\mathscr{F})$. Thus, we get induced maps $\lambda^p : \check{H}^p(\mathcal{U},\mathscr{F}) \to \check{H}^p(\mathcal{V},\mathscr{F})$ for each p. Moreover, given a refinement $\mathcal{W} = \{W_k\}_{k\in K}$ of $\mathcal{V}$ and $\mu : K \to J$ such that $W_k \subseteq V_{\mu(k)}$ for all k, the following diagram commutes:

$$\begin{tikzcd} \check{H}^p(\mathcal{U},\mathscr{F}) \arrow[r, "\lambda^p"] \arrow[dr, "(\mu\circ\lambda)^p"'] & \check{H}^p(\mathcal{V},\mathscr{F}) \arrow[d, "\mu^p"] \\ & \check{H}^p(\mathcal{W},\mathscr{F}). \end{tikzcd}$$

So far, λ^p depends on λ.

Lemma. *λ^p is independent of λ, at least for $p \le 1$.*

Proof. If $p = 0$: Let $\alpha \in C^0(\mathcal{U},\mathscr{F})$ be a cocycle (i.e. $d\alpha = 0$), so

$$0 = (d\alpha)_{i_0,i_1} = (\alpha_{i_0} - \alpha_{i_1})|_{U_{i_0,i_1}} \qquad \forall i_0 < i_1$$

All sections are restricted termwise to the appropriate open sets.

Then

$$(\lambda^0(\alpha) - \mu^0(\alpha))_j = (\alpha_{\lambda(j)} - \alpha_{\mu(j)}) = 0$$

so $\lambda^0(\alpha) = \mu^0(\alpha)$ for any λ and μ.

If $p=1$: Let $\alpha \in C^1(\mathcal{U},\mathscr{F})$ be a cocycle, so

$$(d\alpha)_{j_0,j_1,j_2} = \alpha_{j_1,j_2} - \alpha_{j_0,j_2} + \alpha_{j_0,j_1} = 0 \qquad \forall j_0 < j_1 < j_2$$

Given $j_0, j_1 \in J$, let $i_0 = \lambda(j_0), i_1 = \lambda(j_1), i'_0 = \mu(j_0), i'_1 = \mu(j_1)$, then

$$\begin{aligned}(\lambda^1(\alpha) - \mu^1(\alpha))_{j_0,j_1} &= \alpha_{i_0,i_1} - \alpha_{i'_0,i'_1} \\ &= (\alpha_{i_0,i_1} - \alpha_{i_0,i'_1}) + (\alpha_{i_0,i'_1} - \alpha_{i'_0,i'_1}) \\ &= -\alpha_{i_1,i'_1} + \alpha_{i_0,i'_0} && (\alpha \text{ a cocycle}) \\ &= (d\gamma)_{j_0,j_1}\end{aligned}$$

where γ is defined by $\gamma_j = -\alpha_{\lambda(j),\mu_j}|_{V_j}$. Thus, $\lambda^1(\alpha)$ and $\mu^1(\alpha)$ are cohomologous. $\square$

More on $X \cong (X \times_S Y) \times_{Y \times_S Y} Y$, and when fiber products associate. The following always hold:

A neat trick for showing schemes are isomorphic - a glimpse of Yoneda's Lemma

$$\begin{aligned}(A \otimes_S B) \otimes_S C &\cong A \otimes_S (B \otimes_S C) && \text{(for rings)} \\ A \otimes_S B &\cong B \otimes_S A \\ (A \times_S S') \times_{S'} B &\cong A \times_S B && \text{(for base change)} \\ A \times_S S &\cong A\end{aligned}$$

Fix a scheme, S. Recall that $\mathfrak{Sch}(S)$ is the category of S-schemes, whose objects are morphism $X \to S$ and whose arrows are commutative diagrams $\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$. An object $X \in \mathfrak{Sch}(S)$ may also be viewed as the representable contravariant functor $\mathscr{H}om_S(-,X) : \mathfrak{Sch}(S) \to \mathbf{Sets}$, $S' \mapsto \mathscr{H}om_S(S',X) = X(S')$. Then an S-morphism $f: X \to Y$ corresponds to a natural transformation of functors, $\varphi : X \to Y$ given by $\varphi(S') = f \circ -$. Thus, given $S'' \to S$, the diagram

$$\begin{tikzcd} X(S') \arrow[r, "\varphi(S')"] \arrow[d] & Y(S') \arrow[d] \\ X(S'') \arrow[r, "\varphi(S'')"] & Y(S'') \end{tikzcd}$$

commutes. Conversely, given a natural transformation $\varphi : X \to Y$, you can define $f = \varphi(X)(\mathrm{id}_X)$, which is an element of $Y(X) = \mathscr{H}om_S(X,Y)$. If you start with f and produce a natural transformation φ, then one may verify that $\varphi(X)(\mathrm{id}_X) = f$. Likewise, one may verify that the natural transformation associated to $\varphi(X)(\mathrm{id}_X)$ is indeed φ.

$X(X) \xrightarrow{\varphi(X)} Y(X)$

So S-morphisms are the same a natural transformations[13]. Now suppose $X, Y \in \mathfrak{Sch}(S)$ give isomorphic functors, then the natural isomorphism between the functors induces an isomorphism of S-schemes. Thus, to show that the two schemes are isomorphic, it is enough to find a natural transformation between the functors.

[13]There is nothing special about S-schemes. Objects in *any* category may be viewed this way. It is an immediate corollary of Yoneda's Lemma.

One may verify that $\mathscr{H}om_S(S', -)$ behaves as expected[14]. For example $(X \times_Z Y)(S') = X(S') \times_{Z(S')} Y(S') = \{(x, y) \in X(S') \times Y(S') | f \circ x = g \circ y \text{ in } Z(S')\}$.

$$\begin{array}{c} Y \\ \downarrow \Delta \\ Y \times_S Y \\ {\scriptstyle f \times \mathrm{id}_Y} \uparrow \\ X \times_S Y \end{array}$$

Thus, we may calculate

$$\begin{aligned} ((X \times_S Y) \times_{Y \times_S Y} Y)(S') &\cong (X(S') \times Y(S')) \times_{Y(S') \times Y(S')} Y(S') \\ &= \{(\alpha, \beta, \gamma) \in X(S') \times Y(S') \times Y(S') | \\ &\qquad (f(\alpha), \beta) = (\gamma, \gamma) \in Y(S') \times Y(S')\} \\ &= X(S') \qquad\qquad (\beta = \gamma = f(\alpha)) \end{aligned}$$

So $(X \times_S Y) \times_{Y \times_S Y} Y \cong X$.

Lecture 13

Begin Fast Forward

Some stuff about proof in last lecture. in the works.

More on products: If X, Y, Z, S' are S-schemes and $X \to Z, Y \to Z$ are S-morphisms, then

$$(X \times_Z Y) \times_S S' = X' \times_{Z'} Y'$$

where $\Diamond' := \Diamond \times_S S'$.

Proof. For all S-schemes Y, we have that $\Diamond'(T) = \Diamond(T) \times S'(T)$, so

$$\begin{aligned} (X' \times_{Z'} Y')(T) &= \{(\alpha, \gamma, \beta, \gamma') \in X(T) \times S'(T) \times Y(T) \times S'(T) | \\ &\qquad (\alpha, \gamma) \text{ and } (\beta, \gamma') \text{ both lie over the same element of } Z(T) \times S'(T)\} \\ &= \{(\alpha, \gamma, \beta, \gamma') | \alpha \text{ and } \beta \text{ lie over the same element of } Z(T) \times S'(T)\} \\ &= (X \times_Z Y)(T) \times S'(T) \\ &= ((X \times_Z Y) \times_S S')(T) \end{aligned}$$

□

Corollary 13.1. $(X \times_S Y) \times_S S' = X' \times_{S'} Y'$.

We were in the middle of doing Exercise III.4.4, which states that $\varinjlim \check{H}^p(\mathcal{U}, \mathscr{F}) \to H^p(X, \mathscr{F})$ is an isomorphism. in the works.

Ch III §5: Cohomology of Projective space

Let A be a Noetherian ring, and let $X = \mathbb{P}^r_A$ for some $r \in \mathbb{N}$. Then we will compute $H^i(X, \mathscr{O}(n))$ for all $i \in \mathbb{N}$ and $N \in \mathbb{Z}$. Recall from §II.5 that if $Y \subseteq \mathbb{P}^r_A$ is a closed subscheme and $\mathscr{F}$ is a sheaf on Y, then

$$\Gamma_*(\mathscr{F}) := \bigoplus_{n \in \mathbb{Z}} \Gamma(Y, \mathscr{F}(n))$$

where $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y(n)$, $\mathscr{O}_Y(n) = \mathscr{O}_X(n)|_Y = i^* \mathscr{O}_X(n)$ where $i : Y \hookrightarrow X$.

Theorem 13.2. *If $\mathscr{F}$ is quasi-coherent, then there is a natural isomorphism*

$$\widetilde{\Gamma_*(\mathscr{F})} \xrightarrow{\sim} \mathscr{F}$$

(However, if M is a graded S-module (where $Y = \operatorname{Proj} S$), then we may not have $\Gamma_*(\tilde{M}) \cong M$.).

[14] $\mathscr{H}om_S(S', -)$ should be the right adjoint to something like $- \times_S S'$ ... I haven't verified this.

Proof. Let $S = A[x_0, \ldots, x_r]$ and $X = \operatorname{Proj} S$ $(= \mathbb{P}^r_A)$. Let $\mathscr{F} = \bigoplus_{n\in\mathbb{Z}} \mathscr{O}(n)$. This is a quasi-coherent $\mathscr{O}_X$-module (though not coherent), and since cohomology commutes with arbitratry direct sums (Lemma 4.4),

$$H^i(X, \mathscr{F}) = \bigoplus_{n\in X} H^i(X, \mathscr{O}_X(n)).$$

in the works.

proof continued in next lecture

Lecture 14

Proof Continued. in the works. □

This suggests:

Theorem 14.1. *The natural pairing*

$$H^0(X, \mathscr{O}_X(n)) \times H^r(X, \mathscr{O}_X(-n-r-1)) \to H^r(x, \mathscr{O}_X(-r-1)) \cong A$$

is a perfect pairing of finitely generated free A-modules for all n.

Proof. in the works. □

more in the works.
We've just proved

Theorem 14.2 (III.5.1)**.** *Let A be a noetherian ring, let $X = \mathbb{P}^r_A$ with $r \in \mathbb{Z}_{>0}$ and let $n \in \mathbb{Z}$. Then*

(a) *The natural map $S_n \to H^0(X, \mathscr{O}_X(n))$ is an isomorphism. (where $S = A[x_0, \ldots, x_n]$)*
(b) *$H^i(X, \mathscr{O}_X(n)) = 0$ for all $i \neq 0, r$ and for all n.*
(c) *$H^r(X, \mathscr{O}_X(-r-1)) \cong A$*
(d) *$H^0(X, \mathscr{O}_X(n)) \times H^r(X, \mathscr{O}_X(-n-r-1)) \to H^r(X, \mathscr{O}_X(-r-1)) \cong A$ is a perfect pairing of finitely generated A-modules for all n.*

Theorem 14.3 (III.5.2, Serre)**.** *Let A be a noetherian ring, X a projective scheme over A, $\mathscr{O}_X(1)$ a very ample line sheaf (invertible sheaf) on X over A and let $\mathscr{F}$ be a coherent sheaf on X. Then*

(a) *$H^i(X, \mathscr{F})$ is a finitely generated A-module for all i, and*
(b) *$H^i(X, \mathscr{F}(n)) = 0$ for all $n \gg 0$ (depending on $\mathscr{F}$) and for all $i > 0$.*

Proof. in the works. proof continued in next lecture

Lecture 15

Proof Continued. in the works. □

Corollary 15.1. *For any coherent sheaf $\mathscr{F}$ on a projective scheme X, $\Gamma(X, \mathscr{F})$ is finitely generated (this generalizes Thm II.5.19).*

Corollary 15.2. *Let X be a closed subscheme of $\mathbb{P}^r_A = \operatorname{Proj} S$ with $S = A[x_0, \ldots, x_n]$. Then $S_n \to \Gamma(X, \mathscr{O}_X(n))$ is surjective for all $n \gg 0$.*

Proof. in the works. □

Proposition 15.3 (III.5.3)**.** *Let A be a noetherian ring and let X be a proper scheme over A. Then a line sheaf $\mathscr{L}$ on X is ample if and only if:*

$$\forall \mathscr{F} \in \mathfrak{Coh}(X), H^i(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n}) = 0 \quad \forall i > 0, \forall n \gg 0 \text{ (depending on } \mathscr{F}) \tag{*}$$

Proof. in the works $\square$

Definition. if X is a scheme over a field k and $\mathscr{F}$ a sheaf on X, then $h^i(X, \mathscr{F}) := \dim_k H^i(X, \mathscr{F})$.

Exercise (III.5.5) in the works.

Lecture 16

§III.6: Ext Groups and Sheaves

Let $(X, \mathscr{O}_X)$ be a ringed space. We'll be working with the category of $\mathscr{O}_X$-modules, so Hom means $\mathrm{Hom}_{\mathscr{O}_X}$ and $\mathscr{H}om$ means $\mathscr{H}om_{\mathscr{O}_X}$. Recall that $\mathscr{H}om(\mathscr{F}, \mathscr{G}) : U \mapsto \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a sheaf (Ex. II.1.15).

For us, $(X, \mathscr{O}_X)$ will be one of

(i) a scheme, or
(ii) $X = \{point\}$, $\mathscr{O}_X =$ some ring A, so $\mathcal{M}od(X) = Mod(A)$.

Definition. Let $(X, \mathscr{O}_X)$ be a ringed space, and let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. Then

$$\mathrm{Ext}^i(\mathscr{F}, -)$$

are the right derived fuctors of $\mathrm{Hom}(\mathscr{F}, -)$, and

$$\mathscr{E}xt^i(\mathscr{F}, -)$$

are the right derived functors of $\mathscr{H}om(\mathscr{F}, -)$. (Note that $\mathcal{M}od(X)$ has enough injectives)

Motivation:

(a) Hom and $\mathscr{H}om$ are basic functors, so it makes sense to look at thier derived functors.
(b) Used in duality.
(c) (Ex. III.6.1) $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$ parameterizes *extensions* of $\mathscr{F}$ by $\mathscr{G}$. An extension is a sheaf $\mathscr{F}'$ such that

$$0 \to \mathscr{G} \to \mathscr{F}' \to \mathscr{F} \to 0.$$

Lemma 16.1. *If $\mathscr{I}$ is an injective $\mathscr{O}_X$-module and $U \subseteq X$ is open, then $\mathscr{I}|_U$ is an injective $\mathscr{O}_U$-module.*

Proof. Say we have a diagram of $\mathscr{O}_U$-modules with the top row exact:

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{G} \\ & \mathscr{I}|_U & \end{tikzcd}$$

Then we have

$$\begin{tikzcd} 0 \arrow[r] & j_!\mathscr{F} \arrow[r] \arrow[d] & j_!\mathscr{G} \\ & j_!(\mathscr{I}|_U) \arrow[r, "\text{Ex. II.1.19}"] & \mathscr{I} \end{tikzcd}$$

where $j : U \to X$ is the inclusion and $j_!$ is as in (Ex. II.1.19). By injectivity of $\mathscr{I}$, there is a map $\phi : j_!\mathscr{G} \to \mathscr{I}$ extending this diagram. Restricting to U, we have $\phi|_U : (j_!\mathscr{G})|_U = \mathscr{G} \to \mathscr{I}_U$, as desired. $\square$

Proposition 16.2. *For any $U \subseteq X$, $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})|_U = \mathscr{E}xt^i(\mathscr{F}|_U, \mathscr{G}|_U)$ naturally.*

Proof. Let $0 \to \mathscr{G} \to \mathscr{I}^\cdot$ be an injective resolution of $\mathscr{G}$. Then, by the lemma, $0 \to \mathscr{G}|_U \to \mathscr{I}^\cdot|_U$ is an injective resolution of $\mathscr{G}|_U$, so

$$\begin{aligned} \mathscr{E}xt^i(\mathscr{F}, \mathscr{G})|_U &= h^i(\mathscr{H}om(\mathscr{F}, \mathscr{G}))|_U \\ &= h^i(\mathscr{H}om(\mathscr{F}|_U, \mathscr{G}|_U)) \\ &= \mathscr{E}xt(\mathscr{F}|_U, \mathscr{G}|_U). \end{aligned}$$

$\square$

Proposition 16.3 (III.6.3)**.**

(a) $\mathscr{E}xt^i(\mathscr{O}_X, \mathscr{G}) = \begin{cases} \mathscr{G} & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$

(b) $\operatorname{Ext}^i(\mathscr{O}_x, \mathscr{G}) = H^i(X, \mathscr{G})$ *for all i.*

Proof. (a) $\mathscr{E}xt^i(\mathscr{O}_X, -)$ are the right derived functors of $\mathscr{H}om(\mathscr{O}_X, -)$, which is the identity functor, which is exact.

(b) $\operatorname{Ext}^i(\mathscr{O}_X, -)$ are the right derived functors of $\operatorname{Hom}(\mathscr{O}_X, -)$, which is the functor $\Gamma(X, -)$, whose right derived functors are $H^i(X, -)$. $\square$

Proposition 16.4 (III.6.4)**.** *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is a short exact sequence of $\mathscr{O}_X$-modules, then we have long exact sequences*

$$0 \to \operatorname{Hom}(\mathscr{F}'', \mathscr{G}) \to \operatorname{Hom}(\mathscr{F}, \mathscr{G}) \to \operatorname{Hom}(\mathscr{F}', \mathscr{G}) \to \operatorname{Ext}^1(\mathscr{F}'', \mathscr{G}) \to \cdots$$

and

$$0 \to \mathscr{H}om(\mathscr{F}'', \mathscr{G}) \to \mathscr{H}om(\mathscr{F}, \mathscr{G}) \to \mathscr{H}om(\mathscr{F}', \mathscr{G}) \to \mathscr{E}xt^1(\mathscr{F}'', \mathscr{G}) \to \cdots$$

Proof. Let $0 \to \mathscr{G} \to \mathscr{I}^\cdot$ be an injective resolution. Then by injectivity, we the exact sequences

$$0 \to \operatorname{Hom}(\mathscr{F}'', \mathscr{I}^\cdot) \to \operatorname{Hom}(\mathscr{F}, \mathscr{I}^\cdot) \to \operatorname{Hom}(\mathscr{F}', \mathscr{I}^\cdot) \to 0.$$

Applying the Snake Lemma gives the result. Similarly for the second long exact sequence. $\square$

Lemma 16.5 (III.6.6)**.** *If $\mathscr{I}$ is an injective $\mathscr{O}_X$-module and $\mathscr{L}$ is locally free of finite rank, then $\mathscr{I} \otimes \mathscr{L}$ is also injective.*

Proof. Recall from Ex II.5.1 that $\check{\mathscr{L}} = \mathscr{H}om(\mathscr{L}, \mathscr{O}_X)$ and

(a) $\check{\check{\mathscr{L}}} \cong \mathscr{L}$
(b) $\mathscr{H}om(\mathscr{L}, \mathscr{F}) \cong \mathscr{F} \otimes \check{\mathscr{L}}$ for all sheaves $\mathscr{F}$
(c) $\operatorname{Hom}(\mathscr{L} \otimes \mathscr{F}, \mathscr{G}) \cong \operatorname{Hom}(\mathscr{F}, \mathscr{H}om(\mathscr{L}, \mathscr{G}))$ for all $\mathscr{F}, \mathscr{G}$. More generally, it is true that

$$\operatorname{Hom}(\mathscr{F}_1 \otimes \mathscr{F}_2, \mathscr{G}) \cong \operatorname{Hom}(\mathscr{F}_1, \mathscr{H}om(\mathscr{F}_2, \mathscr{L}))$$

for all $\mathscr{F}_1, \mathscr{F}_2, \mathscr{G}$.

