Anton Geraschenko's Topology Notes

About these notes

I took Math 215A, Algebraic Topology, with Peter Teichner in Fall 2006. I took some notes in class and then wrote them up after class. At first, I would do one lecture at a time (so the earlier sections really are one lecture's worth of material). Later in the course, I would wait a bit before writing up a lecture because subsequent lectures often added context. Thus, the later sections are not in bijection with lectures, but are partitioned by topic. I often added proofs for statements that were not proven in class, or modified the proofs from class.

These notes are not self-contained. There is often dependence on results from the homework and occasionally some statements are not proven at all.

All spaces are Hausdorff and all maps are continuous.

2 A Theorem from Point-Set topology

Theorem 2.1. *If X is compact (and Hausdorff) and Y is Hausdorff, then $f : X \to Y$ is closed. In particular, if f is a bijection, then it is a homeomorphism.*

Proof. If $A \subseteq X$ be closed, then it is compact because closed subsets of compact sets are compact. Since the continuous image of a compact set is compact, $f(A)$ is compact. Since compact subsets of Hausdorff spaces are closed, $f(A)$ is closed. $\square$

3 homotopic maps

Definition 3.1. The n-dimensional ball is $D^n = \{x \in \mathbb{R}^n | \|x\|_2 \leq 1\}$. The sphere is $S^{n-1} = \partial D^n = D^n \smallsetminus \mathring{D}^n$.

Theorem 3.2. *If $C \subseteq \mathbb{R}^n$ is convex, compact, and $\mathring{C} \neq \varnothing$, then $C \approx D^n$ and $\partial C \approx S^{n-1}$.*

Proof. Assume $0 \in \mathring{C}$. Define $f : \partial C \to S^{n-1}$ by $f(x) = \frac{x}{\|x\|_2}$. f is surjective because for every $v \in S^{n-1}$, there is a maximal $t \in \mathbb{R}_{>0}$ so that $tv \in \partial C$. By convexity of C, this t is unique (here you use that $\mathring{C}$ contains a ball around 0), so f is injective. Since ∂C is compact and S^{n-1} is Hausdorff, f is a homeomorphism by Theorem 2.1. Similarly, extend f to a homeomorphism $\tilde{f} : C \to D^n$. $\square$

Corollary 3.3. *For any norm, the ball you get is homeomorphic to the ball with the 2-norm.*

Definition 3.4. $f_0, f_1 : X \to Y$ are *homotopic* (written $f_0 \simeq f_1$) if there is a continuous map $F : X \times I \to Y$ such that $F(x, i) = f_i(x)$ for $i = 0, 1$.

If $F(x,t)$ is continuous in t, there is an associated map $F^*: I \to Y^X := \mathrm{Hom}_{\mathrm{Top}}(X,Y)$. In this case, a homotopy of maps is just a path in the mapping space Y^X.

We'll show next time that for F to be continuous in t, you need X to be *locally compact* (every point must have a compact neighborhood[1]). In this case, there is a topology on Y^X which makes F^* continuous.

Definition 3.5. The *compact-open* topology on Y^X has a sub-basis given by sets of the form $M(K,U) = \{g : X \to Y | g(K) \subseteq U\}$, where $K \subseteq X$ is compact and $U \subseteq Y$ is open.

4 Compact-Open topology and Exponential laws

If X is compact and Y is a metric space, then the compact-open topology is given by the metric $d(f,g) = \sup\{d(f(x), g(x)\}$.

Lemma 4.1. *If X is locally compact, then the evaluation map $ev : Y^X \times X \to Y$, given by $(f,x) \mapsto f(x)$, is continuous.*

Proof. If U is a neighborhood of $f(x)$, $f^{-1}(U)$ contains some compact open neighborhood K of x. Then $ev\big(M(K,U) \times K\big) \subseteq U$ by definition, so $M(K,U) \times \mathring{K}$ is an open neighborhood of (f,x) in $ev^{-1}(U)$. $\square$

Theorem 4.2. *Let X be locally compact. Then $f : X \times T \to Y$ is continuous if and only if (1) each $f_t : X \to Y$ is continuous, and (2) $\hat{f} : T \to Y^X$ is continuous.*

Proof. ($\Leftarrow$) f is the composition $X \times T \xrightarrow{\left(\begin{smallmatrix} 0 & \hat{f} \\ \mathrm{Id}_X & 0 \end{smallmatrix}\right)} Y^X \times X \xrightarrow{ev} Y$. By (1), $\hat{f}(t) = f_t$ is in Y^X; $\left(\begin{smallmatrix} 0 & \hat{f} \\ \mathrm{Id}_X & 0 \end{smallmatrix}\right)$ is continuous by (2) and ev is continuous by Lemma 4.1.

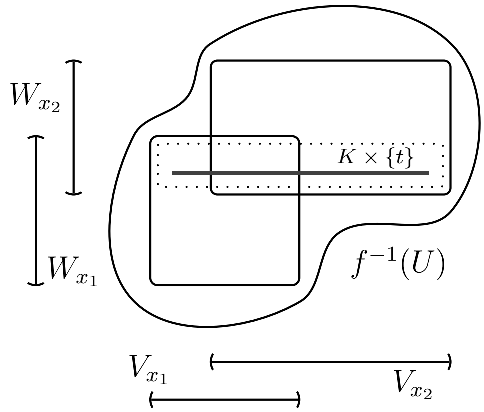
($\Rightarrow$) f_t is the composition of continuous functions $X \xrightarrow{x \mapsto (x,t)} X \times T \xrightarrow{f} Y$, proving (1). To prove (2), we need to prove that $\hat{f}^{-1}\big(M(K,U)\big) = \big\{t \in T | f(K \times \{t\}) \subseteq U\big\} \subseteq T$ is open. Fix some $t \in T$ such that $f(K \times \{t\}) \subseteq U$.

Claim. *There are open sets $W \subseteq T$ and $V \subseteq X$ such that $K \times \{t\} \subseteq V \times W \subseteq f^{-1}(U)$*

By continuity of f, for each $x \in K$, there are open neighborhoods $x \in V_x \subseteq X$ and $t \in W_x \subseteq T$ such that $V_x \times W_x \subseteq f^{-1}(U)$. By compactness, $K \subseteq V_{x_1} \cup \cdots \cup V_{x_n} =: V$.

[1]Since all our spaces are Hausdorff, this is equivalent to saying that every neighborhood of a point contains a compact neighborhood.

Set $W = W_{x_1} \cap \cdots \cap W_{x_n}$. This V and W satisfy the claim.



Now the given W is a neighborhood of t in $\hat{f}^{-1}\big(M(K,U)\big)$. $\square$

Theorem 4.3 (Exponential Law)**.** *If X is locally compact, then $Y^{X\times T} \approx (Y^X)^T$, $f \mapsto \hat{f}$.*

Proof. By the previous theorem, this map is bijective. It is an exercise to show the homeomorphism. Or look at page 531 of Hatcher. $\square$

Lemma 4.4. *If X and T are locally compact, then*

- $Y^X \times W^X \approx (Y \times W)^X$
- $Y^{X \amalg T} \approx Y^X \times Y^T$
- $\big((Y,y_0)^{(X,x_0)}, \mathrm{const}_{y_0}\big)^{(T,t_0)} \approx (Y,y_0)^{(X\times T, X\vee T)} \approx (Y,y_0)^{X\wedge T}$.

where $X \wedge T := X \times T / X \vee T$.

Definition 4.5. $f : X \to Y$ is a *homotopy equivalence* if there is a $g : Y \to X$ so that $g \circ f \simeq \mathrm{Id}_Y$ and $f \circ g \simeq \mathrm{Id}_X$.

5 CW complexes

Definition 5.1. A space X is *homogeneous* if the group of homeomorphisms $X \to X$ acts transitively on the points of X.

Theorem 5.2. *Every connected manifold is homogeneous. Every topological group is homogeneous.*

For example, the Cantor set is homogeneous.
The push-out and pull-back are (respectively)

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B_1 \arrow[d, "g_1"] \arrow[dr] & \\
B_2 \arrow[r, "g_2"] \arrow[rr, bend right] & PO \arrow[ul, phantom, "\lrcorner"] \arrow[r, dashed, "\exists!"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, dashed, "\exists!"'] \arrow[rr, bend left] \arrow[dr] & PB \arrow[r, "g_1"'] \arrow[d, "g_2"] \arrow[dr, phantom, "\ulcorner"] & B_1 \arrow[d, "f_1"] \\
 & B_2 \arrow[r, "f_2"] & A
\end{tikzcd}$$

Pull-backs and push-outs exist in the category of topological spaces (the constructions from Set work). $PO = B_1 \coprod B_2 / f_1(a) \sim f_2(a)$ and $PB = \{(x,y) \in B_1 \times B_2 | f_1(x) = f_2(y)\}$.

A *CW complex* is where you attach cells in order of dimension. That is, define $X^{(-1)} = \varnothing$, and define the *n-skeleton* $X^{(n)}$ by

$$\begin{tikzcd}
\coprod_{\alpha \in I_n} S^{n-1} \arrow[r, hook] \arrow[d, "\coprod \phi_\alpha"'] & \coprod D^n \arrow[d, "\Phi_\alpha"] \\
X^{(n-1)} \arrow[r] & X^{(n)} \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

for some indexing set I_n. The ϕ_α are called *attaching maps* and the Φ_α are called *characteristic maps*. Define $e^n_\alpha = \Phi_\alpha(\mathring{D}^n)$. Define X as $\bigcup X^{(n)}$, with the direct limit topology (a set is open if and only if the intersection with $X^{(n)}$ is open for each n).

6 π_k and more CW stuff

Definition 6.1. $\pi_k(X, x_0) := \big[(S^k, s_0), (X, x_0)\big] = \pi_0\big((X, x_0)^{(S^k, x_0)}\big)$.

Note that $S^k \approx D^k / \partial D^k \approx I^k / \partial I^k$. For $k \geq 1$, $\pi_k(X, x_0)$ has a group structure given by

$$f \cdot g : I^k = I^{k-1} \times I \to I^{k-1} \times 2I = I^k \cup I^k \xrightarrow{f \cup g} X$$

$$\boxed{\;f\;} \cdot \boxed{\;g\;} = \underset{I^{k-1}}{\begin{array}{|c|}\hline f \\ \hline g \\ \hline\end{array}}\, I$$

Lemma 6.2. *For a CW complex X,*

1. $\Phi_\alpha(D^n) = \bar{e}^n_\alpha$ *(closure in X).*
2. *For $A \subseteq X$, $\Phi^{-1}_\alpha(A)$ is closed if and only if $A \cap \bar{e}^n_\alpha$ is closed.*
3. *X has the weak topology with respect to the maps $\Phi_\alpha : D^n \to X$.*

Proof. (1) $\Phi_\alpha(D^n)$ is closed (since D^n is compact and X is Hausdorff (by homework 3)) and contains e^n_α. If A is closed, with $e^n_\alpha \subseteq A \subseteq \Phi_\alpha(D^n)$, then $\Phi^{-1}_\alpha(A)$ is a closed set in D^n which contains $\mathring{D}^n$, so it is all of D^n. It follows that $A = \Phi_\alpha(D^n)$.

(2) Is as easy as (1).

(3) If $A \subseteq X$ has the property that $\Phi^{-1}_\alpha(A) \subseteq D^n$ is closed for all α, then we'd like to show that $A \cap X^{(n)}$ is closed for all n. Well, $A \cap X^{(-1)} = \varnothing$ is closed. Now induct; assume $A \cap X^{(n-1)}$ is closed. Then using the property of push-out, $A \cap X^{(n)}$ is closed. $\square$

7 Van Kampen's Theorem

Chris lectures today because Peter is ill. Remember the group law on $\pi_1(X)$ and that π_1 is a functor, sending homotopy equivalences to isomorphisms. Say we have a cover $j_\alpha : A_\alpha \to X$ of X by open sets A_α, all of which contain the base point x_0. Then it is clear that we have commutative squares

$$\begin{tikzcd} A_\alpha \cap A_\beta \arrow[r, hook, "i_{\alpha\beta}"] \arrow[d, hook, "i_{\beta\alpha}"'] & A_\alpha \arrow[d, hook, "j_\alpha"] \\ A_\beta \arrow[r, hook, "j_\beta"] & X \end{tikzcd} \quad \text{which induce} \quad \begin{tikzcd} \pi_1(A_\alpha \cap A_\beta) \arrow[r, "i_{\alpha\beta*}"] \arrow[d, "i_{\beta\alpha*}"'] & \pi_1(A_\alpha) \arrow[d, "j_{\alpha*}"] \\ \pi_1(A_\beta) \arrow[r, "j_{\beta*}"] & \pi_1(X) \end{tikzcd}$$

where each space has the base point x_0. Thus, we have a map $\Phi : *_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X)$, and it is obvious the relations $j_{\alpha*} \circ i_{\alpha\beta*}(w) = j_{\beta*} \circ i_{\beta\alpha*}(w)$.

Theorem 7.1 (Van Kampen's Theorem)**.** *Let $X = \bigcup A_\alpha$, with X, A_α, and $A_\alpha \cap A_\beta$ all path connected and containing the base point x_0. Then the map $\Phi : *_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ is surjective. Furthermore, if all of the $A_\alpha \cap A_\beta \cap A_\gamma$ are path connected, then the kernel of Φ is generated by elements of the form $i_{\alpha\beta*}(w) \cdot i_{\beta\alpha*}(w)^{-1}$, where $w \in \pi_1(A_\alpha \cap A_\beta, x_0)$.*

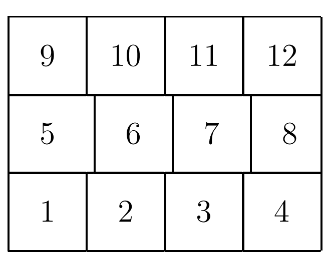
Proof. First we prove surjectivity. Let $f : I \to X$ be a loop at x_0. Choose $0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = 1$ so that $f([s_i, s_{i+1}]) \subseteq A_{\alpha_i} =: A_i$. Define f_i to be the path $I \xrightarrow{\sim} [s_i, s_{i+1}] \xrightarrow{f|_{[s_i,s_{i+1}]}} X$. Choose paths $g_i : I \to A_i \cap A_{i+1}$ from $f(s_i)$ to x_0. Then $f \simeq (f_0 g_1)(g_1^{-1} f_1 g_2) \cdots (g_{n-2} f_{n-2} g_{n-1})(g_{n-1}^{-1} f_{n-1}) \in \operatorname{im} \Phi$.

