# Hamiltonian mechanics

Phase space (q,p)

Hamilton's equations of motion $(\dot{q}_i, \dot{p}_i) = \left( \frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q_i} \right)$

Poisson brackets $\{f, g\} = \sum \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}$

Fundamental Poisson bracket $\{q_i, p_j\} = \delta_{ij}$

$$\dot{F} = \{F, H\} + \frac{\partial F}{\partial t}$$

Constant of motion if Poisson bracket is 0.

# Canonical transformations

We saw that we can change variable

$$q_i \to Q_j(q_i)$$

without changing the form of Euler Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \Leftrightarrow \frac{\partial L}{\partial Q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_i} = 0$$

What happens in Hamiltonian formalism ?

$$L(q, \dot{q}, t)$$

$$H(q, p, t)$$

EOM

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

$$\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q}$$

Change of variables

$$q \to Q$$

$$(p, q) \to (P, Q)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}} = \frac{\partial L}{\partial Q}$$

Hamilton's equations ?

(you can simply see it
as a change of expression
of your DOF)

$$(p, q) \rightarrow (P, Q)$$

Like the Lagrangian case, we would like that the Hamilton's equations stay the same

$$\dot{Q} = \frac{\partial H}{\partial P}, \dot{P} = -\frac{\partial H}{\partial Q}$$

H expressed as a function of new variables P,Q

Only some transformations work, they are called Canonical transformations.

$$q \to Q(q,p)$$
$$p \to P(q,p)$$

$$\dot{q} = H_p = \frac{dH}{dp}, \dot{p} = -H_q$$

$$\dot{Q} = \{Q, H\} = Q_q H_p - Q_p H_q$$

$$H_p = H_Q Q_p + H_P P_p \qquad H_q = H_Q Q_q + H_P P_q$$

$$\implies \{Q, H\} = Q_q(\cancel{H_Q Q_p} + H_P P_p) - Q_p(\cancel{H_Q Q_q} + H_P P_q)$$

$$\implies \{Q, H\} = H_P(Q_q P_p - P_q Q_p)$$

$$\Longrightarrow \{Q,H\} = H_P(Q_q P_p - P_q Q_p)$$

Similarly, we would find $\{P,H\} = -H_Q(Q_q P_p - P_q Q_p)$

Hamilton's equations would be

$$\{Q,H\} = H_P \qquad\qquad \{P,H\} = -H_Q$$

This works iff

$$Q_q P_p - P_q Q_p = \{Q,P\} = 1$$

Canonical transformations :

$$\{Q, P\} = 1$$

Generalization to several DOFs :

$$\{Q_i, P_j\} = \delta_{ij}$$

Or : a canonical transformation is a transformation that preserves Poisson bracket.

# Example 1

$$q_i \to Q_i(q_j)$$

$$p_i \to P_i(q_j, p_j)$$

$$\{Q_i, P_j\} = \sum_k \frac{\partial Q_i}{\partial q_k} \frac{\partial P_j}{\partial p_k} = \delta_{ij}$$ Matrix product

$$\implies \frac{\partial P_j}{\partial p_k} = \left[ \left( \frac{\partial Q}{\partial q} \right)^{-1} \right]_{kj}$$

$$\implies P_j = \left[ \left( \frac{\partial Q}{\partial q} \right)^{-1} \right]_{kj} p_k + f_j(q)$$

Not simply a function of p

# Example 2

$$q \to Q = p$$
$$p \to P = -q + f(p)$$

$$\{Q, P\} = \{p, -q\} = \{q, p\} = 1$$

# Generating functions

Canonical transformations can be tricky to find. Generating functions provide a way to systematically construct some.

