Notes for Geometric Invariant Theory

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Contents

1 Invariants and Quotients

Let $\bar{k} = k$ be an algebraically closed field.

Definition 1.1. An *(affine) algebraic group* G over k is a group object in the category of affine varieties over k. i.e. an affine variety G, together with morphisms of varieties $\mu\colon G \times G \to G$ (multiplication), $i\colon G \to G$ (inverse), and $e\colon \operatorname{Spec} k \to G$ (identity) satisfying the usual relations. ◇

Example 1.2. The additive group $\mathbb{G}_a = (k, +)$, the multiplicative group $\mathbb{G}_m = (k^\times, \times)$, and the general linear group $GL(n) = \{X \in \mathrm{Mat}_{n\times n}(k) | X \text{ invertible}\}$ are examples of algebraic groups. ◇

Let $X = \operatorname{Specm}(k[x_1, \dots, x_n]/I_X)$ be a affine variety (i.e. X is the set of zeros in k^n of some ideal $I_X \subseteq k[x_1, \dots, x_n]$). The *coordinate ring* of X is $k[X] = k[x_1, \dots, x_n]/I_X$. We will usually suppose that $k[X]$ is reduced. Suppose we have an action of an algebraic group G on X. We'd like to construct and study the quotient X/G. We can take the topological space quotient $X/_{\mathrm{top}}G$, but in general, this quotient will not be an algebraic variety. The question is how to make a quotient which is a variety.

Invariants

One approach is to consider the natural action of G on the ring $R = k[X]$. A regular function on X/G should correspond to a regular function on X which is constant on G-orbits. So one candidate for X/G is $\operatorname{Specm} R^G$, where $R^G = \{f \in R | f(gx) = f(x)\}$. Note that we get a map $\psi\colon X/_{\mathrm{top}}G \to \operatorname{Specm} R^G$ by sending a G-orbit Gx to the maximal ideal $\{f \in R^G | f(Gx) = 0\} \subseteq R^G$.

A number of questions arise naturally.

1. Is R^G finitely generated?

Hilbert proved that the answer to the first question is *yes* in case where $G = GL(n)$ and $\operatorname{char}(k) = 0$. More generally, the answer is *yes* for reductive groups (Corollary 3.6), but *no* for arbitrary groups (there is a famous example due to Nagata [[★★★ ref]]).

2. Is the map ψ is an isomorphism of topological spaces? Do points of $\operatorname{Specm} R$ parameterize G-orbits?

In general, the answer to this question is *no*.

Example 1.3. Let $X = \mathbb{A}^n = k^n$ and $G = \mathbb{G}_m = k^\times$ acting on X by homothety: $t \cdot (x_1, \dots, x_n) = (tx_1, \dots, tx_n)$. Then $R = k[X] = k[x_1, \dots, x_n]$, with the action $t \cdot x_i = tx_i$. The only invariant polynomials are the ones where no x_i appears, the constants, so $R^G = k$ and $\operatorname{Specm} R^G$ is a single point. But there is a G-orbit for each direction (and one orbit containing just the origin). ◇

More generally, if Gx is a non-closed orbit, then any invariant regular function must have the same value on the closure $\overline{Gx}$ as it has on the orbit Gx. Thus, if Gx and Gy are two orbits whose *closures* intersect, there is no invariant regular function that separates them, so they have the same image in $\operatorname{Specm} R^G$. So if ψ is to be an isomorphism, all orbits of the action must be closed.

Proj quotients

Instead of looking at the ring of regular functions, we can consider field of rational functions $K = \operatorname{Frac}(R)$. A rational function on X/G should be a rational function on X which is constant on G-orbits, so the rational functions on X/G should be $K^G = \{f \in K | f(gx) = f(x)\}$. This picks out a birational class that X/G should belong to.

Example 1.4 (1.3 continued)**.** In Example 1.3, we get $K^G = \{f(x)/g(x) | f, g \in R$ homogeneous with $\deg(f) = \deg(g)\}$. We can "cover" X with $D(x_i) = \{f(x)/x_i^m | \deg(f) = m\}$ (this misses the origin, so it isn't actually a cover). For these open sets, we can form nice quotient varieties by taking max-spectra of the rings of invariants. Then we can glue the quotients together to get $\mathbb{P}^{n-1}$. ◇

It turns out that this construction generalizes to something called a *GIT quotient* or a *proj quotient*. In our example, the GIT quotient accounts for every orbit except the origin.

Continuing our list of questions,

3. Is K^G the field of fractions of R^G?

Our example shows us that the answer is *no* in general. But in many cases, the answer is *yes*. In particular, the answer is *yes* if X has at least one *stable orbit*[[★★★ ref eventually]].

Definition 1.5. We say that the action of G on X is *closed* if the orbit of any point is closed. ◇

Example 1.6. If G is finite, then the action is always closed. ◇

Example 1.7. $(x, y) \mapsto (x + t, y)$ is a closed action of $\mathbb{G}_a$ on $\mathbb{A}^2$. ◇

Example 1.8. For some action of a group G on a variety X, you can remove all the G-orbits which are in the closures of other G-orbits. Then the action of G on the remaining space is closed. ◇

Example 1.9 (Example of 1.8)**.** Let $X = \operatorname{Mat}_{n\times n}(k)$ with $G = GL(n)$ acting by $g \cdot A = gAg^{-1}$ for $g \in G$ and $A \in X$. Then $R = k[x_{11}, \dots, x_{nn}]$. The coefficents of the characteristic polynomial $\det(tI_n - A) = t^n - \sigma_1 t^{n-1} + \cdots \pm \sigma_n$ are invariant regular functions. For example σ_1 is trace of A and σ_n is the determinant of A. I claim that $R^G = k[\sigma_1, \dots, \sigma_n]$.

The G-orbits correspond to possible Jordan forms. Notice that if you conjugate a Jordan block by a diagonal matrix, you can change the 1s on the first superdiagonal to any other non-zero entries. Any polynomial function that vanishes on all these matrices must also vanish on the matrix where the entries on the superdiagonal are zero. This shows that the orbit corresponding to a non-trivial Jordan block contains the corresponding diagonal matrix in its closure. [[★★★ Charley tells me that more generally, reductive groups have the (characterizing?) property that when they act on an affine scheme, every orbit has a unique closed orbit in its closure. Find ref (in [MFK94]?)]]

Therefore, any invariant function is completely determined by its values on diagonalizable matrices. So on a given matrix, its values are completely determined by the set of eigenvalues (with multiplicity). Any such function must be an algebraic combination of the elementary symmetric functions on the eigenvalues, which are exactly the σ_i, so the σ_i generate R^G. By the way, the σ_i are also algebraically independent [Sta99, Theorem 7.4.4].

Note that in this case, the answer to question 3 is *yes*. ◇

Relationship to Moduli Spaces

Moduli spaces (i.e. parameter spaces for some class of object) can often be constructed as quotients by some group action.

Consider the following problem: llassify degree 2 curves in $\mathbb{A}^2$ up to th action of the Euclidean group $G = SO(2) \rtimes G_0$ (generated by rotations and translations).

A degree 2 curve is given by an equation $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$. This can be represented by a symmetric matrix v by noting that

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = \begin{pmatrix} x & y & 1 \end{pmatrix} \overbrace{\begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}}^{v} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

in which case G may be represented as the group of matrices of the form $\begin{pmatrix} p & q & l \\ -q & p & m \\ 0 & 0 & 1 \end{pmatrix}$ where $p^2 + q^2 = 1$, and the action is given by $v \mapsto g^t v g$. The reader can easily check that this action corresponds to the change of coordinates obtained by applying the rotation matrix $\left(\begin{smallmatrix} p & q \\ -q & p \end{smallmatrix}\right)$ and then translating by (l, m): $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} p & q \\ -q & p \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + \left(\begin{smallmatrix} l \\ m \end{smallmatrix}\right)$.

The ring of regular functions on the quotient should be $R^G = k[a, b, \dots, f]^G$? Since the original space is 6-dimensional and the group is 3-dimensional, we hope to find at least $6 - 3 = 3$ invariants to generate R^G.

Since the determinant of any element of G is 1, the determinant D of v is an invariant regular function. The "translation part" of G (where $p = 1$ and $q = 0$) doesn't change a, b, or c, and the "rotation part" doesn't change trace or determinant of $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$, so $E = \det\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) = ac - b^2$ and $T = \operatorname{tr}\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) = a + c$ are two more invariants.

Proposition 1.10. $R^G = k[D, E, T]$.

Incidently, it's also true that D, E, and T are algebraically independent.

Proof. By an argument similar to the one given in Example 1.9, $k[a, b, c]^{SO(2)} \cong k[E, T]$ [[★★★ I haven't checked this yet]]. On the other hand, a, b, c are invariant with respect to the action of G_0. So it suffices to prove that $k[a, b, c, d, e, f]^{G_0} = k[a, b, c, D]$. We have that

$$k[a, b, c, d, e, f]^{G_0} \subseteq k[a, b, c, d, e, f, 1/E]^{G_0}$$

We will try to find the G_0-invariants of the larger ring and then intersect with the smaller ring to get the invariants of the smaller ring. Using the equation $D = fE + 2bde - cd^2 - ae^2$, we may replace the generator f by D. Now suppose we have a G_0-invariant function h. It must satisfy the relation

$$h(a, b, c, d, e, D, 1/E) = h(a, b, c, d + al + bm, e + bl + cm, D, 1/E)$$

For all values of a, b, c, d, e, D, l, and m (for which $ac - b^2 \neq 0$). Choosing a, b, and c to be distinct non-zero values, we can choose values for l and m to replace d and e by an arbitrary pair of values. So on the dense open subset where a, b, and c are distinct non-zero values, h must be a polynomial that is independent of d and e. But this implies that h must globally be independent of d and e.

So we have shown that $k[a, b, c, d, e, f, 1/E]^{G_0} = k[a, b, c, D, 1/E]$. The intersection of this ring with $k[a, b, c, d, e, f]$ is $k[a, b, c, D]$, so $k[a, b, c, d, e, f]^{G_0} = k[a, b, c, D]$, as desired. □

But two equations that differ by a scalar give the same curve, so we haven't yet found the moduli space of degree 2 curves. Now we consider the bigger group $\tilde{G}$ generated by G and $k^\times$ (acting by scalar matrix). Now we want to compute the ring of invariants $R^{\tilde{G}} = k[D, E, T]^{k^\times}$. D, E, and T are all homogeneous, but they are not at all invariant. The action by $k^\times$ is given by $D \mapsto r^3 D$, $E \mapsto r^2 E$, and $T \mapsto rT$. So $R^{\tilde{G}} = k$. This is not a good quotient, so we try removing some stuff.

Let's restrict to the curves for which $D \neq 0$ (so the corresponding quadratic is non-degenerate). Now let's find $k[D^{\pm 1}, E, T]^{k^\times}$. Now we have invariants $A = E^3/D^2$, $B = T^3/D$, $C = ET/D$.

Exercise. *Show that A, B, C generate the ring of invariants.*

But they are not algebraically independent since $AB = C^3$. I claim that this is the only relation. So the geometric quotient, the moduli space of (non-degenerate) degree 2 curves in $\mathbb{A}^2$, is the singular surface $\operatorname{Specm}(k[A, B, C]/(AB - C^3))$.

Classical binary invariants. Consider $SL(2, \mathbb{C})$. The finite-dimensional irreducible representations V_d are given by non-negative integers. You can think of V_d as $\{f(x, y) \mid \deg f = d\}$. with the obvious action of $SL(2, \mathbb{C})$: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot f(x, y) = f(ax + by, cx + dy)$.

2 Affine Geometric Quotients

An affine algebraic group G over an algebraically closed field k is an affine variety G together with algebraic maps $m\colon G \times G \to G$, $i\colon G \to G$, and $e\colon * \to G$, satisfying the following commutative diagrams:

$$
\begin{tikzcd}
G \times G \times G \arrow[r, "m \times 1_G"] \arrow[d, "1_G \times m"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, shift left, "e \times 1_G"] \arrow[r, shift right, "1_G \times e"'] \arrow[dr, "1_G"'] & G \times G \arrow[d, "m"] \\
& G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times G \arrow[rr, shift left, "1_G \times i"] \arrow[rr, shift right, "i \times 1_G"'] & & G \times G \arrow[d, "m"] \\
G \arrow[u, "\Delta"] \arrow[r] & * \arrow[r, "e"] & G
\end{tikzcd}
$$

This is equivalent to the coordinate ring $k[G]$ being equipped with the structure of a (commutative) *Hopf algebra*: algebra maps $\mu\colon k[G] \to k[G] \otimes k[G]$ (*comultiplication*), $\tau\colon k[G] \to k[G]$ (*antipode*), and $\varepsilon\colon k[G] \to k$ (*counit*) satisfying the following commutative diagrams:

$$
\begin{tikzcd}
k[G] \arrow[r, "\mu"] \arrow[d, "\mu"'] & k[G] \otimes k[G] \arrow[d, "\mu \otimes \mathrm{id}"] \\
k[G] \otimes k[G] \arrow[r, "\mathrm{id} \otimes \mu"] & k[G] \otimes k[G] \otimes k[G]
\end{tikzcd}
\qquad
\begin{tikzcd}
& k[G] \arrow[dl, "\mu"'] \arrow[d, "\mathrm{id}"] \\
k[G] \otimes k[G] \arrow[r, shift left, "\varepsilon \otimes \mathrm{id}"] \arrow[r, shift right, "\mathrm{id} \otimes \varepsilon"'] & k[G]
\end{tikzcd}
\qquad
\begin{tikzcd}
k[G] \arrow[r, "\varepsilon"] \arrow[d, "\mu"'] & k \arrow[r] & k[G] \\
k[G] \otimes k[G] \arrow[rr, shift left, "\mathrm{id} \otimes \tau"] \arrow[rr, shift right, "\tau \otimes \mathrm{id}"'] & & k[G] \otimes k[G] \arrow[u, "m"']
\end{tikzcd}
$$

Basically, if you're working with an affine algebraic group, you can forget the group and just remember the Hopf algebra structure on its ring of regular functions. To go back and forth between the Hopf algebra structure on $k[G]$ and the group structure on G, you use the following relations for $f \in k[G]$ and $g, h \in G$ ($e \in G$ is the identity element).

$$\boxed{\mu(f)(g,h) = f(gh) \qquad \tau(f)(g) = f(g^{-1}) \qquad \varepsilon(f) = f(e)}$$

Example 2.1. If $G = \mathbb{G}_m = k^\times$, then $k[G] = k[t, t^{-1}]$. We have $\mu(t) = t \otimes t$, $\tau(t) = t^{-1}$, and $\varepsilon(t) = 0$. ◇

Example 2.2. If $G = \mathbb{G}_a = k$, then $k[G] = k[t]$. We have $\mu(t) = t \otimes 1 + 1 \otimes t$, $\tau(t) = -t$, and $\varepsilon(t) = 1$. ◇

Representions of Algebraic Groups

If V is a finite-dimensional vector space, then a *representation* of G on V is a morphism of algebraic groups $\rho\colon G \to GL(V)$. If V is infinite-dimensional (and we will need infinite-dimensional representations), then you have to be a bit more delicate. So use the notion of a comodule.

Definition 2.3. A *G-comodule* structure on a vector space V is a k-linear map $\sigma\colon V \to k[G] \otimes V$ (called a *coaction*) satisfying the diagrams

$$\begin{tikzcd} V \arrow[rr, "\sigma"] \arrow[d, "\sigma"'] & & k[G]\otimes V \arrow[d, "\mu\otimes 1_V"] \\ k[G]\otimes V \arrow[rr, "1_G\otimes\sigma"] & & k[G]\otimes k[G]\otimes V \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\sigma"] \arrow[dr, "1_V"'] & k[G]\otimes V \arrow[d, "\varepsilon\otimes 1_V"] \\ & V \end{tikzcd}$$

A morphism of comodules is a k-linear map $\phi\colon V \to W$ that intertwines the coactions: $\sigma_W \circ \phi = (\mathrm{id} \otimes \phi) \circ \sigma_V$. In particular, a *subcomodule* (or *invariant subspace*) is a subspace $W \subseteq V$ such that $\sigma(W) \subseteq k[G] \otimes W$. ◇

Remark 2.4. Let's check that the notion of a finite-dimensional comodule corresponds to the notion of a finite-dimensional representations.[[★★★ what's wrong with the naive notion of an infinite-dimensional representation? Is $GL(V)$ not finite type or is it not even algebraically definable (what is determinant?)?]]

Given a vector space V with basis $\{e_1, \dots, e_n\}$ and comodule structure $\sigma\colon V \to k[G] \otimes V$, we have $\sigma(e_i) = \sum_j f_{ij} \otimes e_j$ for some $f_{ij} \in k[G]$. Then we can define $\rho^*\colon k[GL(V)] \to k[G]$ by $x_{ij} \mapsto f_{ij}$ and verify that the axioms of a comodule imply that the induced map of varieties $\rho\colon G \to GL(V)$ is a group homomorphism.

Conversely, if ρ is a representation, then we can define a coaction $\sigma\colon V \to k[G] \otimes V$ by $e_i \mapsto \sum_j \rho^*(x_{ij}) \otimes e_j$. The fact that ρ is a group homomorphism implies that σ satisfies the axioms of a coaction.

So from now on, we'll use the terms "comodule" and "representation" interchangably. To go back and forth between the two, use the relation that for $g \in G$ and $v \in V$,

$$\boxed{\sigma(v) = \sum f_i \otimes v_i \quad \iff \quad g \cdot v = \sum f_i(g) v_i.}$$ ◇

Example 2.5. Let $G = k^\times$. Given $m \in \mathbb{Z}$, we get a 1-dimensional representation given by the action $t \cdot v = t^m v$. Using Remark 2.4, you may check that this corresponds to the coaction $v \mapsto t^m \otimes v$. ◇

Remark 2.6 (Hom and $\otimes$)**.** For any pair of representations V and W, the tensor product $V \otimes W$ has the structure of a represenation, given by the action $g \cdot (v \otimes w) = gv \otimes gw$ (i.e. the coaction $v \otimes w \mapsto \sum f_i g_j \otimes v_i \otimes w_j$, where $v \mapsto \sum f_i \otimes v_i$ and $w \mapsto \sum g_j \otimes w_j$).

Furthermore, $\mathrm{Hom}_k(V, W)$ has the structure of a representation given by the action $(g \cdot \phi)(v) = g \cdot \big(\phi(g^{-1} \cdot v)\big)$.[1] In particular, the dual of a representation $V^* = \mathrm{Hom}_k(V, k)$ has the structure of a representation (k is interpreted as the trivial representation), making $\mathrm{Hom}_k(V, W) \cong V^* \otimes W$ an isomorphism of representations.

[1] The corresponding coaction is

$$\big(v \mapsto \phi(v)\big) \mapsto \big(v \mapsto (m \otimes \mathrm{id}_V)(\mathrm{id}_{k[G]} \otimes \tau \otimes \mathrm{id}_V)(\mathrm{id}_{k[G]} \otimes \sigma)(\mathrm{id}_{k[G]} \otimes \phi)\sigma(v)\big).$$

There is a string diagram yoga to figuring out these coactions.

The invariants $\mathrm{Hom}_k(V,W)^G$ consist of those linear maps ϕ for which $\phi(v) = g\cdot\phi(g^{-1}\cdot v)$ for all $g \in G$ and $v \in V$. This is exactly the space of G-equivariant maps (i.e. morphisms of representations) $\mathrm{Hom}_G(V,W) = \{\phi \in \mathrm{Hom}_k(V,W) | g\cdot\phi(v) = \phi(g\cdot v)\}$. ◇

Proposition 2.7. *Any representation V of an affine group G is a union of finite-dimensional representations.*

Proof. It is enough to show that any vector $v \in V$ lies in a finite-dimensional representation. We have that $\sigma(v) = \sum_{i=1}^{N} f_i \otimes v_i$, a finite sum in which we can choose the f_i to be linearly independent. Now consider the space $M_v = \langle v_1, \ldots, v_N\rangle$. First of all, $v = \sum \varepsilon(f_i)v_i$ by one of the axioms of a coaction, so $v \in M_v$. Next we'll show that M_v is an invariant subspace. To see that, note that

$$\sum \mu(f_i)\otimes v_i = (\mu\otimes 1_V)\sigma(v) = (\mathrm{id}\otimes\sigma)\sigma(v) = \sum f_i \otimes \sigma(v_i)$$

by the other axiom of a coaction. Since the f_i are linearly independent, we can choose linear functionals $\lambda_i \in \mathrm{Hom}_k(k[G],k)$ such that $\lambda_i(f_j) = \delta_{ij}$. Applying $\lambda_i \otimes \mathrm{id}_{k[G]} \otimes \mathrm{id}_V$ to the left-hand side of the equation, we clearly get an element of $k[G]\otimes M_v$, and applying it to the right-hand side, we get $\sigma(v_i)$. So $\sigma(v_i)$ is in $k[G]\otimes M_v$, as desired. □

Remark 2.8. The linear functional trick at the end of the proof of Proposition 2.7 also shows that if $\sum f_i \otimes v_i = \sum f_i \otimes w_i$ and the f_i are linearly independent, then $v_i = w_i$. ◇

Remark 2.9. The space M_v constructed in the proof of Proposition 2.7 is actually *the smallest* invariant subspace containing v. To see this, it suffices to show that for any invariant subspace W containing v must contain each of the v_i. Since W is invariant, σ sends W into $k[G]\otimes W$, and since $v \in M$, we must have $\sigma(v) = \sum_{j=1}^{S} h_j \otimes w_j$ for some $h_j \in k[G]$ and $w_j \in W$. Applying the linear functional $\lambda_i \otimes \mathrm{id}$ (where $\lambda_i(f_j) = \delta_{ij}$) to the equality $\sum_{i=1}^{N} f_i \otimes v_i = \sum_{j=1}^{S} g_j \otimes w_j$, we get $v_i = \sum_{j=1}^{S} \lambda_i(g_j)w_j$ which is clearly in W. ◇

Corollary 2.10. *Any irreducible representation of G is finite-dimensional.*

Proposition 2.11. *Any representation of $k^\times$ can be written as $V = \bigoplus_{m\in\mathbb{Z}} T_m$, where $T_m = \{v \in V | t(v) = t^m v\} = \{v \in V | \sigma(v) = t^m \otimes v\}$.*

Proof. It is clear that the $T_m \cap T_n = 0$ for $m \neq n$, so we only need to show that every $v \in V$ can be written as a sum of elements of the various T_m. We have that $\sigma(v) = \sum_{i=1}^{N} t^i \otimes v_i$ for some $v_i \in V$. By one of the axioms of a coaction, we have

$$v = (\varepsilon\otimes \mathrm{id}_V)\sigma(v) = \sum \varepsilon(t^i)\otimes v_i = \sum v_i$$

So it is enough to show that $v_i \in T_i$. Using the other axiom of a coaction, we have

$$\sum t^i \otimes t^i \otimes v_i = (\mu\otimes 1_V)\sigma(v) = (\mathrm{id}\otimes\sigma)\sigma(v) = \sum t^i \otimes \sigma(v_i)$$

By Remark 2.8, we get that $\sigma(v_i) = t^i \otimes v_i$, so $v_i \in T_i$ as desired. □

Definition 2.12. A *character* of an algebraic group G is a group homomorphism to $k^\times$. Equivalently, (using the weight 1 representation of $k^\times$) a character is a 1-dimensional representation of G. The set of characters $G^\vee = \mathrm{Hom}_{\mathsf{Gp}}(G, k^\times)$ has a group structure induced by the group structure on $k^\times$. The 1-dimensional representation associated to a character $\chi \in G^\vee$ is denoted by V_χ, and has the action $g \cdot v = \chi(g)v$ (or the coaction $v \mapsto \chi \otimes v$, where $\chi \in \mathrm{Hom}_{\mathsf{Gp}}(G, k^\times) \subseteq \mathrm{Hom}(G, k) = k[G]$ is regarded as a regular function on G). ◇

Example 2.13. By Proposition 2.11, $(k^\times)^\vee \cong \mathbb{Z}$. ◇

Example 2.14 (Algebraic Torus)**.** Let $G = \mathbb{G}_m^r = k^\times \times \cdots \times k^\times$. The group of characters $G^\vee$ is a free abelian group of rank r. ◇

Corollary 2.15 (to Proposition 2.11)**.** *Any representation V of an algebraic torus G may be written as $V = \bigoplus_\chi T_\chi$, where $T_\chi = \{v \in V | g \cdot v = \chi(g)v\} = \{v \in V | \sigma(v) = \chi \otimes v\}$.*

Sketch Proof. We have that $G = \mathbb{G}_m^r$. Restricting to each copy of $\mathbb{G}_m$ and applying Proposition 2.11, we get a direct sum decomposition of V. The desired decomposition is the common refinement of all of those decompositions. □

Proposition 2.16. *Suppose* $\mathrm{char}(k) = 0$*, and let V be a representation of $G = \mathbb{G}_a = k$. Then there exists a locally nilpotent operator $A \in \mathrm{End}_k(V)$ (i.e. for any $v \in V$, $A^{N(v)}v = 0$) such that the representation is given by $t \cdot v = \exp(tA)v = 1 + tAv + \frac{1}{2}t^2A^2v + \cdots$ (which terminates for each v because A is locally nilpotent), or by the corresponding coaction $v \mapsto \sum \frac{1}{p!}t^p \otimes A^p v$.*

Proof. For $v \in V$, $\sigma(v) = \sum_{m=0}^N t^m \otimes v_m$. As usual, we have

$$\sum_m (t \otimes 1 + 1 \otimes t)^m \otimes v_m = (\mu \otimes \mathrm{id}_V)\sigma(v) = (\mathrm{id}_{k[G]} \otimes \sigma)\sigma(v) = \sum_m t^m \otimes \sigma(v_m)$$

By Remark 2.8, we have

$$\begin{aligned}\sigma(v_m) &= \sum_{n=m}^{N} \binom{n}{m} t^{n-m} \otimes v_n = \sum_{p=0}^{N-m} \binom{m+p}{m} t^p \otimes v_{m+p} \\ &= \sum_{p=0}^{N-m} \frac{t^p}{p!} \otimes (m+p)(m+p-1)\cdots(m+1)v_{m+p}.\end{aligned}$$

So we define

$$Av_m = \begin{cases} (m+1)v_{m+1} & m < N \\ 0 & \text{else} \end{cases}$$

[[★★★ but the v_m may not be linearly independent, and they may not span.]][[★★★ You may define A on a space $W = \langle w_1, \ldots, w_N \rangle$ by $Aw_m = (m+1)w_{m+1}$. Then using the action $\exp(tA)$, W is a representation. The calculation above gives a map from W to V. This should show that you can define the operator A without worrying about linear independence of the v_m. Also, the operator A is unique. So you can define it on M_v, then on some M_w, and the operators on the intersection should agree, so you can extend to the sum of the two spaces]]. $\square$

Remark 2.17. Proposition 2.16 is a result about *algebraic* representations of $\mathbb{G}_a$. If you consider the additive group $G = \mathbb{G}_a$ over $\mathbb{C}$ and any endomorphism A of a vector space V (need not be locally nilpotent), having t act by $\exp(tA)$ gives V the structure of a representation of G. But if A is not locally nilpotent, the corresponding map $G \to GL(V)$ is not algebraic. Another way to say this is that the coaction $V \to k[G] \otimes V$ does not send every element of V into $k[G] \otimes V$, but into some *completion* $k[G]\widehat{\otimes}V$ (in this case, the completion with respect to the topology V inherits from $\mathbb{C}$[[★★★ I think]]). $\diamond$

Reductive Groups

Definition 2.18. An affine group G is *(linearly)*[2] *reductive* if any representation of G is completely reducible. That is, one of the following equivalent[3] conditions hold.

1. If $W \subseteq V$ is an invariant subspace, then it has an invariant direct complement (i.e. an invariant subspace $W' \subseteq V$ such that $V = W \oplus W'$).
2. $V = \bigoplus_{i\in I} V_i$ with the V_i irreducible. $\diamond$

Remark 2.19 (Isotypic components)**.** The decomposition $V = \bigoplus_{i\in I} V_i$ with the V_i irreducible is *not canonical*. For example, if the action of G on V is trival, then any direct sum decomposition of V into 1-dimensional vector spaces works. However, there is a canonical decomposition of V, called the decomposition into *isotypic components*.

Let J be the set of all irreducible representations of G. For a given $j \in J$, let V_j be the corresponding irreducible represetation, let T_j be the sum of all subrepresentations $W \subseteq V$ for which $W \cong V_j$. This T_j is called the *V_j-isotypic component of V*. We claim that $T_j \cong \bigoplus_{I_j} V_j$ for some index set I_j, and that there is a canonical decomposition $V = \bigoplus_{j\in J} T_j$ (canonical in the sense that the T_j are uniquely determined and respected by any morphisms of representations).

First, let's show that $T_j \cong \bigoplus_{I_j} V_j$. If $W \subseteq V$ is an irreducible subrepresentation and $U \subseteq V$ is some subrepresentation (may not be irreducible), then the intersection $U \cap W$ is an invariant subspace of W. Since W is irreducible, we either have $W \subseteq U$, in which case $W + U = U$, or $W \cap U = \{0\}$, in which case $W + U \cong W \oplus U$. So we can build up T_j as a direct sum of copies of V_j, one copy of V_j at a time, applying transfinite induction if we need to.

[2] We will usually drop the word "linearly".

[3] To see that (1) implies (2), use transfinite induction on the dimension of V. To see that (2) implies (1), [[★★★ you have to get that any partial decomposition of V into irreducibles can always be continued]].

By the assumption that V decomposes as a sum of irreducible subrepresentations, we know that $V = \sum_{j\in J} T_j$.[4] We will prove by (transfinite) induction that $V \cong \bigoplus_{j\in J} T_j$. For $j \in J$ and $S \subseteq J$ with $j \notin S$, assume (by induction) that $\sum_{i\in S} T_i \cong \bigoplus_{i\in S} T_i$. Let W be an irreducible subrepresentation of the invariant subspace $T_j \cap \bigoplus_{i\in S} T_i$. Then W is an invariant subspace of $T_j \cong \bigoplus_{I_j} V_j$. Composing the inclusion $W \hookrightarrow T_j$ with the projections $T_j \to V_j$, we get maps $W \to V_j$. If $W \neq 0$, one of these maps must be non-zero, so by Schur's Lemma,[5] it must be an isomorphism. So we must have $W \cong V_j$. Similarly, for some $i \in S$, the projection of W onto the T_i must be non-zero, from which we get that $W \cong V_i$, contradicting $j \notin S$. It follows that $T_j \cap \bigoplus_{i\in S} T_i = 0$, so $T_j + \bigoplus_{i\in S} T_i \cong T_j \oplus \bigoplus_{i\in S} T_i$. By (transfinite) induction we build direct sum decomposition $V = \bigoplus_{j\in J} T_j$. [[★★★ This feels more complicated than it has to be]]

Finally, suppose $V = \bigoplus_{j\in J} T_j$ and $W = \bigoplus_{j\in J} R_j$ are the isotypic component decompositions of two representations, then any morphism of representations $f\colon V \to W$ must send T_j to R_j for each j. Otherwise, we would get a non-zero morphism $T_i \to R_j$ for $i \neq j$. Composing with the inclusions $V_i \hookrightarrow T_i$ and the projections $R_j \to V_j$, we get a bunch of morphisms $V_i \to V_j$, at least one of which must be non-zero. By Schur's lemma, we get that $V_i \cong V_j$, contradicting $i \neq j$. ◇

Proposition 2.20. *The following conditions on an algebraic group G are equivalent.*

a. *G is reductive.*

b. *Any representation V decomposes as $V = V^G \oplus W$. (Note this implies $W^G = 0$)*

c. *For any surjection $V \twoheadrightarrow W$ of representations, $V^G \to W^G$ is surjective.*

d. *For any representation V and any $v \in V^G$, there exists $f \in (V^*)^G$ such that $f(v) = 1$.*

Proof. ($a \Rightarrow b$) follows from Remark 2.19: V^G is the trivial isotypic component and W is the sum of the other isotypic components.

($b \Rightarrow c$) Let $V = V^G \oplus V'$ and $W = W^G \oplus W'$. By a Schur's Lemma argument like the one at the end of Remark 2.19, there are no non-zero morphisms of representations $V^G \to W'$ or $V' \to W^G$. So the only way a morphism $V \to W$ can be surjective is if $V^G \to W^G$ is surjective.

($c \Rightarrow a$) Let $W \subseteq V$ be an invariant subspace. We have the surjection of representations $\mathrm{Hom}_k(V, W) \to \mathrm{Hom}_k(W, W)$ given by restriction. Taking invariants, we have $\mathrm{Hom}_G(V, W) \to \mathrm{Hom}_G(W, W)$, which is surjective by (c). So there exists some $\psi \in \mathrm{Hom}_G(V, W)$ that restricts to $\mathrm{id}_W \in \mathrm{Hom}_G(W, W)$. The kernel ψ is a complementary invariant subspace to W.

[4]Note that if V is not completely reducible, it is *not* the sum of its isotypic components.

[5]Schur's Lemma stattes that any morphism $f\colon W \to U$ of irreducible representations is either zero or an isomorphism. To prove it, simply note that $\ker f \subseteq W$ and $\operatorname{im} f \subseteq U$ are invariant subspaces.

($c \Rightarrow d$) Let W be the linear subspace of V spanned by v. Since $v \in V^G$, W is isomorphic to the trivial 1-dimensional representation k. We have a surjection of representations $V^* \cong \mathrm{Hom}_k(V,k) \to \mathrm{Hom}_k(W,k)$. By ($c$), the induced map $(V^*)^G \to \mathrm{Hom}_G(W,k) = \mathrm{Hom}_k(W,k)$ is surjective, so there exists some $f \in (V^*)^G$ lifing the linear functional that is 1 on v.

($d \Rightarrow b$) Choose a basis for V^G, so $V^G = \mathrm{span}\{v_i\}_{i\in I}$. Choose $f_i \in (V^*)^G$ such that $f_i(v_i) = 1$. Then $W = \bigcap_{i\in I} \ker(f_i)$ is an invariant complementary subspace to V^G. □

Example 2.21. We proved that any torus is reductive in any characteristic. ◇

Theorem 2.22 (Maschke's Theorem)**.** *If G is a finite group with* $\mathrm{char}(k) \nmid |G|$, *then G is reductive.*

Proof. Let's prove property (c). Pick a linear functional $f \in V^*$ such that $f(v) = 1$. Then define $\bar{f}(w) = \frac{1}{|G|}\sum_{g\in G} f(g(w))$. This $\bar{f}$ is an invariant functional such that $\bar{f}(v) = 1$. □

Example 2.23. Suppose $k = \mathbb{C}$, and let G be a semi-simple connected Lie group. By the classification of complex semi-simple Lie groups, G is actually an algebraic group (for example, $G = SL(n,C)$). Then Weyl's theorem states that every finite-dimensional representation of G is completely reducible (you have to use that infinite-dimensional representations are unions of finite-dimensional representations). Thus, G is reductive. ◇

Note that $GL(n,\mathbb{C}) = \big(SL(n,\mathbb{C}) \times \mathbb{C}^\times\big)/\mu_n$. It turns out that every reductive group is obtained by taking a semi-simple group, producting with a torus, and quotienting by a finite group.

3 Reductive Groups acting on Affine Varieties

The Reynolds operator and Hilbert's Theorem

Suppose $G \times X \to X$ is an action of a reductive group G on an affine variety X. There is a corresponding ring morphism $\sigma_X\colon k[X] \to k[G] \otimes k[X]$, and the axioms of a group action exactly correspond to this dual morphism being a coaction, so $R = k[X]$ is a representation of G. The fact that this coaction is a ring homomorphism (rather than just a k-linear map) corresponds to the fact that G acts on R is by ring automorphisms (rather than just k-linearly).

If X is a variety of finite type, then R is finitely generated algebra over the ground field: $R = k[x_1, \dots, x_n]/I$. Since G is reductive, then we get the decomposition $R = R^G \oplus R'$, so in addition to the obvious embedding $R^G \hookrightarrow R$, we have a canonical projection $R \to R^G$,[1] called the *Reynolds operator*, which we will denote $f \mapsto \bar{f}$.[[★★★ should mention Reynolds operator earlier ... the term is used for the projection $V \to V^G$ in general]]

Warning 3.1. This projection is not a homomorphism of rings since R' is not an ideal of R.[2] However, the following lemma shows that R' is an R^G-module, so the projection is a homomorphism of R^G-modules. ⌟

Lemma 3.2. *R' is an R^G-module. In particular, the projection $R \to R^G$ is R^G-linear.*

Proof. By definition of R^G, we have that $\sigma_X(f) = 1 \otimes f$. Since σ_X is a ring homomorphism, we have $\sigma(fh) = (1 \otimes f) \cdot \sigma(h)$ for any $h \in R$. That is, multiplication by f is a morphism of representations $R \to R$. By Remark 2.19, it must respect isotypic components. In particular, it must send R^G to R^G and R' to R'. □

Theorem 3.3 (Hilbert)**.** *Suppose $R = k[x_1, \dots, x_n] = \bigoplus_{d \geq 0} R_d$ (the natural grading, where each x_i has degree 1), and G is a reductive group acting on $\operatorname{Spec} R$ such that the action of G respects the grading, so $G \cdot R_d = R_d$ (or $\sigma(R_d) \subseteq k[G] \otimes R_d$). Then R^G is a finitely generated (graded) k-algebra.*

Proof. It is clear that R^G inherits a grading from R. Consider $R^G_{>0} = \bigoplus_{d>0} R^G_d = (\bigoplus_{d>0} R_d)^G$. Let I be the graded ideal in R generated by $R^G_{>0}$. Since R is noetherian, there is a finite set of homogeneous $f_1, \dots, f_n \in R^G_{>0}$ which generate I. We will show that $f_1, \dots, f_n$ generate R^G as a ring.

Given $h \in R^G_d$, we want to show that $h \in k[f_1, \dots, f_n]$. Since $h \in I$, we have $h = \sum f_i r_i$ for some homoegenous $r_i \in R$. Applying the projection to R^G, and using that $h, f_i \in R^G$ and Lemma 3.2, we have $h = \sum f_i \bar{r}_i$. Now each $\bar{r}_i$ is a homogeneous element of R^G of degree less than d, so by induction on d, $\bar{r}_i \in k[f_1, \dots, f_n]$, so $h \in k[f_1, \dots, f_n]$ as desired. □

[1]The projection is canonical because the invariant complement R' is canonical. It is the direct sum of all the non-trival isotypic components of R.

[2]For example, consider the action of $G = \mu_2$ on $R = k[x]$ given by $x \mapsto -x$. Then $R^G = k[x^2]$ and $R' = xk[x^2]$.

Lemma 3.4. *Suppose an algebraic group G (which need not be reductive) acts on an affine variety X. Then there exists a G-equivariant closed embedding $X \hookrightarrow \mathbb{A}^n$ such that the action of G on X extends to a* linear *action of G on $\mathbb{A}^n$.*

Proof. Let $s_1, \dots, s_n$ be generators for $k[X]$ such that the subspace generated by the s_i is an invariant subspace (we proved that each generator lies in a finite-dimensional invariant subspace, so take the sum of all of those). If $\sigma_X(s_i) = \sum_j f_{ij} \otimes s_j$, define an action of G on $k[x_1 \dots, x_n]$ by $\sigma(x_i) = \sum_j f_{ij} \otimes x_j$. This is a linear action of G on $\mathbb{A}^n$, and we have an invariant map $k[x_1, \dots, x_n] \to k[X]$ given by $x_i \mapsto s_i$. This is a surjection since the s_i were chosen to generate $k[X]$, so it induces a closed immersion. □

Remark 3.5. Note that a linear action of G on $\mathbb{A}^n$ is the same thing as a G-comodule structure on $k[x_1, \dots, x_n]$ that respects the grading. ◇

Corollary 3.6 (Hilbert's Theorem + Lemma 3.4)**.** *If a reductive group G acts on an affine variety X, with coordinate ring $R = k[X]$, then the ring of invariants R^G is finitely generated.*

Proof. By Lemma 3.4, we get a surjection of comodules $k[x_1, \dots, x_n] \twoheadrightarrow R$, where the action of G on $k[x_1, \dots, x_n]$ respects the grading. Since G is reductive, we get a surjection $k[x_1, \dots, x_n]^G \to R^G$. By Hilbert's Theorem (3.3), $k[x_1, \dots, x_n]^G$ is finitely generated, so R^G must also be finitely generated. □

The Orbit-closure Relation and Separation Lemma

For an arbitrary algebraic group G acting on a variety X, the inclusion $R^G \hookrightarrow R$ induces a map on spectra $\phi\colon X = \operatorname{Specm} R \to \operatorname{Specm} R^G =: X /\!\!/ G$ whose properties we'd like to study.

Remark 3.7. If G is a reductive group and $I \subseteq R$ is an invariant ideal, then I is a subrepresentation of $R = R^G \oplus R'$, so $I = (I \cap R^G) \oplus (I \cap R')$ (c.f. Remark 2.19). ◇

Lemma 3.8. *If G is a reductive group and $\mathfrak{n} \subseteq R^G$ is an ideal, then $(R\mathfrak{n})^G = \mathfrak{n}$. In particular, if $\mathfrak{n}$ is proper, then $R\mathfrak{n}$ is proper.*

Proof. We have that $R\mathfrak{n}$ is an invariant ideal, so by Remark 3.7, $R\mathfrak{n} = (R\mathfrak{n} \cap R^G) \oplus (R\mathfrak{n} \cap R')$. So $(R\mathfrak{n})^G = R\mathfrak{n} \cap R^G = \mathfrak{n} + (R'\mathfrak{n} \cap R^G)$. By Lemma 3.2, $R'\mathfrak{n} \subseteq R'$, so $(R\mathfrak{n})^G = \mathfrak{n}$. □

Corollary 3.9. *If G is reductive, $\phi\colon X \to X /\!\!/ G$ is surjective.*

Proof. For any maximal ideal $\mathfrak{m} \in \operatorname{Specm} R^G$, $R\mathfrak{m} \cap R^G = \mathfrak{m}$ by Lemma 3.8. So $R\mathfrak{m}$ is contained in some proper maximal ideal $\mathfrak{M} \in \operatorname{Specm} R$. We have that $\mathfrak{M} \cap R^G = \mathfrak{m}$ (since $\mathfrak{m}$ is maximal in R^G and $\mathfrak{M} \cap R^G$ cannot contain 1), so $\phi(\mathfrak{M}) = \mathfrak{m}$. □

Lemma 3.10 (Separation Lemma)**.** *Suppose Z_1 and Z_2 are Zariski-closed G-invariant subsets of X with $Z_1 \cap Z_2 = \varnothing$. Then there is an invariant function $f \in R^G$ such that $f(Z_1) = 0$ and $f(Z_2) = 1$.*

Proof. Let I_1 and I_2 be the ideals of Z_1 and Z_2. Since $Z_1 \cap Z_2 = \varnothing$, $I_1 + I_2 = R$, so we may write $1 = g_1 + g_2$ for $g_i \in I_i$. Applying the projection to R^G, we have $1 = \bar{g}_1 + \bar{g}_2$. Since I_i are invariant ideals, $\bar{g}_i \in I_i$ by Remark 3.7. Take $f = \bar{g}_1$. □

Definition 3.11. If $G \times X \to X$ is an action of an algebraic group on a variety and $x \in X$ is a point, then the *orbit* $G \cdot x$ of x is the image of the restricted map $G \times \{x\} \to X$. [[★★★ I'd like to add a functorial definition, but it's probably not worth it.]] ◇

Note that any invariant function must be constant along an orbit, so each fiber of ϕ is a union of orbits.

Lemma 3.12. *Every orbit Gx is open in its Zariski closure.*

Proof. By Chevalley's constructibility theorem [EGA, Theorem IV$_1$.1.8.4 and Proposition 0$_{\text{III}}$.9.2.2], for any finitely presented morphism of varieties $f\colon A \to B$, $f(A)$ contains an open set of $\overline{f(A)}$. So Gx contains an open subset U of $\overline{Gx}$. But then $Gx \bigcup g(U)$, which is open. □

Remark 3.13. In the differential category, this lemma can be false. For example, you can wrap a 1-parameter subgroup around a 2-dimensional torus so that the subgroup is dense. In the algebraic category, such nasty things can't happen. ◇

Definition 3.14. Two orbits O and O' are *closure equivalent* if there exists a finite set of orbits $O = O_0, O_1 \ldots, O_n = O'$ such that $\overline{O}_i \cap \overline{O}_{i+1} \neq \varnothing$. ◇

If $O \sim O'$, then it is clear that all invariants agree on them, so $\phi(O) = \phi(O')$.

Proposition 3.15. *Suppose a reductive group G acts on an affine variety X, and that O_1 and O_2 are two orbits, then the following conditions are equivalent.*

1. $O_1 \sim O_2$
2. $\phi(O_1) = \phi(O_2)$
3. $\overline{O}_1 \cap \overline{O}_2 \neq \varnothing$

Proof. We've already proven $(1 \Rightarrow 2)$ since all invariants agree on these orbits.

$(2 \Rightarrow 3)$ Suppose $\overline{O}_1 \cap \overline{O}_2 = \varnothing$, then by the Separation Lemma (3.10), there is an inivariant which is 1 on one of them and 0 on the other, contradicting (1).