To prove that the kernel of Φ is what we want, it is enough to show that given an element $f_1 \cdots f_n \in *_\alpha \pi_1(A_\alpha)$, with $f_i \in A_{\alpha_i}$, such that $\Phi(f_1 \cdots f_n) = 0$, we can turn $f_1 \cdots f_n$ into the constant map using a series of the following moves:

1. Replace f_i by an equivalent element of $\pi_1(A_{\alpha_1})$ (i.e. homotope f_i within A_{α_i}), or, if $\alpha_i = \alpha_{i+1}$, replace $f_i \cdot f_{i+1}$ by their product in $\pi_1(A_{\alpha_i})$. These operations don't change the element of $*_\alpha \pi_1(A_\alpha)$.

2. If the image of f_i lies in A_β, then think of it as an element of $\pi_1(A_\beta)$, rather than an element of $\pi_1(A_{\alpha_i})$.

Let $F : I \times I \to X$ be a homotopy from $f_1 \cdots f_n$ to the constant map. Then cut up $I \times I$ into little squares, so that the image of each square lies entirely within some A_α. We can perturb the squares slightly so that each point touches at most three of the squares:



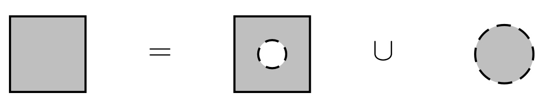
For each vertex v, choose a path g_v from $F(v)$ to x_0. We may choose the path g_v to lie entirely in the three open sets containing the images of the squares adjacent to v. Number the little squares as shown, and let γ_i be the path which has little squares 1 through i below it, so γ_0 is the bottom edge, with $F(\gamma_0) = f_1 \cdots f_n$. Notice that we get a factorization of γ_i as an element of $*_\alpha \pi_1(A_\alpha)$ by looking at the images of the horizontal and vertical edges and concatenating with the g_v. But it is better than that: the image of each edge lies in two of the A_α! That is, we get two versions of each edge, related by operation 2 above.

Start with γ_0, written as some element of $*\pi_1(A_\alpha)$. Apply a homotopy to the first factor that is entirely in A_1 (the open set containing the image of the square labeled 1). Now think of the new edges as living in the adjacent A_α (the is move 2), and repeat until you've reached the constant map. $\square$

Example 7.2. Let $n > 1$. In S^n, let U be a neighborhood of the closed northern hemisphere ($U \approx \mathring{D}^n$), let V be a neighborhood of the closed southern hemisphere ($V \approx \mathring{D}^n$), and let the base point be on the equator. Then $U \cap V$ is path connected (not true if $n = 1$), so by Van Kampen's Theorem, $\pi_1(\mathring{D}^n) * \pi_1(\mathring{D}^n) \cong \{e\}$ surjects onto $\pi_1(S^n)$, so $\pi_1(S^n)$ is trivial. $\bullet$

Example 7.3. If $X = \bigvee_\alpha X_\alpha$, where each X_α is path connected and for each α, $x_\alpha \in X_\alpha$ is a deformation retract of a neighborhood. Then Van Kampen's Theorem applies to tell us that $\pi_1(X) \cong *_\alpha \pi_1(X_\alpha)$. $\bullet$

Example 7.4. Write the torus as the union of a popped torus and a patch:



The first piece, call it U_1, is homotopic to the wedge of two circle, so $\pi_1(U_1) = \mathbb{Z} * \mathbb{Z}$, generated by a and b. The second, call it U_2, is contractible, so $\pi_1(U_2) = 1$. The intersection is homotopic to a circle, $\pi_1(U_1 \cap U_2) = \mathbb{Z}$. The generator of this $\mathbb{Z}$ is sent to 1 is $\pi_1(U_2)$ (of course), and to $aba^{-1}b^{-1}$ in $\pi_1(U_1)$. Thus, $\pi_1(\mathbb{T}^2) = \mathbb{Z} * \mathbb{Z}/(aba^{-1}b^{-1} = 1) \cong \mathbb{Z} \oplus \mathbb{Z}$. $\bullet$

We will see later that given any finitely presented group $G = \langle g_1, \ldots, g_n | r_1, \ldots, r_m \rangle$, one can come up with a topological space (in fact, a CW 2-complex) with the desired group as a fundamental group.

8 Characterization of CW complexes

We mention the following lemma.

Lemma 8.1. *If a finite group G acts freely on a manifold M^n, then M^n/G is a manifold of dimension n.*

Lemma 8.2. *If X is a CW complex and $C \subseteq X$ is compact, then C lies in a finite subcomplex.*

Proof. For each cell intersecting C, choose a point in the intersection. Let S be the union of these points. Since the points are in distinct cells, S is discrete, so closed. Since $S \subseteq C$ must be compact, S must be finite.

Now it remains to show that the closure of each cell lies in a finite subcomplex. The image of the attaching map of an n-cell is a compact and contained in $X^{(n-1)}$, so by induction on the dimension, it lies in a finite subcomplex. Clearly, any 0-cell *is* a finite subcomplex. □

Theorem 8.3 (Characterization of CW complexes)**.** *X is a CW complex if and only if*

0. X is Hausdorff.

“C” $\begin{cases} \text{1. There are } \phi_\alpha : D^n \to X \text{ such that } \phi_\alpha : \mathring{D}^n \xrightarrow{\approx} \phi_\alpha(\mathring{D}^n) =: e^n_\alpha, \text{ with } X = \coprod_{n,\alpha} e^n_\alpha. \\ \text{2. } \phi_\alpha(\partial D^n) \text{ lies in a finite union of lower-dimensional cells. (“closure finite”)} \end{cases}$

“W” $\{$ *3. X has the weak topology with respect to the ϕ_α.*

Proof. ($\Rightarrow$) We’ve checked 0, 1, and 3 already. 2 follows from Lemma 8.2.

($\Leftarrow$) In Hatcher. □

Definition 8.4. $A \subseteq X$ is a *retract* if there is a *retraction* $r : X \to A$, a map so that $r|_A = \mathrm{Id}_A$.

Definition 8.5. $A \subseteq X$ is a *deformation retraction* if there is a *deformation retraction* $r : X \to A$, a retraction such that $i_{A \hookrightarrow X} \circ r \simeq \mathrm{Id}_X$. In particular, $A \simeq X$.

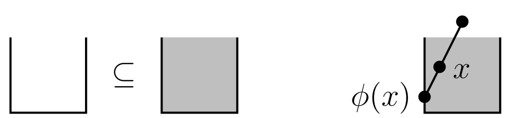
Definition 8.6. $A \subseteq X$ is a *strong deformation retraction* if there is a *strong deformation retraction* $r : X \to A$, a deformation retraction so that $i \circ r \simeq_A \mathrm{Id}_X$. In particular $X \simeq_A A$.

Example 8.7. A semi-circle in a circle is a retract, but not a deformation retract. •

9 The Homotopy Lemma

Lemma 9.1. *$S^{n-1} \times I \cup D^n \times \{0\}$ is a strong deformation retract of $D^n \times I$.*

Proof.



□

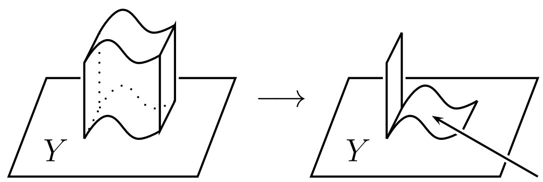
More generally,

Lemma 9.2. *If (Z, B) is a CW pair, then $B \times I \cup Z \times \{0\}$ is a strong deformation retract of $Z \times I$.*

Proof. Apply the previous lemma repeatedly. Induct on dimension. [[★★★ how about if Z is not finite dimensional]] □

Lemma 9.3 (Homotopy Lemma)**.** *If (Z, B) is a CW pair, and if $f_0, f_1 : B \to Y$ are homotopic, then $Y \cup_{f_0} Z$ is homotopic to $Y \cup_{f_1} Z$.*

Proof. Let $F : B \times I \to Y$ be a homotopy $f_0 \simeq f_1$. We wish to show that $Y \cup_{f_1} Z \subseteq Y \cup_F (Z \times I)$ is a strong deformation retract for $i = 0, 1$. By the previous lemma, $B \times I \cup Z \times \{i\}$ is a strong deformation retract of $Z \times I$. This induces a strong deformation retract of $Y \cup_F (Z \times I)$:

$Y \cup_F (Z \times I) =$ 

$= Y \cup_{f_i} Z$

$F(B \times I)$ □

In particular, if B is a space, and $\alpha \in \pi_{n-1}(B, b_0)$, then $B \cup_\alpha D^n$ is well defined up to homotopy equivalence.

Definition 9.4. A pair (X, A) has the *homotopy extension property* (*HEP*) if a homotopy of A, defined at time zero on X, extends to all of X:

$$\begin{tikzcd} (X \times \{0\}) \cup_{A \times \{0\}} (A \times I) \arrow[r, "h"] \arrow[d, hook] & Y \\ X \times I \arrow[ur, dotted, "\tilde{h}"'] & \end{tikzcd}$$

For example, any CW pair has HEP by Lemma 9.2.

Lemma 9.5. *(X, A) has HEP if and only if $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.*

Proof. ($\Rightarrow$) Let $Y = X \times \{0\} \cup A \times I$ with $h = \mathrm{Id}$. Then $\tilde{h}$ is a retract.

($\Leftarrow$) If r is a retract, set $\tilde{h} = h \circ r$. □

Lemma 9.6. *If (X, A) has HEP, then $A \subseteq X$ is closed.*

Some applications:

1. If (X, A) has HEP and the inclusion $A \hookrightarrow X$ is a homotopy equivalence, then A is a deformation retract of X.

2. If (X, A) has HEP and A is contractible, then the canonical map $X \to X/A$ is a homotopy equivalence. To see this, let $h : A \times I \to A$ be a contraction of A, so $h_0 = \mathrm{Id}_A$ and $h_1(A)$ is a point, then extend h_0 to Id_X. By HEP, there is a homotopy $\tilde{h} : X \times I \to X$ so that $h_0 = \mathrm{Id}_X$ and $h_1(A)$ is a point. Then h_1 induces a map $\phi : X/A \to X$, and h induces a homotopy between $\mathrm{Id}_{X/A}$ and the composition $X/A \xrightarrow{\phi} X \to X/A$. Also, h is itself a homotopy between Id_X and the composition $X \to X/A \xrightarrow{\phi} X$.

Example 9.7. In a connected CW complex X, one can find a maximal spanning tree $T \subseteq X$, a contractible subcomplex of X^1. Then application 2 above says that T can be crushed to a point without changing the homotopy type of X. Moreover, it is easy to check that X/T is a CW complex with a single 0-cell.

Later we will see that for a CW complex X, if $\pi_k(X) = 0$, $k > 1$, then X is homotopic to a CW complex with no k-cells. [[★★★ is this right?]] •

10 Some Theorems

Chris lectures.

We will assume we know

- $\pi_i(D^n) = 0$ for all i since D^n is contractible.
- $\pi_1(S^n) = 0$ for $n > 1$.
- $\pi_1(S^1) = \mathbb{Z}$, generated by $\mathrm{Id} : S^1 \to S^1$.
- $\pi_i(S^n) = 0$ for $i < n$.
- $\pi_n(S^n) = \mathbb{Z}$, generated by $\mathrm{Id} : S^n \to S^n$.
- If $f : S^n \to S^n$ satisfies $f(-x) = -f(x)$, then $[f] \neq 0 \in \pi_n(S^n)$; in fact, $[f]$ must represent an odd integer.

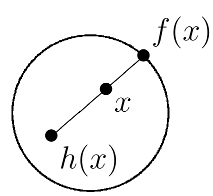
Then we can prove some neat things.

Theorem 10.1 (Fundamental Theorem of Algebra)**.** *Every non-constant $p \in \mathbb{C}[x]$ has a root in $\mathbb{C}$.*

Proof. Assume not. Say p is of degree n. Note that $p_R = p\big(\{z | \|z\| = R\}\big)$ is a loop in $\mathbb{C} \smallsetminus \{0\} \simeq S^1$, so it defines an element of $\pi_1(\mathbb{Z})$.[2] For R large, it is easy to see that it defines the same element as does z^n, which is $n \in \mathbb{Z}$. Slowly shrinking R to zero, we get a homotopy of p_R to p_0, which is a constant map, corresponding to $0 \in \mathbb{Z}$. Thus, $n = 0$, contradicting the assumption that p is non-constant. □

Theorem 10.2 (Brauwer Fixed Point Theorem)**.** *Every $h : D^n \to D^n$ has a fixed point.*

Proof. Assume not, then define $f : D^n \to S^{n-1}$ by the picture



[2]Since we didn't choose a base point, it only defines a conjugacy class, but $\mathbb{Z}$ is abelian, so we get an element.

It is easy to see that f is continuous and that $f|_{S^n} = \mathrm{Id}_{S^n}$. Then we get the following commutative triangle, and its image under π_{n-1}.

$$\begin{tikzcd} S^{n-1} \arrow[r, hook] \arrow[rr, bend right, "\mathrm{Id}_{S^{n-1}}"'] & D^n \arrow[r, "f"] & S^{n-1} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[rr, bend right, "\mathrm{Id}_{\mathbb{Z}}"'] & 0 \arrow[r, "f_*"] & \mathbb{Z} \end{tikzcd}$$

Which is impossible. $\square$

Theorem 10.3 (Invariance of Dimension)**.** $\mathbb{R}^n \approx \mathbb{R}^m \iff n = m$.

