Solutions for Chapter II. Updated November 3, 2016.

Exercise II.1. *Let $f \in R$ be a regular element. Show that $f \in U(R)$ if and only if the localization $R[f^{-1}]$ is module-finite over R.*

[anton@math] If $f \in U(R)$, then $R[f^{-1}] = R$ is clearly finitely generated as a module over R. On the other hand, if $R[f^{-1}]$ is finitely generated as a module over R, say by $\left\{\frac{r_1}{f^{n_1}}, \dots, \frac{r_m}{f^{n_m}}\right\}$, then let $N = \max\{n_i\}$. Now we have that $R[f^{-1}] \subseteq \frac{1}{f^N} R$. In particular, $f^{-N-1} = f^{-N} r$ for some $r \in R$, so $1 = fr$ (in $R[f^{-1}]$). Since f is a regular element, R injects into $R[f^{-1}]$, so the equality $1 = fr$ holds in R, so $f \in U(R)$. ■

Exercise II.2. *(The Rabinowitch Trick). For $J \lhd R$ and $g \in R$, show that $g \in \sqrt{J}$ iff J and $1-tg$ generate the unit ideal in the polynomial ring $R[t]$. Use this to show that the Strong Nullstellensatz (4.17) can be deduced from the Weak Nullstellensatz (4.9) by "adding a variable".*

[David Brown [brownda@math]] "$\Rightarrow$" Suppose $g^n \in J$. Then as $1 - tg$ divides $1 - (tg)^n$, we have $1 = (tg)^n + (1 - (tg)^n)$ is in the ideal generated by J and $1 - tg$.

"$\Leftarrow$" Suppose that J and $1 - tg$ generate the unit ideal. Then modulo J, $1 = (1 - tg)f(t)$ for some $f(t) = \sum a_i t^i \in R[t]$, say of degree n. Then we get $n + 1$ many equations mod J:

$$\begin{array}{rcl} a_0 &=& 1 \\ a_1 &=& a_0 g \\ & \cdots & \\ a_i &=& a_i g \\ & \cdots & \\ a_{n-1} &=& a_{n-2} g \\ a_n g &=& 0 \end{array}$$

and see inductively that $g^n \in J$.

"**Weak Nullstellensatz $\Rightarrow$ Strong Nullstellensatz**" Suppose $g \in \sqrt{J}$. Then for some m, $g^m \in J$, and so for any $a \in V_{\bar{k}}(J)$, $g^m(a) = g(a)^m = 0$. But k is a field so $g(a) = 0$. Conversely, suppose that $I :=< J, 1-tg >$ is a proper ideal of $R[t]$. Then by the Weak Nullstellensatz, we can find $a = (a_1, \cdots, a_n, b) \in V_{\bar{k}}(I)$. Then $1 - bg(a) = 0$, so $g(a_1, \cdots, a_n) \neq 0$ (else $0 = 1$). Furthermore $f(a) = f(a_1, \cdots, a_n) = 0$, so $g \notin V_{\bar{k}}(J)$. ■

Exercise II.3. *For the ideal $J = \big(xy, yz, zx, (x-y)(x+1)\big) \lhd k[x, y, z]$, determine the irreducible k-components of the k-algebraic set $V_{\bar{k}}(J)$.*

[anton@math] I claim that the decomposition $V_{\bar{k}}(J) = V_{\bar{k}}(x, y) \cup V_{\bar{k}}(x + 1, y, z)$ gives the irreducible components of $V_{\bar{k}}(J)$. Since (x, y) and $(x + 1, y, z)$ are prime (the quotients rings are clearly domains), the Nullstellensatz tells us that the sets $V_{\bar{k}}(x,y)$ and $V_{\bar{k}}(x+1,y,z)$ are irreducible, and there is no containment relation between them. Thus, it is enough to verify the equality. It is immediate that $J \subseteq (x,y)$ and $J \subseteq (x+1,y,z)$, so we get the containment $\supseteq$. For the other containment, assume $(a,b,c) \in \bar{k}^3$ is in $V_{\bar{k}}(J)$. Then $ab = bc = ac = 0$, so two of a,b,c are zero. Moreover, we have that either $a = b$ or $a+1 = 0$. If $a = b$, then they must both be zero, so $(a,b,c) \in V_{\bar{k}}(x,y)$. If $a+1 = 0$, then $b = c = 0$, so $(a,b,c) \in V_{\bar{k}}(x+1,y,z)$. ■

Exercise II.4. *Over a field $k = \overline{k}$, let $Y = \{(a^p, a^q) : a \in k\}$, where p, q are fixed positive integers. Show that Y is a k-algebraic set, and determine the ideal $I(Y) \subseteq k[x,y]$.*

[Manuel Reyes; mreyes@math] I claim that it is enough to consider the case where p and q are relatively prime. Indeed, let $d = \gcd(p,q)$ and write $p = dp'$, $q = dq'$. Because k is algebraically closed, every element has a d-th root so that $k^d = k$. Then

$$\begin{aligned} Y &= \{(a^p, a^q) : a \in k\} \\ &= \left\{\left(a^{dp'}, a^{dq'}\right) : a \in k\right\} \\ &= \left\{\left(b^{p'}, b^{q'}\right) : b \in k^d\right\} \\ &= \left\{\left(b^{p'}, b^{q'}\right) : b \in k\right\}. \end{aligned}$$

So we may replace p and q by p' and q' if necessary so that $(p,q) = 1$.