$(3 \Rightarrow 1)$ trivial. □

Corollary 3.16. *So the fibers of $\phi\colon \operatorname{Specm} R \to \operatorname{Specm} R/\!\!/G$ are closure equivalence classes of orbits.*

Example 3.17. $G = k^\times$ acts on $\mathbb{A}^2$ by $t \cdot (x, y) = (tx, t^{-1}y)$. It is not difficut to see that $R^G = k[xy]$. The fibers are $\{xy = C\}$ If $C \neq 0$, there is just one orbit in the fiber (a hyperbola in the real picture). But if $C = 0$, then the fiber consists of three orbits, $\{x \neq 0\}$, $\{y \neq 0\}$, and $\{(0, 0)\}$. But note that the fiber contains only one closed orbit. ◇

Proposition 3.18. *For a reductive group acting on an affine scheme, every closure equivalence class has exactly one closed orbit.*

Proof. (Existence) Pick an orbit O of minimal dimension. By Lemma 3.12, O is open in $\overline{O}$. But $\overline{O}$ is invariant, so it is a union of orbits. If $\overline{O}$ contained any orbit other than O, that orbit would have to be of smaller dimension. Thus, $O = \overline{O}$.

(Uniqueness) Suppose O_1 and O_2 are two closed orbits in the same orbit-closure equivalence class. Then by Proposition 3.15, $O_1 \cap O_2 \neq \varnothing$. But the only way two orbits can intersect is if they are equal. □

In particular, every point in the geometric quotient corresponds to a closed orbit. In Example 1.3 ($k^\times$ acting on k^n by scaling), there is only one closed orbit, so the quotient has only one point.

Corollary 3.19. *If G is reductive with a closed action on an affine variety, then there is a bijection between G-orbits and points of the geometric quotient.*

Example 3.20 (A non-reductive counterexample)**.** Suppose $G \cong \mathbb{G}_a = \mathbb{C}$ acts on $\mathbb{A}^2 = \mathbb{C}^2$ by $t \cdot (x, y) = (x + ty, y)$. In this case, $R^G = k[y]$. The horizontal lines $\{y = c\}_{c \neq 0}$ are orbits; each one corresponds to a point in $X/\!\!/G$. But every point on the line $\{y = 0\}$ is fixed, so there is a line of closed orbits which get sent to the origin in $X/\!\!/G = \operatorname{Specm} k[y]$. This is a closed action where orbits do not correspond to points in the quotient because ϕ identifies some of the closed orbits. ◇

Example 3.21. $G = \mathbb{Z}/2$ acting on $\mathbb{A}^2$ by $(x, y) \mapsto (-x, -y)$. Then $R^G = k[x^2, xy, y^2] = k[u, v, w]/(v^2 - uw)$. Geometrically, $X/\!\!/G$ is a quadratic cone in 3-dimensional space. [[★★★ does this example illustrate something special?]] ◇

Proposition 3.22. *Suppose a reductive group G acts on an affine variety X, then $\phi\colon X \to X/\!\!/G$ is an* open submersion *(i.e. the topology on $X/\!\!/G$ is induced by ϕ; $U \subseteq X/\!\!/G$ is open if and only if $\phi^{-1}(U) \subseteq X$ is open).*

Proof. Since ϕ is surjective (Corollary 3.9), it suffices to show that for an invariant closed subset $Z \subseteq X$, the image $\phi(Z) \subseteq X/\!\!/G$ is closed. Let $I \subseteq R$ be the G-invariant ideal corresponding to Z (we're taking the reduced induced structure on Z). Then the ideal $I^G = I \cap R^G \subseteq R^G$ corresponds to the closure of the image of Z. So we need to show that the map $Z = \operatorname{Specm}(R/I) \to \operatorname{Specm}(R^G/I^G) = \overline{\phi(Z)}$ is surjective. But since G is reductive, invariants is exact, so $R^G/I^G \cong (R/I)^G$, so $\overline{\phi(Z)} \cong Z/\!\!/G$. Applying Corollary 3.9, we have that $Z \to \overline{\phi(Z)}$ is surjective. □

4 Poincaré Series

The first problem set has been posted at math.berkeley.edu/~serganov/274/.

Given an action of an algebraic group G on an affine variety X, we would like to determine R^G and understand its properties. Poincaré (or Hilbert) series are an important tool for doing this.

Assume that G acts on a vector space V linearly. Then $R = k[x_1, \dots, x_n] = \mathrm{Sym}^*(V^*)$, with its natural grading, and G acts in a way that respects the grading, so R^G inherits a grading. [[★★★ move]]

Definition 4.1. Suppose $M = \bigoplus M_d$ a graded k-module. Then the *Poincaré series* of M is $P_M(t) = \sum_{d\geq 0} \dim_l(M_d)t^d$. ◇

Remark 4.2. You may have seen Poincaré series before. A closed subscheme $X \subseteq \mathbb{P}^n$ of projective space corresponds to a graded ideal $I \subseteq k[x_0, \dots, x_n] = S$. Then S/I is a graded k-algebra, and its Poincaré series is usually called the *Hilbert function* of X. The Hilbert function is an important invariant of X. For example, the dimension of X is equal to the degree of its Hilbert function. ◇

Lemma 4.3. *[[★★★ add tensor product here?]] If $0 \to M \to N \to K \to 0$ is a short exact sequence of graded A-modules, then $P_N = P_M + P_K$.*

Proof. $0 \to M_d \to N_d \to K_d \to 0$ is exact, so $\dim(N_d) = \dim(M_d) + \dim(K_d)$. □

Remark 4.4. Using basically the same proof, we see that for a finite exact sequence of graded modules

$$0 \to \cdots \to M_i \to M_{i+1} \to \cdots \to 0$$

the alternating sum $\sum(-1)^i P_{M_i}(t)$ of the Poincaré series is zero. ◇

[[★★★ mention shifting grading here? it's very useful to regard a morphism that shifts grading as a graded morphism to a shifted module]]

Example 4.5. Let $R = k[y_1, \dots, y_n]$, where $\deg y_i = d_i$ for some $d_i > 0$. Then

$$P_R(t) = \frac{1}{(1-t^{d_1})\cdots(1-t^{d_n})}.$$ ◇

[[★★★ example of $\mathrm{Sym}^*(V)$ and $\Lambda^* V$?]]

Theorem 4.6 (Hilbert-Serre). *Suppose $R = k[y_1, \dots, y_n]$ with $\deg y_i = d_i$. If M is a finitely generated graded R-module, then*

$$P_M(t) = \frac{F(t)}{(1-t^{d_1})\cdots(1-t^{d_n})}$$

for some $F(t) \in \mathbb{Z}[t, t^{-1}]$.

Proof. We prove the result by induction on n, the number of generators of R. If $n = 0$, the result is clearly true, with $P_M(t)$ being the "graded dimesion" of M as a graded vector space over k. Now suppose $n > 0$. Let M' and M'' be the kernel and cokernel of multiplication by y_n.

$$0 \to M' \to M \xrightarrow{y_n} M[d_n] \to M'' \to 0$$

By Remark 4.4, we get the identity

$$P_{M'} - P_M + t^{d_n} P_M - P_{M''} = 0. \qquad (*)$$

Since M' and M'' are graded modules over $k[y_1, \ldots, y_{n-1}]$, the induction hypothesis tells us that

$$P_{M'}(t) = \frac{F'(t)}{(1-t^{d_1})\cdots(1-t^{d_{n-1}})} \quad \text{and} \quad P_{M''}(t) = \frac{F''(t)}{(1-t^{d_1})\cdots(1-t^{d_{n-1}})}$$

for some $F'(t), F''(t) \in \mathbb{Z}[t, t^{-1}]$. Solving for P_M in $(*)$, we get the desired result. □

Another way to get a proof is to construct a finite resolution by free modules.

Corollary 4.7. *If G is a reductive group acting on an affine scheme* $\operatorname{Specm} R$, $P_{R^G}(t) = \dfrac{F(t)}{(1-t^{d_1})\cdots(1-t^{d_n})}$ *for some $F(t) \in \mathbb{Z}[t, t^{-1}]$..*

The case of a finite group

Example 4.8. Let $G = \mathbb{Z}/n = \langle g \rangle$ act on $\mathbb{C}^2$ by $g \mapsto \left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{smallmatrix}\right)$ where ω is a primitive n-th root of unity. Then $k[x,y]^G = k[xy, x^n, y^n] = k[u,v,w]/(u^n - vw)$. Regarding R^G as a module over the subring $k[x^n, y^n]$, it is free with generators $1, xy, (xy)^2, \ldots, (xy)^{2n-2}$, so

$$P_{R^G}(t) = \frac{1 + t^2 + \cdots + t^{2n-2}}{(1-t^n)^2}. \qquad \diamond$$

Lemma 4.9. *For a linear representation W of a finite group G,* $\dim W^G = \dfrac{1}{|G|} \displaystyle\sum_g \operatorname{tr} g$.

Proof. Recall the Reynolds operator that we constructed in the proof of Maschke's Theorem (2.22), $\frac{1}{|G|}\sum g \colon W \to W^G$. Since it is a projector, its trace is the dimension of the image. □

Proposition 4.10 (Moilen's formula)**.** *For a finite group G acting linearly on a vector space V,* $P_{R^G}(t) = \dfrac{1}{|G|} \displaystyle\sum_{g \in G} \frac{1}{\det(1 - gt)}$.

Proof. We have $k[V] = \mathrm{Sym}^*(V^*) = \bigoplus_{d\geq 0} \mathrm{Sym}^d(V^*)$, and we'd like to compute

$$P_{R^G}(t) = \sum_{d\geq 0} \dim(\mathrm{Sym}^d(V^*))^G t^d$$

Any element $g \in G$ is of finite order, so it must act diagonalizably. So for some basis $\{x_i\}$ of V^*, g acts by the diagonal matrix $\mathrm{diag}(a_1, \ldots, a_n)$. Then g acts on $\mathrm{Sym}^d(V^*)$ by $g \cdot (x_1^{c_1} \cdots x_n^{c_n}) = (a_1^{c_1} \cdots a_n^{c_n})(x_1^{c_1} \cdots x_n^{c_n})$. Then we have

$$\frac{1}{\det(1 - gt)} = \prod_{i=1}^{n} \frac{1}{1 - a_i t} = \sum_{d\geq 0} \big(\mathrm{tr}\, g|_{\mathrm{Sym}^d(V^*)}\big) t^d$$

Applying Lemma 4.9 completes the proof. $\square$

5 More on Finite Groups and Reflection Groups

Today we'll consider the case of a finite group G acting linearly on a vector space $V = \operatorname{Specm} R[V]$. We'll assume that $\operatorname{char}(k) \nmid |G|$ (so representations are completely reducible). Using Moilen's formula (Proposition 4.10), we can compute the Poincaré series $P_{R^G}(t)$. The idea is that by looking at $P_{R^G}(t)$, you can sometimes guess what R^G is. Before we do an example, let's consider a very interesting class of finite groups that act on $\mathbb{A}^2 = \mathbb{C}^2$.

[[★★★ on the McKay correspondence]] Let $V = \mathbb{C}^2$, with the usual action of $SL(2,\mathbb{C})$. We get an induced action of $SL(2,\mathbb{C})$ on $\operatorname{Sym}^2 V$ so that the "squaring" map $\sigma\colon V \to \operatorname{Sym}^2 V$ given by $v \mapsto v \cdot v$ is $SL(2,\mathbb{C})$-equivariant. Let x and y be the coordinates on V, and let z_1, z_2, and z_3 be the coordinates on $\operatorname{Sym}^2 V$, with σ corresponding to the map $z_1 \mapsto x^2$, $z_2 \mapsto xy$, $z_3 \mapsto y^2$. The image of σ (the cone $z_2^2 - z_1 z_3$) is invariant under the action of $SL(2,\mathbb{C})$, so $SL(2,\mathbb{C})$ respects the quadratic form $z_2^2 - z_1 z_3$ [[★★★ this is where we're using $SL(2,\mathbb{C})$ rather than $GL(2,\mathbb{C})$ (which only respects the form up to scalar))]]. So the induced homomorphism $\gamma\colon SL(2,\mathbb{C}) \to GL(3,\mathbb{C})$ actually factors through $SO(3,\mathbb{C})$. It is easy to see that $\ker\gamma = \{\pm 1\}$. Now consider $SO(3,\mathbb{R}) \subseteq SO(3,\mathbb{C})$.[1] We have that $\gamma^{-1}(SO(3,\mathbb{R})) = SU(2)$ [[★★★ how to see this?]] Givne a finite subgroup $H \subseteq SO(3,\mathbb{R})$, we get a finite subgroup $G = \gamma^{-1}(H) \subseteq SU(2)$. In this way, we can find many finite groups that act on $\mathbb{C}^2$.

Example 5.1. Let $H \subseteq SO(3,\mathbb{R})$ be the group of rotations of the cube; H is abstractly isomorphic to S_4. Now consider $G = \gamma^{-1}(H)$, a non-trivial central extension of S_4. We have $|H| = 24$ and $|G| = 48$.

Now we consider the action of G on V and try to describe the geometric quotient $V/\!\!/G = \operatorname{Specm} k[V]^G$. We'd like to use Moilen's formula to compute $P_{R^G}(t)$, so we need to be able to compute $\det(1 - gt)$ for every $g \in G$. So we make a table, recording the possible diagonal forms of g, the order of g, and the number of elements of the given form: [[★★★ how to make this table?]]

form	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}e^{\frac{2\pi i}{3}} & 0\\ 0 & e^{-\frac{2\pi i}{3}}\end{pmatrix}$	$\begin{pmatrix}-e^{\frac{2\pi i}{3}} & 0\\ 0 & -e^{\frac{2\pi i}{3}}\end{pmatrix}$	$\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$	$\begin{pmatrix}\omega & 0\\ 0 & \omega^{-1}\end{pmatrix}$	$\begin{pmatrix}-\omega & 0\\ 0 & -\omega^{-1}\end{pmatrix}$
order	1	2	3	6	4	8	8
number	1	1	8	8	18	6	6

where $\omega = \frac{1+i}{\sqrt{2}} = e^{2\pi i/8}$. Applying Moilen's formula (4.10), we have that $P_{R^G}(t)$ is

$$\frac{1}{(1-t)^2} + \frac{1}{(1+t)^2} + \frac{8}{1+t+t^2} + \frac{8}{1-t+t^2} + \frac{18}{1+t^2} + \frac{6}{1+\sqrt{2}t+t^2} + \frac{6}{1-\sqrt{2}t+t^2}$$

[1] We're cheating a little bit here. Up to isomorphism, there is only one non-degenerate quadratic form on $\mathbb{C}^2$, but there are four non-isomorphic non-degenerate quadratic forms on $\mathbb{R}^2$, and we want the positive definite one. With our choice of σ, we actually get the form with signature $(1,-1,-1)$ instead of $(1,1,1)$. But we can change coordinates on $\operatorname{Sym}^2 V$ so that σ becomes $z_1 \mapsto \frac{x^2+y^2}{2i}$, $z_2 \mapsto xy$, $z_3 = \frac{x^2-y^2}{2}$. Then the image of σ is the surface $z_1^2 + z_2^2 + z_3^2 = 0$, and the action of $SL(2,\mathbb{C})$ respects the quadratic form $z_1^2 + z_2^2 + z_3^2$.

Simplifying, we get [[★★★ I haven't verified this yet]]

$$P(t) = \frac{1 - t^6 + t^{12}}{(1 + t^6)(1 - t^8)} = \frac{1 + t^{18}}{(1 - t^{12})(1 - t^8)}$$

From this, we see that R^G is a free module over the polynomial ring $k[f_8, f_{12}]$ (for some invariants f_8 and f_{12} of degrees 8 and 12), with generators 1 (of degree 0) and f_{18} (some invariant of degree 18).

So R^G has three generators, but what are the relations? We already know that f_8 and f_12 have no relations among them. But f_{18}^2 must satisfy some relation. By simple degree considerations, we see that the degree $(R^G)_{36}$ is spanned by $f_{12}f_8^3$ and f_{12}^3, so we must have

$$f_{18}^2 = af_{12}f_8^3 + bf_{12}^3$$

for some $a, b \in k$. Later, we'll show that you can make $a = b = 1$ [[★★★ ref once we've done it]]. So $V/\!\!/G \cong \operatorname{Specm} k[x, y, z]/(x^2 - yz^3 - y^3)$ is a surface with a singularity at the origin. This is a so-called *simple singularity.* ◇

Proposition 5.2. *Let G act faithfully on an irreducible affine variety $X = \operatorname{Specm} R$, and let K be the field of fractions of R. Then*

1. *$R^G \subseteq R$ is an integral extension.*
2. *K^G is the field of fractions of R^G.*
3. *$K^G \subseteq K$ is a normal extension, with Galois group equal to G.*

Proof. (1) Given $f \in R$, consider the polynomial $P_f(x) = \prod_{g\in G}(x - g(f))$. It is clear that P_f is a monic polynomial, that it is invariant under the action of G (so $P_f \in R^G[x]$), and that f is a root of P_f. Thus, $R^G \subseteq R$ is an integral extension.

(2) Suppose $f/h \in K^G$, then we need to show that it can be written as the ratio of invariant functions. By (1), h satisfies some monic polynomial $h^n + a_{n-1}h^{n-1} + \cdots + a_0 = 0$ where $a_i \in R^G$ and $a_0 \neq 0$. So $h(h^{n-1} + a_{n-1}h^{n-2} + \cdots + a_1) = -a_0$ is an invariant element of R. Multiplying the numerator and denominator by $(h^{n-1} + a_{n-1}h^{n-2} + \cdots + a_1)$, we have reduced to the case where $f/h \in K^G$, and h is *invariant*. Then since $f/h = g(f/h) = g(f)/h$, we have that $f = g(f)$ for every $g \in G$. [[★★★ here we're using that X is reduced and irreducible. Alternatively, if $\operatorname{char}(k) \nmid |G|$, we can apply the Reynolds operator to f/h to get $f/h = \bar{f}/h$.]]

(3) follows from (1) (as soon as you get an integral extension, ... use the primitive element theorem and the fact that the action is faithful)[[★★★]] □

Groups generated by reflections

In this section, we assume that $k = \bar{k}$ and $\operatorname{char}(k) = 0$.

Definition 5.3. Let V be a vector space. A linear map $r \in \mathrm{End}(V)$ is a *reflection*[2] if $r \neq \mathrm{id}$, r has finite order, and r fixes a hyperplane H_r. ◇

So in some basis, the matrix of r is diagonal, with all but one entry on the diagonal equal to 1, and the remaining entry is a root of unity. The hyperplane H_r is defined by some non-zero linear function $\ell_r \in V^*$, which is determined up to scalar.

Given a reflection r and a function $f \in R = k[V]$, note that f and $r(f)$ agree on H_r, so $f - r(f)$ vanishes along H_r. But the only polynomials that vanish along H_r are those that are divisible by ℓ_r.

Definition 5.4. Given a reflection r, we define the *Demazure operator* $D_r \colon R \to R$ by $f \mapsto \dfrac{f - r(f)}{\ell_r}$. ◇

Lemma 5.5. *If $r \in G$ is a reflection, then $D_r \colon R \to R$ is R^G-linear.*

Proof. Let $f \in R^G$ and $h \in R$, then $r(f) = f$ by assumption, so

$$D_r(fg) = \frac{fh - r(fh)}{\ell_r} = \frac{fh - r(f)r(h)}{\ell_r} = fD_r(h). \qquad \square$$

Lemma 5.6. *Let $G \subseteq GL(V)$ be generated by reflections, and let $I \subseteq R$ be the ideal generated by $R^G_{>0}$. Let $g_1, \ldots, g_m$ and $u_1, \ldots, u_m$ be homogeneous non-zero elements of R, with the $g_i \in R^G$. If $g_1u_1 + \cdots + g_mu_m = 0$ and $u_1 \notin I$, then $g_1 \in R^G g_2 + \cdots + R^G g_m$.*

[[★★★ There should be a good way to think about this lemma in terms of the ring of *coinvariants* $R_G = R/I$.]]

Proof. We will do induction on the degree of u_1. If $\deg u_1 = 0$, then $u_1 = 1$ (up to scalar), so $g_1 = -g_2u_2 - \cdots - g_mu_m$. Applying the Reynolds operator, we have $g_1 = -g_2\bar{u}_2 - \cdots - g_m\bar{u}_m$.

Now suppose $\deg u_1 > 0$. For a reflection $s \in G$, we apply D_s to the relation to get $g_1D_s(u_1) + \cdots g_mD_s(u_m) = 0$, a relation of lower degree. If the conclusion of the lemma is not true, then we must have $D_s(u_1) \in I$ by induction, so $u_1 - s(u_1) \in I$. But this is true for all reflections $s \in G$. Since G is generated by reflections, any $g \in G$ may be written as a product of reflections $g = s_1s_2 \cdots s_b$, and we see that

$$u_1 - g(u_1) = \big(u_1 - s_1(u_1)\big) + s_1\big(u_1 - s_2(u_1)\big) + \quad \cdots \quad + s_1 \cdots s_{b-1}\big(u_1 - s_b(u_1)\big)$$

so $u_1 - g(u_1) \in I$ for all $g \in G$. In particular, $\frac{1}{|G|}\sum_g \big(u_1 - g(u_1)\big) = u_1 - \bar{u}_1 \in I$. But since $\bar{u}_1 \in I$, this implies that $u_1 \in I$. $\square$

Theorem 5.7 (Chevalley-Shephard-Todd)**.** *Suppose $G \subseteq GL(V)$. The ring $k[V]^G$ is isomorphic to a polynomial ring if and only if G is generated by reflections.*

[2]Sometimes called a *pseudoreflection*.

It turns out this is equivalent to saying that the corresponding geometric quotient has no singularities. Today, we'll prove that if G is generated by reflections, then $k[V]^G$ is a polynomial ring.

Proposition 5.8. *If G is generated by reflections, then $k[V]^G$ is a polynomial ring.*

Proof. Let $f_1, \dots, f_r$ be a minimal homogenous generating set for the ideal $I \subseteq R$ generated by $R^G_{>0}$ such that $\deg f_1 \le \cdots \le \deg f_r$. In the proof of Hilbert's theorem (3.3), we showed that the f_p generate R^G as a ring. So we need only to show that there are no algebraic relations among the f_p.

Claim. *If $R = k[x_1, \dots, x_n]$, then for each f_p, there is some i such that $\frac{\partial f_p}{\partial x_i} \notin I$.*

Proof of Claim. Suppose that $\frac{\partial f_p}{\partial x_i} \in I$ for all i. Since $\deg f_1 \le \cdots \le \deg f_r$ and $\deg \frac{\partial f_p}{\partial x_i} < \deg f_p$, we must have $\frac{\partial f_p}{\partial x_i} \in Rf_1 + \cdots + Rf_{p-1}$ for all i. Then we get

$$f_p \cdot \deg f_p = \sum_i x_i \frac{\partial f_p}{\partial x_i} \in Rf_1 + \cdots + Rf_{p-1}$$

Since $\mathrm{char}(k) = 0$, $\deg(f_p)$ is invertible, contradicting the minimality of the set $\{f_1, \dots, f_r\}$. $\boxed{\text{Claim}}$

Now suppose $h(t_1, \dots, t_r) \in k[t_1, \dots, t_r]$ such that $h(f_1, \dots, f_r) = 0$. We may assume h is homogeneous (where $\deg t_i = \deg f_i$) and of minimal degree. By the claim, there is some x_i such that $\frac{\partial f_1}{\partial x_i} \notin I$. By the chain rule, we have

$$0 = \frac{\partial h}{\partial x_i}(f_1, \dots, f_r) = \sum_{p=1}^{r} \frac{\partial h}{\partial t_p}(f_1, \dots, f_r) \cdot \frac{\partial f_p}{\partial x_i}.$$

Since $\frac{\partial f_1}{\partial x_i} \notin I$, Lemma 5.6 tells us that

$$\frac{\partial h}{\partial t_1}(f_1, \dots, f_s) = \sum_{p=2}^{r} c_p \frac{\partial h}{\partial t_p}(f_1, \dots, f_s)$$

for some $c_p \in R^G$. But $\frac{\partial h}{\partial t_1}$ has degree strictly smaller than h, so this is an algebraic relation of smaller degree among the f_p, a contradiction. □

Remark 5.9. Note that we must have $r = n$ because K is a finite extension of K^G, the fraction field of R^G. ◇

6 More CST

There is a classification of complex reflection groups. Of course, the usual reflection groups are Coxeter groups (1934). Shephard and Todd in 1954 classified all finite complex reflection groups. [[★★★ add refs to bibliography and put this comment some place else (after complete proof of CST?)]]

Recall that for a finite group G (with $\mathrm{char}(k) \nmid |G|$) acting linearly on an n-dimensional vector space $V = \operatorname{Specm} R$, Moilen's formula tells us that

$$P_{R^G}(t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - gt)}.$$

If the eigenvalues of the action of g are $a_1, \ldots, a_n$, then $\frac{1}{\det(1-gt)} = \frac{1}{(1-a_1 t)\cdots(1-a_n t)}$. The identity element of G is the only term in the sum that contributes $\frac{1}{(1-t)^n}$, and the only terms that contribute a multiple of $\frac{1}{(1-t)^{n-1}}$ are reflections, the g which have a single non-trivial eigenvalue. So by expanding $P_{R^G}(t)$ as a Laurent series around $t = 1$, we can extract $|G|$ and the number of reflections in G.

Lemma 6.1. *In the Laurent series expansion $P_{R^G}(t) = \sum_{i=-n}^{\infty} c_i (1-t)^i$, $c_{-n} = \frac{1}{|G|}$ and $c_{1-n} = \frac{1}{2|G|}|S_G|$, where $S_G \subseteq G$ is the set of reflections in G.*

Proof. As we've already noted, the identity element is the only term in Moilen's formula that contributes a pole of order n, so the lowest order term in the Laurent series expansion will be $\frac{1}{|G|}(1-t)^{-n}$.

We've also seen that the only elements $g \in G$ that contribute a pole of order $n-1$ are the reflections. If the only non-trivial eigenvalue of g is ε, then the coefficient of $(1-t)^{1-n}$ in the Laurent series expansion of $\frac{1}{\det(1-tg)}$ is

$$\frac{d}{dt}\left(\frac{-(1-t)^n}{(1-t)^{n-1}(1-\varepsilon t)} \right)\bigg|_{t=1} = \frac{(1-\varepsilon t) - \varepsilon(1-t)}{(1-\varepsilon t)^2}\bigg|_{t=1} = \frac{1}{1-\varepsilon}.$$

If g is a reflection of order 2, then $\varepsilon = -1$, so it contributes $\frac{1}{2}$ to the coefficient of $(1-t)^{n-1}$. Otherwise, $g \neq g^{-1}$, and together these two reflections contribute

$$\frac{1}{1-\varepsilon} + \frac{1}{1-\varepsilon^{-1}} = \frac{1 - \varepsilon^{-1} + 1 - \varepsilon}{1 - \varepsilon - \varepsilon^{-1} + \varepsilon\varepsilon^{-1}} = 1$$

to the coefficent of $(1-t)^{1-n}$, so we may think of g and g^{-1} as each contributing $\frac{1}{2}$. Adding these contributions up and multiplying by the $\frac{1}{|G|}$ from Moilen's formula, we get $c_{1-n} = \frac{1}{2|G|}|S_G|$. □

Proposition 6.2. *If*

$$P_{R^G}(t) = \frac{1}{(1-t^{d_1})\cdots(1-t^{d_n})}$$

then $|G| = \prod_i d_i$ and $|S_G| = \sum_i (d_i - 1)$.

Proof. Let the Laurent series expansion be

$$\frac{1}{(1-t^{d_1})\cdots(1-t^{d_n})} = P_{R^G}(t) = \sum_{i=-n}^{\infty} c_i(1-t)^i.$$

Then we compute

$$\begin{aligned}
c_{-n} &= \frac{(1-t)^n}{(1-t^{d_1})\cdots(1-t^{d_n})}\bigg|_{t=1} = \frac{1}{\prod_i(1+t+\cdots+t^{d_i-1})} = \frac{1}{\prod_i d_i} \\
c_{1-n} &= \frac{d}{dt}\left(\frac{-(1-t)^n}{(1-t^{d_1})\cdots(1-t^{d_n})}\right)\bigg|_{t=1} = \frac{d}{dt}\left(-\prod_i \frac{1}{1+t+\cdots+t^{d_i-1}}\right)\bigg|_{t=1} \\
&= \sum_i \frac{d_i(d_i-1)}{2d_i^2}\cdot\frac{1}{\prod_{j\neq i} d_j} = \frac{1}{2\prod_j d_j}\sum_i (d_i-1).
\end{aligned}$$

Now the result follows from Lemma 6.1. □

Proposition 6.3. *If*

$$P_{R^G}(t) = \frac{1}{(1-t^{d_1})\cdots(1-t^{d_n})}$$

then G is generated by reflections.

Proof. [[★★★ For some reason]] R^G must be a polynomial algebra [[★★★ this should probably just be part of the hypothesis of the proposition]]. Suppose $R^G = k[f_1,\dots,f_n]$ with $\deg f_i = d_i$. Let $H \subseteq G$ be the subgroup generated by all the reflections in G. By Proposition 5.8, $R^H = k[h_1,\dots,h_n]$ and $P_{R^H}(t) = \frac{1}{(1-t^{e_1})\dots(1-t^{e_n})}$. We may assume that $d_1 \le \cdots \le d_n$ and $e_1 \le \cdots \le e_n$.

We claim that $e_i \le d_i$. To see this, suppose $d_p < e_p$ for some p. By degree considerations, and using that $R^G \subseteq R^H \subseteq R$, we have that $f_i = Q_i(h_1,\dots,h_{p-1})$ for some polynomials Q_i and all $i \le p$. But $Q_1,\dots,Q_p \in k[h_1,\dots,h_{p-1}]$ must satisfy an algebraic relation since the Krull dimension of $k[h_1,\dots,h_{p-1}]$ is $p-1$. This contradicts the algebraic independence of the f_i.

On the other hand, the number of reflections in G and H must be equal, so $\sum_i(d_i-1) = \sum_i(e_i-1)$. Combining this with the inequality $e_i \le d_i$, we must have $e_i = d_i$ for all i. Then by Lemma 6.1, we have

$$|H| = \prod_i e_i = \prod_i d_i = |G|$$

so $G = H$, so G is generated by reflections. □

Combining Lemma 6.1 with Propositions 5.8 and 6.3, we get the following theorem.

Theorem 6.4 (Chevalley-Shephard-Todd)**.** *Suppose $G \subseteq GL(V)$. The ring $k[V]^G$ is isomorphic to a polynomial ring if and only if G is generated by reflections. If this is the case, with $k[V]^G = k[f_1,\dots,f_n]$ and $\deg f_i = d_i$, then $|G| = \prod_i d_i$ and $|S_G| = \sum_i(d_i-1)$.*

Question: do the other symmetric functions in the d_i give you more information? Answer: I don't see how. Maybe if you write out more terms in the Laurent series expansion, you'll get something.

Lemma 6.5. *For any linearly independent set $\{y_1 + I, \ldots, y_k + I\}$ in R/I (where I is the ideal generated by $R^G_{>0}$), the set $\{y_1, \ldots, y_k\}$ is R^G-linearly independent in R.*

Proof. We use induction on k. If $k = 1$, then the result is clear since R has no zero-divisors.

Suppose you have a relation $h_1y_1+\cdots+h_ky_k = 0$ for some $h_i \in R^G$. By assumption, $y_1 \notin I$, so by Lemma 5.6, we get that $h_1 \in R^Gh_2+\cdots+R^Gh_k$, say $h_1 = u_2h_2+\cdots u_kh_k$. Substituting into the previous relation, we get the relation

$$h_2(y_2 - u_2y_1) + \cdots + h_k(y_k - u_ky_1) = 0.$$

Since the $u_i \in R^G$ reduce to scalars in R/I, $\{y_2 - u_2y_1 + I, \ldots, y_k - u_ky_1 + I\}$ is a linearly independent set in R/I, so we have $h_2 = \cdots = h_k = 0$ by the inductive hypothesis, and then $h_1 = 0$ by the base case. $\square$

Proposition 6.6. *If G is generated by reflections, then R is a free R^G-module.*

We already discussed that we know the rank because we know that K is a degree $|G|$ extension of K^G.

Proof. To choose a set of generators, consider the ideal $I = R^G_{>0} \subseteq R$. Choose homogeneous elements $y_1, \ldots, y_\ell \in R$ such that $y_1+I, \ldots, y_\ell+I$ form a basis for R/I. By induction on degree, the y_i generate R as an R^G-module. So we need only to show independence. [[★★★ they generate by the graded version of Nakayama's lemma: for any graded ideal I and any finitely generated module M, $M = IM \Rightarrow M = 0$ and generators for M/IM lift to generators of M.]]

Independence follows from Lemma 6.5 $\square$

More generally, a finite extension of polynomial algebras is free. [[★★★ the same proof doesn't work ... can we get it to work?]]

Semi-invariants

Suppose G acts on X. Suppose I have a character $\chi\colon G \to k^\times$. A function satisfying the condition $f(gx) = \chi(g)f(x)$ is called *semi-invariant*. So even though f is not invariant, the corresponding line $kf(x)$ is invariant.

Suppose you want some rational function f/g to be invariant, it suffices for f and g to be semi-invariant with the same character.

We are going to construct semi-invariants for groups generated by reflections. Look at the set S_G of all reflections. Clearly G acts on S_G by conjugation. For each

s, we have the associated hyperplane H_s and a linear functional $\ell_s = H^{\perp} \subseteq V^*$. Decompose S_G as a union of orbits. $S_G = O_1 \cup \cdots \cup O_k$. Then define

$$f_{O_i} = \prod_{s \in O_i} \ell_s.$$

We claim that this is semi-invariant. This is easy to check using the identity

$$g(\ell_s) = \ell_{gsg^{-1}}.$$

(up to scalar).

Proposition 6.7. *Any semi-invariant can be written uniquely in the form $f_{O_1}^{a_1} \cdots f_{O_k}^{a_k} f_0$, where $f_0 \in R^G$, and $0 \leq a_i \leq \operatorname{ord}(s)$ where $s \in O_i$.*

Proof. First, induction on degree. Pick a semi-invariant f. If it is not invariant, pick some $s \in S_G$ such that $s(f) = \varepsilon f$ for $\varepsilon \neq 1$. Then I claim that $\ell_s | f$ because ℓ_s divides $f - s(f) = (1 - \varepsilon) f$. Then $\ell_{s'} | f$ for any s' in the orbit of s. Then procede by induction. □

7 Examples of Quotients by Finite Groups

Example 7.1. S_n acts on $\mathbb{A}^n$ by permuting the coordinates: $\pi(x_1, \dots, x_n) = (x_{\pi(1)}, \dots, x_{\pi(n)})$. In this case, the invariants R^G is the algebra of symmetric functions $k[\sigma_1, \dots, \sigma_n]$. S_n is generated by reflections (the transpositions). For the reflection $s = (i\ j)$, we get $\ell_s = x_i - x_j$. We can verify that $d_1 d_2 \cdots d_n = n! = |S_n|$. We also know that $\sum(d_i - 1) = n(n-1)/2$ is the number of reflections.

Now let's look for semi-invariants. S_G has just one orbit O, for which $f_O = \prod_{i<j}(x_i - x_j) = \Delta$, the Vandermonde determinant. We have that $\pi(\Delta) = \operatorname{sgn}(\pi)\Delta$. ◇

Example 7.2. Now consider the subgroup $A_n \subseteq S_n$. Since every element of R^{A_n} is invariant under A_n, the action of S_n is really an action of $\mathbb{Z}/2$. So R^{A_n} decomposes (as a vector space) into a subspace invariant under the action of S_n and a subspace where S_n acts by -1. [[★★★ more generally, if $H \triangleleft G$ with G/H abelian, then the action of G on R^H is really the action of the abelian group G/H, so it decomposes into isotypic components for irreps of an abelian group. Since irreps of an abelian group are 1-dimensional, R^H has a basis of semi-invariants]]. We have that $k[x_1, \dots, x_n]^{A_n} = k[\sigma_1, \dots, \sigma_n] \oplus \Delta k[\sigma_1, \dots, \sigma_n]$. We have that Δ^2 is some specific symmetric polynomial that you can express in terms of symmetric functions. ◇

Example 7.3. Consider the group Γ of symmetries of a cube (not just rotations). We have that $|\Gamma| = 48$. It is generated by reflections, but it should be clear that there are two types (conjugacy classes) of reflections. If the coordinates of the cube are $(\pm 1, \pm 1, \pm 1)$, then one type of reflection is with respect to $x_i^\perp$, and the other with respect to $(x_i \pm x_j)^\perp$.

First let's find invariants and semi-invariants. We know that $d_1 + d_2 + d_3 = 12$ and $d_1 d_2 d_3 = 48$. There is one solution: $(d_1, d_2, d_3) = (2, 4, 6)$. We have one invariant given by the quadratic form: $h_2 = x_1^2 + x_2^2 + x_3^2$. But we also have $h_4 = x_1^4 + x_2^4 + x_3^4$ and $h_6 = x_1^6 + x_2^6 + x_3^6$. These are the invariants. We have semi-invariants $\tilde{h}_3 = x_1 x_2 x_3$, $\tilde{h}_6 = \prod_{i<j}(x_i \pm x_j)$. ◇

Example 7.4. Now let's take $H \subseteq \Gamma$ the subgroup of rotations of the cube. Clearly the invariants h_i remain invariants. But we get one more invariant: $h_9 = \tilde{h}_3 \tilde{h}_6$. You know you have all the invariants because the subgroups is index 2 [[★★★ so the ring of invariants is a module of rank 2 over the other guy?]]. ◇

Example 7.5. Recall that we have a 2-fold cover $\gamma\colon SU(2) \to SO(3, \mathbb{R})$. We got this by considering the squaring map $\mathbb{C}^2 \to \operatorname{Sym}^2 \mathbb{C}^2 \cong \mathbb{C}^3$. Let $G = \gamma^{-1}(H)$ (with H as before). We get three invariants: f_8, f_{12}, and f_{18}.

Since γ is of degree 2, all degrees multipy by 2. $f_4 = \gamma^{-1}(h_2)$, $f_8 = \gamma^{-1}(h_4)$, $f_{12} = \gamma^{-1}(h_6)$, and $f_{18} = \gamma^{-1}(\tilde{h}_9)$. But since $\gamma(V)$ is the vanishing locus of h_2, that invariant goes away. We get the relation

$$f_{18}^2 + f_{12} f_8^3 + f_{12}^3 = 0$$

So the corresponding quotient is the surface $x^2 + yz^3 + y^3 = 0$. This is what is called a *simple singularity.* $\diamond$

Simple singularities. Consider locally maps like $\phi\colon \mathbb{C}^m \to \mathbb{C}$ with $\phi(0) = 0$. We'd like to classify germs of such maps at 0 [[★★★ something to do with classifying orbits]]. If non-singular, then easy, but if singular, then we have to work harder.

Smooth map ϕ is *simple* (or $\phi^{-1}(0)$ is a simple singularity) if whenever we deform it a little bit, there are only finitely many orbits (under the action of the group) in a small neighborhood. That is, a small neighborhood of ϕ intersects non-trivially with a finite number of orbits of G. [[★★★ orbits of the group $\widetilde{\mathrm{Diff}}(\mathbb{C}^m) \times \widetilde{\mathrm{Diff}}(\mathbb{C})$ (the twiddle means you work germy).]]

Finite subgroups of $SU(2)$

Given $G \subseteq SU(2)$, we get an image group in $SO(3)$. Let's classify these groups. The main result is that you get rotation groups, dihedral groups, and rotations of polytopes.

$G \subseteq SO(3, \mathbb{R})$ acts on S^2. Consider $P = \{p \in S^2 | \mathrm{Stab}(p) \neq \{1\}\}$. This is a finite set with a G-action, and they will correspond to vertices, edges, and faces of the polytope.

Write P as a union of orbits $P = P_1 \sqcup \cdots \sqcup P_d$. Let $g = |G|$, and let $g_i = |\mathrm{Stab}(p_i)|$ for any $p_i \in P_i$. Then $\{(g, p) | p \in P, g \in G \smallsetminus \{1\}, gp = p\}$ can be counted in two ways. Counting elements of the group: each non-trivial rotation fixes two points, so we get $2 \cdot (g - 1)$. On the other hand, we can count by orbits, in which case we get $\sum_{i=1}^d \frac{g}{g_i}(g_i - 1)$. So we get

$$2g - 2 = gd - \sum \frac{g}{g_i} \qquad \Rightarrow \qquad d = 2 + \sum_{i=1}^{d} \frac{1}{g_i} - \frac{2}{g}.$$

From this, we can see that d can only take values 1, 2, or 3. The g_i are never 1, so the sum is less than or equal to $\frac{1}{2}|G|$. You can check that $d = 1$ is impossible. For $d = 2$, there is only one possibility: $g_1 = g_2 = g$, which happens when the group is generated by a single rotation, so it is a cyclic group. Finally, if $d = 3$, we can write $1 + \frac{2}{g} = \sum \frac{1}{g_i}$, so the g_i cannot be too big. One of them must be 2, so the possibilities are

- $d = 3$, $g_1 = g_2 = 2$, $g_3 = g/2$. You get the north pole, south pole, and a polygon on the equator. This is the dihedral group.
- $d = 3$, $g_1 = 2$, $g_2 = g_3 = 3$, $g = 12$. Symmetries of a tetrahedron.
- $d = 3$, $\{g_i\} = \{2, 3, 4\}$, $g = 24$. Rotations of the cube, S_4.
- $d = 3$, $\{g_i\} = \{2, 3, 5\}$, $g = 60$. Rotations of the dodecahedron, A_5.

We're interested in subgroups of $SU(2)$. In the first case ($d = 2$) you get the same group, but in all the other cases, you get a non-trivial central extension. For the dihedral case, you get $\langle \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \rangle$

group	invariants	relations	type
$\mathbb{Z}/n$	$f_2 = xy$, $f_n = x^n$, $\bar{f}_n = y^n$	$f_n \bar{f}_n - f_2^n = 0$	A_{n-1}
$\tilde{D}_n$	f_{2n}, f_{2n+2}, f_4	$f_{2n+2}^2 + f_4 f_{2n}^2 + f_4^{n+1}$	D_{n+1}
tetrahedron	f_6, f_8, f_{12}	$f_6^4 + f_8^3 + f_{12}^2$	E_6
cube	f_8, f_{12}, f_{18}	$f_{18}^2 + f_{12} f_8^3 + f_{12}^3$	E_7
dodeca	f_{12}, f_{20}, f_{30}	$f_{30}^2 + f_{20}^3 + f_{15}^5$	E_8

For a simple singularity $\phi(x, y, z) = 0$, call it X_0. We deform to get $\phi(x, y, z) = \varepsilon$, which we call X_ε. Study it in a neighborhood of 0. $H_2(X, \mathbb{Z})$ is a lattice, equiped with a quadratic form (index of intersection). On the other hand, you have a Dynkin diagram, which gives you the root lattice, which will be isomorphic (with it's usual form) to $H_2(X, \mathbb{Z})$.

We've been talking about maps $\mathbb{C}^3 \to \mathbb{C}$. But there is a theorem that for any $\mathbb{C}^m \to \mathbb{C}$, you get the same singularities. Any singularity is of the form $(x^2 + xy + z^3) + (x_1^2 + \cdots x_m^2)$, a surface singularity plus some quadratic part.

In the theory of Lie groups and the theory of algebraic groups, there is a very small difference between the usual Dynkin correspondence. I'd like to point out why the characteristic 0 case is very nice, and why we get many more reductive groups in characteristic 0.

Lie algebras and algebraic groups

Assume that X is an affine variety. We have a tangent space $T_x X = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$, where $\mathfrak{m}_x$ is the maximal ideal in the local ring at x. We can think of it as $T_x X = Der\{\mathcal{O}_x \to k\}$. Or we can think of vector fields $Der(\mathcal{O}(U), \mathcal{O}(U))$ for any open set $U \subseteq X$. (If you don't work with an affine variety, you need to take sheafy derivations). In our case, $X = G$ is an affine non-singular variety (because the dimension of the tangent space is uniform).

For derivations d_1, d_2, the bracket $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$ is again a derivation. We can consider *right-invariant* derivations. Those derivations which commute with the maps $R_g \colon G \to G$, given by multiplication by g on the right. $\mathfrak{g} = \{d | d \circ R_g = R_g \circ d\}$. In terms of hopf algebras, $\Delta \circ d = (d \otimes \mathrm{id}) \circ \Delta$.

8 Chevalley-Jordan decomposition

A new set of exercises has been posted.

I'm going to talk about algebraic groups. The main goal is to describe reductive algebraic groups in any characteristic. Recall that we're only talking about affine algebraic groups. So we may assume $G \subseteq GL(V)$.

Last time I used right-invariant derivations, but I actually prefer left invariant.

We have $\mathfrak{g} = \{d | L_g \circ d = d \circ L_g\} \subseteq \operatorname{Der} k[G]$. This condition is equivalent to $(\operatorname{id} \otimes d) \circ \Delta = \Delta \circ d$ in the Hopf algebra $k[G]$. Algebraically, for $x \in T_e G$, we can realize $x \in \operatorname{Der}(\mathcal{O}_e, k)$. We have $L_g(x) = gx$. The vector field $L_x = (\operatorname{id} \otimes x) \circ \Delta$ is left invariant. Another way to describe this is that $L_x f|_g = L_g(f)|_e$.