Proof. ($\Leftarrow$) is obvious. Let's prove ($\Rightarrow$). Assume $f : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^m$, then we get a homeomorphism $f : \mathbb{R}^n \smallsetminus \{0\} \xrightarrow{\sim} \mathbb{R}^m \smallsetminus \{f(0)\}$. The first is homotopy equivalent to S^n and the second to S^m, so it follows that $S^n \simeq S^m$. We may assume $n \leq m$. Then $\mathbb{Z} = \pi_n(S^n) \cong \pi_n(S^m)$, which is zero if $n < m$. So $n = m$. $\square$

Theorem 10.4 (Borsuk-Ulam)**.** *For any $f : S^n \to \mathbb{R}^n$, there is some $x \in S^n$ so that $f(x) = -f(-x)$.*

Proof. If not, define $g : S^n \to S^{n-1}$ by $g(x) = \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}$. Note that $g(-x) = -g(x)$, so composing with the inclusion $i : S^{n-1} \hookrightarrow S^n$, we get a non-trivial element of $\pi_n(S^n) = \mathbb{Z}$. But if $[i \circ g] \neq 0$, then the image $i \circ g(S^n)$ is not contractible in S^n, so the image $i(S^{n-1})$ is not contractible in S^n. Contradiction. $\square$

Note that we don't really use much about what π_1 *is* in these proofs, we just use some functorial properties. So anything else that behaves kind of like π_1 would work just as well. Later we will define homology and cohomology groups, which serve exactly this purpose.

11 Boring class

Peter was sick, but still in class, and asked people to volunteer information about

– categories and functors

– exact squences

– cofibrations and Puppe sequences

– covering spaces, fiber bundles, and fibrations

We only talked about the first two.

12 More Category stuff, Cofibrations, Fibrations

Definition 12.1. An sequence of pointed sets

$$(A,a) \xrightarrow{f} (B,b) \xrightarrow{g} (C,c)$$

is said to be *exact at B* if $\operatorname{im} f = g^{-1}(c)$.

Note that if we take these to be groups (with the identity element), we get the usual notion of exactness of a sequence of groups.

Definition 12.2. A *natural transformation* $\eta : F \to G$ between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is a morphism $\eta(X)$ for each object X in C so that for every $f : X \to Y$, we have

$$\begin{tikzcd}
F(X) \arrow[r, "\eta(X)"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\circlearrowleft"] & G(X) \arrow[d, "Gf"] \\
F(Y) \arrow[r, "\eta(Y)"] & G(Y)
\end{tikzcd}$$

Example 12.3. For any object X in a category $\mathcal{C}$, we get a functor $h^X : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ given by $h^X(Y) = \operatorname{Hom}(Y, X)$. Similarly, we get a functor $h_X : \mathcal{C} \to \mathbf{Set}$ given by $h_X(Y) = \operatorname{Hom}(X, Y)$. We call a functor *representable* if it is isomorphic to h^X or h_X for some X. •

Theorem 12.4 (Yoneda's Lemma)**.** *For any functor $F : \mathcal{C}^{op} \to \mathbf{Set}$, there is a natural bijection $\operatorname{Nat}(h^X, F) \cong F(X)$. In particular, taking $F = h^Y$, we see that the functor $h^- : \mathcal{C} \to \operatorname{Fun}(\mathcal{C}^{op}, \mathbf{Set})$ is a fully faithful embedding of categories.*

Similarly, we get a fully faithful embedding $h_- : \mathcal{C}^{\text{op}} \to \operatorname{Fun}(\mathcal{C}, \mathbf{Set})$. This is the Yoneda embedding of $\mathcal{C}^{\text{op}}$.

Proof. Given $\eta \in \operatorname{Nat}(h^X, F)$, we have $\eta(X) : \operatorname{Hom}(X, X) \to F(X)$, so we get an element $a = \eta(X)(\mathrm{Id}_X) \in F(X)$. Conversely, given $a \in F(X)$, we construct a natural transformation η which takes $f \in h^X(Y) = \operatorname{Hom}(Y, X)$ to $\eta(Y)(f) = (Ff)(a)$. Check that these are inverses, and that the bijection is natural in F and X. The following diagram should help:

$$\begin{tikzcd}
\mathrm{Id}_X \arrow[ddd, mapsto] & & & a \arrow[ddd, mapsto] \\
& \operatorname{Hom}(X,X) \arrow[d, "-\circ f"'] & F(X) \arrow[d, "Ff"] & \\
& \operatorname{Hom}(Y,X) \arrow[r, "\eta(Y)"] & F(Y) & \\
f \arrow[rrr, mapsto] & & & (Ff)(a)
\end{tikzcd}$$

□

Note that representable functors sometimes factor through some other category on their way to **Set**, i.e. there is a natural group structure on $h^X(Y)$ in such a way that the maps $h^X(Z) \to h^X(Y)$ induced by $Y \to X$ are group homomorphisms.

Definition 12.5. If $\mathcal{C}$ has products, then an object K is a *group object* in $\mathcal{C}$ if there is a maps $m : K \times K \to K$ such that there exist $i : K \to K$ and $1 : * \to K$, where $*$ is the final object of $\mathcal{C}$ (the empty product) satisfying the following diagrams.

$$\begin{tikzcd} * \times K \arrow[r, phantom, "\cong"] \arrow[d, "1\times\mathrm{Id}"'] & K \arrow[r, phantom, "\cong"] \arrow[d, "\mathrm{Id}"] & K \times * \arrow[d, "\mathrm{Id}\times 1"] \\ K \times K \arrow[r, "m"] & K & K \times K \arrow[l, "m"'] \end{tikzcd}
\qquad
\begin{tikzcd} K \times K \arrow[d, "m"'] & K \arrow[l, "i\times\mathrm{Id}"'] \arrow[r, "\mathrm{Id}\times i"] \arrow[d, "\exists!"] & K \times K \arrow[d, "m"] \\ K & * \arrow[l, "1"'] \arrow[r, "1"] & K \end{tikzcd}
\qquad
\begin{tikzcd} K \times K \times K \arrow[r, "m\times\mathrm{Id}"] \arrow[d, "\mathrm{Id}\times m"'] & K \times K \arrow[d, "m"] \\ K \times K \arrow[r, "m"] & K \end{tikzcd}$$

If we change all the products to coproducts (in particular, change the final object to the initial object) and reverse the arrows, we have the definition of a *cogroup object.*

Observe that once i and 1 exist, they are unique. Note that if K is a group (resp. cogroup) object, then h_K (resp. h^K) factors trough $\mathbf{Gp}$.[3] If the multiplication map m is invariant under the "switch factors" map, then we says that K is an *abelian* (co)group object. In this case, the representable functors factor through $\mathbf{Ab}$.

Definition 12.6. An H-group is a group object in $\mathbf{hTop}$. Note that the identity element, inverses, and associativity of the product only work up to homotopy!

Example 12.7. $(S^n, *)$ is an H-cogroup for $n \geq 1$, with comultiplication given by the "crush the equator" map $S^n \to S^n \vee S^n$ and inverse given by "reflect through the equator" map $S^n \to S^n$. If $n \geq 2$, then it is an abelian H-cogroup.

The representable functor $h_{(S^n,*)}$ is $\pi_n : \mathbf{hTop}_{pt} \to \mathcal{C}$ is , where $\mathcal{C} = \mathbf{Set}$, $\mathbf{Gp}$, or $\mathbf{Ab}$ when $n = 0$, 1, or $n \geq 2$, respectively. •

Example 12.8. Later we'll se that for an abelian group A, there are abelian H-groups $K(A, n)$, called *Eilenberg-Maclane* spaces, defined by $\pi_m\big(K(A, n)\big) = \delta_{m,n}A$. We will define *cohomology functors* $H^n(X, A)$ which will turn out to be equal to $h^{K(A,n)} : \mathbf{hTop}_{pt} \to \mathbf{Ab}$. •

Definition 12.9. A map $i : A \to X$ is a *cofibration* if it has the homotopy extension property, i.e. the diagram on the left for all Y, h, and $\tilde{h}_0$. This means that any homotopy of maps from A can be extended to a homotopy of maps from X, given a starting point.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"'] & Y^I \arrow[d, "0\ \text{endpoint}"] \\ X \arrow[r, "\tilde{h}_0"'] \arrow[ur, dashed, "\exists\tilde{h}"] & Y \end{tikzcd}
\qquad\qquad
\begin{tikzcd} Y \arrow[r, "\tilde{h}_0"] \arrow[d, "\mathrm{Id}\times\{0\}"'] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "h"'] \arrow[ur, dashed, "\exists\tilde{h}"] & B \end{tikzcd}$$

A map $p : E \to B$ is a *fibration* if it satisfies the *homotopy lifting property* (HLP), i.e. the diagram on the right for all Y, h, and $\tilde{h}_0$. This means that any homotopy of maps

[3]In fact, h^K factors through $\mathbf{Gp}$ exactly when it is a group object in $\mathrm{Fun}(\mathcal{C}, \mathbf{Set})$, so by Yoneda's Lemma, h^K factors through $\mathbf{Gp}$ if and only if K is a group object. Similarly, h_K factors through $\mathbf{Gp}$ if and only if K is a cogroup object.

to B can be lifted to a homotopy of maps to E, given a starting point. We say that $p : E \to B$ is a *Serre fibration* if it satisfies HLP when $Y = D^n$ (or, equivalently, when Y is a CW-complex).

If B is a pointed space with base point b, and $E \xrightarrow{p} B$ is a fibration, then we define the fiber $F := p^{-1}(b)$. It turns out that a cofibration is always an inclusion of a closed subset.

Lemma 12.10. *If $i : A \to X$ is a cofibration, then for each pointed space (Y, y) we get an exact sequence of sets*

$$[X/A, Y] \to [X, Y] \to [A, Y].$$

Proof. It is clear that the composition of maps always lands in the base point of $[A, Y]$. If $f : X \to Y$ is a map such that $f|_A$ homotopic to the constant map $A \to y$. Then we have a homotopy of maps $f|_A \simeq \mathrm{const}_y$ from A. Since i is a cofibration, we can extend to a homotopy $f \simeq g$, where $g : X \to Y$ and $g|_A = \mathrm{const}_y$, so g induces a map $X/A \to Y$. This proves exactness. $\square$

Lemma 12.11. *If $p : E \to B$ is a fibration with fiber F, then for each Y, we get an exact sequence of sets*

$$[Y, F] \to [Y, E] \to [Y, B].$$

Proof. It is clear that the composition of maps always lands in the base point of $[Y, B]$. If $f : Y \to E$ is a map such that $p \circ f \simeq \mathrm{const}_b$, then we can lift to a homotopy $f \simeq g : Y \to E$, where $p \circ g = \mathrm{const}_b$. That is, the image of g is in F. This proves exactness. $\square$

Lemma 12.12. *A pushout of a cofibration is a cofibration. A pull back of a (Serre) fibration is a (Serre) fibration.*

Proof. The curved dashed arrows exist because $A \to X$ is a cofibration and $E \to B$ is a (Serre) fibration (with Y CW, in the Serre fibration case).

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & Y^I \arrow[d] \\ X \arrow[r] \arrow[urr, dashed, bend left] \arrow[ur, phantom, "\lrcorner", near end] & X' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", near start] & E \arrow[d] \\ Y \times I \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed, bend right] & B' \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

The straight dashed arrows exist by the universal properties of pull-back and push-out. $\square$

13 Fiber bundles and Covering spaces

Definition 13.1. A *fiber bundle* is a surjective map $p : E \to B$ such that every $b \in B$ has an open neighborhood U and a homeomorphism $f : U \times p^{-1}(b) \to p^{-1}(U)$ with $p \circ f$ equal to the projection onto the first coordinate.

$$\begin{tikzcd} U \times p^{-1}(b) \arrow[rr, "\approx", "f"'] \arrow[dr, "p_1"'] & & p^{-1}(U) \arrow[dl, "p"] \\ & U & \end{tikzcd}$$

Note that $\begin{tikzcd}[column sep=small] E' \arrow[rr, "f"] \arrow[dr, "p'"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$ commutes if and only if $f\big((p')^{-1}(b)\big) \subseteq p^{-1}(b)$.

Lemma 13.2. *If B is connected, then all the fibers (inverse images of points) are homeomorphic.*

Proof. Let $b_0 \in B$, then $\{b \in B | F_b \approx F_{b_0}\}$ and $\{b \in B | F_b \not\approx F_{b_0}\}$ are disjoint open sets covering B, and the first is non-empty, so it must be all of B. $\square$

Definition 13.3. A fiber bundle is called a *covering* if the fibers are discrete.

Definition 13.4. A *vector bundle* is a fiber bundle where each fiber comes with a vector space structure, and the local trivialization maps $f : U \times p^{-1}(b) \xrightarrow{\approx} p^{-1}(U)$ are required to be linear on fibers.

For example, if M is a smooth manifold, then the tangent bundle TM is a vector bundle.

Definition 13.5. If G is a topological group, then a *principal G-bundle* is a fiber bundle where each fiber comes with a G-action, and we have local trivialization maps $f : U \times G \xrightarrow{\approx} p^{-1}(U)$ which are G-equivariant on fibers.

Example 13.6. $\mathbb{Z} \to \mathbb{R} \to S^1$ is a principal $\mathbb{Z}$-bundle and a covering. $\mathbb{Z}/n\mathbb{Z} \hookrightarrow S^1 \to S^1$ is principal $\mathbb{Z}/n\mathbb{Z}$-bundle and a covering. $\bullet$

14 Fiber bundles are Serre fibrations

Lemma 14.1. $\big(I^n \times I, I^n \times \{0\}\big) \approx \big(I^n \times I, I^n \times \{0\} \cup \partial I^n \times I\big)$. *That is, there is a homeomorphism $I^n \times I \to I^n \times I$ sending $I^n \times \{0\}$ to $I^n \times \{0\} \cup \partial I^n \times I$.* $\square \approx \square$

Theorem 14.2. *If $p : E \to B$ is a fiber bundle and (X, A) is a relative CW-complex,*[4] *Then*

$$\begin{tikzcd} X \times \{0\} \cup A \times I \arrow[r] \arrow[d, phantom, "\cap|"] & E \arrow[d, "p"] \\ X \times I \arrow[r] \arrow[ur, dashed, "\exists"'] & B \end{tikzcd}$$

In particular, any fiber bundle is a Serre fibration. Moreover, if $p : E \to B$ is a covering, then the dashed arrow is unique.

