“True” form of Liouville theorem

Assume $$\dot{\vec{X}} = \vec{f}(\vec{X}, t)$$

$$\vec{X}(t+dt) = \vec{X}(t) + dt\,\vec{f}(\vec{X}, t)$$

$$(\vec{X} + d\vec{X})(t+dt) = (\vec{X} + d\vec{X})(t) + dt\,\vec{f}(\vec{X} + d\vec{X}, t)$$

$$\implies d\vec{X}(t+dt) = d\vec{X}(t) + dt(\vec{f}(\vec{X} + d\vec{X}, t) - \vec{f}(\vec{X}, t))$$

$$J_f(\vec{X})d\vec{X}$$

$$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_N} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_N}{\partial X_1} & \cdots & \frac{\partial f_N}{\partial X_N} \end{pmatrix}$$

$$d\vec{X}(t+dt) = (I + J_f(\vec{X})dt)d\vec{X}(t)$$

Infinitesimal volume in phase space

$$\delta\Omega(t) \to \delta\Omega(t+dt) = det(I + J_f(\vec{X})dt)\delta\Omega(t)$$

$$det(I+\epsilon A) \simeq 1 + \epsilon Tr(A) + \mathcal{O}(\epsilon^2)$$

$$\implies det(I + dtJ_f) \simeq 1 + dtTr(J_f) + \mathcal{O}(dt^2)$$

If $Tr(J_f) = divf = \sum_i \frac{\partial f_i}{\partial X_i} = 0$ **, then**

$$\delta\Omega(t+dt) = (1 + \mathcal{O}(dt^2))\delta\Omega(t) \qquad \implies \frac{d\Omega}{dt} = 0$$

“True” form of Liouville theorem

Assume $$\dot{\vec{X}} = \vec{f}(\vec{X}, t)$$

$\Omega(t)$: volume in phase space

$$Tr(J_f) = div f = \sum_i \frac{\partial f_i}{\partial X_i} = 0$$

$$\Longrightarrow \frac{d\Omega}{dt} = 0$$

“Conservative system”

Example : Hamiltonian system

$$\frac{d}{dt}\begin{pmatrix} q_i \\ p_i \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p_i} \\ -\frac{\partial H}{\partial q_i} \end{pmatrix}$$

$$J_f = \sum_i \left(\frac{\partial f_{q_i}}{\partial q_i} + \frac{\partial f_{p_i}}{\partial p_i}\right)$$

$$f_{q_i} = \frac{\partial H}{\partial p_i} \qquad\qquad f_{p_i} = -\frac{\partial H}{\partial q_i}$$

$$J_f = \sum_i \left(\frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial q_i \partial p_i}\right) = 0$$

Example 2 :

$$\frac{d}{dt}\begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} = M \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} \qquad M(t) = \begin{pmatrix} 0 & 1 \\ -\omega^2(t) & 0 \end{pmatrix}$$

$Tr(M) = 0$ “Conservative system”

Counter example : friction

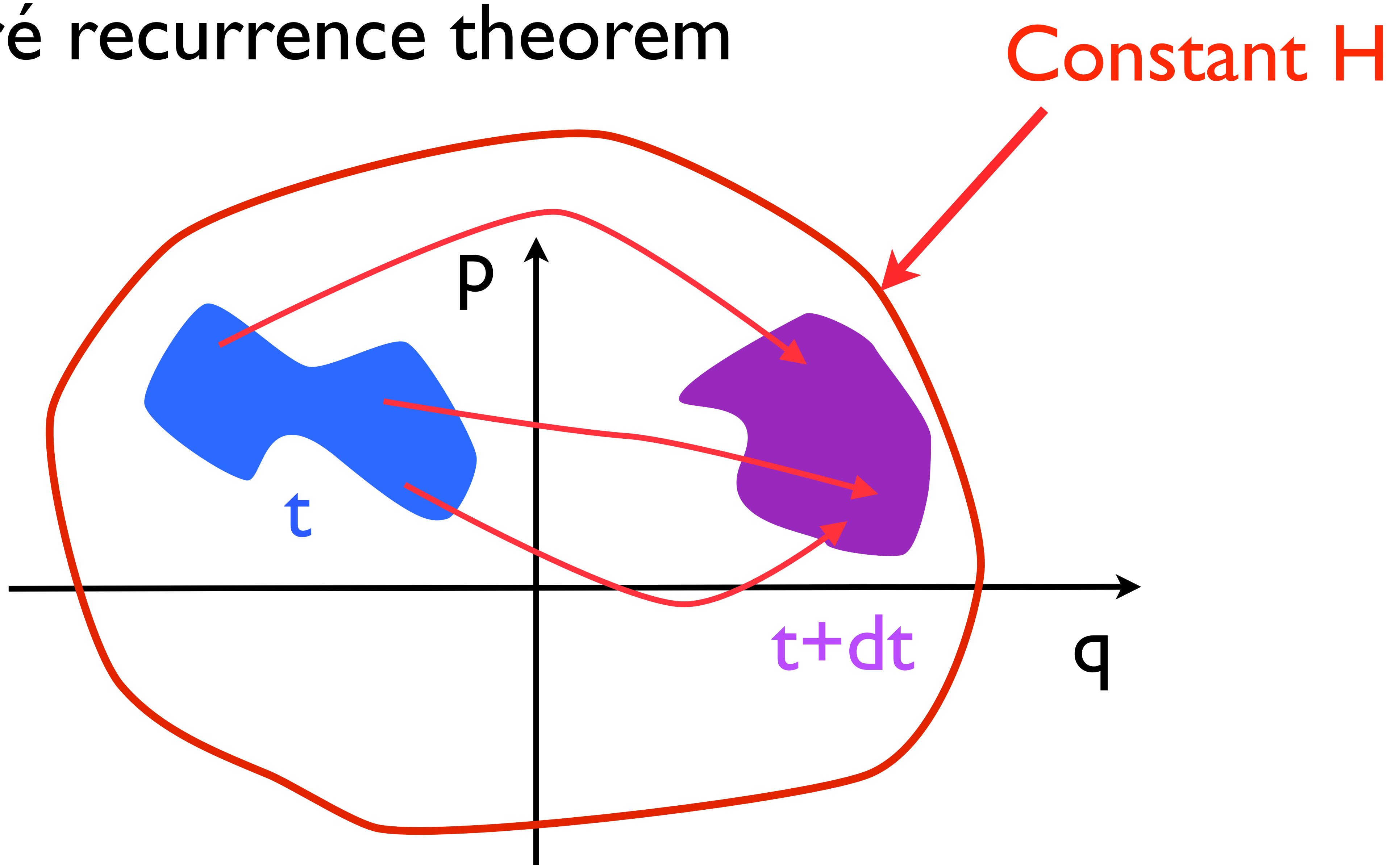
$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0$$

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

$Tr(M) = -\gamma \neq 0$ “Dissipative system”