Thus, $\mathrm{Hom}(-, \mathscr{I} \otimes \mathscr{L})$ is equal to $\mathrm{Hom}(- \otimes \check{\mathscr{L}}, \mathscr{I})$, which is the composite of two exact functors, and is therefore exact. □

Note that

$$\mathrm{Hom}(\mathscr{F} \otimes \mathscr{L}, \mathscr{G}) \overset{(b),(c)}{\cong} \mathrm{Hom}(\mathscr{F}, \mathscr{G} \otimes \check{\mathscr{L}}) \cong \mathscr{H}om(\mathscr{F}, \mathscr{G}) \otimes \check{\mathscr{L}}$$ [15]

Proposition 16.6. *If $\mathscr{L}$ is locally free of finite rank and $\mathscr{F}, \mathscr{G} \in \mathcal{M}od(X)$, then*

(a) $\mathrm{Ext}^i(\mathscr{F} \otimes \mathscr{L}, \mathscr{G}) \cong \mathrm{Ext}^i(\mathscr{F}, \mathscr{G} \otimes \check{\mathscr{L}})$ *and*
(b) $\mathscr{E}xt^i(\mathscr{F} \otimes \mathscr{L}, \mathscr{G}) \cong \mathscr{E}xt^i(\mathscr{F}, \mathscr{G} \otimes \check{\mathscr{L}}) \cong \mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) \otimes \check{\mathscr{L}}$.

Proof. Let $0 \to \mathscr{G} \to \mathscr{I}^{\cdot}$ be an injective resolution. Then (a):

$$\begin{aligned}\mathrm{Ext}^i(\mathscr{F} \otimes \mathscr{L}, \mathscr{G}) &= h^i(\mathrm{Hom}(\mathscr{F} \otimes \mathscr{L}, \mathscr{I}^{\cdot}))\\ &= h^i(\mathrm{Hom}(\mathscr{F}, \mathscr{I}^{\cdot} \otimes \check{\mathscr{L}}))\\ &= \mathrm{Ext}^i(\mathscr{F}, \mathscr{G} \otimes \check{\mathscr{L}}) \qquad (\mathscr{I}^{\cdot} \otimes \check{\mathscr{L}} \text{ inj res of } \mathscr{G} \otimes \check{\mathscr{L}})\end{aligned}$$

And for (b), the first isomorphism is "the same". We also have that

$$\begin{aligned}\mathscr{E}xt^i(\mathscr{F}, \mathscr{G} \otimes \check{\mathscr{L}}) &= h^i(\mathscr{H}om(\mathscr{F}, \mathscr{I}^{\cdot} \otimes \check{\mathscr{L}})\\ &= h^i(\mathscr{H}om(\mathscr{F}, \mathscr{I}^{\cdot})) \otimes \check{\mathscr{L}}\\ &= \mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) \otimes \check{\mathscr{L}}\end{aligned}$$

□

Corollary 16.7. *We know Ext^i and $\mathscr{E}xt^i$ when the first argument is locally free of finite rank:*

$$\begin{aligned}\mathrm{Ext}^i(\mathscr{L}, \mathscr{G}) &= \mathrm{Ext}^i(\mathscr{O}_X \otimes \mathscr{L}, \mathscr{G})\\ &= \mathrm{Ext}^i(\mathscr{O}_X, \mathscr{G} \otimes \check{\mathscr{L}})\\ &= H^i(X, \mathscr{G} \otimes \check{\mathscr{L}})\end{aligned}$$

and

$$\begin{aligned}\mathscr{E}xt^i(\mathscr{L}, \mathscr{G}) &= \mathscr{E}xt^i(\mathscr{O}_X, \mathscr{G}) \otimes \check{\mathscr{L}}\\ &= \begin{cases} \mathscr{G} \otimes \check{\mathscr{L}} & \text{if } i = 0\\ 0 & \text{if } i \neq 0\end{cases}\end{aligned}$$

Proposition 16.8 (III.6.5)**.** *Suppose we have a locally free resolution of $\mathscr{F}$ (i.e. an exact sequence $\cdots \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0$, where $\mathscr{L}_i$ are locally free of finite rank for all i). Then for all $\mathscr{G}$,*

$$\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) \cong h^i(\mathscr{H}om(\mathscr{L}_{\cdot}, \mathscr{G})).$$

Proof. in the works □

Proposition 16.9 (III.6.8)**.** *Let X be a noetherian scheme. Let $\mathscr{F}$ be a coherent sheaf on X, and let $\mathscr{G}$ be any sheaf of $\mathscr{O}_X$-modules. Then for all $x \in X$, and $i \in \mathbb{N}$,*

$$\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})_x = \mathrm{Ext}^i_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x).$$

[15] why is the second isormophism true?

Proof. This is a local question, so we may assume that X is affine, say $X = \operatorname{Spec} A$ with A noetherian, and $\mathscr{F} = \tilde{M}$ where M is a finitely generated A-module. Then ther is a free resolution

$$\cdots \to L_1 \to L_0 \to M \to 0$$

giving the locally free resolution $\tilde{L}. \to \mathscr{F} \to 0$. So

$$\begin{aligned}
\mathscr{E}xt^i_X(\mathscr{F},\mathscr{G})_x &= h^i(\mathscr{H}om(\tilde{L}.,\mathscr{G}))_x \\
&= h^i(\mathscr{H}om(\tilde{L}.,\mathscr{G})_x) \\
&= h^i((\check{\tilde{L}}. \otimes \mathscr{G})_x) \\
&= h^i(((\check{\tilde{L}}.)_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}_x)) \\
&= h^i(\operatorname{Hom}_{\mathscr{O}_{X,x}}((L.)_x, \mathscr{G}_x)) \\
&= \operatorname{Ext}^i_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x).
\end{aligned}$$

□

Proposition 16.10 (III.6.9)**.** *Let X be a projective scheme over a noetherian ring A, let $\mathscr{O}_X(1)$ be a very ample line sheaf on X over A; let $\mathscr{F}$ and $\mathscr{G}$ be coherent sheaves on X, and let $i \in \mathbb{N}$. Then*

$$\Gamma(X, \mathscr{E}xt^i(\mathscr{F},\mathscr{G}(n))) = \operatorname{Ext}^i(\mathscr{F},\mathscr{G}(n))$$

for all $n \gg 0$ (depending on i).

Proof. If $i = 0$, then its true for all $\mathscr{F},\mathscr{G},n$ by definition of $\mathscr{H}om$, so assume $i > 0$.

If $\mathscr{F}$ is locally free of finite rank, then we compute that the left hand side is 0 for all n (Prop 16.3), and the right hand side is zero for $n \gg 0$ (Cor 16.7 and Thm 14.3). In general, by Corollary (II.5.18), there is a short exact sequence

$$0 \to \mathscr{R} \to \mathscr{E} \to \mathscr{F} \to 0$$

where $\mathscr{E}$ is a finite direct sum of twisted structure sheaves. Then for $n \gg 0$, $\check{\mathscr{E}} \otimes \mathscr{G}(n)$ has no higher cohomology (by Thm 14.3), so $\operatorname{Ext}^i(\mathscr{E},\mathscr{G}(n)) = 0$ for all $i > 0$. □

Lecture 17

§II.8: Differentials

Definition (An alternative definition)**.** Let B be an A-algebra ($A \xrightarrow{f} B$), then $\Omega_{B/A}$ is the B-module described by generators db for all $b \in B$ and relations

$$\begin{aligned}
&da = 0 \\
&d(b_1 + b_2) = db_1 + db_2 \\
&d(b_1 b_2) = b_1 db_2 + b_2 db_1
\end{aligned}$$

for all $a \in A$ and $b_i \in B$. Equivalently, d is A-linear and satisfies the Leibniz condition[16].

[16] ⇒ trivial. ⇐: $da = ad1 = ad(1 \cdot 1) = 2ad1$, so $da = 0$.

Proposition 17.1 (II.8.3A: First Exact Sequence)**.** *If $A \to B \to C$ are ring homomorphisms, then*

$$\Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \to 0$$
$$db \otimes c \mapsto cdb, \ \ dc \mapsto dc$$

is an exact sequence of C-modules.

Proof. Surjectivity is obvious (you are simply imposing more relations). The kernel of the second map is generated by the "new" relations, $\{db = 0 | b \in B\}$, which is clearly the image of the first term. □

Example: If B is the polynomial ring $A[x_i]_{i \in I}$, then $\Omega_{B/A}$ is the free B-module generated by dx_i for all $i \in I$[17].

Proof. By the universal property of $\Omega_{B/A}$, we get

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "\partial/\partial x_i"'] & \Omega_{B/A} \arrow[d, dashed, "\exists\ dx_i \mapsto 1"] \\ & B \end{tikzcd}$$

This gives a map $\Omega_{B/A} \to \prod_I B$. The kernel of this map is zero (anything mapping to zero must be d(something w/ no x's) $= 0$. And for any element of the product, it is easy to construct an inverse image[18]. □

Proposition 17.2 (Second Exact Sequence)**.** *If $A \to B \to C$ is a sequence of ring homomorphisms with $B \to C$ surjective, and with kernel I, then*

$$I/I^2 \xrightarrow{b \mapsto db \otimes 1} \underbrace{\Omega_{B/A} \otimes_B C}_{\Omega_{B/A}/I\Omega_{B/A}{}^{19}} \xrightarrow{same} \Omega_{C/A} \to 0$$

is an exact sequence of C-modules.

Proof. Since $B \to C$ is surjective, $\Omega_{B/C} = 0$, so by the first exact sequence, the second map is surjective. The first map is well-defined since

$$d(b_1 b_2) \otimes 1 = db_1 \otimes b_2 + db_2 \otimes b_1 = 0.$$

Now we show exactness in the middle. It is clear that the composition of the two maps is zero. Conversely, if $\sum c_i db_i = 0$, then it is a C-linear
in the works □

Corollary 17.3. *If $C = A[x_i]_{i \in I}/\mathfrak{a}$, then $\Omega_{C/A}$ is the C-module described by generators dx_i and relations $df = 0$ for all $f \in \mathfrak{a}$ (or for all f in a generating set for $\mathfrak{a}$).*

Corollary 17.4 (of the Corollary)**.** *If $A \to B$ and $A \to A'$ are A-algebras and $B' = B \otimes_A A'$, then*

$$\Omega_{B'/A'} \cong \Omega_{B/A} \otimes_B B' \quad (= \Omega_{B/A} \otimes_A A')$$

Proof. in the works (for Anton). □

[17] Anton and Dave say I has to be finite.

[18] Not if I is infinite.

Corollary 17.5. *If S is a multiplicative subset of B, then*

$$\Omega_{S^{-1}B/A} \cong S^{-1}\Omega_{B/A}.$$

Proof. $\Omega_{S^{-1}B/A}$ has the generators $d(s^-1b)$. Then $db = d(s \cdot s^{-1} \cdot b) = s^{-1}b \cdot ds + s \cdot d(s^{-1}b)$, so

$$d(s^{-1}b) = s^{-1}db - s^{-2}bds$$

so you don't really get any new relations (exercise). □

Sheaves of Differentials

Now we pass to sheaves over schemes. If X is a scheme over $\operatorname{Spec} A$, then there is a unique sheaf $\Omega_{X/A}$

Lecture 18

Last time: If $Y = \operatorname{Spec} A$ and $X = \mathbb{P}^n_Y$, then there is an exact sequence

$$0 \to \Omega_{X/Y} \to \mathscr{O}(-1)^{n+1} \to \mathscr{O}_X \to 0.$$

By base change from $\operatorname{Spec}\mathbb{Z}$, this is true for arbitrary Y. By Exercise II.5.16b, we have that $\wedge^{n+1}\mathscr{O}(-1)^{n+1} = \mathscr{O}(-n-1) \cong \wedge^1\mathscr{O}(-1) \otimes \wedge^n\Omega_{X/Y}$.

Definition. Let X be a non-singular variety over an algebraically closed field k. Then the *canonical sheaf* is $\omega_X = \wedge^n\Omega_{X/k}$, where $n = \dim X$.

So if $X = \mathbb{P}^n_k$, then $\omega_X = \mathscr{O}(-n-1)$. Concretely, say $X = \operatorname{Proj} k[x_0, \dots, x_n]$. Then on $D_+(x_0) \cong \operatorname{Spec} k[y_1, \dots, y_n]$ for $y_i = x_i/x_0$, $\Omega_{X/k}|_{D_+(x_0)}$ is free with generators $dy_1, \dots, dy_n$. Thus, $\omega_X|_{D_+(x_0)}$ is free of rank 1, generated by $dy_1 \wedge \cdots \wedge dy_n$.

On $D_+(x_n) = \operatorname{Spec} k[x_0/x_n, \dots, x_{n-1}/x_n]$, $\Omega_{X/k}|_{D_+(x_n)}$ is generated by $d(x_0/x_n), \dots, d(x_{n-1}/x_n)$, so $\omega_X|_{D_+(x_n)}$ is free, generated by $d(x_0/x_n) \wedge \cdots \wedge d(x_{n-1}/x_n)$.

On $D_+(x_0) \cap D_+(x_n)$, this generator is

$$\begin{aligned} d(1/y_n)\wedge & d(y_1/y_n) \wedge \cdots \wedge d(y_{n-1}/y_n) = \\ &= (-y_n^{-2}dy_n) \wedge (y_n^{-1}dy_1 - y_n^{-2}y_1dy_n) \wedge \cdots \wedge (y_n^{-1}dy_{n-1} - y_n^{-2}y_{n-1}dy_n) \\ &= -y_n^{-n-1}dy_n \wedge dy_1 \wedge \cdots \wedge dy_{n-1} \end{aligned}$$

which has a pole of order $n+1$ at $y_n = 0$. So the divisor of $d(x_0/x_n), \dots, d(x_{n-1}/x_n)$ is $-(n+1)\{x_n = 0\}$, and

$$\mathscr{L}(-(n+1)\{x_n = 0\}) = \mathscr{O}(-n-1).$$

§III.7: Serre Duality

In our computations on $\mathbb{P}^n$, we computed $H^i(\mathbb{P}^n_A, \mathscr{O}(q))$ and came up with the perfect pairing

$$\underbrace{\operatorname{Hom}(\mathscr{O}_X, \mathscr{O}(r))}_{\operatorname{Hom}(\mathscr{O}(-r-n-1),\omega)} \times H^n(\mathbb{P}^n_A what???$$

in the works

Theorem 18.1 (III.7.1)**.** *Let k be a field and $X = \mathbb{P}^n_k$. Then*

(a) $H^n(X, \omega_X) \cong k$

(b) *Fix such an isomorphism. For all $\mathscr{F} \in \mathfrak{Coh}(X)$, the pairing*

$$\mathrm{Hom}(\mathscr{F}, \omega) \times H^n(X, \mathscr{F}) \to H^n(X, \omega) \xrightarrow{\sim} k$$

is a perfect pairing of finite dimensional vector spaces over k, and

(c) *For all $i \geq 0$, there is a natural isomorphism*

$$\mathrm{Ext}^i(\mathscr{F}, \omega) \xrightarrow{\sim} H^{n-i}(X, \mathscr{F})'$$

which for $i = 0$ is the isomorphism comming from the pairing (b) (and isn't canonical).

Proof. in the works □

dualizing sheaf!

Definition. Let X be a proper scheme over a field k, of dimension n. A *dualizing sheaf* for X (over k) is a coherent sheaf ω_X° on X which represents the contravariant functor

$$\mathfrak{Coh}(X) \to \mathcal{M}od(k)$$

given by $\mathscr{F} \mapsto H^n(X, \mathscr{F})'$. That is,

$$H^n(X, -)' \cong \mathrm{Hom}(-, \omega_X^\circ).$$

Given $\alpha : \mathrm{Hom}(-, \omega_X^\circ) \xrightarrow{\sim} H^n(X, -)'$, we have

$$\begin{aligned} \alpha(\omega_X^\circ) : \mathrm{Hom}(\omega_X^\circ, \omega_X^\circ) &\to H^n(X, \omega_X^\circ)' \\ \mathrm{id}_{\omega_X^\circ} &\mapsto t \end{aligned}$$

So α gives us a $t : H^n(X, \omega_X^\circ) \to k$. Conversely, given such a t, there is *at most* one α inducing it because for all $\mathscr{F}$ and for all $\phi : \mathscr{F} \to \omega_X^\circ$, the diagram

$$\begin{tikzcd} \omega_X^\circ & & \mathrm{Hom}(\omega_X^\circ, \omega_X^\circ) \arrow[r] \arrow[d, "\mathrm{Hom}(\phi, \omega_X^\circ)"'] & H^n(X, \omega_X^\circ)' \arrow[d, "H^n(X, \phi)"] \\ \mathscr{F} \arrow[u, "\phi"] & & \mathrm{Hom}(\mathscr{F}, \omega_X^\circ) \arrow[r, "\alpha(\mathscr{F})"] & H^n(X, \mathscr{F})' \end{tikzcd}$$

commutes. So

$$\alpha(\mathscr{F})(\phi) = [H^n(X, \mathscr{F}) \xrightarrow{H^n(X,\phi)} H^n(X, \omega_X^\circ) \xrightarrow{t} k] \in H^n(X, \mathscr{F})'. \qquad (*)$$

If α exists, then it gives an isomorphism

$$\alpha(\mathscr{F}) : \mathrm{Hom}(\mathscr{F}, \omega_X^\circ) \xrightarrow{\sim} H^n(X, \mathscr{F})'$$

for all $\mathscr{F}$, and by $(*)$ it is the map associated with the pairing

$$\mathrm{Hom}(\mathscr{F}, \omega_X^\circ) \times H^n(X, \mathscr{F}) \to H^n(X, \omega_X^\circ) \xrightarrow{t} k$$

and conversely[20].

Corollary 18.2. *The dualizing sheaf, if it exists, is unique up to unique isomorphism.*

[20] what?