Next we show that $Y = V_k(x^q - y^p)$. Clearly $Y \subseteq V_k(x^q - y^p)$; for the other inclusion, assume that $(u,v) \in V_k(x^q - y^p)$. If either u or v is zero then $(u,v) = (0,0) \in Y$, so we may assume that $u, v \neq 0$. Because k is algebraically closed, there exists $b \in k$ with $b^p = u$. Consider then that

$$(b^q)^p = b^{pq} = u^q = v^p.$$

It follows that $\zeta := \frac{v}{b^q}$ is a p-th root of unity in k. Because $(p,q) = 1$, $\overline{q}$ is a unit in $\mathbb{Z}/p\mathbb{Z}$. So there exists $m \in \mathbb{Z}$ such that $mq \equiv 1 \pmod{p}$. Then ζ^m is a p-th root of unity with $(\zeta^m)^q = \zeta$. It follows that for $a = \zeta^m b$ we have $a^p = b^p = u$ and $a^q = \zeta b^q = v$. So $(u,v) = (a^p, a^q) \in Y$. This means that $Y = V_k(x^q - y^p)$ is a k-algebraic set.

Now because $k = \overline{k}$ Proposition 1.7 implies that $I(Y) = I(V_k(x^q - y^p)) = \sqrt{(x^q - y^p)}$. I claim that $(x^q - y^p)$ is a prime (hence radical) ideal; this will imply that $I(Y) = (x^q - y^p)$. Consider the k-algebra homomorphism $k[x,y] \to k[z]$ given by $x \mapsto z^p$ and $y \mapsto z^q$. Clearly $(x^q - y^p)$ is contained in the kernel, so this induces a map $\varphi : k[x,y]/(x^q - y^p) \to k[z]$, which we will show is injective. First consider that any element $\overline{f}$ of $A := k[x,y]/(x^q - y^p)$ can be represented by a polynomial of the form $f = \sum_{i=0}^{p-1} f_i(x) y^i$, simply because we have the relation $\overline{y}^p = \overline{x}^q$. Then we have

$$\varphi\left(\overline{f}\right) = \sum_{i=0}^{p-1} f_i(z^p) z^{qi}.$$

Suppose that for positive integers a and c and integers $0 \leq b, d < p$ we have $ap + bq = cp + dq$. It follows that $bq \equiv dq \pmod{p}$, and because q is a unit modulo p we have $b \equiv d \pmod{p}$. But because $|b - d| < p$ we must have $b = d$. It is then easy to see that $a = c$. It follows that for $0 \leq i < j < p$ the monomials in the terms $f_i(z^p) z^{qi}$ are distinct from those in $f_j(z^p) z^{qj}$. Then if $\overline{f} \neq 0$, the representative f_i are not all zero, and we must have $\varphi(\overline{f}) \neq 0$. So φ is injective, and A is isomorphic to a subring of the domain $k[z]$. Hence A is itself an integral domain. So $(x^q - y^p)$ is prime and we are done. ■

Exercise II.5. *Let k be a field that is not algebraically closed. Show that, for any r, there is a polynomial $g(x_1, \cdots, x_r) \in k[x_1, \cdots, x_r]$ such that $V_k(g) = \{(0, \cdots, 0)\} \subset k^r$.*

[David Brown, brownda@math.berkeley.edu] Let $f(x) = (x - \alpha_1) \dots (x - \alpha_n) \in k[x]$ (with $\alpha_i \in \overline{k}$) be irreducible. For $r = 1$, $g(x_1) = x_1$ works. For $r = 2$, $g(x_1, x_2) = (x_1 - x_2\alpha_1) \dots (x_1 - x_2\alpha_n)$ works, because for any non-zero $a \in k$, $g(x, a)$ has the same splitting field as $f(x)$, and thus $g(x, a)$ is never zero if a is non zero (and if $a = 0$ then $x = 0$ is the only solution). For $r > 2$ suppose we have such a polynomial $h(x_1, \dots, x_{r-1})$ for $k - 1$. Then $g(x_r, h(x_1, \dots, x_{r-1}))$ works (by the same argument as the $r = 2$ case). ■

Exercise II.6. *Let k be a field that is not algebraically closed. Use Exercise 5 to show that any k-algebraic set S in k^n can be represented (set-theoretically) as $V_k(f)$ for some $f \in k[x_1, \dots, x_n]$.*

[David Brown, brownda@math.berkeley.edu] Let $I(S)$ be generated by polynomials $f_1, \dots, f_k$ with each $f_i \in k[x_1, \cdots, x_n]$, and let $g \in k[x_1, \dots, x_k]$ be a polynomial such that $g(a_1, \dots, a_k) = 0$ iff each $a_i = 0$ (this exists from problem 5). Then the single polynomial $g(f_1, \dots, f_r) \in k[x_1, \cdots, x_n]$ also defines S. ■

Exercise II.7. *(A generalization of the Weak Nullstellensatz) Let $J \triangleleft k[x_1, \cdots, x_n]$, where k in any field. If $V_k(f) \neq \emptyset$ for every $f \in J$, show that $V_k(J) \neq \emptyset$. Why is this a generalization of the Weak Nullstellensatz?*

[David Brown, brownda@math.berkeley.edu] If $k = \overline{k}$, then this is just the Weak Nullstellensatz, so assume $k \neq \overline{k}$. Let $f_1, \cdots, f_r$ generate J, and let $g(y_1, \cdots, y_r)$ be the polynomial from exercise II.5. Then $h = g(f_1, \cdots, f_r) \in J$, so by our hypothesis there is an $a = (a_1, \cdots, a_n) \in k^n$ such that $h(a) = 0$. By exercise II.5, $f_1(a) = \cdots = f_n(a) = 0$. This is a generalization of the Weak Nullstellensatz because it works when $k \neq \overline{k}$. ■

