

**Definition 1.** For a parameter  $n$ , Let  $L \subset \mathbb{N}^+$  be the set of non-periodic words over the alphabet  $\Sigma = \mathbb{N}$  that are bigger in Arabic (right-to-left) lexicographical order than all of their rotations. Let  $L_n$  be the set of all the words in  $L$  whose length divides  $n$ .

**Definition 2.** For a word  $w = w_1 \cdots w_{n-1} w_n$  let  $R(w) = w_n w_1 \cdots w_{n-1}$  be the rotation of  $w$  to the right. Then, the nested invocation  $R^m(w)$  is the  $m$  letter rotation to the right and its inverse  $R^{-m}(w)$  is the  $m$  letter rotation to the left.

## 1 Forward and backwards transformations

**Definition 3.** For a word  $w$  whose length is smaller or equal than  $n$ , let  $f(w)$  be the transformation defined by successive applications of the following steps to  $w$ :

- $f_1$ : Increase the first letter of the word by one.
- $f_2$ : Pad with zeros on the left to get a word of length  $n$ .
- $f_3$ : Apply the substitution rules  $u(vu)^+ \mapsto vu$  and then  $w^+ \mapsto w$ , with the longest possible  $u$  and the shortest possible  $w$ .

**Definition 4.** For a word  $w$  whose length is smaller or equal than  $n$ , let  $b(w)$  be the transformation defined by successive applications of the following steps to  $w$ :

- $b_1$ : Expand  $w$  to  $uw^m$  where  $m = \lfloor n/|w| \rfloor$  and  $u$  is the suffix of length  $n - m|w|$  of  $w$ .
- $b_2$ : Remove leading zeros.
- $b_3$ : Decrease the first letter by one.

**Observation 1.** For any  $w \in L_n$ ,  $f(b(w)) = b(f(w)) = w$ .

**Proposition 2.** *If we start with  $w(0) = 0$  and generate a sequence of words by  $w(i+1) = f(w(i))$ , we get an enumeration of all the words in  $L$  whose length is smaller or equal to  $n$ .*

*Proof.* This is a version of Duval's algorithm with a reversed order of the alphabet and a reversed order of letters in a word.  $\square$

**Definition 5.** Let  $f^*(w)$  be the first word in  $f(w), f(f(w)), \dots$  whose length divides  $n$  and, similarly, let  $b^*(w)$  be the first word in  $b(w), b(b(w)), \dots$  whose length divides  $n$ .

**Definition 6.** Let  $w(0), w(1), \dots$  be the sequence generated by starting with  $w(0) = 0$  and then continuing ad infinitum by  $w(i+1) = f^*(w(i))$  and let  $w^\infty \in \mathbb{N}^\omega$  be the concatenation of all these words.

## 2 Where can I find $w$ as a sub-word of $w^\infty$ ?

In this section we point at the position of an arbitrary word  $w$  as a sub-word of  $w^\infty$  relative to the position of the a corresponding word in  $L_n$ . This is given in Proposition 7 and in Proposition 8. Towards the proofs of these propositions, we first establish some technical results about the functions  $b$  and  $b^*$  specified, respectively, in Definition 4 and in Definition 5.

**Proposition 3.** *If  $w \in L_n$  and  $|w| \neq n$  then  $b(w) = uw$  for some non-empty word  $u$ .*

*Proof.* The first transformation  $b_1$  extends  $w$  to the left producing the word  $b_1(w) = uw^m$  where  $u$  is a tail of  $w$ . Since  $w \in L_n$  and because it contains a letter  $\sigma$  that is not zero, we have, by maximality of  $w$  among its rotations in right-to-left lexicographical order, that its last letter is not zero. The last letter of  $u$  is the last letter of  $w$  so it is also not zero. This gives us that the next transformation  $b_2$ , that deletes trailing zeros, leaves at least the last copy of  $w$  and the last letter of the before-last (full or partial) copy at the tail of  $b_1(w)$ . Thus,  $b_2(b_1(w)) = uw$  where  $u$  is a non-empty word whose first letter is not zero. Then, the last transformation  $b_3$  only decreases the first letter of  $u$  by one which gives us that  $b(w) = b_3(b_2(b_1(w))) = vw$  for some non-empty word  $v$ .  $\square$

**Proposition 4.** *For any  $w = 0^l \sigma \hat{w} \in L_n$  where  $\sigma$  is a non-zero letter and  $\hat{w}$  is a non-empty word there is a non-empty word  $u$  such that  $b(w) = u\hat{w}$ .*

*Proof.* If  $|w| \neq n$  the proof follows by Proposition 3. If  $|w| = n$  then  $b_1(w) = w$ ,  $b_2(b_1(w)) = \sigma \hat{w}$ , and  $b_3(b_2(b_1(w))) = (\sigma - 1)\hat{w}$  and the claim follows as well.  $\square$

**Proposition 5.** *For any  $w = 0^l \sigma \hat{w} \in L_n$  where  $\sigma$  is a non-zero letter and  $\hat{w}$  is a non-empty word there is a non-empty word  $u$  such that  $b^*(w) = u\hat{w}$ .*