Lemma 18.3. *Let k be a field, $P = \mathbb{P}^N_k$, and let X be a closed subscheme of P of codimension r (i.e. $r = \inf_{Z \subseteq X \text{ irr}} \operatorname{codim} Z$). Then*

$$\mathscr{E}xt^i_P(\mathscr{O}_X, \omega_P) = 0$$

for all $i < r$.

Proof. in the works □

Note that X doesn't have to be equidimensional.

Lecture 19

Recall the lemma from last time: if $P = \mathbb{P}^N_k$, and $X \subseteq P$ has codimension r, then $\mathscr{E}xt^i_P(\mathscr{O}_X, \omega_P) = 0$ for $i < r$.

Lemma 19.1. *Let k, N, P, X and r be as before, and let $\omega^\circ_X = \mathscr{E}xt^r_P(\mathscr{O}_X, \omega_P)$. Then for all $\mathscr{O}_X$-modules $\mathscr{F}$, there is a functorial isomorphism*

$$\operatorname{Hom}(\mathscr{F}, \omega^\circ_X) \cong \operatorname{Ext}^r_P(\mathscr{F}, \omega_P).$$

Proof. in the works □

Theorem 19.2. *Let X be a projective scheme over a field k. Then X has a dualizing sheaf.*

Proof. We may assume $X \neq \varnothing$. Embed $X \hookrightarrow \mathbb{P}^N_k = P$, and let $n = \dim X$, $r = \operatorname{codim}_P X = N - n$. Let $\omega^\circ_X = \mathscr{E}xt^r(\mathscr{O}_X, \omega_P)$. Then for all $\mathscr{O}_X$-modules $\mathscr{F}$,

$$\operatorname{Hom}_X(\mathscr{F}, \omega^\circ_X) \overset{19.1}{\cong} \operatorname{Ext}^r_P(\mathscr{F}, \omega_P) \overset{18.1}{\cong} H^{N-r}(P, \mathscr{F})' \cong H^n(X, \mathscr{F})'$$

functorially in $\mathscr{F}$. □

Theorem 19.3 ((part of) Duality)**.** *Let X be a non-empty projective scheme over a field k; let ω°_X be a dualixing sheaf for X, and let $n = \dim X$. Then for all $i \in \mathbb{N}$ and $\mathscr{F} \in \mathfrak{Coh}(X)$, there are natural maps*

$$\theta^i : \operatorname{Ext}^i_X(\mathscr{F} \omega^\circ_X) \to H^{n-i}(X, \mathscr{F})'$$

which for $i = 0$ reduces to the isomorphism in the definition of the dualizing sheaf.

Proof. Pick a projective embedding $X \subseteq P = \mathbb{P}^N_k$. We have a surjection $\mathscr{E} \to \mathscr{F} \to 0$, where $\mathscr{E}$ is a finite direct sum $\bigoplus \mathscr{O}(-q)$ for $q \gg 0$ (Cor II.5.18). Then

$$\operatorname{Ext}^i(\mathscr{E}, \omega^\circ_X) = \bigoplus \operatorname{Ext}^i(\mathscr{O}(-q), \omega^\circ_X) = \bigoplus H^i(X, \omega^\circ_X(q)) = 0$$

for all $i > 0$, so $\mathscr{E}xt^i(-, \omega^\circ_X)$ is coeffacable for $i > 0$. Also, $H^{n-i}(X, \mathscr{F})'$ are contravariant δ-functors, agreeing with $\operatorname{Ext}^{\cdot}(-, \omega^\circ_X)$ in degree 0. Thus, but Theorem III.1.3A, there are maps of delta functors, as desired, and these are the θ^i we seek. □

Remark. We didn't need k to be algebraically closed.

Definition. A non-empty noetherian scheme X is *equidimensional* (of dimension n) if all of its irreducible components have the same dimension (n).

Definition. A scheme X is *Cohen-Macaulay* if all its local rings are Cohen-Macaulay.

A finite dimensional local ring $(A, \mathfrak{m})$ is Cohen-Macaulay if $\operatorname{depth} A = \dim A$. Here *depth* of A is the maximal length of a regular sequence in A. A *regular sequence* is a sequence $x_1, x_2, \ldots, x_n \in \mathfrak{m}$ such that x_i is not a zero divisor in $A/(x_1, \ldots, x_{i-1})$ for all i.

Facts about Cohen-Macaulay rings and schemes:

(1) A regular scheme is Cohen-Macaulay (II.8.21Aa).

(2) A locally complete intersection inside a regular scheme of finite type over a field is Cohen-Macaulay.

Proof. Let $Y \subseteq X$ be a locally complete intersection with X regular and of finite type over a field. Let $P \in Y$. After shrinking X, we may assume Y is globally a complete intersection, cut out by $X_1, \ldots, X_r$. Thus, $\mathscr{O}_{Y,P} = \mathscr{O}_{X,P}/(x_1, \ldots, x_r)$. By II.8.21Ac, $x_1, \ldots, x_r$ is a regular sequence in $\mathscr{O}_{X,P}$, so by II.8.21Ad, $\mathscr{O}_{Y,P}$ is Cohen-Macaulay.

To apply II.8.21Ad above, we need to check that

$$r = \dim \mathscr{O}_{X,P} - \dim \mathscr{O}_{Y,P}.$$

We always have the inequality $\geq$. We may assume that $X = \operatorname{Spec} A$, and $Y = \operatorname{Spec} A/I$, in which case

$$\begin{aligned} \dim \mathscr{O}_{X,P} &= \operatorname{ht}_A P \\ &= \dim A - \operatorname{depth}_A P \end{aligned}$$

and

$$\begin{aligned} \dim \mathscr{O}_{Y,P} &= \operatorname{ht}_{A/I} P \\ &= \dim A/I - \operatorname{depth}_{A/I} P \\ &= \dim A/I - \dim(A/P) \qquad \text{(since } \mathscr{I} \subseteq P) \end{aligned}$$

$\square$

Lecture 20

Last time we showed that if Y is a locally complete intersection in a regular (or Cohen-Macauley) scheme X, then Y is Cohen-Macaulay.

To find an example of a non-Cohen-Macaulay scheme, we look to contradict facts we know about Cohen-Macaulay rings, like

Theorem 20.1 (Eisenbud Cor 18.10)**.** *In a Cohen-Macaulay ring, all associated primes are minimal.*

If k is a field and $A = k[x, y]/(x^2, xy)$, then is has an embedded point, so $\operatorname{Spec} A$ is not Cohen-Macaulay. To see this directly, let $\mathfrak{m} = (x, y) \subseteq A$. Then $A_\mathfrak{m}$ is not Cohen-Macaulay since $\dim A_\mathfrak{m} = 1$, but $\operatorname{depth} A_\mathfrak{m} = 0$ since all elements of the maximal ideal are zero divisors.

By II.8.21Ab, if A is a local Cohen-Macaulay ring, then any localization of A at a prime ideal is also Cohen-Macaulay. Thus, a scheme X is Cohen-Macaulay if and only if all of its local rings *at closed points* are Cohen-Macaulay.

Remark. Let Y be a locally complete intersection in a Cohen-Macaulay scheme X, and let $\mathscr{I}$ be its ideal sheaf. Assume that Y is equicodimensional (all irreducible components are of the same codimension, r). Then $\mathscr{I}/\mathscr{I}^2$ is a locally free sheaf on Y of rank r.

Proof. We may assume Y is globally a complete intersection in X, cut out by $x_1, \ldots, x_r$, and that X is affine, say $X = \operatorname{Spec} A$. Then $\mathscr{I} = \tilde{I}$ for $I = (x_1, \ldots, x_r)$. We have that

$$(A/I)^r \to I/I^2 \quad , \quad (a_1, \ldots, a_r) \mapsto \sum a_i x_i$$

is well defined and onto. We need to show that it is injective.

Suppose not. Then there is a prime $P \in A$ such that $(\text{kernel})_P \neq 0$. Clearly $I \subseteq P$. Replace A with A_P. Then A is Cohen-Macaulay and local, and $x_1, \ldots, x_r$ is a regular sequence. Let $(b_1, \ldots, b_r)$ be a nonzero element of the kernel in question. We may assume $b_r \notin I$. Then $\sum b_i x_i \in I^2$, so $\sum b_i x_i = \sum c_i x_i$ with $c_i \in I$ for all i. Then we have that $\sum (b_i - c_i) x_i = 0$. Since $b_r \notin I = (x_1, \ldots, x_r)$, it is non-zero in $A/(x_1, \ldots, x_{r-1})$. Thus, we have that x_r is a zero divisor (or 0) in $A/(x_1, \ldots, x_{r-1})$. Contradiction. □

Theorem 20.2. *Let X be a non-empty projective scheme of dimension n over an algebraically closed field k, and let ω_X° be a dualizing sheaf for X (over k). Then TFAE:*

(i) *X is equidimensional and Cohen-Macaulay*
(ii) *for all locally free sheaves $\mathscr{F}$ on X, $H^{n-i}(X, \mathscr{F}(-q)) = 0$ for all $i > 0$ and $q \gg 0$ (depending on $\mathscr{F}$).*
(iii) *the maps $\theta^i : \operatorname{Ext}^i(\mathscr{F}, \omega_X^\circ) \to H^{n-i}(X, \mathscr{F})'$ are isomorphisms for all i and for all coherent sheaves $\mathscr{F}$.*

Proof. in the works □

Corollary 20.3. *Let X be an equidimensional Cohen-Macaulay projective scheme of dimension n over an algebraically closed field k (e.g. a non-singular variety of dimension n). Let $\mathscr{F}$ be a locally free sheaf on X and let ω_X° be the dualizing sheaf. Then there is a natural isomorphism for all i:*

$$H^i(X, \mathscr{F}) \cong H^{n-i}(X, \check{\mathscr{F}} \otimes \omega_X^\circ)'.$$

Proof.

$$H^{n-i}(X, \mathscr{F})' \cong \operatorname{Ext}^i_X(\mathscr{F}, \omega_X^\circ) \cong \operatorname{Ext}^i_X(\mathscr{O}_X, \check{\mathscr{F}} \otimes \omega_X^\circ) \cong H^i(X, \check{\mathscr{F}} \otimes \omega_X^\circ).$$

□

Remark. In proving Lemmas III.7.3 and III.7.4, we only used duality for P, so the proofs actually give

Theorem 20.4. *Let P be an equidimensional Cohen-Macaulay projective scheme of dimension N over an algebraically closed field k, and let X be a non-empty closed subscheme of P of dimension n. Then*

$$\omega_X^\circ = \mathscr{E}xt_P^{N-n}(\mathscr{O}_X, \omega_P^\circ)$$

Definition. Let A be a ring, $f_1, \dots, f_r \in A$. Then the *Kozul complex* of A is the complex $K. = K.(f_1, \dots, f_r)$ defined by $K =$ free A-module of rank r, and $K_i := \wedge^i K$, and $d : K_p \to K_{p-1}$ is given by

$$e_{i_1} \wedge \dots \wedge e_{i_p} \mapsto \sum_{j=1}^{p} (-1)^{j-1} f_{i_j} e_{i_1} \wedge \dots \wedge \hat{e}_{i_j} \wedge \dots \wedge e_{i_p}$$

for all $i_1 < \dots < i_p$, where $\{e_1, \dots, e_r\}$ is the standard basis for A^r. Note that $d^2 = 0$. If M is an A-module, then se define

$$K.(f_1, \dots, f_r, M) = K.(f_1, \dots, f_r) \otimes_A M.$$

Lecture 21

Recall the previous definition.

Proposition 21.1. *If $f_1, \dots, f_r$ is a regular sequence for an A-module M, then*

$$h_i(K.(f_1, \dots, f_r, M)) = \begin{cases} M/(f_1, \dots, f_r)M & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

($i = 0$ is trivial ... you don't even need the regular sequence.) Thus, if M is free, the Kozul complex is a free resolution of $M/(f_1, \dots, f_r)M$.

Theorem 21.2. *Let X be a locally complete intersection closed subscheme of $P = \mathbb{P}^N_k$, with ideal sheaf $\mathscr{I}$. Then*

$$\omega_X^\circ = \omega_P \otimes \wedge^r (\mathscr{I}/\mathscr{I}^2)\check{}.$$

In particular, the dualizing sheaf is a line sheaf on X.

Proof. in the works ∎

Remark. This works for any field.

<u>Next:</u> Compare ω_X° with ω_X when X is a non-singular variety.
We prove half of a theorem from chapter II:

Theorem 21.3 (II.8.17)**.** *Let X be a non-singular variety over an algebraically closed field k. Let $Y \subseteq X$ be an irreducible closed subscheme, and let $\mathscr{I}$ be its sheaf of ideals. Then Y is non-singular if and only if*

(1) *$\Omega_{X/Y}$ is locally free, and*
(2) *the second exact sequence is a short exact sequence*

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega_{X/k} \otimes \mathscr{O}_Y \to \Omega Y/k \to 0.$$

In this case, $\mathscr{I}/\mathscr{I}^2$ is locally free (on Y) of rank $r = \operatorname{codim} Y$ and Y is a locally complete intersection in X.

Half Proof. in the works ∎

Lecture 22

Corollary 22.1 (Adunction formula). *In this situation,*

$$\omega_Y \cong \omega_X \otimes \wedge^r(\mathscr{I}/\mathscr{I}^2)\check{}$$

Proof. By Exercise II.5.16d, $\wedge^n(\Omega_{X/k} \otimes \wedge^r(\mathscr{I}/\mathscr{I}^2)$. The left hand side is $(\wedge^n\Omega_{X/k}) \otimes \mathscr{O}_Y = \omega_X \otimes \mathscr{O}_Y$. Here $n = \dim X$ and $q = \dim Y$. Also, $\wedge^q\Omega_{Y/k} = \omega_Y$, so

$$\omega_Y \cong \omega_X \otimes \mathscr{O}_Y \otimes \wedge^r(\mathscr{I}/\mathscr{I}^2)\check{} \cong \omega_X \otimes \wedge^r(\mathscr{I}/\mathscr{I}^2)\check{}.$$

□

Corollary 22.2. *If X is a non-singular projective variety, the $\omega_X^\circ \cong \omega_X$.*

Proof. Embed X into $P = \mathbb{P}_k^N$ and let $\mathscr{I}$ be the sheaf of ideals. Then

$$\omega_X^\circ \cong \omega_P \otimes \wedge^r(\mathscr{I}/\mathscr{I}^2)\check{} \cong \omega_X.$$

□

Corollary 22.3. *If $Y \subseteq X$ are non-singular varieties over an algebraically closed field k, and $\mathscr{I}$ is the ideal sheaf of Y, with $r = \operatorname{codim} Y$ then*

$$\omega_Y^\circ \cong \omega_X^\circ \otimes \wedge^r(\mathscr{I}/\mathscr{I}^2)\check{}.$$

Proof. $\omega_X^\circ \cong \omega_Y^\circ$, $\omega_Y^\circ \cong \omega_Y$. Actually, if X is a non-singular projective variety over k ($k = \bar{k}$ for now) and Y is a locally complete intersection closed subvariety with ideal sheaf $\mathscr{I}$. Then $\omega_Y \cong \omega_X \otimes \wedge^r(\mathscr{I}/\mathscr{I}^2)\check{}$. Same proof as before. □

in the works

Lecture 23

Homework comments:

1st problem: Ampleness. Given a coherent sheaf $\mathscr{F}$ on a Noetherian scheme X, Supp $\mathscr{F}$ is defined as a **set**, and is closed. Then $\mathscr{F}$ can be viewed as a sheaf on Y, but it is not a sheaf of $\mathscr{O}_Y$ modules (therefore not coherent).

Example: $X = \operatorname{Spec}\mathbb{Z}$, $\mathscr{F} = \widetilde{\mathbb{Z}/4\mathbb{Z}}$. Then $Y = \operatorname{Supp} \mathscr{F} = \{2\}$, but $\mathscr{F}$ is not a sheaf of O_Y modules if we take $Y = \operatorname{Spec}\mathbb{Z}/2\mathbb{Z}$ (since $\mathbb{Z}/4\mathbb{Z}$ is not a module over $\mathbb{F}_2$. We have to take a different subscheme structure on O_Y to get this to work out, and Vojta claims that this can always be done. See his solution to (III ex.4.2).

Lemma (III 2.10) is better than (III ex.4.1).

With induction proof in part (d), need to be sure that the reduction to X, Y integral is "inside" the induction. Why? Because a proper closed subscheme of an irreducible scheme might not be irreducible (you know, take two points).

The proof of (c) was indeed similar to (b), so instead of just saying that, the proper thing to do is to generalize!!!

Lemma 23.1. *Let $\mathscr{I}$ be a coherent sheaf of ideals on X and $\mathscr{F}$ a coherent sheaf on X. Let Y be the closed subscheme corresponding to $\mathscr{I}$. If $H^i(X, \mathscr{I}\mathscr{F}) = 0$ and $H^i(Y, \mathscr{F}/\mathscr{I}\mathscr{F}) = 0$ (actually, the correspondinig sheaf on Y for some i, then $H^i(X, \mathscr{F}) = 0$.*

Then phrase (b) and (c) in terms of the appropriate coherent ideal sheaves.

2nd problem: If I is an injective A-module,then $\widetilde{I}$ is injective in $\mathscr{Q}co$, but not necessarily in $\mathscr{M}od(X)$.

Generally, if $\mathscr{C}$ is a full abelian subcategory of $\mathscr{D}$, and if $X \in \mathscr{C}$ is injective in $\mathscr{D}$, then it is injective also in $\mathscr{C}$. But not necessarily conversely.

Back to III.8.

Theorem 23.2. *Let $f : X \to Y$ be a projective morphism of Noetherian schemes, let $\mathscr{O}(1)$ be a very ample sheaf on X over Y, and let $\mathscr{F}$ be a coherent sheaf on X. Then*

(a) *The natural map $f^* f_* \mathscr{F}(n) \to \mathscr{F}$ is surjective for $n >> 0$.*
(b) *$R^i f_* \mathscr{F}$ is coherent for every i.*
(c) *$R^i f_* \mathscr{F}(n) = 0$ for $i > 0$ and $n >> 0$.*

Proof. The question is local on Y - as Y is quasi compact (it is Noetherian!) we can find an n for each part of a finite affine cover, and then take the largest n - and we can assume that Y is affine, say $Y = \operatorname{Spec} A$.

(a) By (III 8.5), $\mathrm{R}^0 f_* \mathscr{F}(n) = f_* \mathscr{F}(n) = \widetilde{H^0(X, \mathscr{F}(n))}^{Y} = \widetilde{M}^{Y}$. Then for any open affine $\operatorname{Spec} B$ in X, we have by (II prop.5.2) that $f^* f_* \mathscr{F}(n) = \widetilde{M \otimes_A B}$ on X.