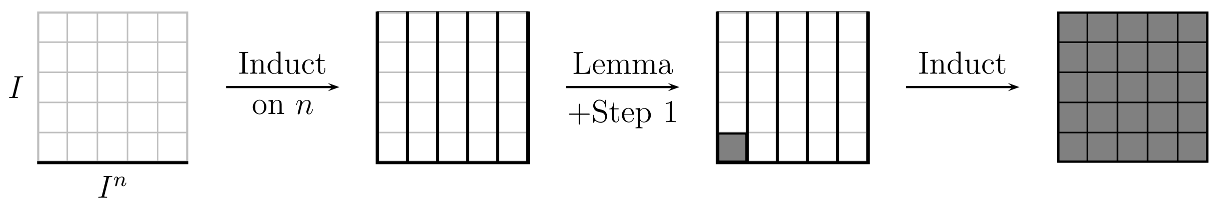
[4] This means that A is an arbitrary topological space, and X is obtained from A by attaching cells of increasing dimensions.

Proof. First we do the non-relative case (i.e. the case $A = \varnothing$).

Step 1: Trivial bundles: If $E = B \times F$, then use
$$\begin{tikzcd} X \times \{0\} \arrow[r, "{(f,g)}"] \arrow[d, phantom, "\cap"] & B \times F \arrow[d, "p_1"] \\ X \times I \arrow[r, "h"] \arrow[ur, dashed, "{(h, g\circ p_1)}"] & B \end{tikzcd}$$
where p_1 always means "project to the first coordinate". Note that if F is discrete, then $F^{X \times I} \approx (F^I)^X = F^X$, so $g \circ p_1$ is the unique map $X \times I \to F$ compatible with $g : X \times \{0\} \to F$.

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \varnothing \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

Step 2: $X = I^n \approx D^n$: We do this by induction on n. If $n = -1$, then $I^n = \varnothing$, so the statement is vacuous (see margin). Now assume we can lift $I^{n-1} \times I$, given the lift on $I^{n-1} \times \{0\}$ (and the lift is unique if F is discrete). Let $\{U_i\}$ be a trivializing open cover of B, and let ε be the Lebesgue number of the pull-back cover under the map $I^n \times I \to B$, so that the image of any little cube of side length ε lies entirely in one of the U_i. Now cut $I^n \times I$ into cubes of side length ε and look at the following picture; the black and dark gray indicates part of the domain where we have lifted $I^n \times I \to B$.



By induction on n, we can lift on a grid of $I^{n-1} \times I$, cutting $I^n \times I$ into narrow columns of width ε. By Lemma 14.1, lifting the ε-size $I^n \times I$ given the lift on $I^n \times \{0\} \cup \partial I^n \times I$ is the same as lifting $I^n \times I$ given the lift on $I^n \times \{0\}$, which we know how to do by Step 1 because the bundle is trivial over each U_i. If F is discrete, then all the intermediate lifts are unique, so the lift is unique.

Step 3: X a CW complex: If X is an arbitrary CW complex, assume by induction that we've lifted $h : X \times I \to B$ to $\tilde{h} : X^{(n-1)} \times I \to E$.

$$\begin{tikzcd} \coprod S^n \times I \cup D^n \times \{0\} \arrow[r] \arrow[d] & E \arrow[d] \\ \coprod D^n \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}
\qquad
\begin{tikzcd} \coprod S^n \times I \arrow[r] \arrow[d] & X^{(n-1)} \times I \arrow[d] \arrow[dr, bend left, "\text{already have}"] & \\ \coprod D^n \times I \arrow[r] \arrow[rr, bend right, "\text{given by Step 2}"'] & X^{(n)} \times I \arrow[r, dashed, "\exists!"] & E \end{tikzcd}$$

We use Step 2 and Lemma 14.1 (keeping in mind that $D^n \approx I^n$) to get the dashed arrow in the left diagram, which is the bottom curved arrow in the right diagram. By the universal property of push-out, we get a lift $X^{(n)} \times I \to E$. Now we use the usual limiting argument for CW complexes to get a lift $X \times I \to E$. As usual, it is clear that uniqueness holds if F is discrete.

Note that in Step 3, we didn't use anything about $X^{(n-1)}$; it could have been completely general, so the proof applies to the relative case. $\square$

Corollary 14.3. *Fiber bundles have the long exact sequence of homotopy groups described in Theorem 15.6.*

Corollary 14.4 (to the Corollary)**.** *If F is discrete, then $p_* : \pi_n(E,e) \xrightarrow{\sim} \pi_n(B,b)$ for $n > 1$. Moreover, if E is connected, then $\pi_1(E) \hookrightarrow \pi_1(B)$ has index $|F|$.*

Note that if F is discrete and E is connected, then the connecting map $\delta : \pi_1(B,b) \to \pi_0(F,e) = F$ is surjective, yielding a transitive action of $\pi_1(B,b)$ on F.

15 Long cofibration and fibration sequences

We denote the suspension of X by $\Sigma X = X \times I/\sim$, with $(x,1) \sim (x',1)$ and $(x,0) \sim (x',0)$. We denote the reduced suspension by $SX = S^1 \wedge X$; this is the suspension, with the additional identification $(x_0, s) \sim (x_0, t)$ for some base point $x_0 \in X$. Note that $\Sigma X \simeq SX$.

Lemma 15.1. *If X and Y are compact, then $X \wedge Y$ is the one point compactification of $(X \smallsetminus \{x\}) \times (Y \smallsetminus \{y\})$.*

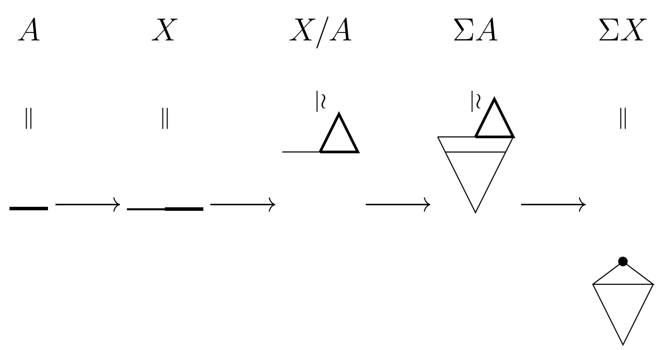
In particular $S^n \wedge S^m \approx S^{n+m}$ because $\mathbb{R}^n \times \mathbb{R}^m \approx \mathbb{R}^{n+m}$.

Lemma 15.2. *If $i : A \to X$ is a cofibration, then there is a long cofibration sequence (with maps defined up to homotopy)*

$$A \xrightarrow{i} X \to X/A \to \Sigma A \xrightarrow{\Sigma i} \Sigma X \to \Sigma(X/A) \to \Sigma^2 A \xrightarrow{\Sigma^2 i} \cdots .$$

That is, any three consecutive terms are homotopic *to a cofibration triple.*

Proof. We know that $A \to X \to X/A$ is a cofibration. We have $X/A \simeq X \cup_A C(A)$, where $C(A)$ is the cone on A. Then it is clear that $X \to X \cup_A C(A) \to \Sigma A$ is a cofibration. Similarly, $\Sigma A \simeq \big(X \cup_A C(A)\big) \cup_X C(X)$, and $\big(X \cup_A C(A)\big) \to \big(X \cup_A C(A)\big) \cup_X C(X) \to \Sigma X$ is a cofibration. Finally, Σ is a functor which preserves cofibrations, so we're done.



$\square$

Corollary 15.3. *For any pointed space Y, by Lemma 12.10, we have a long exact sequence*

$$\cdots \to [S(X/A), Y]_{pt} \to [SX, Y]_{pt} \to [SA, Y]_{pt} \to [X/A, Y]_{pt} \to [X, Y]_{pt} \to [A, Y]_{pt}$$

Remark 15.4. This sequence is called a *Puppe sequence*. If we take $Y = K(A, n)$, then we get the long exact sequence in cohomology for the pair (X, A). To see this, note that $\pi_{i+1}(X) = [S^{i+1}, X] = [S^i, \Omega X] = \pi_i(\Omega X)$. In particular, $\Omega K(A, n) = K(A, n-1)$, so $[S^k X, K(A, n)] = [X, \Omega^k K(A, n)] = [X, K(A, n-k)] = H^{n-k}(X, A)$.[[★★★ clean]]

"Dually", we get

Lemma 15.5. *If $F \to E \to B$ is a fibration, then there is a long fibration sequence*

$$\cdots \to \Omega^3 B \to \Omega^2 F \to \Omega^2 E \to \Omega^2 B \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B$$

That is, any three consecutive terms are homotopic *to a fibration triple.*

Proof. [[★★★ do]] □

Corollary 15.6. *For any pointed space Y, by Lemma 12.11, we get the long exact sequence*

$$\cdots \to [Y, \Omega^2 E] \to [Y, \Omega^2 B] \to [Y, \Omega F] \to [Y, \Omega E] \to [Y, \Omega B] \to [Y, F \to E] \to [Y, B]$$

In particular, taking $Y = S^0$, and noting that $[S^0, \Omega^n X] = [S^n, X]$, we get the long exact sequence in homotopy groups.[[★★★ clean?]]

16 Some equivalences of categories

We'll prove the following theorem later. $\mathbf{hCW}(n)$ is the category of homotopy classes of CW complexes X with $\pi_k(X) = 0$ for $k \neq n$.

Theorem 16.1. *π_n is an equivalence of categories $\mathbf{hCW}(n) \to \mathbf{Ab}$, with inverse $K(-, n)$.*

In particular, $K(A, n)$ is well-defined up to homotopy is always an H-group.

Definition 16.2. A functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence of categories* if there is a functor $G : \mathcal{D} \to \mathcal{C}$ so that $F \circ G$ (resp. $G \circ F$) is naturally isomorphic to $\mathrm{Id}_{\mathcal{D}}$ (resp. $\mathrm{Id}_{\mathcal{C}}$).

Lemma 16.3. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is essentially surjective (the image contains an element from each isomorphism class in $\mathcal{D}$) and for every pair of objects c_0 and c_1 of $\mathcal{C}$, F induces a bijection* $\mathrm{Hom}_{\mathcal{C}}(c_0, c_1) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(Fc_0, Fc_1)$.

Proof. [[★★★]] □

Note that this lemma constructs the inverse functor.

Lemma 16.4. *If $F \to E \to B$ is a covering, then $\pi_1(B)$ acts on F.*

Proof. [[★★★ follows from the LES]] □

Lemma 16.5. *If B is path connected, locally path connected, and semi-locally simply connected,*[5] *then B has a universal covering space.*

Not a Proof. Define $\tilde{E}$ to be the set of paths in B starting at b modulo homotopy rel endpoints. Then we have a map $\tilde{E} \to B$, sending a path to its other endpoint.

Even if B satisfies the conditions of the Lemma, this is not always a covering. Sometimes you have to change the topology on $\tilde{E}$ to make it the universal cover. If B is metrizable, then $\tilde{E}$ does the trick as is. $\square$

Let $\mathbf{Cov}(B)$ be the category of covering spaces of B, and let G-**Set** be the category of sets with a G action (and morphisms are equivariant set maps).

Theorem 16.6. *If B is path connected, locally path connected, and semi-locally simply connected, then there is an equivalence of categories $\mathbf{Cov}(B) \to \pi_1(B)$-$\mathbf{Set}$, given by $E \mapsto F$.*

"Proof". The inverse functor is given by $F \mapsto \tilde{E} \times_{\pi_1(B)} F$. $\square$

Remark 16.7. Any bundle $E \to B$ with fiber F is isomorphic to $P \times_G F$ for some group $G \subseteq \mathrm{Homeo}(F)$ and some principal G bundle $P \to B$. Thus, if you understand principal bundles, you understand all bundles.

Corollary 16.8. *If B is as above, then E is connected if and only if the $\pi_1(B)$ action on F is transitive.*

Proof. The connected coverings of B are exactly the ones which cannot be written as coproducts (disjoint unions) of others. The $\pi_1(B)$-sets that cannot be written as coproducts (disjoint unions) of others are exactly those for which the action is transitive. $\square$

Note that if E is connected, then the action of $\pi_1(B)$ on F is transitive, so $F \cong \pi_1(B)/U$, where U is the subgroup of $\pi_1(B)$ that stabilizes $e_0 \in F$.

$\rho : G \to GL(V)$ a representation, and $P \to B$ a principal G-bundle, then $P \times_G V \to B$ is a vector bundle! [[★★★]]

Corollary 16.9. *$E \to B$ has a section if and only if F has a $\pi_1(B)$-fixed point. In fact, there is a bijection between sections and fixed points.*

Proof. A section is a bundle map from the trivial bundle $B \xrightarrow{\mathrm{Id}} B$ to $E \to B$. Under the equivalence of categories, this is a $\pi_1(B)$-equivariant map from the one point $\pi_1(B)$-set to F, i.e. a fixed point. $\square$

Corollary 16.10. *If $\pi_1(B)$ is trivial, then all covers of B are trivial.*

Corollary 16.11. *Isomorphism classes of connected covers of B are in bijection with subgroups of $\pi_1(B)$ up to conjugation.*

[5] For every point $x \in B$ and every neighborhood V of x, there is a neighborhood $U \subseteq V$ of x so that any loop in U is homotopic to a constant loop *in* B (the homotopy may go outside of U). A connected CW complex will have all these connectivity conditions.

Proof. The first is $\pi_0\big(\mathbf{Cov}_{\text{conn}}(B)\big)$, and the second is $\pi_0\big(\pi_1(B)\text{-}\mathbf{Set}_{\text{trans}}\big)$. These sets are in bijection by the equivalence of categories. $\square$

Corollary 16.12. *Isomorphism classes of connected covers of B (with base point) are in bijection with subgroups of $\pi_1(B)$.*

The subgroup corresponding to $E \to B$ is the stabilizer of the base point. This is sometimes called "Galois theory for coverings".

17 Finally Homology

We'll define simplicial homology on Δ-sets, then singular homology, which will work for all topological spaces, but will be monstrous. Then we'll define cellular homology, which is the awesomest homology theory for calculating (but only for CW complexes). We won't cover Čech cohomology or its generalization, sheaf cohomology.