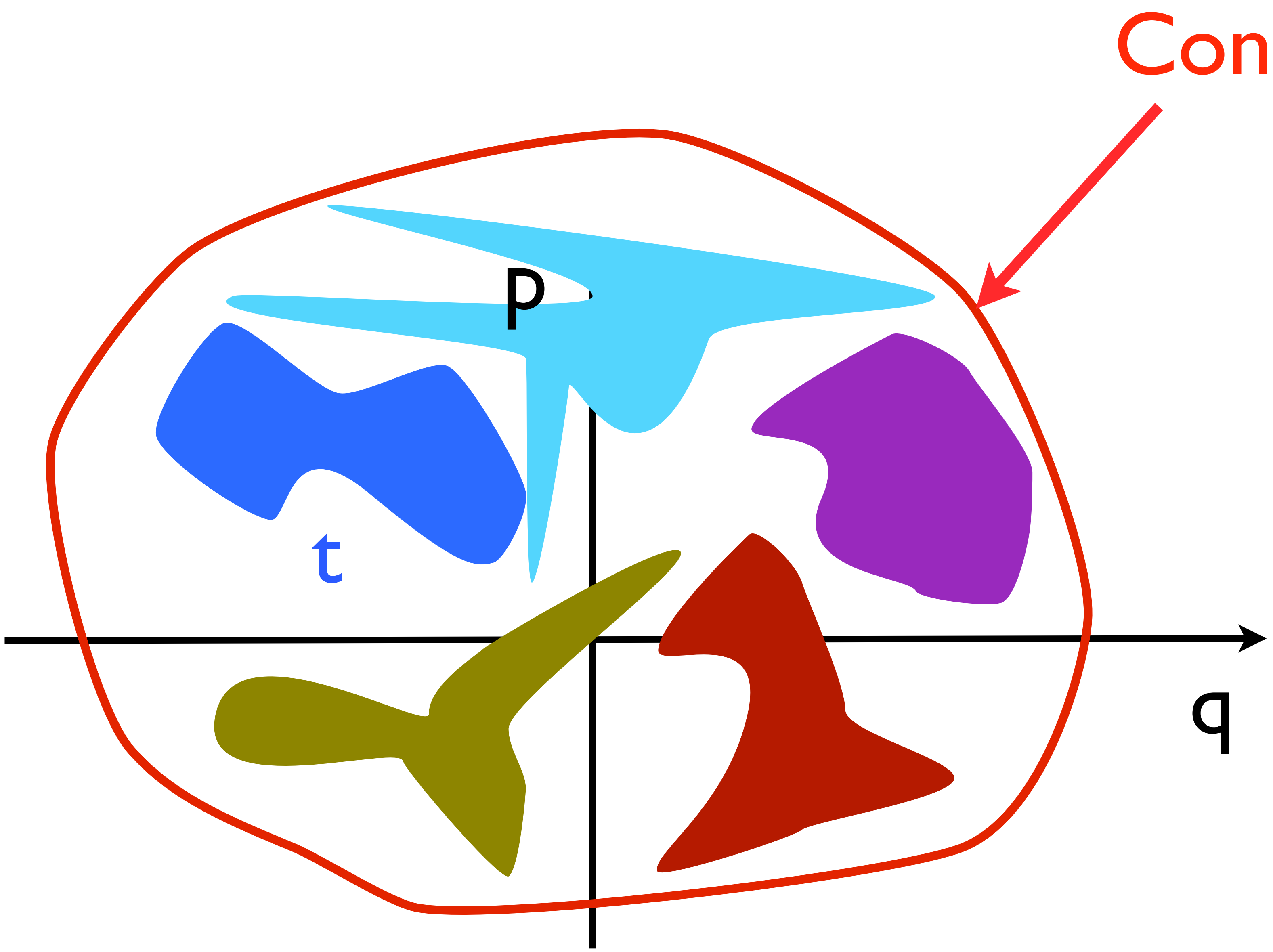
Advanced mechanics I:
Poincaré recurrence theorem
Hamilton-Jacobi formalism
Action-Angle formalism

Poincaré recurrence theorem

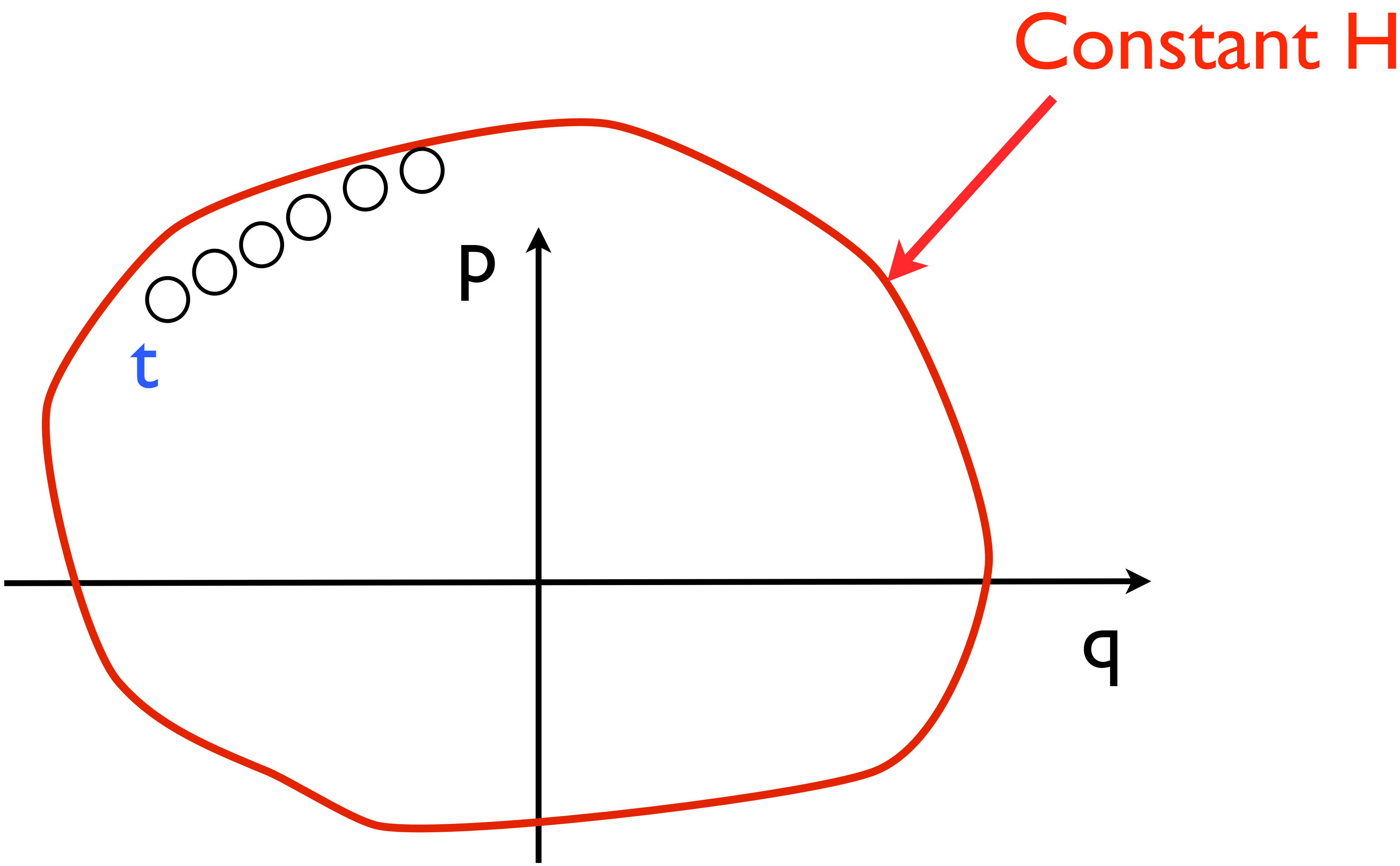


Liouville theorem : area does not change.

Assume constant Hamiltonian so that area enclosed is finite. What happens to the blob for long times ?



Since the area of the blob is constant, at some point, because the area inside the curve is finite, two blobs will overlap.



Consider a very small region in phase space dA. Whatever dA is, at some point, we will "fill" the phase space inside the line of constant H.

It means that there exists two times so that :

$$dA_{t_1} \cap dA_{t_2} \neq \emptyset$$

$$dA_{t_1} \cap dA_{t_2} = E$$

Assume $t_2 > t_1$

Let us go back in time of t_1 steps

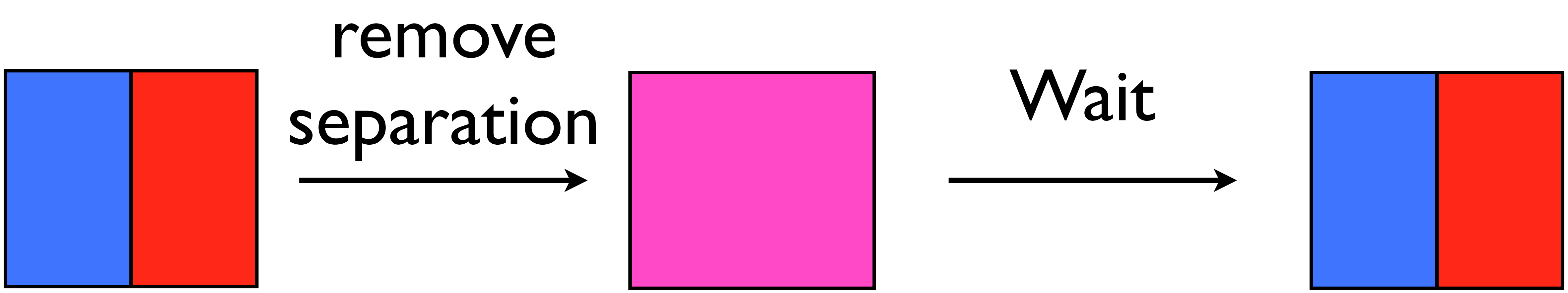
- first flow : E comes back inside the original dA_0
- second flow : E comes back inside $dA_{t_2 - t_1}$

$$\Longrightarrow \; dA_0 \cap dA_{t_2 - t_1} \neq \emptyset$$

Since dA is arbitrary, this means that this system can come back arbitrary close to its initial condition. This is Poincaré recurrence theorem.