So our map $f^* f_* \mathscr{F}(n) \to \mathscr{F}(n)$ is defined (at the global section level) by

$$M \otimes_A B \longrightarrow M \xrightarrow{\quad\alpha\quad} \Gamma(\operatorname{Spec} B, \mathscr{F}(n))$$

$$m \otimes b \longrightarrow mb.$$

By prop (II 5.17), for large enough n the sheaf is generated by global sections, so the map is surjective at the stalk level, and thus surjective.

(b) By (III 8.5), $\mathrm{R}^i f_* \mathscr{F}(n) = \widetilde{H^i(X, \mathscr{F}(n))}^{Y}$ is coherent because $H^i(X, \mathscr{F}(n))$ is finitely generated (III 5.2(a))for all i and n.

(c) By (III 5.2(b)), $\mathrm{R}^i f_* \mathscr{F}(n) = \widetilde{H^i(X, \mathscr{F}(n))}^{Y} = 0$ for $i > 0$ and large enough n. $\square$

§III.9: Flat Morphisms

Definition. Let M be an A-module. We say that M is flat (over A) if the functor $M \otimes_A -$ is exact, i.e.

$$0 \to N' \to N \to N'' \to 0$$

exact implies

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

exact.

Also, M is faithfully flat if the converse of the above is also true.

Examples:

- A free module is flat,a nd is faithfully flat iff it's non-zero. In particular, if A is a field, then everything is flat.
- $\mathbb{Z}/n\mathbb{Z}$ is not flat over $\mathbb{Z}$. Indeed, let $p|n$ and tensor

$$0 \to \mathbb{Z} \to^p \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

with $\mathbb{Z}/n\mathbb{Z}$ to get

$$\mathbb{Z}/n\mathbb{Z} \to^p \mathbb{Z}/n\mathbb{Z} \to ? \to 0$$

Example: $S^{-1}A$ is flat over A for any multiplicative system S (i.e. 'localization is exact').

Properties of Flatness

Theorem 23.3. *Let B be an A-algebra, and let M, N be A and B modules, resp. Then*

(a) *M is flat over A iff the map $M \otimes_A \mathfrak{a} \to M$ is injective for every finitely ideal $\mathfrak{a}$ A.*
(b) *If M is flat over A, then $M \otimes_a B$ is flat over B.*
(c) *Transitivity: if N is flat over B and B is flat over A, thhen N is flat over A.*
(d) *M is flat over A iff M_p is flat over A_p for all $p \in \operatorname{Spec} A$*
(e) *Let $0 \to M' \to M \to M'' \to 0$ be an exact seq of A modules. Then M' and M'' flat imply M flat. M and M'' flat imply M' flat.*
(f) *If M if a finitely generated module over a Noetherian local ring A, then M is flat iff its free.*

Proof. (a) "$M \otimes_A -$" is a right exact functor, and $\mathscr{M}od(X)$ has enough projectives (in fact, every module has a free resolution). The right derived functors of $M \otimes_A -$ are called $\operatorname{Tor}_i^A(M, -)$. It is known that M is flat over A iff $\operatorname{Tor}_i^A(M, N) = 0$ for all $i > 0$ and N, iff $\operatorname{Tor}_i^A(M, \mathfrak{a}) = 0$ for all $i > 0$ and finitely generated ideals $\mathfrak{a}$ of A.

(b,c) (M flat over $A \Rightarrow M \otimes_A B$ flat over B)$\Leftarrow N \otimes_B (M \otimes_A B)) \cong N \otimes_A M$.

(d) True because Tor commutes with localization.

(e) Comes from the left exact sequence in Tor:

$$\cdots \longrightarrow \operatorname{Tor}_1^A(M', N) \longrightarrow \operatorname{Tor}_1^A(M, N) \longrightarrow \operatorname{Tor}_1^A(M'', N) \longrightarrow$$

$$\longrightarrow M' \otimes N \longrightarrow M \otimes N \longrightarrow M'' \otimes N \longrightarrow 0$$

We see that if $\operatorname{Tor}_i^A(M', N) = \operatorname{Tor}_i^A(M'', N) = 0$ (i.e. M' and M'' are flat), then $\operatorname{Tor}_i^A(M, N) = 0$ and M is flat too (similarly if M and M'' are flat, M' is also).

(f) □

Definition. Let $f : X \to Y$ be a morphism of schemes and let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. For $x \in X$, we say that $\mathscr{F}$ is flat over Y at x if $\mathscr{F}_x$ is a flat $\mathscr{O}_{Y,f(x)}$ module (via the map $f^\# : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$. We say that $\mathscr{F}$ is flat over Y if it is flat over Y at x for all $x \in X$.

We say that X is flat over Y if $\mathscr{O}_X$ is flat over Y. We say that f is flat if X is flat over Y.

Proposition 23.4. (a) *An open immersion is flat.*
- (b) *Flatness is preserved under base change.*
- (c) *Composition of flat morphisms is flat.*
- (d) *A morphism $f : X \to Y$ is flat iff for all open affines, $\operatorname{Spec} A \subset Y$ and for all $\operatorname{Spec} B \subset f^{-1}(\operatorname{Spec} A)$, B is flat over A (i.e. flatness is local).*
- (e) *If X is noetherian and $\mathscr{F}$ is coherent, then $\mathscr{F}$ is flat over X iff it is locally free.*

Proof. (a) The stalks are the same.

(b)-(e) follow from the algebraic properties above.

Also, flatness is local on the base (because its local upstairs). If $f : X \to Y$ and $f' : X' \to Y'$ are flat morphisms over S, then the product is flat. Indeed, see Vojta's solution of II ex.4.8 - (a) is not needed for (d). □

Examples:
- Closed immersion are generall not flat: $\operatorname{Spec} \mathbb{Z}/p\mathbb{Z} \to \operatorname{Spec} Z$ is not flat. Indeed, by (d) above, it is flat iff $\mathbb{Z}/pZ$ is flat over $\mathbb{Z}$.
- Blowing ups are generally not flat. I'm not going to try to tex this. Just look at the simplest blow up possible.

Lecture 24

Addendum: why part (b) of flatness lemma is true - (M flat over $A \Rightarrow M \otimes_A B$ flat over B)$\Leftarrow N \otimes_B (M \otimes_A B)) \cong N \otimes_A M$.

Cohomology commutes with flat base change:

Proposition 24.1. *Let*

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"] & Y \end{tikzcd}$$

Be a cartesian square of noetherian schemes, with f separated and of finite type, and u flat. Also, let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then there is a natural isomorphism (as sheaves on Y')

$$u^* R^i f_*(\mathscr{F}) \cong R^i g_*(v^* \mathscr{F})$$

Proof. By locality and naturalness, we may assume Y and Y' are affine, say $Y = \operatorname{Spec} A$ and $Y' = \operatorname{Spec} A'$. Then

$$R^i f_*(\mathscr{F}) \cong \mathrm{H}^i(X, \mathscr{F})\widetilde{\ }^{Y}$$

$$u^* R^i f_*(\mathscr{F}) \cong (\mathrm{H}^i(X, \mathscr{F}))\widetilde{\ }^{Y'} \otimes_A A' \cong (H^i(X, \mathscr{F}) \otimes_A A')\widetilde{\ }^{Y'}$$

and
$$\mathrm{R}^i g_*(v^*\mathscr{F}) = \mathrm{H}^i(X', v^*\mathscr{F})\widetilde{\ }^{Y'}$$
So we need a natural isomorphism
$$H^i(X,\mathscr{F}) \otimes_A A' \cong H^i(X', v^*\mathscr{F})$$
Use Cech cohomology: let $\mathcal{U}$ be an open affine cover of X. Say $\mathcal{U} = (U_i)_{i\in I}$ with $U_i = \operatorname{Spec} B_i$. Then let $\mathcal{U}' = v^{-1}\mathcal{U} = \mathcal{U} = (v^{-1}(U_i))_{i\in I} = (\operatorname{Spec} B_i \otimes_A A')_{i\in I}$. This is an open affine cover of X'.

Then we have
$$H^i(X,\mathscr{F})\otimes_A A' \cong h^i(\mathscr{C}^\cdot(\mathcal{U},\mathscr{F}))\otimes_A A' \cong_{flatness} h^i(\mathscr{C}^\cdot(\mathcal{U},\mathscr{F})\otimes_A A') = h^i(\mathscr{C}^\cdot(\mathcal{U}, v^*F)) = H^i(X', v^*\mathscr{F})$$
□

Flat Families We want to exclude things like blow ups, where the dimension of fibres is not constant. For flat morphisms, this cannot happen.

Proposition 24.2. *Let $f : X \to Y$ be a flat morphism of schemes of finite type over a field k. Let $x \in X$, let $y = f(x)$, and let X_y be the fibre of f over y. Then $dim_x X_y = dim_x X$ - $dim_y Y$, where $dim_x X = dim \mathscr{O}_{X,x}$*

Proof. **Step 1**: Reduce to the situation where Y is affine and there is a unique closed point, and y is that point. *(say a local ring)*. Let $Y' = \operatorname{Spec} \mathscr{O}_{Y,y}$, and let $X' = X \times_Y Y'$. Then X_y is unchanged, and so are all the local rings involved. (Now X and Y are covered by open affines which are localizations of k-algebras of finite type.

Step 2: Reduce to Y reduced. Base change to Y_{red}. All of the local rings are replaced by quotients by ideals contained in the nilradical. So essentially, nothing is changed.

Step 3: The rest. Prove by induction on $\dim Y$. Note that $\dim Y = \dim_y Y$.

Base case: If $\dim Y = 0$, then Y has a unique minimal prime, which is also maximal. Thus Y is spec of an artin ring (it has a unique maximal ideal, which is also minimal), and since Y is reduced *(this excludes products of fields)*, $Y = \operatorname{Spec} E$ for some field E. But then $X_y = X$, so we're done.

Inductive Step: If $\dim Y > 0$, then the maximal ideal of $\mathscr{O}_{Y,y}$ contains a non-zero element t, which is not a zero divisor. (Later). Thus,
$$0 \to \mathscr{O}_{Y,y} \to^t \mathscr{O}_{Y,y}$$
is exact. By flatness(?),
$$0 \to \mathscr{O}_{X,x} \to^{f^\# t} \mathscr{O}_{X,x}$$
is exact. Let $Y' = \operatorname{Spec} \mathscr{O}_{Y,y}/(t)$. By the Hauptidealsatz and the fact that $\mathscr{O}_{Y,y}$ is catenary (?)
$$\dim{}_y Y' = \dim{}_y Y - 1$$
Let $X' = X \times_Y Y'$. Then $\mathscr{O}_{X',x} = \mathscr{O}_{X,x}(f^\# t)$. By flatness, $f^\# t$ is not a zero divisor (or 0), so $\dim{}_x X' = \dim{}_x X - 1$.

Finally, $X'_y = X_y$, since $t \in \mathfrak{m}_y$. So we're done by induction. □

Corollary 24.3. *Let $f : X \to Y$ be as in the statement of the proposition. Assume also that Y is irreducible. The the conditions*

(i) *X is equidimensional of dimension $dimY + n$*
(ii) *X_y is equidimensional of dimension n for all $y \in Y$ (closed or not).*

are equivalent.

Proof. $(i) \to (ii)$. Pick $y \in Y$, let Z be an irreducible component of X_y, and let $x \in Z$ be a closed point not lying in any other irreducible component. By the proposition,

$$\dim_x Z = \dim Z = \dim_x X_y = \dim_x X - \dim_y Y$$

Also,

$$\dim_x X = \dim X - \dim \overline{\{x\}}$$

and likewise

$$\dim_y Y = \dim Y - \dim \overline{\{y\}}.$$

But now, since x is a closed point of X_y, $k(x)$ is a finite (and therefore algebraic) extension of $k(y)$. Thus,

$$\dim \overline{\{x\}} = \text{tr. deg}(k(x)/k) = \text{tr. deg}(k(y)/k) = \dim \overline{\{y\}}$$

Therefore the right hand side is $\dim X - \dim Y$, which by (i) is $n = \dim Z$.

$(ii) \to (i)$: Let Z be an irreducible component of X, let $x \in Z$ be a closed point not lying in any other irreducible component, and let $y = f(x)$. Then y is a closed point of Y, because $k \subset k(y) \subset k(x)$ (which are algebraic extensions). By the proposition (and assumption (ii)),

$$n = \dim_x X_y = \dim_x X - \dim_y Y = \dim_x Z - \dim Y$$

□

Associated Primes: Assume Noetherian throughout.

Definition. Let A be a ring. An associated prime of A is a prime ideal equal to the annihilator of some element of A. *Not every element of A has its annihilator equal to some prime. You can even do this more generally for modules, just take associated primes to be the annihilators of elements of M.*

Theorem 24.4. *(Eisennbud Thm 3.1): Let A be a noetherian ring. Then*

(a) *it has only finitely many associated primes*
(b) *the union of the associated primes is the set of zero divisors of $A(\cup\{0\})$*

Remark. All minimal primes of A are also associated primes. And these associated primes localize nicely: If S is a multiplicative subset of A, then a prime of $S^{-1}A$ is an associated prime iff the corrsponding prime of A is an associated prime. In other words:

$$\text{Ass}(S^{-1}A)\text{“} = \text{”}(\text{Ass}A) \cup \text{Spec}\, S^{-1}A$$

Definition. Let X be a (locally noetherian) scheme. Then an associated point of X is a point $x \in X$ such that the maximal ideal of $\mathscr{O}_{X,x}$ is an associated prime of $\mathscr{O}_{X,x}$.

If A is noetherian, then the set of associated points of $\operatorname{Spec} A$ corresponds to the set of associated primes of A (by this comment about localization).

Can almost fill in the 'later' above, but need one more thing. Okay, several more things.

Proposition 24.5. ***Primary Decomposition***: *If A is a noetherian rind, then we can write $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$, where the $\mathfrak{q}_i$ are primary ideals (*this is more natural using Eisenbud's definition, which we don't know offhand*). A primary ideal satisfies (for $xy \in \mathfrak{q}_i, x \notin \mathfrak{q}_i \Rightarrow y^n \in \mathfrak{q}_i$ for some n).*

Remark. Note that if $\mathfrak{q}$ is primary, then $\sqrt{\mathfrak{q}}$ is prime. If $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r = (0)$ is a minimal primary decomposition (r minimal), then $\sqrt{\mathfrak{q}_i}$ are distinct and are exactly the set of associated primes of A.

Primary decompositions localize well:

Definition. An **embedded prime** in a noetherian ring A is a non-minimal associated prime. An **embedded point** in a noetherian scheme is an associated point that is not the generic point of an irrreducible component.

Proposition 24.6. *A reduced noetherian scheme X has no embedded points.*

Proof. This is a local question, so we may assume that $X = \operatorname{Spec} A$ is affine (where by assumption A is reduced).

Then $0 = \operatorname{nil}(A) =$ intersection of the minimal primes of A. *In fact it is the intersection of all primes of A, but if a prime is not minimal its not needed in the intersection.* This gives a primary decomposition of 0 in A. So all associated primes are minimal. $\square$

The union of the minimal primes in not the maximal ideal because its of dimension greater than one, and by prime something, if you have the maximal ideal, and you have some primes that are strictly contained in that, then their union is not the whole maximal ideal because, its an exercise. But its basically prime something... "later"...

Actually, a little more is true:

Remark. (Eisenbud Ex. 11.10) A noetherian ring is reduced iff it has no embedded primes and its localization at each minimal prime is a field.

Intuition: Non-zero nilpotents in a ring A correspond to 'fuzz' in $\operatorname{Spec} A$. 'Fuzz' on a dense open subset of an irreducible component occurs iff the localization at the corresponding minimal prime is not a field. Embedded points correspond to fuzz nt spread over an irreducible component. Ex. $\operatorname{Spec} k[x,y]/(xy,y^2)$

This embedded stuff might be useful for the homework...

You'd like to represent a number theory problem as a problem over $\operatorname{Spec}\mathbb{Z}$ *or spec of a ring of integers in a number field.*

Proposition 24.7. *Let $f : X \to Y$ be a morphism of schemes. Assume that Y is integral and regular of dim 1. Then f is flat iff all associated points of X lie over the generic point of Y.*

Proof. "$\Rightarrow$" Let $x \in X$ be a point lying over a closed point $y \in Y$. We want to show that x is not an associated point. Then $\mathscr{O}_{Y,x}$ is a discrete valuation ring, so its maximal ideal contains a nonzero element t, which is a nonzerodivisor since Y is integral. By flatness, $f^{\#}t$ is not a zero divisor in $\mathscr{O}_{X,x}$, so x is not an associated point of X.

Because if it was that would mean that the maximal ideal would be an associated prime. Multiplication by t is injective in $\mathscr{O}_{Y,y}$, so it is in $\mathscr{O}_{X,x}$ too.

Remark: We didn't need Y to be regular, or even dimension 1. We just needed Y to be integral of dimension greater than zero. □

Lecture 25

[[The last proposition of the last lecture was missing some hypothesis - like noetherian.]]

Proposition 25.1. *Let $f : X \to Y$ be a morphism of noetherian schemes. Assume that Y is integral and regular of dim 1. Then f is flat iff all associated points of X lie over the generic point of Y.*

In particular, if X is reduced, then f is flat iff all irreducible components of X dominate Y

Proof. '$\Rightarrow$' last time.

'$\Leftarrow$' Let $x \in X$, and let $y = f(x)$. We need to show that the local rings are flat, i.e. $\mathscr{O}_{x,X}$ is flat over $\mathscr{O}_{y,Y}$.

Claim. A module M over a PID R is flat iff it is torsion free.

Proof '$\Rightarrow$' Suppose M has torsion, say $t = 0$ with $x \in R$, $m \in M$, both $\neq 0$. Then M is not flat because multiplication by x is an injection from R to R (since R is a domain and x is non-zero, but it aquires a kernel after tensoring wiht M.

'$\Leftarrow$' We need to show that $\mathfrak{a} \otimes M \to M$ is injective for all ideals $\mathfrak{a}$ [[which are all finitely generated since A is a PID]]. We may assume $\mathfrak{a} \neq (0)$. Then $\mathfrak{a} = (x)$ for some $x \in R$, and so $\mathfrak{a} \otimes M \cong M$, and the map is multiplication by x on M. That's injective because M is torsion free. □

Case 1. If y is the generic point then $\mathscr{O}_{Y,y}$ is a field so there's nothing to show.

Case 2. here, y is a closed point. Then $\mathscr{O}_{Y,y}$ is a d.v.r. Suppose $\mathscr{O}_{X,x}$ is not flat over $\mathscr{O}_{Y,y}$. Then it has torsion: there exists some nonzero $t \in \mathscr{O}_{Y,y}$ s.t. multiplication by $f^\# t$ is not injective in $\mathscr{O}_{X,x}$. Then $f^\# t$ is a zero divisor, so it lies in some associated prime $\mathfrak{p}$ of $\mathscr{O}_{X,x}$. This gives an associated point $x' \in X$. But $f(x') = y$ since [[as there are only 2 prime ideals in a PID]] $\mathfrak{p} \cap \mathscr{O}_{Y,y}$ contains $t \neq 0$. Therefore $\mathfrak{p} \cap \mathscr{O}_{Y,y} \neq (0)$, so $f(x') \neq$ the generic point, contradicting our assumption that all associated points lie over the generic point.