A homology theory is usually defined by using a topological space to produce a *chain complex*, a sequence of abelian groups

$$\cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

such that $d_n \circ d_{n+1} = 0$ for all n. We call the d_n *boundary operators.* We define the n-th *homology* of the chain complex to be $\ker d_n / \operatorname{im} d_{n+1}$, and call this the n-th homology of the topological space we started with.

Relation to homotopy

We seek to define functors $H_n : \mathbf{hTop} \to \mathbf{Ab}$ which satisfy some nice properties (the Eilenberg-Steenrod axioms). From these properties, one can prove that $H_n(S^n) \cong \mathbb{Z}$. This allows us to define the *Hurewicz map* $\phi : \pi_n(X) \to H_n(X)$. Given some $f : S^n \to X$, we get an induced map $f_* : H_n(S^n) \cong \mathbb{Z} \to H_n(X)$. We define $\phi([f])$ to be $f_*(1) \in H_n(X)$.

Theorem 17.1 (Hurewicz)**.** *$H_0(X)$ is the free abelian group on $\pi_0(X)$. If X is 0-connected, $H_1(X)$ is the abelianization of $\pi_1(X)$. If X is $(n-1)$-connected with $n \geq 2$, then the Hurewicz map $\pi_n(X) \to H_n(X)$ is an isomorphism.*

In fact, there is a (stronger) relative version.

Theorem 17.2 (Hurewicz)**.** *If $f : X \to Y$ induces isomorphisms on π_i for $i \leq n$, then it induces isomorphisms on H_i for $i \leq n$. If X and Y are 1-connected, then the converse is true.*

18 Simplicial Homology

A *simplicial complex* is a set obtained by gluing together simplices along their faces. An n-dimensional simplex has $n+1$ faces which are subsimplices. We can encode the gluing information in a Δ-*set*.

Definition 18.1. A Δ-*set* is a sequence of sets S_n, with maps $d_i^n : S_n \to S_{n-1}$ for $0 \le i \le n$ that satisfy the relation $d_j \circ d_i = d_{i-1} \circ d_j$ for $j < i$.

From a Δ-set, we can produce a *geometric realization* $|S_\bullet|$ as in the homework.
Some facts:

- Whitehead's Theorem: If X and Y are CW complexes and $f : X \to Y$ is a *weak equivalence* (f induces isomorphisms $\pi_n(X) \to \pi_n(Y)$ for all n), then f is a homotopy equivalence.

- Any CW complex is homotopy equivalent to some $|S_\bullet|$.

- Given *any* topological space X, there is a canonical CW complex X' and a weak homotopy equivalence $X' \to X$.

Given a Δ-set $S_\bullet$, we define a chain complex by setting $C_n = \mathbb{Z} \cdot S_n$, the free abelian group on the "n-simplices", with $d_n = \sum_{i=0}^n (-1)^i d_i^n$, where the maps d_i^n have been extended linearly. It is a standard exercise to check that $d_n \circ d_{n-1} = 0$. We define the *simplicial homology* of $S_\bullet$ (or $|S_\bullet|$) to be the homology of this chain complex.

It is true but not at all obvious that if $|S_\bullet| \approx |T_\bullet|$, then $S_\bullet$ and $T_\bullet$ produce the same homology groups. One can show that if $T_\bullet$ is a refinement of $S_\bullet$, then this holds. For a long time, people conjectured that any two simplicial decompositions of a space, there exists a common refinement. This conjecture, called the Hauptvermutung, would prove the result. However, the Hauptvermutung is false! This was demonstrated by Milnor around 1960.

19 Singular Homology

The standard n-simplex is $\Delta^n = \{\mathbf{x} \in \mathbb{R}^{n+1} | x_i \ge 0 \text{ for all } i \text{ and } \sum x_i = 1\}$.

Definition 19.1. The singular Δ-set $\Delta_\bullet(X)$ of a topological space X has $\Delta_n(X) = \{\text{continuous maps } \sigma : \Delta^n \to X\}$ with the obvious boundary maps.

Definition 19.2. The *singular homology* of a topological space X is the simplicial homology of $\Delta_\bullet(X)$.

Thus, H_n is the composition of functors

$$H_n : \mathbf{Top} \xrightarrow{\Delta_\bullet} \mathbf{\Delta\text{-Set}} \xrightarrow{\text{“free”}} \mathbf{Chain} \xrightarrow{n\text{-th homology}} \mathbf{Ab}.$$

We will denote by $S_*(X)$ the singular chain complex $C_*(\Delta_\bullet(X))$ of X.

Lemma 19.3. $H_0(X) = \mathbb{Z} \cdot \pi_0(X)$.

Proof. $H_0(X) = \operatorname{coker}\big(\mathbb{Z}[\sigma : I \to X] \xrightarrow{d} \mathbb{Z}[pt \to X]\big)$, where $d(\sigma) = \sigma(1) - \sigma(0)$. Define a map $H_0(X) = S_0(X)/\operatorname{im} d \to \mathbb{Z} \cdot \pi_0(X)$ by sending $x \in X$ (viewed as a map $pt \to X$) to the connected component of x. The map is clearly surjective. If there is some linear combination in the kernel, then each path component has as many pluses as minuses, so you can match them up with paths, so that combination is in the image of d. Thus, the map is an isomorphism. $\square$

Example 19.4. If X is a point, then $S_i(X) = \mathbb{Z}$ for each $i \geq 0$ because there is only one map from Δ^i to a point. d^n is an alternating sum of $n+1$ identity maps, so the singular chain complex is

$$\cdots \xrightarrow{0} \overset{S_4}{\mathbb{Z}} \xrightarrow{1} \overset{S_3}{\mathbb{Z}} \xrightarrow{0} \overset{S_2}{\mathbb{Z}} \xrightarrow{1} \overset{S_1}{\mathbb{Z}} \xrightarrow{0} \overset{S_0}{\mathbb{Z}}$$

so $H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$ •

Definition 19.5. An n-dimensional manifold M is *orientable*[6] if $H_n(M) \cong \mathbb{Z}$. In this case, an *orientation* of M is a choice of generator for $H_n(M)$.

Theorem 19.6. *There is a natural adjunction*

$$\operatorname{Hom}_{\mathbf{hTop}}(|S_\bullet|, X) \cong \operatorname{Hom}_{\mathbf{\Delta\text{-}Set}}\big(S_\bullet, \Delta_\bullet(X)\big).$$

The natural map $S_\bullet \to \Delta_\bullet(|S_\bullet|)$ induces isomorphisms on homology, and the natural map $|\Delta_\bullet(X)| \to X$ is a weak equivalence.

It follows that the simplicial homology of $S_\bullet$ is equal to the singular homology of $|S_\bullet|$, a handy fact for calculation. Also, for any space X, we get a CW complex $X' := |\Delta_\bullet(X)|$ and a weak equivalence $X' \to X$.

20 Eilenberg-Steenrod Axioms

It turns out that any sequence of functors satisfying the following axioms are naturally isomorphic to singular homology. In practice, one can use these axioms to compute homology. We will have to prove that singular homology satisfies these axioms to verify that such functors exist.

1. (Homotopy axiom) The functors $H_n : \mathbf{Top} \to \mathbf{Ab}$ factor through $\mathbf{hTop}$.

2. (Mayer-Vietoris axiom) If $U, V \subseteq X$ are open sets, then there is a natural long exact sequence

$$\begin{tikzcd}[column sep=small] & U \cap V \arrow[dl, "j_U"'] \arrow[dr, "j_V"] & \\ U \arrow[dr, "i_U"'] & & V \arrow[dl, "i_V"] \\ & U \cup V & \end{tikzcd}$$

$$\cdots \to H_n(U \cap V) \xrightarrow{(j_U, j_V)} H_n(U) \oplus H_n(V) \xrightarrow{i_U - i_V} H_n(U \cup V) \xrightarrow{\delta} H_{n-1}(U \cap V) \to \cdots$$

[6]This is really the definition of $\mathbb{Z}$-orientable. See Definition 25.7 for a more general definition.

3. (Dimension axiom) $H_0(pt) = \mathbb{Z}$ and $H_n(pt) = 0$ for $n > 0$.
4. (Additivity axiom) The inclusions $X_\alpha \hookrightarrow \coprod X_\alpha$ induce an isomorphism $\bigoplus H_n(X_\alpha) \xrightarrow{\sim} H_n\big(\coprod X_\alpha\big)$.
5. (Weak homotopy axiom) A weak equivalence induces an isomorphism on homology.

For any homology H_n, we can define *reduced homology* $\widetilde{H}_n(X) := \ker\big(H_n(X) \to H_n(pt)\big)$. It will satisfy the following axioms:

1. (Homotopy axiom) $\widetilde{H}_n : \mathbf{Top} \to \mathbf{Ab}$ factor through $\mathbf{hTop}$.
2. (Mayer-Vietoris axiom) If $U, V \subseteq X$ are open sets, then there is a natural long exact sequence
$$\cdots \to \widetilde{H}_n(U \cap V) \xrightarrow{(j_U, j_V)} \widetilde{H}_n(U) \oplus \widetilde{H}_n(V) \xrightarrow{i_U - i_V} \widetilde{H}_n(U \cup V) \xrightarrow{\delta} \widetilde{H}_{n-1}(U \cap V) \to \cdots$$
3. (Dimension axiom) $\widetilde{H}_n(pt) = 0$ for all n.
4. (Additivity axiom) The inclusions $X_\alpha \hookrightarrow \bigvee X_\alpha$ induce an isomorphism $\bigoplus \widetilde{H}_n(X_\alpha) \xrightarrow{\sim} \widetilde{H}_n\big(\bigvee X_\alpha\big)$.
5. (Weak homotopy axiom) A weak equivalence induces an isomorphism on reduced homology.

Lemma 20.1. *$\widetilde{H}_n(S^n) = \mathbb{Z}$ and $\widetilde{H}_k(S^n) = 0$ for $k \neq n$.*

Proof. Induct on n. It is true for $n = 0$ by the dimension and additivity axioms (for H_n). For $n \geq 1$, write $S^n = U \cap V$, with $U \simeq V \simeq *$ and $U \cap V \simeq S^{n-1}$. By the Mayer-Vietoris property, we get
$$\cdots \to \overbrace{\widetilde{H}_k(U) \oplus \widetilde{H}_k(V)}^{0} \to \widetilde{H}_k(S^n) \xrightarrow{\sim} \widetilde{H}_{k-1}(\overbrace{U \cap V}^{\simeq S^{n-1}}) \to \overbrace{\widetilde{H}_{k-1}(U) \oplus \widetilde{H}_{k-1}(V)}^{0} \to \cdots$$
which proves the result. ☐

Proposition 20.2. *Singular homology satisfies the weak homotopy axiom.*

Proof. If $X \to Y$ is a weak equivalence, it induces isomorphisms on all homotopy groups, so by the relative Hurewicz theorem, it induces isomorphisms on all homology groups. ☐

Proposition 20.3. *Singular homology satisfies the additivity axiom.*

Proof. This follows from the fact that the functors $\Delta_\bullet$, C_*, and n-the homology of a chain all preserve coproducts.
$$\Delta_n(\textstyle\coprod X_\alpha) = \{\sigma : \Delta^n \to \coprod X_\alpha\} = \coprod \{\sigma : \Delta^n \to X_\alpha\} = \coprod \Delta_n(X_\alpha)$$
$$C_n(\textstyle\coprod S_{\bullet,\alpha}) = \mathbb{Z} \cdot \coprod S_{n,\alpha} = \bigoplus \mathbb{Z} \cdot S_{n,\alpha} = \bigoplus C_n(S_{\bullet,\alpha})$$
$$H_n\big(\bigoplus C_{*,\alpha}\big) = \bigoplus H_n(C_{*,\alpha})$$
☐

We've already verified the dimension axiom in Example 19.4.

Lemma 20.4. *Given a short exact sequence of chain complexes* $0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$, *there is a long exact sequence of homology groups*

$$\cdots \to H_n(A_*) \xrightarrow{f_*} H_n(B_*) \xrightarrow{g_*} H_n(C_*) \xrightarrow{\delta} H_{n-1}(A_*) \to \cdots$$

Proof. A diagram chase. □

Proposition 20.5 (Mayer-Vietoris for Δ-sets)**.** *If $S_\bullet$ is a Δ-set with sub-Δ-sets $U_\bullet$ and $V_\bullet$, then*

$$0 \to C_*(U_\bullet \cap V_\bullet) \xrightarrow{(j_U, j_V)} C_*(U_\bullet) \oplus C_*(V_\bullet) \xrightarrow{i_U - i_V} C_*(U_\bullet \cup V_\bullet) \to 0$$

is a short exact sequence. In particular, since homology preserves direct sums, you get a long exact sequence in homology by the lemma.

Proposition 20.6. *Singular homology satisfies the Mayer-Vietoris axiom.*

Proof. It is clear that we get the exact sequence

$$0 \to S_*(U \cap V) \to S_*(U) \oplus S_*(V) \to S_*(U \cup V)$$

The result follows from the following claim.

Claim. *The inclusion $S_*(U) + S_*(V) \hookrightarrow S_*(U \cup V)$ is a homotopy equivalence.*

To see the claim, consider the chain map $b : S_*(U \cup V) \to S_*(U \cup V)$ given by barycentric subdivision. We get that $b \simeq \mathrm{Id}$ [[★★★ is there an easy way to see it?]], and for any simplex σ, $b^N(\sigma) \in S_*(U) + S_*(V)$ for large enough N. [[★★★ this somehow works out]] □

The hardest axiom to verify is the homotopy axiom. We will do it after we develop cellular homology.

In the process, we will show that the Eilenberg-Steenrod axioms completely determine homology. We will define cellular homology in terms of a homology homology and then show that the cellular homology is isomorphic to the original homology. Then we will show that cellular homology is independent of the homology theory you started with.

21 Cellular Homology and Uniqueness of Homology

Definition 21.1. If $A \subseteq X$, the *relative singular homology* $H_n(X, A)$ is the n-th homology of the chain complex $S_*(X)/S_*(A)$.