This can be generalized for 2N phase space. This typically applies to Hamiltonian systems.

Classical example : box filled with two gas, 2x10^23 dimensional space



Thermodynamical paradox !

(classical answer : takes close to infinite time to have overlap in phase space)

Quantum recurrence theorem

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Quantum Recurrence Theorem

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(Received October 9, 1956)

A recurrence theorem is proved, which is the quantum analog of the recurrence theorem of Poincaré. Some statistical consequences of the theorem are stressed.

In this paper we shall show that a similar recurrence theorem holds in quantum theory; it can be formulated as follows: "Let us consider a system with *discrete* energy eigenvalues E_n; if $\Psi(t_0)$ is its state vector in the Schrödinger picture at the time t_0 and ϵ is any positive number, at least one time T will exist such that the norm $\|\Psi(T)-\Psi(t_0)\|$ of the vector $\Psi(T)-\Psi(t_0)$ is smaller than ϵ."[2]

Discrete spectrum is a condition equivalent to finite volume in phase space in classical mechanics.

The proof of this theorem is simple and can be sketched in the following way: The equation of motion is

$$i(\partial\Psi(t)/\partial t)=H\Psi(t); \tag{1}$$

the formal solution is

$$\Psi(t)=\sum_{n=0}^{\infty} r_n \exp(i\varphi_n - iE_n t)u(E_n), \tag{2}$$

(the r_n's being real positive numbers). From (2),

$$\|\Psi(T)-\Psi(t_0)\|=2\sum_{n=0}^{\infty} r_n^2(1-\cos E_n\tau); \quad (\tau\equiv T-t_0), \tag{3}$$

and, if N is suitably chosen,

$$\sum_{n=N}^{\infty} r_n^2(1-\cos E_n\tau)<\epsilon. \tag{4}$$

Consequently, it is sufficient to prove that there is a value of τ such that

$$\sum_{n=0}^{N-1}(1-\cos E_n\tau)<\epsilon. \tag{5}$$

But this is actually the case according to a standard result of the theory of the almost-periodic functions.[3]

Hamilton-Jacobi formulation of mechanics

Lagrangian	N 2nd order ODEs
Hamiltonian	2N 1st order ODEs
Hamilton-Jacobi	1 equation on N variables+ time

Recall the action

$$\mathcal{S}[q_i] = \int_0^T L(q_i, \dot{q}_i, t)dt$$

Define the following functional for a physical trajectory

$$\omega(q_i^{init}, q_i^{final}, T) = \mathcal{S}[q_i]$$

Fix the initial condition, how does it depend on T, q_i^{final} ?

$$\omega(q_i^{final}, T) = \mathcal{S}[q_i]$$

$$\omega(q_i^{final}, T) = \mathcal{S}[q_i]$$

$$\delta\mathcal{S} = \sum_i \int_0^T \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) \right) \delta q_i dt + \left. \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right|_0^T$$

With fixed initial conditions, for a solution of EOM, we therefore have

$$\delta\mathcal{S}|_T = \sum_i \left. \frac{\partial L}{\partial \dot{q}_i} \right|_T \delta q_i^{final} = \sum_i p_i^{final} \delta q_i^{final}$$

$$\implies \frac{\partial \omega}{\partial q_i^{final}} = p_i^{final}$$

$$\omega(q_i^{final}, T) = \mathcal{S}[q_i] = \int_0^T L(q_i, \dot{q}_i, t)dt$$

$$\frac{d\omega}{dT} = L(q_{final}, \dot{q}_{final}, T)$$

$$= \sum_i \frac{\partial \omega}{\partial q_i^{final}} \dot{q}_i^{final} + \frac{\partial \omega}{\partial T}$$

$$\frac{\partial \omega}{\partial q_i^{final}} = p_i^{final}$$

$$\implies L(q_{final}, \dot{q}_{final}, T) = p_i^{final} \dot{q}_i^{final} + \frac{\partial \omega}{\partial T}$$

$$L(q_{final}, \dot{q}_{final}, T) = p_i^{final} \dot{q}_i^{final} + \frac{\partial \omega}{\partial T}$$

$$\implies \frac{\partial \omega}{\partial T} = -H(q_{final}, p_{final}, T)$$

With $\frac{\partial \omega}{\partial q_i^{final}} = p_i^{final}$

$$\boxed{\frac{\partial \omega}{\partial T} = -H(q_i, \frac{\partial \omega}{\partial q_i}, T)}$$

Hamilton-Jacobi equation

$$\boxed{\frac{\partial \omega}{\partial T} = -H(q_i, \frac{\partial \omega}{\partial q_i}, T)}$$

Strongly reminiscent of Schrodinger equation. Action plays the same role as wave function.

If we solve this equation, how do we recover trajectories ?

Example : 1-D system $H = \dfrac{p^2}{2m} + V(q)$

$$\Longrightarrow \omega_t = -V(q) - \frac{\omega_q^2}{2m}$$

Hamilton-Jacobi equation

Ansatz : $\omega = w(q) - Et$ "Separation of variables"

$$\Longrightarrow E = V(q) + \frac{1}{2m} w'(q)^2$$

$$\Longrightarrow \frac{dw}{dq} = \pm\sqrt{2m(E - V(q))}$$

$$\Longrightarrow w(q) = \pm \int^q \sqrt{2m(E - V(x))}dx$$

So a general solution of H-J equation is :

$$\omega(q,t) = \pm \int^q \sqrt{2m(E - V(x))}dx - Et$$

This is expressed as a function of a constant E, which obviously is a new extra parameter for the Jacobi function. Let us treat E as a VARIABLE for now.

Claim : $\frac{\partial \omega}{\partial E} = \beta$ second constant of motion

$$\implies \frac{d\omega}{dt} = L(q, \dot{q})$$

Explicit dependency on E (via initial/final conditions)

No explicit dependency in E

$$\partial_E \implies \frac{d}{dt}\frac{\partial \omega}{\partial E} = 0$$

$\dfrac{\partial \omega}{\partial E} = \beta$ second constant of motion

This is a general feature in Hamilton-Jacobi formalism : partial derivative of solution with respect to integration constants are constant of motion.

$$\omega(q,t) = \pm \int^{q} \sqrt{2m(E - V(x))}dx - Et \qquad \frac{\partial \omega}{\partial E} = \beta$$

$$\beta = \pm \int^{q(t)} \frac{m}{\sqrt{2m(E - V(x))}}dx - t$$

We recognize the equation for a 1D system we had derived in the beginning of this class. There are two integration constants, so this solves the problem.

Parameter E introduced happens to be the energy. This is one of the advantage of this formalism : constant of motions appear directly during integration.

$$\frac{\partial \omega}{\partial E} = \beta$$

Partial derivative with respect to integration constants are constants of motion.

Reminder :

For time independent canonical transformation $F(q, Q)$

$$H(Q, P) = H(q(Q, P), p(Q, P))$$

With $F(q, Q, t)$ (we add a time dependency to F)

$$K(Q, P, t) = H(q(Q, P), p(Q, P), t) + \frac{\partial F}{\partial t}$$

$$\frac{\partial \omega}{\partial T} = -H(q_i, \frac{\partial \omega}{\partial q_i}, T) \qquad \frac{\partial \omega}{\partial q_i} = p_i$$

$$\Longrightarrow \; H(q_i, p_i, T) + \frac{\partial \omega}{\partial T} = 0$$

Canonical transformation cancelling the Hamiltonian are generated by the action !

$$H(q_i, p_i, T) + \frac{\partial \omega}{\partial T} = 0 = K(Q_i, P_i, T)$$

$$H(q_i, p_i, T) + \frac{\partial \omega}{\partial T} = 0 = H(Q, P)$$

Canonical transformation :

$$\frac{\partial \omega}{\partial q_i} = p_i$$ already derived $$\frac{\partial \omega}{\partial Q} = -P$$?