Consider a function $F(q, Q)$

1. Define $Q(q, p)$ so that $p = \dfrac{\partial F}{\partial q}$

2. Define $P(q, p)$ so that $P = -\dfrac{\partial F}{\partial Q}$

Claim : this implies that $\{Q, P\} = 1$

Proof :

$$p = \frac{\partial F}{\partial q} \implies \begin{array}{l} \partial_q \\ \partial_p \end{array} \qquad \begin{array}{l} 0 = F_{qq} + F_{qQ} Q_q \\ 1 = F_{qQ} Q_p \end{array}$$

$$P = -\frac{\partial F}{\partial Q} \implies \begin{array}{l} \partial_q \\ \partial_p \end{array} \qquad \begin{array}{l} P_q = -F_{QQ} Q_q - F_{qQ} \\ P_p = -F_{QQ} Q_p \end{array}$$

$$\{Q, P\} = Q_q P_p - Q_p P_q$$

$$= 1$$

We can also generate canonical transformations in 3 other different ways

$$F(q,P) \implies p = \frac{\partial F}{\partial q}, Q = \frac{\partial F}{\partial P}$$

$$F(p,Q) \implies q = -\frac{\partial F}{\partial p}, P = -\frac{\partial F}{\partial Q}$$

$$F(p,P) \implies q = -\frac{\partial F}{\partial P}, Q = \frac{\partial F}{\partial P}$$

# How does the Lagrangian change for canonical transformation ? $F(q, Q)$

$$p = \frac{\partial F}{\partial q} \qquad\qquad P = -\frac{\partial F}{\partial Q}$$

$$H(P, Q) = P\dot{Q} - L'(Q, \dot{Q}, t) = -\frac{\partial F}{\partial Q}\dot{Q} - L'(Q, \dot{Q}, t)$$

$$H(p, q) = p\dot{q} - L(q, \dot{q}, t) = \frac{\partial F}{\partial q}\dot{q} - L(q, \dot{q}, t)$$

$$0 = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial Q}\dot{Q} + L'(Q, \dot{Q}, t) - L(q, \dot{q}, t)$$

$$L'(Q, \dot{Q}, t) = L(q, \dot{q}, t) - \frac{dF}{dt}$$

Now let us consider a change of variable in Lagrangian formalism

$$q \to Q$$

So that the Lagrangian changes in the following way :

$$L'(Q, \dot{Q}, t) = L(q, \dot{q}, t) - \frac{dF}{dt}$$

With $F(q, Q, t)$ (we add a time dependency to F)

Claim : this implies that hamiltonian K for Q,P is now

$$K(Q, P) = H(q, p) + \frac{\partial F}{\partial t}$$

Proof : let us consider a general case.

We take two independent descriptions of the world, Q and q.

They describe the same dynamics, least action principle then implies that their Lagrangian are related :

$$L'(Q, \dot{Q}, t) = L(q, \dot{q}, t) - \frac{dF}{dt}$$

Assume using our two descriptions we choose

$$F(q, Q, t)$$

$$L'(Q,\dot{Q},t) = L(q,\dot{q},t) - \frac{dF}{dt}$$

$$P\dot{Q} - K = p\dot{q} - H - \dot{q}\frac{\partial F}{\partial q} - \dot{Q}\frac{\partial F}{\partial Q} - \frac{\partial F}{\partial t}$$

$$\left(P + \frac{\partial F}{\partial Q}\right)\dot{Q} - \left(p - \frac{\partial F}{\partial q}\right)\dot{q} + H + \frac{\partial F}{\partial t} - K = 0$$

$$\left(P + \frac{\partial F}{\partial Q}\right)dQ - \left(p - \frac{\partial F}{\partial q}\right)dq + \left(H + \frac{\partial F}{\partial t} - K\right)dt = 0$$

Now Q and q are independent descriptions, so necessary we have :

$$P = -\frac{\partial F}{\partial Q} \qquad p = \frac{\partial F}{\partial q} \qquad K = H + \frac{\partial F}{\partial t}$$

Summary :

With $F(q,Q,t)$ (we add a time dependency to F)

$$K(Q,P,t) = H(q(Q,P),p(Q,P),t) + \frac{\partial F}{\partial t}$$

- write hamiltonian H as function of q,p

- express q,p as functions of Q,P using F

- Hamiltonian for Q,P is K(Q,P) defined above.

