Definition 1. For a parameter n, Let $L \subset \mathbb{N}^+$ be the set of non-periodic words over the alphabet $\Sigma = \mathbb{N}$ that are bigger in Arabic (right-to-left) lexicographical order than all of their rotations. Let L_n be the set of all the words in L whose length divides n.

Definition 2. For a word $w = w_1 \cdots w_{n-1} w_n$ let $R(w) = w_n w_1 \cdots w_{n-1}$ be the rotation of w to the right. Then, the nested invocation $R^m(w)$ is the m letter rotation to the right and its inverse $R^{-m}(w)$ is the m letter rotation to the left.

1 Forward and backwards transformations

Definition 3. For a word w whose length is smaller or equal than n, let $f(w)$ be the transformation defined by successive applications of the following steps to w:

f_1: Increase the first letter of the word by one.

f_2: Pad with zeros on the left to get a word of length n.

f_3: Apply the substitution rules $u(vu)^+ \mapsto vu$ and then $w^+ \mapsto w$, with the longest possible u and the shortest possible w.

Definition 4. For a a word w whose length is smaller or equal than n, let $b(w)$ be the transformation defined by successive applications of the following steps to w:

b_1: Expand w to uw^m where $m = \lfloor n/|w| \rfloor$ and u is the suffix of length $n - m|w|$ of w.

b_2: Remove leading zeros.

b_3: Decrease the first letter by one.

Observation 1. For any $w \in L_n$, $f(b(w)) = b(f(w)) = w$.

Proposition 2. *If we start with $w(0) = 0$ and generate a sequence of words by $w(i+1) = f(w(i))$, we get an enumeration of all the words in L whose length is smaller or equal to n.*

Proof. This is a version of Duval's algorithm with a reversed order of the alphabet and a reversed order of letters in a word. □

Definition 5. Let $f^*(w)$ be the first word in $f(w), f(f(w)), \ldots$ whose length divides n and, similarly, let $b^*(w)$ be the first word in $b(w), b(b(w)), \ldots$ whose length divides n.

Definition 6. Let $w(0), w(1), \ldots$ be the sequence generated by starting with $w(0) = 0$ and then continuing ad infinitum by $w(i+1) = f^*(w(i))$ and let $w^\infty \in \mathbb{N}^\omega$ be the concatenation of all these words.

2 Where can I find w as a sub-word of w^∞?

In this section we point at the position of an arbitrary word w as a sub-word of w^∞ relative to the position of the a corresponding word in L_n. This is given in Proposition 6 and in Proposition 7. Towards the proofs of these propositions, we first establish some technical results about the functions b and b^* specified, respectively, in Definition 4 and in Definition 5.

Proposition 3. *If $w \in L_n$ and $|w| \neq n$ then $b(w) = uw$ for some non-empty word u.*

Proof. The first transformation b_1 extends w to the left producing the word $b_1(w) = uw^m$ where u is a tail of w. Since $w \in L_n$ and because it contains a letter σ that is not zero, we have, by maximality of w among its rotations in right-to-left lexicographical order, that its last letter is not zero. The last letter of u is the last letter of w so it is also not zero. This gives us that the next transformation b_2, that deletes trailing zeros, leaves at least the last copy of w and the last letter of the before-last (full or partial) copy at the tail of $b_1(w)$. Thus, $b_2(b_1(w)) = uw$ where u is a non-empty word whose first letter is not zero. Then, the last transformation b_3 only decreases the first letter of u by one which gives us that $b(w) = b_3(b_2(b_1(w))) = vw$ for some non-empty word v. $\square$

Proposition 4. *For any $w = 0^l\sigma\hat{w} \in L_n$ where σ is a non-zero letter there is a non-empty word u such that $b(w) = u\hat{w}$.*

Proof. If $|w| \neq n$ the proof follows by Proposition 3. If $|w| = n$ then $b_1(w) = w$, $b_2(b_1(w)) = \sigma\hat{w}$, and $b_3(b_2(b_1(w))) = (\sigma - 1)\hat{w}$ and the claim follows as well. $\square$

Proposition 5. *Let w be an arbitrary word in $\mathbb{N}^n$ and let and let $\bar{w} = f_3(w)$. Let l be the (possibly zero) number of trailing zeros (from the left) in w. Then, for all $0 \leq i \leq |w| - l - 1$, the word $R^i(\bar{w})$ comes $i + n - |w|$ letters before w as a sub-word of w^∞.*

Proposition 6. *For a given $w \in L_n$, let l be the number of trailing zeros (from the left) in w and let $\bar{w} = b_1(w)$. Then, for all $0 \leq i \leq |w| - l - 1$, the word $R^i(\bar{w})$ comes $i + n - |w|$ letters before w as a sub-word of w^∞.*