Lemma 21.2. *If $A \subseteq X$ is a cofibration, then $H_n(X, A) \cong \widetilde{H}_n(X/A)$.*

Proof. Note that $S_*(X)/S_*(A) \cong \widetilde{S}_*(X)/\widetilde{S}_*(A)$. By Lemma 20.4, we get the long exact sequence shown as the top row in the diagram.

$$\begin{tikzcd}
\widetilde{H}_n(A) \arrow[r] \arrow[d, "\mathrm{Id}"] & \widetilde{H}_n(X) \arrow[r] \arrow[d, "\mathrm{Id}"] & H_n(X,A) \arrow[r] \arrow[d, "\phi"] & \widetilde{H}_{n-1}(A) \arrow[r] \arrow[d, "\mathrm{Id}"] & \widetilde{H}_{n-1}(X) \arrow[d, "\mathrm{Id}"] \\
\widetilde{H}_n(A) \arrow[r] & \widetilde{H}_n(X) \arrow[r] & \widetilde{H}_n(X/A) \arrow[r] & \widetilde{H}_{n-1}(A) \arrow[r] & \widetilde{H}_{n-1}(X)
\end{tikzcd}$$

To get the bottom exact sequence, observe that $X/A \simeq X \cup_A CA$, so we may choose open sets $U, V \subseteq X \cup_A CA$ so that $U \simeq X$ and $V \simeq *$. Then the bottom row is just the Mayer-Vietoris sequence in reduced homology.

The map ϕ is induced by the obvious map $\widetilde{S}_*(X)/\widetilde{S}_*(A) \to \widetilde{S}_*(X/A)$ and makes the above diagram commute. By the 5-lemma, we get the desired result. $\square$

Definition 21.3. For a CW complex X, we define the group of cellular n-chains to be $C_n(X) = H_n(X^{(n)}, X^{(n-1)})$. The boundary map is defined as the composition d^{CW} : $H_n(X^{(n)}, X^{(n-1)}) \to H_{n-1}(X^{(n-1)}) \to \widetilde{H}_n(X^{(n-1)}, X^{(n-2)})$. The *cellular homology* $H_n^{CW}(X)$ is the n-th homology of this chain complex.

Note that the lemma shows that $C_n(X) = H_n(X^{(n)}/X^{(n-1)}) = \widetilde{H}_n(\bigvee_{I_n} S^n) \cong \mathbb{Z} \cdot I_n$, where I_n indexes the n-cells of X.

To prove that cellular homology agrees with singular homology, we'll need a few lemmas.

Lemma 21.4. *$H_k(X^{(n)}) = 0$ for $k > n$.*

Proof. Induct on n. For $n = 0$, it is true by additivity because $X^{(0)}$ is a collection of points. In general, we get an exact sequence

$$\underbrace{H_k(X^{(n-1)})}_{0 \text{ by induction}} \to H_k(X^{(n)}) \to \underbrace{H_k(X^{(n)}, X^{(n-1)})}_{\cong \widetilde{H}_k(X^{(n)}/X^{(n-1)}) = \widetilde{H}_n(\bigvee S^n) = 0}$$

so the middle term is zero. $\square$

Lemma 21.5. *The inclusion $i : X^{(k)} \to X$ induces isomorphisms $H_n(X^{(k)}) \to H_n(X)$ for $n < k$.*

Proof. We have the sequence

$$\underbrace{H_{n+1}(X^{(n+\ell+1)}, X^{(n+\ell)})}_{0} \to H_n(X^{(n+\ell)}) \xrightarrow{i} H_n(X^{(n+\ell+1)}) \to \underbrace{H_n(X^{(n+\ell+1)}, X^{(n+\ell)})}_{0}.$$

It follows that for $\ell > 0$, inclusion induces isomorphisms $H_n(X^{(n+\ell)}) \cong H_n(X^{(n+\ell+1)})$.

To show that the inclusion $X^{(k)} \to X$ induces isomorphisms, we can take two approaches:

(Bad) We know that singular homology is represented by maps $\sigma : \Delta^n \to X$. Since Δ^n is compact, the image of σ lies in a finite skeleton. This shows that $H_n(X^{(k)}) \to H_n(X)$ is surjective (given the isomorphisms already constructed). Similarly, if some σ is a boundary of $\tilde{\sigma}$ in X, then the image of $\tilde{\sigma}$ is in some finite skeleton, showing that the map is injective. This approach is bad because it explicitly uses the definition of singular homology.

(Good) From the Eilenberg-Steenrod axioms, one can prove that homology respects filtered colimits. This is the good version because it proves the lemma for arbitrary homology theories. □

Theorem 21.6. *Cellular homology is naturally isomorphic to singular homology.*

Proof. In the following diagram, all the horizontal and vertical sequences are exact. By the way, this makes it clear that $(d^{CW})^2 = 0$.

$$\begin{tikzcd}
 & & 0 \overset{21.4}{=} H_n(X^{(n-1)}) \arrow[d] & \boxed{H_n(X)} \arrow[d, equal, "\wr\ 21.5"] & \widetilde{H}_n(\bigvee S^{n+1}) = 0 \arrow[d, equal, "\wr"] \\
H_{n+1}(X^{(n+1)}) \arrow[r] & H_{n+1}(X^{(n+1)}, X^{(n)}) \arrow[r, "\delta"] \arrow[d, equal] \arrow[dr, "d^{CW}"'] & H_n(X^{(n)}) \arrow[r] \arrow[d, "j_n"] & H_n(X^{(n+1)}) \arrow[r] & H_n(X^{(n+1)}, X^{(n)}) \\
 & C_{n+1}(X) & H_n(X^{(n)}, X^{(n-1)}) \arrow[d, "\delta"'] \arrow[dr, "d^{CW}"] \arrow[r, phantom, "="] & C_n(X) & \\
 & 0 \overset{21.4}{=} H_{n-1}(X^{(n-2)}) \arrow[r] & H_{n-1}(X^{(n-1)}) \arrow[r, "j_{n-1}"'] & H_{n-1}(X^{(n-1)}, X^{(n-2)}) &
\end{tikzcd}$$

Now we compute the $H_n(X)$ which is boxed.

$$\begin{aligned}
H_n(X) &\cong H_n(X^{(n)}) / \operatorname{im}\delta && \big(H_n(X^{(n+1)}, X^{(n)}) = 0\big) \\
&= \operatorname{im} j_n / \operatorname{im} d^{CW} && (j_n \text{ injective}) \\
&= \ker\delta / \operatorname{im} d^{CW} && (\text{vertical sequence exact}) \\
&= \ker d^{CW} / \operatorname{im} d^{CW} && (j_{n-1} \text{ injective})
\end{aligned}$$

□

Example 21.7. Now it is easy to compute the homology of S^n for $n \geq 2$ because the cellular chain complex has zeros everywhere except in dimensions n and 0. Thus, $H_n(S^n) \cong \mathbb{Z} \cong H_0(S^n)$ and $H_k(S^n) = 0$ for $k \neq 0, n$. •

In general, we can compute cellular homology using the following result.

Proposition 21.8. *If X is a CW complex, let ϕ_i be the attaching map of the i-th n-cell, and let p_j be the projection $X^{(n)} \to X^{(n)}/\big(X^{(n)} \smallsetminus e^n_j\big)$. Then the boundary map $d^{CW} : C_n(X) = \mathbb{Z} \cdot I_n \to \mathbb{Z} \cdot I_{n-1} = C_{n-1}(X)$ is a matrix with (i, j)-th entry $\deg(p_j \circ \phi_i)$.*

Proof. Note that we have the diagram

$$\begin{tikzcd}
 & H_n(X^{(n)}, X^{(n-1)}) \arrow[r, "\delta"] \arrow[d, equal, "\sum(\Phi_i)_* \wr"'] & \widetilde{H}_{n-1}(X^{(n-1)}) & \\
\widetilde{H}_n(D^n) = 0 \arrow[r] & \bigoplus_{I_n} H_n(D^n, S^{n-1}) \arrow[r, "\sim", "\delta"'] & \bigoplus_{I_n} \widetilde{H}_{n-1}(S^{n-1}) \arrow[r] \arrow[u, "\sum(\phi_i)_*"'] & 0 = \widetilde{H}_{n-1}(D^n)
\end{tikzcd}$$

which fits (one and a half times) into the diagram

$$\begin{tikzcd}
H_n(X^{(n)}, X^{(n-1)}) \arrow[r] \arrow[d, equal, "\sum(\Phi_i)_* \wr"'] & \widetilde{H}_{n-1}(X^{(n-1)}) \arrow[r] \arrow[ddr, bend left, "\sum(p_j)_*"] & H_{n-1}(X^{(n-1)}, X^{(n-2)}) \arrow[d, equal, "\wr \sum(\Phi_i)_*"] & \\
\mathbb{Z} \cdot I_n \arrow[r, "\sim", "\delta"'] \arrow[drr, bend right, "d^{CW}"'] & \bigoplus \widetilde{H}_n(S^n) \arrow[u, "\sum(\phi_i)_*"'] & \bigoplus H_{n-1}(D^{n-1}, S^{n-2}) \arrow[r, "\sim", "\delta"'] \arrow[d, equal] & \bigoplus \widetilde{H}_{n-1}(S^{n-1}) \\
 & & \mathbb{Z} \cdot I_{n-1} &
\end{tikzcd}$$

from which we see that $d^{CW} = \left(\sum(\phi_i)_*\right) \circ \left(\sum(p_j)_*\right)$. This proves the result. $\square$

Corollary 21.9. *Any homology theory on CW complexes which is homotopy invariant, additive, and satisfies the dimension and Mayer-Vietoris axioms is isomorphic to singular homology.*

Proof. The results in this section show that any homology theory with these properties is isomorphic to *its own version of CW homology*. That is, the matrix coefficients are the degrees of $\phi_i \circ p_j$, computed in the given homology theory. We will show that degree can be defined independent of homology theory.

We have a Hurewicz map $\pi_n(S^n) \to h_n(S^n) \cong \mathbb{Z}$, which is a group homomorphism (using the distributivity trick that was on one of the homeworks) and onto because the identity map on S^n gets sent to a generator of $\mathbb{Z}$. Assuming the Hurewicz theorem, we get that $\pi_n(S^n) \cong \mathbb{Z}$. Notice that a map $f : S^n \to S^n$ induces a map $f_* : \pi_n(S^n) \to \pi_n(S^n)$, so we can define degree via π_n. Thus, degree is independent of the homology theory h. $\square$

An issue you have to deal with: If X and Y are CW complexes and $f : X \to Y$ is a map, then you don't get any natural map $H_n^{CW}(X) \to H_n^{CW}(Y)$. To get functoriality of H^{CW}, you need to prove the cellular approximation theorem (Theorem 4.8 of Hatcher).

Theorem 21.10. *Every map $f : X \to Y$ is homotopic to a cellular map, a map that sends $X^{(n)}$ to $Y^{(n)}$ for each n.*

Morse (Floer) homology on finite-dimensional smooth manifolds

Choose a sufficiently nice function f on a smooth manifold M. At a critical point of f, define the index of f to be the number of negative eigenvalues of the Hessian (that is, the number of downward curving directions). Define $C_n^{\mathrm{Morse}} = \mathbb{Z} \cdot I_n$, where I_n is the set of index n critical points of f. The boundary map counts the number of flow lines from one point to another. Now one can check that $d^2 = 0$, so we get a homology theory.

One can check that the result is independent of the choice of f. In fact, f dictates a CW structure on M, with one n-cell for each index n critical point.

In Floer homology, where M is allowed to be infinite-dimensional, the function f is somehow given to you by the geometry of the situation. You no longer get independence of f.

22 Homotopy invariance

If $f_0, f_1 : X \to Y$ are homotopic maps, then we have the picture $X \xrightarrow{i_0, i_1} X \times I \xrightarrow{F} Y$, with $f_j = F \circ i_j$ for $j = 0, 1$. We would like to show that f_0 and f_1 induce the same map in homology. To prove that, we will actually show that they induce homotopic chain maps, and we will use F to construct the homotopy. However, to do this, we must understand what the chain complex associated to a product looks like. This is the content of the Eilenberg-Zilber Theorem 22.6.

Definition 22.1. If C_* and D_* are two chain complexes, then their tensor product is defined by $(C_* \otimes D_*)_n = \bigoplus_{p+q=n} C_p \otimes_{\mathbb{Z}} D_q$, with boundary map $d_n^{C\otimes D} = \bigoplus_{p+q=n} d_p^C \otimes 1 + (-1)^p \otimes d_q^D$. The sign ensures that the result is a chain complex; it is a standard exercise to check this.

$$\begin{tikzcd}
C_p \otimes D_q \arrow[r, "d^C \oplus 1"] \arrow[d, "(-1)^p \otimes d^D"'] \arrow[dr, phantom, "\text{anti-commutes}"] & C_{p+1} \otimes D_q \arrow[d, "(-1)^{p+1} \otimes d^D"] \\
C_p \otimes D_{q+1} \arrow[r, "d^C \otimes 1"'] & C_{p+1} \otimes D_{q+1}
\end{tikzcd}$$

Lemma 22.2. *If X and X' are CW complexes, then $C_*(X \times X') = C(X) \otimes C(X')$*

Proof. This follows from the fact that $\partial(D^p \times D^q) = \partial D^p \times D^q \cup D^p \times \partial D^q$. $\square$

Definition 22.3. If $S_\bullet$ is a Δ-set, the *cone on $S_\bullet$* is the Δ-set defined by $(CS)_0 = S_0 \cup \infty$, $(CS)_n = S_n \cup S_{n-1}$ for $n > 0$. The boundary maps are defined by $d_i : \begin{array}{l} S_n \xrightarrow{d_i} S_{n-1} \\ S_{n-1} \xrightarrow[d_i]{} S_{n-2} \end{array}$ for $i \leq n$ and $d_{n+1} : \begin{tikzcd}[row sep=small] S_n \arrow[r, "d_{n+1}"] & S_{n-1} \\ S_{n-1} \arrow[ur, "\mathrm{Id}"'] & S_{n-2} \end{tikzcd}$, where $S_{-1} = \{\infty\}$.