Exercise II.8. *Let K/k be a field extension, and let $Y \subseteq K^n$ be a k-algebraic set. For any $a \in Y$, let $\lambda_a : k[Y] \to K$ be "evaluation at a"; that is, $\lambda_a(f) = f(a)$ for any k-polynomial function f on Y. Show that λ_a is a k-algebra homomorphism, and that $a \mapsto \lambda_a$ defines a bijection from Y to $\mathrm{Hom}_{k\text{-}alg}(k[Y], K)$ (the set of k-algebra homomorphisms from $k[Y]$ to K).*

[anton@math] For the constant polynomial $c \in k[Y]$, we have $\lambda_a(c) = c(a) = c$, so λ_a maps k to k. Moreover, we have $\lambda_a(f \cdot g) = f(a) \cdot g(a) = \lambda_a(f) \cdot \lambda_a(g)$ and $\lambda_a(f + g) = f(a) + g(a) = \lambda_a(f) + \lambda_a(g)$, so λ_a is a k-algebra homomorphism.

If a and a' are two points in $Y \subseteq K^n$, then they differ in some coordinate, say the i-th. Let $\bar{x}_i$ be the image of the polynomial x_i in $k[Y]$. Then $\lambda_a(\bar{x}_i) \neq \lambda_{a'}(\bar{x}_i)$, so $\lambda_a \neq \lambda_{a'}$. We've shown that the map $\lambda_- : Y \to \mathrm{Hom}_{k\text{-alg}}(k[Y], K)$ is injective.

If $f : k[Y] \to K$ is a k-algebra homomorphism, then I claim it is λ_a for $a = \big(f(\bar{x}_1), \ldots, f(\bar{x}_n)\big)$. To see this, note that $\lambda_a(\bar{x}_i) = f(\bar{x}_i)$ for all i. Since the $\bar{x}_i$ generate $k[Y]$ as a k-algebra, this implies that $\lambda_a = f$, proving that λ_- is surjective. By the way, note that $a \in Y$ because $g \in I(Y) \Rightarrow \bar{g} = 0 \in k[Y] \Rightarrow f(g) = 0$, so $g(a) = g\big(f(\bar{x}_1), \ldots, f(\bar{x}_n)\big) = f(g) = 0$. The last equality follows from the fact that f is a k-algebra homomorphism, so $f(\bar{x}_i) \cdot f(\bar{x}_j) = f(\bar{x}_i \bar{x}_j)$, and so on. ■

Exercise II.9. *True or False: for any field extension K/k, the only closed points in K^n in the Zariski k-topology are the k-points*

[anton@math] False. Consider the extension $K = \mathbb{Q}(\sqrt[3]{2})$ of $k = \mathbb{Q}$. Then $V_K(x^3 - 2) = \{\sqrt[3]{2}\}$ is a closed point which is not a k-point. ■

[lam@math] *Comment.* It doesn't pay to add the assumption that K/k is normal either. If $\mathrm{char}(k) = p > 0$ and k is not perfect, we can take an element $\alpha \notin k$ with $\alpha^p = a \in k$. For $K = k(\alpha)$, we have $V_K(x^p - a) = \{\alpha\} \subseteq K^1$, so $\{\alpha\}$ is a closed point in K^1 (w.r.t. the k-topology) that is not a k-point. How about assuming K/k is (finite) Galois? ■

Exercise II.9'. *(This exercise amplifies the former Ex. 9.) Let K/k be fields such that $K \subset \overline{k}$. Show that (1) in the k-topology, any closed poin in K^n is algebraic, and (2) if K/k is a Galois extension, the only closed points in K^n are the k-points.*

[David Brown, brownda@math] (1) Let $X = \{(a_1, \cdots, a_n)\} \subset K^n$ be algebraic. Let $X_{\mathrm{alg}} = X \cap \overline{k}^n$. By exercise II.21, $\overline{X_{\mathrm{alg}}} = X$. This is only possible if V_{alg} is non-empty (else its closure would be empty), so $(a_1, \cdots, a_n)$ is in fact algebraic.

(2) Suppose $P = (a_1, \cdots, a_n) \in K^n$, and assume $X = \{P\}$ is closed, i.e. there exists $J \lhd A = k[x_1, \cdots, x_n]$ with $X := V_K(J)$. Let J be generated by $f_1, \cdots, f_r$. Then $f_i(P) = 0$. Let $\sigma \in \mathrm{Gal}(K/k)$. Then $P^\sigma = (a_1^\sigma, \cdots, a_n^\sigma)$, and $0 = 0^\sigma = (f_i(P))^\sigma = f_i(P^\sigma)$ (since the coefficients of f_i are in k, hence fixed by σ). Thus, $P^\sigma \in V_K(J) = \{P\}$, and we conclude that $P^\sigma = P$ for every σ, (thus $P \in k^n$). ■

Exercise II.10. *For a nonempty space X, the following are equivalent:*

1. *X is not the union of two proper closed sets.*
2. *Any two nonempty open sets in X intersect.*

3. Any nonempty open set in X is dense.

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1. $\Rightarrow$ 2. Let $U, V \subset X$ be open, nonempty and assume $U \cap V = \varnothing$. Passing to the complement shows $X \setminus U \cup X \setminus V = X$ which contradicts the irreducibility of X, so $U \cap V \neq \varnothing$.

2. $\Rightarrow$ 3. Let $U \subset X$ be open and nonempty. Let $x \in X$ be an arbitrary point, then for every open neighborhood U_x of x we have $U_x \cap U \neq \varnothing$ by assumption, so $x \in \bar{U}$, which proves $\bar{U} = X$.