Proof. By Proposition **??** the words that come before w ends with the last $|w|-l$ letters of w. In particular, the n letter word that starts $i + n - |w|$ before w is $R^i(\bar{w})$. $\square$

Proposition 7. *For a given $w \in L_n$, let l be the number of trailing zeros (from the left) in w and let $\bar{w} = b_1(w)$. Then, for all $|w| - l \leq i \leq n - 1$ the word $R^i(\bar{w})$ comes $i - (n - |f_3(u)| \pmod n)$ letters before the first $u \in \langle 0^{m-1}(\bar{w}_m + 1)\bar{w}_{m+1} \cdots \bar{w}_n\rangle_{m=i+1}^n$ that is in L_n.*

Proposition 8. *The word w^∞ contains all the words in $\mathbb{N}^n$ as subwords.*

Proof. Any word of length n is a rotation of the expansion of a word in L_n. $\square$

Proposition 9. *For any k the prefix $w_1^\infty \cdots w_{k^n}^\infty$ is an n-order de Bruijn sequences. Moreover, it is the reversed of the n-order prefer-max sequence on the alphabet $\langle 0, \ldots, k-1 \rangle$ (in this order).*

Proof. Counting argument + arguing that if $|w| = n-1$ and $\sigma_1 < \sigma_2$ then $w\sigma_1$ comes before $w\sigma_2$ as subwords of w^∞. $\square$

Proposition 10. *For $w \in \mathbb{N}^n$, let i be the minimal index such that $R^{-i}(w) \in L$ and let $\bar{w} = R^{-i}(w)$. Let $\bar{w}^+ = \bar{w}_{1..i}(\bar{w}_{i+1}+1)\bar{w}_{(i+2)..n}$, i.e., the word obtained by increasing the $(i+1)$th letter of $\bar{w}$ by one. Then, the function*

$$next(w) = \begin{cases} f^*(f_3(w))_1 & \text{if } w \in L; \\ w_1 + 1 & \text{if } \bar{w}_{1..i} = 0^i \wedge \bar{w}^+ \in L \wedge \max(\bar{w}^+_{1..(n-1)}) \le \max(w); \\ 0 & \text{if } \bar{w}_{1..i} = 0^i \wedge (\bar{w}^+ \notin L \vee \max(\bar{w}^+_{1..(n-1)}) > \max(w)); \\ w_1 & \text{otherwise.} \end{cases}$$

represents the mapping of a word w to the letter that follows the (one and only) occurrence of w as a subword of w^∞.

Definition 7. Let $w(0) = 0, w(1) = f^*(w(0)), \ldots, w(i) = f^*(w(i-1)), \ldots$ be our enumeration of all the words in L_n. Let $w^{(i)} = w(0) \cdots w(i)$ be the concatenation of the first i words in this enumeration and let $u(j) = w^{(i)}_{j-n+1} \cdots w^{(i)}_j$ be the "window" of length n before the jth letter in $w^{(i)}$.

Proposition 11. *For a word $w = w(i)$ be the ith word in the above enumeration. Let l be the number of leading zeros in w and let $\hat{w} = R^{-l}(b_1(b^*(w)))$. Then, inserting the cycle $\langle R^{-s}(b_1(w)) \rangle_{s=0}^{n-1}$ after the word $\hat{w}_{1..(n-l)}0^l$ in $\langle u(j) \rangle_{j=0}^{i-1}$ yields the sequence $\langle u(j) \rangle_{j=0}^{i}$.*

3 Where can I find w as a sub-word of w^∞? (second try...)

Definition 8. For a word w, $max(w)$ is the maximal digit in w.

Definition 9. A word $u \in \mathbb{N}$ corresponds to $w \in L_n$ if u is a rotation of $w^{\frac{n}{|w|}}$. Note that each $u \in \mathbb{N}$ corresponds to exactly one word $w \in L_n$.

Proposition 12. *If $w \in L_n$ and $|w| < n$, then $f^*(w) = f(w) = 0^{|w|}u$ for some word u.*

Proof. Write $\frac{n}{|w|} = c > 1$. Thus, $f_1(w) = x$, $f_2(x) = 0^{(c-1)|w|}x$ and $f_3(0^{(c-1)|w|}x) = 0^{c-1|w|}x = f(w)$. Since $|x| = |w|$, $|f(w)| = n$ and hence, $f^*(w) = f(w)$. In addition, $(c-1)|w| \ge |w|$. $\square$

Proposition 13. *If $|w| < n$ and $w = w'k$ where $0 < k = max(w)$, then $b(w) = uw$ and $max(u) \le max(w)$.*