Example. $\mathbb{Q}$ is flat over $\mathbb{Z}$, therefore it is torsion free. But it's not free.

□

Hard Work - Scheme Theoretic Closure, Schematic Denseness, and Associated Points.

Recall (II ex.3.11(d)): Let $f : Z \to X$ be a morphism with Z noetherian. Then there exists a unique closed subscheme Y of X such that f factors through Y, and such that if f factors through any other closed subscheme Y', then $Y \subset Y'$ schematically ($\mathscr{I} \supset \mathscr{I}'$). This is the **closed scheme theoretic image of** f.

Definition. Let $i : Z \to X$ be a subscheme of a noetherian scheme X. Then the **scheme theoretic closure** of Z is the closed scheme-theoretic image of i.

Definition. Z is **scheme-theoretically dense** if its scheme-theoretic closure is all of X (as a scheme). An open set is scheme theoretically dense if the corresponding open subscheme is.

Example $X = \operatorname{Spec} k[x,y]/(x^2, xy)$ (i.e. the y-axis with an embedded point at (0,0)). Let $U = X - \{(0,0)\}$. Let $A = k[x,y]/(x^2, xy)$. Then $U = \operatorname{Spec} A_y$. But $A_y = (k[x,y]/(x))_y \cong k[y]_y$. Since U is reduced, its scheme theoretic closure is $X_{\text{red}} = \operatorname{Spec} k[y]$, so U is not schematically dense. (Note that U is dense set-theoretically). (You can compute the scheme-theroetic closure of $\operatorname{Spec} B \to \operatorname{Spec} A$ by finding $\ker(A \to B)$).

Proposition 25.2. *An open subset of a noetherian scheme is chematicallly dense iff it contains all of the associated points.*

Proof. This is a local question, so we may assume $X = \operatorname{Spec} A$ is affine. Let U be an open subscheme [[equivalently an open subset]]. Lets let $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ be a minimal primary decomposition of (0) in A. [[This is where we use the fact that A is noetherian.]] Let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$, so $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ are the associated primes of A.

'$\Leftarrow$'. Say U contains $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$, and let $\mathfrak{a}$ be the ideal associated to the schematic closure $\overline{U}$ of U. Since $\mathfrak{p}_i \in U$, and $U \subset \overline{U}$, we have $(A/\mathfrak{a})_{\mathfrak{p}_i/\mathfrak{a}} \cong A_{\mathfrak{p}_i}$. Therefore, for any $f \in \mathfrak{a}$, $f = 0$ in $A_{\mathfrak{p}_i}$, so $\operatorname{Ann}(f)$ meets $A/\mathfrak{p}_i$. By prime avoidance [an actual term in Eisenbud's book, or just by general messing around], there is some $x \in \operatorname{Ann}(f)$ such that $x \notin \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$. Therefore x is not a zero divisor (and $x \neq 0$), so $f = 0$. Therefore $\mathfrak{a} = (0)$, so $\overline{U} = X$

and therefore U is schematically dense.

'$\Rightarrow$' Suppose its false. Then U is schematically dense but without loss of generality $\mathfrak{p}_n \notin U$. We may assume $\{i : \mathfrak{p}_i \not\supset \mathfrak{p}_n\} = \{1, \cdots, r\}$; $r < n$. Let $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$. By minimality, $\mathfrak{a} \neq (0)$. So it will suffice to show that $U \subset \operatorname{Spec}(A/\mathfrak{a})$.

Let $\mathfrak{p} \in U$. Since $\mathfrak{p}_n \notin U$, $\mathfrak{p}_n \not\to \mathfrak{p}$ [make this arrow squiggly], so $\mathfrak{p}_n \not\subset \mathfrak{p}$. Therefore $\mathfrak{p} \not\supset \mathfrak{p}_i$, for all $i > r$. Let $S = A/\mathfrak{p}$; then S meets $\mathfrak{p}_i$ for all $i > r$, so $S^{-1}\mathfrak{q}_i = (1)$ for all $i > r$. Therefore $(0) = \bigcap_{i=1}^n S^{-1}\mathfrak{q}_i = \bigcap_{i=1}^r S^{-1}\mathfrak{q}_i = S^{-1}\mathfrak{a}$, so $S^{-1}\mathfrak{a} = (0)$. Therefore $\mathfrak{p} \in \operatorname{Spec}(A/\mathfrak{a})$. ($f \in \mathfrak{a} \Rightarrow \operatorname{Ann}(f)$ meets $S \Rightarrow f \in \mathfrak{p}$ primeness). also $(A/\mathfrak{a})_{\mathfrak{p}/\mathfrak{a}} \cong A_\mathfrak{p}$, since LHS $= A_\mathfrak{p}/\mathfrak{a}/\mathfrak{p} = A_\mathfrak{p}/(0)$. So $U \subset \operatorname{Spec}(A/\mathfrak{a})$ as schemes [[here he drew a big frowny face with x'ed out eyes]]. $\square$

Proposition 25.3. *Lelt Y be an integral, regular, noetherian scheme of dimension 1, let $P \in Y$ be a closed point, and let $X \subset \mathbb{P}^n_{Y/P}$ be a closed subscheme which is flat over y/P. Then there exists a unique closed sobscheme $\overline{X}$ of $\mathbb{P}^n_Y$ such that $\overline{X} \cap \mathbb{P}^n_{Y/P} = X$ and $\overline{X}$ is flat over Y.*

Proof. **Existence** Let $\overline{X}$ be the schematic closure of X in $\mathbb{P}^n_Y$. We didn't add any associated points, so its still flat.

Uniqueness. Suppose X' also saatisfies the condition. We have to have $X' \subset_{\text{subschemes}} \overline{X}$ (by definition of schematic closure). since $\overline{X}$ is not schematically dense in X', X' contains some associated point not in $\overline{X}$. Then that point lies over P (because where else could it lie), contradicting flatness. $\square$

[[We didn't really need the fact that we were in $\mathbb{P}^n_Y$. We just wanted to be concrete, and in applications we only use $\mathbb{P}^n_Y$. In practice you could use any noetherian scheme in place of $\mathbb{P}^n_Y$.]]

The last application has to do with bases of arbitrary dimension.

Theorem 25.4. *Let T be an integral noetherian scheme, and let $X \subset \mathbb{P}^n_T$ be a closed subscheme. Then for each point $t \in T$ (closed or not), lets let $P_t \in \mathbb{Q}[z]$ be the Hilbert polynomial of the fiber $X_t = X \times_T k(t)$ (which is a closed subscheme of $\mathbb{P}^n_{k(t)}$ - calculate the Hilbert polynomial by calculating dimensions of things over $k(t)$).*

Then X is flat over T iff P_t is independent of t.

Proof. Recall the definition of Hilbert Polynomial: P_t is characterized by $P_t(m) = \dim_{k(t)} H^o(X_t, \mathscr{O}(m))$ for all $m \gg 0$. Also, $k(t)$ is not necessarily closed, but by a flat base change, we can pass to the algebraic closure.

We will prove this theorem by first generalizing it: More generally let $\mathscr{F}$ be a coherent sheaf on X. Then we can write $\mathscr{F}_t = \mathscr{F}|_{X_t} = i^*\mathscr{F}$, where $i : X_t \to X$ is the inclusion map. So we have a Hilbert polynomial of $\mathscr{F}_t$"

$$P_t(m) = h^0(X_t, \mathscr{F}_t(m)), m \gg 0.$$

So we may assume $X = \mathbb{P}^n_T$ (let $\mathscr{F} = \mathscr{O}_{\text{old } X}$ [[literally waved hands and said 'usual trick']]).

Also, we may assume $T = \operatorname{Spec} A$, with A a local noetherian ring.

So the situation is $T = \operatorname{Spec} A$ as above, $X = \mathbb{P}^n_A$, and $\mathscr{F}$ is coherent on X. [[Now we prove some claims]].

Claim 1. $\mathscr{F}$ is flat over T iff $H^0(X, \mathscr{F}(m))$ if a free A module (of finite rank - this part always holds) for $m \gg 0$.

Proof. '$\Rightarrow$'. Let $\mathscr{U}$ be the standard open affine cover of $\mathbb{P}^n_A$ [[offhand I can't think of why it has to be the standard one, but why not?]]. Then $H^i(\mathcal{U}, \mathscr{F}(m)) = H^i(X, \mathscr{F}(m)) = 0$ for all $m \gg 0$ So the sequence (imagine the kernels and cokernels)

$$0 \to H^0(X, \mathscr{F}(m)) \to C^0(\mathcal{U}, \mathscr{F}(m)) \to C^1(\mathcal{U}, \mathscr{F}(m)) \to \cdots \to C^{n-1}(\mathcal{U}, \mathscr{F}(m)) \to C^n(\mathcal{U}, \mathscr{F}(m)) \to 0$$

(no verb either).

All the $C^i(\mathcal{U}, \mathscr{F}(m))$ are flat over A. [[Twisting by m doesn't affect flatness - you can check it locally, and it is unaffected locally. So then you have a short exact sequence on the right where everything is flat, and working your way back all the kernels and cokernels are flat too.]] Splitting this into short exact sequences, you get that $\mathscr{K}_i$ is flat over A for all i, so $H^0(X, \mathscr{F}(m))$ is flat over A by (III 9.1A(e)). Also, $H^0(X, \mathscr{F}(m))$ is finitely generated, so by (III 9.1A(f)), its free (of finite rank).

$\square$

Lecture 26

We were showing: A projective morphism is flat if and only if the Hilbert polynomial is independent of $y \in Y$.

<u>The Setup:</u> $T = \operatorname{Spec} A$, where A is local noetherian domain. $X = \mathbb{P}^n_T$, and $\mathscr{F}$ is a coherent sheaf on X. It will suffice to show that TFAE:

(i) $\mathscr{F}$ is flasque over T.
(ii) $H^0(X, \mathscr{F}(m))$ if a free A-module (of finite rank) for all $m \gg 0$.
(iii) The Hilbert polynomial, P_t of $\mathscr{F}_t$ on X_t is independent of $t \in T$.

We have already shown that (i)$\Rightarrow$ (ii)

(ii)$\Rightarrow$ (i): Choose m_0 such that $H^0(X, \mathscr{F}(m))$ is free for all $m \geq m_0$. Let $M = \bigoplus_{m \geq m_0} H^0(X, \mathscr{F}(m))$. Then $M \cong \Gamma_*(\mathscr{F})_{m \geq m_0}$, so $\tilde{M} \cong \widetilde{\Gamma_*(\mathscr{F})_{m \geq m_0}} \cong \mathscr{F}$. Since M is free, it's flat over A, so $\mathscr{F}$ is flat over A (on open affines it's $\sim$ of localizations of M).

Lemma 26.1. *Let T, A, X as above, and let $\mathscr{F}$ be coherent on X. Then $H^0(X_t, \mathscr{F}_t(m)) \cong H^0(x, \mathscr{F}(m)) \otimes_A k(t)$ for all $t \in T$ and $m \gg_t 0$.*

Proof. We may assume that t is the closed point of T (replace T with $T' := \operatorname{Spec} \mathscr{O}_{T,t}$ and note that T' is flat over T, then LHS unaffected by base change and RHS is tensored with $A' := \mathscr{O}_{T,t}$). Given a finite generating set for the maximal ideal of A, we have an exact sequence

$$A^n \to A \to k(t) \to 0 \qquad (*)$$

so we can get the exact sequence "$(*) \otimes_A \mathscr{F}$":

$$\mathscr{F}^n \to \mathscr{F} \to \mathscr{F}_t \to 0.$$

Therefore,

$$\begin{tikzcd}
H^0(X, \mathscr{F}(m)^n) \arrow[r] \arrow[d, "\wr"] & H^0(X, \mathscr{F}(m)) \arrow[r] \arrow[d, equal] & H^0(X_t, \mathscr{F}_t(m)) \arrow[r] \arrow[d, dotted, "\exists" ', "\wr"] & 0 \\
H^0(X, \mathscr{F}(m))^n \arrow[r] & H^0(X, \mathscr{F}(m)) \arrow[r] & H^0(X, \mathscr{F}(m)) \otimes_A k(t) \arrow[r] & 0
\end{tikzcd}$$

The top row is exact by some homework exercise for $m \gg 0$, and the bottom row is exact for all m because it is $(*) \otimes_A H^0(X, \mathscr{F}(m))$ (and tensor is right exact). And the diagram commutes. Therefore, you get the isomorphism on the right. □

(ii)⇒(iii): The Lemma implies that the rank of $H^0(X, \mathscr{F}(m))$ is $\dim_{k(t)} H^0(X- t, \mathscr{F}_t(m))$ for all $t \in T$ and $m \gg_t 0$. The RHS is $P_t(m)$ for all $m \gg 0$, and the LHS is independent of t.

(iii)⇒(ii): Recall II.8.9: A finitely generated module over A (as above) is free if and only if $M \otimes_A K$ and $M \otimes_A k$ have the same dimensions over the fraction field K and the residue field k, respectively. In our case, $H^0(X, \mathscr{F}(m))$ free is implied by $\dim_{k(\tau)} H^0(X, \mathscr{F}(m) \otimes_A k(\tau)) = \dim_{k(t)} H^0(X, \mathscr{F}(m)) \otimes_A k(t)$ where τ and t are the generic and special points of T, respectively. By the Lemma, the LHS is $P_t(m)$ for $m \gg 0$, and the RHS is $P_t(m)$ for $m \gg 0$.

So we're done.

Remark. To prove flatness, it's enough to compare Hilbert polynomials at closed and generic points.

Corollary 26.2. *Let T be a connected noetherian scheme, and let X be a closed subscheme of $\mathbb{P}^n_T$, flat over T. Then the degree, dimension, and arithmetic genus of X_t is independent of $t \in T$.*

Proof. By base change to an irreducible component of T (with reduced induced subscheme structure), we may assume that T is integral. Then use the fact that the Hilbert polynomial is independent of $t \in T$ (all of these things are determined by the Hilbert polynomial). □

<u>Exercise III.9.1:</u> *Let $f : X \to Y$ be a flat morphism of finite type of noetherian schemes. Then f is an open[21] morphism.*

Proof. Let $U \subseteq X$ be open. Then $f(U) \subseteq Y$ is constructable (Ex. II.3.19), meaning that it is a closed set minus a constructable set of lower dimension. By Ex. II.3.18c, it will suffice to show that $f(U)$ is stable under generization. That is, if $y \in f(U)$ and $\eta \in Y$ such that $\eta \rightsquigarrow y$, then $\eta \in f(U)$. To see this, let $\operatorname{Spec} A \subseteq Y$ be an open affine neighborhood of y (thus, $\eta \in \operatorname{Spec} A$). Let $x \in f^{-1}(y)$, and let $\operatorname{Spec} B$ be an open affine neighborhood of x in

[21]For all open $U \subseteq X$, $f(U) \subseteq Y$ is open.

$U \cap f^{-1}(\operatorname{Spec} A)$. Then B is flat over A. Let $\mathfrak{p}, \mathfrak{p}', \mathfrak{q}$ be prime ideals of A, A, B (resp.) corresponding to y, η, x (resp.)

$$\begin{tikzcd} & \mathfrak{q} \subseteq B \arrow[d, dash] \arrow[d, dash, shift left=4ex, "\text{flat}"] \\ \mathfrak{p}' \subseteq & \mathfrak{p} \subseteq A \end{tikzcd}$$

There is a prime $\mathfrak{q}'$ of B lying over $\mathfrak{p}'$ such that $\mathfrak{q}' \subseteq \mathfrak{q}$. let $\xi \in U \subseteq X$ be the corresponding point. Then $\eta = f(\xi) \subseteq f(U)$, as was to be shown. $\square$

Exercise III.9.4: *Let $f : X \to Y$ be a morphism of finite type of noetherian schemes. Then $\{x \in X | f \text{ flat at } x\} \subseteq X$ is open.*

§III.10: Smoothness

This is a relative version of non-singularity (*almost*). For this section, k is any field (not necessarily algebraically closed), and all schemes considered will be assumed to be of finite type over k (or locally of finite type).

Definition. Let $f : X \to Y$ be a morphism of schemes over k (as above), then f is *smooth* of relative dimension n if

(1) f is flat
(2) if X' and Y' are irreducible components of X and Y (resp.) such that $f(X') \subseteq Y'$, then $\dim X' = \dim Y' + n$.
(3) for all $x \in X$ (closed or not), $\dim_{k(x)}(\Omega_{X/Y} \otimes k(x)) = n$.

Remark.

(i) By (1) and Cor III.9.6, (2) is equivalent to
 (2') for all $y \in Y$ (closed or not), X_y is equidimensional of dimension n.
 Also, (3) is equivalent to
 (3') for all $x \in X$, $\dim_{k(x)}(\Omega_{X_y/k(y)} \otimes k(x)) = n$ where $y = f(x)$. $\Omega_{X_y/k(y)}$ is really $\Omega_{X/Y}$ pulled back to X_y.
 So (2) and (3) just concern the fibers X_y.
(ii) Smoothness of relative dimension n is local on X (therefore local on the base) in the sense that $f : X \to Y$ is smooth of relative dimension n if and only if there exists an open cover $(U_i)_{i \in I}$ of X such that f_{U_i} is smooth of relative dimension n for all i. So we can define $f : X \to Y$ is smooth of relative dimension n *at x* if there is an open neighborhood $U \subseteq X$ such that $f|_U$ is smooth of relative dimension n. Then f is smooth of relative dimension n if and only if it is at each point in X. Also, $\{x \in X | f \text{ smooth of relative dimension } n\} \subseteq X$ is open.

Examples:

(i) For any Y, $\mathbb{P}^n_Y$ and $\mathbb{A}^n_Y$ are smooth over Y of relative dimension n. It is enough to show it for $\mathbb{A}^n_Y$ since $\mathbb{P}^n_Y$ is covered by open affines isomorphic to $\mathbb{A}^n_Y$.

 Proof. (1) Flatness is ok. (2') is easy: $\dim_{k(y)} \mathbb{A}^n_Y = n$. (3') $\Omega_{\mathbb{A}^n_Y/Y}$ is free of rank n; ditto for its fibers. $\square$

(ii) Let $Y = \operatorname{Spec} k$ with k algebraically closed. Then X is smooth over Y of relative dimension n if and only if X is non-singular of pure dimension n if and only if X is regular of pure dimension n.

Proof. The second equivalence comes from Chapter I. The first condition is equivalent to the third is given by II.8.15. □

Caution: We do need k algebraically closed here (see Ex III.10.1).