3. $\Rightarrow$ 2. For open nonempty $U, V \subset X$ we have $\bar{U} = X = \bar{V}$, i.e. $\bar{U} \supset V$, so $V \cap U \neq \varnothing$.

2. $\Rightarrow$ 1. Assume that $X = A \cup B$ for $A, B \subsetneqq X$ closed, then we have $X \setminus A \cap X \setminus B = \varnothing$, which contradicts the assumption 2.

■

Exercise II.11. *(1) The irreducible closed subsets of* $\operatorname{Spec}(R)$ *are precisely sets of the form* $\mathcal{V}(\mathfrak{p})$*, where* $\mathfrak{p} \in \operatorname{Spec}(R)$*. (2) The irreducible components of* $\operatorname{Spec}(R)$ *are precisely sets of the form* $\mathcal{V}(\mathfrak{p})$*, where* $\mathfrak{p} \in \operatorname{Min}(R)$*.*

[Manuel Reyes; mreyes@math] (1) It is straightforward to verify that a subset of a topological space is irreducible iff, whenever it is contained in the union of two closed subsets, it belongs to one of the two subsets. Let $\mathcal{V}(I) \neq \varnothing$ be an arbitrary closed subset of $\operatorname{Spec}(R)$. We may assume that $I \neq R$ is radical. Then $\mathcal{V}(I)$ is irreducible iff for any ideals $J_1, J_2 \subseteq R$ such that $\mathcal{V}(I) \subseteq \mathcal{V}(J_1) \cup \mathcal{V}(J_2) = \mathcal{V}(J_1 J_2)$ we have that $\mathcal{V}(I) \subseteq \mathcal{V}(J_i)$ for some i. Recalling that I is radical, this holds iff $\sqrt{J_1 J_2} \subseteq I$ implies that one of the $\sqrt{J_i} \subseteq I$, iff $J_1 J_2 \subseteq I$ implies that one of the $J_i \subseteq I$. This is true iff I is prime (see the solution to Exercise II.15).

(2) Consider that for two irreducible closed sets $\mathcal{V}(\mathfrak{p}), \mathcal{V}(\mathfrak{q})$, with $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(R)$, we have $\mathcal{V}(\mathfrak{p}) \subseteq \mathcal{V}(\mathfrak{q})$ iff $\mathfrak{q} \subseteq \mathfrak{p}$. It follows that

$$\begin{aligned}
\{F \subseteq \operatorname{Spec}(R) : F \text{ is closed and irreducible}\}^* &= \{\mathcal{V}(\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec}(R)\}^* \\
&= \{\mathcal{V}(\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec}(R)_*\} \\
&= \{\mathcal{V}(\mathfrak{p}) : \mathfrak{p} \in \operatorname{Min}(R)\}.
\end{aligned}$$

Hence the irreducible components of $\operatorname{Spec}(R)$ are the subsets of the form $\mathcal{V}(\mathfrak{p})$ with $\mathfrak{p} \in \operatorname{Min}(R)$. ■

Exercise II.12. *(1) For any $I \lhd R$, let $\pi : R \to R/I$ be the projection map. Show that the map $\pi^* : \operatorname{Spec}(R/I) \to \operatorname{Spec}(R)$ induces a homeomorphism from* $\operatorname{Spec}(R/I)$ *onto* $\mathcal{V}(I)$*. (In particular, if $I \subseteq \operatorname{Nil}(R)$, π^* is a homeomorphism from* $\operatorname{Spec}(R/I)$ *onto* $\operatorname{Spec}(R)$*.*
(2) For any multiplicative set $S \subseteq R$, show that the localization map $f : R \to R_S$ induces a homeomorphism f^* from $\operatorname{Spec}(R_S)$ *onto the subspace* $\{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \cap S = \emptyset\}$ *of* $\operatorname{Spec}(R)$*.*