So we have $\mathfrak{g} \cong T_e G$. In particular, since the group G is non-singular, $\dim G = \dim \mathfrak{g}$. In fact, we've constructed a functor from the category of affine algebraic groups to the category of finite-dimensional Lie algebras. To see this, you have to check that for a homomorphism $\phi\colon G \to H$ of algebraic groups, the induced $D\phi|_e\colon \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras. We'll call this functor Lie.

Remark 8.1. If G is a Lie group, or analytic complex group, then if G is connected, then $\mathfrak{g}$ almost determines G. For a Lie algebra $\mathfrak{g}$, there is a unique connected simply-connected group $\tilde{G}$ such that $\operatorname{Lie}(\tilde{G}) = \mathfrak{g}$. And any other connected Lie group G with the same Lie algebra is a quotient of $\tilde{G}$ by some central discrete subgroup Γ.

However, this is not true in the algebraic category. We've discussed the algebraic groups $k^\times$ and k. Both have the same Lie algebra (the unique 1-dimensional Lie algebra), but neither is a quotient of the other. Basically, in differential geometry, we have the exponential map, which is very powerful. But in the algebraic category, we have some other tools which are perhaps even better. ◇

In our case, we always have $G \subseteq GL(V)$. So first, we'll compute the Lie algebra of $GL(V)$. It is $\mathfrak{gl}(V)$, the matrix algebra, with the bracket $[X, Y] = XY - YX$. Since G is a closed subgroup, we just have to compute the tangent space to the identity.

How do you compute this? Consider $A = k[\varepsilon]/\varepsilon^2$. We have that $\operatorname{Lie}(G) = \{X \in \mathfrak{gl}(V) | 1 + \varepsilon X \in G(A)\}$.

Example 8.2. We have the group $SL(V) = \{g \in GL(V) | \det g = 1\}$. So $\mathfrak{sl}(V) = \{X | \det(1 + \varepsilon X) = 1 + \varepsilon \operatorname{tr}(X) = 1\} = \{X | \operatorname{tr}(X) = 0\}$. ◇

Given the Lie algebra, what can you say about the group? Like reductivity or some other properties?

Representations

If we have a representation $G \to GL(V)$, we automatically get a representation $\mathfrak{g} \to \mathfrak{gl}(V)$. This works if V is finite-dimensional. Otherwise, we have $\sigma\colon V \to k[G] \otimes V$. Then any $x \in \mathfrak{g}$ acts by $V \xrightarrow{\sigma} k[G] \otimes V \xrightarrow{x \otimes \operatorname{id}} V$.

Given a representation V of G, we have $V^G = \{v | gv = v \text{ for all } g \in G\}$. Similarly, we get $V^{\mathfrak{g}} = \{v | Xv = 0 \text{ for all } X \in \mathfrak{g}\}$.

Exercise. $V^G \subseteq V^{\mathfrak{g}}$.

The inverse is not true! For example, if G is finite, then $\mathfrak{g}$ is trivial. The inverse is true if $\mathrm{char}(k) = 0$ and G is connected (which means irreducible). This is very useful because often checking that something is invariant under an infinitessimal action is easier than checking that it is invariant under a global action.

Proposition 8.3. *G connected and $\mathrm{char}(k) = 0$, thne $V^G = V^{\mathfrak{g}}$.*

Proof. Suppose $v \in V^{\mathfrak{g}}$. Then consider the map $\phi\colon G \to V$ given by $g \mapsto gv$. We have $D\phi|_e = 0$ by assumption. Since ϕ is equivariant with respect to the action of G, $D\phi = 0$ [[★★★ Use that Dg never sends non-zero things to zero and the commutative square you get from applying D to the equivariance square]]. Since G is connected, the image must be a point. [[★★★ need a result: in characteristic zero, the kernel of $d\colon A \to \Omega_{A/k}$ is exactly k. This uses that A is faithfully flat over k (actually, we have a section, so we don't need the faithfully flat yoga)]] □

Note that we need the fact that if the differential of a map is zero, then the map is constant, which is clearly not true in characteristic p. To see this, consider the action of $k^\times$ on V by $t \cdot v = t^p v$. Then the action of $\mathrm{Lie}(k^\times)$ is trivial.

The same is true with invariant subspaces. Suppose $W \subseteq V$ is a G-invariant subspace, then it is invariant with respect to the action of $\mathfrak{g}$. But the converse is only true if $\mathrm{char}(k) = 0$ and G is connected. The argument is similar.

Since $G \subseteq GL(V)$, consider the stabilizer of some $v \in V$, $G_v = \{g \in G | gv = v\}$, and $\mathfrak{g}_v = \{X \in \mathfrak{g} | Xv = 0\}$. We always have $\mathrm{Lie}(G_v) \subseteq \mathfrak{g}_v$, but other containment is not always true. If $\mathrm{char}(k) = 0$, then $\mathrm{Lie}(G_v) = \mathfrak{g}_v$ and $T_v(G \cdot v) = \mathfrak{g}v$. Note that it doesn't make a difference if G is connected in this case.

Example 8.4 (Adjoint representation)**.** A group G can act on itself by conjugation: $x \mapsto gxg^{-1}$. The identity is preserved, so it preserves $\mathfrak{g} = T_e G$. Thus, we have a representation $\mathrm{Ad}\colon G \to GL(\mathfrak{g})$. In the case $G = GL(V)$, then $\mathfrak{g} = \mathfrak{gl}(V)$, and the action is honestly action by conjugation: $X \mapsto gXg^{-1}$. In general, this is an automorphism of $\mathfrak{g}$ as a Lie algebra, not just a linear automorphism!

We get a corresponding representation $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. This is given by $\mathrm{ad}(X)(Y) = [X, Y]$. ◇

If $\mathrm{char}(k) = 0$ and G is connected, then $\ker \mathrm{Ad} = Z(G)$. If G is connected and $\mathfrak{g}$ is abelian, then G is abelian as well. This is not true in characteristic p, and here is an example.

Example 8.5. $G = \left\{ \left(\begin{smallmatrix} a & 0 & 0 \\ 0 & a^p & b \\ 0 & 0 & b \end{smallmatrix}\right) | a \in k^\times, b \in k \right\}$ then $\mathfrak{g} = \left\{ \left(\begin{smallmatrix} t & 0 & 0 \\ 0 & 0 & s \\ 0 & 0 & 0 \end{smallmatrix}\right) | s, t \in k \right\}$. $\mathfrak{g}$ is abelian, but G is not. We have that $\ker \mathrm{Ad} = \{\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix}\right)\}$, but $Z(G) = \{1\}$. ◇

Chevalley-Jordan decomposition

This is nice and works in all characteristics. However, we will assume $k = \bar{k}$.

Theorem 8.6. *Any operator $x \in \mathrm{End}_k(V)$ on a finite dimensional space can be written uniquely as a sum of a semi-simple (diagonalizable) operator and a nilpotent operator $X = X_s + X_n$ such that*

1. $[X_s, X_n] = 0$
2. *There exist polynomials $p(t)$ and $q(t)$ with zero constant coefficients such that $X_s = p(X)$ and $X_n = q(X)$.*

This is basically Jordan normal form of the operator, plus something else. If X is invertible, then X_s is also invertible. Then $X = X_s(1 + X_s^{-1}X_n) = X_sX_u$. The operator $X_s^{-1}X_n$ is not nilpotent, but $1 + X_s^{-1}X_n$ is *unipotent*, meaning that $(x-1)^N = 0$. This is *Chevalley decomposition.* The following doesn't work for Lie groups, but is true for algebraic groups.

Theorem 8.7. *If $G \subseteq GL(V)$ is a closed algebraic subgroup, then $g \in G \Rightarrow g_s, g_u \in G$, and $x \in \mathfrak{g} \Rightarrow x_s, x_u \in \mathfrak{g}$.*

The proof follows from a simple observation.

Lemma 8.8. *Suppose $H \subseteq G$ are affine groups, and suppose I_H is the ideal corresponding to H. Then $H = \{g \in G | g(I_H) \subseteq I_H\}$ and $\mathfrak{h} = \{x \in \mathfrak{g} | x(I_H) \subseteq I_H\}$.*

Proof. The containments $\subseteq$ should be clear.

Suppose $g \in G$ such that $g(I_H) \subseteq I_H$, and let $f \in I_H$. It is enough to show that $f(g^{-1}) = 0$ [[★★★ this shows that $g^{-1} \in H$, so $g \in H$]]. But $f(g^{-1}) = (gf)(e) = 0$ (since $gf \in I_H$ and $e \in H$). □

From this lemma, we get the theorem because g_s and g_u are polynomials in g! As soon as g preserves some space, g_s and g_u must also preserve it. [[★★★ we're taking $G = GL(V)$ and $H = G$, and regarding $GL(V)$ as sitting inside of $End(V)$ to apply the Jordan form theorem]]

It looks like the decomposition depends on the choice of embedding $G \subseteq GL(V)$, but in fact the decomposition is natural.

Remark 8.9. For any representation $\rho \colon G \to GL(V)$, $\rho(g_s)$ is always semi-simple and $\rho(g_u)$ is always unipotent.

Suppose we define G as being in $GL(W)$, so $G \subseteq GL(W)$, then g_s and g_u act semi-simply and unipotently on W by construction. So g_s and g_u are semi-simple (resp unipotent) operators on W^*, so they are semi-simple (resp unipotent) on $\mathrm{Sym}^*(W^*) = k[W]$, so they are semi-simple (resp unipotent) on $k[G] = k[W]/I_G$. $V \otimes V^* \to k[G]$, given by $v \otimes \psi \mapsto \langle \psi, gv \rangle$. If I let G act only on the V and not the V^*, then the map is equivariant. So any finite-dimensional representation appears in $k[G]$ [[★★★ For the map to be injective, you need V to be irreducible, so we're only showing that g_s and g_u act as expected on irreducible representations]]. If you like, $k[G]$ is an injective generator for the category of algebraic G-modules. $\diamond$

For Lie groups, this doesn't hold. For example, consider the Lie algebra $\left(\begin{smallmatrix} t & t \\ 0 & t \end{smallmatrix}\right)$. Expoentiating, we get a Lie group. First, note that the Lie algebra is not closed under taking semisimple and nilpotent parts.

Q: is the problem that we can have Lie subgroups that aren't closed? A: no, that's not the only problem. You can find a closed subgroups which is not a Zariski closed subgroup, in which case the decomposition result fails. Moreover, you can take bad representations which violate the decomposition result.

9 Classifying Reductive Groups, Part I

Today we'll try to classify reductive groups in characteristic p.

Last time, we showed that any $g \in G$ has a natural decomposition $g = g_s g_u$. Let $G_s \subseteq G$ be the set of semi-simple elements and G_u be the set of unipotent elements. In general $G_u \subseteq G$ is a Zariski closed set (the vanishing locus of $(1-x)^N$ for some N).

Fact from linear algebra: any commuting set of semi-simple (diagonalizable) matrices can be simultaneously diagonalized.

Consider the case where G is an abelian group. In this case, $G_s G_s = G_s$ and $G_u G_u = G_u$, so we get a decomposition $G = G_s \times G_u$ as a group. We already know that G_u is closed, and we get that G_s is closed. This gives us a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{g}_n \oplus \mathfrak{g}_u$.

Proposition 9.1. *Suppose $G = G_s$ is an abelian group. Then $G = \Gamma \times G_0$, where Γ is a finite group (with $\mathrm{char}(k) \nmid \Gamma$) and G_0 is isomorphic to a torus.*

Proof. We have $G \subseteq GL(V)$ for some V. Since all elements of G are diagonalizable and they commute, they are simultaneously diagonalizable, so in some basis for V, G is a closed subgroup of the group of diagonal matrices, so we are describing closed subgroups of a torus. So we have $G \subseteq T$ a closed subgroup in a torus. $R = k[T] \cong k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Consider the character lattice $T^\vee$. Then $R = \bigoplus_{\chi \in T^\vee} k \cdot t^\chi$, where t^χ is the monomial such that $g \cdot t^\chi = \chi(g) t^\chi$. Define $L = \{\chi \in T^\vee | t^\chi(G) = 1\} \subseteq T^\vee$.

$T^\vee / L = \Gamma^\vee \times \mathbb{Z}^m$ for some finite group $\Gamma^\vee$. So $G = \Gamma \times G_0$, where G_0 is a torus of rank m. [[★★★ We're using that $G^{\vee\vee} = G$ and that $-^\vee$ is exact here]]

[[★★★ next three paragraphs are to show that $p \nmid |\Gamma|$]] Consider $Q \subseteq R$, the linear subspace spanned by all expressions of the form $t^\chi - t^{\chi'}$ where $\chi, \chi' \in L$, and let I^L be the ideal generated by Q. We have $I^L = \bigoplus_{\chi \in T^\vee / L} Q_\chi$, where $Q_\chi = t^\chi Q$. Note that $Q_1 = Q$. It is clear that this ideal is invariant under the action of G since Q is invariant under the action of G. It is also clear that $I^L \subseteq I_G$ because it is generated by Q and every element of Q vanishes on G.

Now $R = \bigoplus_{\chi \in T^\vee / L} R_\chi$, where $R_\chi = \{f \in R | f(gx) = \chi(g) f(x) \text{ for all } g \in G\} = \bigoplus_{\chi' - \chi \in L} k \cdot t^{\chi'}$. Each Q_χ has codimension 1 in R_χ because R_χ / Q_χ is a 1-dimensional vector space (the generators of Q_χ identify all the basis vectors of R_χ). If J is some G-invariant ideal, then $J = \bigoplus_{\chi \in T^\vee / L} (J \cap R_\chi)$ [[★★★ present this better]]. So the ideal I^L is a *maximal* G-invariant proper ideal in R because any ideal that contains R_χ must be all of R since $t^{\chi'} R_\chi = R_{\chi' + \chi}$. So we must have $I_G = I^L$.

Finally, we'd like to show that $p \nmid |\Gamma|$. This is equivalent to showing that $T^\vee / L$ doesn't have an element of order p. If there is an element of order p, then you can find $\chi \notin L$ such that $\chi^p \in L$. But then $\chi^p - 1 \in I_G$, so $\chi - 1 \in I_G$ (because I_G is radical (since G is required to be a variety), so $\chi - 1 \in I_G$, so $\chi \in L$. [[★★★ more generally, if we don't require G to be reduced, I think we get that Γ is a diagonalizable group scheme.]] ∎

Lemma 9.2. *Suppose G is a reductive group, then $Z(G)_s = Z(G)$.*

Proof. Suppose $u \in Z(G)_u$ with $u \neq 1$, and suppose $G \to GL(V)$ is a representation on which u acts non-trivially. Then $\ker(u-1)$ is a G-invariant subspace of V, but it doesn't split because $(u-1)$ is a nilpotent operator. □

Proposition 9.3. *If G is a reductive group, then $\mathfrak{g}_s = \mathfrak{g}$.*

Proof. We can assume G is not abelian, because in the abelian case, it's already clear. Pick a representation V of G. Consider $\mathrm{Sym}^p(V) \supseteq W = \{u^p | u \in V\}$. W is an invariant subspace.

Claim: If $\mathfrak{g}$ has a non-zero nilpotent element, then there is no $\mathfrak{g}$-invariant subspace W' such that $\mathrm{Sym}^p V = W \oplus W'$. The action on $\mathrm{Sym}^p V$ is $g(x_i) = \sum a_{ij} x_i \frac{\partial}{\partial x_j}$. So $W \subseteq (\mathrm{Sym}^p V)^G$. Suppose we have a non-zero nilpotent element $A \in \mathfrak{g}$. Pick x, y such that $Ax = y$ and $Ay = 0$. Then $Ax^p = 0$ and $A(xy^{p-1}) = y^p$. Any W' must contain a vector of the form $xy^{p-1} + z^p$ [[★★★ because in the quotient $\mathrm{Sym}^p(V)/W$, we have some element xy^{p-1}, so take a lift]], and when we apply A to it, we get into W, so W' is not invariant. □

Proposition 9.4. *Suppose G is connected and $\mathfrak{g}_s = \mathfrak{g}$. Then G is abelian (and must therefore be a torus).*

Proof. Let $G \subseteq GL(V)$. We will induct on the dimension of G *and* the dimension of V. The idea is to find a subgroup which is connected, so it must be a torus, and then procede.

I will assume $k \neq \overline{\mathbb{F}}_p$, but note that if $k = \overline{\mathbb{F}}_p$, then you can change base to some transcendental extension and apply the proof. The conclusion is stable under base change. I want to have elements of infinite order, and elements in an algebraic group over $\overline{\mathbb{F}}_p$ are of finite order.[1]

Step 1. G has a dense set of elements of infinite order. If $g \in G$ is of finite order, its characteristic polynomial $p_g(t) \in \overline{\mathbb{F}}_p[t]$ [[★★★ or $\overline{\mathbb{Q}}[t]$ if in characteristic zero]]. The coefficients are $\sigma_i(g) \in \overline{\mathbb{F}}_p$, the elementary symmetric functions. Since G is connected, it is impossible because the set of values of a regular function is a constructible set [[★★★ a constructible set contains an open subset of its closure, so the only dense constructible sets are those that contain an open subset, but $\overline{\mathbb{F}}_p \subseteq \mathbb{A}^1_k$ contains no open subset, so the only constructible subsets of $\overline{\mathbb{F}}_p$ are finite sets.]]. So each $\sigma_i(g)$ is constant, so every g must have the same characteristic polynomial as the identity element of G, $(1-t)^n$, so all elements are unipotent. So the closed subgroup G_u is equal to G, implying $\mathfrak{g} = \mathfrak{g}_u$, a contradiction. Density follows similarly [[★★★ you get that the σ_i are constant on an open neighborhood of the identity]].

Step 2.

[1] Note that since $GL_n(\mathbb{F}_q)$ is a finite group, all its elements are of finite order. It follows that all elements of $GL_n(\overline{\mathbb{F}}_p)$ are of finite order, so all closed points of a group G over $\overline{\mathbb{F}}_p$ are of finite order.

Lemma 9.5. *For* $s \in G_s$*, let* $C_G(s) = \{g \in G | sgs^{-1} = g\}$ *and* $C_{\mathfrak{g}}(s) = \{x \in \mathfrak{g} | \mathrm{Ad}_s(x) = x\}$*. Then* $\mathrm{Lie}\, C_G(s) = C_{\mathfrak{g}}(s)$*.*

Proof. $\mathrm{Lie}\, C_G(s) \subseteq C_{\mathfrak{g}}(s)$ is clear, so we only need to show that the dimensions are equal. The result is true in the case $G = GL(n)$ because the condition of being the in the centralizer looks exactly the same in the group and the Lie algebra, and it is some linear condition $C_G(s) = \{g | gs = sg\}$ (even if s is not semi-simple, btw). So all the p-th power problems don't appear here at all. [[★★★ $C_G(s)$ may be non-reduced in general, and the GL_n argument works in general. The argument that follows is for $C_G(s)_{\mathrm{red}}$ rather than $C_G(s)$.]]

Now consider two locally-closed Zariski sets. $Y = \{gsg^{-1}s^{-1} | g \in G\}$ and $S = \{gsg^{-1}s^{-1} | g \in GL(n)\}$ (remember that $G \subseteq GL(n)$ is a closed subgroup). $Y \subseteq S \cap G$, so $T_eY \subseteq T_eS \cap \mathfrak{g}$. Define $\mathfrak{n} = T_eS$, then $\mathfrak{gl} = C_{\mathfrak{gl}}(s) \oplus \mathfrak{n}$ [[★★★ s is semisimple, so it always acts diagonalizably. Choose a basis where Ad_s is diagonal. Then $1 - \mathrm{Ad}_s$ is some diagonal operator for which T_eS is the image and $C_{\mathfrak{gl}}(s)$ is the kernel, so you get a direct sum decomposition]]. We have the decomposition $\mathfrak{g} = C_{\mathfrak{g}}(s) \oplus \mathfrak{m}$ [[★★★ again using the diagonal $1 - \mathrm{Ad}_s$]]. Then $\mathfrak{m} = \mathfrak{g} \cap \mathfrak{n}$ [[★★★ $\mathfrak{g} \subseteq \mathfrak{gl}$ is a subspace invariant under the action of the diagonal operator $1 - \mathrm{Ad}_s$]]. This gives $\dim T_eY \le \dim \mathfrak{m}$. But we also have $\dim T_eY + \dim C_G(s) = \dim \mathfrak{g} = \dim C_{\mathfrak{g}}(s) + \dim \mathfrak{m}$ by the usual dim group is dim orbit plus dim stabilizer. But this tells me that $\dim C_G(s) \ge \dim C_{\mathfrak{g}}(s)$, which is what we wanted.

The moral is that if an element is semi-simple, you get the same sort of thing you usually get in characteristic 0. □

□

10 Classifying Reductive Groups, Part II

Statements we proved last time:

1. If $G = G_s$ and G abelian, then $G = \Gamma \times G_0$ where G_0 is a torus and Γ is a finite group with $\mathrm{char}(k) \nmid |\Gamma|$.
2. $s \in G_s$, then $\mathrm{Lie}(C_G(s)) = C_{\mathfrak{g}}(s)$. From which we get the corollary: If $C_{\mathfrak{g}}(s) = \mathfrak{g}$ and G is connected, then $s \in Z(G)$.
3. If G is reductive, then $Z(G) = Z(G)_s$.
4. If G is reductive and $\mathrm{char}(k) > 0$, then $\mathfrak{g} = \mathfrak{g}_s$.

Then we were in the middle of the following

Proposition 10.1. *Suppose $\mathfrak{g} = \mathfrak{g}_s$ and G is connected. Then G is abelian (and therefore a torus).*

So we get a description of reductive groups in characteristic p.

Proof. We were doing induction on $\dim G$ and $\dim V$, where $G \subseteq GL(V)$. By extending our ground field, we showed that the elements of infinite order is a dense set.

Let $g \in G$ be of infinite order, and let $H = [C_G(g)]_0$ (connected component). Since g is of infinite order, $\dim H \geq 1$ (since it isn't finite since it contains the powers of g). If for any infinite order element g we get $H = G$, then we are done since elements of infinite order form a dense set. So we may assume $H \neq G$. Then by induction on $\dim G$, H is abelian and hence is a torus.

Since H is a torus, we get the decomposition $V = \bigoplus_{\chi \in P} V_\chi$, where $P \subseteq H^\vee$ and $V_\chi = \{v \in V | hv = \chi(h)v\}$. Similarly, we get $\mathfrak{g} = \bigoplus_{\chi \in Q} \mathfrak{g}_\chi$ for some $Q \subseteq H^\vee$. Now it is easy to check that $\mathfrak{g}_\chi V_\eta \subseteq V_{\chi\eta}$. This tells me that $\mathfrak{g}_\chi^n V_\eta \subseteq V_{\chi^n \eta}$. But P is finite, so if $\chi \neq 1$, we must have that $\mathfrak{g}_\chi$ is nilpotent. But since $\mathfrak{g} = \mathfrak{g}_s$, there are no nilpotent elements, so $\mathfrak{g}_\chi = 0$ for $\chi \neq 1$. Thus, the adjoint action of H on $\mathfrak{g}$ is trivial. So $H \subseteq Z(G)$.

Suppose $|P| > 1$. Then since the action of G commutes with the action of H, each V_χ is G-invariant. We have projections $\pi \colon G \to G_\chi \subseteq GL(V_\chi)$. But by induction on $\dim V$, each G_χ is abelian. So G is abelian.

So we may assume $|P| = 1$. So H consists only of scalar matrices. Then $\tilde{G} = [G \cap SL(V)]_0$ has dimension smaller than G, and G is generated by $\tilde{G}$ and H. By induction on $\dim G$, $\tilde{G}$ is abelian. ☐

Corollary 10.2. *If G is a reductive connected group and* $\mathrm{char}(k) = p > 0$, *then G is a torus.*

Theorem 10.3. *G is reductive in characteristic* $\mathrm{char}(k) = p > 0$ *if and only if G_0 is a torus and $p \nmid |G/G_0|$.*

Proposition 10.4. *G is reductive if and only if G_0 is reductive and G/G_0 is reductive.*

Proof. Suppose G is reductive, and let V be a representation of the quotient group G/G_0. It lifts to a representation of G, and the representation must split completely because G is reductive.

Now suppose V is a not completely reducible representation of G_0, then $W = k[G] \otimes_{k[G_0]} V$ is a finite-dimensional representation of G which is not completely reducible. [[★★★ check for yourself]]

Now suppose G_0 and G/G_0 are reductive. Recall that it is enough to show that $V = V^G \oplus W$. We have that $V = V^{G_0} \oplus W'$ since G_0 is reductive, and we have $V^{G_0} = V^G \oplus W''$ as a representation of G/G_0. [[★★★ it is easy to check that W' is invariant under the action of G]] □

By the way, we had a little question about why it was enough to check something on finite-dimensional representations. One of the criteria was that if $V \to W$ is surjective, then so is $V^G \to W^G$. Since every representation is a union of finite-dimensional representations, we get what we want.

Remark 10.5 (Restricted Lie algebras)**.** Suppose $\mathrm{char}(k) = p > 0$. You can check that if $D \in \mathrm{Der}(k[G])$, then D^p is again a derivation (follows from the binomial theorem). So we have a homomorphism of derivations $D \mapsto D^p$ (it plays well with the bracket). Left invariant derivations are sent to left invariant derivations, so we get a homomorphism $\mathfrak{g} \to \mathfrak{g}$, denoted by $x \mapsto x^{(p)}$. Such a Lie algebra (there are some axioms) is called a *restricted Lie algebra*. ◇

Example 10.6. If $G = k$, then $k[G] = k[t]$ and $\mathfrak{g} = k \cdot \frac{\partial}{\partial t}$. We have that $(\frac{\partial}{\partial t})^p = 0$, so our map is $x^{(p)} = 0$.

On the other hand, if $G = k^\times$, then $k[G] = k[t, t^{-1}]$ and $\mathfrak{g} = k \cdot t\frac{\partial}{\partial t}$. Now we have $(t\frac{\partial}{\partial t})^p = t\frac{\partial}{\partial t}$.

So even though these algebras are isomorphic, they are distinguishable as restricted Lie algebras. ◇

Reductive groups in characteristic zero

Definition 10.7. A Lie algebra $\mathfrak{g}$ is *simple* if it is not abelian and $\mathfrak{g}$ has no non-trivial proper ideals. We say $\mathfrak{g}$ is *semi-simple* if it is a direct sum of simple Lie algebras. ◇

The point of 261 is that semi-simple algebras are possible to classify in characteristic 0 over an algebraically closed field. In characteristic p, I think it is also possible, but there are more of them and it is more difficult.

Definition 10.8. A group G is *simple* (resp. *semi-simple*) if its Lie algebra $\mathfrak{g}$ is. ◇

Theorem 10.9 (Weyl's Theorem)**.** *Every finite-dimensional representation over a semi-simple Lie algebra is completely reducible.*

Theorem 10.10. *Any semi-simple algebraic group G is reductive in characteristic zero.*

Proof. It is sufficent to check for G connected since finite groups are reductive. We showed that in this situation $W \subseteq V$ is an invariant subspace if and only if $W \subseteq V$ is $\mathfrak{g}$-invariant. □

Remark 10.11. If a module is completely reducible over an algebraic closure, then it is completely reducible over the original field, so you don't need algebraically closed. ◇

Now assume G is reductive. Then the adjoint representation Ad_G is completely reducible, so $\mathrm{ad}_\mathfrak{g}$ is completely reducible. Thus, we have $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. Each $\mathfrak{g}_i$ is a submodule (ideal) with no proper non-trivial ideals, so each $\mathfrak{g}_i$ is either simple or 1-dimensional. Thus, we may rewrite $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r \oplus Z(\mathfrak{g}) = \mathfrak{g}^{ss} \oplus Z(\mathfrak{g})$, where each $\mathfrak{g}_i$ is semi-simple and $Z(\mathfrak{g})$ is the center. Note that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}^{ss}$.

Fact: [[★★★ exercise]] If G is a connected algebraic group, then $G' = [G, G] = \{ghg^{-1}h^{-1} | g, h \in G\}$ is a closed connected subgroup with Lie algebra $\mathrm{Lie}\, G' = [\mathfrak{g}, \mathfrak{g}]$.

Now assume G is connected and reductive. Then $G' \times Z(G)_0 \to G$ is surjective. By fact (3), $Z(G)_0$ is a torus. So G is a quotient of a product of a semi-simple group G' with a torus by a finite central subgroup.

Typical example: $GL_n = (SL_n \times k^\times)/\mu_n$.

Theorem 10.12. *If* $\mathrm{char}(k) = 0$*, then G is reductive if and only if G is a quotient of $G^{ss} \times T$ (G^{ss} semi-simple and T a torus) by some finite subgroup.*

It's clear that any such quotient is reductive. I've implicitly used the following.

Exercise. *If G_1 and G_2 are reductive, then $G_1 \times G_2$ is reductive.*

11 Stability in the affine case

There is some confusion of terminology. Depending on which books you read, "stable points" means slightly different things.

Suppose G is an affine algebraic groups that acts on an affine variety X. We know that for a point $x \in X$, the orbit $G \cdot x$ is locally closed. The stabilizer G_x is a closed subgroup, and we have

$$\dim G_x + \dim G \cdot x = \dim G.$$

We can define a function $d(x) = \dim G_x$. We say that x is *regular* if $d(y)$ is constant in an open neighborhood of x. This definition works perfectly well for any scheme.

It's clear that $X_{\geq d} = \{x \in X | d(x) \geq d\}$ is a closed set [[★★★ we're applying some kind of semi-continuity result here. If G is flat, then the action map $G \times X \to X$ is flat, so $G \times X \to X \times X$ is flat, so by semi-continuity, the dimensions of the fibers is upper semi-continuous. The fibers along the diagonal are exactly the stabilizers of points]]. Therefore, the set of all regular points, X^{reg}, is open. It is also clear that $X^{\text{reg}} = X_0^{\text{reg}} \sqcup X_1^{\text{reg}} \sqcup \cdots \sqcup X_k^{\text{reg}}$, where $X_i^{\text{reg}} = \{x \in X^{\text{reg}} | d(x) = i\}$. If X is irreducible, then $X^{\text{reg}} = \{x | \dim G_x \text{ is minimal}\}$, which is the union of the largest-dimensional orbits.

Proposition 11.1. *Suppose G is affine and X is irreducible with $X = X^{\text{reg}}$. Then the action of G is closed.*

Proof. Consider $\phi \colon X \to Y = \operatorname{Specm} k[X]^G$. The fibers $\phi^{-1}(y)$ are closure equivalence classes and contain a unique closed orbit. The closed orbit has the minimum possible dimension. But each orbit has the same dimension, so each closure equivalence class is just one orbit.

[[★★★ Alternative: if $\overline{G \cdot x} \neq G \cdot x$, then $\overline{G \cdot x}$ contains $G \cdot y$ of smaller dimension. You don't need X irreducible or G affine.]] □

Definition 11.2. $x \in X$ is *stable* if $G \cdot x$ is closed and x is regular. $x \in X$ is *properly stable* if it is stable and G_x is finite.[1] ◇

Remark 11.3. Alternative definition: $x \in X$ is stable if $G \cdot x$ is closed and not contained in the closure of any other orbit. [[★★★ If x is in the closure of another orbit, it's clearly not stable. On the other hand, if it's not stable, then any open neighborhood intersects an orbit of higher dimension (by that same semi-continuity result).]] [[★★★ you need to use reductive somewhere; otherwise, we have the example of the $\mathbb{G}_a$ action $t \cdot (x, y) = (x, tx + y)$.]] ◇

Suppose G is reductive, and consider $\phi \colon X \to Y = \operatorname{Specm} k[X]^G$. Let $X^{\text{irreg}} = X \smallsetminus X^{\text{reg}}$, and let $Z = \phi(X^{\text{irreg}})$. [[★★★ Z is closed because the topology on $X /\!\!/ G$ is induced by the topology on X]]

Lemma 11.4. *The stable points are $X^s = X \smallsetminus \phi^{-1}(Z)$.*

[1] Sometimes, when people say "stable," they mean properly stable.

Proof. Suppose $x \notin \phi^{-1}(Z)$. Then $G \cdot x$ is in some fiber $\phi^{-1}(y)$ for some $y \in Y$. If it is not closed, then there is a smaller-dimensional orbit $G \cdot z$ in its closure equivalence class (so z is not regular). Then $\phi(x) \in \phi(Z)$.

On the other hand, let $x \in \phi^{-1}(Z)$. Suppose $x \in \phi^{-1}(z)$, so its closure equivalence class contains a non-regular point (by definition!). Either x is not regular, or it's orbit is not closed. □

Example 11.5. $k^\times$ acts on $\mathbb{A}^2$ by $t \cdot (x, y) = (tx, t^{-1}y)$. Then the only invariant is xy. $(A^2)^{\text{reg}} = \mathbb{A}^2 \smallsetminus \{(0,0)\}$, but $(\mathbb{A}^2)^s$ is the complement of the axes. Indeed, every orbit except the axes is closed and regular. So $(\mathbb{A}^2)^s = \{(x,y) | xy \neq 0\}$. ◇

Later on, we'll study stable points for non-affine quotients as well.

For any $f \in k[X]$, let $X_f = \{x \in X | f(x) \neq 0\}$.

Proposition 11.6. *$x \in X$ is stable if and only if there exists an invariant function $f \in k[X]^G$ such that $x \in X_f$ and the action of G on X_f is closed.*

Proof. Let $I_Z \subseteq k[X]^G$ be the ideal of functions vanishing on $Z \subseteq X/\!\!/G$. Since the fibers of ϕ are orbit-closure equivalence classes, it is not hard to check that for an invariant function f, the action of G on X_f is closed if and only if $f \in I_Z$ [[★★★ if the action on X_f is closed, how do you get that $f \in I_Z$?]]. Given $x \in X^s$, we have that $\phi(x) \notin Z$ by the Lemma. So there is some invariant function $f \in I_Z$ that doesn't vanish on $\phi(x)$. Then X_f is a neighborhood of x on which the action is closed. □

The point is that the notion of an affine quotient only makes sense if you have stable points. Otherwise, you may as well throw out the notion. For example, if $k^\times$ acts on $\mathbb{A}^n$ by homothety, then there are no stable points, and the quotient is correspondingly bad.

Definition 11.7. A quotient $X \to X/G$ is a *geometric quotient* if the fibers are G-orbits. ◇

So consider the restriction $\phi\colon X^s \to X^s/\!\!/G \subseteq X/\!\!/G$. This will be a geometric quotient (meaning that the fibers are orbits). If X^s is non-empty, then it is an open set for which there is a good quotient.

This leads to another definition.

Definition 11.8. $x \in X$ is *pre-stable* if it has a G-invariant affine open neighborhood U such that the action of G on U is closed. ◇

Note that in the example of $k^\times$ acting on $\mathbb{A}^n$ by homothety, everything except the origin is pre-stable. By the Proposition, any stable point is prestable (take $U = X_f$).

So we can take the geometric quotients $U_x/\!\!/G$ and glue these quotients together to get the *prestable quotient* $X^{\text{pre}}/\!\!/G$. This is a geometric quotient (in the sense that the fibers are orbits) [[★★★ in order for the gluing construction to make sense, we need to prove that for a geometric quotient $\phi\colon U \to U/\!\!/G$ and an invariant open subset $W \subseteq U$, $\phi(W) \cong W/\!\!/G$]][[★★★ Actually, the property of a map being an affine quotient is stable under arbitrary base change since $(R \otimes_{R^G} S)^G \cong S$]]

Example 11.9. Take $k^\times$ acting on $\mathbb{A}^2$ by homothety. The prestable points are $\mathbb{A}^2 \smallsetminus \{0\}$. We have the open cover $U_1 = \{x \neq 0\}$ and $U_2 = \{y \neq 0\}$. Then $U_1/G = \operatorname{Specm} k[y/x]$ and $U_2/G = \operatorname{Specm} k[x/y]$. ◇

Example 11.10. Let $k^\times$ act on $\mathbb{A}^2$ by $t\cdot(x,y) = (tx, t^{-1}y)$. $(\mathbb{A}^2)^{\text{pre}} = \mathbb{A}^2 \smallsetminus \{0\}$. Take the same cover U_1 and U_2. Then $U_1/G = \operatorname{Specm} k[yx]$ and $U_2/G = \operatorname{Specm}[yx]$. When we glue them together, we get the non-separated line! The two origins correspond to the two non-closed orbits. ◇

Proposition 11.11. *Let X be affine and irreducible, and let G be reductive. Assume that $X^s \neq \varnothing$. Let $R = k[X]$ and $K = K(X) = \operatorname{Frac}(R)$. Then $K^G = \operatorname{Frac}(R^G)$.*

[[★★★ We showed something similar for a finite group before.]] This proposition somehow tells you that if you have stable points, then the affine quotient is pretty good.

Proof. Suppose $h/f \in K^G$ is an irreducible fraction (no non-units divide both h and f) [[★★★ not clear you can get such a thing]]. [[Then we want to show that $f \in R^G$.]] We want to show that $h/f = b/a$ for some $b, a \in R^G$.

$$\begin{tikzcd} R^G \arrow[r, hook] \arrow[d, hook] & R^G[h/f] \arrow[d, hook] \\ R \arrow[r, hook] & R_f \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "\phi"'] & X_f \arrow[l, hook'] \arrow[d] \\ Y & Y' \arrow[l, "\psi"', "\text{dominant}"] \end{tikzcd}$$

where $Y' = \operatorname{Specm} R^G[h/f]$. If $y \in Y^s = \phi(X^s)$, then $\psi^{-1}(y)$ is just one point. This tells me that h/f is algebraic over R^G, for otherwise, the dimension of Y' would be bigger than the dimension of Y. Suppose it satisfies some polynomial of degree n. Then the preimage of a generic point would be n points. But you only get one point on the stable locus, so the polynomial is of degree 1, so we have $a\frac{h}{f} = b$. [[★★★ so we didn't need that irreducible fraction business after all. But we have to note that in general, the closure of an orbit is a union with smaller-dimensional orbits.]] □

Next time, we'll do some examples, like the moduli space of smooth surfaces. Then we'll do the proj quotient.

12 Degree d Hypersurfaces in $\mathbb{P}^n$

Consider the space of homogeneous degree d polynomials in $k[x_0, \dots, x_n]$, $\mathbb{V}_{n,d} = \{f(x_0, \dots, x_n) = \sum_{i_0+\cdots+i_n=d} a_{i_0,\dots,i_n} x_0^{i_0} \cdots x_n^{a_n}\}$. Then the space of degree d hypersurfaces in $\mathbb{P}^d$ is $\mathcal{H}_{n,d} = \mathbb{P}(\mathbb{V}_{n,d})$. Ultimately, we'll want to understand these surfaces up to isomorphism (or projective equivalence), so we'll be interested in the quotient $\mathbb{V}^{\times}_{n,d}/GL(n+1) = \mathcal{H}_{n,d}/GL(n+1)$.

First of all, consider the degree d polynomials corresponding to *smooth* hypersurfaces $\mathbb{V}^{sm}_{n,d} = \{f | \frac{\partial f}{\partial x_0} = \cdots = \frac{\partial f}{\partial x_n} = 0$ has only the trivial solution$\}$, and the projectivization $\mathcal{H}^{sm}_{n,d}$, the space of smooth hypersurfaces of degree d. Let $\mathcal{H}^{sing}_{n,d}$ be the complementary space, corresponding to $\mathbb{V}^{sing}_{n,d}$. [[★★★ it should be clear that $\mathbb{V}^{sing}_{n,d} \subseteq \mathbb{V}_{n,d}$ and $\mathcal{H}^{sing}_{n,d} \subseteq \mathcal{H}_{n,d}$ are closed immersions, but I don't see a good way to show it.]]

Lemma 12.1. *$\mathcal{H}^{sing}_{n,d}$ is a hypersurface in $\mathbb{P}(\mathbb{V}_{n,d})$.*

Proof. Consider $M = \{(x, h) | x \in \mathbb{P}^n, h \in \mathcal{H}_{n,d}, h$ singular at $x\}$. Then we have

$$\begin{tikzcd} & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^n & & \mathcal{H}^{sing}_{n,d} \end{tikzcd}$$

We get $\dim \pi_1^{-1}(x) \geq \dim \mathcal{H}_{n,d} - (n+1)$ because saying that the hypersurface cut out by f is singular at x amounts to imposing the $n+1$ conditions $\frac{\partial f}{\partial x_0} = \cdots = \frac{\partial f}{\partial x_n} = 0$ (it follows that $f(x) = 0$). Since all the points of $\mathbb{P}^n$ are identical, we have that π_1 is locally a product, so we get

$$\dim M = \dim \mathbb{P}^n + \dim \pi^{-1}(x) \geq \dim \mathcal{H}_{n,d} - 1.$$

[[★★★ moreover, this inequality holds for every component of M since the "$(n+1)$ conditions" argument works locally]] A generic singular surface will have a zero-dimensional (so finite) singular locus [[★★★ how to see this easily?]], so there is an open subset of $\mathcal{H}^{sing}_{n,d}$ over which π_2 is finite. It follows that

$$\dim \mathcal{H}^{sing}_{n,d} \geq \dim \mathcal{H}_{n,d} - 1.$$

But since $\mathcal{H}^{sing}_{n,d}$ is a closed subscheme and there exist non-singular surfaces, we get that $\dim \mathcal{H}^{sing}_{n,d} = \dim \mathcal{H}_{n,d} - 1$. □

So we have some homogeneous polynomial on $\mathbb{V}_{n,d}$, called the *discriminant*, D, which vanishes exactly on those degree d forms f that give singular surfaces. [[★★★ make this a definition]]

Since the action of $GL(n+1)$ on $\mathbb{V}_{n,d}$ doesn't change whether or not a form corresponds to a singular surface (and it respects the degree), the action of $GL(n+1)$

must act on D by a scalar. That is, D is a semi-invariant of the group $GL(n+1)$. Since the only characters of $GL(n+1)$ are powers of the determinant [[★★★ good way to see this?]], D is an invariant of $SL(n+1)$.

We have a finite-to-one (where finite is $\deg D$) projection $\{f \in \mathbb{V}_{n,d} | D(f) = 1\} \to \mathcal{H}^{sm}_{n,d}$,[1] equivariant with respect to the action of $SL(n+1)$.

Remark 12.2. In the case $d = 2$, we know what $\mathcal{H}_{n,d}/SL(n+1)$ is: quadratic forms are determined by their rank, so the quotient space is discrete. ◇

Lemma 12.3. *If $d > 2$, then the stabilizer of any $h \in \mathcal{H}^{sm}_{n,d}$ in $SL(n+1)$ is finite.*

Proof. By considering the finite-to-one cover[[★★★ make this better]], it is enough to show that the stabilizer of any degree d form f (with $D(f) = 1$) in $GL(n+1)$ (and so $SL(n+1)$) is finite. Since we are in characteristic 0, this is equivalent to computing the stabilizer in the Lie algebra [[★★★ exactly what result are we using here?]]. We have $\mathfrak{g} = \mathfrak{gl}(n+1)$. We have the action given by $(a_{ij}) \mapsto \sum a_{ij} x_i \frac{\partial}{\partial x_j}$. We want to show that $Stab_{\mathfrak{g}}(f) = 0$, so we must show that there is no non-zero (a_{ij}) such that $\sum a_{ij} x_i \frac{\partial f}{\partial x_j} = 0$. We may rewrite the equation as

$$\ell_0 \frac{\partial f}{\partial x_0} + \cdots + \ell_n \frac{\partial f}{\partial x_n} = 0$$

for some linear forms ℓ_i. Assume that such linear forms exist, with some of the ℓ_i non-zero (assume $\ell_0 \neq 0$).

I claim that $\frac{\partial f}{\partial x_0}$ is not a zero divisor on $C = \operatorname{Spec}\big(k[x_0, \dots, x_n]/(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})\big)$. It is clear that the dimension of all components of C is at least one. If $\frac{\partial f}{\partial x_0}$ were a zero-divisor on C, then $V_C(\frac{\partial f}{\partial x_0}) = \operatorname{Spec}\big(k[x_0, \dots, x_n]/(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})\big)$ would be at least 1-dimensional. But this is exactly the affine cone on the singular locus of the surface corresponding to f. Since the surface was assumed to be non-singular, the affine cone must consist of just the origin.

Thus, we must have $\ell_0 \in (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$, which is impossible since $\deg f > 2$. □

We've actually proved

Proposition 12.4. *Every $h \in \mathcal{H}^{sm}_{n,d}$ is properly stable with respect to the action of $SL(n+1)$.*

So we can consider the geometric affine quotient $\mathcal{H}^{sm}_{n,d}/SL(n+1)$. The point is how to find the invariants, which is not easy at all.

First, let's consider the case $n = 1$. The problem of finding all the invariants is still unsolved (the first person who worked on this was Cayley?). [[★★★ I think this was done recently by Ben Howard, John Millson, Andrew Snowden, and Ravi Vakil]]

[1] This is finite because all the coefficents can be multiplied by any $\deg(D)$-th root of unity to give another form corresponding to the same surface with $D = 1$

Classical binary invariants (the case $n = 1$)

We're looking for $SL(n+1)$-invariants in $\mathbb{V}_{n,d}$. We have the action on the d-th symmetric power of the standard representation, and we're looking for invariants. If $n = 1$, we have $V_d = \mathbb{V}_{1,d}$. Geometrically, this is d points on $\mathbb{P}^1$. Smoothness means that no two points coincide. In this case, we can explicitly write the discriminant.