[[★★★ picture here]]

Lemma 22.4. *Cones are acyclic: $H_*\big(C_*(CS)\big) = 0$.*

Proof. If $(a, b) \in \mathbb{Z}CS_n = \mathbb{Z}S_n \oplus \mathbb{Z}S_{n-1}$ maps to zero, then it is the image of $\big(0, (-1)^{n+1}a\big)$.

$$\begin{tikzcd}[row sep=small]
\mathbb{Z}S_{n+1} \arrow[r, "d"] & \mathbb{Z}S_n \arrow[r, "d"] & \mathbb{Z}S_{n-1} \\
\oplus & \oplus & \oplus \\
\mathbb{Z}S_n \arrow[r, "d"] \arrow[uur, "(-1)^{n+1}"] & \mathbb{Z}S_{n-1} \arrow[r, "d"] \arrow[uur, "(-1)^n"] & \mathbb{Z}S_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small]
0 & a \arrow[r, mapsto] & da - (-1)^n b = 0 \\
\oplus & \oplus & \oplus \\
(-1)^{n+1}a \arrow[r, mapsto] \arrow[uur, mapsto] & b \arrow[r, mapsto] \arrow[uur, mapsto] & db = 0
\end{tikzcd}$$

$\square$

Corollary 22.5 ($\Delta^p \times \Delta^q$ is "acyclic")**.** $\widetilde{H}_*(\Delta^p \times \Delta^q) = 0$.

Proof. Singular homology agrees with simplicial homology by Theorem 19.6, so we use simplicial homology. We can break the product of in to simplices. Then by repeated application of Mayer-Vietoris for Δ-sets, it is enough to show that $\widetilde{H}_*(\Delta^k) = 0$, which is true by the lemma. $\square$

Theorem 22.6 (Eilenberg-Zilber)**.** *For topological spaces X and Y, there is a* natural *chain homotopy equivalence $h_{XY} : S_*X \otimes S_*Y \xrightarrow{\simeq} S_*(X \times Y)$ such that $(h_{XY})_0 : S_0(X) \otimes S_0(Y) \to S_0(X \times Y)$ is the map $\sigma \otimes \sigma' \mapsto \sigma \times \sigma'$.*[7] *Moreover, h_{XY} is unique up to chain homotopy.*

This may be rephrased as "$S_* : (\mathbf{Top}, \times) \to (\mathbf{Chain}, \otimes)$ is a monoidal functor." We saw already that "$S_* : (\mathbf{Top}, \sqcup) \to (\mathbf{Chain}, \oplus)$ is a monoidal functor."

Proof (via "acyclic models"). Induct on degree. The start of the induction is part of the hypothesis. Assume we have h_{XY} defined in degree $< p + q$ for all spaces X and Y. We would like to define $h_{XY} : S_pX \otimes S_qY \to S_{p+q}(X \times Y)$. Let $a \in S_pX$ and $b \in S_qY$, so $a : \Delta^p \to X$ and $b : \Delta^q \to Y$. If naturality is to hold, the following diagram must commute.

$$\begin{tikzcd}
S_pX \otimes S_qY \arrow[r, dotted, "h_{XY}?"] & S_{p+q}(X \times Y) \\
S_p\Delta^p \otimes S_q\Delta^q \arrow[u, "S_*a \otimes S_*b"] \arrow[d, "d"] \arrow[r, dotted, "h_{\Delta^p\Delta^q}?"] & S_{p+q}(\Delta^p \times \Delta^q) \arrow[u, "S_*(a \times b)"'] \arrow[d, "d"] \\
(S_{p-1}\Delta^p \otimes S_q\Delta^q) \oplus (S_p\Delta^p \otimes S_{q-1}\Delta^q) \arrow[r, "h_{\Delta^p\Delta^q}", "\text{induction!}"'] & S_{p+q-1}(\Delta^p \times \Delta^q)
\end{tikzcd}$$

$$\begin{tikzcd}
a \otimes b \arrow[r, dotted, mapsto] & h_{XY}(a \otimes b) \\
\mathrm{Id}_{\Delta^p} \otimes \mathrm{Id}_{\Delta^q} \arrow[u, mapsto, "S_*a \otimes S_*b"] \arrow[d, mapsto, "d"] \arrow[r, dotted, mapsto] & y? \arrow[u, mapsto, "S_*(a \times b)"'] \arrow[d, mapsto, "d"] \\
\Big(\big(\sum_{i=0}^{p}(-1)^i\delta_i\big) \otimes \mathrm{Id}, \mathrm{Id} \otimes \big(\sum_{j=0}^{q}(-1)^j\delta_j\big)\Big) \arrow[r, mapsto, "h_{\Delta^p\Delta^q}"] & x
\end{tikzcd}$$

---

[7] Note that this makes sense only because $\Delta^0 \times \Delta^0 \approx \Delta^0$.

Now we see that a wonderful thing has happened. So long as we can find a $y \in S_{p+q}(\Delta^p \times \Delta^q)$ so that $dy = x$, we can define $h_{XY}(a \otimes b)$ to be $S_*(a \times b)(y)$, and this will automatically be natural! We know that $dx = 0$ because $dx = d \circ (h_{\Delta^p \Delta^q})_{p+q-1} \circ d(\mathrm{Id} \otimes \mathrm{Id}) = (h_{\Delta^p \Delta^q})_{p+q-2} \circ d \circ d(\mathrm{Id} \otimes \mathrm{Id}) = 0$ (note that we only used the degrees for which h is already defined). Since $\Delta^p \times \Delta^q$ is acyclic $H_{p+q-1}(\Delta^p \times \Delta^q) = 0$, so such a y exists.

It remains to show that the h_{XY} only depends on the choice of y up to homotopy. If we use y' instead of y, we have $d(y - y') = 0$. Since $H_{p+q}(\Delta^p \times \Delta^q) = 0$, there is some $z \in S_{p+q+1}(\Delta^p \times \Delta^q)$ so that $dz = y - y'$. This z induces a homotopy between the two choices of h_{XY}. $\square$

Now we are ready to prove the desired result.

Theorem 22.7 (Homotopy invariance of Homology)**.** *If f_0 and f_1 are homotopic maps from X to Y, then they induce chain homotopic maps $S_*X \to S_*Y$. In particular, they induce the same map on homology.*

Proof. Let $F : X \times I \to Y$ be a homotopy from f_0 to f_1. We have a canonical inclusion $C_*I \to S_*I$. Thus, we get the chain map

$$S_*X \otimes C_*I \longrightarrow S_*X \otimes S_*I \xrightarrow[h_{XI}]{\simeq} S_*(X \times I) \xrightarrow{S_*F} S_*(Y)$$

$$(S_{n+1}X \otimes C_0 I) \oplus (S_n X \otimes C_1) I \longrightarrow S_{n+1}Y$$

$$\begin{array}{ccc}
(a_0 \otimes v_0 + a_1 \otimes v_1, b \otimes e) & \longmapsto & S_*f_0(a_0) + S_*f_1(a_1) + h_{n+1}(b) \\
\big\downarrow {\scriptstyle d^{S_*X \otimes C_*I}} & & \big\downarrow {\scriptstyle d^Y} \\
(d^X a_0 \otimes v_0 + d^X a_1 \otimes v_1 + (-1)^n b \otimes (v_0 - v_1), d^X b \otimes e) & & d^Y S_*f_0(a_0) + d^Y S_*f_1(a_1) + d^Y h_{n+1}(b) \\
& \searrow & \| \\
& & S_*f_0(d^X a_0) + S_*f_1(d^X a_1) + (-1)^n (S_*f_0 - S_*f_1)(b) + h_n(d^X b)
\end{array}$$

From the fact that it is a chain map, we get the equalities

$$\begin{aligned}
d^Y S_*f_0(a_0) + d^Y S_*f_1(a_1) + d^Y h_{n+1}(b) &= S_*f_0(d^X a_0) + S_*f_1(d^X a_1) + (-1)^n (S_*f_0 - S_*f_1)(b) + h_n(d^X b) \\
d^Y h_{n+1} &= (-1)^n (S_*f_0 - S_*f_1) + h_n d^X \qquad (d^Y \circ S_*f_i = S_*f_i \circ d^Y) \\
d^Y \circ (-1)^n h_{n+1} + (-1)^{n-1} h_n \circ d^X &= S_*f_0 - S_*f_1
\end{aligned}$$

Thus, if we twist h by some minus signs as above, we get a homotopy from S_*f_0 to S_*f_1, as desired. $\square$

23 The Generalized Jordan Curve Theorem

The following theorem looks obvious until you think about it.

Theorem 23.1 (Jordan Curve Theorem)**.** *If $S^1 \hookrightarrow \mathbb{R}^2$ is an embedding (continuous injection), the the complement of the image has two path components.*

We will prove a more general version.

Theorem 23.2. *If $i : S^r \hookrightarrow S^n$ with $r < n$, then* $\widetilde{H}_k\big(S^n \smallsetminus i(S^r)\big) = \begin{cases} \mathbb{Z} & k = n - r - 1 \\ 0 & \text{else} \end{cases}$.

That is, homology is independent of the embedding. This is kind of sad because it tells us that we cannot use homology to understand knots.

You can guess the answer by considering linear embeddings, where we think of S^n as $\mathbb{R}^n \cup \infty$. Then we have

$$\begin{aligned} S^n \smallsetminus S^r &= \mathbb{R}^n \smallsetminus \mathbb{R}^r \\ &= (\mathbb{R}^{n-r} \times \mathbb{R}^r) \smallsetminus (0 \times \mathbb{R}^r) \\ &= (\mathbb{R}^{n-r} \smallsetminus 0) \times \mathbb{R}^r \simeq S^{n-r-1}. \end{aligned}$$

However, you have to be careful. It is not true that for any embedding i, $S^n \smallsetminus i(S^r) \simeq S^{n-r-1}$. For example, we have the Alexander horned sphere in S^3, whose complement is not homotopic to S^0.

For the proof of Theorem 23.2, we'll need the following lemma.

Lemma 23.3. *Let Y be compact and assume it has the property that for every $i : Y \hookrightarrow S^n$, $\widetilde{H}\big(S^n \smallsetminus i(Y)\big) = 0$. Then $Y \times I$ has the same property.*

Proof. Let $f : Y \times I \hookrightarrow S^n$ be an embedding. Let $U_0 = S^n \smallsetminus f(Y \times [0, 1/2])$ and let $U_1 = S^n \smallsetminus f(Y \times [1/2, 1])$. Then $U_0 \cup U_1 = S^n \smallsetminus f(Y \times \{1/2\})$, so it has trivial homology by assumption, and $U_0 \cup U_1 = S^n \smallsetminus f(Y \times I)$. By Mayer-Vietoris, we get the isomorphism

$$\widetilde{H}_k\big(S^n \smallsetminus f(Y \times I)\big) \xrightarrow{\sim} \widetilde{H}_k(U_0) \oplus \widetilde{H}_k(U_1).$$

Any non-zero homology class would have to give a non-zero homology class in one of the U_i. Then we can induct to get non-zero homology classes in $S^n \smallsetminus f(Y \times [p - \varepsilon, p + \varepsilon])$ for some $p \in I$ and arbitrarily small ε. Since homology commutes with filtered colimits, we get a non-zero homology class in $S^n \smallsetminus f(Y \times \{p\})$, a contradiction. □

In particular, $D^0 = pt$ has the property in the lemma because $S^n \smallsetminus pt \cong \mathbb{R}^n \simeq pt$. By the lemma, $D^r \cong D^{r-1} \times I$ has the property for all r. This seems obvious, but you have to watch out, it is not true that $S^n \smallsetminus i(D^r) \simeq pt$ for all embeddings i. For example, we have the Fox-Artin wild arc in S^3.

Proof of Theorem 23.2. Induct on r. For $r = 0$, our naïve calculation works; $S^n \smallsetminus i(S^0) \simeq S^{n-1}$ for all i. Now assume the result for $r - 1$, and let $i : S^r \hookrightarrow S^n$. Then we can write $S^r = D^r_+ \cup_{S^{r-1}} D^r_-$, and we have open sets $U = S^n \smallsetminus i(D^r_+)$ and $V = S^n \smallsetminus i(D^r_-)$ in S^n, with

$$\begin{aligned} S^n \smallsetminus i(S^{r-1}) &= U \cup V \\ S^n \smallsetminus i(S^r) &= U \cap V. \end{aligned}$$

By the Lemma, U and V have trivial homology. Mayer-Vietoris and the induction step immediately give the desired result. □

There are some related results.

Theorem 23.4 (Schoenfließ Conjecture, proven by Mazur and Brown)**.** *If* $i : S^{n-1} \times (-\varepsilon, \varepsilon) \hookrightarrow S^n$*, then* $S^n \smallsetminus i\big(S^{n-1} \times (-\varepsilon, \varepsilon)\big) \approx D^n \sqcup D^n$*. For* $n = 2$*, you don't even need the collar neighborhood.*

Theorem 23.5 (Annulus Conjecture, proven by Kirby for $n \geq 5$, Quinn for $n = 4$, and somebody for $n \leq 3$)**.** *If* $i : \big(S^{n-1} \times (-\varepsilon, \varepsilon)\big) \sqcup \big(S^{n-1} \times (-\varepsilon, \varepsilon)\big) \hookrightarrow S^n$*, then the (appropriate component of) the complement of* $\operatorname{im} i$ *is homeomorphic to* $S^{n-1} \times I$*.*

24 Lefshetz, Alexander, and Poincaré Dualities

Definition 24.1. Let H_* be a homology theory obtained from a chain complex, $H_n : \mathbf{hTop} \to \mathbf{Chain} \xrightarrow{n\text{-th homology}} \mathbf{Ab}$. Then we define the corresponding *cohomology* theory by dualizing the chain complex in the middle, $H^n : \mathbf{hTop} \to \mathbf{Chain} \xrightarrow{\mathrm{Hom}(-,\mathbb{Z})} \mathbf{Chain} \xrightarrow{n\text{-th homology}} \mathbf{Ab}$.

In homework 13, we showed uniqueness of cohomology.

