Definition 1. For a parameter n, Let $L \subset \mathbb{N}^+$ be the set of non-periodic words over the alphabet $\Sigma = \mathbb{N}$ that are bigger in Arabic (right-to-left) lexicographical order than all of their rotations. Let L_n be the set of all the words in L whose length divides n.

Definition 2. For a word $w = w_1 \cdots w_{n-1} w_n$ let $R(w) = w_n w_1 \cdots w_{n-1}$ be the rotation of w to the right. Then, the nested invocation $R^m(w)$ is the m letter rotation to the right and its inverse $R^{-m}(w)$ is the m letter rotation to the left.

1 Forward and backwards transformations

Definition 3. For a word w whose length is smaller or equal than n, let $f(w)$ be the transformation defined by successive applications of the following steps to w:

f_1: Increase the first letter of the word by one.

f_2: Pad with zeros on the left to get a word of length n.

f_3: Apply the substitution rules $u(vu)^+ \mapsto vu$ and then $w^+ \mapsto w$, with the longest possible u and the shortest possible w.

Definition 4. For a a word w whose length is smaller or equal than n, let $b(w)$ be the transformation defined by successive applications of the following steps to w:

b_1: Expand w to uw^m where $m = \lfloor n/|w| \rfloor$ and u is the suffix of length $n - m|w|$ of w.

b_2: Remove leading zeros.

b_3: Decrease the first letter by one.

Observation 1. For any $w \in L_n$, $f(b(w)) = b(f(w)) = w$.

Proposition 2. *If we start with $w(0) = 0$ and generate a sequence of words by $w(i+1) = f(w(i))$, we get an enumeration of all the words in L whose length is smaller or equal to n.*

Proof. This is a version of Duval's algorithm with a reversed order of the alphabet and a reversed order of letters in a word. □

Definition 5. Let $f^*(w)$ be the first word in $f(w), f(f(w)), \ldots$ whose length divides n and, similarly, let $b^*(w)$ be the first word in $b(w), b(b(w)), \ldots$ whose length divides n.

Definition 6. Let $w(0), w(1), \ldots$ be the sequence generated by starting with $w(0) = 0$ and then continuing ad infinitum by $w(i+1) = f^*(w(i))$ and let $w^\infty \in \mathbb{N}^\omega$ be the concatenation of all these words.

2 Where can I find w as a sub-word of w^∞?

In this section we point at the position of an arbitrary word w as a sub-word of w^∞ relative to the position of the a corresponding word in L_n. This is given in Proposition 6 and in Proposition 7. Towards the proofs of these propositions, we first establish some technical results about the functions b and b^* specified, respectively, in Definition 4 and in Definition 5.

Proposition 3. *If $w \in L_n$ and $|w| \neq n$ then $b(w) = uw$ for some non-empty word u.*

Proof. The first transformation b_1 extends w to the left producing the word $b_1(w) = uw^m$ where u is a tail of w. Since $w \in L_n$ and because it contains a letter σ that is not zero, we have, by maximality of w among its rotations in right-to-left lexicographical order, that its last letter is not zero. The last letter of u is the last letter of w so it is also not zero. This gives us that the next transformation b_2, that deletes trailing zeros, leaves at least the last copy of w and the last letter of the before-last (full or partial) copy at the tail of $b_1(w)$. Thus, $b_2(b_1(w)) = uw$ where u is a non-empty word whose first letter is not zero. Then, the last transformation b_3 only decreases the first letter of u by one which gives us that $b(w) = b_3(b_2(b_1(w))) = vw$ for some non-empty word v. $\square$

Proposition 4. *For any $w = 0^l\sigma\hat{w} \in L_n$ where σ is a non-zero letter there is a non-empty word u such that $b(w) = u\hat{w}$.*

Proof. If $|w| \neq n$ the proof follows by Proposition 3. If $|w| = n$ then $b_1(w) = w$, $b_2(b_1(w)) = \sigma\hat{w}$, and $b_3(b_2(b_1(w))) = (\sigma-1)\hat{w}$ and the claim follows as well. $\square$