Proposition 26.3.
(a) *An open immersion is smooth of relative dimension 0.*
(b) *(Base Change) If $f : X \to Y$ is smooth of relative dimension n, and $Y' \to Y$ is any morphism, then $f' : X' := X \times_Y Y' \to Y'$ is also smooth of relative dimension n.*
(c) *(Composition) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are smooth of relative dimension n and m, resp., then $g \circ f : X \to Z$ is smooth of relative dimension $n + m$.*
(d) *(Products) if $f : X \to Y$ and $f' : X' \to Y'$ are smooth S-morphisms of relative dimensions n and n', resp., then $f \times_S f' : X \times_S X' \to Y \times_S Y'$ is smooth of relative dimension $n+n'$. (Book's special case: $Y = Y' = S$)*

Proof. (a) is trivial. (d) follows from (b) and (c) by published proof of Exercise II.4.8d.

(b): Flatness is immediate. For (2') and (3'), let $y' \in Y'$ and let $y \in Y$ be the image of y'. Then $X'_{y'} = X_y \times_{k(y)} k(y')$. $k(y')$ is a finite extension of $k(y)$, so it doesn't affect dimension[22]. And it doesn't affect (3') □

Lecture 27

Comments on the homework:

- $\Gamma(-)$ does *not* commute with $\otimes$ (in general).
- Exercises 9.3cd give ways in which 9.7 fails when $\dim Y > 1$.
- You *do* need to prove that $x \otimes z \pm y \otimes w \neq 0$ in $(x, y) \otimes A$
- In 9.3c, to show that $I \subseteq k[x, y, z, w]$ is primary, you can use the fact that it's homogeneous in z and w, but you cannot immediately reduce to working with homogeneous elements.

All schemes today are assumed to be of finite type over appropriate field.

Loose end from last time: Let X be a scheme over k, let k' be an extension of k, and let $X' = X \times_k k'$, then $\dim X' = \dim X$

Proof. Let $\operatorname{Spec} A$ be an open affine in X, and let $A' = A \otimes_k k'$, so that $\operatorname{Spec} A' = \pi^{-1}(\operatorname{Spec} A)$, where $\pi : X' \to X$ is the projection. By Noether's normalization lemma, there is a subring $B \subseteq A$ with $k \subseteq B$, $B \cong k[x_1, \ldots, x_r]$, and A is finite over B.

[22] see next lecture

$$\begin{array}{ccc} A' & \text{---} & A \\ | & & | \\ B' & & B \\ | & & | \\ k' & \text{---} & k \end{array}$$

Then $B' := B \otimes_k k'$ is $\cong k'[x_1, \ldots, x_r]$, and A' is finite over B'. Then $\dim A' = \dim B' = r = \dim B = \dim A$ (Note that A is integral over B; then $\dim A \geq \dim B$ by lying over and going up for integral extensions, and $\dim A \leq \dim B$ by incomparability for integral extensions). Then $\dim X = \max_A \dim A = \max_A \dim A' = \dim X'$. □

Also, $\pi : X' \to X$ is onto. To see this, we may assume that $X = \operatorname{Spec} A$. We have that k' is faithfully flat over k, so a' is faithfully flat over A (by base change: $M \otimes_k k' = M \otimes_A (A \otimes_k k') = M \otimes_A A' = 0$, so $M = 0$). Then $\operatorname{Spec} A' \to \operatorname{Spec} A$ is surjective by homework.

Also, all irreducible components of X' dominate irreducible components of X. This follows from going down for flat extensions.

Back to smooth morphisms. We were proving part (c) of the proposition: if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are smooth of relative dimension n and m, resp., then $g \circ f$ is smooth of relative dimension $n + m$.

Proof. (1) flatness is obvious.

(2) Let X', Y', and Z' be irreducible components of X, Y, Z, respectively, such that $f(X') \subseteq Y'$ and $g(Y') \subseteq Z'$. Then $\dim X' = \dim Z' + n + m$ by (2)

(3) Use the first exact sequence: let $x \in X$, let $y = f(x)$, and $z = g(f(x))$. Then

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

is exact, so

$$\underbrace{f^*\Omega_{Y/Z} \otimes k(x)}_{(\Omega_{Y/Z} \otimes k(y)) \otimes k(x)} \to \Omega_{X/Z} \otimes k(x) \to \underbrace{\Omega_{X/Y} \otimes k(x)}_{\dim_{k(x)} = n} \to 0$$

dimension of first term over $k(x)$ is m, so the dimension over $k(x)$ of the middle is $\leq n + m$.

Show the other inequality. Note that $x \in X_z$, so

$$\Omega_{X/Z} \otimes k(x) = \Omega_{X_z/k(z)} \otimes k(x).$$

By (2'), for $g \circ f$, X_z has pure dimension $n + m$. Let X' be an irreducible component of X_z containing x. By the second exact sequence:

$$i^*\Omega_{X_z/k(z)} \to \Omega_{X'/k(z)} \to 0$$

is exact where $i : X' \to X_z$, so

$$\underbrace{i^*\Omega_{X_z/k(z)} \otimes k(x)}_{\Omega_{X_z/k(z)} \otimes k(x)} \to \Omega_{X'/k(z)} \otimes k(x) \to 0$$

is exact, so it suffices to show that $\Omega_{X_z/k(z)} \otimes k(x)$ has dimension $\geq n+m$. We know that $\Omega_{X'/k(z)}$ has rank $\geq n+m$ at the generic point, and it is coherent, so the rank does not decrease when you specialize to x (use Nakayama's Lemma). □

Theorem 27.1. *Let $f : X \to Y$ be a morphism of schemes (of finite type) over k. Then f is smooth of relative dimension n, if and only if*

(1) *f is flat.*
(2) *for all $y \in Y$, the geometric fiber $X_{\bar{y}} = X_y \times_{k(y)} \overline{k(y)}$ is regular of pure dimension n.*

Proof. ($\Rightarrow$) (1) is obvious. (2): by base change, $X_{\bar{y}}$ is smooth over $\overline{k(y)}$, so it is regular by (II.8.8 or II.8.15).

($\Leftarrow$) (1) implies condition (1) for smoothness. (2) implies (2') for smoothness (all fibers have the same dimension, so geometric fibers have the same dimension). (2) also implies (3') for smoothness. To see this, note that $\Omega_{X_{\bar{y}}/\overline{k(y)}}$ is locally free of rank n by II.8.8, which implies that $\Omega_{X_{\bar{y}}/\overline{k(y)}} \otimes \overline{k(y)}$ has dimension n for all y, which implies that $\Omega_{X_y/k(y)} \otimes k(x)$ has dimension n for all $x \in X_y$ since $\Omega_{X_{\bar{y}}/\overline{k(y)}} = \Omega_{X_y/k(y)} \otimes_{k(y)} \overline{k(y)}$ □

Corollary 27.2. *Let $f : X \to Y$ be a morphism of schemes (of finite type) over k. Then f is smooth of relative dimension n if and only if f is flat and X_y is smooth of relative dimension n over $k(y)$ for all $y \in Y$. (second part equivalent to (2))*

<u>Smoothness over a field:</u>

Theorem 27.3 (EGA IV 6.74a)**.** *Let X be a scheme of finite type over a field k, and let k' be an extension field of k, let $X' = X \times_k k'$, let $\pi : X' \to X$ be the projection, let $x' \in X'$ and let $x = \pi(x')$. Then*

(a) *If X' is regular at x', then X is regular at x.*
(b) *The converse holds if k' is separable over k.*

Corollary 27.4. *Since π is surjective,*

(a) *X' is regular implies that X is regular, and*
(b) *the converse holds if k' is separable over k.*

Note that (b) is false in general (Ex 10.1).

Corollary 27.5. *Let X be a scheme of finite type over a perfect field k. If X is regular, then it is smooth over k.*

Proof. $X \times_k \bar{k}$ is regular, so X is smooth. □

Corollary 27.6. *If X is smooth over an arbitrary field k, then it is regular by (b) ($\bar{k}$ is separable over k).*

Proof. $X \times_k \bar{k}$ is smooth over $\bar{k}$, so $X \times_k \bar{k}$ i s regular, so X is regular by (a) (of the first corollary). □

A "more self-contained" proof:

Lemma 27.7. *Let k be a field. Then $\mathbb{A}^n_k$ is regular.*

Proof. We need to show that $k[x_1, \ldots, x_n]$ is a regular ring. This follows from Matsumura, Commutative Ring Theory [1986], Theorem 19.5[23]. It is also Eisenbud, Exercise 19.3 $\square$

Proposition 27.8. *Let X be a smooth scheme of finite type over a field k. Then X is regular.*

Proof. We may assume $X = \operatorname{Spec} A$ is affine. Choose a generating set $x_1, \ldots, x_n \in A$, so $k[x_1, \ldots, x_n] \twoheadrightarrow A$, so $X \hookrightarrow \mathbb{A}^n_k$. By Matsumura Thm 19.3 (a localization of a regular local ring at a prime ideal is regular), it suffices to show that X is regular at x for all closed points $x \in X$. Let $\mathscr{O}$ be the local ring $\mathscr{O}_{\mathbb{A}^n_k,x}$, let $\mathfrak{m}$ be its maximal ideal, and let I be the kernel of the surjection $\mathscr{O} \twoheadrightarrow \mathscr{O}_{X,x}$. Then $I \subseteq \mathfrak{m}$ ($x \in X$). Let $d = \dim \mathscr{O}_{X,x} = \dim_x X$. Use the second exact sequence:

$$I/I^2 \xrightarrow{\delta} \Omega_{\mathscr{O}/k} \otimes_{\mathscr{O}} \mathscr{O}_{X,x} \xrightarrow{\alpha} \Omega_{\mathscr{O}_{X,x}/k} \to 0$$

is exact, so

$$(I/I^2) \otimes k(x) \xrightarrow{\delta'} \underbrace{\Omega_{\mathscr{O}/k} \otimes_k (x)}_{\dim n \text{ over } k(x)} \xrightarrow{\alpha'} \underbrace{\Omega_{\mathscr{O}_{X,x}/k} \otimes k(x)}_{\dim d \text{ over } k(x)} \to 0$$

is exact. Let $r - n - d$, and pick $f_1, \ldots, f_r \in I$ such that $\delta'((f_i \text{mod } I^2 \otimes 1)$ generate $\ker \alpha'$. Let $\mathscr{O}' = \mathscr{O}/(f_1, \ldots, f_r)$. It is local with maximal ideal $\mathfrak{m}' = \mathfrak{m}/(f_1, \ldots, f_r)$. Then $\mathfrak{m}'/\mathfrak{m}'^2 \cong \mathfrak{m}/(\mathfrak{m}^2 + (f_1, \ldots, f_r))$ has dimension $\geq n - r = d$ over $k(x)$. But also $\mathscr{O}'$ maps surjectively to $\mathscr{O}_{X,x}$, so

$$\dim_{k(x)} \mathfrak{m}'/\mathfrak{m}'^2 \geq \dim \mathscr{O}' \geq \dim \mathscr{O}_{X,x} = \text{Stuck}$$

is also exact. $\square$

Next: Étale morphisms

Definition (Ex 10.3)**.** Let $f : X \to Y$ be a morphism of schemes (of finite type) over k. Then

(a) f is *étale* if it is smooth of relative dimension 0.
(b) f is *unramified* if for all $x \in X$, letting $y = f(x)$, we have $\mathfrak{m}_y \cdot \mathscr{O}_{X,x} = \mathfrak{m}_x$ and $k(x)$ is a separable algebraic extension of $k(y)$.

Example: Let $R = \mathbb{Z}[\sqrt{2}]$, then $\operatorname{Spec} R$ is étale over $\operatorname{Spec} \mathbb{Z}$ except at the prime $(\sqrt{2}) \subseteq R$. (These are not schemes over a field, but the same principles apply.) At other primes, there may be residue field extensions, but they are separable.

Lecture 28

All schemes still of finite type over a field.

Continued Proof. X smooth of relative dimension d over k ... we were showing that X is regular. All notation as in last lecture.

[23] If R is regular and noetherian, then so is $R[x]$

We need $\dim \mathfrak{m}'/\mathfrak{m}'^2 \le n - r = d$.

$$\begin{tikzcd}
(I/I^2) \otimes k(x) \arrow[r] \arrow[d] & \Omega_{\mathscr{O}/k} \otimes k(x) \arrow[r, "\alpha'"] \arrow[d, equal] & \Omega_{\mathscr{O}_{X,x}/k} \otimes k(x) \arrow[r] \arrow[d, two heads] & 0 \\
\mathfrak{m}/\mathfrak{m}^2 \arrow[r] & \Omega_{\mathscr{O}/k} \otimes k(x) \arrow[r] & \Omega_{k(x)/k} \arrow[r] & 0
\end{tikzcd}$$

may be 0

By arrow chasing, the images of $f_1, \dots, f_r$ in $\mathfrak{m}/\mathfrak{m}^2$ ar linearly independent over $k(x)$. Thus, the dimension of $\mathfrak{m}'/\mathfrak{m}'^2$ is $n - r = d$. Then by the list of inequalities from the last lecture, equality holds. So $\mathscr{O}'$ is a regular local ring. So $\mathscr{O}'$ is entire (II.6.11.1A), so (0) is prime in $\mathscr{O}'$, and it is the unique minimal prime, so it doesn't go away in $\mathscr{O}_{X,x}$. Thus, $\mathscr{O}' = \mathscr{O}_{X,x}$. So X is regular at x. $\square$

Next: Instead of étaleness, we'll do the jacobian criterion for smoothness.

Lemma 28.1. *Let* $\begin{tikzcd} Z \arrow[r, "j"] \arrow[dr, "\phi"'] & X \arrow[d, "\psi"] \\ & Y \end{tikzcd}$ *, be a commutative diagram of schemes (of finite type) over k. Let $z \in Z$ be a closed point, with $x = j(z)$. Assume that j is an immersion and that ϕ and ψ are smooth of relative dimensions d and n, respectively (at z and x, resp.). Let $\mathscr{I}$ be the sheaf of ideals defining Z in some open neighborhood of $x \in X$. Let $r = n - d$ (the codimension of Z in X). Then*

(a) *There is an open neighborhood U ov x in X and sections $f_1, \dots, f_r \in \mathscr{I}(U)$ such that the images $df_1, \dots, df_r$ in $\Omega_{X/Y} \otimes k(x)$ are linearly independent, and*
(b) *Any such $f_1, \dots, f_r$ generated $\mathscr{I}$ in a (possibly smaller) open neighborhood of x.*

Proof. We may assume that $Z \hookrightarrow X$ is a closed immersion. Let $\mathscr{O} = \mathscr{O}_{X,x}$, $\mathfrak{m} =$ maximal ideal in $\mathscr{O}$, $I = \ker(\mathscr{O}_{X,x} \to \mathscr{O}_{Z,z}) = \mathscr{I}_x$; note that $I \subseteq \mathfrak{m}$. We have a commutative diagram with exact rows:

$$\begin{tikzcd}
(I/I^2) \otimes k(x) \arrow[r] \arrow[d] & \Omega_{X/Y} \otimes k(x) \arrow[r, "\alpha'"] \arrow[d, "\wr", equal] & \Omega_{Z/Y} \otimes k(x) \arrow[r] \arrow[d, two heads] & 0 \\
\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2 \arrow[r] & \Omega_{X_y/k(y)} \otimes k(x) \arrow[r] & \Omega_{k(x)/k(y)} \arrow[r] & 0
\end{tikzcd}$$

where $y = \psi(x) = \phi(z)$. Here, $\bar{\mathscr{O}} = \mathscr{O} \otimes k(y) = \mathscr{O}_{X_y,x} = \mathscr{O}/\mathfrak{m}_y\mathscr{O}$, and $\bar{m} = \mathfrak{m}\bar{\mathscr{O}}$. For part (a), pick $f_1, \dots, f_r \in I$ so that their images form a basis for $\ker \alpha'$. Let $\bar{I} = I\bar{\mathscr{O}} = \ker(\bar{O} \to \mathscr{O}_{Z_y/k(y)})$, and let $\bar{f}_1, \dots, \bar{f}_r \in \bar{I}$ be the images of $f_1, \dots, f_r$. If we put bars on everything in the top row of the diagram, then as in the previous proof, $\bar{f}_1, \dots, \bar{f}_r$ generate $\bar{I}$, so by Nakayama's Lemma, $f_1, \dots, f_r$ generate I, so some open neighborhood $U \ni$

x in X, $f_1, \ldots, f_r$ extend to sections of $\mathscr{I}(U)$ (by defn of a stalk), and generated $\mathscr{I}(U)$ by coherence (after shrinking U). $\square$

Corollary 28.2. *In this situation (one smooth thing, Z, inside another smooth thing, X), Z is a locally complete intersection in X of codimension r.*

In the proof of the Lemma, we assumed that z is a closed point, but we don't really need that. If $\Omega_{X/Y}$ is locally free (e.g. if X is integral, (III.10.0.2), or $X = \mathbb{A}^n_Y$, then the lemma holds for all $z \in Z$. To see this, let $z \in Z$ be any point. Shrink to open neighborhoods of z and x such that $Z \hookrightarrow X$ is a closed immersion, and $\Omega_{X/Y}$ is free. Then at a closed point z' such that $z \rightsquigarrow z'$, the conclusions of the Lemma hold.

But actually, we don't the assumption on X because we just need z to be a closed point *in its fiber* Z_y, and then use the fact that $\Omega_{X/Y} \otimes k(x)$ is a localization of $\Omega_{X/Y} \otimes k(x')$.

Lemma 28.3. *Let $\phi : X \to Y$ be a smooth morphism of schemes over k of relative dimension n, and let $f \in \Gamma(X, \mathscr{O}_X)$. Let $x_0 \in X$ a point such that $x_0 \in Z(f)$ (i.e. $f \in \mathfrak{m}_{x_0}$), and the image of $df \in \Omega X/Y \otimes k(x_0)$ is nonzero. Then $Z := Z(f)$ is smooth of relative dimension $n-1$ over Y at x_0. Also, $\Omega_{Z/Y} \otimes k(x_0) \cong (\Omega_{X/Y} \otimes k(x_0))/(\text{image of } f)$.*

Proof. The condition on df is equivalent to $x \notin \operatorname{Supp}(\ker(\mathscr{O}_X \xrightarrow{\cdot df} \Omega_{X/Y}))$, so replace X with an open neighborhood of x_0 such that this condition (on df) holds for all $x \in Z$ (in place of x_0). Also,

$$((f)/(f)^2) \otimes k(x) \xrightarrow{\delta'} \Omega_{X/Y} \otimes k(x) \to \Omega_{Z/Y} \otimes k(x) \to 0$$

is exact and the image of δ' has rank 1, so (3) holds at x for all $x \in Z$, and the last sentence in the lemma is true.