First of all, $V_d = \{\xi_0 x^d + d\xi_1 x^{d-1}y + \binom{d}{2}\xi_2 x^{d-2}y^2 + \cdots + \xi_d y^d\}$. The action is given by $f(x,y) \mapsto f(ax+by, cx+dy)$, where $ad - bc = 1$. If you have two polynomials, you can measure if they have a common zero (this is called the resultant)[[★★★ look stuff up about the resultant]]. Consider

$$Res\Big(\frac{\partial f}{\partial x}\Big|_{y=1}, \frac{\partial f}{\partial y}\Big|_{y=1}\Big).$$

This is a polynomial of degree $2d-2$ and it is clearly an invariant. [[★★★ I don't see why you can specialize to $y = 1$ ... you might have many points come together at infinity]]

For example, if $d = 2$, then $D(\xi) = \left|\left(\begin{smallmatrix}\xi_0 & \xi_1\\ \xi_1 & \xi_2\end{smallmatrix}\right)\right|$. For $d = 3$, we have

$$D(\xi) = \begin{vmatrix} \xi_0 & 2\xi_1 & \xi_2 & 0 \\ 0 & \xi_0 & 2\xi_1 & \xi_2 \\ \xi_1 & 2\xi_2 & \xi_3 & 0 \\ 0 & \xi_1 & 2\xi_2 & \xi_3 \end{vmatrix}$$

Next we calculate $P_V(t) = \sum_{k\geq 0} \dim \operatorname{Sym}^k(V)^G t^k$.

[[★★★ resume editing here]]

V_d are all the irreducible representations of $SL(2) \ni \left\{\left(\begin{smallmatrix} q & 0\\ 0 & q^{-1}\end{smallmatrix}\right)\right\} = \mathbb{G}_m$. Let $g_q = \left(\begin{smallmatrix} q & 0\\ 0 & q^{-1}\end{smallmatrix}\right)$ acting on V. For $d \geq 0$, we define the *character* of V is $\operatorname{ch} V = \operatorname{tr}_V\left(\begin{smallmatrix} q & 0\\ 0 & q^{-1}\end{smallmatrix}\right) = \sum_{m\in\mathbb{Z}} a_m q^m \in \mathbb{C}[q, q^{-1}]$. Characters behave very well with tensor products and direct sums. For any representation, we have $\operatorname{ch} V = \sum c_d \operatorname{ch} V_d$, and $\operatorname{ch} V_d = q^d + q^{d-2} + \cdots q^{-d}$. Suppose $\operatorname{ch} V = \sum b_m q^m$. Then $\dim V^G = Res_0(q - q^{-1}) \cdot \operatorname{ch} V$. You check this by [[★★★]]

In fact, we have the following formula.

$$\sum \operatorname{ch} \operatorname{Sym}^k(V) t^k = \frac{1}{(1 - t\det_V(q - q^{-1}))}$$

You get this by looking at eigenvalues of this matrix and you get what you want.

Therefore, we get

$$P_V(t) = \sum \dim(\operatorname{Sym}^k V)^G t^k = -Res_0(q - q^{-1})\frac{1}{1 - t\det\left(\begin{smallmatrix} q & 0\\ 0 & q^{-1}\end{smallmatrix}\right)}.$$

This is a Moilen formula for G. The proof is exactly the same as for the Moilen formula.

Remark 12.5. If K is a compact topological group (maximal compact group in G), like $SU(2) \subseteq SL(2,\mathbb{C})$, then we get

$$P_V(t) = \int_K \frac{1}{1 - t\det g}\, dg$$

where dg is the invariant volume form on K such that $\int_K dg = 1$. Every element of $SU(2)$ is diagonalizable, so it is conjugate to $\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{smallmatrix}\right)$. But the formula is constant on conjugacy classes. If you think of $SU(2)$ as S^3, then the conjugacy classes are S^2s. If you do the calculation, you can reduce the integral to

$$\frac{1}{\pi}\int_0^{2\pi} \frac{\sin^2\theta}{1 - t\det\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{smallmatrix}\right)}\, d\theta$$

where $q = e^{i\theta}$. You can reduce this to the contour integral

$$-\frac{1}{2\pi i}\oint \frac{q - q^{-1}}{1 - t\det\left(\begin{smallmatrix} q & 0 \\ 0 & q^{-1}\end{smallmatrix}\right)} dq$$

$\diamond$

Now our problem is to compute

$$P_d(t) = P_{V_d}(t) = -Res_0(q - q^{-1})\prod_{i=0}^{d} \frac{1}{1 - tq^{d-2i}}$$

$\left(\begin{smallmatrix} q & 0 \\ 0 & q^{-1}\end{smallmatrix}\right)$ acts on V_d by $diag(q^d, q^{d-2}, \ldots, q^{-d})$.

Now introduce quantum binomial coefficients. We define

$$\begin{aligned} [d]_q &= \frac{q^d - q^{-d}}{q - q^{-1}} \\ [d]_q! &= [d]_q[d-1]_q \cdots [1]_q \\ \begin{bmatrix} p \\ d \end{bmatrix}_q &= \frac{[p]_q!}{[d]_q![p-d]_q!} \end{aligned}$$

Claim. $P_d(t) = \sum_{k\geq 0} Res_0(q - q^{-1}) \left[\begin{smallmatrix} d+k \\ k \end{smallmatrix}\right]_q t^k$

which is just a calculation

I define

$$\Phi(q,t) = \prod_{i=0}^{d} \frac{1}{1 - tq^{d-2i}} = \sum_{k\geq 0} \begin{bmatrix} d+k \\ k \end{bmatrix}_q t^k$$

and get

$$\Phi(q, q^2 t) = \frac{1 - q^{-d}t}{1 - q^{d+2}t}\Phi(q,t)$$

Then you solve to get that if $\Phi(q,t) = \sum c_k(q)t^k$, we get the recurrence

$$c_k = c_{k-1}\frac{q^{k+d} - q^{-k-d}}{q^k - q^{-k}}.$$

Immediately from that, we get the Cayley-Sylvester formula. If

$$m(d,k) = \dim(\operatorname{Sym}^k V_d)^G$$

then we get

$$m(d,k) = \begin{cases} 0 & d_k \text{ is odd} \\ \text{coeff of } u^{dk/2} \text{ in } \frac{(1-u^{k+1})\cdots(1-u^{k+d})}{(1-u^2)\cdots(1-u^d)} & \text{else} \end{cases}$$

Next time we'll do the case $d = 4$ and $n = 2$, $d = 3$ (elliptic curves).

13 Lecture 13

Last time somebody asked if in characteristic p the stabilizer is also finite. The answer is of course yes because $\dim \mathfrak{g}_x \geq \dim G_x$. Since we showed that the stabilizer in the Lie algebra is zero, so the group is finite. Of course, the problem is that $GL(n+1)$ is not reductive in finite characteristic.

Question: Does the the quotient space we've been working with represent the correct functor? Answer: We'll discuss all this stuff next week.

Recall what we did last time. We proved the Cayley-Sylvester formula. We had $V_d = \mathbb{V}_{1,d}$, and we showed that

$$\dim(\operatorname{Sym}^k V_d)^{SL(2)} = \begin{cases} 0 & dk \text{ odd} \\ \text{coef of } u^{dk/2} \text{ in } \frac{\cdots}{\cdots} \end{cases}$$

Example 13.1 ($d = 4$)**.** The is the case of four points in $\mathbb{P}^1$. We are looking for the coefficient of u^{2k} in $\frac{(1-u^{k+1})\cdots(1-u^{k+4})}{(1-u^2)(1-u^3)(1-u^4)}$. This will be the same as the coefficient of u^{2k} in $\frac{1-u^{k+1}-u^{k+2}-u^{k+3}-u^{k+4}}{(1-u^2)(1-u^3)(1-u^4)}$ (even though these things are completely equal). We can rewrite this as

$$\begin{aligned} \frac{1}{(1-u^2)(1-u^3)(1-u^4)} &- \frac{u^{k+1}(1+u+u^2+u^3)}{(1-u^2)(1-u^3)(1-u^4)} \\ &= \frac{1}{(1-u^2)(1-u^3)(1-u^4)} - \frac{u^k u}{(1-u)(1-u^2)(1-u^3)} \end{aligned}$$

The u^{2k} coefficient of this is the same as the u^k coefficient of

$$\frac{1}{(1-u)(1-u^{3/2})(1-u^2)} - \frac{u}{(1-u)(1-u^2)(1-u^3)}$$

[[★★★ subbing $u \mapsto u^{1/2}$ for the first term and taking out a u^k in the second part]]

Eventually, I should get $\frac{1}{(1-u^2)(1-u^3)}$. To get this, I multiply by $(1+u^{3/2})$ to get that I want the coefficient of u^k in

$$\frac{1+u^{3/2}-u}{(1-u)(1-u^2)(1-u^3)}$$

and then I can forget about [[★★★ something ... the $u^{3/2}$?]]

So we have $P_u(t) = \frac{1}{(1-t^2)(1-t^3)}$. So we have two algebraically independent invariants f_2 and f_3 (of degree 2 and 3 respectively).

I have $\operatorname{Sym}^2(V_2) = V_4 \oplus V_0$. We may regard V_2 as having basis $x^2 = u$, $2xy = v$, and $y^2 = w$. Then $Q = 4uw - v^2$ is invariant under $SL(2)$, so it spans the V_0.

The form $A = \xi_0 x^4 + 4\xi_1 x^3 y + 6\xi_2 x^2 y^2 + 4\xi xy^3 + \xi_4 y^4$ is some element of $\operatorname{Sym}^2 V_2$. I may rewrite $A = \xi_0 u^2 + 2\xi_1 uv + 2\xi_2 uw + \xi_2 v^2 + 2\xi_3 vw + \xi_4 w^2$.

I get a map $SL(2,\mathbb{C}) \to SO(3) = \{x \in SL(3) | x^t Q x = Q\}$ as we've seen before. We have $\mathrm{tr}(AQ) = 0$, which is the condition that cuts out V_4 [[★★★ ?]].

In matrix form $Q = \left(\begin{smallmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{smallmatrix}\right)$ and $A = \left(\begin{smallmatrix} \xi_0 & \xi_1 & \xi_2 \\ \xi_1 & \xi_2 & \xi_3 \\ \xi_2 & \xi_3 & \xi_4 \end{smallmatrix}\right)$.

We have $\det(A + \lambda Q) = 4\lambda^3 - f_2(\xi)\lambda - f_3(\xi)$ remains invariant under element of $SO(3)$ and therefore under $SL(2,\mathbb{C})$. Explicitly

$$
\begin{aligned}
f_2(\xi) &= \det\left(\begin{smallmatrix} \xi_0 & \xi_2 \\ \xi_2 & \xi_4 \end{smallmatrix}\right) - 4\det\left(\begin{smallmatrix} \xi_1 & \xi_2 \\ \xi_2 & \xi_3 \end{smallmatrix}\right) \\
f_3(\xi) &= \det\left(\begin{smallmatrix} \xi_0 & \xi_1 & \xi_2 \\ \xi_1 & \xi_2 & \xi_3 \\ \xi_2 & \xi_3 & \xi_4 \end{smallmatrix}\right)
\end{aligned}
$$

and we get the discriminant is the resultant $D = f_2^3 - 27f_3^2$. ◇

The next example is more interesting. $\mathbb{V}_{2,3} = \{f = \sum_{i+j+k=3} a_{ij} x^i y^j z^k\}$ is the space of degree 3 forms in $2+1$ variables. We have $\dim \mathbb{V}_{2,3} = 10$. These are cubic curves in $\mathbb{P}^2$. In the case of complex geometry, the smooth ones are just tori.

The idea is that every smooth curve of degree 3 is in fact an abelian group, of the form $\mathbb{C}/\Gamma$ for some lattice $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$.

We define $W(z) = \frac{1}{z^2} + \sum_{\gamma \in \Gamma \smallsetminus 0} \frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2}$. This is a doubly periodic function, with $W(z+\omega_1) = W(z+\omega_2) = W(z)$. The integral around the parallelogram has to be zero, so $W(z)$ cannot have a single pole in the parallelogram. We write out the Laurent series

$$
\begin{aligned}
W(z) &= \frac{1}{z^2} + 3G_2 z^2 + 5G_3 z^4 + \cdots \\
G_2 &= \sum_{\gamma \in \Gamma \smallsetminus 0} \frac{1}{\gamma^4} \\
G_3 &= \sum_{\gamma \in \Gamma \smallsetminus 0} \frac{1}{\gamma^6} \\
W'(z) &= -\frac{2}{z^3} + 6G_2 z + 20G_3 z^3
\end{aligned}
$$

So we get $(W')^2 - 4W^3 + g_2 W + g_3 = 0$ [[★★★ by some reason]] with $g_2 = 60G_4$ and $g_3 = 140G_6$.

We get the map $z \mapsto (W : W' : 1)$ and $0 \mapsto (0 : 1 : 0)$. The image is a cubic curve in $C\ \mathbb{P}^2$.

We get the curve $4X^3 - g_2 XZ^2 - g_3 Z^3 = Y^2 Z$. This is called the Wierstrauss normal form of the curve. We suspect that g_2 and g_3 are the invariants.

For $Z = 1$, we get $4x^3 - g_2 x - g_3 = y^2$. This is non-singular when something something, which gives me an expression for the discriminant $D = g_2^3 - 27g_3^2$.

Whatever Γ you pick, it turns out that D will not vanish, so you'll get a non-singular curve. This should remind you of the previous example.

We notice that C has the structure of an abelian group. Pick a point and call it O. Given two points a and b, you draw the line through them; it intersects in a third point c. Then draw a line connecting O to c, and this intersects in a third point, which we call $a+b$.

From this we see that every smooth cubic has 9 inflection points. To see this, consider the tangent line to O, it will intersect at some other point O'. An inflection point is a solution to the equation $3a = O'$. But the group is $\mathbb{C}/\Gamma$, so we know that there are 9 solutions to this equation.

In particular, the curve has an inflection point. In Weirstrauss normal form, we get a special inflection point $(0:1:0)$, with tangent line $z=0$. Given any curve, you transform it to one that satisfies these conditions. Once you have these conditions, the curve is given by an equation of the form

$$f = ax^3 + \overbrace{bx^2z} + cxz^2 + dz^3 + ezy^2 + \overbrace{hxyz + fyz^2}$$

Using the transformations $y \mapsto \alpha y + \beta x + \gamma z$ and $z \mapsto x + \delta z$, you can get rid of the overbraced terms.

Next we'll show that $\mathcal{H}_{1,4}^{sm}/PSL(2)$ and $\mathcal{H}_{2,3}^{sm}/PSL(3)$ are isomorphic. Given a smooth cubic curve C, we map it to $\mathbb{P}^1$ as follows. Given a point a, you map it to the line connecting O and a (O gets mapped to the tangent line). It is clear that this map $C \to \mathbb{P}^1$ is a double cover of $\mathbb{P}^1$. The branch points are exactly the solutions to $a+a=O'$, the 2-torsion points (of which there are 4). We claim that these branching points determine C uniquely (the choice of O is actually unimportant once you take the group actions into account). It turns out that a $PSL(3)$ orbit in $\mathcal{H}_{2,3}^{sm}$ corresponds exactly to a $PSL(2)$ orbit in $\mathcal{H}_{1,4}^{sm}$. The best way to do this is by a calculation.

First you have to check that orbits are sent to orbits. You check that when you change your O, the branch points are related by a projective transformation. This is again the result that a doubly periodic function with a double pole in the parallelogram is defined almost uniquely.

[[★★★ this is Gale duality]]

Question: where to read more? Answer: In [Muk03].

What are the degrees of g_2 and g_3? We didn't compute them. You pick some linear transformation with a given determinant and see what it does to g_2 and g_3. You use the transformation $x \mapsto x$, $y \mapsto t^{-1}y$, and $z \mapsto t^2 z$ (which has determinant t). Then you see that g_2 must have degree 4 and g_3 must have degree 6.

Proj quotients

Let $X \subseteq \mathbb{P}(V)$ be a projective variety. Then $X = \operatorname{Proj} R$ for some graded ring R. Suppose a reductive group G acts linearly on V and induces an action on X. If you take the $GL(V)$-invariants of the whole ring, you clearly don't get any invariants (because of scalar action). You may as well assume $G = G \cap SL(V)$ since we're interested in everything being projective. The the most natural quotient to consider is $X /\!\!/ G = \operatorname{Proj} R^G$.

In the affine case, we tried to find when invariants separate orbits. In the projective spectrum, we get all rational functions f/g where $f, g \in R^G$. The trouble comes up when all invariants are zero.

The plan is to define stable and semi-stable points and discuss the Hilbert-Mumford criterion.

14 Stability in the Projective case

We assume we have a faithful representation V of a reductive group $G \subseteq GL(V)$. Suppose $X \subseteq \mathbb{P}(V)$ is an invariant subvariety under the action of G. Then $X = \operatorname{Proj} R$ for some graded ring $R = \bigoplus R_n$. We define $X/\!\!/G = \operatorname{Proj} R^G$.

Definition 14.1. A point $x \in X$ is *semi-stable* if there exists an invariant homogeneous function $f \in R^G_{>0}$ such that $x \in X_f$ (where X_f is the open set where f does not vanish). We say x is *stable* if it is semi-stable and the action of G on $\mathbb{P}(V)_f$ is closed.

A point which is not semi-stable is called *unstable*, and the affine cone on X^{unst} is sometimes called the *nilpotent cone*. ◇

Remark 14.2. Note that the definition of stability depends on the choice of embedding of X into $\mathbb{P}(V)$. [[★★★ it has to be that way because there is no equivalence of categories between graded rings and projective varieties, so you can't define a canonical quotient like we did in the affine case. When you choose the graded ring that you're taking invariants of, you're effectively chooseing a line bundle with a linearization.]] Note that the notion of semi-stability does not depend on the embedding. ◇

Remark 14.3. Note that the invariant function f is constant along G-orbits (and hence is constant on closures of G-orbits). So as soon as a G-orbit is contained in X_f, the closure of that G-orbit is also contained in X_f (as soon as f is non-zero on the orbit, it is non-zero on the closure). ◇

Every point is actually a line in V. Let $X_a \subseteq V$ be the affine cone over X (without 0), and let $x_a = kv \smallsetminus \{0\}$ be the line corresponding to x (without 0). [[★★★ We don't use this next lemma ... I thought we would need it, so I wrote it up]]

Lemma 14.4. *Suppose $0 \notin \overline{G \cdot v}$, then the stabilizers G_x and G_v have the same dimension.*

Proof. It is clear that $G_v \subseteq G_x$. Given and element $g \in G_x$, we have that $g \cdot v = \lambda_g v$ for some $\lambda_g \in k^\times$. So we get an exact sequence

$$0 \to G_v \to G_x \xrightarrow{\lambda} \mathbb{A}^1$$

The kernel of λ is G_v by definition of G_v. The image of λ is essentially $G \cdot v \cap kv$, so since $0 \notin \overline{G \cdot v}$, we must have that the image of λ does not contain 0 in its closure. It follows that the image of λ (which is isomorphic to G_x/G_v) is finite, so zero-dimensional. □

Proposition 14.5. *Let $x_a = kv$. Then x is semi-stable if and only if $0 \notin \overline{G \cdot v}$, and x is stable if and only if $G \cdot v$ is closed and v is regular (i.e. v is stable with respect to the action of G on V).*

Proof. Suppose $0 \notin \overline{G \cdot v}$. By the Separation Lemma, there is some $f \in R^G$ such that $f(0) = 0$ and $f(v) = 1$. Then $f = f_1 + \cdots + f_n$ for homogeneous f_i (of positive degree). Since the action of G on V is linear, the action of G on R respects the grading, so since f is invariant, each of the f_i must be invariant. At least one of these f_i is non-zero on v, so X_{f_i} witnesses semi-stability of v.

On the other hand, if $0 \in \overline{G \cdot v}$, then for any homogeneous invariant function f of positive degree, we have $f(v) = f(0) = 0$, so x cannot be stable.

Now suppose x is stable, so there is an invariant homogeneous function f such that X_f is a neighborhood of x on which the action is closed. Since all the points $y \in X_f$ are stable (so semi-stable), we have that $0 \notin \overline{G \cdot v_y}$ for all y. By the lemma, we get that

Now suppose $G \cdot v$ is closed and v is regular. As before, we get a homogeneous invariant function f such that X_f

If $G \cdot v$ is closed, then $x \in X_f$ for some f. In X_f, we have a closed orbit, but X_f is already affine. Using the remark, we can check stability on X_f. We already proved this equivalence when we talked about the affine case. [[★★★ I don't see why v is regular. Where did we prove that if the action on an open neighborhood is closed, then the points in that neighborhood are regular? It must use that G is reductive.]] □

From the point of view of invariants, unstable points are invisible. If you want to construct $\operatorname{Proj} R^G$, you can take some homogeneous $f \in R^G_n$, then you have $R^G_f = \{h/f^n | \deg h = n \cdot \deg f\}$. We can make the quotient $\operatorname{Specm} R^G_f$, and glue all these affine pieces together. The problem is that unstable points are missing in this construction.

So we only get a rational map $X \dashrightarrow X/\!\!/G$, but we get an honest map $X^{ss} \to X/\!\!/G := X^{ss}/\!\!/G$, and this map is a surjective submersion (topology is induced on the target). In that sense, it is a categorical quotient. However, several orbits are still glued together.

If we consider $X^s \to X^s/\!\!/G$, then the preimage of every point is a single orbit; it is a geometric quotient.

Now let us consider the case where X is a quasi-projective variety. Suppose $X \subseteq \overline{X} \subseteq \mathbb{P}(V)$, with $\overline{X} = \operatorname{Proj} R$. In this case, we need to change the definitions of stable and semi-stable points. We say $x \in X$ is *semi-stable* if there exists an $f \in R^G_n$ such that $x \in X_f$ and X_f *is affine*. It is stable if furthermore the action on X_f is closed.

We can construct $X^{ss}/\!\!/G$ by gluing together the $X_f/\!\!/G$. In this situation, we always get a separable scheme [[★★★]]. In this way, I can recover the definition of the affine quotient.

Suppose we have an affine variety X, then we can find an equivariant closed immersion $X \subseteq V$. Let $W = V \oplus ku$, where the action of G on u is trivial, then $X \subseteq \overline{X} \subseteq \mathbb{P}(W)$. We see immediately from the definition that $X^{ss}/\!\!/G$ is the affine quotient.

Example 14.6. All this works for any reductive Lie group and reductive Lie algebra.

Consider that adjoint representation $\mathrm{Ad}\colon G = GL(n) \to \mathrm{Aut}(\mathfrak{gl}(n))$. So $GL(n)$ acts on the set of all matrices by conjugation. $\mathbb{P}(\mathfrak{g})/G$. We have $\det(A - \lambda I) = (-1)^n \lambda^n + \sigma_1(A)\lambda^{n-1} + \cdots + \sigma_n(A)$. The nilpotent cone is indeed the cone of all nilpotent elements (when all the $\sigma_i(A) = 0$).

What are the stable points. They are exactly the matrices with distinct eigenvalues. In other words, the discriminant of the characteristic polynomial should be non-zero.

So the projective quotient in this case is going to be $\operatorname{Proj} k[\sigma_1, \dots, \sigma_n]$. This is usually called weighted projective space. [[★★★ is this isomorphic to usual projective space?]] ◇

Example 14.7. $G = SL(2)$ and $V = V_4 = \{\xi_0 x^4 + 4\xi_1 x^3 y + 6\xi_2 x^2 y^2 + 4\xi_3 x y^3 + \xi_4 y^4\}$. In this case, we computed the invariants, f_2 and f_3, last time.

We already proved that the smooth points are stable, but perhaps we can have more stable points. We'd also like to compute the nilpotent cone.

Each element of V gives me four points in $\mathbb{P}^1$. If these are distinct, then we're in the smooth case. So suppose we're in the situation where two of the points coincide. Since the form must be invariant under $SL(2)$, we can assume the form is of the form $x^2(ax^2 + bxy + xy^2)$, so $\xi_3 = \xi_4 = 0$. The form is unstable if and only if $f_2 = f_3 = 0$, which only happens if $\xi_2 = 0$ (i.e. when $c = 0$). Thus, the unstable points are those which have one point with multiplicity three (i.e. either 3 and 1, or all 4 points together).

To see that the form $x^2(ax^2 + bxy + xy^2)$ is unstable, apply the element $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1}\end{smallmatrix}\right)$ to get $at^2x^4 + btx^3y + cx^2y^2$. As $t \to 0$, we get cx^2y^2 in the closure, which is where the points are broken up as 2 and 2. So in this case, stable is equivalent to smooth. But in general, this will not be the case. ◇

Remember that a point is *properly stable*, which means stable with finite stabilizer.

Suppose an algebraic torus T of dimension n acts on a vector space V of dimension N. Suppose further that there are properly stable points. The action of T can be diagonalized, so in some set of coordinates, $t \cdot (x_1, \dots, x_N) = (\chi_1(t)x_1, \dots, \chi_N(t)x_N)$ for some characters $\chi_i \in T^\vee$ with $\bigcap_i \ker \chi_i$ finite.

We have $\mathbb{Z}^n \cong T^\vee \subseteq \mathbb{Q}^n \subseteq \mathbb{R}^n$. Each torus has two lattices associated to it. One is the lattice of characters. The other lattice is the lattice of 1-parameter subgroups $P = \{\lambda\colon k^\times \to T\}$. By composition ($k^\times \xrightarrow{\lambda} T \xrightarrow{\chi} k^\times$) we get a natural pairing $P \times T^\vee \to \mathbb{Z}$, which is a non-degenerate pairing.

Consider our set of N characters as a set of vectors in $\mathbb{Z}^n \subseteq \mathbb{R}^n$. For $x \in V$, let the support (with respect to the given set of characters) be $\operatorname{Supp}(x) = \{\chi_i | x_i \neq 0\}$. Given $\mu_1, \dots, \mu_k \in \mathbb{R}^n$, we let $C(\mu_1, \dots, \mu_k) = \{\sum a_i\mu_i | \text{not all } a_i = 0 \text{ and } a_i \geq 0\}$.

Proposition 14.8. *Under all the assumptions we have in place. Given $x \in V$ with $\operatorname{Supp}(x) = \{\chi_1, \dots, \chi_k\}$. Then x is semi-stable if and only if $0 \in C(\chi_1, \dots, \chi_k)$. x is properly stable if*

1. $\dim\langle \chi_1, \ldots, \chi_n \rangle = n$,

2. 0 *is an interior point of* $C(\chi_1, \ldots, \chi_k)$.

If x is unstable, then 0 is not in C, so all the χ_i must be in some open half-space. But if x is semi-stable, then the χ_i must all be in some closed subspace. If x is stable, then there is no closed half-space which contains all the χ_i.

Next time, we'll prove the proposition. Then we'll start proving the Hilbert-Mumford criterion for stability.

15 The Hilbert-Mumford numerical criterion

Remark 15.1. The quotient we construct *depends* on the embedding $X \to \mathbb{P}(V)$ because the notion of stability depends on the embedding. $\diamond$

Proposition 15.2. *Let T be a torus of dimension n acting on $\mathbb{A}^N$, with the stabilizer of a generic point being finite. Let the action be given by $t \cdot (x_1, \ldots, x_N) = (\chi_1(t)x_1, \ldots, \chi_N(t)x_N)$. We regard the χ_i as vectors in $T^\vee \cong \mathbb{Z}^n \subseteq \mathbb{Q}^n \subseteq \mathbb{R}^n$. We defined* $\mathrm{Supp}(x) = \{\chi_i | x_i \neq 0\} = \{\chi_1, \ldots, \chi_k\}$

x is semi-stable if $0 \in C(\mathrm{Supp}(x))$ and it is stable if 0 is an interior point.

Proof. Suppose $0 \notin C(\chi_1, \ldots, \chi_k)$, then there exists a $\lambda \in (\mathbb{R}^N)^*$ such that $\langle \lambda, \chi_i \rangle > 0$ for $1 \le i \le k$. In fact, since there are finitely many χ_i, we may assume $\lambda \in (\mathbb{Q}^N)^*$, so after rescaling, we may assume $\lambda \in (\mathbb{Z}^N)^* = (T^\vee)^*$. So λ defines a 1-parameter subgroup $\lambda \colon k^\times \to T$.

Now we consider $\lim_{t \to 0} \lambda(t)x$. By the condition that λ is positive on the χ_i, we have that $\lambda(t)x = (t^{a_1}x_1, \ldots, t^{a_N}x_N)$ for strictly positive a_i [[★★★ at least for those i for which $x_i \neq 0$]]. So the limit is zero, so $0 \in \overline{T \cdot x}$, so x is unstable.

If $0 \in C(\chi_1, \ldots, \chi_k)$, then we can choose $a_1, \ldots, a_k \in \mathbb{R}_{\ge 0}$ not all zero such that $\sum a_i \chi_i = 0$. Consider the exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow K & \longrightarrow & \mathbb{Q}^k & \longrightarrow & T^\vee \otimes \mathbb{Q} \\ & & (a_1, \ldots, a_k) & \longmapsto & \sum a_i \chi_i \end{array}$$

Since $\mathbb{R}$ is flat over $\mathbb{Q}$, we get an exact sequence

$$0 \to K \otimes \mathbb{R} \to \mathbb{R}^k \to T^\vee \otimes \mathbb{R}.$$

By assumption, we have an element of $K \otimes \mathbb{R}$ for which all the a_i are non-negative and not all zero. Since K is dense in $K \otimes \mathbb{R}$, we can find *rational* non-negative a_i (not all zero) such that $\sum a_i \chi_i = 0$. After scaling, we may assume they are integers. Then the function $f(x) = x_1^{a_1} \cdots x_k^{a_k}$ is T-invariant and non-zero at x (by the definition of $\mathrm{Supp}(x)$). So x is semi-stable.

Note that the dimension of the span of $\{\chi_1, \ldots, \chi_k\}$ is n if and only if the stabilizer of x is finite. This is because a 1-parameter subgroup λ is in the stabilizer of x if and only if $\{\chi_1, \ldots, \chi_k\} \subseteq \lambda^\perp$, and a subgroup of the torus is finite if and only if it contains no non-trivial 1-parameter subgroups.

If 0 is in the interior of $C(\chi_1, \ldots, \chi_n)$, then the dimension of the span of the χ_i must be n; otherwise, $C(\chi_1, \ldots, \chi_k)$ would have no interior. So the stabilizer of x must be finite. We can choose the a_i to be *strictly* positive integers. Then $V_f = \{x | \text{all } x_i \neq 0\}$. Then every point in V_f has the same support as x, so every point in V_f has finite stabilizer, so all points are regular, so the action is closed on V_f. $\square$

This result allows you to draw pictures to find out which points are stable and which are not.

We have the following corollary, which says that stability can be checked on 1-parameter subgroups. If x is unstable, then we may find the 1-parameter subgroup $\lambda\colon k^\times \to G$ which demonstrates instability. On the other hand, in the other cases (when x is (semi)-stable) I can't find such a 1-parameter subgroup.

Corollary 15.3. *In the case $G = T$, proper stability can be checked on 1-parameter subgroups.*

Theorem 15.4 (Hilbert-Mumford numerical criterion)**.** *If G is reductive, then*

1. *x is semi-stable if and only if it is semi-stable for any 1-parameter subgroup, and*
2. *x is properly stable if and only if it is properly stable for any 1-parameter subgroup.*

Semi-stable for a 1-parameter subgroup means that 0 is not in the closure. Stable means that the orbit is closed and the stabilizer is finite.

For any reductive group, you get a maximal torus. All maximal tori are conjugate, and any 1-parameter subgroup lies in a maximal torus. Checking stability for all 1-parameter subgroups is annoying, but checking it for maximal tori is easier.

I used [MFK94], and there is one statement that isn't completely clear to me. Other people do it differently.

Lemma 15.5. *For $x \in V$, suppose $y \in \overline{G\cdot x} \smallsetminus G\cdot x$. Let $O = k[\![t]\!]$ and $K = k(\!(t)\!)$, with $\mathfrak{m} = (t) \subseteq O$. Then there exists $\gamma \in G(K)$ such that $\gamma\cdot x \in O \otimes V$ and $\gamma \cdot x \equiv y$ modulo $\mathfrak{m}$.*

$\gamma(t) \in G$ is a laurent series. When I apply it to x, I get a formal power series $\gamma(t)\cdot x \in k[\![t]\!]V$, and $\lim_{t\to 0}\gamma(t)x = y$.

Proof. Let $Z = \overline{G\cdot x}$. Consider $\phi\colon G \to Z$ given by $g \mapsto g\cdot x$. This is a dominant open immersion (recall that the orbit is open in its closure [[★★★ ref]]). I claim that there is a curve $C \subseteq Z$ such that $y \in C$ and $C \cap G\cdot x \neq \varnothing$ [[★★★ this is like checking valuative criteria on an open set]]. Suppose $Z \smallsetminus G\cdot x \subseteq \{h(x) = 0\}$. If $\dim \mathfrak{m}_y/\mathfrak{m}_y^2 > 1$, then there is an $f \in \mathfrak{m}_y$ such that $f \not\equiv h$ modulo $\mathfrak{m}_y^2$. Take $\{f(x) = 0\}$. Keep decreasing the dimension to get a curve.

We may take some curve C_1 whose image is in C, so we have

$$\begin{tikzcd} s \in \overline{C}_1 \arrow[d, "\phi"'] & C_1 \arrow[l] \arrow[r, hook] \arrow[d, "\phi|_{C_1}"'] & G \arrow[d, "\phi"] \\ y \in \overline{C} & C \arrow[l] \arrow[r, hook] & Z \end{tikzcd}$$

The map $\phi|_{C_1}$ is not surjective, but it is dominant. We can take the projective completions of the affine curves C and C_1 and extend the map. Let $s \in \overline{C}_1$ be a point that maps to y. We may assume $\overline{C}_1$ is non-singular at s by taking normalization if needed. Let U be an open neighborhood of s. We have $\gamma\colon U \smallsetminus s \to G \xrightarrow{\phi} C$. γ has a pole at s. $\phi(\gamma(U)) \subseteq C$, $\phi(s) = y$. [[★★★ once you go from G to C, the pole goes away]]

Let O_s be the local ring of $s \in \overline{C}_1$, and K_s the field of fractions. We've constructed $\gamma \in G(K_s)$ and the condition at the end of the previous paragraph means that $\gamma \cdot x \in O_s \otimes_k V$ and $\gamma \cdot x \equiv y$ modulo $\mathfrak{m}_s$.

Now we just complete with respect to the maximal ideal and since s was a smooth point, you get $k[\![t]\!]$. □

The next result I'll only prove for $SL(n)$, because it involves some structure theory of semi-simple groups.

Theorem 15.6 (Iwahori)**.** *Let G be a reductive algebraic group. Then each double coset in $G(O)\backslash G(K)/G(O)$ contains a (unique![[★★★ probably]]) 1-parameter subgroup $\lambda(t)$.*

Consider the case of $SL(n)$. Start with a matrix $X = (x_{ij}) \in G(K)$, so each x_{ij} is a Laurent series. We have a valuation (the smallest power of t that appears; the order of the zero/pole). Choose $v(x_{ij})$ minimal. By multiplying on the left and right by permutation matrices, we may assume $(i, j) = (1, 1)$, so $v(x_{11})$ is minimal. Multiplying X on the left by some matrix, we can do Gaussian elimantion (we can divide because we chose $v(x_{11})$ minimal) to get the first column to be all zeros below x_{11}. Similarly, we can get all zeros to the right of x_{11}. Now repeat on the smaller matrix until we get a diagonal matrix by induction. So we may assume we have the matrix $diag(z_{11}, \dots, z_{nn})$. Since we are in $SL(n)$, the product of the z_{ii} is 1. Each z_{ii} is of the form $t^{a_i} \cdot f_i$ where the $f_i \in k[\![t]\!] = O$ are invertible. So we have $\sum a_i = 0$ and $\prod f_i = 1$. So we multiply our matrix on the left by $diag(f_1^{-1}, \dots, f_n^{-1})$ to get our 1-parameter subgroup.

16 Lecture 16

Recall the setup: We have $K = k((t)) \supseteq O = k[\![t]\!] \supseteq \mathfrak{m} = (t)$. We proved that

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] & G \arrow[d, "\phi"] \\ & \overline{G \cdot x} \ni y \end{tikzcd}$$

if $\phi(g) = g \cdot x$, there is a point $g \in G(K)$ such that $\lim_{t\to 0} g(t)x = y$ and such that $\phi(g) \in O \otimes_k V$.

We proved Iwahori's theorem (kinda): If G is reductive, and $g \in G(K)$ is a point, then $g(t) = A(t)\lambda(t)B(t)$ where $A, B \in G(O)$ and λ is a 1-parameter subgroup. The action of the 1-torus λ is diagonalizable, so you can think of the action of λ as $\lambda(t) = diag(t^{a_1}, \dots, t^{a_n})$.

Lemma 16.1. *Suppose $\lambda(t)$ is a 1-parameter subgroup of G, $x \in X$, and $B \in G(O)$ such that the limit $\lim_{t\to 0} \lambda(t)B(t)x = y$ exists. Then the limit $\lim_{t\to 0} \lambda(t)B(0)x = z$ exists and $z \in \overline{G \cdot y}$.*

Proof. □

Theorem 16.2. *x is semi-stable (resp. properly stable) with respect to the action of G if and only if it is semi-stable (resp. properly stable) for any 1-parameter subgroup λ of G.*

Proof. It is clear that if the point is semi-stable with respect to G, then it is semi-stable with respect to a subgroup.

Now suppose x is semi-stable for any 1-parameter subgroup λ, but x is not stable with respect to G. Since x is unstable, $0 \in \overline{G \cdot x}$. So there is a point $g \in G(K)$ such that $\lim_{t\to 0} g(t)x = 0$. By Iwahori, we have $g(t) = A(t)\lambda(t)B(t)$ for some 1-parameter subgroup λ and $A, B \in G(O)$. There is a natural homomorphism $G(O) \to G(k)$ given by evaluating at $t = 0$. We may write $A(t) = A_1(t)A_0$ and $B = B_1(t)B_0$, where A_0 and B_0 are constant and A_1 and B_1 are 1 modulo t. Suppose $\lambda(t) = diag(t^{b_1}, \dots, t^{b_n})$ in some basis.

$$\begin{aligned} 0 &= \lim_{t\to 0} A(t)\lambda(t)B(t)x \\ 0 &= \lim_{t\to 0} \lambda(t)B_1(t)B_0x && \text{(left multiply by } A(t)^{-1}) \end{aligned}$$

If all $b_i > 0$, this is no condition. If $b_i \leq 0$, then we get that $(B_0x)_i = 0$. So we conclude that

$$\lim_{t\to 0} \lambda(t)B_0x = 0.$$

[[★★★ we can ignore $B_1(t)$ because it goes to 1 as $t \to 0$. This implication only works in one direction: if $\lim_{t\to 0} \lambda(t)B_1(t)B_0x = 0$, then $\lim_{t\to 0} \lambda(t)B_0x = 0$. The other way doesn't work]] But $B_0x \in G \cdot x$, so B_0x is a point in the orbit of x which is not stable. That means that x is unstable with respect to the 1-parameter subgroup $B_0^{-1}\lambda(t)B_0$.

Assume that x is not properly stable with respect to G. We have to consider two cases.

Case (a). $G \cdot x \neq \overline{G \cdot x}$.[[★★★ Remark: we actually show that the orbit of x is closed if and only if it is closed under the action of all 1-parameter subgroups]] Let $y \in \overline{G \cdot x} \smallsetminus G \cdot x$. Then we get $y = \lim_{t\to 0} A_1(t)A_0\lambda(t)B(t)B_0x$, so we have

$$\lim_{t\to 0} \lambda(t)B_1(t)B_0x = A_0^{-1}y.$$

By the same sort of condition as in the first part of the proof, we have a limit $z = \lim_{t\to 0} \lambda(t)B_0x$. $A_0^{-1}y$ may have more zero coordinates than z. If $b_i > 0$, then we get $(A_0^{-1}y)_i = 0$. If $b_i = 0$, we get $(A_0^{-1}y)_i = (B_0x)_i$, and if $b_i < 0$, we get $(B_0x)_i = 0$. It is easy to check that $\lim_{t\to 0} \lambda(t^{-1})(A_0^{-1}y) = z$ componentwise. So we have that $z \in \overline{G \cdot y}$. This implies that $z \notin G \cdot x$, showing that the orbit of x under the action of a 1-parameter subgroup is not closed.

Case (b). $G \cdot x = \overline{G \cdot x}$, but $Stab(x)$ is not finite. The stabilizer is the fiber of the map $G \to G \cdot x$. If the fiber is not finite, then it is affine. By the same argument as before, there is a point $g(t) \in G(K) \smallsetminus G(O)$ such that $g(t)x = x$ (because the stabilizer is not proper). Letting $g(t) = A(t)\lambda(t)B(t)$, we compute $\lim_{t\to 0} \lambda(t)B_1(t)B_0x = A_0^{-1}x$. By the same argument as before, we have that the limit $\lim_{t\to 0} \lambda(t)B_0x = z$ exists. The implies that $\lambda(t) \in Stab(z)$. z lies in the closure of the orbit, but the orbit is closed, and is stabilized by λ (looking at coordinates of z ... the only non-zero ones are those for which $b_i = 0$), so z is not properly stable with respect to a 1-parameter subgroup, so niether is x. □

Applications

In the case of $SL(n)$. Any 1-parameter subgroup is diagonalizable. In that basis, the diagonal matrices form a maximal torus. All maximal tori are conjugate. So we can check stability with respect to G by checking it for all maximal tori using the condition we proved before.

First let's consider the case of $\mathcal{H}_{n,d}$.

Proposition 16.3. *$f \in \mathbb{V}_{1,d}$ ($d \geq 2$) is semi-stable if the multiplicity of each point is $\leq d/2$, and f is properly stable is the multiplicity of each point is $< d/2$.*

Proof. In this case, $G = SL(2)$. Any 1-parameter subgroup in some basis is $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$. I'll draw the picture of characters in the case $d = 6$. [[★★★ draw $x^{6-i}y^i$ at position $2i - 6$; the usual 7-dimensional representation of $SL(2)$]]

If $f = \sum a_i x^{d-i}y^i$, then $\mathrm{Supp}(f) = \{x^{d-i}y^i | a_i \neq 0\}$. The unstable situation is when $x^4|f$ or $y^4|f$ (that's when the convex hull of the support misses 0). Similarly, the semi-stable situation is where $x^3|f$ or $y^3|f$. □

Now let's consider plane cubics. $\mathbb{V}_{2,3}$, which is 10-dimensional. Here we have $G = SL(3)$. The maximal torus is diagonal matrices $diag(t_1, t_2, t_3)$ such that $t_1 t_2 t_3 = 1$. Now it is convinient to draw the torus as 2-dimensional, with three weights that add up to zero, induced from the 3-dimensional torus. [[★★★ triangle with 10 monomials of degree 3, x^3, y^3, z^3 are the vertices. two monomials in the interior of each edge]]. The zero is the monomial xyz.

Unstable forms (so-called nil forms) are all basically the same, the forms supported at $y^3, xy^2, x^2y, x^3, y^2z$ (up to renaming the variables). Then the form can be written as

$$zy^2 = ay^3 + bxy^2 + cx^2y + dx^3.$$

If $d \neq 0$, we can apply the transformation $x \mapsto x + \alpha y$ to get

$$zy^2 = ay^3 + bxy^2 + dx^3$$

applying $z \mapsto z + ay + bx$ and rescaling, we get that the curve is of the form

$$zy^2 = x^3.$$

So the nil curves are exactly the cuspidal cuves.

But there are degenerations. If $d = 0$, then we have

$$zy^2 = ay^3 + bxy^2 + cx^2y$$

this is a line tangent to a quadratic curve.

We can have further degeneration (the quadric can degenerate). Then you get three lines meeting at a point, and the lines could lie on top of each other.

We already proved in general that smooth curves are stable.

Now let's consider the semi-stable forms. The maximal support is $y^3, xy^2, x^2y, x^3, y^2z, zx^2, xyz$ (up to permuting the variables). This corresponds to the form

$$zp(x, y) = q(x, y)$$

where p is a quadratic form. After some change of coordinates, we can get rid of the coefficient of y^2z, so we get that p is non-degenerate. It can be written as

$$z(x^2 + y^2) = q(x, y)$$

and after $z \mapsto z + \alpha x + \beta y$ and some more dancing, we can get

$$zy^2 = x^3 - zx^2$$

A nodal cubic.

There is one more situation (which is in the closure of this one), where the support is $y^3, y^2x, y^2z, x^2y, xyz, yz^2$. This must be of the form $yp(x, y, z) = 0$ which is a line and a quadric (not tangent). It can further degenerate to three lines, not all meeting at a point (corresponding to $xyz = 0$ in some basis). I think we've now listed all the orbits of the action. Question: if all these are semi-stable, why do only some of them appear in the quotient? Answer: only the most general one appears ... the quotient identifies the things in the closures of the semi-stable orbits.

We have the $SL(3)$-invariants g_2 (of degree 4) and g_3 (of degree 6). We get the quotient $\operatorname{Proj} k[g_2, g_3]$, which I believe is isomophic to $\mathbb{P}^1$, as we can see from the charts g_2^3/g_3^2 and g_3^2/g_2^3.

17 Stability of Hypersurfaces

Even though we can't find invariants, we can now tell which hypersurfaces are stable with the numerical criterion.