- This means that the EOM can be computed from the Hamilon's equation either with q,p,H or with Q,P,K

$$K(Q,P,t) = H(q(Q,P),p(Q,P),t) + \frac{\partial F}{\partial t}$$

Why do we do all of this ?

Solving differential equations is a difficult problem.

It is worth finding algebraic transformations to simplify hamiltonian and renders time integration trivial.

We will see later that we can even cancel out Hamiltonian ! New variables are then constant of motions.

# Infinitesimal transformations

$$q_i \to Q_i = q_i + \epsilon E_i(q,p)$$

$$p_i \to P_i = p_i + \epsilon F_i(q,p)$$

$$\{Q_i, P_j\} = \delta_{ij} = \{q_i, p_j\} + \epsilon(\{E_i, p_j\} + \{q_i, F_j\}) + \mathcal{O}(\epsilon^2)$$

$$\delta_{ij}$$

$$\{E_i, p_j\} + \{q_i, F_j\} = 0$$

$$\{E_i, p_j\} + \{q_i, F_j\} = 0$$

$$\frac{\partial E_i}{\partial q_j} + \frac{\partial F_j}{\partial p_i} = 0$$

$$\implies \exists G / F_j = -\frac{\partial G}{\partial q_j}, E_i = \frac{\partial G}{\partial p_i}$$

We therefore have the canonical transformation

$$q_i \to Q_i = q_i + \epsilon \frac{\partial G}{\partial p_i}$$

$$p_i \to P_i = p_i - \epsilon \frac{\partial G}{\partial q_i}$$

G: generator of the infinitesimal canonical transformation

# One first important example : time evolution

$$q_i \to Q_i = q_i(t+dt) = q_i + dt\dot{q}_i = q_i + dt\frac{\partial H}{\partial p_i}$$

$$p_i \to P_i = p_i(t+dt) = p_i + dt\dot{p}_i = p_i - dt\frac{\partial H}{\partial q_i}$$

Time evolution is the canonical transformation generated by the Hamiltonian

# What about changes of a function F ?

$$q_i \to Q_i = q_i + \epsilon \frac{\partial G}{\partial p_i}$$

$$p_i \to P_i = p_i - \epsilon \frac{\partial G}{\partial q_i}$$

$$F(q_i, p_i) \to F(Q_i, P_i) = F(q_i, p_i) + \frac{\partial F}{\partial q_i}\delta q_i + \frac{\partial F}{\partial p_i}\delta p_i$$

$$= F(q_i, p_i) + \epsilon(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i})$$

$$F(q_i, p_i) \to F(Q_i, P_i) = F(q_i, p_i) + \epsilon\{F, G\}$$

# Time translation

$$q_i \to Q_i = q_i(t+dt) = q_i + dt\dot{q}_i = q_i + dt\frac{\partial H}{\partial p_i}$$

$$p_i \to P_i = p_i(t+dt) = p_i + dt\dot{p}_i = p_i - dt\frac{\partial H}{\partial q_i}$$

$$dF = dt\{F,H\} \implies \dot{F} = \{F,H\}$$

# Hamiltonian change for an infinitesimal transformation

$$q_i \rightarrow Q_i = q_i + \epsilon \frac{\partial G}{\partial p_i}$$

$$p_i \rightarrow P_i = p_i - \epsilon \frac{\partial G}{\partial q_i}$$

$$H(q_i, p_i) \rightarrow H(Q_i, P_i) = H(q_i, p_i) + \epsilon\{H, G\}$$

$$-\epsilon \dot{G}$$

$$H(q_i, p_i) \to H(Q_i, P_i) = H(q_i, p_i) - \epsilon \dot{G}$$

If G is a constant of motion, H is invariant

This is Noether's theorem !

Constant of motions are generator of infinitesimal symmetries that leaves H invariant

or

a symmetry is a canonical transformation generated by G so that $\{G, H\} = 0$

More generally, even if $\{G, H\}$ is not zero, some special G generate interesting geometrical transformations.