[los@math] (1) The map $K \mapsto \pi^{-1}(K)$ is a bijection from the set of ideals of R/I to the set of ideals of R containing I, and its inverse is given by $J \mapsto J/I$. For any $J \triangleleft R$ containing I, the rings $(R/I)/(J/I)$ and R/J are isomorphic, hence J is prime in R if and only if J/I is prime in R/I. This establishes that π^* is a bijection. The map π^* is also continuous. Therefore, to show it is a homeomorphism, it is enough to show it is closed. Any closed subset of $\mathrm{Spec}(R/I)$ is of the form $\mathcal{V}(J/I)$ for some $J \triangleleft R$ containing I. We have $\pi^*(\mathcal{V}(J/I)) = \mathcal{V}(J)$. The latter set is closed in $\mathrm{Spec}(R)$, thus π^* is a closed map. This proves that π^* is a homeomorphism from $\mathrm{Spec}(R/I)$ onto $\mathcal{V}(I)$. The last statement results from the fact that $\mathrm{Nil}(R)$ is contained in every prime ideal of R.
(2) Let W_S denote the subspace $\{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \cap S = \emptyset\}$ of $\mathrm{Spec}(R)$. If I (resp. J) is an ideal in R (resp. R_S), write I^e (resp. J^c) for the ideal $IR_S = I_S$ of R_S (resp. the ideal $f^{-1}(J)$ of R). (In particular, for $\mathfrak{q} \in \mathrm{Spec}(R_S)$, we have $f^*(\mathfrak{q}) = \mathfrak{q}^c$ by definition.) Because f is a localization map, we have $J = J^{ce}$ for every ideal J of R_S. First, we show that $f^*(\mathrm{Spec}(R_S)) \subseteq W_S$. Let $\mathfrak{q} \in \mathrm{Spec}(R_S)$, and assume there exists $s \in \mathfrak{q}^c \cap S$. Then the invertible element $f(s) = s/1$ of R_S belongs to $\mathfrak{q}^{ce} = \mathfrak{q}$. This is absurd, since $\mathfrak{q}$, being prime, cannot be all of R_S. Therefore $\mathfrak{q}^c \cap S = \emptyset$, which shows that $\mathfrak{q}^c \in W_S$. Now we show that f^* is a bijection from $\mathrm{Spec}(R_S)$ onto W_S, with inverse given by $\mathfrak{p} \mapsto \mathfrak{p}^e$. For this it is enough to prove that $\mathfrak{p}^e = \mathfrak{p}_S$ is prime in R_S and $\mathfrak{p}^{ec} = \mathfrak{p}$ whenever $\mathfrak{p} \in W_S$. Let $\pi : R \to R/\mathfrak{p}$ denote the projection. Then we have natural maps $R/\mathfrak{p} \to R_S/\mathfrak{p}^e = R_S/\mathfrak{p}_S \xrightarrow{\sim} (R/\mathfrak{p})_S \xrightarrow{\sim} (R/\mathfrak{p})_{\pi(S)}$ the composite of which is the localization homomorphism g relative to the multiplicative subset $\pi(S)$ of $R/\mathfrak{p}$. By hypothesis, $\mathfrak{p} \cap S = \emptyset$, hence $\pi(S) \subseteq (R/\mathfrak{p}) \setminus \{0\}$. Furthermore, $R/\mathfrak{p}$ is a domain. Thus $(R/\mathfrak{p})_{\pi(S)}$ is a domain and the localization homomorpism g is injective. In particular, $\mathfrak{p}^e$ is prime and $\mathfrak{p}^{ec}/\mathfrak{p} = \ker(g) = 0$. This shows that f^* is a bijection onto W_S. Finally, we show f^* is a homeomorphism onto its image in $\mathrm{Spec}(R)$. Since f^* is continuous, it is enough to show that every closed subset Z of $\mathrm{Spec}(R_S)$ is $(f^*)^{-1}(Z')$ for some closed subset Z' of $\mathrm{Spec}(R)$. By definition of the topology on $\mathrm{Spec}(R_S)$, we have $Z = \mathcal{V}(J)$ for some ideal J of R_S. Now for $Z' = \mathcal{V}(J^c) \subseteq \mathrm{Spec}(R)$ we have $(f^*)^{-1}(Z') = \mathcal{V}(J^{ce}) = \mathcal{V}(J) = Z$, which completes the proof. ■

Exercise II.13. *For any ring R, show that the following are equivalent:*

1. $\mathrm{Spec}\, R$ *is a discrete space*
2. $\mathrm{Spec}\, R$ *is finite and discrete*
3. R *is* 0*-dimensional and semilocal*

Show that these conditions are not equivalent to $|\mathrm{Spec}\, R| < \infty$. If R is a noetherian ring, show that 1.-3. are equivalent to: 4. R is artinian.

[Lars Kindler, lars_k@berkeley.edu]

1. $\Leftrightarrow$ 2. $\mathrm{Spec}\, R$ is discrete iff all points are open, iff $\mathrm{Spec}\, R$ is finite and discrete, since $\mathrm{Spec}\, R$ is compact.

2. $\Rightarrow$ 3. It is clear that R is semilocal. Assume there are $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec} R$ such that $\mathfrak{p}_1 \subsetneqq \mathfrak{p}_2$. Then every open neighborhood of $\mathfrak{p}_2$ contains $\mathfrak{p}_1$, since if $D(f)$ is a basic open set with $\mathfrak{p}_2 \in D(f)$, then $\mathfrak{p}_1 \in D(f)$. This contradicts the assumption that $\operatorname{Spec} R$ is discrete.

3. $\Rightarrow$ 1. Let $\operatorname{Spec} R = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ and let R be 0-dimensional. Then for every $j \neq i$ there is a $x_j \in \mathfrak{p}_j \setminus \mathfrak{p}_i$, so $x := \prod_{j \neq i} x_j \in \mathfrak{p}_k$ for all $k \neq i$, which means $D(x) = \{\mathfrak{p}_i\}$, i.e. $\operatorname{Spec} R$ is discrete.

If R is noetherian, then R is 0-dimensional iff R is artinian, i.e. 3. $\Leftrightarrow$ 4. To see that the above are not equivalent to $|\operatorname{Spec} R| < \infty$, consider $R = \mathbb{F}_2[x,y]/(x^2, y^2)$. R has only finitely many primes since R is a finite ring, but $(x) \subsetneqq (x,y)$ is a chain of prime ideals, so R is not 0-dimensional, which proves $|\operatorname{Spec} R| < \infty \not\Rightarrow 3$. ■

Exercise II.14. *For $\mathfrak{p}_1 \neq \mathfrak{p}_2$ in $\operatorname{Spec} R$, show that there is no prime $\mathfrak{p} \subset \mathfrak{p}_1 \cap \mathfrak{p}_2$ iff there exist disjoint open sets $X_1, X_2 \subseteq \operatorname{Spec} R$ such that $\mathfrak{p}_i \in X_i$ for $i = 1, 2$.*