Recall that we have $\mathbb{V}_{n,d} = \mathrm{Sym}^d(k^{n+1})$, the space of forms of degree d in $n+1$ variables. We have an action of $SL(n+1)$ on $\mathbb{P}(\mathbb{V}_{n,d}) = \mathcal{H}_{n,d}$, the space of hypersurfaces of degree d in $\mathbb{P}^n$. If we choose homogeneous coordinates $(x_0, \dots, x_n)$, then any f is some degree d form in the x_i. We can define $\mathrm{Supp}(f) = \{(i_0, \dots, i_n) \in \mathbb{Z}^{n+1}_{\geq 0} | a_{i_0,\dots,i_n} \neq 0\}$, the *Newton polytope* of f.

In this case, the support is a subset of $\Delta_d = \{a = (a_0, \dots, a_n) \in \mathbb{Z}^{n+1}_{\geq 0} | \sum a_i = d\}$.

We've shown that f is unstable if and only if for some choice of coordinates, the convex hull $C(\mathrm{Supp}(f))$ does not contain zero. However, it get's more and more complicated to draw pictures, so another condition is desirable. Let $T = \{(t_0, \dots, t_n) | t_0 \cdots t_n = 1\}$ be the torus of $SL(n+1)$. We have the set of 1-parameter subgroups $\Lambda = \{\lambda = (\lambda_0, \dots, \lambda_n) \in \mathbb{Z}^{n+1} | \sum \lambda_i = 0\}$. Then f is unstable if and only if for some choice of coordinates, there exists $\lambda \in \Lambda$ such that $\langle \lambda, a \rangle > 0$ for any $a \in \mathrm{Supp}(f)$. Similarly, f is semi-stable if and only if for any choice of coordinates and for any $\lambda \in \Lambda$, there exists $a \in \mathrm{Supp}(f)$ such that $\langle \lambda, a \rangle \leq 0$. f is stable if and only if for any choice of coordinates and for any $\lambda \in \Lambda$, there exists $a \in \mathrm{Supp}(f)$ such that $\langle \lambda, a \rangle < 0$. [[★★★ stability is the same as proper stability here]]

The main idea is that stability is a geometric property. Everything is described by the badness of the singularity on your hypersurface. We'll consider the case of cubic surfaces.

For a projective hypersurface cut out by f, let P be a point (in affine coordinates $x_1, \dots, x_n$). We say that P is singular if $f(P) = 0$ and all the first derivatives of f vanish at P. Then we can write $f = p(x_1, \dots, x_n) + \cdots$, where p is a quadratic form.

Definition 17.1. P is an *ordinary double point* if p is non-degenerate quadratic form. Multiplicity k means that all the (up to) k-th partial derivatives of f vanish at P. $\diamond$

In the case of a cubic surface with homogeneous coordinates x, y, z, w, let's assume $(0, 0, 0, 1)$ is a singular point. Then we have

$$f(x, y, z, w) = wp(x, y, z) + q(x, y, z)$$

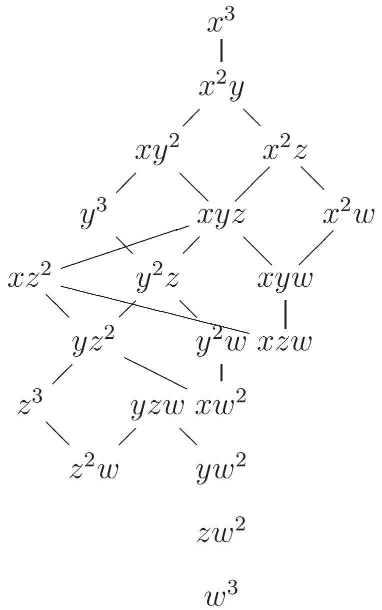
where p is quadratic and q is cubic. If we have a double point, then we could have $rk(p) = 3$ (ordinary double point), $rk(p) = 2$ (we have $f = wxy + q(x, y, z)$, so p looks like the intersection of two planes; the line $x = y = 0$ is called the *axis* of the singularity), or $rk(p) = 1$ (we have $f = wx^2 + q(x, y, z)$), or $rk(p) = 0$ (triple point, in which case f is independent of w, so you just have a product of a line and an elliptic curve)

Theorem 17.2. *A cubic surface X in $\mathbb{P}^3$ is stable if and only if X has only finitely many singularities, all of which are ordinary double points. X is semi-stable if and* *only if it has finitely many singularities, all of which are either ordinary double points or rank 2 double points with axis not contained in X.[[★★★ like $wxy + z^3$]]*

Proof. In this case, our Δ_3 is a tetrahedron, like this [[★★★ picture with vertices x^3, y^3, z^3, w^3, two more on each edge and one more on each face]]. We have already changed coordinates so that we get no w^3 or w^2 terms. If a surface has a double point of rank $2k < 3$, then it is clearly unstable because you only have one element of the support at "height 1" [[★★★ height being the number of w terms]].

If X has a double point of rank 2 and is semi-stable, then z^3 must be in the support of the form, so we get that the axis cannot be in X.

For the other direction, we draw the picture. Let $\Lambda^+ = \{(\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n)\}$, which we can always get to by permuting the coordinates. This Λ^+ is a convex cone in the lattice. We introduce a partial order on Δ_d by setting $a \leq b$ if $\langle \lambda, a \rangle \leq \langle \lambda, b \rangle$ for any $\lambda \in \Lambda^+$. [[★★★ incomplete picture]]



Suppose X is not stable, then we will try to find a point which is not an ordinary double point. Since X is unstable, there is some choice of coordinates and some λ such that $\langle \lambda, a \rangle < 0$ for all $a \in \operatorname{Supp}(f)$. We may reorder coordinates so that $\lambda \in \Lambda^+$. Since $\lambda_0 + \cdots + \lambda_3 = 0$, we get that the monomials $w^3, zw^2, yw^2, z^2w, yzw$ do not appear in the support of f.

First case: Suppose $z^3 \in \operatorname{Supp} f$. Then xyw is not in the support because $z + x + y + w = 0$ [[★★★]]. Then everything below this point is also not in the support. Then the only place w can appear is in x^2w, so I get that $(0,0,0,1)$ is a double point of rank 1.

Next case: suppose $xzw \in \operatorname{Supp} f$, then $y^3 \notin \operatorname{Supp} f$, so nothing below it is in the support, so everything is divisible by x, so the surface is reducible, so it has many singular points.

Final case: suppose $z^3, xzw \notin \operatorname{Supp} f$, then nothing below them is in the support either. Then w appears in three places, but there is no term with a z, so $(0,0,0,1)$ is a double point of rank 2.

Now suppose X is semi-stable. It might happen that $\langle \lambda, a \rangle \geq 0$ for all $a \in \operatorname{Supp} f$, so we have to consider some more cases. Now λ could be zero on both xzw and y^3 for example. These cases are boundary cases. We only need to consider the cases where $\lambda_1 = 0$ or $\lambda_2 = 0$. It's a little bit more work, and there is one tricky point that I haven't quite figured out.

Suppose $\lambda_1 > 0$ and $\lambda_2 = 0$. Then $xzw \notin \operatorname{Supp} f$. In this case, we remove xzw and everything below it, which leaves me with a rank 2 singularity at $(0,0,0,1)$.

Another case: $\lambda_1 = \lambda_2 = 0$. Then $xw^2, y^2w \notin \operatorname{Supp} f$. Again, once I remove these two points and the things below them, I'm left with a singularity of the form [[★★★]] which is again a double point of rank 2

Finally, there is a special case: $\lambda = (2, 0, -1, -1)$. In this case, you remove xw^2, and the maximal support gives you f of the form $ay^3 + xp(x, y, z, w)$. In this case, there is a singularity which is not an ordinary double point. Take $x = y = 0$. On this line, I have a solution to $p(0, 0, z, w) = 0$. There are two solutions, which give you singular points with sub-maximal rank. □

18 Points in $\mathbb{P}^n$. Linearization

The plan is to do one more example about points and then talk about linearization.

We already discussed the case of points on a line. Now we'll talk about d unordered points in $\mathbb{P}^n$. This is the same as hyperplanes in the dual space. So we can consider the closed set of completely decomposible forms of degree d in $n+1$ variables. That is, $f(x) = f_1(x)\cdots f_d(x)$ where each f_i is of degree 1.

Proposition 18.1. *A set $S \subseteq \mathbb{P}^n$ with $|S| = d$ is stable (resp. semi-stable) with respect to the action of $SL(n+1)$ if and only if for any linear projective subspace Z, we have*

$$|S \cap Z|/d < (\dim Z + 1)/(n+1) \text{ (resp. } |S \cap Z|/d \leq (\dim Z + 1)/(n+1)). \qquad (*)$$

Example 18.2. For 6 points in $\mathbb{P}^2$, you have a stable set only if no two points coincide. Otherwise, we get $2/6 = 1/3$ (taking Z to be a point), so we don't get strict inequality. Moreover, in a stable configuration no 4 points can be colinear because then we get $4/6 = 2/3$ (here Z is a line).

For semi-stability, no 3 points can coincide and no 5 points can be on a line. ◇

Proof. Choosing a coordinate basis, we define $\operatorname{Supp} f$ to be those monomials which appear in f. f is unstable if for some choice of coordinates and for some $\lambda \in \Lambda^+$, we get $\langle \lambda, a\rangle > 0$ for all $a \in \operatorname{Supp} f$. f is stable (semi-stable) if for any choice of coordinates and any $\lambda \in \Lambda^+$, there is some $a \in \operatorname{Supp} f$ such that $\langle \lambda, a\rangle < 0$ ($\langle \lambda, a\rangle \leq 0$).

We define a partial order on monomials, given by $a \leq b$ if $\langle \lambda, a\rangle \leq \langle \lambda, b\rangle$ for all $\lambda \in \Lambda^+$.

Claim. *If f is completely reducible, then* $\operatorname{Supp} f$ *has a smallest element with respect to this partial order.*

Proof. We have $f(x) = f_1(x)\cdots f_d(x)$, with $f_i = \sum_j a^i_j x_j$. Let x_{j_i} be the smallest element in $\operatorname{Supp} f_i$, then the product of these is the smallest element in $\operatorname{Supp} f$. $\square_{\text{Claim}}$

Now we pick a set of fundamental weights of Λ^+ (the terminology comes from representation theory). Let $\omega_i = (n+1-i, \ldots, n+1-i, -i, \ldots, -i)$ (where there are i copies of $n+1-i$ and $n+1-i$ copies of $-i$). The ω_i generate Λ^+ in the sense that any $\lambda \in \Lambda^+$ is a positive linear combination of the ω_i. It is clear that it is enough to check the stability (or semi-stability) condition ($\langle \lambda, a\rangle < 0$ or $\langle \lambda, a\rangle \leq 0$) on the fundamental weights.

Suppose there is a Z such that $(*)$ fails. Z is given by the equation $x_{p+1} = \cdots = x_n = 0$ in some coordinate system ($\dim Z = p+1$). Let $k = |S \cap Z|$. If $a \in \operatorname{Supp} f$, then $a_0 + \cdots + a_p \geq k$ and $a_{p+1} + \cdots + a_n \leq d - k$. We compute

$$\langle \omega_{p+1}, a\rangle \geq (n-p)k - (d-k)(p+1) = (n+1)k - d(p+1) > 0$$

where $\operatorname{Supp} f \subseteq \Delta_d = \{(a_0, \ldots, a_n) | \sum a_i = d\}$. So S is unstable.

Suppose $(*)$ holds for all Z. We may assume Z is cut out by $x_{p+1} = \cdots = x_n = 0$. Let $k = |S \cap Z|$. We have $k/d \leq (\dim Z + 1)/(n+1)$. Then we'd like to show that $\min_{a \in \operatorname{Supp} f}\{\langle \omega_{p+1}, a\rangle\} \leq 0$. I can find a point $a \in \operatorname{Supp} f$ such that $a_0 + \cdots + a_p = k$ and $a_{p+1} + \cdots + a_n = d - k$. This check is for all p [[★★★]], so the point is semi-stable (or stable) □

The quotient comes from the embedding $X \hookrightarrow \mathbb{P}(V)$, where G acts linearly on V. One question is why we can do this in all cases. The other question is how the quotient depends on the embedding.

Recall that if X is affine, then we proved (very easily) that there is a closed immersion $X \hookrightarrow V$ and a linear action on V such that the embedding is equivariant. Now we drop the assumption that X is an affine scheme.

Suppose X is an algebraic variety. To embed it into projective space, we start with a line bundle L. Suppose $W \subseteq \Gamma(X, L)$ is a linear subspace which is base-point free (i.e. there is no $x \in X$ such that all $s \in W$ vanish at x). Then we get an induced map $X \to \mathbb{P}(W^*)$, given by sending x to the hyperplane in W of sections which are zero at x. If X is a projective variety, then you usually take $W = \Gamma(X, L)$. If the map is a closed embedding, then L is called *very ample.*

Let $\operatorname{Pic} X = \{\text{line bundles on } X\}/\cong$, with the group structure given by $\otimes$ and $L^{-1} = L^*$. A line bundle is sometimes also called an *invertible sheaf.*

Suppose $\sigma\colon G \times X \to X$ is an action. A *linearization* of L is an action $\overline{\sigma}\colon G \times L \to L$ so that the following square commutes and the action is linear on fibers.

$$\begin{tikzcd} G \times L \arrow[r, "\bar\sigma"] \arrow[d, "\mathrm{id}\times\pi"'] & L \arrow[d, "\pi"] \\ G \times X \arrow[r, "\sigma"] & X \end{tikzcd}$$

Given a linearization, you can twist it by a character of the group. Suppose $\mathcal{L}$ is the sheaf of sections of L. G acts linearly on $\mathcal{L}$ (a sheaf of $\mathcal{O}_X$-modules) and it acts on $\mathcal{O}_X$. We must have $g^* f(x) = f(gx)$ and $g \cdot gs = g^*(f) g \cdot s$.

Example 18.3. $PGL(n+1)$ acts on $\mathbb{P}^n$, but the line bundle $\mathcal{L} = \mathcal{O}(1)$ is not linearizable. The sections of $\mathcal{O}(1)$ would have to be an $(n+1)$-dimensional representation. If we had the group $SL(n+1)$, we could act on the representation. Another thing we could do it take a big tensor power of the sheaf. The tangent bundle has an action of $PGL(n+1)$. If I take the top exterior power of the tangent bundle, I have $\bigwedge^{\text{top}} T(\mathbb{P}^n) \cong \mathcal{O}(n+1)$. ◇

We define $\operatorname{Pic}^G X$ to be the group of line bundles with G-linearization (a vector bundle with G-linearization is sometimes called a G-bundle). We have a homomorphism $\alpha\colon \operatorname{Pic}^G X \to \operatorname{Pic} X$. The kernel tells us how many linearizations there are on a given line bundle.

To compute the kernel of α, we just need to find all linearizations on $\mathcal{O}_X$. We have the standard linearization. Any other action must be given by $g \cdot f(x) = \Phi(g, x) g^* f(x)$

where $\Phi(g,x) \in \mathcal{O}(G \times X)^\times$. We must have $\Phi(e,x) = 1$. We also get the condition

$$\Phi(gh,x)f(ghx) = (gh){\cdot}f(x) = h{\cdot}(g{\cdot}f)(x) = \Phi(h,x)g{\cdot}f(hx) = \Phi(h,x)\Phi(g,hx)f(ghx)$$

[[★★★ is the action of G on $\Gamma(X, \mathcal{O}_X)$ a left or a right action? I think it's a left action, in which case we should have $g^*f(x) = f(g^{-1}x)$]]which tells us that

$$\Phi(gh,x) = \Phi(h,x)\Phi(g,hx).$$

Moreover, two functions Φ and Φ' give isomorphic linearizations if there is some $\phi \in \mathcal{O}(X)^\times$ such that

$$\phi(x)\Phi(g,x) = \Phi'(g,x)\phi(gx).$$

So $\Phi(g,x) = \frac{\phi(gx)}{\phi(x)}\Phi'(g,x)$. We define $Z^1(G, \mathcal{O}(X)^\times) = \{\Phi \in \mathcal{O}(G \times X)^\times | \Phi$ satisfies the cocycle condition$\}$ and $B^1(G, \mathcal{O}(X)^\times) = \{\Phi | \Phi(g,x) = \frac{\phi(gx)}{\phi(x)}$ for some $\phi \in \mathcal{O}(X)^\times\}$. Then $\ker\alpha$ is given by $H^1 = Z^1/B^1$.

19 Lecture 19

We started talking about linearization last time, and there was a little discussion. If the zero section of L is G-invariant, then the action is linear. If you have a map $\mathbb{A}^1 \to \mathbb{A}^1$ which is an isomorphism sending zero to zero, then it must be linear.

On the elements of L, it will be a left action, but on sections, it's a right action.

Last time, we discussed the homomorphism $\alpha\colon \operatorname{Pic}^G X \to \operatorname{Pic} X$. We described the kernel of α as $H^1 = Z^1(G, \mathcal{O}(X)^\times)/B^1(G, \mathcal{O}(X)^\times)$, where $Z^1(G, \mathcal{O}(X)^\times) = \{\Phi \in \mathcal{O}(G \times X)^\times | \Phi \text{ satisfies the cocycle condition}\}$ and $B^1(G, \mathcal{O}(X)^\times) = \{\Phi | \Phi(g, x) = \frac{\phi(gx)}{\phi(x)}$ for some $\phi \in \mathcal{O}(X)^\times\}$.

Theorem 19.1 (Rosenlicht)**.** *Suppose X and Y are irreducible varieties. Then the natural map $\mathcal{O}(X)^\times \times \mathcal{O}(Y)^\times \to \mathcal{O}(X \times Y)^\times$ is surjective. Namely, $f(x, y) \in \mathcal{O}(X \times Y)^\times$ can always be written as $f(x, y) = \gamma(x)\beta(y)$.*

Proof. This is a local statement [[★★★ γ and β are unique upto scalar, so we can glue]], so we may assume that X and Y are affine. We choose an embedding $X \hookrightarrow \overline{X}$ into a proper normal variety (normalizing if needed) [[★★★ do the reduction to the normal case]]. For a given $y \in Y$, consider $\phi_y(x) = f(x, y)$ on $\overline{X}$, which is invertible on X, but it could give a divisor on $\overline{X}$. Since $\overline{X} \smallsetminus X = \bigcup Z_i$ divisors Z_i [[★★★ and some higher codimension stuff]]. So we have $\operatorname{div}(\phi_y(x)) = \sum m_i(y) Z_i$. Since the m_i are continuous functions [[★★★ why are they continuous?]], they must be constant. So $\operatorname{div}(\phi_y(x)) = \operatorname{div}(\phi_{y_0}(x))$ so $\phi_y(x)/\phi_{y_0}(x) = \beta(y)$ [[★★★ here we're using properness of $\overline{X}$ to say that $\phi_y/\phi_{y_0} \in k^\times$, and normalness to make divisors behave nicely]], and we have $f(x, y) = \phi_{y_0}(x)\beta(y)$.

[[★★★ $\operatorname{div}(\phi_y)$ should be equal to the intersection of $\operatorname{div}(f)$ with the fiber over y. But $\operatorname{div}(f)$ is supported on $(\overline{X} \smallsetminus X) \times Y = \bigcup Z_i \times Y$, so it is of the form $\sum m_i Z_i \times Y$.]] [[★★★ We assume that $\overline{X} \smallsetminus X$ consists of codimension 1 pieces, but if there are pieces of higher codimension, they don't mess anything up]]

[[★★★ another approach: We have $\operatorname{div}(f) = \sum m_i Z_i \times Y$ for some integers m_i. Choose a rational function $\gamma \in \mathcal{O}(\overline{X})$ such that $\operatorname{div}(\gamma) = \sum m_i Z_i$. Then when $\gamma(x)$, regarded as a function on $\overline{X} \times Y$ has the same divisor as $f(x, y)$, so the ratio is an invertible function $\beta(x, y)$. Since $\overline{X}$ is proper, β must be constant along fibers, so it is really $\beta(y)$.]] □

Now I have a cocycle $\Phi(g, x) = \chi(g)\beta(x)$. We may rescale χ and β (inversely) so that $\chi(e) = 1$. The cocycle condition becomes

$$\chi(gh)\beta(x) = \chi(g)\beta(hx)\chi(h)\beta(x).$$

It follows that $\beta(x)$ is constant [[★★★ after you cancel the $\beta(x)$'s, the $\beta(hx)$ is the only dependence on x]], so $\beta(hx) = 1$ since $\beta(x) = \chi(e)\beta(x) = \Phi(e, x) = 1$. So we have $\Phi(g, x) = \chi(g)$. Thus, any cocycle is a character since $\chi(gh) = \chi(g)\chi(h)$.

Now we have

$$0 \to G_X^\vee \to G^\vee \to \ker\alpha \to 0$$

the right map is surjective because we have a linearization of $\mathcal{O}_X$, and we've shown that any other differs by a character. $G_X^\vee = \{\chi(g) = \phi(gx)/\phi(x) | \phi \in \mathcal{O}(X)^\times\}$, which is the group of characters arrising from semi-invariants on X.

Example 19.2. If $G^\vee = 1$ (like in the case of SL), we get that $\ker \alpha$ is trivial, so linearizations are unique.

If $\mathcal{O}(X)^\times = k^\times$ (like in the projective case), then $G_X^\vee = 1$, so then $\ker \alpha = G^\vee$. ◇

Theorem 19.3. *Let G be a connected affine algebraic group. Then $\mathcal{O}(G)^\times = k^\times G^\vee$.*

We've already seen this for a torus.

Proof. We have the multiplication $m\colon G \times G \to G$. Given $f \in \mathcal{O}(G)^\times$, we have $m^* f(g,h) = f(gh)$. On the other hand, I know that $f(gh) = f_1(g) f_2(h)$. Rescale f so that $f(e) = 1$ and rescale f_1 and f_2 (inversely) so that $f_1(e) = 1$ (it follows that $f_2(e) = 1$). Then $f_1(g) = f_2(g) = f(g)$ by substituting $h = e$ or $g = e$. □

Given a linearization, we have

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"] \arrow[d, "p_2"'] & X \\ X & \end{tikzcd}$$

the two pullbacks $\sigma^* L$ and $p_2^* L$ are isomorphic. The data of a linearization is exactly the data of this isomorphism (satisfying a cocycle condition). We have that for $g \in G$, $g^* L = \sigma^* L|_{X \times e}$.

Proposition 19.4. *In the case where G is connected, L has a G-linearization if and only if $\sigma^* L \cong p_2^* L$. [[★★★ that is, if an isomorphism exists, you can find an iso satisfying the cocycle condition]]*

Proof. By assumption, there is $\psi_g \colon L \to g^* L$, and we have to check that it can be made into an action. First we normalize so that $\psi_e = \mathrm{id}$. Now I have to check the following diagram

$$\begin{tikzcd} L \arrow[r, "\psi_{gh}"] \arrow[d, "\psi_h"'] & (gh)^* L \\ h^* L \arrow[ur, "h^* \circ \psi_g \circ (h^*)^{-1}"'] & \end{tikzcd}$$

The two maps differ by an automorphism. There is a function $F(g,h) \in \mathcal{O}(X)^\times$ such that $\psi_{gh} \circ F(g,h) = h^* \circ \psi_g \circ (h^*)^{-1} \circ \psi_h$. Since $F(g,h) \in \mathcal{O}(X)^\times$, I can think of it as a function of three variables: $F(g,h,x) = F_1(g) F_2(h) F_3(x)$ by the Theorem.

We know that $F(e,h,x) = F(g,e,x) = 1$, so we get $F_2(h)F_3(x) = 1$ and $F_1(g)F_3(x) = 1$, so F_1, F_2, and F_3 must be constant. [[★★★ this proof can probably be rewritten more cleanly with everything taking place on $G \times G \times X$]] □

This is a strange result. It is related to the structure of affine algebraic groups in general.

Now our goal is to construct the following sequence

$$0 \to \ker \alpha \to \operatorname{Pic}^G X \to \operatorname{Pic} X \to \operatorname{Pic} G$$

Remark 19.5. Given $f \in \mathcal{O}_X^\times$, we get a nonvanishing function $F(g, x) = f(gx)/f(x)$. By the theorem, we get $F(g, x) = \chi(g)\beta(x)$, where we may rescale so that $\chi(e) = 1$. Then it is easy to see that β is identitcally 1, so $f(gx)/f(x) = \chi(g)$. It is easy to check that χ is a character. Thus, we've constructed a map $\mathcal{O}_X^\times(X) \to G^\vee$. The image of this map is exactly $G_X^\vee$, and the kernel is $(\mathcal{O}_X^\times)(X))^G$, G-invariant invertible regular functions. So we're shooting for a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{O}_X^\times(X))^G \arrow[r] & \mathcal{O}_X^\times(X) \arrow[r] & G^\vee \arrow[dll, rounded corners] \\
& \operatorname{Pic}^G X \arrow[r] & \operatorname{Pic} X \arrow[r] & \operatorname{Pic} G
\end{tikzcd}$$

This is certainly a long exact sequence in some cohomology theory. The left column is probably happening on the quotient stack $[X/G]$, the middle one on X, and the right one on G. $\diamond$

Proposition 19.6. *Let X be non-singular, with G an affine connected algebraic group acting on X. Let F be a line bundle on $G \times X$. Fix some $x_0 \in X$. Then $F \cong p_1^*(F|_{G\times x_0}) \otimes p_2^*(F|_{e\times X})$.*

Lemma 19.7. *If G is a connected affine group, then there is an open set $U \subseteq G$ such that $U \cong (k^\times)^n \times k^m$.*

We'll prove this lemma later, but for now, we'll do some examples and use it to prove the proposition.

Example 19.8. $G = SL(n)$. Then we have the subgroups $T \subseteq G$ (the diagonal matrices), $N^+ \subseteq G$ (strictly upper triangular matricies), and $N^- \subseteq G$ (strictly lower triangular matricies). Then $U = N^- T N^+$ is clearly an open subset of G. Since the three subgroups don't intersect, there is a unique way to write an element of U as a product, so $U \cong N^- \times T \times N^+$. The T is a torus, and the $N^\pm$ are affine spaces. $\diamond$

There is a statement in Hartshorne which I'll use: $Cl(X \times \mathbb{A}^1) \cong Cl(X)$. Another statement we'll use: if $Z \subseteq X$ is a subvariety of codimension 1, we have an exact sequence

$$\mathbb{Z} \cdot Z \to Cl(X) \to Cl(X \smallsetminus Z) \to 0.$$

In particular, if $Cl(X) = 0$, then $Cl(X \smallsetminus Z) = 0$. And if Z is of codimension bigger than 1, it doesn't change Cl whne you remove it.

Now we have $U \times X \subseteq G \times X$. Let $Z = G \smallsetminus U$. Then we have $Cl(U \times X) = Cl(X)$ (since you're just removing stuff of high codimension from $X \times \mathbb{A}^n$). We have $F(U \times X) = p_2^*(F_1)$. Let $F_2 = p_2^*(F_1) \otimes F^{-1}$. Then the divisor of F_2 will be supported on $Z \times X$, which has some irreducible components $Z_i \times X$. So $F_2 = p_1^*(F_3)$ for some line bundle F_3 on G.

This allows me to define the map $\operatorname{Pic} X \to \operatorname{Pic} G$ in the sequence

$$0 \to \ker \alpha \to \operatorname{Pic}^G X \xrightarrow{\alpha} \operatorname{Pic} X \to \operatorname{Pic} G.$$

It is defined by sending L to $p_2^* L \otimes \sigma^*(L)^{-1}$ on $G \times X$, which should then come from something on G [[★★★ will be added next time]]. We want a map so that the map is exact [[★★★ we've already proven this because linearizable if and only if $p_2^* L \cong \sigma^* L$]].

I'll also have to show that $\operatorname{Pic} G$ is a finite group.

20 Lecture 20

Theorem 20.1. *If G is an affine algebraic group, then $\operatorname{Pic} G$ is finite.*

We can assume G is connected (since $\operatorname{Pic} G \cong G/G^\circ \times \operatorname{Pic} G^\circ$ and G/G° is finite).

Proposition 20.2. *Suppose L is a line bundle on G. Let $L^\times$ be the complement of the zero section. Then we can define a group structure on $L^\times$ such that we have the following exact sequence of groups:*

$$1 \to k^\times \to L^\times \xrightarrow{\delta} G \to 1.$$

Moreover, the pullback of L along δ to $L^\times$ is $L^\times$-linearizable.

Proof. Let $\pi\colon L \to G$ be the projection.

$$\begin{tikzcd} L \times L \arrow[r] \arrow[d, "\pi\times\pi"'] & p_1^*L \otimes p_2^*L \arrow[r, "\phi", "\sim"'] \arrow[d] & m^*L \arrow[r] \arrow[d] & L \arrow[d, "\pi"] \\ G \times G \arrow[r, "\mathrm{id}"] & G \times G \arrow[r, "\mathrm{id}"] & G \times G \arrow[r, "m"] & G \end{tikzcd}$$

[[★★★ in general, we showed that any bundle on $G \times X$ is a product of a bundle on G and one on X. This allows us to construct the isomorphism ϕ. Note that $m^*L|_{e\times G} \cong L$.]][[★★★ the map $L \times L \to p_1^*L \otimes p_2^*L$ is the map $V \times W \to V \otimes W$ on fibers.]] We want to show that the composition across the top row, $\mu\colon L \times L \to L$ is a group structure. But μ is only determined up to scalar. We identify $L_e \cong k \ni 1$, and consider

$$\begin{tikzcd} L \arrow[r] \arrow[d] & L \times \{1\} \arrow[r, "\mu"] \arrow[d] & L \arrow[d] \\ G \arrow[r] & G \times \{e\} \arrow[r] & G \end{tikzcd}$$

I have $\mu(u, 1) = \chi(\pi(u))u$, where $\chi(g) \in \mathcal{O}(G)^\times$. Similarly, I get $\mu(1, v) = \eta(\pi(v))$. So I rescale ϕ, by chaning it to $\phi \circ (\chi^{-1} \otimes \eta^{-1})$.

Then I need to check associativity of μ. We want $\mu \circ (\mathrm{id} \times \mu) = \mu \circ (\mu \otimes \mathrm{id})$. I know that if we apply π, the thing is associative. Last time, we proved (using the Rosenlicht result) that the cocycle condition is automatically satisfied. We have $\mu \circ (\mathrm{id} \times \mu)(u, v, w) = \psi(\pi(u), \pi(v), \pi(w))\mu \circ (\mu \times \mathrm{id})(u, v, w)$. By Rosenlicht, we have $\psi(g, h, k) = \psi_1(g)\psi_2(h)\psi_3(k)$. But we have by construction $\psi(e, e, e) = 1$, so $\psi_i(e) = 1$ (after rescaling). Then we get $1 = \psi(g, e, e) = \psi_1(g)$ and similarly, ψ_3 and ψ_2 are identically 1.

So $\mu\colon L^\times \times L^\times \to L^\times$ is a group. The second statement (about linearization) comes from the same $\mu\colon L^\times \times L \to L$. ◻

Lemma 20.3. *Let $L \in \operatorname{Pic} G$. Then there exists a algebraic group G' and a finite cover $\gamma\colon G' \to G$ such that γ^*L is is trivial (and therefore linearizable).*

Remark 20.4. In fact, if you have a linearizable line bundle L on G, it must be trivial as a line bundle. This is because the linearization canonically identifies all the fibers of L, so you get an isomorphism $L \cong L_e \times G$. ⋄

Proof. Consider the sequence

$$1 \to k^\times \to L^\times \xrightarrow{\pi} G \to 1.$$

We know that $\pi^* L$ is $L^\times$-linearizable. Since $L^\times$ is an algebraic group, there is a faithful representation V, so $L^\times \subseteq GL(V)$. The action of $k^\times = T$ breaks V up as $V = V_{\eta_1} \oplus \cdots \oplus V_{\eta_k}$ where $V_\eta = \{v \in V | tv = \eta(t)v\}$. For any $t \in k^\times$ and $g \in L^\times$, we have $tgv = g(g^{-1}tg)v$. But $\eta(g^{-1}tg) = \eta(t)$ because $L^\times$ is a connected group (the conjugation acts trivially on the space of characters $T^\vee \cong \mathbb{Z}$).

So $L^\times(V_\eta) = V_\eta$. Suppose $\eta_1 \neq 1$. Then we have $\rho\colon L^\times \to GL(V_{\eta_1})$. I can take $G' = \rho^{-1}(SL(V_{\eta_1}))$. Since η_1 is non-trivial, G' has the same dimension as G. Since $\pi^* L$ was $L^\times$-linearizable, it is G'-linearizable, and hence trivial. □

I claim this means that some power of L in $\operatorname{Pic} G$ will be trivial. Consider the exact sequence

$$1 \to \Gamma \to G' \to G \to 1.$$

I claim that $L^{\otimes |\Gamma|}$ is G-linearizable because the Γ action in the G'-linearization is trivial.

Corollary 20.5. *Every element of* $\operatorname{Pic} G$ *has finite order.*

Lemma 20.6. $\operatorname{Pic} G$ *is a finitely generated abelian group.*

Proof. Recall that we have a dense open subset $U \subseteq G$ of the form $U \cong (k^\times)^n \times \mathbb{A}^m$. Let $Z = G \smallsetminus U = \bigcup Z_i$, then we get an exact sequence

$$\bigoplus \mathbb{Z} \cdot Z_i \to Cl(G) \to Cl(U) = 0.$$

So $\operatorname{Pic} G = Cl(G)$ is finitely generated. □

Recall that we constructed the exact sequence

$$0 \to \ker \alpha \to G^\vee \to \operatorname{Pic}^G X \to \operatorname{Pic} X \to \operatorname{Pic} G.$$

Since $\operatorname{Pic} G$ is finite, we get the following corollary.

Corollary 20.7. *Let L be a line bundle on a smooth variety X and let G be a connected group.*[1] *Then there exists an $n > 0$ such that $L^{\otimes n}$ is G-linearizable.*

Remark 20.8. If G is not connected, then you can linearize over the connected part and then deal with the finite part. ⋄

[1]We used G connected in proving that if $p_2^* L \cong \sigma^* L$, then L linearizable. We need connected because we use the Rosenlicht result.

Remark 20.9. We need X smooth because we want the Weil divisors to be the same as Cartier divisors. $\diamond$

Theorem 20.10 (Linearization Theorem)**.** *Let G be a connected affine algebraic group acting on a smooth quasi-projective variety X. Then there exists a representation $G \to GL(V)$ and a G-equivariant embedding $X \hookrightarrow \mathbb{P}(V)$.*

This is the analogue of the result for X affine. The reason this was harder is that we didn't have functions on X, so we had to choose a line bundle.

Proof. Take any very ample line bundle L. By taking some power, we may assume L is G-linearizable. Then $\Gamma(X, L)$ is a representation of G. Since L is ample, there is a finite-dimensional subspace $W \subseteq \Gamma(X, L)$ such that $X \to \mathbb{P}(W^*)$ is an embedding (mapping x to $h_x = \{s \in W | s(x) = 0\}$). As we've shown before, W is inside a finite-dimensional invariant subspace (so we may assume W is G-invariant). $\square$

For X projective and a very ample G-linearized line bundle L, we can construct the quotient $X /\!\!/_L G = \operatorname{Proj}(\bigoplus \Gamma(X, L^{\otimes n}))$.

We can define $X^s(L)$ (resp. $X^{ss}(L)$) to be the stable (resp. semi-stable) locus with respect to the embedding coming from L. That is, $x \in X$ is L-semi-stable if there is an invariant section $f \in \Gamma(X, L^{\otimes n})$ such that $f(x) \neq 0$, and L-stable if furthermore the action of G on X_f is closed. Note that this implies that the orbit of x is closed.

If X is projective, then $x \in X$ is stable.

21 More on Stability

I'm going to talk a bit more about stability. We assume that G is always reducitve.

Let L be a G-linearized line bundle on X. We define $X^{ss}(L) = \{x \in X | \exists f \in \Gamma(X, L^{\otimes n})^G, f(x) \neq 0\}$. If X is only quasi-projective, then I also require that X_f be an affine variety (which is automatic if X is projective). We define $X^s(L) = \{x \in X^{ss}(L) |$ the G-action on X_f is closed$\}$. Note that these sets don't change if we replace L by $L^{\otimes r}$.

If X is projective and L is ample, we have the following criterion. Consider $X \hookrightarrow \mathbb{P}(V)$, where $V = \Gamma(X, L^{\otimes d})$. Then x is semi-stable if and only if for any $v \in kx$, $0 \notin \overline{G \cdot v}$. Moreover, x is stable if and only if $G \cdot v$ is closed and x is a regular point in X [[★★★ equivalently, v is regular in X_a.]].

For the semi-stability criterion: the condition $0 \notin \overline{G \cdot v}$ implies (by the Separation Lemma) that there is a homogeneous polynomial $f \in k[V]^G$ which does not vanish at v. Conversely, if $0 \in \overline{G \cdot v}$, then any invariant homogeneous function must vanish at v.

For the stability criterion: let f be a homogeneous invariant function such that $x \in X_f$. Note that X_f is affine. If x is stable, then $G \cdot x$ is closed in X_f. I claim that this implies that $G \cdot v$ is closed in V. For any point $w \in \overline{G} \cdot v$, it is automatically in X_f, so the line $y = kw$ is in X_f. We have $y \in \overline{G \cdot x}$, so there is some point on the same line where f doesn't vanish, so f doesn't vanish on the whole line. Conversely, if $G \cdot v$ is closed, then $G \cdot x$ is closed. Now we have to do regularity. If we have X_f on which the action of G is closed, then each point must be regular.

Recall that in the affine situation, the stable points are regular points with closed orbits. We had the quotient map $\phi\colon X \to Y$. We showed that $X^s = X \smallsetminus \phi^{-1}(\phi(X^{\text{irreg}}))$. The action of G is closed on this open set. If there is a closed orbit with small dimension, then it is in the closure of some larger orbit. The fibers of ϕ are closure equivalence classes. We need a semi-continuity result to get that the fibers of ϕ have upper semi-continuous dimensions.

Now suppose X is projective. We define $X /\!\!/_L G = \operatorname{Proj} \bigoplus_{n \geq 0} \Gamma(X, L^{\otimes n})^G = X^{ss} /\!\!/ G$ (piecewise gluing) [[★★★ semi-stable or stable?]]. If X is not projective, there is another way to define it by gluing. We have $X^{ss}(L) = \bigcup_{i=1}^k X_{f_i}$, where we can assume that f_i are invariant sections of the same power of L, $L^{\otimes r}$. Let $U_i = X_{f_i}$. Then $\mathcal{O}(U_i) = \{g/f_i^n | g, f^n \in \Gamma(X, L^{\otimes rn})\}$. We have $\mathcal{O}(U_i)^G = \{g/f_i^n | g, f^n \in \Gamma(X, L^{\otimes rn})^G\}$. Let $Y_i = \operatorname{Specm} \mathcal{O}(U_i)^G$. We can glue them together in the natural way. [[★★★ I'm worried you might need *stable* points to get geometric quotients to glue ... no, it should work]]. We can define $Y_{ij} \subseteq Y_i, Y_j$ as a localization, then glue. It's a bit of work to show that this $X^{ss} /\!\!/ G$ does not depend on the choice of open cover. This is called a *categorical quotient*, Y. We have a map $X^{ss} \to Y$. This map is surjective and an open submersion.

Categorical quotients. Suppose G acts on a variety X. Suppose ϕ is a G-invariant map $\phi\colon X \to Y$. Suppose that for any G-invariant map $q\colon X \to Z$, there exists a unique factorization $q\colon X \xrightarrow{\phi} Y \to Z$. The categorical quotient is uniquely deteremined. People usually require more.

Good quotients. (1) We require ϕ to additionally be a surjective open submersion ($U \subseteq Y$ is open if and only if $\phi^{-1}(U)$ is open). (2) We also require that for any open set $U \subseteq Y$, $\phi^*(\mathcal{O}_Y(U)) \cong \mathcal{O}_X(\phi^{-1}(U))^G$ [[★★★ LHS means fuctions which are constant on fibers: $\phi^{-1}\mathcal{O}_Y(\phi^{-1}(U))$. maybe this is called $\phi^\#$]].

Geometric quotients. We additionally impose (3) the fibers of ϕ are orbits of the G-action.

These three conditions actually imply by themselves that ϕ is a categorical quotient.

Proof. Suppose $q\colon X \to Z$ is G-invariant. Cover Z by affine sets: $Z = \bigcup V_i$. Consider $\phi(q^{-1}(V_i)) = U_i$, which are open sets on Y by (1). We have $q^{-1}(V_i) \subseteq \phi^{-1}(U_i)$. Since the fibers are orbits (3), this should actually be an equality. Since ϕ is surjective (1), we have $Y = \bigcup U_i$. Since q is G-invariant, we have a homomorphism $\mathcal{O}(V_i) \to \mathcal{O}(q^{-1}(V_i))^G = \mathcal{O}(\phi^{-1}(U_i)) = \mathcal{O}(U_i)$. This gives a map $Y \to Z$. □

Charley: there is a proof in Borel that shows that (1) and (2) imply that ϕ is a categorical quotient.[[★★★ nevermind, it's slightly different]] Vera: I think he uses something else too. Suppose you have (1) and (2) and (3′) if W_1 and W_2 are closed G-invariant sets in X with $W_1 \cap W_2 = \varnothing$, then $\phi(W_1) \cap \phi(W_2) = \varnothing$. Then you can get that ϕ is a categorical quotient.

Proof. Let V_i be as in the previous proof. Define $W_i = X \smallsetminus q^{-1}(V_i)$ (these are closed sets). Define $U_i = Y \smallsetminus \phi(W_i)$. Since $\bigcap W_i = \varnothing$, the U_i are an open cover of Y. Then procede exactly as in the previous proof. □

The main point is that $X^{ss}/\!\!/G$ is a categorical quotient and $X^s/\!\!/G$ is geometric quotient. Then we'll talk about linearizations. And toric varieties.

22 Some toric examples

Last time we talked about categorical, good, and geometric quotients.

Proposition 22.1. *If X is affine and G is reductive, then $\phi\colon X \to Y = \operatorname{Specm} k[X]^G$ is a categorical quotient. Moreover, the restriction to the stable locus, $\phi\colon X^s \to Y^s$, is a geometric quotient.*

Proof. We already proved that ϕ is a surjective open submersion. The categorical quotient property follows from the Separation Lemma: if $W_1, W_2 \subseteq X$ are closed G-invariant subsets that don't intersect, then $\phi(W_i)$ are closed subsets of Y which don't intersect. The proof is at the end of the last lecture.

The action of G on X^s is closed (almost by definition), so the fibers of the restricted map are orbits. □

Now suppose X is an arbitrary algebraic variety (prefered irreducible, but not always needed). Suppose $L \in \operatorname{Pic}^G X$, then we constructed $X^{ss}(L)/\!\!/G$ by gluing together the affine quotients for the $X_{f_1}, \dots, X_{f_k}$, where the f_i are invariant sections of some tensor power of L. Since each of the $X_{f_i}/\!\!/G$ is a categorical quotient, you can check that gluing them together you still get a categorical quotient.

Theorem 22.2. *Suppose G is a reductive group acting on an algebraic variety X, with $L \in \operatorname{Pic}^G X$. Then $\phi\colon X^{ss}(L) \to X^{ss}(L)/\!\!/G$ is a good categorical quotient, and the restriction $\phi\colon X^s(L) \to X^s(L)/\!\!/G$ is a geometric quotient. There exists an ample line bundle M on $X^{ss}/\!\!/G$ such that $\phi^*(M) \cong L^{\otimes r}$, so the quotient is a quasi-projective variety. [[★★★ you don't even need L to be ample, though it will be ample on X^{ss} almost by hypothesis.]]*

Recall that we have the $Y_i = X_{f_i}/\!\!/G$. One way to define M is to say what the transition functions between the Y_i are. We take the transition function to be $f_i/f_j = \phi^*(\alpha_{ij})$. Then you can get that $\phi^* M$ is a power of L.

All this depends on how you choose your L and the linearization.

<u>Certain class of examples</u>. Let $X = \mathbb{A}^n$ and G a reductive group acting linearly on X. Even in such a simple situation, we get many differnt quotients. We have to calculate $\operatorname{Pic}^G X$. We have that $\operatorname{Pic} X = \{1\}$ and $\mathcal{O}(X)^\times = k^\times$. From the exact sequence, we get that $\operatorname{Pic}^G X = G^\vee$.

Take $x \in \mathbb{A}^n$ and $u \in \mathbb{A}^1$. The total space of the bundle is $\mathbb{A}^n \times \mathbb{A}^1$. Given $\chi \in G^\vee$, we define the linearization $g(u, x) = (\chi^{-1}(g)u, gx)$. Call this linearized bundle L_χ.

If $\chi = 1$, then all the invariants are of the form $u^n f(x)$, where $f(x) \in k[\mathbb{A}^n]^G$. So when we take the proj quotient, we get the usual affine quotient $X/\!\!/G = X/\!\!/_{L_{\chi=1}}G = X^{ss}(L_{\chi=1})/\!\!/G$. But in other cases, we get other quotients.

Example 22.3. Consider the action of $G = k^\times$ by homothety: $t \cdot (x_1, \dots, x_n) = (tx_1, \dots, tx_n)$. We have different linearizations, $t \cdot (u, x_1, \dots, x_n) = (t^a u, tx_1, \dots, tx_n)$.

When $a = 0$, the only invariant section is the zero section, so the quotient is $\mathbb{A}^n /\!\!/ G = *$. If $a > 0$, then $\mathbb{A}^n /\!\!/ G = \varnothing$.