Proof. Write $w = w'k$. Thus, $b_1(w) = xk(w'k)^r$, $r > 0$. $b_2(xk(w'k)^r) = y(w'k)^r$. $b_3(y(w'k)^r) = uw'k = uw$. It is easy to see that $\max u \leq k$. $\square$

Proposition 14. *If $w \in L_n$, $|w|^m = n$, $m > 1$ and $w = 0^l\sigma\hat{w}$ such that $\sigma \neq 0$, then $b^*(w) = u\hat{w}w^{m-1}$ for some u. Note that in particular, $|b^*(w)| = n$.*

Proof. Since $w \in L_n$ and $w \neq 0$, $b_1(w) = w^m = (0^l\sigma w'k)^m$.

$$b(w) = b_3(b_2((0^l\sigma\hat{w})^m)) = b_3((xk)(w'k)^{m-1}) = y(w'k)^{m-1} = y\sigma - 1\hat{w}w^{m-1}.$$

Now, if $|y\sigma - 1\hat{w}w^{m-1} = n$, then $y\sigma - 1\hat{w}w^{m-1} = b^*(w)$ and we are done. Otherwise,

$$b^*(w^{m-1}) = b^r(y\sigma - 1\hat{w}w^{m-1}).$$

Hence, the words $b(y\sigma - a\hat{w}w^{m-1}), b^2(y\sigma - \hat{w}w^{m-1}), \ldots, b^{r-1}(y\sigma - 1\hat{w}w^{m-1})$ are of length less than n. We apply Proposition 13 on these words and we get that for each $l \leq r-1$, $b^l(y\sigma - 1\hat{w}w^{m-1}) = u_1u_2\ldots u_l y\sigma - 1\hat{w}w^{m-1}$. $\square$

Proposition 15. *Assume that $u \in \mathbb{N}^n$ corresponds to $w \in L_n$ such that $|w| < n$. then, u is a subword of w^∞.*

Proof. If $u = 0^n$, then u is a prefix of w^∞ and we are done. Otherwise, $w = 0^l\sigma\hat{w}$ where $\sigma \neq 0$ and take m such that $|w|^m = n$. Note that $m > 1$. By Propositions 12 and 14, $b^*(w)wf^*(w) = x\hat{w}w^{m-1}w0^{|}w|y$ is a subword of w^∞. Hence,

$$\hat{w}(0^l\sigma\hat{w})^m0^l \text{ is a subword of } w^\infty.$$

u is a rotation of w^m thus u is a subword of $\hat{w}(0^l\sigma\hat{w})^m0^l$ which implies that u is a subword of w^∞. $\square$

Proposition 16. *Assume that $u = yx \in \mathbb{N}^n$ corresponds to $w = xy \in L_n$ where $|w| = n$. If $x \neq 0^r$, then u is a subword of w^∞.*

Proof. We show that $u = yx$ is a subword of $b^*(w)w$. Write $x = 0^l\sigma z$ where $\sigma \neq 0$. Thus, since $|w| = n$, $b(w) = \sigma - 1zy$. If $b(w) = b^*(w)$, then

$$b^*(w)w = v(\sigma - 1)zyx$$

and we get that u is a subowrd of $b^*(w)b(w)$. Otherwise, $|(\sigma - 1)zy|$ does not divides n, and in particular, $|\sigma - 1zy| < n$. By applying Proposition 13 several times, we get that $b*(w) = v(\sigma - 1)zy$ for some v, and $u = yx$ is a subword of $b^*(w)w = v(\sigma - 1)zyxy$. $\square$

Lemma 17. *Assume that $w = 0^lv \in L_n$ and $|w| = n$. Write $w = 0^lz_1\sigma z_2$ where σ is the first digit in such that $0^{l+|z_1|}(\sigma+1)z_2$ is lexicographically maximal among its rotations. Take $k \in \mathbb{N}$ such that $|z_1| \leq |(\sigma z_2)^k|$. Then, z_1 is a suffix of $(\sigma z_2)^k$.*

Proof. Assume for a contradiction that z_1 is not a suffix of $(\sigma z_1)^k$, and hence there are $\tau \neq \tau'$ in $\mathbb{N}$ and a word y, such that $\tau' y$ is a suffix of σz_2, and

$$z_1 = x\tau y(\sigma z_2)^r,\ (\sigma z_2)^k = x'\tau' y(\sigma z_2)^r.$$

Clearly, $\tau < \tau'$ since otherwise, $\tau' < \tau$, and we get that $w = 0^l z_1 \sigma z_2 = 0^l x\tau y(\sigma z_2)^{r+1}$. However, if we assume that $\tau' < \tau$, $w' = (\sigma z_2)^r 0^l x\tau y$ is lexicographically larger than w, in contradiction to $w \in L_n$. □

Corrolary 18. Assume that $w = 0^l v \in L_n$ and $|w| = n$. Write $w = 0^l z_1 \sigma z_2$ where σ is the first digit in such that $0^{l+|z_1|}(\sigma+1)z_2$ is lexicographically maximal among its rotations. Then, there are words x, y such that $z_2 = xy$, $w = 0^l y(\sigma xy)^{r+1}$ and $z_1 = y(\sigma xy)^r$.