Q, P ?

$$\dot{Q} = \frac{\partial H(Q, P)}{\partial P} \qquad \dot{P} = -\frac{\partial H(Q, P)}{\partial Q}$$

H is now zero, so P and Q are constant of motions !

$$\omega(q,t) = \pm \int^{q} \sqrt{2m(E - V(x))}dx - Et$$

$$= \omega(q, Q, t)$$

$$\implies Q = E$$

This is the fundamental reason why we can differentiate with respect to E : E is like the canonical variable Q.

$$\frac{\partial \omega}{\partial Q} = -P$$

P constant of motion.

$$\boxed{\frac{\partial \omega}{\partial T} = -H(q_i, \frac{\partial \omega}{\partial q_i}, T)}$$

Can we connect this more explicitly to Schrodinger equation ?

$$H = \frac{p^2}{2m} + V(q)$$

$$i\hbar \partial_t \psi = H\psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial q^2} + V(q)\psi$$

Let's look for a solution of the form:

$$\psi(q,t) = R(q,t)e^{i\omega(q,t)/\hbar}$$

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial q^2} + V(q)\psi \qquad \psi(q,t) = R(q,t)e^{i\omega(q,t)/\hbar}$$

$$\psi_t = (R_t + \frac{i}{\hbar}\omega_t R)e^{i\omega/\hbar} \qquad \psi_q = (R_q + \frac{i}{\hbar}\omega_q R)e^{i\omega/\hbar}$$

$$\psi_{qq} = (R_{qq} + 2\frac{i}{\hbar}\omega_q R_q + \frac{i}{\hbar}\omega_{qq}R_q - \frac{\omega_q^2}{\hbar^2}R)e^{i\omega/\hbar}$$

$$(i\hbar R_t - \omega_t R) = -\frac{1}{2m}(\hbar^2 R_{qq} + 2i\hbar\omega_q R_q + i\hbar\omega_{qq}R_q - \omega_q^2 R) + VR$$

$$\hbar \to 0 \qquad -\omega_t = \frac{1}{2m}\omega_q^2 + V(q) = H(\omega,\omega_q)$$

Classical limit for Schrodinger gives H-J for the phase

$$\psi(q,t) = R(q,t)e^{i\omega(q,t)/\hbar} \qquad -\omega_t = \frac{1}{2m}\omega_q^2 + V(q) = H(\omega, \omega_q)$$

Classical limit for Schrodinger gives H-J for the phase

This motivates the Feynman path integral formulation of quantum mechanics :

$$\psi(q_{final}, t) = \int_{q_{init}}^{q_{final}} d[q(t)] e^{\frac{i}{\hbar}\mathcal{S}[q_i]}$$

action = HJ solution=phase

Action-angle variables

Goal : find a change of variable

$$(q_i, p_i) \to (\theta_i, I_i)$$

So that :

$$H(q_i, p_i) \to H(I_i)$$

In that case, we will have

$$\{H, I_i\} = 0$$

i.e. I_i are constant of motion.

The system is said to be "integrable" .

In that case, Hamilton' s equations of motion get trivial

$$\dot{I}_i = 0$$

$$\dot{\theta}_i = \frac{\partial H}{\partial I_i} = \omega_i$$

$$\implies \theta_i = \omega_i t$$

Now for bounded motion, we can scale things so that

$$0 \leq \theta_i \leq 2\pi$$

Assignment 7, small oscillations for heavy top

$$L = \frac{I_1}{2}(\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{I_3}{2}(\dot\psi + \dot\phi\cos\theta)^2$$

$$\theta_0, \dot\phi = \omega_p, \dot\Psi = \Omega$$

Integrable system

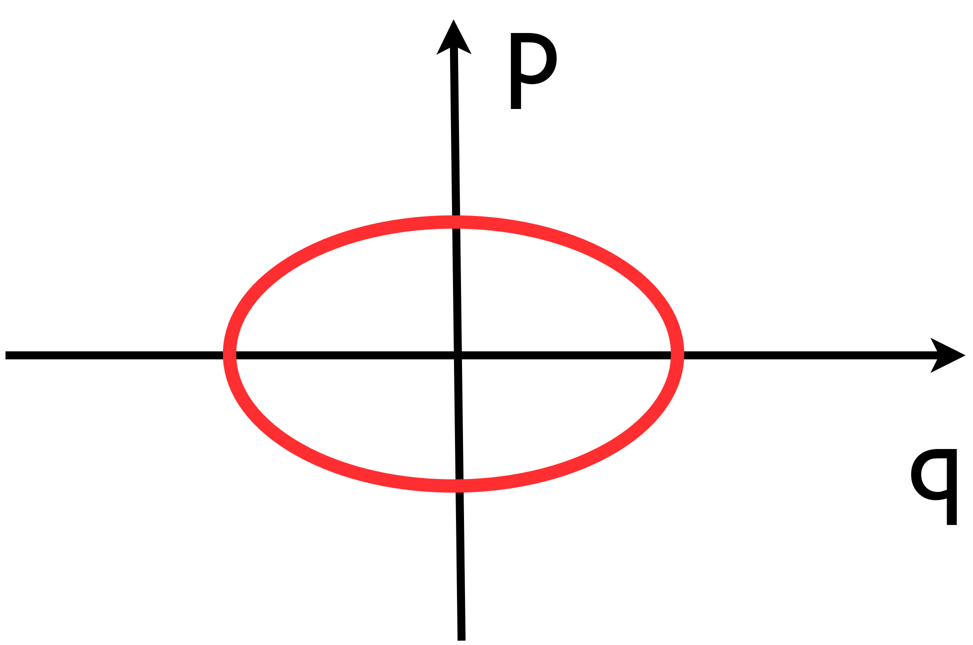
Example : 1-D system $$H = \frac{p^2}{2m} + V(q)$$

We know that
- H is conserved
- Movement is periodic

$$\Longrightarrow \quad \dot{\theta} = \frac{\partial H}{\partial I} = \omega = \frac{2\pi}{T}$$

Consider a trajectory in phase space, and let

$$I = \frac{1}{2\pi} \oint p dq$$



This is area in phase space !

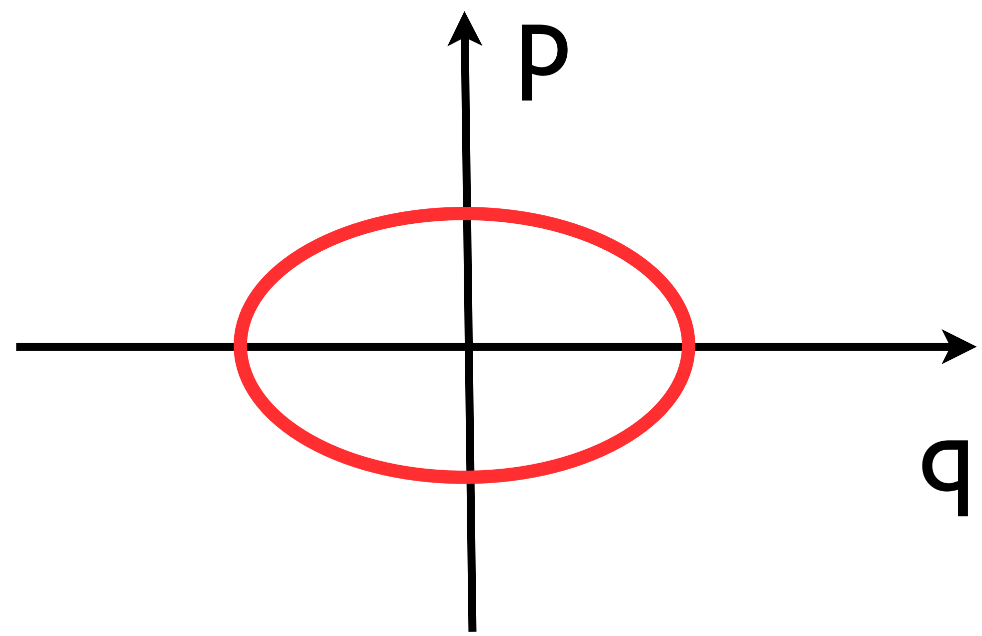
$$\frac{dI}{dE} = \frac{1}{2\pi}\oint \frac{dp}{dE}dq$$