[Lars Kindler, lars_k@berkeley.edu] First, let $X_1, X_2 \subset \operatorname{Spec} R$ be disjoint open sets with $\mathfrak{p}_i \in X_i$. Then there are $f_1, f_2 \in R$ such that $\mathfrak{p}_i \in D(f_i) \subset X_i$, so if there is a prime $\mathfrak{p} \subset \mathfrak{p}_1 \cap \mathfrak{p}_2$ this implies $f_i \notin \mathfrak{p}$, $i = 1, 2$, so $\mathfrak{p} \in D(f_1) \cap D(f_2) = \varnothing$, which is absurd.
Conversely, assume that there is no prime $\mathfrak{p} \subset \mathfrak{p}_1 \cap \mathfrak{p}_2$. We want to show that there are $f_1 \in R \setminus \mathfrak{p}_1, f_2 \in R \setminus \mathfrak{p}_2$ with $f_1 f_2 = 0$, since that would give us $D(f_1) \cap D(f_2) = D(f_1 f_2) = \varnothing$ and $\mathfrak{p}_i \in D(f_i)$, $i = 1, 2$. Assume that there are no such elements f_1, f_2. Let S and T denote the multiplicative sets $R \setminus \mathfrak{p}_1$ and $R \setminus \mathfrak{p}_2$, then ST is also a multplicative set because $0 \notin ST$ by assumption. Moreover note that $S \subset ST$ and $T \subset ST$, since $1 \in S \cap T$. Now there is a prime ideal $\mathfrak{p}$, maximal with respect to being disjoint from ST (Kaplansky's Theorem 1), but this means $\mathfrak{p} \subset R \setminus ST \subset R \setminus S \cap R \setminus T = \mathfrak{p}_1 \cap \mathfrak{p}_2$ which is a contradiction. ■

Exercise II.15. *Show that $\operatorname{Spec}(R)$ is an irreducible space iff $R/\operatorname{Nil}(R)$ is an integral domain. (Conceptually, this is the same as Exercise 21(2) in Ch. I.)*

[mreyes@math] As a preliminary we prove the fact that an ideal $\mathfrak{p} \subsetneqq R$ is prime iff, for ideals $I, J \subseteq R$, $IJ \subseteq \mathfrak{p}$ implies that one of I or J is contained in $\mathfrak{p}$. Indeed, if $\mathfrak{p}$ is an ideal with this property, then $ab \in \mathfrak{p}$ implies that $(a)(b) = (ab) \subseteq \mathfrak{p}$. Then, without loss of generality, we have $a \in (a) \subseteq \mathfrak{p}$ and $\mathfrak{p}$ is prime. Now suppose that there exist ideals $I, J \nsubseteq \mathfrak{p}$ with $IJ \subseteq \mathfrak{p}$. Then there exist $a \in I \smallsetminus \mathfrak{p}$ and $b \in J \smallsetminus \mathfrak{p}$, with $ab \in IJ \subseteq \mathfrak{p}$. Hence $\mathfrak{p}$ is not prime.

This means that a prime lies above IJ iff it lies above either I or J, or $\mathcal{V}(IJ) = \mathcal{V}(I) \cup \mathcal{V}(J)$. As another side note, consider that $\mathcal{V}(I) = \operatorname{Spec}(R)$ iff $I \subseteq \operatorname{Nil}(R)$.

We are now ready to solve the problem. The space $\operatorname{Spec}(R)$ is irreducible iff, for any ideals $I, J \subseteq R$ with $\operatorname{Spec}(R) = \mathcal{V}(I) \cup \mathcal{V}(J) = \mathcal{V}(IJ)$, we have that $\operatorname{Spec}(R)$ is equal to one of $\mathcal{V}(I), \mathcal{V}(J)$. But this is true iff $IJ \subseteq \operatorname{Nil}(R)$ implies that one of I or J is contained in $\operatorname{Nil}(R)$. This happens iff $\operatorname{Nil}(R)$ is prime, iff $R/\operatorname{Nil}(R)$ is an integral domain ■

Exercise II.16. *Let $S \subseteq T$ where S is Hilbert and T is ring-finite over S. Show that any maximal ideal in T contracts to a maximal ideal in S.*

[Jonah (jblasiak@math)] Let m be maximal in T and let ϕ be the inclusion of S into T. The composition $S \hookrightarrow T \to T/m$ factors through $A := S/\phi^{-1}(m)$, yielding the injection $A \overset{\alpha}{\hookrightarrow} T/m$. A is a domain and it follows easily from the definition that A is Hilbert. Now we'll assume that A is not a field and arrive at a contradiction.

Let i be the inclusion of A into its field of fractions $Q(A)$. There exists a unique map β from $Q(A)$ to T/m such that $\beta \circ i = \alpha$, since all elements of A are sent by α to something invertible. The map ϕ being ring-finite easily implies the same for α and β. $Q(A)$ is a field so β is injective and lemma 4.4 implies that β is a finite field extension. We will show below that i is ring-finite. Assuming this, let $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_k}{b_k}$ be generators for $Q(A)$ as an A-algebra (where the a_i and b_i are in A). Then $\frac{1}{b_1 b_2 \dots b_k}$ generates $Q(A)$ as an A-algebra. Since A is a Hilbert domain, we can choose a maximal ideal $m' \subseteq A$ such that $b_1 b_2 \dots b_k \notin m'$. Then localizing at m' yields $Q(A)$ since $b_1 b_2 \dots b_k$ is inverted, but contains the maximal ideal m' as well as the non-maximal ideal (0), contradiction.