Proof. This is a consequence of the previous Lemma and the fact that $|0^l z_1| = |x(\sigma z_2)|^m$. □

Proposition 19. *Assume that $v0^l \in \mathbb{N}^n$ corresponds to $w = 0^l v \in L_n$ where $|w| = n$ and $l > 0$. Then, $v0^l$ is a subword of w^∞.*

Proof. Write $w = 0^l z_1 \sigma z_2$ where $\sigma \in \mathbb{N}$ is the first digit in w so that $0^{l+|z_1|}(\sigma+1)z_2$ is lexicographically maximal among its rotations. Note that such a digit exists since the last digit in w satisfies this requirement. Hence, $v = z_1\sigma z_2$.

By Corollary 18, $z_2 = xy$ and $z_1 = y(\sigma xy)^r$. Now, since $|0^{l+|z_1|}(\sigma+1)z_2| = n$ and $0^{l+|z_1|}(\sigma+1)z_2$ is lexicographically maximal among its rotations, $0^{l+|z_1|}(\sigma+1)z_2 = (w')^{k+1}$ where $w' \in L_n$. Note that $0^{l+|z_1|}$ is a prefix of w'. We consider three possibilities

Case 1. $\sigma z_2 \in L_n$. We show that in this case, $v0^l$ is a subword of $b^*(b^*(w'))(b^*(w'))w'$, which is a subword of w^∞.

$b_1(w') = w'^{k+1} = 0^{l+|z_1|}(\sigma+1)z_2$. Hence, $b(w') = b_3(b_2(0^{l+|z_1|}(\sigma+1)z_1)) = \sigma z_2$. Since $\sigma z_2 \in L_n$, $b(w') = b^*(w') = \sigma z_2$ and in particular, $|(\sigma z_2)^{m+1}| = n$ for some $m \in \mathbb{N}$. Observe that $|z_1| \leq |\sigma z_2|^m$ and use Lemma 17 to conclude that z_1 is a suffix of $(\sigma z_2)^m$.

By invoking Proposition 13 several times, $b^*(\sigma z_2) = u(\sigma z_2)^m$ for some u. Hence, $v0^l = z_1\sigma z_2 0^l$ is a subword of

$$b^*(b^*(w'))b^*(w')b(w') = u(\sigma z_2)^{m+1}0^{l+|z_1|}x'.$$

Case 2.

Now, $b_1(\sigma z_2) = (\sigma z_2)^{m+1}$ and by Corollary 18, z_1 is a suffix of $(\sigma z_2)^m$. Since the last digit in (σz_2) is not zero, by applying Proposition 13 several times (possibly none), we get that $b^*\sigma z_2 = x(\sigma z_2)^m$. As a result we have,

$$b^*(b^*(w'))(b^*(w'))w' = x(\sigma z_2)^m(\sigma z_2)0^{l+|z_1|}x'.$$

Therefore, $v0^l = z_1\sigma z_2 0^l$ is a subword of $b^*(b^*(w'))(b^*(w'))w'$ as required.

Before we deal with the other cases, we note that $b_1(\sigma z_2) = b_1(\sigma xy) = x'y(\sigma xy)^{r+1}$ for some x' that satisfies $|x'| = l > 0$.

Case 2. $\sigma z_2 \notin L_n$ and $x' \neq 0^l$. We show that in this case, $v0^l$ is a subword of $b^*(w')w'$, which is a subword of w^∞.

Recall that $b(w') = \sigma z_2$ which is, by assumption, not a key-word. Since $x' \neq 0^l$, several invocations of Proposition 13 imply that $b^*(\sigma z_2) = x''y(\sigma z_2)^{r+1}$. Since $v = z_1\sigma z_2 = y(\sigma z_2)^{r+1}$, we get that $v0^l$ is a subword of

$$b^*(w')w' = x''y(\sigma z_2)^{r+1}0^{l+|z_1}u.$$

Case 3. $\sigma z_2 \notin L_n$ and $x' = 0^l$. In this case, $b_1(\sigma z_2) = 0^l y(\sigma xy)^{r+1}$. Write $w'' = (\sigma xy)^{r+1}0^l y$ (is it a key-word?). Then, $v0^l$ is a subword of $b^*(w'')w''$ (really?).

$\square$