If $a < 0$, then it doesn't matter which negative a you take because L and $L^{\otimes r}$ give the same quotient. In fact, as soon as the set of semi-stable points is the same, the quotients are isomorphic because the quotient is the categorical quotient. This gives us invariants $ux_1, \dots, ux_n$. So the quotient is $\operatorname{Proj} k[ux_1, \dots, ux_n] \cong \mathbb{P}^{n-1} = \mathbb{A}^n /\!\!/_{L_q} G$. ⋄

Q: is there always some "best" choice where the quotient has maximal dimension? A: you still get different birational models.

You want $\dim X /\!\!/ G = \dim X - \dim G \cdot x$ for generic x. If you have such a thing, then $X^s(L)$ is non-empty. If you have two such linearized bundles, you have $X^s(L_1) \cap X^s(L_2) = U$ is an open set. So the two quotients will be birationally equivalent, with generic point given by the fraction field of $\mathcal{O}(U)^G$.

Example 22.4. Consider $G = k^\times$ acting on $\mathbb{A}^4$ by $t \cdot (x_1, x_2, x_3, x_4) = (tx_1, tx_2, t^{-1}x_3, t^{-1}x_4)$. The possible linearizations are $t \cdot (u, x_1, x_2, x_3, x_4) = (t^{-a}u, tx_1, tx_2, t^{-1}x_3, t^{-1}x_4)$.

We get three cases as before. If $a = 0$, the generating invariants are $z_{13} = x_1x_3, z_{14} = x_1x_4, z_{23} = x_2x_3, z_{24} = x_2x_4$, and they satisfy the relation $z_{13}z_{24} - z_{14}z_{23}$. Let $Y_0 = X /\!\!/_{L_0} G$. We have that Y_0 is a quadatic cone with a singularity at 0. What are the semi-stable points in this case? Define $V^+ = \{x_1 = x_2 = 0\}$ and $V^- = \{x_3 = x_4 = 0\}$. The semistable locus is $X^{ss}(L_0) = \mathbb{A}^4 \smallsetminus (V^+ \cup V^-)$.

If $a = 1$, then we have all the old invariants $(z_{13}, z_{14}, z_{23}, z_{24})$, but also $t_1 = ux_1$ and $t_2 = ux_2$. Now $X^{ss}(L_1) = \mathbb{A}^4 \smallsetminus V^+$. This X^{ss} is not affine, but I can glue together X_{t_1} and X_{t_2}. We get $Y_+ \subseteq \mathbb{P}^1 \times \mathbb{A}^4$. Relations are $z_{13}t_2 - z_{23}t_1 = 0$, $z_{14}t_2 - z_{24}t_1 = 0$, $z_{13}z_{24} - z_{23}z_{14} = 0$. We get a map $p_+ \colon Y_+ \to Y_0$. We get that $p_+^{-1}(y)$ is a point if $z_{ij} \neq 0$ and $p_+^{-1}(0) = \mathbb{P}^1$. This is sort of a "partial blow-up." [[★★★ this is one of the resolutions of the toric variety which is a cone on a square]] You can check that Y_+ is non-singular. This is called a small resolution because the fibers are codimension bigger than 1. The projectivization of the tangent space is 2-dimensional (it's the $\mathbb{P}^1 \times \mathbb{P}^1$ sitting inside the $\mathbb{P}^3$ which is really the projectivization of the tangent space). Y_+ is a blow up of the subvariety $z_{12} = z_{23} = 0$.Note that $V^+ /\!\!/ G \cong \mathbb{P}^1$.

If $a = -1$, then everything is very similar, but we get $t_3 = ux_3$ and $t_4 = ux_4$. The relation you get is the "transpose" of the one in the case $a = 1$. Again, we get a resolution $p_- \colon Y_- \to Y_0$.

By the symmetry of the situation, Y_+ and Y_- are isomorphic, but *not as varieties over* Y_0. ⋄

Toric Varieties as GIT quotients

We're in the situation $X = \mathbb{A}^n$ and $G = T$ is a torus acting linearly on $\mathbb{A}^n$. Then we have a large group of linearizations (it is $T^\vee$). Whenever we take $\mathbb{A}^n /\!\!/_{L_\chi} T$, it will have the action of a bigger torus. $t \cdot (x_1, \dots, x_n) = (\chi_1(t)x_1, \dots, \chi_n(t)x_n)$. χ stands for linearization. The semi-invariants are all monomials. So in $\mathbb{A}^n$, we have a big torus, $U = (\mathbb{A}^1 \smallsetminus 0)^n$. On the quotient $\mathbb{A}^n /\!\!/_{L_\chi} T$, the torus U acts, giving the quotient the structure of a toric variety.

Next time, we'll consider the following example, $\left(\begin{smallmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{smallmatrix}\right)$, with the action of $\left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right)$ on the left and $diag(s_1, s_2, s_3)$ on the right. So we have an action of a 5-dimensional torus (except that a 1-dimensional subtorus acts trivially). The quotient turns out to be a projective surface.

23 Lecture 23

If the group G acts on $\mathbb{A}^n$, then a linearized line bundle is determined by a choice of character. The action on the line bundle (which must be trivial) is given by $g\cdot(f(x),x) = (\chi(g)f(x), g\cdot x)$, where f is a regular function. So the invariant sections are those regular functions f for which $\chi(g)f(x) = f(g\cdot x)$, the semi-invariants with character χ. The invariant sections of the n-th tensor power are semi-invariants with character χ^n. The proj quotient is $\operatorname{Proj}\bigoplus_{n\geq 0}\Gamma(\mathbb{A}^n, \mathcal{O}_X^{\otimes n})^G$.

Example 23.1. Consider matrices $\left(\begin{smallmatrix} x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{smallmatrix}\right)$ modulo the action of diagonal matrices on the left and right (call this torus T). Each coordinate has a weight with respect to the torus action. If I draw them all together, I get the picture [[★★★ prism on a triangle with x's on top and y's on bottom. actually a 4-dimensional picture with the prism in a plane (not through zero)]]

The quotient depends on the choice of a character $\chi = (a_1, a_2|b_1, b_2, b_3)$. For example, the character of the action on x_1 is $(1,0|1,0,0)$. That is, x_1 is a semi-invariant function with character $(1,0|1,0,0)$. Even though T is 5-dimensional, the action factors through a 4-dimensional torus, but these coordinates are convinient. All the coordinates have characters that lie on one side of a plane, so the quotient is projective [[★★★ The proj quotient is always projective over $R^G = \Gamma(\mathbb{A}^n, \mathcal{O}^{\otimes 0})^G$, the degree zero part of the graded ring that defines the proj quotient. I think this comment means that $R^G = k$, so the quotient will always be projective over $\operatorname{Spec} k$.]].

If $\chi = 0$, then the quotient is a point.

If I take $\chi = (3,0|1,1,1)$. Then the quotient is $\operatorname{Proj} k[x_1x_2x_3]$ [[★★★ really $u^3x_1x_2x_3$, but we'll ignore the u's because that's just to get the grading]].

In general, we get the invariants $R^\chi = \bigoplus R^\chi_d$ where $R^\chi_d = \{f \in k[\mathbb{A}^n] | f(tx) = \chi^d(t)f(x)\}$, the semi-invariants with character χ.

Suppose $\chi = (1,1|1,1,0)$. Then we get $\operatorname{Proj} k[x_1y_2, x_2y_1]$, which is $\mathbb{P}^1$. We've increased the dimension, but we'd really like to get a 2-dimensional quotient because the orbits are 4-dimensional.

Let $\chi_1 = (2,1|1,1,1)$. Then invariants are $\alpha_1 = y_1x_2x_3$, $\alpha_2 = x_1y_2x_3$, and $\alpha_3 = x_1x_2y_3$, which generate all invariants. The quotient is $X/\!\!/_{L_\chi} T = \operatorname{Proj} k[\alpha_1,\alpha_2,\alpha_3] = \mathbb{P}^2 = Y_1$. Consider the open set where all the x_i are non-zero. Then by the action of the torus, we can make $x_1 = x_2 = x_3 = 1$. Then we have $y_i = \alpha_i$, and we can multiply the second row by any unit t_2, so we get a copy $\mathbb{P}^2$. Note that if all the $\alpha_i = 0$, then the point is unstable because we can get the orbit to have 0 in its closure. We have three special lines, given by $\alpha_i = 0$ for $i = 1,2,3$.

If $\alpha_2 = \alpha_3 = 0$, then we have the orbit corresponding to $\left(\begin{smallmatrix} 1 & 1 & 1\\ * & 0 & 0\end{smallmatrix}\right)$. But we see that if we evaluate the α_i on $\left(\begin{smallmatrix} 0 & * & *\\ * & * & *\end{smallmatrix}\right)$, we get $\alpha_2 = \alpha_3 = 0$, so this whole set gets sent to a single point in $\mathbb{P}^2$.

Let $\chi_2 = (1,2|1,1,1)$. As before, we get three invariants $\beta_3 = y_1y_2x_3$, $\beta_2 = y_1x_2y_3$, $\beta_1 = x_1y_2y_3$, with the identification $\left(\begin{smallmatrix} \beta_1 & \beta_2 & \beta_3\\ 1 & 1 & 1\end{smallmatrix}\right)$. We have the identification $\beta_i = 1/\alpha_i$. Except we see that over the point P we found in the previous paragraph, we get the line $\beta_1 = 0$. That is, every element of the form $\left(\begin{smallmatrix} 0 & * & * \\ * & * & * \end{smallmatrix}\right)$ gives you a different point on the line $\beta_1 = 0$. Call the quotient $Y_2 \cong \mathbb{P}^2$.

Now let's add these characters together to consider the case $\chi_3 = (3,3|2,2,2)$. Here there are plenty of semi-invariants. $z = x_1x_2x_3y_1y_2y_3$, $v_1 = x_1^2x_2y_2y_3^2$, $v_2 = y_1^2x_2y_2x_3^2$, $w_1 = x_1y_1y_2^2x_3^2$, $w_2 = x_1y_2x_2^2y_3^2$, $u_1 = y_1^2x_2^2x_3y_3$, and $u_2 = x_1^2y_2^2x_3y_3$. The proj of the ring generated by these is a surface in $\mathbb{P}^6$. The relations are given by $z^2 = v_1v_2 = w_1w_2 = u_1u_2$ and $z^3 = u_1v_1w_1 = u_2v_2w_2$. Call the quotient Y_3.

You can see that $X^{ss}(L_{\chi_3})$ is the intersection of the semi-stable points of χ_1 and χ_2. The set of semi-stable points drops, and the size of the quotient goes up, which is what normally happens.

Outside of the three special lines on Y_1 and Y_2, we get that the two are isomorphic. Consider the closure of the rational map from Y_1 to Y_2, which is a surface in $\mathbb{P}^2 \times \mathbb{P}^2$. This surface will be Y_3. Over any point other than the three distinguished points, you get a unique point, and over the special points, you get a $\mathbb{P}^1$. To see that, you can glue out of two pieces $\left(\begin{smallmatrix} 0 & 1 & 1 \\ 1 & \alpha_2 & \alpha_3 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & \beta_1 & \beta_2 \\ 1 & 1 & 1 \end{smallmatrix}\right)$.

The upshot is that Y_3 is the blowup of Y_1 at the three special points. Y_3 is a del Pezzo surface. In this case, I believe this is "the biggest" quotient. In general, there need not be a biggest one, like in the example from last time.

You can get the same surface if you blow up two points on $\mathbb{P}^1 \times \mathbb{P}^1$. Perhaps for a suitable choice of character, you can get the quotient $\mathbb{P}^1 \times \mathbb{P}^1$. $\diamond$

General $\mathbb{A}^n /\!\!/ T$

Fix a torus T of dimension r acting on $\mathbb{A}^n = \{(x_1, \ldots, x_n) | x_i \in k\}$, given by $t \cdot (x_1, \ldots, x_n) = (\chi_1(t)x_1, \ldots, \chi_n(t)x_n)$, with $\chi_1, \ldots, \chi_n \in T^\vee$. Let $a \in \mathbb{Z}^n$ be $a = (a_1, \ldots, a_n)$, and define $x^a = x_1^{a_1} \cdots x_n^{a_n}$. I'm interested in monomials because all the semi-invariants will be monomials. We define $\operatorname{Supp}(a) = \{i | a_i \neq 0\}$. Given $I \subseteq \{1, \ldots, n\}$, we define $x^I = \prod_{i \in I} x_i$. We define $M = \{a \in \mathbb{Z}^n | \sum a_i\chi_i = 0\} \subseteq \mathbb{Z}^n \subseteq \mathbb{R}^n$. We can define the group-algebra of M, which will be a subring of Laurent polynomials in $t_1, \ldots, t_n$. More generally, for a submonoid $S \subseteq \mathbb{Z}^n$, we define $k[S]$ to be the ring $k[S] = \{x^a | a \in S\}$.

Fix a character χ. Let's describe the quotient $\mathbb{A}^n /\!\!/_{L_\chi} T$. In our case, $S_k = \{a \in \mathbb{Z}^n_{\geq 0} | \sum a_i\chi_i = k\chi\}$. Then $k[S_k]$ is the space of semi-invariants corresponding to the character $k\chi$, and $k[S_0]$ is exactly the ring of T-invariant polynomials. Let $S = \bigcup_{k \geq 0} S_k$ and consider the ring $k[S] = \bigoplus k[S_k]$.

To construct the quotient, I'll glue it out of X_f's as always. I have the ideal $k[S]_{>0} \subseteq k[S]$ generated by positive degree monomials. Choose a minimal set of monomial generators $\{f_1, \ldots, f_p\}$ of this ideal. Define $U_i = \{x \in \mathbb{A}^n | f_i(x) \neq 0\}$. These are all semi-stable points by definition, and the U_i cover X^{ss}. I define $R_i = \{g/f_i^k | \deg g = k \deg f_i\}$.

Each f_i is a monomial, so $f_i = x^{m_i}$. Let $J_i = \operatorname{Supp} m_i$. Then $U_i = \{x \in \mathbb{A}^n | x_j \neq 0$ for all $j \in J_i\}$. Define $\sigma_i^\vee = \{a \in \mathbb{R}^n | a_j \geq 0$ for all $j \notin J_i\}$.

Claim. $R_i = k[\sigma_i^\vee \cap M]$.

Proof. It is clear that any $g/f_i^k \in k[M]$. But for any element $h \in \sigma_i^\vee \cap M$, by multiplying by a sufficiently high power of f_i, I can make all the exponents for coordinates in J_i non-negative. Say hf_i^k has positive exponents for all coordinates in J_i are non-negative. $\square$

We have the inclusion $M \subseteq \mathbb{Z}^n$. We get the dual map $(\mathbb{Z}^n)^* \to M^*$. Next time we'll talk about fans.

24 Lecture 24

We were considering the action of T on $\mathbb{A}^n$ by the characters $\chi_1, \dots, \chi_n$. We fix a character χ, which gives us a linearization of the structure sheaf. We defined $M = \{a \in \mathbb{Z}^n | \sum a_i\chi_i = 0\}$. The quotient is given by $Y = \mathbb{A}^n /\!\!/_{L_\chi} T = \operatorname{Proj} k[S]$. We chose monomial generators $f_1, \dots, f_p$ for $k[S]_{>0}$. Each f_i is of the form x^{m_i}, and we defined $J_i = \operatorname{Supp} m_i$.

We defined polyhedral rational cones $\sigma_i^\vee = \{a \in \mathbb{R}^n | a_j \geq 0 \text{ for all } j \in J_i\}$. $Y = \bigcup Y_i$, where $\mathcal{O}(Y_i) = R_i = k[\sigma_i^\vee \cap M]$.

I can think of T as sitting inside the larger torus $U \cong (\mathbb{A} \smallsetminus \{0\})^n$. U has a natural action on $\mathbb{A}^n$, given by $(s_1, \dots, s_n) \cdot (x_1, \dots, x_n) = (s_1x_1, \dots, s_nx_n)$, so we get a natural map $T \to U$. We get another torus $Q = (U/T)_0$ (connected component of the identity, so it's really a torus). We have that $M \cong Q^\vee$, and $k[M] = k[Q] \supset R_i$ [[★★★ LHS is monoid algebra, and RHS is coordinate ring]]. We have the comultiplication $\Delta\colon k[M] \to k[M] \otimes k[M]$ comming from the group structure on Q. On the other hand, we can restrict to R_i to get $\Delta\colon R_i \to k[M] \otimes R_i$. So Q acts on each Y_i, and they glue together equivariantly, so we get an action of Q on Y. This isn't too surprising; we quotiented by a subgroup, so the quotient group should still act. Moreover, Y has one open dense orbit.

Consider $(\mathbb{Z}^n)^* \subseteq (\mathbb{R}^n)^*$. The natural embedding $M \hookrightarrow \mathbb{Z}^n$ gives the dual map $(\mathbb{Z}^n)^* \to M^*$. Let $M^*_\mathbb{R} = M^* \otimes_\mathbb{Z} \mathbb{R}$. Then we get a map $(\mathbb{R}^n)^* \to M^*_\mathbb{R}$.

We introduce a basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ for $(\mathbb{Z}^n)^*$. By $\bar\varepsilon_i$, we denote the images of the ε_i in $M^*_\mathbb{R}$. We have the collection $J_1, \dots, J_p$. We say that σ_i is the cone generated by $\{\bar\varepsilon_j | j \notin J_i\}$. We define Σ to be the collection $\sigma_1, \dots, \sigma_p$. It is clear (almost by definition) that σ_i is dual to $\sigma_i^\vee \cap M$.

Lemma 24.1. *$\sigma_k \cap \sigma_\ell$ is a face of σ_k and σ_ℓ.*

A face is a subcone such that there is a linear functional which is zero on the subcone and positive on the rest of the cone.

Proof. We have that $\mathcal{O}(Y_k \cap Y_\ell) = R_k(x^{-c}] = R_\ell[x^c]$ for $c \in M$, with $\operatorname{Supp} c \subseteq J_k \cup J_\ell$. Then $\langle c, \sigma_k\rangle \subseteq \mathbb{R}_{\geq 0}$ and $\langle c, \sigma_\ell\rangle \subseteq \mathbb{R}_{\leq 0}$. It follows that $\langle c, \sigma_k \cap \sigma_\ell\rangle = 0$. □

Example 24.2. $\mathbb{P}^{n-1} = \mathbb{A}^n /\!\!/ k^\times$. In this case, we have $J_1, \dots, J_n$, where J_i are one-element sets (since the generators for $k[S]_{>0}$ are the x_i). We have one relation $\varepsilon_1 + \dots + \varepsilon_n = 0$. σ_i is generated by all the ε_j for which $j \neq i$. In the case $n = 3$, we get a partition of the plane by three cones. ◇

Example 24.3. $t \cdot (x_1, x_2, x_3, x_4) = (tx_1, tx_2, t^{-1}x_3, t^{-1}x_4)$. Now I get $\bar\varepsilon_1 + \bar\varepsilon_2 = \bar\varepsilon_3 + \bar\varepsilon_4$. The fan will be in three dimensions. It will be the cone on a square, with ε_1 and ε_2 opposite corners of the square.

In the case of Y_0, I just get $J_1 = \varnothing$, so I get a single cone, which is not simplicial. This gives a quotient with a singularity.

In the case of Y_+, I get $J_1 = \{1\}$ and $J_2 = \{2\}$, so I break my square cone into two simplicial cones (one excluding ε_1 and the other excluding ε_2). Similarly, Y_- breaks the square cone into two simplicial cones, but the other way. $\diamond$

Example 24.4. Recall the action of the 5-dimensional torus on the 6-dimensional space $\left(\begin{smallmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{smallmatrix}\right)$. In this case, the natural basis is $\varepsilon_1, \varepsilon_2, \varepsilon_2$ (corresponding to the x's) and $\delta_1, \delta_2, \delta_3$ (corresponding to the y's). They satisfy the relations $\bar{\varepsilon}_i + \bar{\delta}_i = 0$ and $\sum \bar{\varepsilon}_i = 0$.

Recall that in the biggest quotient, I have 6 pieces that are being clued together. The two other quotients we got were obtained by taking just the $\bar{\varepsilon}_i$ and we got the other quotient by taking just the $\bar{\delta}_i$. We should also be able to remove $\bar{\varepsilon}_3$ and $\bar{\delta}_3$ to get $\mathbb{P}^1 \times \mathbb{P}^1$ as a quotient. To do that, we'd have to choose a character which would give something like $J_1 = \{\varepsilon_3, \delta_3, \delta_1, \varepsilon_2\}$. $\diamond$

Geometric meaning of Σ. I can identify M^* with the dual lattice to M, so M^* is the lattice of 1-parameter subgroups of Q. If you think about $M^\perp \subseteq (\mathbb{Z}^n)^*$, you get the lattice of 1-parameter subgroups in T.

There is an open Q-orbit $Y^0 \subseteq Y$ which is isomorphic to Q itself if none of the σ contains a line (such a line would correspond to a subtorus that stabilizes every point). [[★★★ how can such a bad cone ever happen?]]

Given a generic point $y \in Y^0$, we can define $M_0^* = \{\lambda \in M^* | \lim_{t\to 0} \lambda(t)y \text{ exists}\}$. We say that $\lambda \sim \mu$ if $\lim_{t\to 0} \lambda(t)y = \lim_{t\to 0} \mu(t)y$. Note that this is independent of choice of $y \in Y^0$. Each equivalence class is the interior of a face. To see this, identify $k[M]$ with $k[y_1^{\pm 1}, \dots, y_\ell^{\pm 1}] \supset R_\sigma$. Pick a monomial y^b.

$$\lim_{t\to 0} \lambda(t) y^b = \begin{cases} y^b & \text{if } \langle b, \lambda\rangle = 0 \\ 0 & \text{if } \langle b, \lambda\rangle > 0 \\ DNE & \text{if } \langle b, \lambda\rangle < 0 \end{cases}$$

So we're checking locally in each open affine which coordinates become zero.

If Y is complete, then $M_0^* = M^*$ because limits must always exist. In other words, $\bigcup \sigma_i = M^*_{\mathbb{R}}$.

Corollary 24.5. *The faces are in bijection with the orbits of Q in Y. Moreover, the dimension of a face is equal to the dimension of the stabilizer of a point in the orbit. So if the fan is non-degenerate, the origin corresponds to the open orbit.*

σ is simplicial if it is spanned by linearly independent rays (i.e. it's generated by a partial basis of $\mathbb{R}^n$). A fan is simplicial if every cone in it is simplicial.

Proposition 24.6. *Suppose that* $\ker \chi_1 \cap \dots \cap \ker \chi_n$ *is finite.* Σ *is simplicial if and only if* $\mathbb{A}^n(L_\chi)^{ss} = \mathbb{A}^n(L_\chi)^s$.

Proof. Suppose σ is not simplicial, then we have some relation $\sum_{i\notin J_\sigma} c_i \bar{\varepsilon}_i = 0$. Then $\sum c_i \varepsilon_i \in M^\perp$. This $\sum c_i\varepsilon_i$ corresponds to a 1-parameter subgroup $\lambda(t)$ of T. Choose $x \in \mathbb{A}^n$ such that $x_i = 1$ for $i \in J_\sigma$ and $x_i = 0$ for $i \notin J_\sigma$. Then $\lambda(t) \in Stab(x)$, so x is semi-stable but not stable. $\square$

25 Lecture 25

No class on Monday. I'll also be away Thanksgiving week, so there won't be class then.

Obtaining a toric variety as a quotient. Think of $\mathbb{Z}^s \subseteq \mathbb{R}^s \supseteq \Sigma = \{\sigma\}$, where the σ are rational polyhedral cones.

We pick $\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_n \in \mathbb{Z}^s$ which are the generators of cones. Assume $\{\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_n\}$ generate all of $\mathbb{Z}^s$ (as a lattice). For a cone $\sigma \in \Sigma$, we associate the set $J_\sigma = \{i | \bar{\varepsilon}_i$ is not a generator of $\sigma\}$. We define $V = \mathbb{A}^n \smallsetminus \{x^{J_\sigma} = 0 | \sigma \in \Sigma\}$ [[★★★ we're taking an intersection in that last set]].

Claim. *The toric variety Y_Σ is a categorical quotient $V /\!\!/ T$, where T is defined as below. Moreover, if Σ is simplicial, then the quotient is a geometric quotient.*

We define an $s \times n$ matrix A whose columns are $\bar{\varepsilon}_i$. We find an $n \times r$ matrix B such that $AB = 0$. Let χ_j be the rows of B (the χ_j are a basis for the kernel of A). Then we define the action of an r-dimensional torus T by $t \cdot (x_1, \dots, x_n) = (\chi_1(t)x_1, \dots, \chi_n(t)x_n)$.

The map $\mathbb{Z}^n \to \mathbb{Z}^s$ is surjective (by our assumption). The kernel is another free abelian group

$$0 \to \mathbb{Z}^r \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^s \to 0.$$

We regard these free abelian groups as lattices of 1-parameter subgroups of the tori in the short exact sequence

$$0 \to T \to U \to Q \to 0.$$

We also get the short exact sequence of character lattices

$$0 \to Q^\vee \to U^\vee \to T^\vee \to 0$$

where $Q^\vee$ is what we called M before. Alsmost by construction, $k[M] = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^T$. I can write $V = \bigcup_{\sigma \in \Sigma} V_{x^{J_\sigma}}$. By construction, V is covered by these open sets. What are invariant functions $\mathcal{O}(V_{x^{J_\sigma}})^G$? it is exactly $k[\sigma^\vee \cap M]$. Each of the affines is a categorical quotient, so when we glue them together, we get a categorical quotient.

Remark 25.1. It may not be true that this is a Mumford quotient (exists a line bundle with linearization so that this is the semi-stable quotient). It is known that there are toric varieties that are proper but not projective, and Mumford quotients are always quasi-projective. ◇

Remark 25.2. If we drop the assumption that the $\bar{\varepsilon}_i$ generate $\mathbb{Z}^s$, but generate a finite-index subgroup, then Y_Σ is the quotient of V by $T \times \Gamma$, where Γ is some finite abelian group (Γ will be the cokernel of $A\colon \mathbb{Z}^n \to \mathbb{Z}^s$?). We get a morphism $U \to Q$, and the kernel of this morphism will contain Γ as a subgroup. ◇

Example 25.3. [[★★★ $\bar{\varepsilon}_1 = (0,1)$, $\bar{\varepsilon}_2 = (1,0)$ and their negatives, with the upper left and lower right maximal cones missing]] This is the fan of $\mathbb{P}^1 \times \mathbb{P}^1$ minus two points. we have

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We have $(t,s) = (x_1, x_2, y_1, y_2) = (tx_1, sx_2, ty_1, sy_2)$, $\chi(t,s) = ts$. We have $V = \mathbb{A}^4 \smallsetminus \{x_1x_2 = 0, y_1y_2 = 0\}$. The points where $x_1 = y_1 = 0$ or $x_2 = y_2 = 0$ are unstable, so they are always removed. We've also removed $x_1 = y_2 = 0$ and $x_2 = y_1 = 0$. ◇

Example 25.4. [[★★★ cone generated by $\bar{\varepsilon}_1 = (2,1)$ and $\bar{\varepsilon}_2 = (1,2)$]] We get $\mathbb{Z}^2/\langle \bar{\varepsilon}_1, \bar{\varepsilon}_2 \rangle = \mathbb{Z}/3$. The acting torus is actually trivial here. The map $U \to Q$ is given by $(t_1, t_2) \mapsto (t_1^2 t_2, t_1 t_2^2)$. This is surjective [[★★★]], with kernel $\{e, (\varepsilon, \varepsilon), (\varepsilon^2, \varepsilon^2)\}$ where $\varepsilon = \sqrt[3]{1}$. The quotient is $\mathbb{A}^2/(\mathbb{Z}/3)$. ◇

Chow quotient and Hilbert quotient

Here's a way to compactify quotients.

Idea: Suppose X is projective and irreducible, and G acts on X. The main problem with the quotient is that there is some open space of good orbits (e.g. stable with respect to some linearization). So in general, there exists some Zariski open set $U \subseteq X$ with the follwoing property. For $x \in U$, $\overline{G \cdot x}$ is a closed subvariety of X, so we can regard it as an algebraic cycle (say its dimension is r). For $x, y \in U$, the closures of the orbits will be rationally equivalent cycles, so they represent the same homology class in $H_r(X, \mathbb{Z})$ [[★★★ we'll prove that there is a U with this property later]].

$U /\!\!/ G \hookrightarrow C_r(X, \delta)$, the Chow variety (the variety of algebraic cycles with cohomology class δ), where $\delta \in H_r(X, \mathbb{Z})$. If you like, you can think about $U /\!\!/ G$ as being in $\mathrm{Hilb}_r(X)$.

Definition 25.5. The *Chow quotient* $X /\!\!/_C G$ is the closure of $U /\!\!/ G$ in $C_r(X, \delta)$. ◇

Note that no linearization is involved so far.

Realization of the Chow variety. Consider $C_r(\mathbb{P}^n, d)$, the variety of subvarieties of $\mathbb{P}^n$ of dimension r and degree d. Then we'll restrict to X. Classically, this was understood with *Chow forms*. Pick $r+1$ linear forms $\ell_0, \dots, \ell_r$ on $\mathbb{P}^n$. For an algebraic cycle $Z = \sum c_i Z_i$, there exists a polynomial $R_Z(\ell_0, \dots, \ell_r)$ with the following properties. $(*)$ $R_{Z_1+Z_2} = R_{Z_1} R_{Z_2}$, and $(**)$ If Z is an irreducible variety, $R_Z(\ell_0, \dots, \ell_r) = 0$ if and only if $Z \cap \{\ell_0, \dots, \ell_r = 0\} \neq \varnothing$.

$C_r(\mathbb{P}^n, d)$ is the projectivization of the space of homogeneous polynomials in $\ell_0, \dots, \ell_r$ of degree d.

Example 25.6. Consider two points in $\mathbb{P}^2$, with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2). A form is of the form $ax+by+cz$. The form R_Z will be $(ax_1+by_1+cz_1)(ax_2+by_2+cz_2)$, which is in the space of quadratic forms in a, b, c. So we are interested in those forms which are the product of two linear forms. This is the condition that it's rank is at most 2. $C_0(\mathbb{P}^2, 2)$ is then the projectivization of the space of quadratic forms with zero discriminant, which is of dimension 4. This is singular when the rank drops to 1, which corresponds to the case when two points coincide. The space of forms of rank 1 is a $\mathbb{P}^2$ embedded by the Veronese map. $\diamond$

If you do the same with Hilbert schemes, what will happen? The Hilbert scheme and the Chow variety will agree when two points do not coincide, but they will be different on the degenerate locus.

Consider $\mathbb{A}^2 \subseteq \mathbb{P}^2$. You are looking for ideals $I \subseteq k[x, y]$ such that the quotient is 2-dimensional. If two points don't coincide, then this ideal is the intersection of two maximal ideals. If the two points coincide, then I cannot be the square of a maximal ideal (the quotient would be 3-dimensional), so the Hilbert scheme records the "direction of collision", whereas the Chow variety doesn't see this. So the Hilbert scheme is non-singular, whereas the Chow variety was singular. The Hilbert scheme in this case is exactly the resolution of the singularities of the Chow variety. In general, there is a map from Hilbert to Chow.

Next we'll show that for any Mumford quotient, you get a map to the Chow quotient. The Chow quotients are difficult to compute in general, but it has been done in the case of toric varieties, for example. This Chow quotient is not a categorical quotient, which is bad, but there are many good sides too.

26 Chow quotients

Next the plan is to talk about Chow quotients. Then we'll need moment maps. That will probably take at least a week.

Let X be a projective variety with an action of a reductive group G. We defined the Chow quotient $X/\!\!/_C G$. There exists a Zariski open set $U \subseteq X$ such that for any two points $x, y \in U$, the orbit closures $\overline{Gx}$ and $\overline{Gy}$ represent the same homology class as algebraic cycles. Such orbits are called generic orbits. You can look on an affine set, then look at invariant functions $f_i(x) = c_i$ cutting out the orbit. If you perturb them a bit, then you'll get the same homology class. You can take U as small as you like, so you can just take the regular orbits. Then you get a geometric quotient U/G. [[★★★ U may not be G-invariant. you just restrict the equivalence relation generated by G to U, and U/G is the quotient by this relation]]

We defined the Chow variety $C_r(X, \delta)$, the projective variety parameterizing all algebraic cycles of dimension r which represent homology class δ. Later we'll give a construction of this variety.

We get an embedding $U/G \hookrightarrow C_r(X, \delta)$. The Chow quotient is defined as the closure of the image. [[★★★ it's probably obvious that the map is quasi-compact and quasi-separated so you get a scheme-theoretic closed image]]

Example 26.1. Consider $\mathbb{A}^4 \subseteq \mathbb{P}^4$ with the action of $k^\times$ given by $t(x_1, x_2, x_3, x_4) = (tx_1, tx_2, t^{-1}x_3, t^{-1}x_4)$. We discussed before that we have three quotients Y_0, Y_+, and Y_-, where Y_0 is the usual affine quotient, given by $z_1z_2 = z_3z_4$. We had two maps $Y_+ \to Y_0$ and $Y_- \to Y_0$, each of which is kind of a "partial blowup" of the cone point. This is related to the fact that we have the decomposition $k^4 = V_+ \oplus V_-$. Depending on the linearization, we get that the semi-stable points are the complement of V_+ or V_-.

What is the Chow quotient in this case? Every orbit $G \cdot x$ can be projected onto V_+ or V_-. It's not hard to see that under each projection, the image is a line. So associated to each orbit are two lines ℓ_1 and ℓ_2, with $G \cdot x \subseteq \ell_1 \oplus \ell_2$. It's actually a hyperbola, given by $uv = c$, where u and v are coordinates on ℓ_1 and ℓ_2. As soon as $c \neq 0$, all these orbits are rationally equivalent, so we take our U to be the union of all of those. If $c = 0$, then the hyperbola degenerates and you get a "limit algebraic cycle". This cycle $\ell_1 + \ell_2$ is actually the union of three orbits, $\ell_i \smallsetminus \{0\}$ and $\{0\}$. Notice that $\ell_1 \smallsetminus \{0\}$ is a semi-stable orbit in one case, and $\ell_2 \smallsetminus \{0\}$ is semi-stable in the other case. Note that the degenerate cross is rationally equivalent to the hyperbola, so the cross is the limit in the Chow variety. The Chow variety is very large. The Chow quotient has a point for each hyperbola and an extra point for the degenerate cross for each pair of lines. Q: you don't have a canonical choice of coordinates on the ℓ_i. A: yes, but it doesn't matter.

So the (affine part of the) Chow quotient Y has maps to Y_+ and to Y_-. [[★★★ why?]] We also have a map $Y \to Y_0$. There is a general theory that these affine maps can be extended. The preimage of the singular point in Y_0 is $\mathbb{P}^1 \times \mathbb{P}^1$. This is in fact the blowup of the singular point of Y_0. It's a good exercise to check what's going on at infinity, but I won't do it now.

There is a paper by Sturmfels-Kapranov-somebody else where they describe Chow quotients for toric varieties in general. $\diamond$

Remark 26.2. Note that the Chow quotient *is not* a categorical quotient. That is, there is no regular morphism $\overline{X} \to \overline{Y}$. In the example, the orbit $\ell_0 \smallsetminus \{0\}$ appears in many limit cycles, so there is no natural way to send it to a single limit cycle. You should also be able to see it from the toric description. $\diamond$

The following is from a paper [[★★★ from Kapranov, Chow quotient of Grassmanians, I.M. Gelfand seminar http://arxiv.org/abs/alg-geom/9210002, Theorem 0.3.1. There is also a paper of Hu]]

Proposition 26.3. *Let X be a smooth projective variety and G a reductive group acting on X. Assume the stabilizer G_x is finite for generic $x \in X$. Moreover, assume that G_x is not unipotent for any $x \in X$. Let $Z = \sum n_i Z_i$ be a cycle in the chow quotient (i.e. a point on the Chow variety). Then every irreducible component Z_i is the closure of exactly one orbit.*

In the example, we have the cross, which has two components, each of which is the closure of an orbit.

Proof. [[★★★ proof in the case of a torus]] Let G be a torus. Suppose the statement is false, so there is some Z_i which is not the closure of a single orbit. Then there is a rational invariant function f on Z_i which is not constant [[★★★ linearize the action, look at lattice of laurent polynomials, so there must be an invariant laurent polynomial because the dimension of the variety is greater than that of the torus]][[★★★ For a torus, you can always find an affine open neighborhood of any orbit. Throw away some orbits to get a non-constant regular function. You can always throw away something so that the action is closed. Cover by affines, and separate orbits by invariants. Invariants on open sets give you rational functions on the whole space.]][[★★★ the invariant rational function on Z can be lifted because it is a ratio of semi-invariants. Semi-invariants can be lifted by complete reducibility; the same reason you can lift invariant regular functions]]. But since G is a torus, the rational function is a ratio of two semi-invariants [[★★★ was one of the exercises]], $f = p(x)/q(x)$, and every semi-invariant is a regular semi-invariants on Z_i. So we can lift them to get $p(x), q(x) \in k[V]^G$ (where $X \subseteq \mathbb{P}(V)$). $Z = \lim_{t\to 0} Z(t)$, where $Z(t)$ is the closure of one orbit, so f is constant on $Z(t)$ for any $t \neq 0$, so by continuity, it must be constant on Z, a contradiction. $\square$

Remark 26.4. In general, the proposition is not true. Consider the diagonal action of $SL(2)$ on $\mathbb{C}^2 \times \mathbb{C}^2 = \{(v_1, v_2) | v_i \in \mathbb{C}^2\}$. We can naturally embed into $\mathbb{P}^2 \times \mathbb{P}^2$ and extend the action in the natural way.

What are the generic orbits? $SL(2)$ preserves determinant and can map one pair to another pair, so we get an invariant $\det(v_1, v_2)$. If v_1 is not proportional to v_2, we get a generic cycle in the Chow quotient, and it degenerates to the case $\det(v_1, v_2) = 0$, in which case we get that the two are proportional, a 3-dimensional set. In this set, we have a 1-parameter family of orbits. $v_1 = \lambda v_2$ is an orbit. Clearly the proposition is not true in this case because the cycle $\det(v_1, v_2)$ cannot be the closure of an orbit because the orbits are 2-dimesional.

The point is that in this case, the stabilizer of $(e_1, \lambda e_1)$ is $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$, which is unipotent. $\diamond$

How do we finish the proof of this proposition? There's a proof with Luna's slice theorem. Pick up a generic point $x \in Z_i$. If the proposition fails, then the dimension of the stabilizer of x is positive by dimension reasons, so there is a non-trivial torus T in the stabilizer since the stabilizer is not unipotent. For the torus, the proposition is already proven. Pick a curve $x(t)$ such that $x(0) = x$ and $x(t)$ is generic (in the sense of Chow quotient) if $t \neq 0$. Now we can tak the Chow quotient with respect to T. Consider the cycle $C = \sum m_i C_i$ which is the limit of $\overline{T \cdot x(t)}$ as $t \to 0$. This cycle is contained in Z. So we can find $y_i \in C_i$ such that $x \in \overline{Ty}$. From this, we get that $x \in \overline{Gy}$, but we have to check that $x \notin Gy$ [[★★★ this is the part I don't have an argument for]]. Then we are done by dimension considerations.

27 The Moment Map

Suppose X is a projective variety with an action of a reductive group G, with finite generic stabilizer, and $Stab(x)$ never unipotent. If $Z \in X /\!\!/_C G$, then each component of Z is the closure of a single G-orbit.

Proof. Suppose not. Let $x \in Z_i$ generic, so its orbit Gx is maximal-dimensional among orbits in Z_i. If the result is not true, we know that $\dim(Stab(x)) > 0$. Since this stabilizer is not unipotent, there is a non-trivial torus $T \subseteq Stab(x)$. Suppose $X \subseteq \mathbb{P}^n$, with the action on $\mathbb{P}^n$ linear. Consider $T_x(\mathbb{P}^n) = T_x(Gx) \oplus N$, with N a T-invariant subspace of $T_x(\mathbb{P}^n)$. Next, take $S = x + N \subseteq \mathbb{P}^n$. By definition, S is transverse to Gx. So there is an open neighborhood $x \in U$ such that S will be transverse to any orbit Gy for $y \in U$. If $y \in Z_i$, then by dimension and transversality, Gy intersects S in finitely many points. On the other hand, S is a T-invariant subvariety by construction, so the action of T on $Z_i \cap S$ is trivial.

Now consider a curve $x(t) \subseteq S \cap X$ with $x(0) = x$ and $Stab(x(t))$ finite for $t \neq 0$ [[★★★ why can we find such a curve in $S \cap X$?]]. Now I define $C(t) = \overline{T \cdot x(t)}$. Outside of Z, T acts with finite stabilizer, so $\dim C(t) = \dim T$. It degenerates to a cycle C, which lies in $Z_i \cap S$ (on which the action of T is trival). But we showed that the components of C must be closures of single T-orbits, a contradiction. □

[[★★★ for transversality, it looks like we really used that X is smooth.]]

Let me introduce the next theorem, which I'll only prove later, after we've done the symplectic moment map.

Theorem 27.1. *Suppose $k = \mathbb{C}$, and X is projective with an action of a reductive group G. Let $L \in \operatorname{Pic}^G X$. Then there exists a morphism of algebraic varieties $X /\!\!/_C G \to X /\!\!/_L G$. If $X^s(L) \neq \varnothing$, then this is morphism is birational.*

[[★★★ btw, can we characterize when Mumford quotients have maps between them? It should be some kind of chamber decomposition of $\operatorname{Pic}^G X$. Is the Chow quotient the fiber product of all the Mumford quotients?]]

Definition of the moment map

Let M be a real symplectic manifold of dimension $2n$ with symplectic form $\omega \in \Omega^2 M$, $d\omega = 0$, and ω non-degenerate at all points. Assume K is a connected real Lie group which acts on M and preserves ω. Since ω is non-degenerate, we get an identification $\omega\colon T^*M \cong TM$. Let $v \in Vect(M)$ such that $L_v(\omega) = 0$ (i.e. v is a Hamiltonian vector field).[1] Then (at least locally), there exists a function H_v such that $\langle dH_v, w\rangle = \omega(v, w)$ for all $w \in Vect(M)$. This follows from the Darboux lemma, which gives us coordinates $p_1, \dots, p_n, q_1, \dots, q_n$ such that $\omega = \sum p_i \wedge q_i$. Then

[1]For a differential k-form α, $d\alpha$ is a $(k+1)$-form and $i_v\alpha = \alpha(v, -)$ is a $(k-1)$-form. $L_v\alpha = i_v d\alpha + d(i_v\alpha)$.

$v = \sum \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}$ (possibly in the other order). If such a function exists globally (e.g. if M is simply connected), then the field is Hamiltonian. Denote the set of such vector fields by $HVect(M)$

$C^\infty(M)$ has a Poisson bracket $\{f, g\} = \omega(df, dg) = \omega(df) \cdot g$, where we're thinking of ω as an isomorphism $T^*M \xrightarrow{\sim} TM$. Then we get a short exact sequence

$$0 \to \mathbb{R} \to C^\infty(M) \xrightarrow[f\mapsto\{f,-\}]{\omega\circ d} HVect(M) \to 0$$

which is in general a non-trivial central extension of Lie algebras.

Let $\mathfrak{k} = Lie(K)$. From the action, we have a map $\mathfrak{k} \to HVect(M)$. Assume we can lift this to a Lie algebra homomorphism $\mathfrak{k} \to C^\infty(M)$. This is always possible, for example, if $\mathfrak{k}$ is a semi-simple Lie algebra (because semi-simple Lie algebras don't have non-trivial central extensions). If $u \in \mathfrak{k}$, we will write L_u for the image in $HVect(M)$. Assume every L_u is Hamiltonian (which will always be true if M is simply connected). [[★★★ notice that the lift is not unique; you can replace all the H_u by $H_u + const$ and still get a lift. Actually, since $\mathfrak{k} \to C^\infty(M)$ is required to be Lie algebra homomorphism, it's only determined up to an element of $\mathrm{Hom}_{Lie}(\mathfrak{k}, \mathbb{R})$. In particular, if $\mathfrak{k}$ is semi-simple, the map is unique]]

Under all these assumptions, we can define a map $\mu\colon M \to \mathfrak{k}^*$, given by $\langle \mu(x), u\rangle = H_u(x)$ for $x \in M$ and $u \in \mathfrak{k}$. [[★★★ This μ is defined only up to shift, since any two lifts $\mathfrak{k} \to C^\infty(M)$ will differ by an element of $\mathrm{Hom}(\mathfrak{k}, \mathbb{R}) = \mathfrak{k}^*$.]]

Remark 27.2. *K acts on $\mathfrak{k}$ by the adjoint representation. It also acts on $HVect(M)$. The map μ is K-equivariant. Infinitesimally, you can write it as $\langle d\mu(x)(\xi), u\rangle = \omega(\xi, L_u)$ for $\xi \in T_xM$.* ◇

Example 27.3. If X is any smooth variety, there is a canonical symplectic variety associated to it, namely $M = T^*X$. If $x_1, \dots, x_n$ are local coordinates on M, then the coordinates on the fibers are $p_i = \frac{\partial}{\partial x_i}$, and $\omega = \sum dp_i \wedge dx_i$. If I have any vector field $v = \sum f_i \frac{\partial}{\partial x_i}$ on X, it automatically induces a vector field on M.