Ex : how to generate a spatial translation ?

$$G = \sum c_i p_i$$

$$q_i \to q_i + \epsilon \frac{\partial G}{\partial p_i} = q_i + \epsilon c_i$$

# Example 2 : rotation

$$\vec{l} = \vec{r} \times \vec{p} \qquad G = \sum \alpha_k l_k \qquad l_k = \sum_{ij} \epsilon_{ijk} r_i p_j$$

$$\frac{\partial G}{\partial p_j} = \sum_{ik} \epsilon_{ijk} \alpha_k r_i = (\vec{\alpha} \times \vec{r})_j \quad \frac{\partial G}{\partial r_i} = \sum_{ik} \epsilon_{ijk} \alpha_k p_j = -(\vec{\alpha} \times \vec{p})_i$$

$$\vec{r} \to \vec{r} + \epsilon \vec{\alpha} \times \vec{r}$$

$$\vec{p} \to \vec{p} + \epsilon \vec{\alpha} \times \vec{p}$$

Infinitesimal rotation around axis defined by $\vec{\alpha}$

$$d\vec{r} = \epsilon \vec{\alpha} \times \vec{r}$$

Phase space rotates !

# Going further (not required for final)

$$F(q+c) = \sum \frac{c^n}{n!}\,\frac{d^n F}{dq^n}\bigg|_q$$

$$= e^{c\partial_q} F(q)$$

Translation operator

$c \to 0$

$$F(q+c) = F(q) + c\frac{\partial F}{\partial q} = F(q) + c\{F, p\}$$

Infinitesimal generator

$F \to \{F, p\}$ Linear operator

$$F(q+c) = \sum \frac{c^n}{n!} \left. \frac{d^n F}{dq^n} \right|_q$$

$$= \sum_n \frac{c^n}{n!} \{\ldots\{\{F,p\},p\}\ldots,p\}$$

Power n of the operator

$$= e^{c\{\cdot,p\}} F$$

Infinitesimal generator

$p_i$

$\partial_{q_i}$

Symmetry operator

$$e^{c_i \partial_{q_i}} = e^{c_i \{\cdot, p_i\}}$$

c : continuous parameter

# Noether's theorem (exponential form)

Consider the symmetry operator

$$e^{s\{.,I\}}$$

Assume it leaves Hamiltonian Invariant

$$e^{s\{.,I\}}H = H(q(s),p(s)) = H(q(0),p(0))$$

$$\frac{d}{ds}\left[e^{s\{.,I\}}H\right]\Big|_{s=0} = 0 \quad = \{.,I\}e^{s\{.,I\}}H\Big|_{s=0}$$

$$= \{H,I\}$$

(clearly easier to work infinitesimally)

## Lie group

Continuous symmetries defined by a continuous parameters. (should be differentiable)

In general, we will have

$$S(t) = e^{tA}$$

Symmetry

Generator of the symmetry

A belongs to the Lie Algebra defined by Lie group.

# Example : infinitesimal generators for rotations in 3D

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = exp\left(\theta\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[A_i, A_j] = \epsilon_{ijk} A_k$$