$$p = \sqrt{2m(E-V)} \implies \frac{dp}{dE} = \frac{m}{\sqrt{2m(E-V)}}$$

$$\implies \frac{dI}{dE} = \frac{1}{2\pi}\oint \frac{mdq}{\sqrt{2m(E-V)}}$$

$$\implies \frac{dI}{dE} = \frac{1}{\omega}$$

$$\implies \frac{dI}{dE} = \frac{1}{\omega}$$

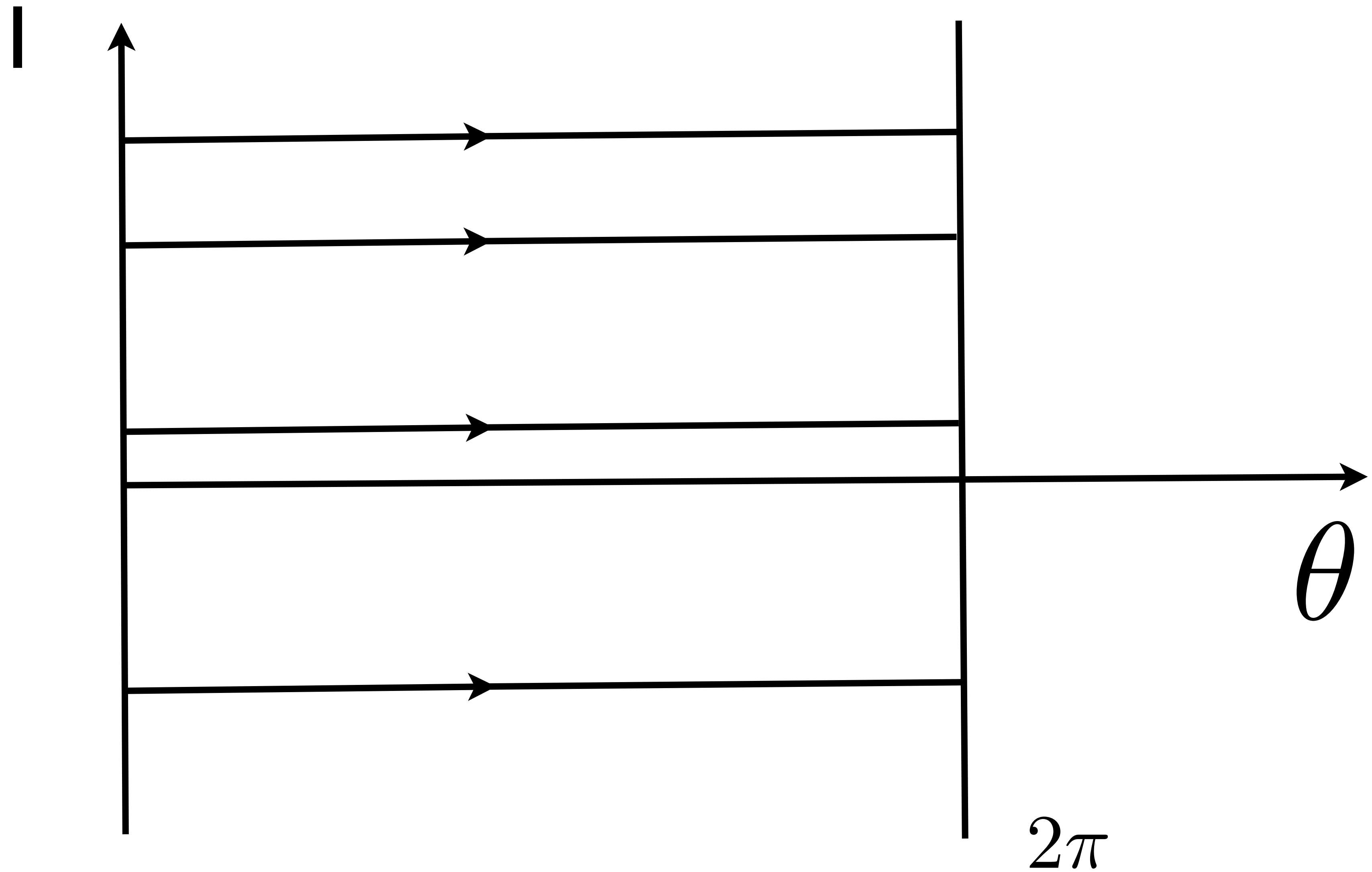


Since H=E, this implies that in phase space

$$\omega = \frac{\partial H}{\partial I}$$

as required.

NB : no need to work out θ



Lines of constant H, “untangled” phase space.

Planck : quantization = 1 state per unit of area.

So all 1 D systems are integrable.

In higher D most systems are not.

The other Liouville' s theorem :

A D-dim system is integrable iff there exists D Poisson commuting constants of motion

$$\{I_i, I_j\} = 0$$

If all the omegas are rational multiple of one another, the system comes back to its initial value at some point and is periodic.

If not, frequencies are " incommensurate", the system never comes back to its initial value (but arbitrarily close).

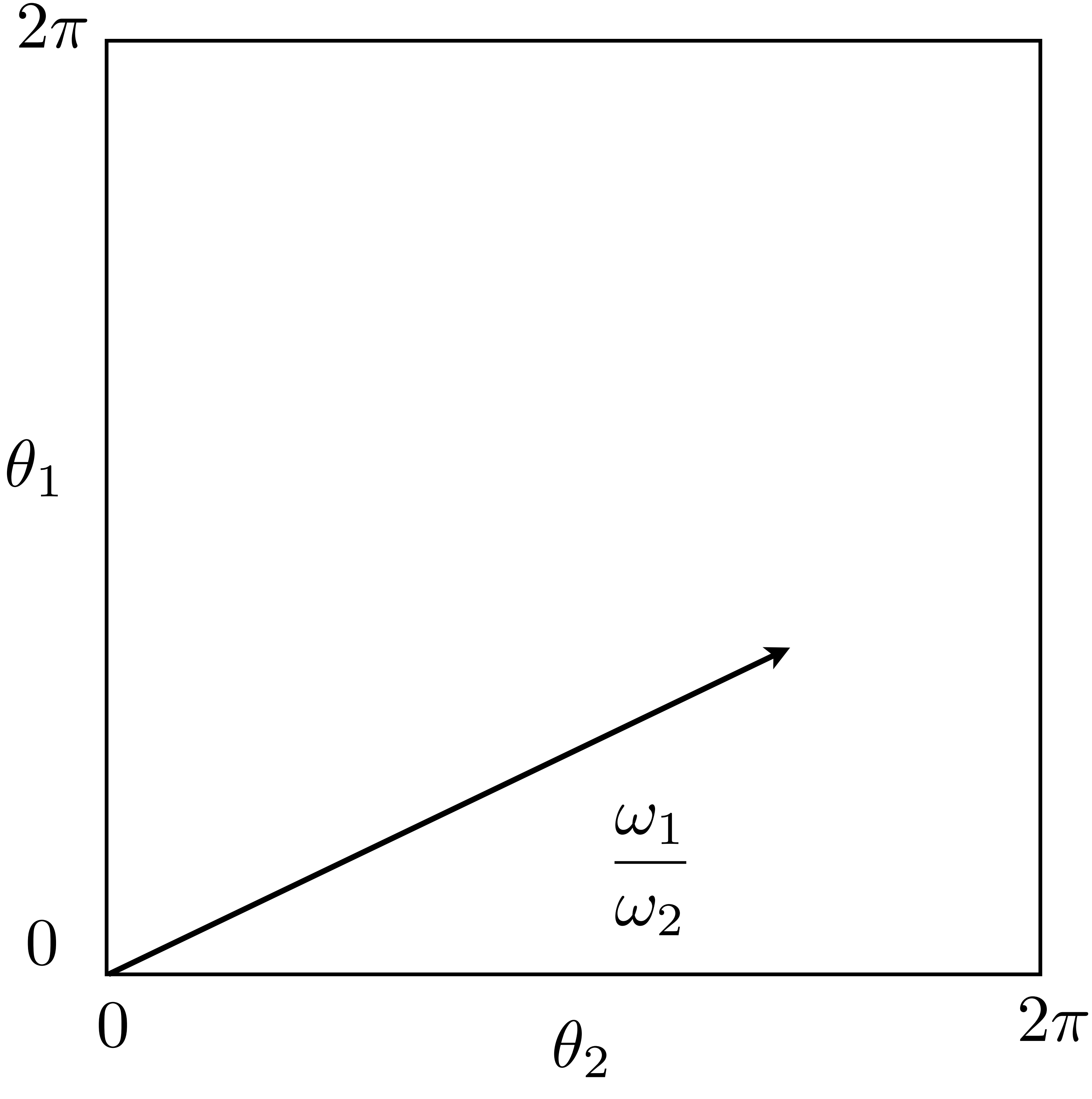
e.g. Two oscillators.