*Proof.* By definition,  $b^*$  consists of repeated applications of  $b$  until a word whose length divides  $n$  is obtained. In particular, only the first of these applications of  $b$  act on a parameter whose length may be  $n$ . By Proposition 4, this first application produces a word that ends with  $\hat{w}$  and by Proposition 3 each of the remaining applications only add prefixes.  $\square$

**Proposition 6.** *Let  $w$  be an arbitrary word in  $\mathbb{N}^n$  and let  $\bar{w} = f_3(w)$ . Let  $l$  be the (possibly zero) number of trailing zeros (from the left) in  $w$ . Then, for all  $0 \leq i \leq |w| - l - 1$ , the word  $R^i(\bar{w})$  comes  $i + n - |w|$  letters before  $w$  as a sub-word of  $w^\infty$ .*

**Proposition 7.** *For a given  $w \in L_n$ , let  $l$  be the number of trailing zeros (from the left) in  $w$  and let  $\bar{w} = b_1(w)$ . Then, for all  $0 \leq i \leq |w| - l - 1$ , the word  $R^i(\bar{w})$  comes  $i + n - |w|$  letters before  $w$  as a sub-word of  $w^\infty$ .*

*Proof.* By Proposition 5 the words that come before  $w$  ends with the last  $|w| - l$  letters of  $w$ . In particular, the  $n$  letter word that starts  $i + n - |w|$  before  $w$  is  $R^i(\bar{w})$ .  $\square$

**Proposition 8.** For a given  $w \in L_n$ , let  $l$  be the number of trailing zeros (from the left) in  $w$  and let  $\bar{w} = b_1(w)$ . Then, for all  $|w| - l \leq i \leq n - 1$  the word  $R^i(\bar{w})$  comes  $i - (n - |f_3(u)| \pmod{n})$  letters before the first  $u \in \langle 0^{m-1}(\bar{w}_m + 1)\bar{w}_{m+1} \cdots \bar{w}_n \rangle_{m=i+1}^n$  such that  $u \in L_n$  and  $\max\{u_1, \dots, u_{n-1}\} \leq \max(w)$ .

**Proposition 9.** The word  $w^\infty$  contains all the words in  $\mathbb{N}^n$  as subwords.

*Proof.* Any word of length  $n$  is a rotation of the expansion of a word in  $L_n$ .  $\square$

**Proposition 10.** For any  $k$  the prefix  $w_1^\infty \cdots w_k^\infty$  is an  $n$ -order de Bruijn sequences. Moreover, it is the reversed of the  $n$ -order prefer-max sequence on the alphabet  $\langle 0, \dots, k - 1 \rangle$  (in this order).

*Proof.* Counting argument + arguing that if  $|w| = n - 1$  and  $\sigma_1 < \sigma_2$  then  $w\sigma_1$  comes before  $w\sigma_2$  as subwords of  $w^\infty$ .  $\square$

**Proposition 11.** For  $w \in \mathbb{N}^n$ , let  $i$  be the minimal index such that  $R^{-i}(w) \in L$  and let  $\bar{w} = R^{-i}(w)$ . Let  $\bar{w}^+ = \bar{w}_{1..i}(\bar{w}_{i+1} + 1)\bar{w}_{(i+2)..n}$ , i.e., the word obtained by increasing the  $(i + 1)$ th letter of  $\bar{w}$  by one. Then, the function

$$\text{next}(w) = \begin{cases} f^*(f_3(w))_1 & \text{if } w \in L; \\ w_1 + 1 & \text{if } \bar{w}_{1..i} = 0^i \wedge \bar{w}^+ \in L \wedge \max(\bar{w}_{1..(n-1)}^+) \leq \max(w); \\ 0 & \text{if } \bar{w}_{1..i} = 0^i \wedge (\bar{w}^+ \notin L \vee \max(\bar{w}_{1..(n-1)}^+) > \max(w)); \\ w_1 & \text{otherwise.} \end{cases}$$

represents the mapping of a word  $w$  to the letter that follows the (one and only) occurrence of  $w$  as a subword of  $w^\infty$ .

**Definition 7.** Let  $w(0) = 0, w(1) = f^*(w(0)), \dots, w(i) = f^*(w(i-1)), \dots$  be our enumeration of all the words in  $L_n$ . Let  $w^{(i)} = w(0) \cdots w(i)$  be the concatenation of the first  $i$  words in this enumeration and let  $u(j) = w_{j-n+1}^{(i)} \cdots w_j^{(i)}$  be the “window” of length  $n$  before the  $j$ th letter in  $w^{(i)}$ .

**Proposition 12.** For a word  $w = w(i)$  be the  $i$ th word in the above enumeration. Let  $l$  be the number of leading zeros in  $w$  and let  $\hat{w} = R^{-l}(b_1(b^*(w)))$ . Then, inserting the cycle  $\langle R^{-s}(b_1(w)) \rangle_{s=0}^{n-1}$  after the word  $\hat{w}_{1..(n-l)}0^l$  in  $\langle u(j) \rangle_{j=0}^{i-1}$  yields the sequence  $\langle u(j) \rangle_{j=0}^i$ .