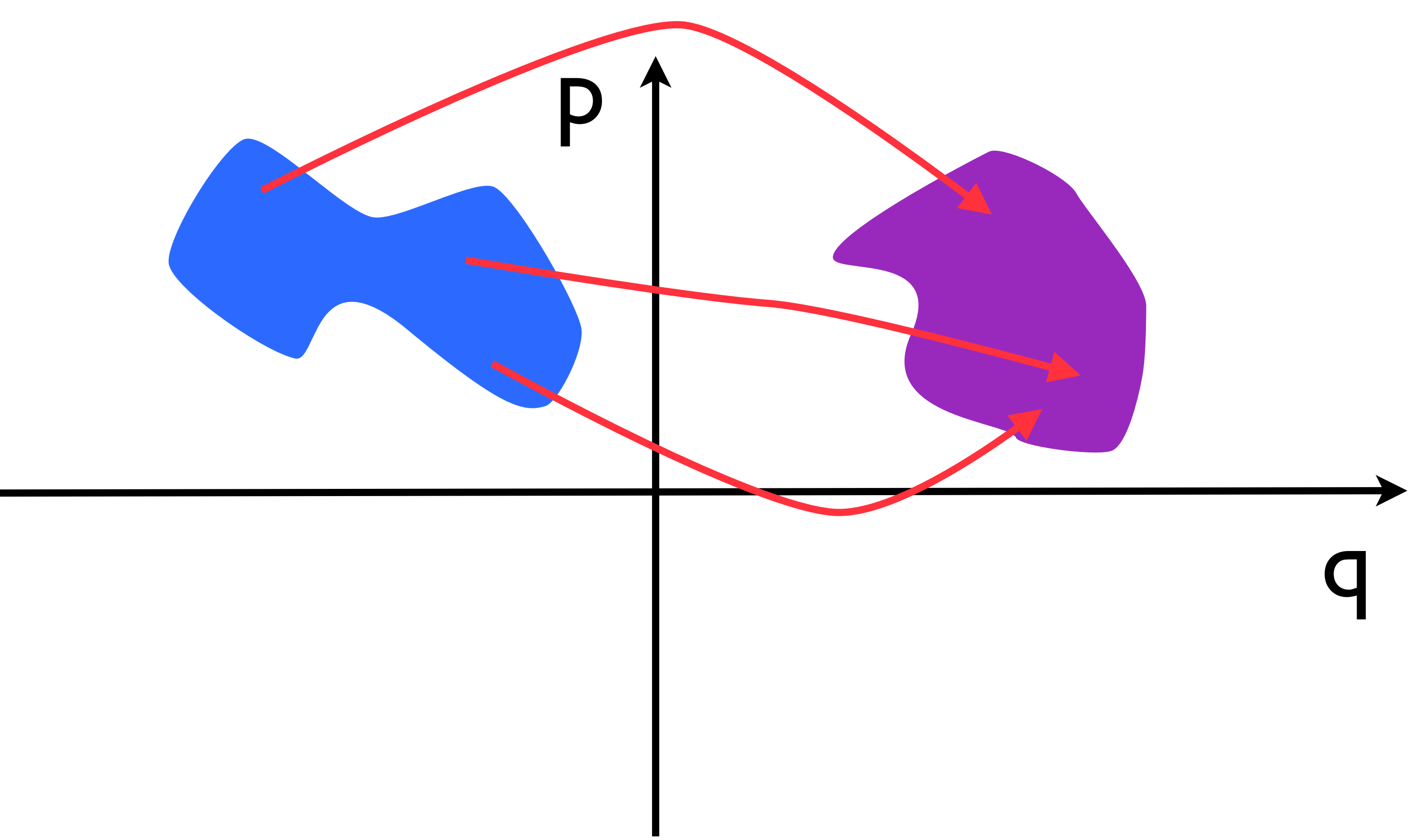
These commutations relationships on the Lie Algebra express the fact that the members of the Lie Group do not commute (and tell you how to commute them).