Theorem 24.2 (Lefschetz Duality)**.** *If* M *is an* n*-dimensional compact orientable manifold, then* $H_k(M, \partial M) \cong H^{n-k}(M)$ *and* $H_k(M) \cong H^{n-k}(M, \partial M)$*.*

Theorem 24.3 (Alexander Duality)**.** *If* K *is a finite CW complex and* $i : K \hookrightarrow \mathbb{R}^n$ *is an embedding, then* $\widetilde{H}_k\big(\mathbb{R}^n \smallsetminus i(K)\big) \cong H^{n-k-1}(K)$*.*

Remark 24.4 (Excision axiom). Originally, the axioms of homology were stated for pairs of spaces, and one of the axioms was *excision*: if $\overline{B} \subseteq \mathring{A}$, then $H_n(X, A) \cong H_n(X \smallsetminus B, A \smallsetminus B)$. It is easy to check that this holds in our formulation of homology. We define $H_n(X, A)$ as $\widetilde{H}_n(X/A)$ [[★★★ at least when $A \subseteq X$ is a cofibration ... what if it isn't?]]

Proof. There is a regular ε-neighborhood $N(K)$ of $i(K)$. That is, there is a compact n-manifold $N(K)$ which retracts to $i(K)$. $N(K)$ is oriented because it is an n-submanifold in the oriented n-manifold $\mathbb{R}^n$. Let $N'(K)$ be an $\varepsilon/2$-neighborhood of $i(K)$. Now we can compute

$$\begin{aligned}
H^{n-k-1}(K) &\cong H^{n-k-1}\big(N(K)\big) \\
&\cong H_{k+1}\big(N(K), \partial N(K)\big) && \text{(Lefschetz)} \\
&\cong H_{k+1}\big(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \mathring{N}'(K)\big) && \text{(excision)} \\
&\cong H_{k+1}\big(\mathbb{R}^n, \mathbb{R}^n \smallsetminus i(K)\big) \\
&\cong \widetilde{H}_k\big(\mathbb{R}^n \smallsetminus i(K)\big)
\end{aligned}$$

where the last isomorphism follows from the long exact reduced pair sequence and the fact that $\widetilde{H}_*(\mathbb{R}^n) = 0$. $\square$

Note that we can now compute $H_k\big(S^n \smallsetminus i(K)\big)$. In particular, the Generalized Jordan Curve Theorem is a corollary of Alexander Duality. Use Mayer-Vietoris, with U an neighborhood of infinity that doesn't intersect $i(K)$, and V a big ball containing the (missing) image of K. Then we have that $U \simeq *$, $V \simeq \mathbb{R}^n \smallsetminus i(K)$, and $U \cap V \simeq S^{n-1}$.

$$H_k(S^{n-1}) \to H_k\big(\mathbb{R}^n \smallsetminus i(K)\big) \to H_k\big(S^n \smallsetminus i(K)\big) \to H_{k-1}(S^{n-1})$$

If $k \neq n, n-1$, then we get $H_k\big(\mathbb{R}^n \smallsetminus i(K)\big) \cong H_k\big(S^n \smallsetminus i(K)\big)$. For the remaining cases, we use naturality of Mayer-Vietoris. We have inclusions $\mathbb{R}^n \smallsetminus i(K) \hookrightarrow \mathbb{R}^n \smallsetminus pt$ and $S^n \smallsetminus i(K) \hookrightarrow S^n \smallsetminus pt$. The corresponding map on S^{n-1} is the identity. Thus, we get

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{H}_n\big(\mathbb{R}^n \smallsetminus i(K)\big) \arrow[r] \arrow[d] & \widetilde{H}_n\big(S^n \smallsetminus i(K)\big) \arrow[r, "b"] \arrow[d] & \widetilde{H}_{n-1}(S^{n-1}) \arrow[r, "a"] \arrow[d, "\wr"] & \widetilde{H}_{n-1}\big(\mathbb{R}^n \smallsetminus i(K)\big) \arrow[r] \arrow[d, "j"] & \widetilde{H}_{n-1}\big(S^n \smallsetminus i(K)\big) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{H}_n(\mathbb{R}^n \smallsetminus pt) \arrow[r] & \widetilde{H}_n(S^n \smallsetminus pt) \arrow[r] & \widetilde{H}_{n-1}(S^{n-1}) \arrow[r, "\sim"] & \widetilde{H}_{n-1}(\mathbb{R}^n \smallsetminus pt) \arrow[r] & \widetilde{H}_{n-1}(S^n \smallsetminus pt) \arrow[r] & 0
\end{tikzcd}$$

We can deduce that a must be an injection (since $j \circ a$ is an isomorphism), so b must be the zero map. In fact, one can show that a splits. Thus,

$$\widetilde{H}_k\big(\mathbb{R}^n \smallsetminus i(K)\big) \cong \begin{cases} \widetilde{H}_k\big(S^n \smallsetminus i(K)\big) & k \neq n-1 \\ \widetilde{H}_k\big(S^n \smallsetminus i(K)\big) \oplus \mathbb{Z} & k = n-1 \end{cases}$$

and the extra factor of $\mathbb{Z}$ is represented by a big sphere near infinity.

Another corollary of Lefshetz duality is Poincaré duality. The proof is immediate.

Theorem 24.5 (Poincaré Duality)**.** *If M is a closed orientable n-manifold, then $H_k(M) \cong H^{n-k}(M)$.*

Note that Poincaré duality implies that $H^r(M) = 0 = H_r(M)$ for r larger than the dimension of M.

The idea of Poincaré duality is this. Assume $M = |S_\bullet|$ for some $S_\bullet$ (in fact, not every manifold is triangulatable, so this is a non-trivial assumption). Then $H_*(M) = H_*(S_\bullet)$. We get a cellular chain $C_*(S_\bullet)$, and we can construct M as the dual CW complex, in which every k-cell is replaced by an $(n-k)$-cell.[8] It is easy to see that $C_k \cong \mathrm{Hom}(C^{\mathrm{dual}}_{n-k}, \mathbb{Z})$. Thus, taking homologies, we get $H_k(M) \cong H^{n-k}(M)$.

25 Coefficients and the Künneth Theorem

Definition 25.1. Homology (resp. cohomology) with coefficients in some abelian group A is defined by the Eilenberg-Steenrod axioms, with the dimension axiom replaced by $H_n(pt; A) = \delta_{n0} A$ (resp. $H^n(pt; A) = \delta_{n0} A$).

[8] In the case where M is a surface, so the CW structure is a graph on M, the dual decomposition is the usual dual graph.

The proofs of uniqueness work just fine. For singular homology, we define $H_n(X;A)$ as the n-th homology of $S_*(X)\otimes_{\mathbb{Z}} A$ and $H^n(X;A)$ as the n-th homology of $\operatorname{Hom}_{\mathbb{Z}}\big(S_*(X), A\big)$. The replacement of the Hurewicz map is a map $\pi_n(X) \otimes_{\mathbb{Z}} A \to H_n(X;A)$.

Note that we cannot define $H_n(X;A)$ as $H_n(X) \otimes_{\mathbb{Z}} A$ because $\otimes_{\mathbb{Z}} A$ is not exact, so the Mayer-Vietoris axiom would not hold. However, the Universal Coefficients Theorem says we wouldn't be wrong by too much.

Definition 25.2. Given two complexes C_* and D_*, we have the inclusion $C_p \otimes D_q \to (C_* \otimes D_*)_{p+q}$. Because of how the boundary maps in a tensor product are defined, this induces a map $H_p(C_*) \otimes H_q(D_*) \xrightarrow{\times} H_{p+q}(C_* \otimes D_*)$.

The *cross product* in homology is the map $H_p(X) \otimes H_q(Y) \xrightarrow{\times} H_{p+q}(X \times Y)$. The cross product in cohomology is the map $H^p(X) \otimes H^q(Y) \xrightarrow{\times} H^{p+q}(X \times Y)$.

Theorem 25.3 (Algebraic Künneth Theorem)**.** *If C_* and D_* are free chain complexes over a PID R, then for each n there is a natural short exact sequence*

$$0 \to \bigoplus_{p+q=n} \Big(H_p(C_*) \otimes_R H_q(D_*)\Big) \xrightarrow{\times} H_n(C_* \otimes_R D_*) \to \bigoplus_{p+q=n-1} \operatorname{Tor}_R\big(H_p(C_*), H_q(D_*)\big) \to 0.$$

This sequence splits, but not naturally.

This is Theorem 3B.5 of Hatcher. [[★★★ If R is not a PID, then there are higher Tor groups]] Properties of Tor_R:

1. $\operatorname{Tor}_R(M, N) = \operatorname{Tor}_R(N, M)$
2. $\operatorname{Tor}_R(\bigoplus_i M_i, N) \cong \bigoplus_i \operatorname{Tor}_R(M_i, N)$
3. $\operatorname{Tor}_R(M, N) = \operatorname{Tor}_R\big(T(M), N\big)$, where $T(M)$ is the torsion part of M
4. $\operatorname{Tor}_R(R/I, N) = \ker(I \otimes_R N \to R \otimes_R N)$

Corollary 25.4 (Universal Coefficients for Homology)**.** *There is a natural exact sequence (which splits un-naturally)*

$$0 \to H_n(X) \otimes A \to H_n(X;A) \to \operatorname{Tor}_{\mathbb{Z}}\big(H_{n-1}(X), A\big) \to 0.$$

Just take A to be a chain complex concentrated in degree 0 and apply the Künneth Theorem.

Corollary 25.5 (Topological Künneth Theorem)**.** *There is a natural short exact sequence which splits un-naturally*

$$0 \to \bigoplus_{p+q=n} \big(H_p(X) \otimes H_q(Y)\big) \xrightarrow{\times} H_n(X \times Y) \to \bigoplus_{p+q=n-1} \operatorname{Tor}\big(H_p(X), H_q(Y)\big) \to 0.$$

Theorem 25.6 (Universal Coefficients for Cohomology)**.** *There is a natural short exact sequence which splits un-naturally*

$$0 \to \mathrm{Ext}_{\mathbb{Z}}\big(H_{n-1}(X), A\big) \to H^n(X; A) \to \mathrm{Hom}_{\mathbb{Z}}\big(H_n(X), A\big) \to 0.$$

[[★★★ maybe there should be something about the algebraic Dual Künneth theorem here]] Properties of Ext:

1. $\mathrm{Ext}(\bigoplus_i A_i, B) \cong \prod_i \mathrm{Ext}(A_i, C)$
2. $\mathrm{Ext}(A, \prod_i B_i) \cong \prod_i \mathrm{Ext}(A, B_i)$
3. $\mathrm{Ext}(A, B) = 0$ if A is free
4. $\mathrm{Ext}(\mathbb{Z}/n, \mathbb{Z}) \cong \mathbb{Z}/n$
5. $\mathrm{Ext}(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathbb{Z}/gcd(m, n)$

[[★★★ the rest of this section needs to be organized]]

Definition 25.7. [[★★★ What is the definition of A-orientable for an abelian group A?]]

Definition 25.8. An orientation of a real finite-dimensional vector space V is an ordered basis up to positive determinant change. This is the same as a generator for $H_n(V, V \smallsetminus 0) \cong \widetilde{H}_{n-1}(V \smallsetminus 0) \cong \mathbb{Z}$.

Definition 25.9. An orientation of a manifold M^n is a compatible choice of generators $\mu_x \in H_n(M, M \smallsetminus x) \cong \mathbb{Z}$ for all points $x \in M$. Compatible means that there is an open cover so that $\mu_U \in H_n(M, M \smallsetminus U)$ restricts to μ_x for all $x \in U$.

Remark 25.10. Any manifold is $\mathbb{Z}/2\mathbb{Z}$-oriented, so we can apply all the duality theorems to get information, so long as we use $\mathbb{Z}/2\mathbb{Z}$ coefficients.

26 Cup product in Cohomology

Definition 26.1. The cup product in cohomology is defined as $\smile: H^p(X) \otimes H^q(X) \xrightarrow{\times} H^{p+q}(X \times X) \xrightarrow{\Delta^*} H^{p+q}(X)$, where $\Delta : X \to X \times X$ is the diagonal map.

Theorem 26.2. $H^*(X) = \bigoplus H^p(X)$ *is naturally a graded commutative ring under cup product. Recall that commutativity for a graded ring means* $a \smile b = (-1)^{|a| \cdot |b|} b \smile a$.

Remark 26.3. If the Tor terms in the Künneth formula are zero, then $\times$ is an isomorphism, so we get coring structure on $H_*(X)$.

Remark 26.4. If $a \in H^p(X)$ and $b \in H^q(Y)$, then $a \times b = p_1^* a \smile p_2^* b$, where $\times$ is the cross product, and p_1 and p_2 are the projections from $X \times Y$ to X and Y, respectively. This is a handy way to compute cup products.

If some Tor terms are zero [[★★★ which?]], then $H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$ as a graded commutative ring, where we define $(a \otimes b) \cdot (c \otimes d) = (-1)^{|b| \cdot |c|} ac \otimes bd$.

Remark 26.5. The 1 in the ring $H^*(X)$ is given by the natural map $\mathbb{Z} \cong H^*(pt) \to H^*(X)$. In particular, since $H^*(pt)$ is concentrated in degree 0, the 1 in $H^*(X)$ is in degree zero.

Example 26.6. $H^*(S^1) \cong \mathbb{Z} \oplus x\mathbb{Z} \cong \mathbb{Z}[x]/(x^2)$, where $x \in H^1(S^1)$ is a generator. Note that $x \smile x = 0$ because it must lie in degree 2.

More generally, $H^*(S^n) \cong \mathbb{Z}[x]/(x^2)$, where $|x| = n$. •

Example 26.7. $H^*(S^n \times S^m) \cong \mathbb{Z}[x, y]/(x^2, y^2)$, with $|x| = n$ and $|y| = m$. This follows from Remark 26.3. •

Example 26.8. What is the ring $H^*(\mathbb{CP}^2)$? We claim it is $\mathbb{Z}[z]/(z^3)$, with $z \in H^2(\mathbb{CP}^2) \cong \mathbb{Z}$. [[★★★ how does one prove this?]] •