$$\theta_1 = \omega_1 t$$

$$\theta_2 = \omega_2 t$$

$$\Longrightarrow \quad \frac{\theta_1}{\theta_2} = \frac{\omega_1}{\omega_2}$$

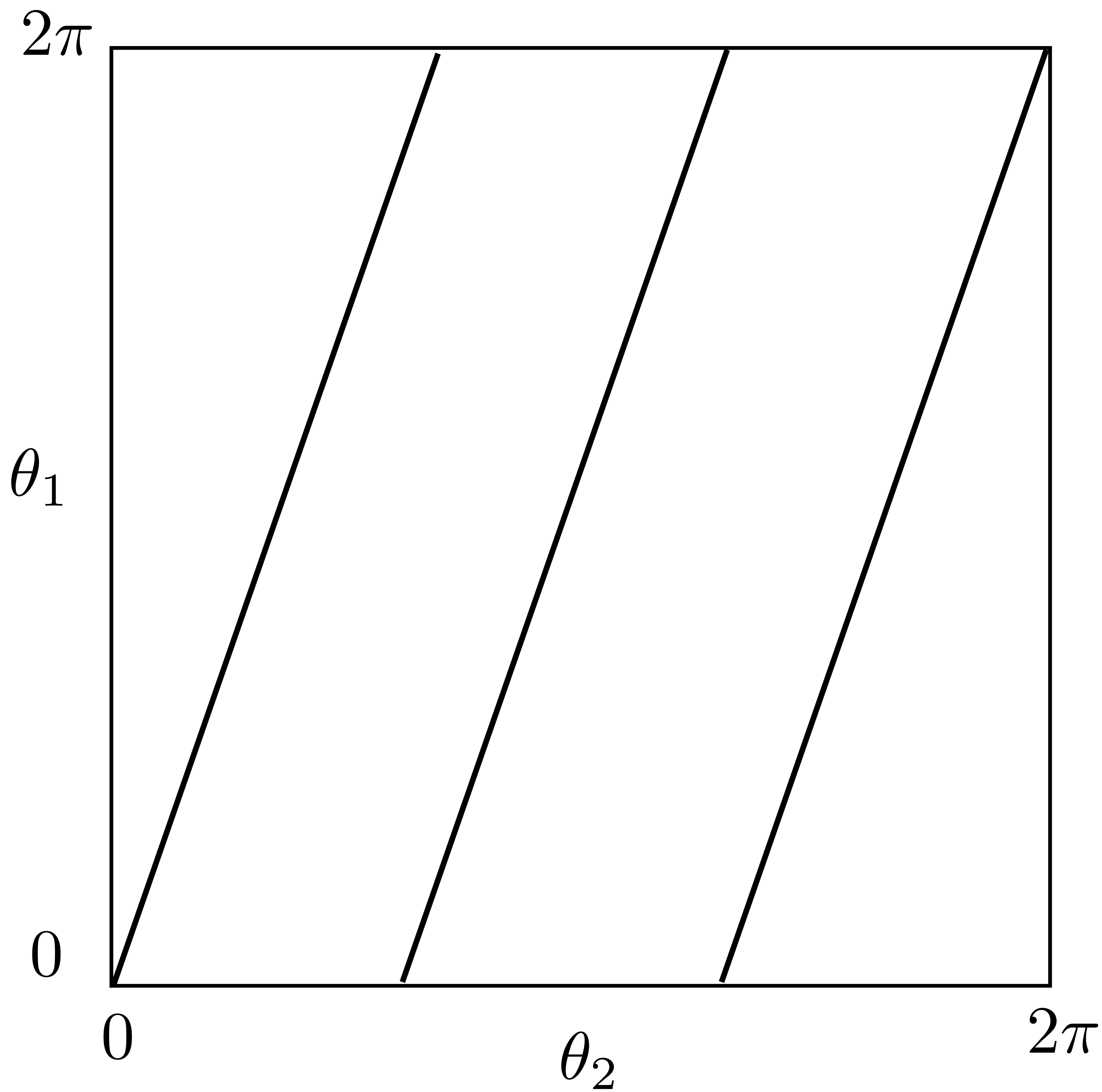
"winding number"