Exercise. *Check that $H_v = \sum p_i f_i$.*

◇

Example 27.4. Let $K = U(n+1)$ act on $\mathbb{C}^{n+1}$. This extends to an action on $\mathbb{P}^n_{\mathbb{C}}$ (thought of as a real manifold!). It is easy to check that the action on $\mathbb{P}^n$ is transitive. By definition, $U(n+1)$ are those transformations that preserve the Hermitian form (v, w) on $\mathbb{C}^{n+1}$. In coordinates, $(v, w) = \sum v_i \overline{w}_i$. Such a form defines a Kahler form on $\mathbb{P}^n$, which can be written in homogeneous coordinates $\sum_{i=0}^n \frac{dz_i\, \overline{dz_i}}{\sum |z_i|^2}$, called the Fubini-Study form. On the imaginary part of $\mathfrak{k}$ this form is a symplectic form ω.

We get a moment map $\mu\colon \mathbb{P}^n \to \mathfrak{k}^*$. $\mathfrak{k}$ is the space of skew-hermitian matrices, which I identify with hermitian matrices ($\overline{A}^t = A$) by multiplying by $\sqrt{-1}$.

If $x \in \mathbb{P}^n$, pick a vector $v \in \mathbb{C}x$. Define $\langle u, \mu(v)\rangle = \frac{\overline{v}^t uv}{\overline{v}^t v}$, where v is regarded as a column vector. It is not hard to see that this is an equivariant map. If $g \in U(n+1)$, then $\langle u, \mu(gv)\rangle = \frac{\overline{v}^t \overline{g}^t ugv}{\overline{v}^t \overline{g}^t gv}$, but $\overline{g}^t g = 1$, so this is $\langle \overline{g}^t ug, \mu(v)\rangle$. ◇

Next time we'll talk about the relation of this stuff to Mumford quotients.

28 Lecture 28

Last time, we considered the case $K = U(n+1)$ acting on $\mathbb{P}^n$ in the standard way. We have a Hermitian form which induces the symplectic form, so we get a moment map $\mu\colon \mathbb{P}^n \to \mathfrak{k}^*$ defined by

$$\langle \mu(v), u\rangle = \frac{(uv, v)}{(v, v)}$$

where we've identified $\mathfrak{k}^*$ with the space of hermitian matrices. The pairing $(-,-)$ is the hermitian form on $V = \mathbb{C}^{n+1}$. In the formula, $v \in V$, and $u \in \sqrt{-1}\mathfrak{k}^*$.

If the group is reductive, we can identify $\mathfrak{k}$ with $\mathfrak{k}^*$ by the Killing form [[★★★ doesn't the group need to be semi-simple for that?]]. In this case, we can write the moment map as $\mu(v)w = w - \frac{(u,w)}{(v,v)}v$, the projection onto the orthogonal complement of v. Here, I'm assuming the form is skew-linear in the first coordinate and linear in the second one.

Since everything we're going to study is related to this example, I want to talk more about it.

If K is a compact group [[★★★ btw, compact groups always complexify to reductive groups]], and V is a complex linear representation of dimension $n+1$. Then by compactness, there is a positive definite hermitian form $(-,-)$ on V. So I get a map $K \to U(n+1)$, giving the dual map $\mathfrak{u}(n+1)^* \to \mathfrak{k}^*$. So for an arbitrary compact group, the moment map is induced by the one in the example.

Consider the case $K = T$, the maximal torus of $GL(V)$, so $\mathfrak{t}_K^* \cong \mathbb{R}^{n+1}$. If I choose an orthonormal basis $\{e_0, \dots, e_n\}$ for V, then

$$\mu(z_0, \dots, z_n) = \frac{(|z_0|^2, \dots, |z_n|^2)}{\sum |z_i|^2}.$$

So the image is the simplex $\{(a_0, \dots, a_n) \in \mathbb{R}^{n+1} | \sum a_i = 1\}$. In general, when K is a torus, the image of the moment map is a convex polytope.

If G is an arbitrary reductive group over $\mathbb{C}$, then by Cartan's theorem, it has one compact real form $K \subseteq G$ (up to conjugation), with $K_{\mathbb{C}} \cong G$. If G is connected, it can be proven that K is connected. See, for example, Helgason's book. Here are the classical cases:

G	K
$\mathbb{C}^\times$	S^1
$SL(n,\mathbb{C})$	$SU(n)$
$SO(n,\mathbb{C})$	$SO(n,\mathbb{R})$
$Sp(2n,\mathbb{C})$	$SU(n,\mathbb{H})$

If X is a projective complex variety, we find an ample line bundle $L \in \operatorname{Pic}^G X$. This gives us an immersion $X \hookrightarrow \mathbb{P}(V)$, which is G-equivariant ($V = \Gamma(X, L^{\otimes N})$ for some N). Since $K \subseteq G$, we get a map $\mu\colon X \to \mathfrak{k}^*$.

Theorem 28.1. *(1) A point $x \in X$ is semi-stable if and only if $0 \in \mu(\overline{G \cdot x})$ [[★★★ we're using the* particular *moment map we defined at the beginning of the lecture;*

actually, we require the map $\mathfrak{k}^* \to C^\infty(M)$ to be a Lie algebra map]]. (2) $\mu^{-1}(0)$ is K-invariant (since 0 is invariant). Every semi-stable orbit either does not meet $\mu^{-1}(0)$ or they meet in a single K-orbit. We get an isomorphism of topological spaces $\mu^{-1}(0)/K \xrightarrow{\sim} X/\!\!/_L G$.

Example 28.2. $G = T$ the maximal torus in $SL(V)$. The the image of the moment map is a simplex whose center is at 0. The preimage $\mu^{-1}(0) = \{(z_0, \dots, z_n) | |z_0| = |z_1| = \cdots |z_n|\}$. This is exactly one K-orbit, since $K = \{diag(t_0, \dots, t_n) | |t_i| = 1\}$. The semi-stable orbit is the open orbit. There are other orbits corresponding to the faces of the simplex. ◇

This theorem gives you a nice criterion for semi-stability, and it reduces to an "easier" case where the group is compact.

The proof of the Theorem is due to Kempf and Ness. Consider $f(v) = (v, v) = \|v\|^2$ as a function on V. Restrict to some orbit $O = G \cdot v$.

Proposition 28.3.

1. *$v \in O$ is a critical point (of the restriction $f : O \to \mathbb{R}$) if and only if $\mu(v) = 0$.*
2. *Every critical point is a minimum.*
3. *If f has a minimum on O, then it attains it at a single K-orbit (f is clearly K-invariant).*
4. *f has a minimum on O if and only if O is closed.*

Proof. (1) Suppose v is a critical point on $O = G \cdot v$. $T_v G \cdot v = \mathfrak{g} \cdot v$, so it is a critical point if and only if

$$\frac{d}{dt}(e^{ut}v, e^{ut}v)\Big|_{t=0} = 0$$

for all $u \in \mathfrak{g}$. We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{k}$ are skew-hermitian matrices and $\mathfrak{m}$ are hermitian. When you calculate the LHS, you only need to consider the linear term in t, so the condition becomes

$$(v + uvt + \cdots, v + uvt + \cdots) = (v, v) + ((uv, v) + (v, uv))t + \cdots$$

But skew-hermitian means that $(av, v) = (v, av)$, so we get that the condition for being a critical point is that $(bv, v) = 0$ for all b, which is equivalent to $\mu(v) = 0$.

The idea for (2-4) i to do it first for the case where G is a torus, and then do the general case using a bit of structure theory. In all these proofs, we can basically reduce to the case of a torus because of the Hilbert-Mumford criterion. So let's consider the case $G = T$.

Then the action is diagonalizable: $t(x_1, \dots, x_n) = (\chi_1(t)x_1, \dots, \chi_n(t)x_n)$. Since the form is invariant under the action of the torus, we can choose the T-eigenbasis to be orthonormal with respect to the form. $T = (\mathbb{C}^*)^m$. We have a surjection

$\mathbb{C}^m \xrightarrow{\exp} T$. We have that $(tv, tv) = \sum |x_i|^2 |\chi_i(t)|^2$, which we regard as a function on $\mathbb{C}^n$. We have $(t_1, \ldots, t_n) = (e^{s_1}, \ldots, e^{s_n})$, so the form looks like $\sum |x_i|^2 |e^{\ell_i(s)}|^2$, where ℓ_i are some linear functions on $\mathbb{C}^m$. This function is invariant with respect to K, so if we write $\mathbb{C}^m = \mathbb{R}^m + \sqrt{-1}\mathbb{R}^m$, the imaginary part does not contribute to the form. If I define $\phi(s) = \|\exp(s \cdot v)\|$, we get the factorization $\phi \colon \mathbb{R}^m \to \mathbb{R}$. The restriction of ϕ to any line (parameterized by τ) is given by $\sum c_i e^{a_i \tau}$. We can see that if the function is not constant on the line, it has a positive second derivative. That is, ϕ is convex when restricted to a line, and strictly convex if it is not constant. How can it happen that ϕ is constant? Only if each ℓ_i is zero. $\phi(s_0 + \tau\vec{r})$ constant implies that $\ell_i(\vec{r}) = 0$ for all $i = 0, \ldots, n$, which implies that $\vec{r}$ is in the stabilizer of $e^{s_0}v$. If v is a critical point, then I know that there are some directions where ϕ is constant, and these will form a subspace. $\phi(\tau\vec{r})$ constant if $\vec{r} \in Stab(v)$, and the function $\phi \colon \mathbb{R}^m / Stab(v) \to \mathbb{R}$ is strictly convex, so it has a single critical point, which then must be a minimum. From this, we get (2) and (3) for a torus. We'll finish proving this theorem next time. $\square$

29 Lecture 29

We had the map $\mu^{-1}(0)/K \to X/\!\!/_L G$. We were working on the criterion for semi-stabilty: x is semi-stable if and only if $\mu(\overline{G \cdot x})$ contains 0. The argument was based on the result of Kempf-Ness that if you consider $f(v) = (v, v) = \|v\|^2$ restricted to an orbit $G \cdot v$, we have

1. If $v \neq 0$, then v is a critical point of f is and only if $\mu(v)$=0. We proved this last time.
2. Every critical point is a minimum.
3. If minimum is obtained on $G \cdot v$, then it obtained on a single K-orbit.
4. A minimum is attained if and only if the orbit $G \cdot v$ is closed.

(2) and (3) were proven as follows. First restrict to the case $G = T$ is a torus. Let $\mathfrak{t} = \mathrm{Lie}(T)$ and $\mathfrak{t}_K$ be the lie algebra of the compact subgroup, and $Stab_{\mathfrak{t}}(v)$ the stabilizer of v. We have

$$\phi\colon \mathfrak{t} \xrightarrow{\exp} T \xrightarrow{f(tv)} \mathbb{R}$$

We have that $\phi = \sum a_i e^{\ell_i(s)}$. Since all the stabilizers are conjugate, it's clear that ϕ is constant along $Stab_{\mathfrak{t}}(v)$ and along $\mathfrak{t}_K$. So I can consider $\overline{\phi}\colon \mathfrak{t}/(Stab_{\mathfrak{t}} v + \mathfrak{t}_K) \to \mathbb{R}$. This $\overline{\phi}$ is not constant along any line, so it has strictly positive second derivative. From this, you get (2) and (3) for a torus.

To get (2) and (3) in general, we need some knowledge about complex reductive groups. See Helgason's book "Symmetric spaces and Lie algebras".

Proposition 29.1. *Fix a redictive group G over $\mathbb{C}$ and a maximal compact subgroup $K \subseteq G$. Let $\mathcal{T}$ be the set of all maximal tori T in G such that $T \cap K$ is a maximal compact subgroup of K. Then $G = \bigcup_{T \in \mathcal{T}} KT$.*

It's pretty clear how we're going to get (2) and (3) for G from this result. Let's prove the Proposition in the case $G = GL(n)$.

Proof. The maximal compact subgroup of $G = GL(n)$ is $K = U(n)$. A maximal torus T can always be diagonalized in $\mathbb{C}^n$. We have that $T \in \mathcal{T}$ if and only if the eigenbasis can be chosen to be orthogonal. The "polar decomposition" says that $X \in GL(n, \mathbb{C})$ can be written uniquely as $X = UH$, where U is unitary and H is positive definite hermitian. This is easy to see: $\overline{X}^t X$ is a positive definite hermitian operator, so take $H = \sqrt{\overline{X}^t X}$ and $U = XH^{-1}$.

But hermitian operators are diagonalizable, and the eigenbasis can be chosen to be orthogonal. $\square$

Now let's prove (2) and (3). Suppose $v, w \in G \cdot v$ and $f(v) = f(w)$ is a minimum. We have $w = g \cdot v$ for some $g = k \cdot t$ where $k \in K$ and $t \in T \in \mathcal{T}$. Since f is

K-invariant, we have that $f(v) = f(tv)$, and for T we know that the theorem is true. So v and tv are in the same K-orbit, so $t \in T_K Stab_v(t)$ ($T_K = T \cap K$), so $w \in K \cdot v$.

If v is a critical point on $G \cdot v$, then it is a critical point on $K \cdot Tv$, where $T \in \mathcal{T}$. It is a minimum on $K \cdot Tv$ for all T, so it is a minimum on $\bigcup KTv = Gv$.

Now let's prove (4). I need another fact about reducitive groups. The main idea is to use the Hilbert-Mumford criterion. Recall we proved that if $G \cdot v$ is not closed, then one can find a 1-parameter subgroup $\lambda(t) \subseteq G$ such that $\lim_{t\to 0} \lambda(t)v = w \notin G \cdot v$. This $\lambda(t)$ lies in some torus T. Assume for now that we may choose $T \in \mathcal{T}$. Choose an orthogonal basis $v_1, \dots, v_N$ for V such that $\lambda(t)v_i = t^{b_i}v_i$ and $b_1 \geq b_2 \geq \dots \geq b_N$. If $v = \sum c_i v_i$, then we see that $c_i = 0$ whenever $b_i < 0$. Then we see that

$$(\lambda(t)v, \lambda(t)v) = \sum |c_i|^2 |t|^{b_i}.$$

with all $b_i \geq 0$. So as t goes to zero, we see that the length of v decreases, so it cannot be a minimum of f. The only way the length doesn't change is if $\lambda(t)v = v$ for all v. So it remains to show that we may choose $T \in \mathcal{T}$ such that $\lambda(t) \subseteq T$.

Every 1-parameter subgroup canonically defines a parabolic subgroup. Consider the adjoint action of G on $\mathfrak{g}$. The action $\mathrm{Ad}_{\lambda(t)}$ on $\mathfrak{g}$ is diagonalizable. It defines a $\mathbb{Z}$-grading on $\mathfrak{g}$, $\mathfrak{g} = \bigoplus \mathfrak{g}_i$, where $\mathfrak{g}_i = \{x \in \mathfrak{g} | \mathrm{Ad}_{\lambda(t)}x = t^i x\}$. We have that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$, so $\mathfrak{p} = \bigoplus_{i\geq 0} \mathfrak{g}_i \subseteq \mathfrak{g}$ is a Lie subalgebra. The corresponding Lie subgroup $P \subseteq G$ is a parabolic subgroup. In this way, you get all parabolic subgroups. The algebro-geometric characterization of parabolic subgroups is that G/P is projective if and only if P is projective.

Proposition 29.2. *For any 1-parameter subgroup $\lambda(t) \subseteq G$, there is an element $g \in P$ (the associated parabolic subgroup) such that $g\lambda(t)g^{-1}$ is contained in some $T \in \mathcal{T}$.*

Proof. Again, let's prove this only in the case $G = GL(n)$. We can choose a basis $e_1, \dots, e_n$ in which $\lambda(t)$ is diagonal, say $\lambda(t)e_i = t^{a_i}e_i$, with $a_1 \geq a_2 \geq \dots \geq a_n$. The problem is that the e_i may not be orthogonal. Define $\tilde{e}_1 = e_1$, $\tilde{e}_2 = e_2 - (e_2, \tilde{e}_1)\tilde{e}_1$, $\tilde{e}_i = e_i - \sum_{j<i}(e_i, \tilde{e}_j)\tilde{e}_j$. This change of basis is given by some upper triangular matrix g, which lies in P. In general, you have to be careful so that you choose g to lie in your algebraic group. □

Now let's prove (4). If the orbit Gv is closed, then f attains a minimum. Suppose f has a minium at v on the orbit $G \cdot v \neq \overline{G \cdot v}$. By the Hilbert-Mumford criterion, we can find $\lambda(t)$ such that $\lim_{t\to 0} \lambda(t)v = w \notin G \cdot v$. Choose g as in the proposition, and you get $\lim_{t\to 0} \lambda(t)g^{-1}v = g^{-1}w$. You can think of the parabolic as a block upper triangular matrix. Where $b_i < 0$, v must have zero coordinates, and when you multiply by by g^{-1}, you don't distrub that. So we get $\lim_{t\to 0} g\lambda(t)g^{-1}v = 0$, and we saw that f being minimum of v implies the 1-parameter subgroup fixes v.

Theorem 29.3.

(a) *$x \in X$ is semi-stable if and only if $0 \in \mu^{-1}(Gx)$.*

(b) $\mu^{-1}(0)/K \to X/\!\!/_L G$ *is a homeomorphism.*

Proof. Recall the definition of semi-stability: $x \in \mathbb{P}(V)$ is semi-stable if and only if $0 \notin \overline{G \cdot v}$, where v is a vector on the line defined by x. The point is that $\overline{G \cdot v}$ has only one closed orbit Z because V is affine. So x is semi-stable if and only if $Z \neq \{0\}$.

Consider f on the orbit closure $\overline{G \cdot v}$. This minimum exists, and by (4), this minimum must be at some $w \in Z$. If $w \neq 0$, then $\mu(w) = 0$ by (1). If $w = 0$, then $0 \notin \mu(\overline{G \cdot x})$ (μ is not defined at 0, so if the minimum of f is at 0, there is no other ciritical point, so by (1), the image doesn't contain 0).

The closure equivalence class of $G \cdot v$ meets $\mu^{-1}(0)$ at most at one K-orbit. It meets it only if the corresponding point $x \in \mathbb{P}(V)$ is semi-stable. So we get an embedding $\mu^{-1}(0) \hookrightarrow X^{ss}(L)$, and we have the map $X^{ss}(L) \to X/\!\!/_L G$. The composition is K-equivariant, so we have a continuous bijection from a compact space to a hausdorff space (use the complex topology on $X/\!\!/_L G$), so it's a homeomorphism. □

Next time, we'll prove convexity of the image of μ in the case G is a torus. Then we'll return to Chow quotients.

30 Lecture 30

Example 30.1. Suppose we have n points in $\mathbb{P}^1$. We haven't discussed this problem yet, but we've considered ordered n-tuples. We have an $SL(2)$ action on $(\mathbb{P}^1)^n$. There are lots of linearizations, but there is one which is symmetric, the one coming from the standard action $(\mathbb{C}^2)^{\otimes n}$ (you can get more via the Veronese embeddings). We have a natural map $(\mathbb{C}^2)^n \to (\mathbb{C}^2)^{\otimes n}$. We checked the stability condition in the case of numbered points, but stability only depends on the connected component of the group, so we have that the semi-stable points are the ones where the multiplicity of each point is $\leq n/2$.

We have $K = SU(2) \subseteq SL(2)$. Let $\nu\colon \mathbb{P}^1 \to \mathfrak{su}(2)^* \cong \mathbb{R}^3$ be the moment map for the action on $\mathbb{P}^1$. Think of $\mathfrak{su}(2)$ as hermitian 2×2 matrices with trace zero. The image of the moment map will be the sphere in $\mathbb{R}^3$. The image of ν will be the set of matrices X such that $\overline{X}^t = X$ and $\operatorname{tr}(X) = 0$. This means that the eigenvalues of X must be $\{1/2, -1/2\}$, and X is of the form $\left(\begin{smallmatrix} a & b \\ \bar{b} & a \end{smallmatrix}\right)$ where $-a^2 - b\bar{b} = -1/4$. This is exactly the usual identification if $\mathbb{P}^1$ with S^2.

So for $(x_1, \dots, x_n) \in (\mathbb{P}^1)^n$, we have $\mu(x_1, \dots, x_n) = \sum \nu(x_i)$. So $\mu^{-1}(0)$ is the set of sets of n points on the sphere such that the sum of them is zero. We have to quotient this by the action of $SU(2)$, which is just given by rotation of the sphere (via the map $K = SU(2) \to SO(3, \mathbb{R})$).

If $n = 4$, we know the quotient is $X/\!\!/G \cong \mathbb{P}^1$, given by the cross-ratio. We want to classify sets of four points $x_1, \dots, x_4 \in S^2$ such that $\sum x_i = 0$, up to rotation. Let $y = x_1 + x_2$ and $x_3 + x_4 = -y$. We can rotate so that y perpendicular to the equator. Fixing the vertical axis, we can still rotate x_3 to be on some fixed meridian. Under these conditions, everything is determined uniquely by x_1. It lies on some line of latitude, so x_2 must be on at the same latitude, and opposite longitude. Now we know that x_3 is on the meridian, and x_4 must be at the opposite longitude. $\diamond$

Convexity Theorems

Suppose M is a compact connected symplectic manifold and $T \cong (S^1)^n$ is a real torus acting on M, preserving the symplectic form ω. Suppose we have a moment map $\mu\colon M \to \mathfrak{t}^* = \mathbb{R}^n$.

Theorem 30.2.

(a) *$\mu^{-1}(c)$ is connected for all c.*

(b) *$\mu(M)$ is convex.*

(c) *Let $Z = \{x \in M | d\mu(x) = 0\} = Z_1 \sqcup \cdots \sqcup Z_N$ (the Z_i are the connected components). Each Z_i is a non-singular manifold, and $\mu(Z_i) = c_i$ is a single point. Moreover, $\mu(M)$ is the convex hull of the c_i. In particular, $\mu(M)$ is a convex polytope.*

Theorem 30.3. *Suppose M is a complex variety with Kähler structure, and $T_{\mathbb{C}}$ is a complex torus acting on M such that the real torus T_K preserves the Kähler structure. Given $y \in M$, let $Y = T_{\mathbb{C}} \cdot y$ and $\overline{Y}$ its closure. Suppose a moment map μ exists.*

(a) $\mu(\overline{Y}) = P$ is a convex polytope.

(b) If S is an open face of P, then $\mu^{-1}(S)$ is exactly one orbit of $T_{\mathbb{C}}$ whose complex dimension is equal to the real dimension of S.

(c) $\mu\colon \overline{Y}/T_K \to P$ is a homeomorphism of topological spaces.

It is clear that a projective toric variety is exactly such a situation. On the other hand, we have the fan of the variety. The relationship between the fan Σ of $\overline{Y}$ and P is that Σ is the dual fan of P.

We can regard $P \subseteq \mathbb{R}^n$ and $\Sigma \subseteq (\mathbb{R}^n)^*$. For a face S of P, the corresponding cone in Σ is $\sigma = \{\lambda \in (\mathbb{R}^n)^* | \lambda \text{ attains its maximum (on } P) \text{ at } S\}$. The moment map is not unique (because you can shift by scalar of scale), but the fan is uniquely determined. For example, [[★★★ picture for $\mathbb{P}^2$, but equalateral rather than what I'm used to]]. It is often more conviniant to work with the polytope rather than the fan.

Let $\mathcal{H}_n$ be the $n \times n$ hermitian matrices. We have a natural action of $U(n)$ (since $\mathcal{H}_n$ is naturally $\mathfrak{u}(n)$), given by $g \cdot X = g^{-1}Xg = \bar{g}^t Xg$. The orbits are given by eigenvalues $(\lambda_1, \dots, \lambda)n)$. Each orbit M is a real manifold, but it's actually a complex manifold. The isotropy group is $U(n_1) \times \cdots \times U(n_k)$, where $n = n_1 + \cdots n_k$ and the n_i are the multiplicities of the eigenvalues. So the quotient by the isotropy group is a flag variety. What is the moment map of the orbit M [[★★★ with respect to the action of $T^n = \{diag(e^{i\theta_1}, \dots, e^{i\theta_n})\} \subseteq U(n)$]]? It is just the projection onto the diagonal: $\mu(X) = (x_{11}, x_{22}, \dots, x_{nn})$. We get that $\mu(M)$ is the convex hull of $\{(\lambda_{s(1)}, \dots, \lambda_{s(n)} | s \in S_n\}$. Such convex hulls are called permutohedrons. These orbits are actually all the same variety. You have a lot of choice in the eigenvalues, but all the combinatorial and topological characteristics will be the same. All that matters is the multiplicities $n_1, \dots, n_k$.

Let's start the proof of the first theorem.

We have (M, ω) and $f_1, \dots, f_n$ are commuting functions (in the sense that $\{f_i, f_j\} = 0$) such that the corresponding vector fields $L_{f_i} = \omega(df_i)$ generate some torus $\operatorname{Lie} T = \mathfrak{t}$ (which may have larger dimension than n. [[★★★ the flow of the Hamiltonian may not correspond to a *closed* 1-parameter subgroup]]. Then we have $f\colon M \to \mathbb{R}^n$ factoring as $M \xrightarrow{\mu} \mathfrak{t}^* = \mathbb{R}^m \xrightarrow{projection} \mathbb{R}^n$. [[★★★ now we've started denoting the dimension of the torus in the theorem by m.]]

Now we can rephrase the theorem as

(a) $f^{-1}(c)$ is connected.

(b) $f(M)$ is convex.

If $n = 1$, then (b) is trivial. In general, $(a) \Rightarrow (b)$. Suppose by induction that I've proven it up to n. Now given $g\colon M \xrightarrow{f} \mathbb{R}^{n+1} \xrightarrow{\pi} \mathbb{R}^n$ for any projection π. By induction, the preimage under g of any point is connected. We have $f(M) \cap \pi^{-1}(c) = f(g^{-1}(c))$ is connected because f is continuous and $g^{-1}(c)$ is connected. But $\pi^{-1}(c)$ is a line. But this means that $f(M)$ is convex.

Now let's do the proof of (a) in the case where $n = 1$. For this, we use some Morse theory. f is *non-degenerate* (in the Morse sense) if it has isolated critical points and the hessian $D^2 f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ is non-degenerate.

If the signature of the hessian $D^2 f$ is never 1 or $n-1$, then the preimage $f^{-1}(c)$ is always connected. This is because the preimage is a quadric which can only have two connected components if the signature is ± 1. The basic idea is to prove that the Hamiltonian is a non-degenerate Morse function.

31 Convexity Theorems

Proposition 31.1. *Let M be a compact connected symplectic variety. Suppose $f \in C^\infty(M)$ such that $\omega(df) = L_\zeta \in Vect(M)$ where $\zeta \in \mathfrak{t}_f$ (the Lie algebra of some compact torus T with a Hamiltonian action on M). Let $Z = \{x \in M | df(x) = 0\} = Z_1 \sqcup \cdots \sqcup Z_N$, where each Z_i is a connected component of the cricical locus Z. Then*

(a) Each Z_i is a smooth manifold.

(b) For $x \in Z_i$, the hessian $D^2 f(x)|_{T_xM/T_xZ_i}$ is a non-degenerate quadratic form.

(c) The form in (b) has even signature (even number of positive eigenvalues, so also an even number of negative eigenvalues, since $\dim M$ is even).

(d) The restriction of ω to Z_i is non-degenerate, so Z_i is symplectic.

Remark 31.2. Say K is compact (or reductive, in the algebraic case), and let Z be the set of fixed points. Then Z is non-singular. To see this, consider the action $\sigma\colon K \times M \to M$ and $d\sigma(x)\colon \mathfrak{k} \oplus T_xM \to T_xM$. For a fixed point $x \in Z$, the group (and the Lie algebra) acts on the tangent space. The tangent space T_xZ is just $(T_xM)^{\mathfrak{k}} = (T_xM)^K$. This proof will work in the algebraic setting if the group is reductive. We have a decomposition $T_xM = (T_xM)^{K_0} \oplus N$, where N is the normal bundle to Z. If x is in a connected component of Z, $\dim(T_xM)^{K_0}$ is fixed because representations of a reductive or compact group are discrete, so the dimension of the normal bundle is fixed, so Z is smooth. This proves part (a). ◇

Proof. Let $\mathfrak{t}_f = \operatorname{Lie} T_f$. Then Z is fixed by T_f since the 1-parameter subgroup generated by L_ζ fixes Z.

Let $x \in Z_i$, so $T_xM = T_xZ \oplus N$. We know that ζ infinitesimally preserves the form because the action is symplectic: $\omega(\zeta v, w) + \omega(v, \zeta w) = 0$. We also know that the eigenvalues of ζ are purely imaginary because if we exponentiate, we get a unitary matrix. Consider $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$. If $\zeta v = pv$ and $\zeta w = qw$ are eigenvectors, then $\omega(v, w) \neq 0$ implies that $p + q = 0$. So we get a decomposition (after complexifying) $V_\mathbb{C} = V_0 \oplus V_1 \oplus \cdots \oplus V_k$ where $V_0 = \ker \zeta$ and $\omega(V_i, V_j) = 0$ for $i \neq j$. So $\dim V_i = 2$ and the restriction of ζ is $\left(\begin{smallmatrix} p_i & 0 \\ 0 & -p_i \end{smallmatrix}\right)$. $T_xM = T_xZ \oplus V_1^\mathbb{R} \oplus \cdots \oplus V_k^\mathbb{R}$. Since p_i was purely imaginary, the action of ζ on $V_i^\mathbb{R}$ is $\left(\begin{smallmatrix} 0 & q_i \\ -q_i & 0 \end{smallmatrix}\right)$ where $q_i = \sqrt{-1}p_i$; we can choose a basis so that $\omega(e_1, e_2) = 1$. Locally, we can choose coordinates $x_1, \dots, x_k, y_1, \dots, y_k$ such that $H_\zeta = \sum_{i=1}^k q_i(x_i^2 + y_i^2) + O(higher)$, so we know the terms that contribute to the signature of the hessian, and we see that the signature is even. Something about $q_i x_i \frac{\partial}{\partial y_i} - q_i y_i \frac{\partial}{\partial x_i}$. From the decomposition of T_xM, we see that the restriction of ω to T_xZ is non-degenerate since $\omega(T_xZ, V_i) = 0$. □

So we've proven that the preimage of a point under the moment map of a projection to a 1-dimensional space is connected.

Proof of Theorem. Assume the preimage of a point is connected for n commuting Hamiltonians. Suppose we have $n+1$ commuting Hamiltonians $f_1, \ldots, f_{n+1}$. We want to show $f_1^{-1}(c_1) \cap \cdots \cap f_{n+1}^{-1}(c_{n+1})$ is connected. We know that $N = f_1^{-1}(c_1) \cap \cdots \cap f_n^{-1}(c_n)$ is T-invariant and connected. We want to show that $f_{n+1} \colon N \to \mathbb{R}$ satisfies (a), (b), and (c). Since the number of connected components of the fiber is semi-continuous, it is enough to prove the result for a generic c_{n+1}. So we may assume $df_1(x), \ldots, df_n(x)$ are linearly independent for $x \in N$ (which we can assume because they generate the torus; if they aren't independent generically, we can remove one of them). Let $x \in N$ be a critical point of $f_{n+1}|_N$, so $df_{n+1}(x) + \sum_{i=1}^n b_i df_i(x) = 0$. Consider $\phi = f_{n+1} + \sum b_i f_i$. Define $Z_\phi = \{y \in M | d\phi(y) = 0\}$. Then by definition, $x \in N \cap Z_\phi$. By transversality, we have that $T_x(N \cap Z_\phi) = T_x N \cap T_x Z_\phi$. $\omega(df_1)(x), \ldots, \omega(df_n)(x) \in T_x N$ because these are vector fields of the action of the group and N is T-invariant. Since the df_i are linearly independent, they are linearly independent when restricted to $T_x N$. All the arguments in the proof of the Proposition can now be repeated. ϕ is the hamiltonian of some vector field: $\omega(d\phi) = L_\zeta$. This ζ has purely imaginary eigenvalues and xxxx, so the argument works the same way.

Once you know that the indexes of critical points are even, you get connectivity of the fibers. This is some result from Morse theory (it's work of Bott?). If you have an isolated critical point, when you pass a critical point, it looks like $x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2 = \varepsilon$, whose fibers are connected unless $k = 1$ or $k = n-1$ (depending on the sign of ε).

To finish the Theorem, we have the moment map $\mu \colon M \to \mathfrak{t}^* = \mathbb{R}^n$. We have (a) $\mu^{-1}(c)$ is connected. From this, we get (b) $\mu(M)$ is convex as we explained last time. Moreover, (c) if $Z = Z_1 \sqcup \cdots Z_N = \{x \in M | d\mu(x) = 0\}$, then $\mu(Z_i) = c_i$ (a single point!), and $\mu(M)$ is the convex hull of the c_i.

Recall that $\mu = (f_1, \ldots, f_n)$, where the f_i are commuting Hamiltonians. Take an arbitrary linear combination $f = \sum b_i f_i$ for $b_i \in \mathbb{R}$. If this combination is generic, then $T_f = T$ and $df = 0$ if and only if $d\mu = 0$. That means that the Z_i go to one point since they are fixed points of f. And $\max_{m \in M} f$ is attained at one of the Z_i. So $\max_{x \in \mu(M)}(b_1 x_1 + \cdots b_n x_n)$ is attained at some c_i. So we have a convex set, and for a generic linear functional, the maximum is attained at one point, and there are a finite number of such points. This implies that the convex set is a polytope, the convex hull of those points. □

Let me talk a little bit about the second theorem. Suppose M is a complex variety (think $\mathbb{P}^n$) with Kähler structure. Let T be a complex torus acting on M, preserving the Kähler structure. Again we get a moment map. For $y \in M$, consider $\overline{Y} = \overline{Ty}$, which is a toric variety. In most cases, this $\overline{Y}$ is singular, so we can't use the previous theorem. The moment map $\mu \colon \overline{Y} \to \mathfrak{t}_K^*$ (the Lie algebra of a compact form of the torus). Then we get a bijection between faces of the image of μ and T-orbits in $\overline{Y}$. And the quotient $\overline{Y}/T_K$ is homeomorphic to the polytope $\mu(\overline{Y})$.

The point is that a Kähler manifold has a symplectic structure and a Riemannian structure. We have two isomorphisms $\omega, g : T^*M \to TM$ and $\omega^*, g^* \colon TM \to T^*M$, with $\omega \circ \omega^* = -1$ (since it is skew-symmetric) and $g \circ g^* = 1$ (since it is symmetric). The Kähler condition says that $J = g \circ \omega^* = \omega \circ g^*$ and it satisfies $J^2 = -\operatorname{id}$.

We can consider T as the product of T_K, the compact torus, and H. For $T = \mathbb{C}^\times$, $H = \mathbb{R}^+$ and $K = S^1$. Consider $\mu \colon M \to \mathbb{R}^n$. Acting by K doesn't change the image under μ, but acting by an element of H moves you around in the image of μ. For some $h \in \operatorname{Lie} H$ and $y \in \overline{Y}$, $\exp(hs)y$ is a curve in M whose image we can look at under μ. For generic h, the curve goes to a vertex. But if h is perpendicular to a face, the curve flows to the face.

32 The moment map for toric varieties

Bott-Morse theory reference for the fact we didn't prove last time. Non-degenerate critical manifold (Annals of Mathematics 60 1954)

We have a Kähler manifold M (has compatible symplectic form ω and Riemannian form g). Let T be a complex torus acting on M so that the unique maximal compact subgroup K preserves the Käher structure. Suppose we have a moment map $\mu\colon M \to \mathfrak{k}^*$.

Let $Y \subseteq M$ be a T-orbit, and let $\overline{Y}$ be its closure. Now consider $\mu\colon \overline{Y} \to \mathfrak{k}^*$. If the action is algebraic and M is a smooth algebraic variety, $\overline{Y}$ may still be a singular variety.

We can write $T = H \cdot K$, where $K \cong (S^1)^n$ ($n = \dim_{\mathbb{C}} T$) and $H \cong (\mathbb{R}_{>0})^n$. We get a corresponding decomposition of the Lie algebra $\mathfrak{t} = \mathfrak{k} \oplus \mathfrak{h}$. If we regard $\mathfrak{t}$ as a complex space, $\mathfrak{k} = \sqrt{-1}\mathfrak{h}$, so we may identify the two: $\mathfrak{h} \cong \mathfrak{k}$.

Without loss of generality, we may assume T acts with finite stabilizer on Y, because we can quotient by the connected component of the identity of the stabilizer (which is a torus, so the quotient is a torus). Then the stabilizer T_y is a finite group, so it must be in K. [[★★★ because K is the maximum compact subgroup of T, not just a maximal compact subgroup.]]

More or less by definition of the moment map, $\langle d\mu_x(v), a\rangle = \omega_x(v, L_a(x))$, where $x \in M$, $v \in T_xM$ and $x \in \mathfrak{k}$. We have $\omega_x(v, L_a(x)) = g_x(Jv, L_a(x))$, where J is the complex structure on M. Using the identification of $\mathfrak{h}$ with $\mathfrak{k}$, we have the relation $\langle d\mu_x(v), h\rangle = g_x(v, L_h)$, where $h \in \mathfrak{h}$ (regarded as being in $\mathfrak{k}$ by $\sqrt{-1}$).

Consider the function $\phi_h(x) = \langle \mu(x), h\rangle$. On the other hand, I can construct the flow $\exp(hs)x$, where $x \in M$ and s is a parameter. This will be the gradient flow of ϕ_h. This is because

$$\frac{d}{ds}\langle \mu(\exp(hs)x, h\rangle\bigg|_{s=0} = g_x(L_h, L_h).$$

This will bring you to a (local) maximum of ϕ. Now let $y \in Y$. Define $y^h = \lim_{s\to\infty} \exp(hs)y$. This limit exists because we are on a projective variety, and this point is a local maximum of the functions ϕ_h. Q: it might happen that you flow to some critical point which is not a maximum. A: something about being able to approach things using 1-parameter subgroups. For now, let's just say it's a critical point. I think it won't matter for the proof.

If $t \in T$, then $(ty)^h = t \cdot y^h$. If h is chosen generically, then $\sqrt{-1}h$ will generate all of K. Then y^h must be a fixed point of K, h must be in $Stab_{\mathfrak{h}}(y^h)$, and ϕ_h must be maximal. In this situation y^h is independent of y because we are working on the closure of a single torus orbit (you can flow to any point from the big T-orbit), so y^h must be a maximum of ϕ_h. $\{y^h | y \in Y\}$ is a single T-orbit.

$\langle \mu(x), h\rangle = \phi_h(x)$, when h varies, this has maxima at finitely many points. Conider $\langle a, h\rangle$, a linear functional associated to some $a \in \mathfrak{h}$. The maximum (on $\mu(\overline{Y})$) of this function is attained on $\mu(\{\text{fixed points of } T \text{ in } \overline{Y}\})$. There are finitely many points

$\mu(Z_i \cap \overline{Y})$ (the Z_i are connected components of the critical locus of μ). $\mu(\overline{Y})$ lies in the convex hull P of the points $c_i = \mu(Z_i \cap \overline{Y})$.

Now pick $y \in Y$. Consider the composition $\nu \colon \mathfrak{h} \xrightarrow{h \mapsto (\exp h)y} H \cdot y \xrightarrow{\mu} \mathfrak{h}^* \cong \mathfrak{h} \cong \mathbb{R}^n$ (with the standard inner product). Then $d\nu_y = d\mu_y$. The first map is a diffeomorphism. $\mu(H \cdot y) = \mu(T \cdot y)$ because the moment map is K-equivariant. $d\nu$ is therefore invertible, and ν is a diffeomorphism onto its image. The pre-image of every point is just one point. If there were two points, there's a 1-parameter subgroup that joins them, and ϕ_h increases along 1-parameter subgroups.

Claim. *$\mu(Y) = P \smallsetminus \partial P$.*

We may assume that $\mu(y) = 0$ by shifting if neccesary. We take $h \in \mathfrak{h}$ such that $\|h\| = 1$. then $(\mu((\exp hs)s), h)$ is an increasing function (it's the restriction of ϕ_h to a Hamiltonian flow), and the limit as $s \to \infty$ will be $\mu(y^h) \in \partial P$.

Let $r = dist(0, \partial P)$. Then $\|y^h\| \geq r$. By Cauchy-Schwartz, $\|\mu((\exp hs)y)\|$ is big. So for some $s = s_h$, we have $\|\mu((\exp hs)y)\| = r/2$. So we get a star-shaped neighborhood of the origin $0 \in U \subseteq \mathfrak{h}$ such that $\mu(\partial U)$ is the sphere of radius $r/2$. So $\mathrm{im}(\nu)$ contains a ball of radius $r/2$ with center $\mu(y)$. Now I can move the image around, so this is true for any point inside of P, so the image is the whole interior. You pick a point in the interior, look at the distance to the boundary, and you can show that the ball of half the radius is in the image.

We have that $\mu^{-1}(\mu(y)) = Ky$ is a single K-orbit because we have a diffeomorphism and $Y = H \cdot Ky$. If $\alpha \in \partial P$, we have that $\mu^{-1}(\alpha)$ is connected, and therefore a single K-orbit. This tells us that the map $\overline{Y}/K \to P$ is a homeomorphism of topological spaces.

$$\begin{tikzcd} \overline{Y} \arrow[d] \arrow[dr, "\mu"] & \\ \overline{Y}/K \arrow[r] & P \end{tikzcd}$$

We need a bijection between T-orbits on $\overline{Y}$ and faces of P. If $S_h \subseteq P$ is a face, let $h \in \mathfrak{h}$ be such that the maximum of (h, ζ) (as ζ runs over P) is reached on S, I claim $S = \{\mu(y^h) | y \in Y\}$. You might think there are several T-orbits going to the same face, but then the fibers of boundary points would have multiple K-orbits.

Now let's check that the dimensions agree. Given $Tz \in \overline{Y}$, we can repeat all the arguments inductively. But for the big orbit, we know that the dimensions agree.

We see also that the polytope is dual to the fan, exactly because $S_h = \{\mu(y^h) | y \in Y\}$. The dual fan to $\mu(\overline{Y})$ is Σ, where $\sigma_S = \{h | \max_{\zeta \in P}(h, \zeta)$ is attained at the face $S\}$.

Example 32.1. Consider the grassmannian $Gr(4, 2)$ of 2-dimensional subspaces of $\mathbb{C}^4$. Let T be the maximal torus in $SL(4)$ (which is 3-dimensional). Consider the grassmannian as 2×4 matrices (z_{ij}) of full rank modulo the action of $GL(2)$. We have plüker coordinates $p^{ij} = \det\left(\begin{smallmatrix} z_{1i} & z_{1j} \\ z_{2i} & z_{2j} \end{smallmatrix}\right)$. The equation for $Gr(4,2)$ in $\mathbb{P}^5$ is $p^{12}p^{34} - p^{13}p^{24} + p^{14}p^{23} = 0$. We get the moment map

$$\mu((z_{ij})) = \frac{\sum |p^{ij}|^2(\varepsilon_i + \varepsilon_j)}{\sum |p^{ij}|^2}$$

For the standard $\varepsilon_1, \dots, \varepsilon_4 \in \mathbb{R}^4$. The fixed point of the torus action correspond to choosing two of the columns to be zero. The image of the moment map is an octahedron [[★★★ once you project out one dimension, which corresponds to quotienting the 4-dimensional torus by the 1-dimensional torus that acts trivially ... the image of the moment map lies in the plane $x+y+z+w=3$.]], with the vertices corresponding to opposite plücker coordinates.

Since you know that the orbits are faces and the vertices are fixed points, the stable T-orbits are the ones which correspond to the whole octahedron. The semi-stable orbits are the ones that correspond to either a pyramid or a square slice [[★★★ how does a pyramid correspond to an orbit? It's not a face.]] [[★★★ We're applying the Hilbert-Mumford criterion ..., what's the relationship between possible supports of points and the image of the moment map?]]. In principle, it could happen that you get the origin on a single edge, but that never happens, because such an edge would correspond to two of the three terms in the relation being zero, and this cannot happen without the third also being zero. Unstable orbits correspond to triangular faces, edges, and vertices.

We will see that this is equivalent to the question of 4-points in $\mathbb{P}^1$. ◇

33 Moment maps and GIT quotients

There will be no class Friday, Monday, Wednesday, but we'll have two extra classes after Nov. 4. I'll send out an email.

Let's recall some things. Suppose G is a reductive group acting on a projective variety X, and $L \in \operatorname{Pic}^G X$, which automatically gives an equivariant embedding $X \hookrightarrow \mathbb{P}^n$, and we have a canonical map $\bar{\mu}\colon \mathbb{P}^n \to \mathfrak{su}(n+1)^*$. Let $K \subseteq G$ be a maximal compact subgroup, so we get a map $\mathfrak{su}(n+1)^* \to \mathfrak{k}^*$. Composing, we get a map $X \to \mathfrak{k}^*$. We have $\bar{\mu}(z)(v) = v - \frac{(z,v)}{(z,z)} z$ for $z, v \in \mathbb{C}^{n+1}$. We showed that $X /\!\!/_L G \cong \mu^{-1}(0)/K$ is an isomorphism of topological spaces.