To show (2'), note that $\bar{f}$ is non-zero in $\mathscr{O}_{X_y,x}$, which is a regular local ring, and therefore entire, so $\bar{f}$ is not a zero divisor, and therefore (by Hauptidealsatz)

$$\begin{aligned}
\dim_x Z &= \dim \mathscr{O}_{Z_y,z} \\
&= \dim \mathscr{O}_{Z_y,z}/(\bar{f}) \\
&= \dim \mathscr{O}_{X_y,z} - 1 \\
&= \dim_z X_y - 1 = n - 1
\end{aligned}$$

Thus, (2') for Z holds for all $x \in Z$. here $y = \phi(x)$

To show that Z is flat over Y, pick $x \in Z$, and let $y = \phi(x)$. Then by smoothness, X_y is regular at x, so $\mathscr{O}_{X_y,x}$ is a regular local ring, and therefore entire. And again, f is non-zero in this ring, so it is not a zero divisor, so

$$0 \to \mathscr{O}_{X_y,x} \xrightarrow{f} \mathscr{O}_{X_y,x} \to \mathscr{O}_{Z_y,x} \to 0$$

is exact, so

$$0 \to \mathscr{O}_{X,x} \otimes k(y) \to \mathscr{O}_{X,x} \otimes k(y) \to \mathscr{O}_{Z,x} \otimes k(y) \to 0$$

is exact. Also,

$$\mathscr{O}_{X,x} \xrightarrow{f} \mathscr{O}_{X,x} \to \mathscr{O}_{Z,x} \to 0$$

is exact, and forms part of a flat resolution for $\mathscr{O}_{Z,x}$ over $\mathscr{O}_{Y,y}$. So $\operatorname{Tor}_1^{\mathscr{O}_{X,x}}(\mathscr{O}_{Z,x}, k(y)) = \ker(\mathscr{O}_{X,x}\otimes k(x) \xrightarrow{\cdot f} \mathscr{O}_{X,x}\otimes k(y))/(\textit{something})$. Therefore, $\operatorname{Tor}_1^{\mathscr{O}_{X,x}}(\mathscr{O}_{Z,x}, k(y)) = 0$. Therefore, $\mathscr{O}_{Z,x}$ is flat over $\mathscr{O}_{Y,y}$ (Bourbaki; Comm Alg III§5 No. 2 Thm 1 and some proposition a few pages later). This holds for all $x \in Z$, so Z is flat over Y.

So Z is smooth at x over Y of relative dimension $n-1$. □

Jacobian Criterion for Smoothness

Theorem 28.4 (Jacobian Criterion for Smoothness)**.** *Let Z be a subscheme of codimension r is a scheme X, which is smooth over Y of relative dimension n. Then Z is smooth over Y at a point $z \in Z$ if and only if: locally at $z \in X$, the sheaf of ideals defining Z can be generated by r elements $f_1, \dots, f_r$, and the differentials $df_1, \dots, df_r$ are linearly independent over $k(z)$ in $\Omega_{X/Y}\otimes k(z)$. Moreover, if this hold, then Z is smooth of relative dimension $n-r$ over Y.*

Proof. ($\Rightarrow$) This is given to us by Lemma (28.1)

($\Leftarrow$) This follows from repeated applications of Lemma (28.3). □

Next: Étale morphisms.

Lemma 28.5. *Let k be a field, and let $(R, \mathfrak{m})$ be a noetherian local k-algebra. Assume that the residue field $k' := R/\mathfrak{m}$ is a finite separable extension of k, and that $\mathfrak{m}^2 = 0$. Then $\dim_{k'} \Omega_{R/k} \otimes k' = \dim_{k'} \mathfrak{m} \quad (= \dim_{k'} \mathfrak{m}/\mathfrak{m}^2)$.*

Proof. Let $a_1, \dots, a_r$ be a basis for $\mathfrak{m}$ over k'. Pick $t \in R$ such that $\bar{t} \in R/\mathfrak{m}$ is a primitive element for k' over k, and let $f \in k[T]$ be its irreducible polynomial. Note that $f \in R[T]$, and $f(t) \in \mathfrak{m}$, but $f'(t) \notin \mathfrak{m}$ (since k' is separable over k).

We have a map $\phi : k[T, X_1, \dots, X_r] \to R$ defined by $T \mapsto t$, $X_i \mapsto a_i$ for all i, and k is fixed. Note that ϕ maps (T)[24] onto $R/\mathfrak{m}$ and that $(X_1, \dots, X_r)$ onto $\mathfrak{m}$. So ϕ is onto. Since $f(t) \in \mathfrak{m}$, there are $g_1, \dots, g_r \in k[T, X_1, \dots, X_r]$ such that $f - \sum g_i X_i \in \ker\phi$. Also, $X_i X_j \in \ker\phi$ for all i, j (including $i = j$). let $\mathfrak{a}$ be the ideal generated by these elements.

Claim. $\mathfrak{a} = \ker\phi$

Proof. $\mathfrak{a} \in \ker\phi$ by construction. Then the other inclusion follows by noting that $\dim_k k[T, X_1, \dots, X_r]/\mathfrak{a} \le \dim_k R = (r+1)[k' : k]$

$\square_{\text{Claim}}$

Then $\Omega_{R/k} \otimes k' = \Omega_{R/k}/\mathfrak{m}\Omega_{R/k}$ is described by generators $dt, da_1, \dots, da_r$ over k' and relations $\underbrace{f'(t)}_{\text{unit}}\, dt = \sum(\underbrace{a_i dg_i}_{\in\mathfrak{m}} + g_i da_i)$ and $a_i da_j + a_j da_i \in \mathfrak{m}$, so $\Omega_{R/k} \otimes k'$ has basis $da_1, \dots, da_r$. □

[24] ?

Lecture 29

Homework(12): For the last problem, you may assume that k is algebraically closed of characteristic different from 2 and 3.

Comment on last homework(11): Z' should be noetherian.

Comment of homework 10: $X \subseteq \mathbb{P}^n$, then there may be line sheaves on X which do not come from line sheaves on $\mathbb{P}^n$. For example, take the 2-uple embedding $i : \mathbb{P}^1 \to \mathbb{P}^2$. Then $i^*\mathscr{O}(1) = \mathscr{O}(2)$, so you can only get the $\mathscr{O}(2n)$'s on $\mathbb{P}^1$. Thus, $\operatorname{Pic}\mathbb{P}^n \to \operatorname{Pic} X$ may not be onto.

A reference for what we're doing on smoothness: Bosch, Lutkebohnert, and Raynand, Neron Models §2.2.

Last time: If $(R, \mathfrak{m})$ is a local noetherian k-algebra such that $\mathfrak{m}^2 = 0$ and $k' = R/\mathfrak{m}$ is finite separable over k, then

$$0 \to \underbrace{\mathfrak{m}/\mathfrak{m}^2}_{\mathfrak{m}} \to \Omega_{R/k} \otimes k' \to \underbrace{\Omega_{k'/k}}_{0} \to 0$$

is exact.

Corollary 29.1. *If $(R, \mathfrak{m})$ is a local noetherian k-algebra such with $k' = R/\mathfrak{m}$ is finite separable over k, then $\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{R/k} \otimes k'$ is an isomorphism (cf. II.8.7)*

Proof. Apply the earlier lemma to $R/\mathfrak{m}^2$. Note that $\mathfrak{m}^2/\mathfrak{m}^4 \to \Omega_{R/k} \otimes (R/\mathfrak{m}^2) \to \Omega_{(R/\mathfrak{m})/k} \to 0$ is exact. This gives

$$(\mathfrak{m}/\mathfrak{m}^2) \otimes k' \xrightarrow{0} \Omega_{R/k} \otimes k' \xrightarrow{\sim} \underbrace{\Omega_{(R/\mathfrak{m}^2)/k}}_{\mathfrak{m}/\mathfrak{m}^2} \otimes k' \to 0$$

□

For today, all schemes are of finite type over k. Recall that étale means smooth of relative dimension 0. $f : X \to Y$ is unramified if for all $x \in X$, writing $y = f(x)$, we have $k(x)$ is separable algebraic over $k(y)$ and $\mathfrak{m}_y \cdot \mathscr{O}_{X,x} = \mathfrak{m}_x$ (X of finite type implies $k(x)$ is finite separable over $k(y)$).

Exercise III.10.3 *Let $f : X \to Y$ be a morphism, then TFAE:*

(i) *f is étale*
(ii) *f is flat and $\Omega_{X/Y} = 0$*
(iii) *f is flat and unramified*

Proof. (i⇒ ii) flat is obvious. Also, (3) in smoothness condition implies that $\Omega_{X/Y} \otimes k(x) = 0$ for all x., so by Nakayama, we have $\Omega_{X/Y} = 0$.

(ii⇒ iii) Let $x \in X$ and $y = f(x)$. By the second exact sequence,

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \to \Omega_{X/Y} \otimes k(x) \to \Omega_{k(x)/k(y)} \to 0$$

and $\Omega_{X/Y} = 0$ implies that $\Omega_{k(x)/k(y)} = 0$ which implies that $k(x)$ is separated and algebraic over $k(y)$ (II.8.6A)

Assume $\mathfrak{m}_y \cdot \mathscr{O}_{X,x} \neq \mathfrak{m}_x$ (i.e. $\subsetneq$). We have $\mathscr{O}_{X_y,x} = \mathscr{O}_{X,x} \otimes k(y) = \mathscr{O}_{X,x}/\mathfrak{m}_y\mathscr{O}_{X,x}$. This is a noetherian local k-algebra as in the corollary, with maximal ideal not zero. Thus, $\Omega_{\mathscr{O}_{X_y,x}/k(y)} \otimes k(x) \neq 0$. Thus, $0 =_{ii} \Omega_{X/Y} \otimes k(x) = \Omega_{X_y/k(y)} \otimes k(x) \neq 0$. contradiction.

(iii$\Rightarrow$ i)Check (1),(2'),and (3') in the definition of smoothness. (1) is obvious. For (2'), (3'), note that $\mathscr{O}_{X_y,x} = \mathscr{O}_{X,x}/\mathfrak{m}_y\mathscr{O}_{X,x} =_{iii} \mathscr{O}_{X,x}/\mathfrak{m}_x = k(x)$. (2') holds because $\dim_x X_y = \dim \mathscr{O}_{X_y,x} = \dim k(x_{=}0$. (3') holds because $\Omega_{X_y/k(y)} \otimes k(x) = \Omega_{\mathscr{O}_{X_y,x}/k(y)} \otimes k(x) = \Omega_{k(x)/k(y)} \otimes k(x) =_{sep\ alg} 0$ □

<u>Eisenbud 4.4:</u>*Let R be a ring. If n elements of the R-module R^n generate it, then they form a basis.*

Corollary 29.2. *Let R be a ring, and let*

$$M' \xrightarrow{\alpha} M \to M'' \to 0$$

be and exact sequence of R-modules. If M' and M'' can be generated by n and m elements, resp, and if M is free of rank $n+m$, then α is injective, and the given generating sets are bases of M' and M''.

Proof. exercise. □

Proposition 29.3. *If $f : X \to Y$ is smooth of relative dimension n, then $\Omega_{X/Y}$ is locally free of rank n.*

Proof. The question is local on X, so we may assume that $X = \operatorname{Spec} A$. Pick a closed immersion $i : X \hookrightarrow \mathbb{A}^N_Y$. The second exact sequence gives

$$\mathscr{I}/\mathscr{I}^2 \xrightarrow{\delta} i^*\Omega_{\mathbb{A}^N_Y/Y} \to \Omega_{X/Y} \to 0$$

where $\mathscr{I}$ is the sheaf of ideals defining X in $\mathbb{A}^N_Y$. Taking stalks, we have

$$(\mathscr{I}/\mathscr{I}^2)_x \xrightarrow{\delta_x} (i^*\Omega_{\mathbb{A}^N_Y/Y})_x \to (\Omega_{X/Y})_x \to 0$$

The middle thing is free of rank $N-n$, and the two other things can be generated by $N-n$ and n elements, respectively. By the corollary, $(\Omega_{X/Y})_x$ is free, so $\Omega_{X/Y}$ is locally free of rank n. □

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$$

Lemma 29.4. *Let $f : X \to Y$ be a smooth S-morphism of smooth S-schemes. Then the canonical sequence*

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

is exact and locally split.

Proof. For all $x \in X$, the first exact sequence gives an exact sequence

$$(f^*\Omega_{Y/S})_x \xrightarrow{\alpha_x} (\Omega_{X/S})_x \to (\Omega_{X/Y})_x \to 0$$

in which the terms are locally free and the ranks add up, so by the corollary, α_x is injective, and the short exact sequence is split. □

Proposition 29.5 (Another Jacobian Criterion)**.** *Let $f : X \to Y$ be an S-morphism of smooth S-schemes. Consider the conditions*

(i) *f is smooth*
(ii) *the canonical map $f^*\Omega_{Y/S} \to \Omega_{X/S}$ is locally left invertible*
(iii) *for all $x \in X$, the map $(f^*\Omega_{Y/S}) \otimes k(x) \to \Omega_{X/S} \otimes k(x)$ is injective*

Then (i)$\Rightarrow$(ii)$\Rightarrow$(iii). If $Y = \mathbb{A}^N_S$, then (iii)$\Rightarrow$(i) (actually, this assumption is unnecessary).

Proof. (i⇒ii) follows from the most recent lemma.

(ii⇒iii) obvious.

(iii⇒i) Since $Y = \mathbb{A}^N_S$, f is given by coordinates $\bar{f}_1, \dots, \bar{f}_N \in \Gamma(X, \mathscr{O}_X)$. Then (iii) is equivalent to the $d\bar{f}_i$ being linearly independent in $\Omega_{X/S} \otimes k(x)$ for all $x \in X$. We may assume that X is affine. Embed $i : X \hookrightarrow \mathbb{A}^m_S$, let $\mathscr{I}$ be the sheaf of ideals, and let r be the relative dimension of X over S. Pick $x \in X$.

Lift $\bar{f}_1, \dots, \bar{f}_N$ to functions $f_1, \dots, f_N$ on a neighborhood of x in $\mathbb{A}^m_S$. Also, let $h_1, \dots, h_{m-r}$ be a generating set for $\mathscr{I}$ near x (by the Jacobian Criterion).

Consider the composite morphism

$$X \xhookrightarrow{\Gamma_f} X \times_S Y \hookrightarrow \mathbb{A}^m_S \times_S Y = \mathbb{A}^m_Y$$

Here Y has relative dimension N over S and X has codimension $m - r$ in $\mathbb{P}^m_S$, so the image of the composition has codimension $N + m - r$.

Let $t_1, \dots, t_N$ be the coordinate functions on $Y = \mathbb{A}^N_S$. Then (locally) $X \subseteq \mathbb{A}^m_Y$ has local defining equations $\underbrace{h_1, \dots, h_{m-r}}_{\text{first factor}}$ and $\underbrace{t_1 - f_1, \dots, t_N - f_N}_{\text{graph morphism}}$.

Claim. *These functions satisfy the earlier Jacobian criterion for smoothness at the image, x', of x in $\mathbb{A}^m_Y$.*

Proof. The images of $dh_1, \dots, dh_{m-r}$ form a basis for the kernel of $\Omega_{\mathbb{A}^m_S/S} \otimes k(x) \to \Omega_{X/S} \otimes k(x)$, which is isomorphic to the kernel of $\Omega_{\mathbb{A}^m_Y/Y} \otimes k(x') \to \Omega_{X \times_S Y/Y} \otimes k(x')$. So we need to know whether $d(t_i - f_i)$ are linearly independent in $\Omega_{X \times Y/Y} \otimes k(x')$. The dt_i are zero (t_i are functions on Y), so we are looking at $-df_i$, which are linearly independent in $\Omega_{X/S} \otimes k(x)$ by (iii). $\square_{\text{Claim}}$

$\square$

Corollary 29.6. *Let X be a smooth scheme over S, and let $f : X \to Y$ be an S-morphism, where Y is smooth over S. If f is étale, then the natural map*

$$f^*\Omega_{Y/S} \to \Omega_{X/S}$$

is an isomorphism, and the converse holds if $Y = \mathbb{A}^N_S$ (actually, it always holds).

Proof. If f is étale, then it is smooth, so

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

is exact, but $\Omega_{X/Y} = 0$, so we get our isomorphism.

For the converse, use (ii)⇒(i) to get that f is smooth, and then $\Omega_{X/Y} = 0$ from the first exact sequence to get relative dimension 0. $\square$

Theorem 29.7 ("étale over affine")**.** *A morphism $f : X \to Y$ is smooth of relative dimension n at a point $x \in X$ if an only if there is an open* *neighborhood $x \in U \subseteq X$ and an étale morphism $g : U \to \mathbb{A}^n_Y$ such that*

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[dr, "f|_U"'] & \mathbb{A}^n_Y \arrow[d] \\ & Y \end{tikzcd}$$

commutes

We are saying that locally, near X, it looks like the affine fibers. previous corollary is like implicit function theorem.

Lecture 30

<u>Last time:</u> We did Corollary (29.6). Compare this with the *inverse* function theorem.
We also did Theorem (29.7)

Proof. ($\Leftarrow$) $f|_U$ is a composition of smooth morphisms, so it is smooth, of relative dimension $0 + n = n$.
($\Rightarrow$) Pick local sections $g_1, \dots, g_n$ of $\mathscr{O}_X$ near x such that $dg_1, \dots, dg_n$ form a basis for $\Omega_{X/Y}$ near x. Let U be an open neighborhood of x, where this happens. Let $g : U \to \mathbb{A}^n_Y$ be the map given by $(g_1, \dots, g_n)$ (think of the g_i as maps $X \to \mathbb{A}^1_Y$). Then note that $\Omega_{\mathbb{A}^n_Y/Y}$ is free with basis $dx_1, \dots, dx_n$. Then $g^*\Omega_{\mathbb{A}^n_Y/Y}$ is free with basis $g^*dx_1, \dots, g^*dx_n$, but these are just $dg_1, \dots, dg_n$ because $g^*dx_i = d(g^*x_i) = dg_i$. So

$$g^*\Omega_{\mathbb{A}^n_Y/Y} \to \Omega_{X/Y}$$

is an isomorphism, so g is étale. □

Riemann-Roch (for curves)

<u>Riemann-Roch (for curves):</u> For this section, *curves* (varieties of dimension 1, thus over an algebraically closed field) are assumed to be projective and non-singular (= smooth). If X is such a curve, we have that $p_a(X) = p_g(X)$; call this value $g(X)$ or the *genus*. We have that $g \geq 0$ (since one of these is the dimension of some H^1), and all integers ≥ 0 occur.

<u>Divisors and line sheaves on curves:</u> Weil divisors and Cartier divisors are the same thing. Recall that a Weil divisor is of the form $\sum n_P P$, where the P are closed points, and n_P are integers, almost all of which are zero. A Cartier divisor is a pull-back for a dominant (i.e. finite) morphism of curves. If $f \in K(X)^\times$ and $P \in X$ a closed point[25]. Then $\mathscr{O}_{X,P}$ is a dvr with fraction field $K(X)$, so we can get $n_P = v(F) \in \mathbb{Z}$. note that $n_P = 0$ for almost all P. Thus, we get a principal divisor $(f) = \sum n_P P$. Two divisors are equivalent ($D_1 \sim D_2$) if the difference is principal. We get a homomorphism $K(X)^\times \to \mathrm{Div} X$. The cokernel is the group of divisor classes, $\mathrm{Cl} X$. For all D, we can define a line sheaf $\mathscr{O}(D)$ (or $\mathscr{L}(D)$), defined as a subsheaf of the constant sheaf $K(X)$. Then $1 \in K(X)$ corresponds to a non-zero rational section of $\mathscr{O}(D)$, called 1_D.