To see that i is ring-finite, let $t_1, \dots, t_r$ be generators for T/m as an A-algebra. Let $v_1, \dots, v_s$ be a vector space basis for T/m over $Q(A)$, where $v_1 = 1$. Now we can write $t_i = \sum_j c_{ij} v_j$, with $c_{ij} \in Q(A)$, and $v_i v_j = \sum_k d_{ijk} v_k$, with $d_{ijk} \in Q(A)$. Any $u \in Q(A)$ can be written as $\sum_\nu a_\nu \mathbf{t}^\nu$ with the $a_\nu \in A$, $\nu \in \mathbb{N}^s$. Since each $\mathbf{t}^\nu$ can be multiplied out and expressed as a linear combination of the v_i using the above relations, we have $u = \sum_i e_i v_i$, where e_i is a polynomial in the c_{ij} and d_{ijk} with coefficients in A . Then we have $u = \sum_i e_i v_i = e_1 v_1 = e_1$ because $u \in Q(A)$ and the v_i are a basis. Therefore the c_{ij} and d_{ijk} generate $Q(A)$ as an A-algebra. ∎

Exercise II.17. *Let $f : R \to S$ be a ring homomorphism. Then $\mathfrak{p} \in \operatorname{Spec} R$ is in the image of $f^* : \operatorname{Spec} S \to \operatorname{Spec} R$ iff $\mathfrak{p} = \mathfrak{p}^{ec}$.*

[Lars Kindler, lars_k@berkeley.edu] First let $\mathfrak{p} \in \operatorname{Im}(f^*)$, which means there is a $\mathfrak{q} \in \operatorname{Spec} S$, such that $\mathfrak{q}^c = f^{-1}(\mathfrak{q}) = \mathfrak{p}$. Then we have $\mathfrak{p}^e = (\mathfrak{q}^c)^e \subset \mathfrak{q}$ and thus $\mathfrak{p} \subset \mathfrak{p}^{ec} \subset \mathfrak{q}^c = \mathfrak{p}$.
Conversely, let $\mathfrak{p} = \mathfrak{p}^{ec} = f^{-1}(f(\mathfrak{p})S)$. Then we have $\mathfrak{p} \subseteq f^{-1}(f(\mathfrak{p})) \subseteq f^{-1}(f(\mathfrak{p})S) = \mathfrak{p}^{ec} = \mathfrak{p}$, and thus $f(\mathfrak{p}) = f(\mathfrak{p})S = \mathfrak{p}^e$, and since $\mathfrak{p}$ is prime this implies $\mathfrak{p}^e$ is prime, hence $\mathfrak{p} \in \operatorname{Im}(f^*)$. ∎

Exercise II.18. *Show that a topological space X is noetherian iff all subspaces (respectively, all open subspaces) of X are compact.*

[Manuel Reyes; mreyes@math] If all subspaces of X are compact, then all open subspaces of X are certainly compact.

Suppose that all open subspaces of X are compact, and let $U_1 \subseteq U_2 \subseteq \cdots$ be an ascending chain of open subsets of X. Then $U = \bigcup U_i$ is an open subset and thus is compact. This means that the open cover $\{U_i\}$ has a finite subcover.

Then choosing a maximal element U_n of that subcover, it is clear that $U = U_n = U_{n+1} = \cdots$. So X satisfies the ACC on open subsets and thus is noetherian.

Finally suppose that X is noetherian, and let $Y \subseteq X$ be any subspace. To show that Y is compact, let $\{U_i\}$ be any nonempty open cover of Y. Denote by $\mathcal{F}$ the family of finite unions of elements of the cover $\{U_i\}$. Because $\mathcal{F} \neq \varnothing$ it has a maximal element, say $V := U_{i_1} \cup \cdots \cup U_{i_n}$. Assume for contradiction that $Y \subsetneq V$. Then choose some point $x \in Y \smallsetminus V$, and choose some U_j containing x. Then $V \subsetneq V \cup U_j \in \mathcal{F}$, contradicting the maximality of V. So $U_{i_1}, \ldots, U_{i_n}$ is a finite subcover of Y and thus Y is compact. ■

Exercise II.19. *Let M be a f.g. R-module. What are the irreducible components of* Supp(M) *(as a subspace of* Spec(R)*)? Show that the number of irreducible components is finite if M is a noetherian module. (***Hint.** *Let $I = \operatorname{ann}(M)$, and use Exercise 12 in conjunction with (I.3.12) and (II.3.12)!)*

[los@math] Let $I = \operatorname{ann}(M)$. By (I.3.12), we have $\operatorname{Supp}(M) = \mathcal{V}(I)$. The set $\mathcal{V}(I)$ is homeomorphic to $\operatorname{Spec}(R/I)$ by Exercise 12, and for any ideal J of R containing I, the set $\mathcal{V}(J) \subseteq \mathcal{V}(I)$ corresponds to $\mathcal{V}(J/I) \subseteq \operatorname{Spec}(R/I)$ under this identification. By (II.3.12(2)) then, the irreducible components of $\mathcal{V}(I)$ are the the sets $V(\mathfrak{p})$ for each prime $\mathfrak{p}$ of R minimal over I. If R/I is noetherian, then by (I.4.13) applied to the ideal (0) of R/I, there will be only finitely many primes of R minimal over I, and therefore only finitely many irreducible components in $\operatorname{Supp}(M)$. Thus in order to prove the last statement, it will be enough to show that if M is a faithful noetherian module over a ring A (which in our case will be R/I), then the ring A is itself noetherian. Assume such an A and M given. Then M is generated by finitely many elements, say $m_1, m_2, \ldots, m_n$. We define a linear map $\varphi : A \to M^n$ by $\varphi(a) = (am_1, am_2, \ldots, am_n)$. Because the elements $m_1, m_2, \ldots, m_n$ generate M we have $\ker(\varphi) \subseteq \operatorname{ann}(M)$. But $\operatorname{ann}(M) = 0$ because M is faithful. Therefore A is isomorphic to an A-submodule of the noetherian A-module M^n, and hence is a noetherian A-module itself. By definition, A is a noetherian ring. This completes the proof. ■