Proposition 5. *Let w be an arbitrary word in $\mathbb{N}^n$ and let and let $\bar{w} = f_3(w)$. Let l be the (possibly zero) number of trailing zeros (from the left) in w. Then, for all $0 \leq i \leq |w| - l - 1$, the word $R^i(\bar{w})$ comes $i + n - |w|$ letters before w as a sub-word of w^∞.*

Proposition 6. *For a given $w \in L_n$, let l be the number of trailing zeros (from the left) in w and let $\bar{w} = b_1(w)$. Then, for all $0 \leq i \leq |w| - l - 1$, the word $R^i(\bar{w})$ comes $i + n - |w|$ letters before w as a sub-word of w^∞.*

Proof. By Proposition **??** the words that come before w ends with the last $|w|-l$ letters of w. In particular, the n letter word that starts $i + n - |w|$ before w is $R^i(\bar{w})$. $\square$

Proposition 7. *For a given $w \in L_n$, let l be the number of trailing zeros (from the left) in w and let $\bar{w} = b_1(w)$. Then, for all $|w| - l \leq i \leq n - 1$ the word $R^i(\bar{w})$ comes $i - (n - |f_3(u)| \pmod n)$ letters before the first $u \in \langle 0^{m-1}(\bar{w}_m + 1)\bar{w}_{m+1} \cdots \bar{w}_n\rangle_{m=i+1}^n$ such that $u \in L_n$ and $\max\{u_1, \ldots, u_{n-1}\} \leq max(w)$.*

Proposition 8. *The word w^∞ contains all the words in $\mathbb{N}^n$ as subwords.*

Proof. Any word of length n is a rotation of the expansion of a word in L_n. $\square$

Proposition 9. *For any k the prefix $w_1^\infty \cdots w_{k^n}^\infty$ is an n-order de Bruijn sequences. Moreover, it is the reversed of the n-order prefer-max sequence on the alphabet $\langle 0, \ldots, k-1 \rangle$ (in this order).*

Proof. Counting argument + arguing that if $|w| = n-1$ and $\sigma_1 < \sigma_2$ then $w\sigma_1$ comes before $w\sigma_2$ as subwords of w^∞. □

Proposition 10. *For $w \in \mathbb{N}^n$, let i be the minimal index such that $R^{-i}(w) \in L$ and let $\bar{w} = R^{-i}(w)$. Let $\bar{w}^+ = \bar{w}_{1..i}(\bar{w}_{i+1}+1)\bar{w}_{(i+2)..n}$, i.e., the word obtained by increasing the $(i+1)$th letter of $\bar{w}$ by one. Then, the function*

$$next(w) = \begin{cases} f^*(f_3(w))_1 & \text{if } w \in L; \\ w_1 + 1 & \text{if } \bar{w}_{1..i} = 0^i \wedge \bar{w}^+ \in L \wedge \max(\bar{w}^+_{1..(n-1)}) \leq \max(w); \\ 0 & \text{if } \bar{w}_{1..i} = 0^i \wedge (\bar{w}^+ \notin L \vee \max(\bar{w}^+_{1..(n-1)}) > \max(w)); \\ w_1 & \text{otherwise.} \end{cases}$$

represents the mapping of a word w to the letter that follows the (one and only) occurrence of w as a subword of w^∞.

Definition 7. Let $w(0) = 0, w(1) = f^*(w(0)), \ldots, w(i) = f^*(w(i-1)), \ldots$ be our enumeration of all the words in L_n. Let $w^{(i)} = w(0) \cdots w(i)$ be the concatenation of the first i words in this enumeration and let $u(j) = w^{(i)}_{j-n+1} \cdots w^{(i)}_j$ be the "window" of length n before the jth letter in $w^{(i)}$.

Proposition 11. *For a word $w = w(i)$ be the ith word in the above enumeration. Let l be the number of leading zeros in w and let $\hat{w} = R^{-l}(b_1(b^*(w)))$. Then, inserting the cycle $\langle R^{-s}(b_1(w)) \rangle_{s=0}^{n-1}$ after the word $\hat{w}_{1..(n-l)}0^l$ in $\langle u(j) \rangle_{j=0}^{i-1}$ yields the sequence $\langle u(j) \rangle_{j=0}^{i}$.*