Given a rational section $s \neq 0$ of a line sheaf $\mathscr{L}$, form a divisor as follows: Given $P \in X$, let s_0 be a local generator for $\mathscr{L}$ near P, then $s/s_0 \in K(X)^\times$,

[25]We're usually using closed points ... if I don't say it, I probably meant to.

so let $n_P = v(s/s_0)$. This is well-defined, so define $(s) = \sum n_P P$. For two rational sections $s, t \neq 0$ of $\mathscr{L}$, $(s) - (t)$ is a principal divisor, equal to the principal divisor (s/t) (note that s/t is a non-zero rational function). More generally, if s_1, s_2 are rational sections of $\mathscr{L}_1$ and $\mathscr{L}_2$, resp., then $s_1 \otimes s_2$ is a non-zero rational section of $\mathscr{L}_1 \otimes \mathscr{L}_2$, and $(s_1 \otimes s_2) = (s_1) + (s_2)$. Also, recalling 1_D as a rational section of $\mathscr{O}(D)$, we have $(1_D) = D$. Recall also that $\mathscr{O}(D_1+D_2) \cong \mathscr{O}(D_1) \otimes \mathscr{O}(D_2)$, so we get an isomorphism $\mathrm{Cl} X \xrightarrow{\sim} \operatorname{Pic} X$, taking D to $\mathscr{O}(D)$. The reverse map is obtained by taking any non-zero rational section, and taking its divisors. All of this is functorial.

A divisor $D = \sum n_P P$ is *effective* if $n_P \geq 0$ for all P. Let $s \neq 0$ be a rational section of $\mathscr{L}$, then (s) is effective if and only if $s \in \Gamma(X, \mathscr{L})$.

Definition. Let D be a divisor on X. Then the *complete linear system*, written $|D|$, is the set of effective divisors linearly equivalent to D.

$$|D| \overset{\leftrightarrow}{1 - 1} (H^0(X, \mathscr{O}(D)) \smallsetminus \{0\})/k^*$$

given by $(s) \leftarrow s$, and $E = D + (f) \mapsto f$ (in $K(X)$, this is $\{f \in K(X)^\times : D + (f) \geq 0\}$). Then $|D|$ has an algebraic structure of dimension $\underbrace{h^0(X, \mathscr{O}(D))}_{=:l(D)} - 1$

A *linear system* associated to D is a subset of a complete linear system corresponding to a linear subspace of $H^0(X, \mathscr{O}(D)) \smallsetminus \{0\}$. $D_1 \sim D_2$ implies that $l(D_1) = l(D_2)$.

All of this behaves nicely with respect to pull-backs (via dominant morphisms of curves).

Definition. If $D = \sum n_P P$, then $\deg D = \sum n_P$. If D is principal, say (f), then $\deg D = 0$.

Proof. If $\phi : X \to Y$ is a finite (= dominant) morphism of curves and E is a divisor on Y, then

$$\begin{aligned}\deg(\phi^* D) &= \underbrace{(\deg \phi)}_{[K(X):K(Y)]}(\deg E) \\ &= k - \text{length of fibers over closed points}\end{aligned}$$

The function f gives a finite map (provided $f \notin k$; $f \in k \Rightarrow (f) = 0$, so $\deg D = 0$) $\phi : X \to \mathbb{P}^1$ and $f = \phi^* z$ so $(f) = (\phi^* z) = \phi^*(z) = \phi^*([0] - [\infty])$, so $\deg D = (\deg \phi) \underbrace{(\deg(z))}_{0} = 0$. $\square$

Thus, the degree gives well-defined group homomorphisms:

$$\begin{array}{ccc} \mathrm{Cl} X & \longrightarrow & \mathbb{Z} \\ \| \wr & & \\ \operatorname{Pic} X & \longrightarrow & \mathbb{Z} \end{array}$$

Since X is a curve, $\omega_X = \wedge^1 \Omega_{X/k} = \Omega_{X/k}$. Any divisor of X associated to ω_X is called "the" canonical divisor of X, written K.

Theorem 30.1 (Riemann-Roch for curves)**.** *Let X be a curve over k, let K be a canonical divisor of X, and let D be any divisor on X. Then*

$$l(D) - l(K-D) = \deg D + 1 - g.$$

where $g = g(X)$ (Note that the LHS is independent of the choice of K)

Proof. Note that $l(D) = h^0(X, \mathscr{O}(D))$ and

$$\begin{aligned} l(K-D) &= h^0(X, \mathscr{O}(K-D)) \\ &= h^0(X, \omega_X \otimes \mathscr{O}(D)^\vee) \\ &=_{\text{duality}} h^1(X, \mathscr{O}(D)) \end{aligned}$$

so LHS $= \chi(\mathscr{O}(D))$

Now show:

(1) $\chi(\mathscr{O}(D)) = \deg D$ is independent of D
(2) compute $\chi(\mathscr{O}(D))$ for one particular D

Start with (2): if $D = 0$, then $\mathscr{O}(D) = \mathscr{O}_X$, and $\chi(\mathscr{O}_X) = h^0(X, \mathscr{O}_X) - h^1(X, \mathscr{O}_X) = 1 - g$. So $\chi(\mathscr{O}(D)) = \deg D = 1 - g$ if $D = 0$.

Now we'll do (1): $\chi(\mathscr{O}(D+P)) = \underbrace{\deg(D+P)}_{\deg D+1} = \chi(\mathscr{O}(D)) - \deg D$ for all D, P. That is, $\chi(\mathscr{O}(D+P)) = \chi(\mathscr{O}(D)) + 1$

Regard P as a reduced closed subscheme of X; $i : P \to X$. Its sheaf of ideals is $\mathscr{O}(-P)$, so

$$0 \to \mathscr{O}(-P) \to \mathscr{O}_X \to \underbrace{k(P)}_{\text{skyscraper}} \to 0$$

is exact. Tensor with $\mathscr{O}(D+P)$, and note that $k(P) \otimes \underbrace{\mathscr{O}(D+P)}_{\substack{\text{locally triv} \\ \text{near } P}} \cong k(P)$ (non-canonically),so

$$0 \to \mathscr{O}(D) \to \mathscr{O}(D+P) \to k(P) \to 0$$

is exact, so

$$\chi(\mathscr{O}(D+P)) = \chi(\mathscr{O}(D)) + \chi(k(P)).$$

$$\begin{aligned} \chi(k(P) &= h^0(X, i_*\mathscr{O}_P) - h^1(X, i_*\mathscr{O}_P) \\ &= h^0(P, \mathscr{O}_P) - h^1(P, \mathscr{O}_P) \\ &= 1 - 0 = 1 \end{aligned}$$

□

What happens if $D = K$? $\underbrace{l(K) - l(0)}_{-\chi(\mathscr{O}_X) = g-1} = \deg(K) + 1 - g$, so we have that $\deg(K) = 2g - 2$.

To compute K, pick any $f \in K(X) \smallsetminus k$. Compute (df) (where df is the rational section of $\Omega_{X/k}$).

Lemma 30.2. *Let D be an effective divisor. Then $\deg D \geq 0$, and if $\deg D = 0$, then $D = 0$.*

Lemma 30.3. *Let D be a divisor on X with $l(D) \neq 0$. Then* $\deg D \geq 0$, *and if* $\deg D = 0$*, then D is principal.*

Proof. Apply the earlier lemma to some effective divisor E, linearly equivalent to D (such an E exists by the assumption). □

Proposition 30.4. *If X is a curve of genus 0, then $X \cong \mathbb{P}^1$.*

Proof. let $P \neq Q$ be two distinct points on X, and apply RR to $D = P - Q$. Then $l(P-Q) - l(\underbrace{K - P + Q}_{\deg = -2}) = \underbrace{\deg(P-Q)}_{0} + \underbrace{1-g}_{1}$. So $l(P-Q) = 1$, so since $\deg(P-Q) = 0$, $P-Q$ is principal. Then f gives a morphism $X \to \mathbb{P}^1$, which is thus an isomorphism. □

Lecture 31

Comments on Homework 12:

(i) Common mistake (#1) was failing to realize the difference between a closed point and a rational point. X defined by $y^2 = x^p - t$. There is a point in X corresponding to the point $(t^{1/p}, 0)$, namely $P = \operatorname{Spec} k[x,y]/(y, x^p - t)$. Then $k(P) = k(t^{1/p})$. This is a closed, but not rational point.

(ii) (#2) This can be used to *define* étale (provided you rephrase it not to require that $k(x) \hookrightarrow \hat{\mathscr{O}}_x$

(iii) (#3) This is an example of the general principal that irreducible components in $\operatorname{Spec} \hat{\mathscr{O}}_{X,x}$ can be separated in an étale cover.

Back to Riemann-Roch. Again, curves are smooth and projective over an algebraically closed field k, until we say otherwise.

Recall that we have $\deg : \operatorname{Pic} X \to \mathbb{Z}$ well-defined surjective group homomorphism.

Definition. $\operatorname{Pic}^0 X =$ the kernel of deg, which is $\{\mathscr{L} \in \operatorname{Pic} X | \deg(\mathscr{L}) = 0\}$.

If X has genus 0, then $X \cong \mathbb{P}^1$, so $\operatorname{Pic} X = \mathbb{Z}$ (via $n \leftrightarrow \mathscr{O}(n)$, so $\operatorname{Pic}^0 X$ is trivial. In genus > 0, $\operatorname{Pic} X$ is non-trivial. Pick two distinct points P, Q. Then $\mathscr{O}(P - Q) \ncong \mathscr{O}_X$ (lest $X \cong \mathbb{P}^1$), so $\operatorname{Pic}^0 X \neq 0$.

Genus 1 curves: Let X be a curve of genus 1 and fix $P_0 \in X$. Then we have a map $\phi : X(k) \to \operatorname{Pic}^0 X$ given by $P \mapsto \mathscr{O}(P - P_0)$.

Remark. The canonical divisor K is ~ 0. By RR applied to the divisor 0, $l(0) - l(K) = 1 - l(k) = 0$. So $l(K) = 1$. But $\deg K = 2g - 2 = 0$, so $K \sim 0$ by an earlier lemma.

Proposition 31.1. *ϕ as above is a bijection.*

Proof. 1-1: $\phi(P) = \phi(Q) \Leftrightarrow P - P_0 \sim Q - P_0 \sim P - Q \sim 0$, which cannot happen (lest $X \cong \mathbb{P}^1$)

Onto: Pick $\mathscr{L} \in \operatorname{Pic}^0 X$, and let D be a corresponding divisor. Apply RR to $D + P_0$:

$$l(D + P_0) - l(K - D - P_0) = 1 + 1 - g = 1.$$

so $l(D + P_0) = 1$, so $D + P_0 \sim Q$, for an effective divisor Q. Then $D \sim Q - P_0$, so $\mathscr{L} = \phi(Q)$ □

Corollary 31.2. *There is a canonical (depending on choice of P_0) group structure on $X(k)$ given by ϕ and the group structure on $\operatorname{Pic}^0 X$.*

Definition. An *elliptic curve* is a smooth curve of genus 1 over a field k together with a choice of rational point $P_0 \in X(k)$.

Note that $\phi(P_0) = \mathscr{O}(P_0 - P_0) = \mathscr{O}_X$, so P_0 is the identity element. Let's call it 0 from now on.

Apply RR to some multiples of [0]:

$$l([0]) - \underbrace{l(K - [0])}_{0} = 1 + 1 - g = 1 = \dim(H^0(X, \mathscr{O}([0])) = k)$$

where we think of $\mathscr{O}([0])$ as a subsheaf of $K(X)$. Similarly

$$l(2[0]) = 2$$

A basis of $H^0(X, \mathscr{O}(2[0]))$ is $1, x$.
$l(3[0]) = 3$ ... basis is $1, x, y$.
$l(4[0]) = 4$ ... basis is $1, x, y, x^2$.
$l(5[0]) = 5$ ... basis is $1, x, y, x^2, xy$.
$l(6[0]) = 6$ ... basis is $1, x, y, x^2, xy$, (x^2 or y^2).

So $1, x, y, x^2, xy, x^2, y^2$ are linearly dependent over k, and the resulting function f *does* involve y^2 and x^3. By Exercise 1.3, $X \setminus \{0\}$ is affine, and x, y generate its affine ring: ($H^0(X, \mathscr{O}(n \cdot [0]))$ has basis $1, x, y, x^2, xy, x^3, x^2y, ...$), so $k[x, y]$ maps onto the affine ring, and the kernel is (f). So X is the projective closure of $\operatorname{Spec} k[x, y]/(f)$ in $\mathbb{P}^2$. So it is a non-singular cubic in $\mathbb{P}^2$, and it has only one point at infinity, namely 0.

If the characteristic of k is not 2 or 3, then we can do a linear coordinate change so that f is $y^2 = 4x^3 + ax + b$ with $a, b \in k$. This is called Weierstrass form.

Let $P = (x_0, y_0) \in X$, and let $P' = (x_0, -y_0)$. Then $[P] + [P'] \sim 2 \cdot [0]$ because $(x - x_0) = [P] + [P'] - 2[0]$. So $P' = -P$ in the group. Also, if P, Q, R are colinear, then

$$[P] + [Q] + [R] - 3[0] = \text{the principal divisor (equation of the line)}$$

insert picture around here

so $P + Q + R = 0$ in the group law, so $Q + R = P'$.

Varieties over arbitrary fields: now k may not be algebraically closed.

Examples:

(i) Let k be a field. On $\mathbb{A}^1_k = \operatorname{Spec} k[x]$, you have the generic point, plus the closed points; also, $\{\text{closed points}\} \leftrightarrow \{\text{irreducible monics}\}/\text{action of } \operatorname{Aut}_k(\bar{k})$

(ii) $\sqrt{2} \in \mathbb{A}^1_{\mathbb{Q}}$, only it is paired with $-\sqrt{2}$. $\operatorname{Spec} \mathbb{Q}[x]/(x^2 - 2)$. $kP = \mathbb{Q}[\sqrt{2}]$. it is a closed point, but not a rational point.

(iii) Let $k = \mathbb{F}_p(t)$. Then $\sqrt[p]{t} \in \mathbb{A}^1_k$, realized as $\operatorname{Spec} k[x]/(x^p - t)$. When we base change to $\bar{k}$, we get $\operatorname{Spec} \bar{k}[x]/(x - t^{1/p})^p$, which is not reduced.
So the point $t^{1/p}$ in $\mathbb{A}^1_k$ is not geometrically reduced.
Likewise $\operatorname{Spec} \mathbb{Q}[x]/(x^2 - 2)$ becomes $\bar{\mathbb{Q}}[x]/(x - \sqrt{2})(x + \sqrt{2})$ which is reducible, so our point is not geometrically irreducible. (See Exercise II.3.15)

What about $\mathbb{A}^1_k$? This situation is similar:

$$\{\text{closed points}\} \text{“}=\text{''}\ \bar{k}^n/(\text{action of } \operatorname{Aut}_k \bar{k})$$

Grothendieck Topologies: (reference: Vistoli, Notes on Grothendieck topologies fibered categories, and descent theory[26])

General idea: If you had a smooth map of complex manifolds $f : X \to Y$, you can work near $P \in X$ by finding a local section of f near $f(P)$. You can't necessarily do this with schemes because the Zariski topology is too coarse (exceptions: $\mathbb{P}(\mathscr{E})$, etc.). One way to appoximate the classical topology is to work in $\hat{\mathscr{O}}_{X,x}$. Then you have

$$\begin{tikzcd} \operatorname{Spec} \hat{\mathscr{O}}_{X,x} \arrow[d] \\ \operatorname{Spec} \hat{\mathscr{O}}_{Y,f(x)} \arrow[u, dotted, bend left, "\exists"] \end{tikzcd}$$

The image is cut out by $f_1, \ldots, f_r \in \hat{\mathfrak{m}}_x/\hat{\mathfrak{m}}_x^2$. These can be lifted to $\mod \hat{\mathfrak{m}}^2$ to $f_1, \ldots, f_r \in \mathfrak{m}_x \smallsetminus \mathfrak{m}_x^2$. Lift to $f_1, \ldots, f_r \in \mathscr{O}_X(U)$ for some open neighborhood. Take the closure $Z(f_1, \ldots, f_r)$ to get $Z \subseteq X$ closed subscheme.
Z may have many sheets over Y, and you can't keep one and throw out the others. But you do have $f|_Z$ is étale at x ($x = P$). You can base change to Z :

$$\begin{tikzcd} X' = X \times_Y Z \arrow[d] \\ Z \arrow[u, bend left] \end{tikzcd}$$

Of course $f|_Z$ is not étale everywhere, but it is in a Zariski-open neighborhood of x.
This is called working locally in the étale topology.

Definition. Let $\underline{\mathcal{C}}$ be a category. A *Grothendieck topology* on $\underline{\mathcal{C}}$ is, for each object U, a collection of covers of U, where a cover is a set of morphisms $\{U_i \to U\}$ in $\underline{\mathcal{C}}$ satisfying:

(i) if $V \to U$ is an isomorphism in $\underline{\mathcal{C}}$, then $V \to U$ is a cover.
(ii) if $\{U_i \to U\}$ is a covering and $V \to U$ is a morphism, then $U_i \times_U V$ exists in $\underline{\mathcal{C}}$, and $\{U_i \times_U V\}$ is a covering (of V)
(iii) if $\{U_i \to U\}$ is a covering, and $\{V_{i,j} \to U_i\}_{j \in J}$ is a covering for each i, then the composites $\{V_{i,j} \to U\}$ form a covering.

Example: Let $\underline{\mathcal{C}}$ = category of schemes. Given $U \in \underline{\mathcal{C}}$, a covering is a collection $\{U_i \to U\}$ of open immersions whose images cover U.

(i) ok
(ii) if $\{U_i\}$ cover U, then $\{f^{-1}(U_i)\}$ cover V.
(iii) trivial

This is how the Zariski topology is realized as a Grothendieck topology.

[26] This is on the arXiv.

Definition. The Grothendieck topology on U is the collection of open covers.

Remark. Grothendieck topologies versus the earlier kind:

- open sets are now morphisms
- intersections are now fibered products over U
- unions are now obsoleted by looking at covers

Definition. A *site* is a category with a Grothendieck topology.

Another example of a site: fix a field k. Let $\underline{\mathcal{C}}$ = category of schemes of finite type over k. Given $U \in \underline{\mathcal{C}}$, a covering of U is a finite collection $\{U_i \to U\}$ of étale morphisms such that the union of the images is all of U. This "is" the étale site.

Exercise 10.5 says: if a (Zariski) sheaf is locally free as a sheaf on the étale topology, then it is locally free under the Zariski topology.