2π
θ_1
0
$\frac{\omega_1}{\omega_2}$
0
θ_2
2π

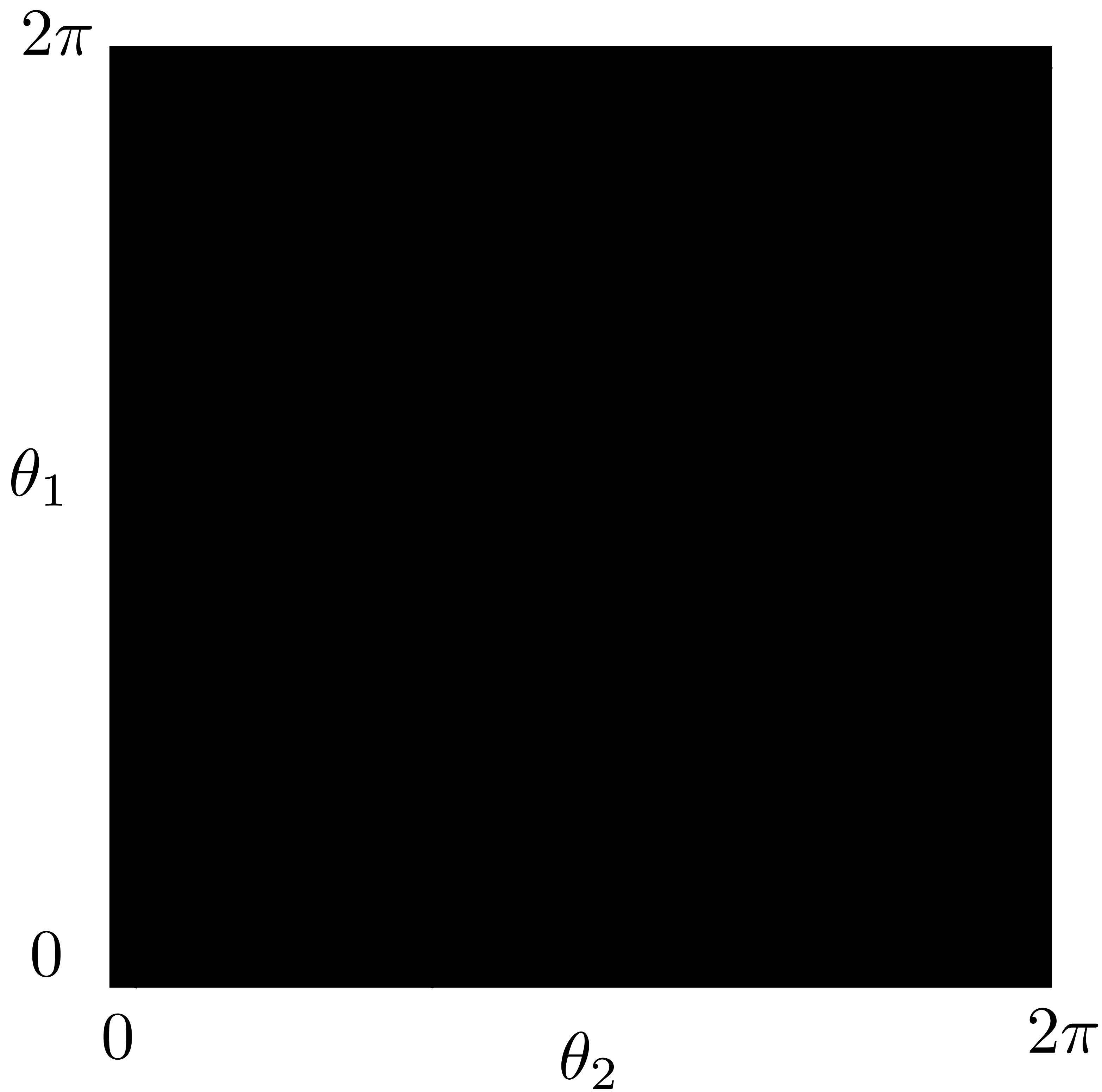
Two cases

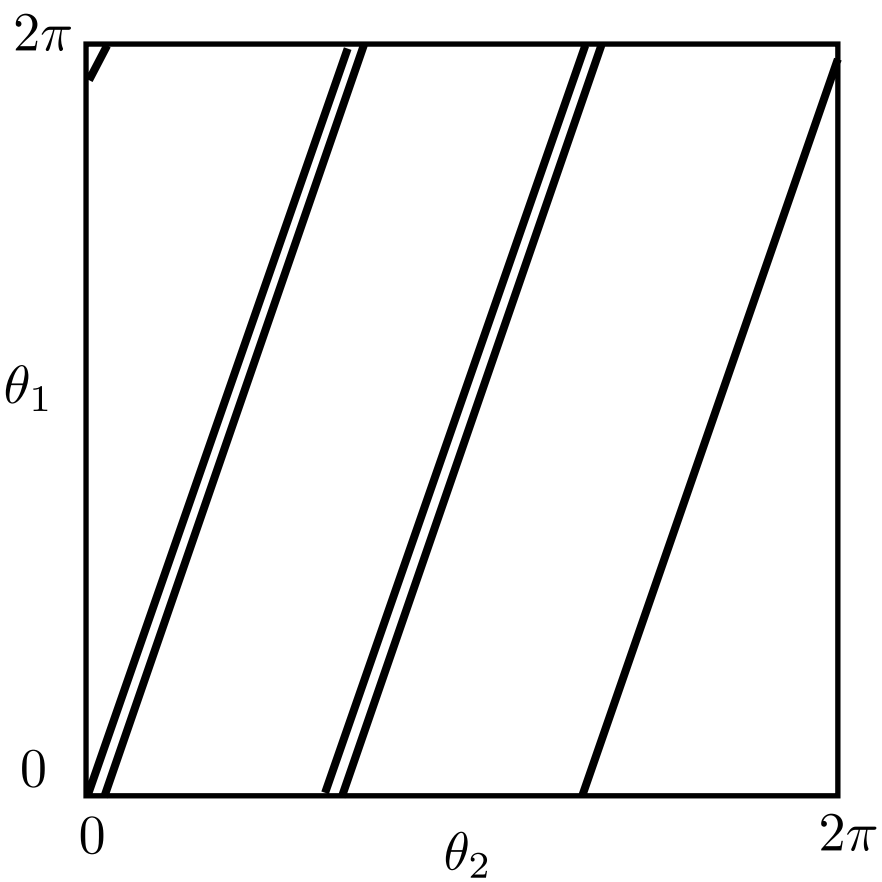
$\frac{\omega_1}{\omega_2}$ rational, periodic motion, only a couple of lines



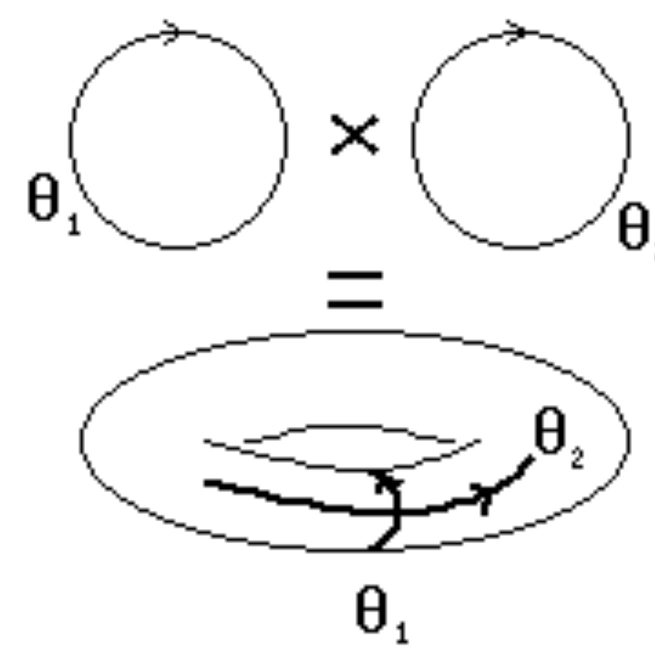
Two cases

$\frac{\omega_1}{\omega_2}$ irrational : one “fills”the square





Topologically:
Torus

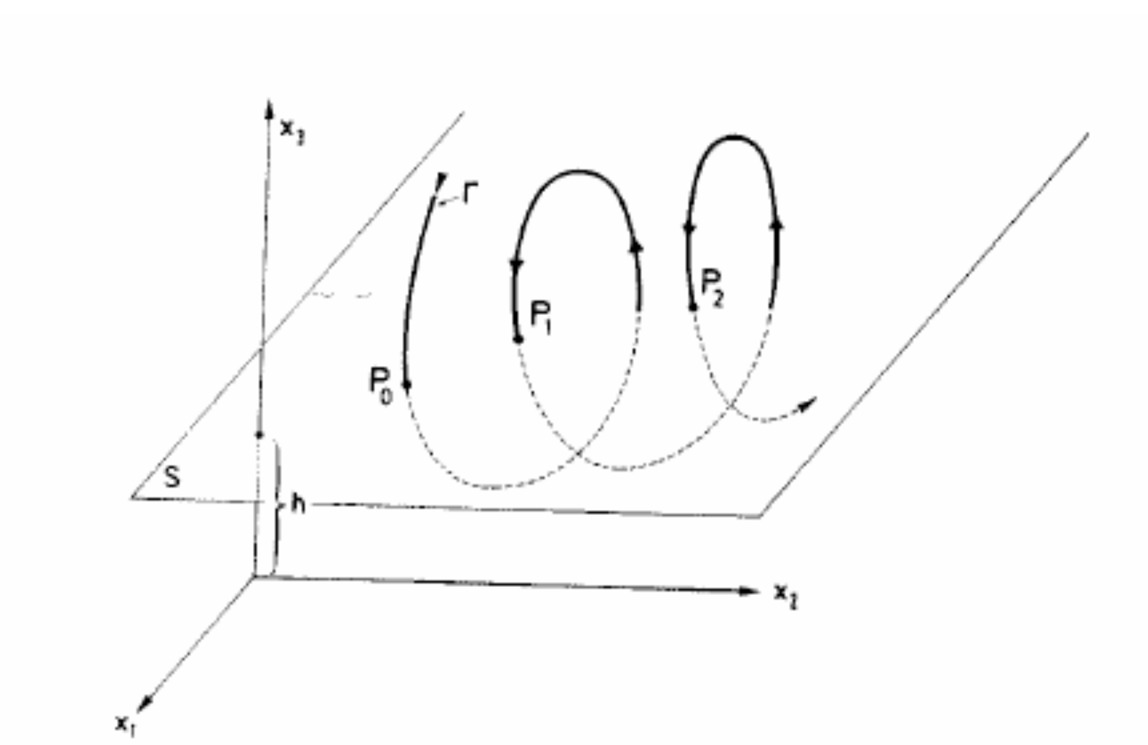


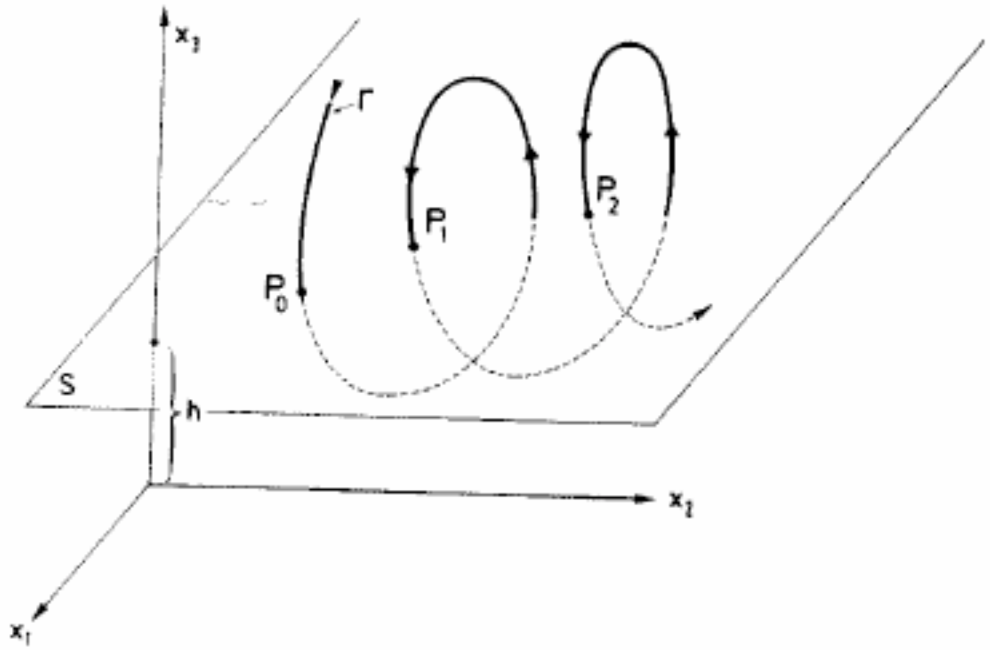
Action angles with several DOFs :“invariant tori”

Poincaré section

Intersection of a trajectory in dimension N with a given hypersurface of dimension N-1.