Exercise II.20. *For any field k, show that any non-maximal prime ideal in $A = k[x, y]$ is principal. [If necessary, you may use the fact (to be proved in Chap. III) that prime chains in A have length ≤ 2.] Assuming k is algebraically closed, give a 1-minute running commentary on the following picture of* Spec(A) *in Mumford's "Little Red Book": (picture omitted)*

[los@math] We will show that if R is any principal ideal domain, then any non-maximal prime ideal of $R[t]$ is principal. The result of the exercise will follow, taking $R = k[x]$.

Let $\mathfrak{p}$ be any non-maximal prime ideal of $R[t]$. $\mathfrak{p} \cap R$ is a prime ideal $\mathfrak{p}'$ of R. Assume first $\mathfrak{p}'$ is not zero. Then it is (p) for some irreducible $p \in R$. Since R is a principal ideal domain, $R/(p)$ is a field. Therefore, any nonzero prime ideal of $R[t]/pR[t] \cong (R/(p))[t]$ is maximal. $\mathfrak{p}$ is a prime ideal of $R[t]$ which contains $pR[t]$ and is non-maximal, hence is equal to $pR[t]$. Assume therefore that $\mathfrak{p} \cap R = \{0\}$. We may assume $\mathfrak{p} \neq 0$. If Q is any nonzero element of $\mathfrak{p}$, then we may write Q as a product of irreducible factors, one of which, say P, must belong to $\mathfrak{p}$. We will show that $\mathfrak{p} = (P)$, completing the proof. Otherwise, we have an ascending chain of three prime ideals $\{0\} \subset (P) \subset \mathfrak{p}$, the ideal (P) being prime because P is irreducible and $R[t]$ is a unique factorization domain. The intersection of each of these prime ideals with the multiplicative subset $S = R \setminus \{0\}$ of $R[t]$ is empty. Let K denote the fraction field of R. By the proof of Exercise 12, the localizations in $S^{-1}R[t] \cong K[t]$ of these ideals form a chain of prime ideals of length 2 in $K[t]$. But this is absurd, since $K[t]$ is a principal ideal domain.

Thus the prime ideals of $A = k[x, y]$ are of three kinds: $\{0\}$, (f) for some irreducible $f \in A$, and maximal ideals. The last two kinds are mutually exclusive since, by the Nullstellensatz, if $\mathfrak{m}$ were a maximal ideal of A, then $A/\mathfrak{m}$ would be an algebraic field extension of k. However, it is clear that for an irreducible nonzero $f \in A$, the transcendence degree of $A/(f)$ over k is 1. Now assume that k is algebraically closed. The closed points in $\mathrm{Spec}(A)$ correspond bijectively to the points of k^2. The ideal $\{0\}$ is the generic point of the plane $\mathrm{Spec}(A)$. The remaining prime ideals of A are of the form (f) for some irreducible f, and are the generic points of the curves $f(x, y) = 0$. Here f is determined up to a nonzero constant in k. ■

Exercise II.21. *Let K be a field containing $\overline{k}$. For any k-algebraic set $Y \subset K^n$, show that, in the k-topology, Y_{alg} is dense in Y.*

[David Brown, browndа@math] Note that $Y_{\mathrm{alg}} = V_{\overline{k}}(J)$. By the Nullstellensatz, $I(V_{\overline{k}}(J)) = \sqrt{J}$. Thus, $\overline{Y_{\mathrm{alg}}} = V_K(I(Y_{\mathrm{alg}})) = V_K(\sqrt{J}) = V_K(J) = Y$. ■

[lam@math] *Comment.* Those who dig characteristic p would want to further amend this exercise into the following: "Y_{sep} is dense in Y" (where Y_{sep} denotes the set of $(y_1, \ldots, y_n) \in Y$ with all y_i separably algebraic over k). The proof takes another couple of lines. ■

Exercise II.22. *It was pointed out (after (4.12)) that 0-dimensional rings are Hilbert. Coversely, show that semilocal Hilbert rings are 0-dimensional.*

[mreyes@math] Let R be a semilocal Hilbert ring. Then any prime $\mathfrak{p}$ of R is an intersection of finitely many maximal ideals and thus contains the product of these finitely many maximal ideals. Because $\mathfrak{p}$ is prime it must then contain one of these maximal ideals (see the solution to problem II.15), so that $\mathfrak{p}$ itself is maximal. So every prime of R is maximal, and R is 0-dimensional. ■

Exercise II.23. *Show that a 1-dimensional noetherian domain is Hilbert iff it is not semilocal.*

[Jonah (jblasiak@math)] Corollary 5.7 states that a ring R is semilocal iff $R/\mathrm{rad}(R)$ is artinian. A 1-dimensional noetherian domain R is Hilbert iff (0) is an intersection of maximal ideals iff $\mathrm{rad}(R) = 0$ iff $R/\mathrm{rad}(R)$ is not artinian. This last iff is because artinian is equivalent to 0-dimensional and neotherian, and R is 1-dimensional. The exercise then follows from the corollary. ■

Exercise II.24. *Discuss the Ping Pong Paddle Picture in more detail in the case where* $k = \mathbb{Q}$*,* $K = \mathbb{R}$*, and* Y *is the real parabola* $V_K(y - x^2)$*.*