In the case $G = T \subseteq (\mathbb{C}^*)^{n+1}$ is the torus. Then I have $\mu\colon X \to \mathbb{P}^n \to \mathbb{R}^{n+1} \cong ((\mathbb{C}^*)^{n+1})^\vee_\mathbb{R} \xrightarrow{\gamma} T^\vee_\mathbb{R}$, where $T^\vee_\mathbb{R} = T^\vee \otimes_\mathbb{Z} \mathbb{R}$. If $X = \overline{T \cdot x}$, then $\mu(X) = P$ is a convex polytope. We have an isomorphism of posets between orbits and faces of the image of μ. The embedding $T \subseteq (\mathbb{C}^*)^{n+1}$, we have the characters $\chi_0, \dots, \chi_n$. We choose the standard basis $\varepsilon_1, \dots, \varepsilon_n$ in $\mathbb{R}^{n+1}$, then the formula for $\bar{\mu}$ is

$$\bar{\mu}(z) = \frac{\sum |z_i|^2 \varepsilon_i}{\sum |z_i|^2}.$$

In particular, $\bar{\mu}(\mathbb{P}^n) = \Delta_n$ is the convex hull of $\varepsilon_0, \dots, \varepsilon_n$. So we have $X \to \mathbb{P}^n \to \Delta_n \xrightarrow{\gamma} P$, with $\gamma(\varepsilon_i) = \chi_i$. We have $P = \mu(X)$ is the weight polytope or moment polytope of X. It is the convex hull of $\chi_0, \dots, \chi_n$. Some of these characters may be interior points. The ones on the outside are the ones that correspond to fixed points of the action.

Suppose $H \subseteq T$ is a subtorus. We want to consider $X /\!\!/_L H$, which has an action of T. It is a toric variety with torus T/H. How do we relate the polytopes of these two toric varieties? Under the moment map, the moment polytope will have vertices in the lattice of characters. The inclusion $H \subseteq T$ induces a map $\pi\colon T^\vee_\mathbb{R} \to H^\vee_\mathbb{R}$. Let $Q = \pi(P)$.

Proposition 33.1. *$X /\!\!/_L H$ is a toric variety with moment polytope equal to $\pi^{-1}(0)$. If $0 \notin \pi^{-1}(0)$, then the geometric quotient is trivial.*

Proof. Consider $\tilde{\mu}\colon X \to H^\vee_\mathbb{R}$. By the theorem of Kirwan, if $K_H \subseteq H$ is a maximal compact subgroup, $\tilde{\mu}^{-1}(0)/K_H \cong X /\!\!/_L H$. Now consider the group that preserves the fiber of $\tilde{\mu}$ over 0. First of all, all of K_H preserves the fiber. In fact, $Stab_T(\tilde{\mu}^{-1}(0)) = K \cdot S$, where $S = T/H$. Pick any point $x \in \tilde{\mu}^{-1}(0)$. The restriction $\mu\colon \overline{S \cdot x} \to \widehat{H}^\perp_\mathbb{R} = \{\chi \in G^\vee | \chi|_H = 1\}$. Now I can use the theorem again to say that this is the moment map restricted to the fiber. □

Suppose we've fixed $L \in \operatorname{Pic} X$, but I allow the linearization to vary (by a character). We get for each $\chi \in \widehat{H}$ an element $L_\chi \in \operatorname{Pic}^G X$. We can also take $L^{\otimes m}$ for any m, so I can take $\chi \in \widehat{H}_\mathbb{Q}$. If $m\chi \in \widehat{H}$, we can take the linearization χ on $L^{\otimes m}$. [[★★★ hats and checks are the same thing, right?]]

Proposition 33.2. *$X/\!\!/_{L_\chi} H$ is a toric variety whose fan is the dual fan to $\pi^{-1}(\chi) \cap P$.*

Corollary 33.3. *The quotient $X/\!\!/_{L_\chi} H$ is non-degenerate (has stable orbits, so has the right dimension) if and only if $\pi^{-1}(\chi) \cap P$ has full dimension, or equivalently, χ is an interior point of Q.*

It's clear that you get chambers in Q where the preimage polytope is combinatorially the same. Therefore, we get a nice unerstanding of how the quotient changes based on linearization. Note that we've fixed L. If we change L, P changes. In the case $X = \mathbb{P}$, then there is only one choice of L, so we get a complete understanding in that case.

Proof of Proposition. $\tilde{\mu}_m \colon X \to \mathbb{P}^n \to \mathbb{P}(\mathrm{Sym}^m(\mathbb{C}^{n+1})) \to \mathbb{R}^{n+1} \to \widehat{T}_{\mathbb{R}}P \to \widehat{H}_{\mathbb{R}}$. Taking the Veronese map just gives you $m \cdot \Delta_n$, which maps to $m \cdot Q$. We have $\varepsilon_i \mapsto \chi_i + \chi$, so $m\varepsilon_i \mapsto m(\chi_i + \chi)$. So taking the preimage of chi under this map $\tilde{\mu}_m$ of zero is $m \cdot \tilde{\mu}^{-1}(\chi)$. □

Chow quotients

Next time I'll talk about this paper of Kapranov, Sturmels, somebody else. For now, let me prove a result I promised you before.

Let X be a complex projective variety (though I think it works for characteristic zero, since you may be able to reduce to the case of a subfield of $\mathbb{C}$). We can talk about algebraic cycles of homology class $\delta \in H_{2k}(X, \mathbb{Z})$. The space of these cycles is denoted $C_k(X, \Delta)$, which turns out to be a projective variety. Pick a generic orbit of a reductive group G, and consider it's closure $\overline{G \cdot x}$. For generic x, we get $\overline{G \cdot x} \in C_k(X, \delta)$. Remember that the Chow quotient is the closure of the image of this open set of X under the map $x \mapsto \overline{G \cdot x} \in C_k(X, \delta)$. Note that this definition of the quotient does not depend on any linearization.

Theorem 33.4. *If $L \in \mathrm{Pic}^G X$ is a non-degenerate linearization (i.e. there exist stable orbits), then there is a regular birational map $X/\!\!/_C G \to X/\!\!/_L G$.*

It's clear that if I take the closures of stable orbits, I get a bijection. Now consider a limit cycle. Let $Z(t) \in C_k(X, \delta)$, with $Z(t) \in U$ for $t \neq 0$ (these are irreducible), and $Z(0)$ is a limit cycle. [[★★★ I should probably say that G is connected.]] The support of $Z(0)$ contains at least one semi-stable orbit. For this I just have to check that $0 \in \mu(Z(0))$. But $0 \in \mu(Z(t))$ for any $t \neq 0$, so by continuity of μ, we get $0 \in \mu(Z(0))$. It might contain several semi-stable orbit, but they all go to the same point of $X/\!\!/_L G$. That is, any equivariant rational function must take the same value on all these orbits. To see this, suppose it is false. That is, suppose there is a rational invariant function f which takes different values on different obrits. But f is constant on $Z(t)$ for $t \neq 0$, so by continuity, we get that $f|_{Z(0)}$ must be constant.

34 Chow quotients in the toric case

I put some exercises on the web which you can do while I'm away.

Recall that we have the situation that X is a projective toric variety, and we consider the quotient $X/\!\!/H$, where $H \subseteq T$ is a subtorus. We have the map π from $P \subseteq \widehat{T}_{\mathbb{R}}$ to $Q \subseteq \widehat{H}_{\mathbb{R}}$.

We first define the *fiber polytope* $\Sigma(P,Q)$. We define a volume form dqq on $\widehat{H}_{\mathbb{R}}$ (such that the volume of the lattice parallelopiped is 1). Let $s\colon Q \to P$ be a continuous section, then we get $I_s = \int_Q s(q)\,dq \in \widehat{T}_{\mathbb{R}}$. Define $\Sigma(P,Q) = \{I_s | s$ a continuous section$\}$. This polytope is contained in the fiber over the center of mass of Q. The main result is that this fiber polytope is the polytope of the Chow quotient.

Proposition 34.1. *The moment polytope of $X/\!\!/_C H$ is $\Sigma(P,Q)$.*

Proposition 34.2 (Billera-Sturmfels)**.** *The dual fan to $\Sigma(P,Q)$ is the common refinement of all dual fans to polytopes that arrise as fibers of $\pi\colon P \to Q$.*

For the proof of 34.1, we can restrict to the case $X = \mathbb{P}^n$ and $T = (\mathbb{C}^\times)^{n+1}$. In general, we have $\Delta_n \xrightarrow{\gamma} P \xrightarrow{\pi} Q$, and the notion of fiber polytope is well-behaved. So we may assume $\Delta_n = P$.

$\pi\colon \Delta_n \to Q$. We have the corners $\varepsilon_0, \dots, \varepsilon_n$, which are mapped to $\pi(\varepsilon_i) = \chi_i$; let $\mathcal{A} = \{\chi_i\}$. Some of these are vertices of Q, and some are interior points. Look at all triangulations of Q with vertices in $\mathcal{A}$ (not all elements of $\mathcal{A}$ need to be involved). For a triangulation τ, consider $\phi_\tau = \sum_{\sigma\in\tau} vol(\sigma(i_0,\dots,i_k))(\varepsilon_{i_0} + \cdots + \varepsilon_{i_k})$.

Claim. *$\Sigma(P,Q)$ is the convex hull of the ϕ_τ.*

The idea is that the extreme sections give you points on the vertices of your fiber polytope, and triangulations give you these extremal sections.

Chow variety. Let $Z \subseteq \mathbb{P}^n$ be an irreducible subvariety of dimension k. Consider $H_Z = \{L \in Gr(n{+}1, n{-}k) | \mathbb{P}(L) \cap Z \neq \varnothing\}$. If Z is irreducible, this H_Z is an irreducible hypersurface in $Gr(n+1, n-k)$ [[★★★ exercise]]. The equation $R_Z(\ell_0,\dots,\ell_k) = 0$ giving H_Z is called the *Chow form*. If you have a cycle, the chow forms multiply. Think of L as given by $\ell_0 = \cdots = \ell_k = 0$ (the ℓ_i are linear forms). Since R_Z is a form on the grassmannian, it is a form in the minors of the corresponding matrix, with coefficients ℓ_i^j. In other words, R_Z depends only on the plüker coordinates, and of course it only is defined up to scalar, so we consider $R_Z \in \mathbb{P}(\mathrm{Sym}^d \bigwedge^{k+1}(\mathbb{C}^{n+1}))$.

The group $GL(n+1)$ (and it's maximal torus T) acts on $\mathbb{C}^{n+1}$, so it acts on $\mathbb{P}(\mathrm{Sym}^d \bigwedge^{k+1}(\mathbb{C}^{n+1}))$. This action is compatible with the action on something.

To find the polytope of $\mathbb{P}^n/\!\!/H$, pick a generic point (like $[1 : \cdots : 1]$), and let Z be the closure of the H-orbit. The Chow quotient $\mathbb{P}^n/\!\!/H$ will be a toric variety with the torus T/H. Consider $\overline{T\cdot R_Z}$. The vertices of the corresponding polytope $\mu(\overline{T\cdot R_Z})$ are T-weights of the representation $\mathrm{Sym}^d(\bigwedge^{k+1}(\mathbb{C}^{n+1}))$, so it's clear that the have the same weights as ϕ_τ. The weights are of the form $\sum_{i_0<\cdots<i_k} m_{i_0\dots i_k}(\varepsilon_{i_0} + \cdots + \varepsilon_{i_k})$, with $\sum m_{i_0\dots i_k} = d$, so combinatorially what I said makes perfect sense.

How to actually find R_Z? In general, you can use the Koszul complex. Fix some $r \gg 0$. Let Z be an irreducible projective variety. The Koszul complex is $K^i_r(\ell_0,\dots,\ell_k) = \Gamma(\mathcal{O}_Z(i+r)) \otimes \bigwedge^i(\mathbb{C}^{k+1})$. Fix a basis $e_0,\dots,e_k$ for $\mathbb{C}^{k+1}$. The differential will be given by $\partial(f\otimes\omega) = \sum \ell_i f \otimes e_i \wedge \omega$. It turns out that $K^\bullet_r(\ell_0,\dots,\ell_k)$ is exact if and only if $L \in H_Z$. To see this, consider the corresponding complex of sheaves (don't take global sections), which is the complex of forms, and the differential is wedging with the form $\sum \ell_i e_i$. If $L \notin H_Z$, this form is not zero at any point of the variety, so locally the complex is exact because it is just muliplication by a 1-form. If there is a point where the form vanishes, then it's clear you get cohomology. The idea is to make r large enough that the sheaf is generated by global sections. This is a finite-dimensional complex with finite lenght. Generically, the complex is exact. You can measure this using the *determinant* of the complex. For the complex $0 \to A \xrightarrow{\partial} B \to 0$, you get $\det\partial \in \bigwedge^{\mathrm{top}} A^* \otimes \bigwedge^{\mathrm{top}} B$. In general, $\det\partial \in \bigwedge^{\mathrm{top}}(K^0)^* \otimes \bigwedge^{\mathrm{top}}(K^1) \otimes \bigwedge^{\mathrm{top}}(K^2)^* \cdots$. [[★★★ somehow the idea is to divide by the images of the maps. Locally, you write the matrices, take a suitable minor, and take the first minor, divide by the determinant of the next minor, then multiply by the determinant of the next minor, and so on. This is a functorial ratio of minors]]. The complex is not exact if and only if you have a pole or a zero. From this, it's clear that this hypersurface H_Z is either the set of poles or the set of zeros of $\det\partial$. If you want, you can look at the paper. The main claim is

Claim. $R_Z(\ell_0,\dots,\ell_k) = \det(K^\bullet_r(\ell_0,\dots,\ell_k))^{(-1)^{k+1}}$.

Now you have a formula for computing the Chow form.

Now I take 1-parameter subgroups in T, written as t^λ. $\lambda\colon \mathcal{A} \to \mathbb{Z}$ you can assume the weights are positive because you're working in projective space. A regular triangulation is one so that there exists a strictly convex function $Q \to \mathbb{R}$ which is linear on simplices $\sigma \in \tau$. The height at each vertex is λ_i. So to each t^λ, we can associate a special triangulation. It's not hard to see that in the fiber polytope, only the regular triangulations give you vertices ... the others give you interior points. $\lim_{t\to\infty} t^\lambda Z = \sum vol(\sigma) L_\sigma$, where $L_\sigma = span\{e_{i_0},\dots,e_{i_k}\}$ is the span of the vertices of σ. To do this, you should study the complex $K^\bullet_r(t^\lambda\ell_0,\dots,t^\lambda\ell_k)$. There is a filtration, and it is possible to calculate the associated graded complex. The limit will be something like $\lim_{t\to\infty} t^\lambda R_Z = \prod(\sigma\text{-minor})^{vol(\sigma)}$. This is a technical thing done in the work of Koushinerenko. If you want to show the degree of the cycle Z, it is clearly the volume of Q because the degree is the sum of the volumes.

Corollary 34.3. $\deg(\overline{H\cdot[1:\cdots:1]}) = vol(Q)$.

Part of this stuff I'll put in the exercises.

After the break, I'd like to talk about Luna's slice theorem and stable vector bundles.

No class Nov. 20,23,25, but we'll have classes Nov. 30, Dec. 2, 4, 7, 9.

35 Luna's slice theorem

I was out sick, but I got Pablo's notes, which I'll try to transcribe later.

36 More about Luna's slice theorem

Theorem 36.1 (Luna's slice theorem)**.** *Let X be an affine variety with an action of a reductive group G, and let $x \in X$ be a point with closed orbit $G \cdot x$. Then there exists an affine subvariety $S \subseteq X$ (called an* étale slice*) such that*

(a) $x \in S$

(b) S *is* G_x*-invariant*

(c) $\phi\colon G *_{G_x} S \to X$*, given by sending* (g, s) *to* gs*, is excellent*

(d) $\operatorname{im}\phi$ *is affine.*

Recall that being excellent means that

$$\begin{tikzcd} G *_{G_x} S \arrow[r, "\phi"] \arrow[d] & X \arrow[d] \\ S /\!\!/ G_x \arrow[r, "\phi_G"'] & X /\!\!/ G \end{tikzcd}$$

is cartesean and the horizontal arrows are etale.

Example 36.2. Let G be a reductive group (thing $G = SL(n)$ if you like) with Lie algebra $\mathfrak{g}$, and consider the adjoint action. $x \in \mathfrak{g}$, then $G \cdot x$ is closed if and only if x is semi-simple. Let $x \in \mathfrak{g}^{reg} = \{x \in \mathfrak{g}^{ss} | Z_{\mathfrak{g}}(x) = \mathfrak{t}$ the Lie algebra of the maximal torus$\}$. In this example, the slice is given by $\mathfrak{t}^{reg} = \mathfrak{t} \cap \mathfrak{g}^{reg}$. I have $G *_T \mathfrak{t} \to \mathfrak{g}$ and $G *_T \mathfrak{t}^{reg} \to \mathfrak{g}^{reg}$. The second map is not an isomorphism; it's a covering. The preimage of a point is the intersection of the orbit with the maximal torus. What is true that that $\mathfrak{t}^{reg}/W \to \mathfrak{g}^{reg} /\!\!/ G$ is an isomorphism, where $W = N(T)/T$ is the Weyl group. ◇

Proof of Luna. Step 1. Reduce to the case where $X = V$ is a linear representation of G. We have $X \subseteq V$ is a closed immersion for some V. Suppose we have a slice $G *_{G_x} S \to V$, then I claim that $G *_{G_x} (S \cap X) \to X$ is also a slice.

Exercise. *If $\phi\colon Y \to X$ is excellent and $Z \subseteq X$ is a G-invariant closed subvariety, then $\phi^{-1}(Z) \to Z$ is excellent.*

Step 2. $x \in V$, and $T_xV = T_x(G \cdot x) \oplus N$ where N is G_x-invariant because G_x is reductive (because the orbit is closed; see last lecture). I have a natural map $\phi\colon G *_{G_x} N \to V$, given by $\phi(g, n) = g(x + n)$.

(a) ϕ is G-equivariant

(b) $G \cdot (e, 0)$ has the minimal dimension, and is therefore closed.

(c) the restriction of ϕ to this orbit $G \cdot (e, 0)$ is an isomorphism onto the orbit of x

(d) $d\phi|_{(e,0)}$ is an isomorphism, so ϕ is étale at $(e,0)$.

Lemma 36.3 (Fundamental Lemma). *Let G be reductive, with X and Y affine. Let $\phi\colon Y \to X$ be a G-equivariant map with $\phi(y) = x$ such that Gx and Gy are closed orbits, $\phi|_{Gy}\colon Gy \to Gx$ is an isomorphism, and ϕ is étale at y. Then there exists an open affine set U in Y such that*

(a) $y \in U$,

(b) $U = p_Y^{-1}(p_Y(U))$,

(c) $\phi|_U : U \to X$ is excellent

(d) $\phi(U)$ is affine.

$$\begin{tikzcd} Y \arrow[r, "\phi"] \arrow[d, "p_Y"'] & X \arrow[d, "p_X"] \\ Y/\!\!/G \arrow[r, "\phi_G"'] & X/\!\!/G \end{tikzcd}$$

It should be clear that applying this lemma will finish the proof, so we just need to prove it.

Let $R = k[X]$ and $S = k[Y]$. let r and s be the ideals of Gx and Gy respectively. Define $\widehat{R} = \varprojlim R/r^n$. We have $\phi^*\colon R \to S$, with the induced map $R^G \to S^G$. We have $\widehat{R^G} = \varprojlim R^G/(r \cap R^G)^n$.

The map is étale at y (and so at each point of the orbit), so it is étale in some neighborhood. From this, we see that $\phi^*\colon R/r^n \to S/s^n$ is an isomorphism, so we get an induced isomorphism $\widehat{R} \to \widehat{S}$.

I need to do a digression. Let R be a k-algebra with an algebraic action of G. Then $R = \bigoplus R(M)$ is a sum of isotypic components, where M runs over irreducible representations of G up to isomorphism, and $R(M)$ is a direct sum of (possibly infinitely many) copies of M. Each isotypic component is an R^G-module.

Theorem 36.4 (finite generation of coinvariants). *$R(M)$ is a finitely generated R^G-module.*

This follows from

Theorem 36.5. *Suppose R is noetherian. If N is any R-module (and G-module, compatibly) finitely generated over R, then N^G is finitely generated over R^G.*

To prove this, pick generators $n_1, \dots, n_k \in N^G$ for $RN^G \subseteq N$. For $n \in N^G$, I have $n = \sum a_i n_i$, so $n = \sum \bar{a}_i n_i$ after applying the Reynolds operator.

To get the first theorem, apply this theorem to $N = R \otimes M^*$ and use the fact that $(R \otimes M^*)^G \cong R(M)$.

$\widehat{R}(M) = \varprojlim R(M)/(r^n \cap R(M))$. On the other hand, $\widehat{R(M)} = R(M) \otimes_{R^G} \widehat{R^G}$.

Lemma 36.6. *$\widehat{R}(M) \cong \widehat{R(M)}$. In particular, $\widehat{R}^G \cong \widehat{R^G}$.*

Follows from the following. There exists $m_0 \geq 1$, $n_0 \geq 0$ such that for any $n \in \mathbb{Z}$, we have $r^{m_0 n+n_0} \cap R(M) \subseteq (r^G)^n R(M) \subseteq r^n \cap R(M)$. I'll give you a sketch and you can fill in the details yourself.

$A = R \oplus \bigoplus r^n t^n \subseteq R[t]$. Then A^G is finitely generated over R^G, say with generators $a_1 t^{m_1}, \dots, a_s t^{m_s}$, and $A(M)$ is finitely generated over A^G, say with generators $b_1 t^{n_1}, \dots, b_\ell t^{n_\ell}$. Let $m_0 = \max m_i$ and $n_0 = \max n_i$. For $\bar{a} \in r^{m_0 n+n_0} \cap R(M) \Rightarrow \bar{a} = at^{m_0 n+n_0}$. $at^{m_0 n+n_0} = \sum \underbrace{p_j(a_1 t^{m_1}, \dots, a_s t^{m_s})}_{\in (r^G)^n t^n} b_j t^{n_j}$.

So I get $R \otimes_{R^G} \widehat{R^G} \cong S \otimes_{S^G} \widehat{S^G}$. Taking invariants, I get $\widehat{R^G} \cong \widehat{S^G}$. So ϕ_G is étale at $p_Y(y)$.

$\overline{R} = R \otimes_{R^G} S^G$ is regular functions on the fiber product, $k[X \times_{X/\!\!/G} Y]$. Using the results we have, we can prove $\overline{R} \otimes_{S^G} \widehat{S^G} = R \otimes_{R^G} S^G \otimes_{S^G} \widehat{S^G} = R \otimes_{R^G} \widehat{R^G} \cong S \otimes_{S^G} \widehat{S^G}$. By a theorem from commutative algebra, I can put a local ring downstairs because the completion of a local ring is flat. $\overline{R} \otimes_{S^G} S_{loc} \cong S \otimes_{S^G} S^G_{loc}$. So we can find $f \in S^G$ such that after localization of S^G, I get what I need, an isomorphism $\overline{R}_f \cong S_f$. Take $U_f = Y_f \cap p_Y^{-1}(V)$, where V is the open set where ϕ_G is étale, so we get the property of the fiber product for U_f.

Now we need the image to be affine. Pick f_1 which is zero on $X \setminus \phi(U_f)$ and pull it back. $\square$

37 Consequences of the slice theorem

There will be class Monday and Wednesday of next week.

The proof from last time was from a paper of Knope.

We have

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & X \\ \downarrow & & \downarrow \\ Y/\!\!/G & \xrightarrow[\phi_G]{} & X/\!\!/G \end{array}$$

For $y \in Y$ if ϕ is étale, then it will be étale i a neighborhood. We get that ϕ_G is étale in a neighborhood V. Let D be the set of points where ϕ is not étale. We can separate it by some invariant (since y is a point with closed orbit). There exists $f \in k[Y]^G$ such that $Y_f \supseteq Gy$, and on Y_f, ϕ is étale.

Since we know ϕ_G is étale, I know orbit closure equivalence classes go to orbit closure equivalence classes, and I want to prove that ϕ restricts to isomorphisms on the fibers. On each orbit, it could be a covering map. We know it's an isomorphism on the orbit Gy, and we want to conclude that it is an isomorphism on all orbits. If ϕ were a finite map, it would be clear [[★★★]].

If you have $\phi\colon Y \to X$ a G-equivariant map of normal varieties, sending closed orbits to closed orbits, with the preimage of each point finite, then there is Z and a factorization $Y \hookrightarrow Z$ (open immersion) and $Z \to X$ finite. If in addition $Y/\!\!/G \to X/\!\!/G$ is finite, $Z = Y$. So by taking a smaller neighborhood, we can make the downstairs map finite, making the upper map finite.

We always assume G is a reductive group acting on an affine variety X. Let $p\colon X \to X/\!\!/G$. A subset U of X is saturated if $U = p^{-1}(p(U))$.

Proposition 37.1. *If the orbit of $x \in X$ is closed, then there is a p-saturated neighborhood U of x such that for any $y \in U$, G_y is conjugate to some subgroup of G_x.*

Proof. Any point can be moved by G to $(e, s) = z \in G *_{G_x} S$. Then $G_z \subseteq G_x$. □

Example 37.2. Let $SL(2)$ act on binary cubic forms V_3. A generic closed orbit has a stabilizer (of order 3), but $x^2y \in V_3$ has no stabilizer. So you really need the orbit of the point to be closed for the proposition to hold. ◇

Proposition 37.3. *Suppose $x \in X$ as before (Gx closed) and suppose x is a smooth point. Then for some étale slice S there exists an excellent G_x-equivariant morphism $\psi\colon S \to T_xS$ (which can be identified with the normal bundle to the G-orbit).*

Proof. To construct the slice, we started with $X \subseteq V = T_x(Gx) \oplus N$, so I get $S \hookrightarrow N \to T_xS$, the later map being the G_x-invariant projection. This map satisfies the conditions of the Fundamental Lemma, so we can apply it to get the proposition. □

As a consequence, over $\mathbb{C}$, we have analytic slices.

Theorem 37.4. *Suppose X is a complex affine variety and all as before, with x a smooth point with closed orbit. Then there exists a G-invariant analytic neighborhood of Gx which is isomorphic to some G-invariant analytic neighborhood of the zero section of the normal bundle to Gx.*

The most interesting application is that you can describe the fibers of $p\colon X \to X/\!\!/G$. Each fiber has a unique closed orbit Gx. We can locally identify $X/\!\!/G$ and $S/\!\!/G_x$, so we get an isomorphism $p^{-1}(p(Gx)) \cong G*_{G_x}p^{-1}_{S/G_x}(p_{S/G_x}(x))$ by carteseanness of the square in Luna's slice theorem.

If x is non-singular, I have $T_x(Gx) \oplus N_x = T_xX$, where N_x is a representation of G_x. Let $\mathfrak{n}_x$ be the nil cone in N_x, the closure equivalence class of 0. Then $p^{-1}(p(Gx)) \cong G *_{G_x} \mathfrak{n}_x$.

Example 37.5. For a Lie algebra, the nil cone is indeed the cone of all nilpotent elements. The closed orbits are the orbits of semi-simple elements. Let $\mathfrak{g} = \mathfrak{gl}(n)$ and $G = GL(n)$. Since Gx is closed, x is semisimple, so in some basis it is diagonal. Say the eigenvalues are λ_i with multiplicity m_i. Then $G_x = GL(m_1) \times \cdots \times GL(m_k)$. What is N_x? $T_x(Gx)$ is the space of matrices with zeros in those blocks, so $N_x = \mathfrak{gl}(m_1) \oplus \cdots \oplus \mathfrak{gl}(m_k)$. So the fiber $p^{-1}(p(x)) = \{x + n_1 + \cdots + n_k | n_i$ nilpotent matrix in the i-th block$\}$. If you think about this, this is a fancy way to prove the Jordan normal form theorem. $\diamond$

If V is a linear representation of a reductive group G and $p\colon V \to V/\!\!/G$, with $v \in V$, then $v = s + n$, where s has closed G-orbit in V and n lies in the nil cone G_s-orbit in V.

Consider the case $X/\!\!/G$ is a single point, so $k[X]^G = k$. Then X has a unique closed orbit Gx. So there exists an affine G_x variety Y such that

(1) $k[Y]^{G_x} = k$, with the closed orbit of Y a fixed point.

(2) $X \xrightarrow{\sim} G *_{G_x} Y$.

Suppose x is also a non-singular point, then Y is a linear representation of G_x (since the nil cone is the whole space). [[★★★ !]]

Corollary 37.6. *Suppose X smooth with $k[X]^G = k$, and the closed orbit is a fixed point. Then $X \cong V$ with a linear action.*

Problem: Suppose you have the action of a reductive group on $\mathbb{A}^n$. Is this action linear? In general, no! But if $k[\mathbb{A}^n]^G = k$ and the closed orbit is a fixed point, then it's true.

38 Lecture 38

Two more applications of the slice theorem.

Lemma 38.1. *If a reductive group G acts on an affine variety X, then for any $y \in X$ there exists a 1-parameter subgroup $\lambda(t) \in G$ such that $\lim_{t\to 0} \lambda(t)y = y_0$ and Gy_0 is closed.*

Proof. Let $Y = \overline{Gy}$, and let Gx be the unique closed orbit in Y (this orbit is affine, so the stabilizer G_x is reductive). By what we proved last time, I have the isomorphism $Y \cong G *_{G_x} Z$, where Z has a unique closed G_x-orbit. Z may be singular, but we can G_x-equivariantly embed it into a vector space V so that x goes to 0. For any $v \in V$, there is a 1-parameter subgroup $\tilde{\lambda}(t) \in G_x$ such that $\lim \tilde{\lambda}(t)v = 0$ by Hilbert-Mumford. $y = (g, v)$ and take $\lambda(t) = g\tilde{\lambda}(t)g^{-1}$ and it works. □

Luna stratification: assume X is smooth affine and G is reductive. Recall that if $x \in X$ has closed orbit, then by the Luna slice theorem, some étale neighborhood is isomorphic to a fiber bundle. $T_xX = T_x(Gx) \oplus N_x$, where N_x is G_x-invariant. We know that some étale neighborhood of x is isomorphic to $G *_{G_x} N_x$.

Consider $\mathcal{M} = \{G *_H M | M$ a linear representation of H, a reductive subgroup of $G\}/\sim$, where $\sim$ is G-equivariant isomorphisms. If $x \in X$ and Gx is closed, we associate to x the isomorphism class $[G *_{G_x} N_x]$. Any point in the same orbit gives an isomorphic fiber bundle. So we have, $X \xrightarrow{p} X/\!\!/G \xrightarrow{\gamma} \mathcal{M}$. So we can define $(X/\!\!/G)_\mu = \gamma^{-1}(\mu)$ and $X_\mu = p^{-1}(X/\!\!/G)_\mu$.

Theorem 38.2.

1. *$\mathcal{M}_X = \operatorname{im}\gamma$ is a finite set.*
2. *$X/\!\!/G = \bigsqcup_{\mu\in\mathcal{M}_X} (X/\!\!/G)_\mu$, and each $(X/\!\!/G)_\mu$ is a non-singular locally closed subvariety of $X/\!\!/G$.*
3. *All fibers of $p\colon X_\mu \to (X/\!\!/G)_\mu$ are isomorphic (in fact, it's a locally isotrivial fibration; there is an étale cover making it the trivial fibration).*

Example 38.3. Consider the case $X = \mathfrak{g} = \mathfrak{gl}(n)$ and $G = GL(n)$ with the adjoint action. Then $p\colon X \to \mathbb{A}^n \cong \mathbb{A}^n/S_n$ (the coefficients of the characteristic polynomial), and I have the étale cover $\mathbb{A}^n \to \mathbb{A}^n$ given by taking roots (this is the quotient by S_n).

Let $x \in X$ be semi-simple, so the orbit is closed. In some basis, x is block identity (with eigenvalues λ_i with multiplicities m_i). So $\mathcal{M}_X$ can be identified with the set of partitions of n. When they are all different (all multiplicites 1), you get the open stratum. This stratification is S_n-invariant, so it descends to the quotient $\mathbb{A}^n/S_n$. ◇

Sketch of Proof. Gx is closed. $G *_{G_x} N_x$ is an étale neighborhood. In this neighborhood, we consider $Y = \{y \in G *_{G_x} N_x | G_y \text{ is conjugate to } G_x\}$. It is enough to consider elements of the form (e, v) and use G to move them around. We see that $(e, v) \in Y$ is and only if v is fixed by G_x. We have $N_x = N_x^G \oplus M_x$, and $Y \cong G *_{G_x} N_x^G$, so Y

looks like a subspace of $G *_{G_x} N_x$. X_λ is (étale) locally isomorphic to $Y = G *_{G_x} N_x^{G_x}$. It's clear that this thing is locally closed. So after the étale cover, the stratum is a subspace. If S is a slice, then there is an étale map $p\colon (S/G_x)_\lambda \to N_x^{G_x}$. The fiber is isomorphic to the nil cone $m_x \in M_x$. □

Remark 38.4. If $X/\!\!/G$ is irreducible, then we have a single open stratum $(X/\!\!/G)_p$, usually called the principal stratum. Then we have $X_p \to (X/\!\!/G)_p$. This is the only stratum such that the fibers are non-singular. If $x \in X_p$ has closed orbit, then $p^{-1}(p(x)) \cong G *_{G_x} m_x$, and $k[M_x]^G = k$, so $m_x = M_x$, and $G *_{G_x} M_x$ is non-singular. For all other strata, the fibers are nil cones, which can never be non-singular.

In the example of the adjoint representation, the fibers are closed orbits over the principal stratum. ◇

Exercise. *If M is a linear representation of G, then the nil cone is non-singular if and only if $M = M^G \oplus W$, where $k[W]^G = k$.*

We always have the assumption that G is reductive, and in characteristic zero, this cannot be improved. We want the results to work for Chevalley groups (like $GL(n, F)$ and $SL(n, F)$) in finite characteristic, and it does! We can replace linearly reductive by geometrically reductive.

Definition 38.5. G is *geometrically reductive* if for any linear representation V and any $v \in V^G \smallsetminus 0$, there exists $m > 0$ and $F \in \operatorname{Sym}^m(V^*)$ such that $F(v) = 1$. ◇

This is equivalent to the following. Suppose you have an exact sequence of finite-dimensional representations

$$0 \to W \to V \to k \to 0$$

where k has the trivial G-action. Then there exists an $m > 0$ such that

$$0 \to \operatorname{Sym}^{m-1} V \otimes W \to \operatorname{Sym}^m V \to \operatorname{Sym}^m k = k \to 0$$

splits. [[★★★ exercise]]

There are two key results.

Theorem 38.6 (Nagata?)**.** *If G is geometrically reductive and R is a finitely generated k-algebra (or just noetherian ring?) with a G-action, then R^G is a finitely generated algebra.*

This allows us to define $X/\!\!/G = \operatorname{Specm} k[X]^G$.

Theorem 38.7. *If a geometrically reductive group G acts on an affine variety X, and $Z_1, Z_2 \subseteq X$ are two disjoint closed G-invariant subsets, then there exists an $f \in k[X]^G$ such that $f(Z_i) = i$.*

39 Lecture 39

Assume G is geometrically reductive and acts on some ring R. Suppose $I \subseteq R$ is a G-invariant ideal. [[★★★ when G was reductive, we had $R^G/I^G \cong (R/I)^G$]] Then $R^G/I^G \subseteq (R/I)^G$.

Lemma 39.1. *If $r \in (R/I)^G$ then for some m, $r^m \in R^G/I^G$.*

Proof. r is contained in some finite-dimensional G-invariant subspace $V \subseteq R$. So we get the exact sequence

$$0 \to W = I \cap V \to V \to kr \to 0$$

where $W \subseteq V$ is invariant and kr is the trivial representation. As we showed last time, if we take a symmetric power, the sequence splits:

$$0 \to \mathrm{Sym}^{m-1}(W)V \to \mathrm{Sym}^m(V) \to kr^m \to 0$$

splits. □

Now we'll prove the first theorem from the end of the last lecture.

Remark 39.2. Suppose $(R/I)^G$ is finitely generated. Then by the lemma, $R^G/I^G \subseteq (R/I)^G$ is an integral extension of rings, so R^G/I^G is finitely generated as well. ◇

We'll prove the theorem first for graded rings (like with Hilbert's theorem), and use induction on Krull dimension.

Step 1. $R = \bigoplus_{i \geq 0} R_i$ graded with $R_0 = k$. We induct on the Krull dimension. Consider the ideal $R \cdot R^G_{>0}$. Since R is noetherian, we can choose finitely many generators for the ideal (which we may assume are invariant) $f_1, \dots, f_k$. Let $f = f_1$. By induction on Krull dimension, we may assume $(R/Rf)^G$ is finitely generated. There are two cases: either f is a zero-divisor or it's not.

(a) Suppose f is not a zero-divisor. Then $(Rf)^G = R^G f$. Pick $\alpha_1, \dots, \alpha_s$ representatives of generators for $R^G/(Rf)^G$ in R^G. Then $f, \alpha_1, \dots, \alpha_s$ generate R^G. To see this, given $x \in R^G$, we have $x = (\alpha_1, \dots, \alpha_k) + fy$, with $\deg y < \deg x$; now induct on degree of x.

(b) Suppose f is a zero-divisor, then $J = \mathrm{Ann}(f)$ is a G-invariant ideal. Then $R^G/(Rf)^G$ and R^G/J^G are finitely generated by the inductive hypothesis (and induction on number of generators). Pick $\alpha_1, \dots, \alpha_p$ representatives of generators of $R^G/(Rf)^G$ in R^G and $\beta_1, \dots, \beta_q$ representatives of generators of R^G/J^G. Now consider $B = k[\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q]$. Let $c_1, \dots, c_n$ be representatives of generators of $(R/J)^G$ as a $B/B \cap J$-module. Then $fc_i \in R^G$ because $f(gc_i - c_i) = 0$ for all $g \in G$.

Claim. $R^G = B[fc_1, \dots, fc_n]$

Let $x \in R^G$, then there is a $b \in B$ such that $x - b \in fR$ because the natural map $B \to R^G/(fR)^G$ is surjective. So $x - b = fc$, fc is G-invariant so $c \in (R/J)^G$. So $x = b + fc \in B[fc_1, \dots, fc_n]$.

Note that for this argument (b), we did not use the assumption that R is graded.

Step 2. General case. Let $A = k[x_1, \dots, x_n]$ with $R = A/I$ for some G-invariant ideal I. I know that A^G/I^G is finitely generated by the graded case. We have $A^G/I^G \subseteq (A/I)^G$ is an integral extension. If R^G is an integral domain, then it is sufficient to check that the field of fractions $Q((A/I)^G)$ if finitely generated over $Q(A^G/I^G)$. If it is not an integral domain, then I can use the same argument as in part (b) of Step 1.

Now let's prove the second theorem from the end of last lecture (the separation lemma). We have invariant ideals $I(Z_1)$ and $I(Z_2)$ such that $I(Z_1) + I(Z_2) = k[X]$. Let $1 = \alpha + \beta$ with $\alpha \in I(Z_1)$ and $\beta \in I(Z_2)$. Then $\beta(Z_1) = 1$ and $\beta(Z_2) = 0$, but β is not invariant. There is a G-invariant finite-dimensional subspace $V \subseteq k[X]$ containing β. Let $\phi_1, \dots, \phi_n$ be a basis for V. We may assume $\phi_i = g_i \cdot \beta$ for some $g_i \in G$. We get a map $\phi\colon X \to \mathbb{A}^n$. By invariance of Z_1 and Z_2, we get that $\phi(Z_1) = (1, \dots, 1)$ and $\phi(Z_2) = (0, \dots, 0)$. By geometric reductivity, there is a G-invariant homogeneous polynomial $F(\phi_1, \dots, \phi_n)$ such that $F(Z_1) = 1$ and $F(Z_2) = 0$.

In one book I used, I found this theorem. There is a third notion of a reductive group. A group G is *algebraically reductive* if its unipotent radical is trivial. The unipotent radical is the maximal normal unipotent subgroup, which is the intersection of the kernels of all irreducible representations. In characteristic zero, this is equivalent to linearly reductive.

Theorem 39.3 (Popov)**.** *Suppose* $\mathrm{char}(k) = 0$*[[★★★ maybe not needed]]. If on every affine variety X with G-action, $k[X]^G$ is finitely generated, then G is algebraically reductive.*

A couple of books I found later: Invariant Theory by Popov and Vinberg, Birkhauser DMV seminar Algebraic Transformation Groups and Invariant Theory, and Lectures on Invariant Theory by Dolgachev.

Nagata's example

$G = C \cdot G'$ acts on $\mathbb{A}^n \times \mathbb{A}^n = \mathbb{A}^{2n}$ (coordinates x_i and y_i). The G' action is given by $(x_k, y_k) \mapsto (x_k + \alpha_k y_k, y_k)$ for $\sum a_{ij}\alpha_j = 0$ and $i = 1, 2, 3$, $\alpha_i \in k$. The C action is $c \cdot (x_i, y_i) = (c_i x_i, c_i y_i)$ where $c_1 \cdots c_n = 1$ and $c_i \in k^\times$.

In some basis, this is block diagonal with blocks $\left(\begin{smallmatrix} c_i & \alpha_i \\ 0 & c_i \end{smallmatrix}\right)$, with determinant 1. This group has $\dim G = 2n - 4$. Take $n = 9$. For a suitable choice of a_{ij}, we don't get a finitely generated ring of invariants $k[x, y]^G$.

Step 1. The a_{ij} are a 3×9 matrix. Suppose $\det(a_{ij})_{i,j=1,2,3} \neq 0$ (the first minor is non-singular). Let $z_i = \sum_{j=1}^n a_{ij}(x_j t / y_j)$ where $t = y_1 \cdots y_n$ for $i = 1, 2, 3$.

$k(x,y)^G = k(t, z_1, z_2, z_3)$. Checking that they are invariant uses the calculation $g(x_j t/y_j) = x_j/y_j + \alpha_j$.

$k(x_1, \ldots, x_n, y_1, \ldots, y_n) = k(z_1, z_2, z_3, x_4, \ldots, x_n, y_1, \ldots, y_n) = k(t, z_1, z_2, z_3, x_4, \ldots, x_n, y_1, \ldots, y_n$. If I pick $H \subseteq G$ such that $\alpha_5 = \cdots \alpha_n = 0$, we get that x_4 is not invariant, and that's the only thing H acts on, so we can eliminate the x_4. Procede inductively to get

$$k(t, z_1, z_2, z_3, x_4, \ldots, x_n, y_1, \ldots, y_{n-1} = \cdots = k(t, z_1, z_2, z_3)$$

The nine columns of (a_{ij}) can be regarded as nine points $p_1, \ldots, p_9$ in $\mathbb{P}^2$. Let R_m be the set of homogeneous polynomials $f(z_1, z_2, z_3)$ which have multiplicity at least m at each p_i.

Step 2. We have that $k[x,y]^G = \{\sum f_m(z_1, z_2, z_3)t^{-1} | f_m \in R_m\}$.

$k[x,y]^G = k[z_1, z_2, z_3, t, t^{-1}] \cap k[x,y]$. I claim you can only invert t and that the z_i don't get inverted. This is because $k[x_1, \ldots, x_n, y_1^{\pm 1}, \ldots, y_n^{\pm 1}] = k[z_1, z_2, z_3, x4, \ldots, x_n, y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$, and then we just impose the condition that it's actually a polynomial in the y_i.

$f = \sum f_{i_1, i_2, i_3, m} z_1^{i_1} z_2^{i_2} z_3^{i_3} t^{-m}$. Each $z_i = \sum a_{ij} x_j y_1 \ldots \hat{y}_j \ldots y_n$. When I divided by t, everything is okay except y_j.

Multiplicity condition follows from the fact that all y_i must be in non-negative powers.

Step 3. Cubic curve C comes into the picture. A cubic curve is an abelian group (depending on a choice of zero, which we choose to be an inflection point).

Claim. *The order of $p_1 + \cdots + p_9$ is m if and only if there exists a homogeneous polynomial $f(z_1, z_2, z_3)$ non-zero on C which has multiplicity m at each p_i.*

Proof. Let $p + q + r = 0$ on C. In the group of divisors, we have $[p] + [q] - [r] - [0]$ is the divisor of a rational function. This is equivalent to saying that $r + p + q = 0$. Then $m[p_1] + m[p_2] + \cdots + m[p_9] - 9m[0]$ is the divisor of a rational function. The numerator of this rational function must have zeros of multiplicity m at each p_i. The rational function is F/ℓ^{3m}, where ℓ is the equation of the tangent line at 0. $\square$

We're going to choose the p_i such that $p_1 + \cdots + p_9$ does not have finite order.

Consider $f = \sum f_m(z_1, z_2, z_3)t^{-m}$ as a polynomial in x and y. If $\deg f = d$, then $\deg f_m = k$ and I have the condition $nk - mn = d$ because each z_i has degree n in x and y. So $k = m + d/n$ (and $n = 9$). So we have a double grading, and we're going to calculate the degree of each element in the double grading.

Let $R_{k,m} = \{f(z_1, z_2, z_3)$ of degree k and multiplicity at each p_i at least $m_i\}$. This is a finite-dimensional space and I can estimate its dimension. It is the dimension of all polynomials of degree k minus the dimension of polynomials with small multiplicities.

$$\dim R_{k,m} \geq \frac{1}{2}(k+1)(k+2) - \frac{9}{2}m(m+1)$$

The last term comes from looking at $f(p_1, z_2, z_3)$, which is a polynomial in two variables of degree less than m, and there are 9 such conditions.

$$= \frac{1}{2}(k - 3m)(k + 3m + 3).$$

This $R_{k.m}$ is a piece in the invariant ring. The dimension is positive if $k > 3m$. On the line $k = 3m$, I have dimension 1. The polynomial must divide the polynomial which defines my cubic (suppose it's given by $h(z_1, z_2, z_3) = 0$), so we can decrease the degree, which shows that we have dimension is exactly 1 on that line. When I draw a parallel line, the dimension increases along the line just from the formula. Now we can see that there cannot be finitely many generators. If there were, they would correspond to some points in the picture.

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