Practically : dimension 3, plane of dimension 2.





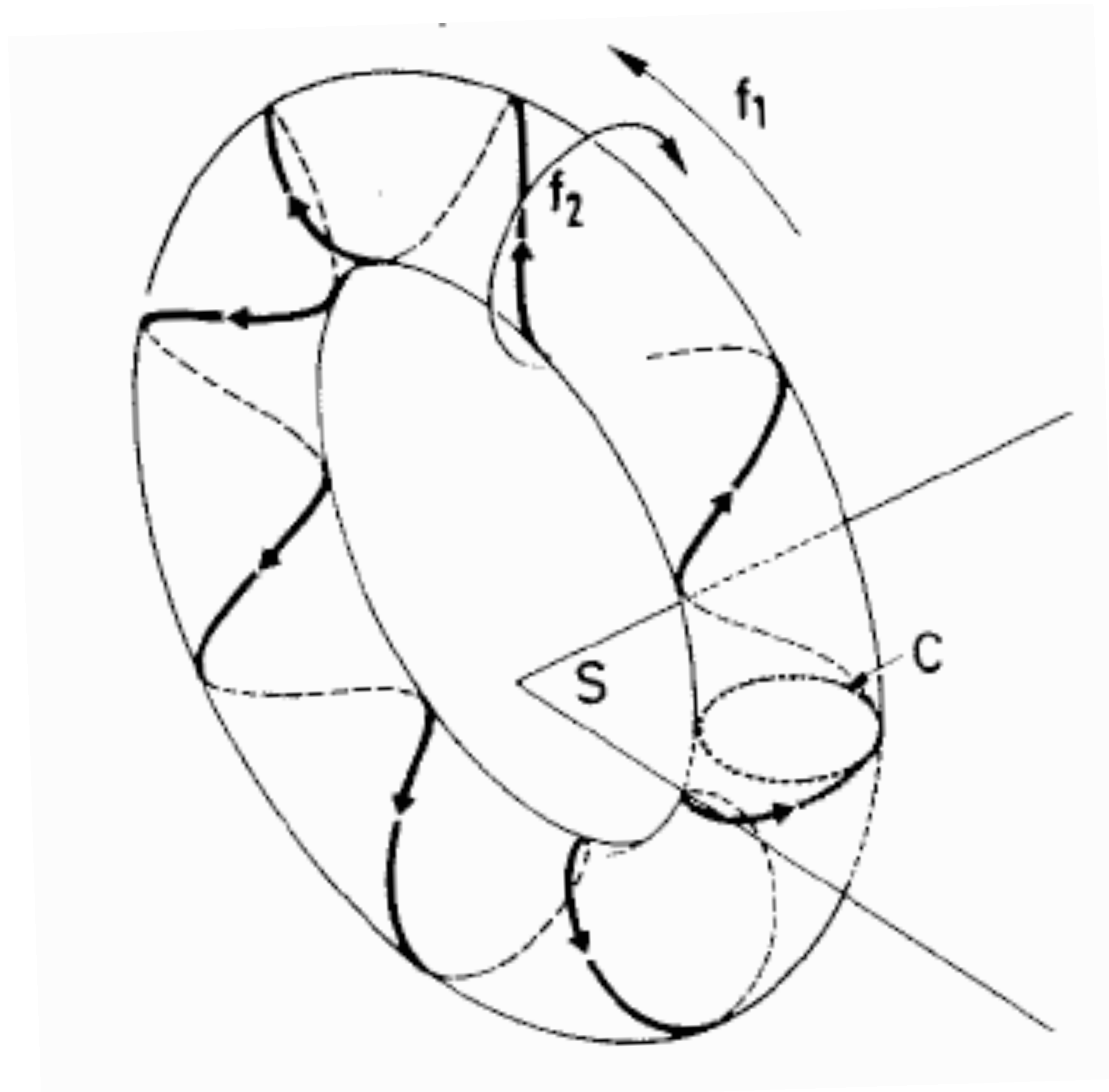
Advantages :
- Poincaré sections define a mapping of the plane

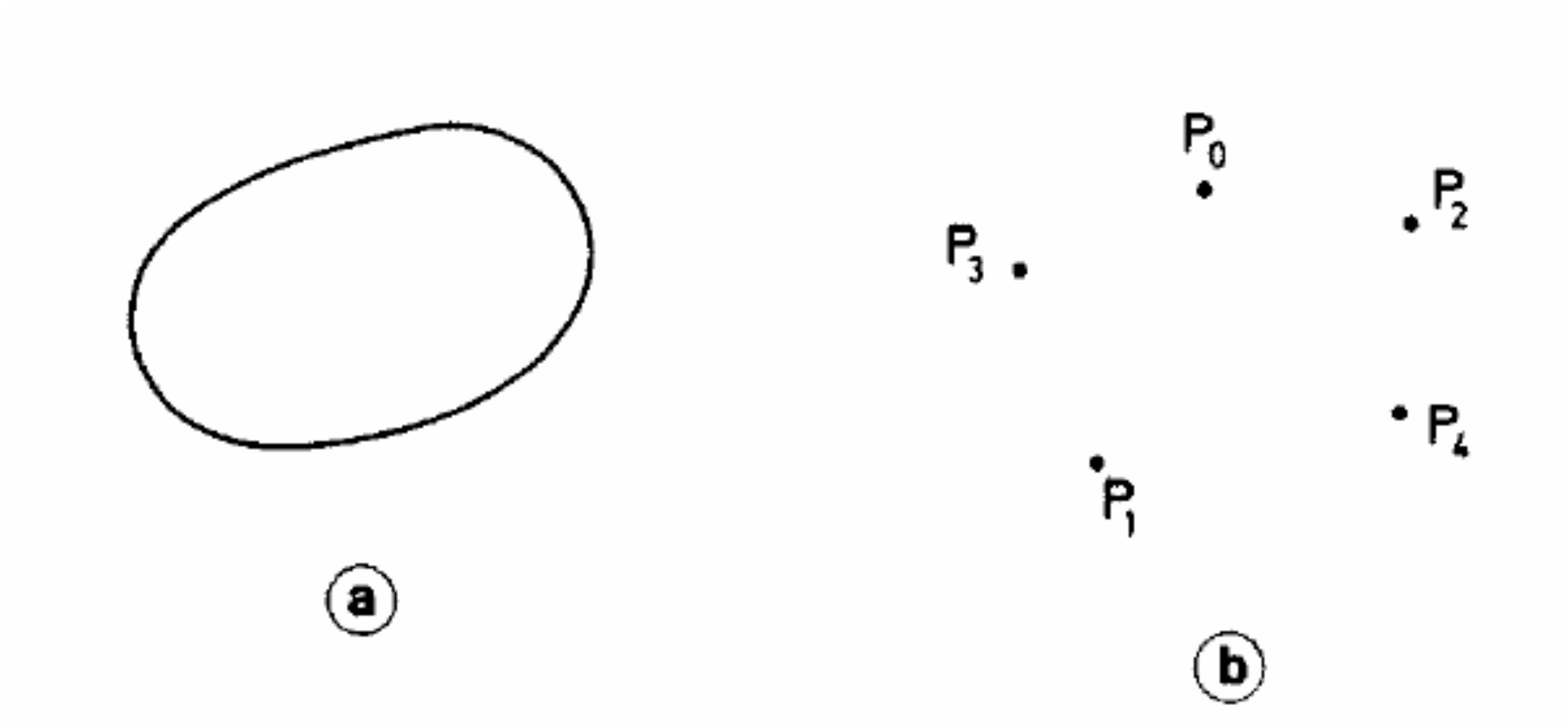
$$x_i(k+1) = T(x_i(k))$$

This way characterization of dynamics can be reduced to study of numerical sequences.

- The nature of these Poincaré sections help characterizing the motion.

Example : Poincaré section for two oscillators





Irrational

Rational

Chaotic system