# Liouville's theorem

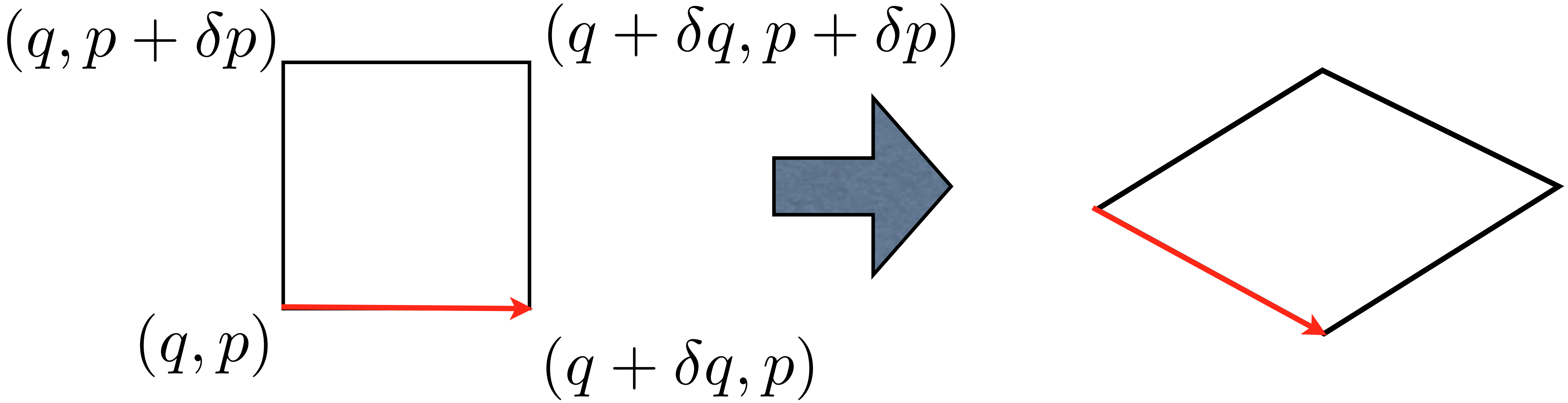
Consider some mechanical system

$$(q_i, p_i) \qquad\qquad H(q_i, p_i)$$

Consider evolution in time of a set of initial conditions in phase space




Claim : the area of the blob is a constant of time !




$$(q, p) \to (q + dtH_p, p - dtH_q)$$

$$(q + \delta q, p) \to (q + \delta q + dtH_p|_{q+\delta q}, p - dtH_q|_{q+\delta q})$$

$$(q + \delta q + dt(H_p + \delta q H_{pq}), p - dt(H_q + \delta q H_{qq}))$$

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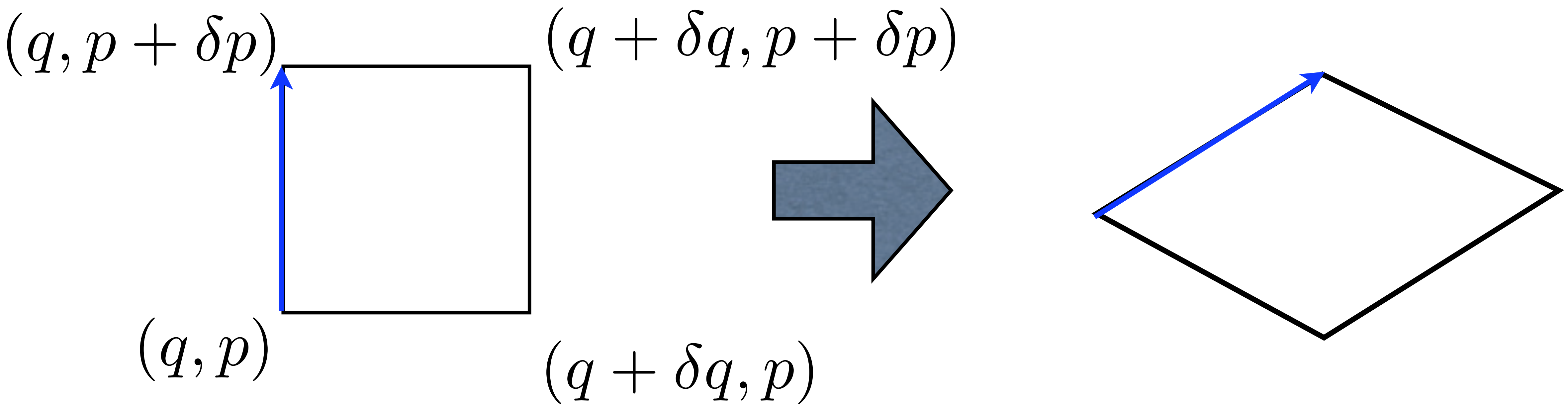
$$(\delta q, 0) \to (\delta q(1 + dtH_{pq}), -\delta q dtH_{qq})$$




$$(q,p) \to (q + dtH_p, p - dtH_q)$$

$$(q, p+\delta p) \to (q + dtH_p|_{p+\delta p}, p + \delta p - dtH_q|_{p+\delta p})$$

$$(q + dt(H_p + \delta p H_{pp}), p + \delta p - dt(H_q + \delta p H_{pq}))$$

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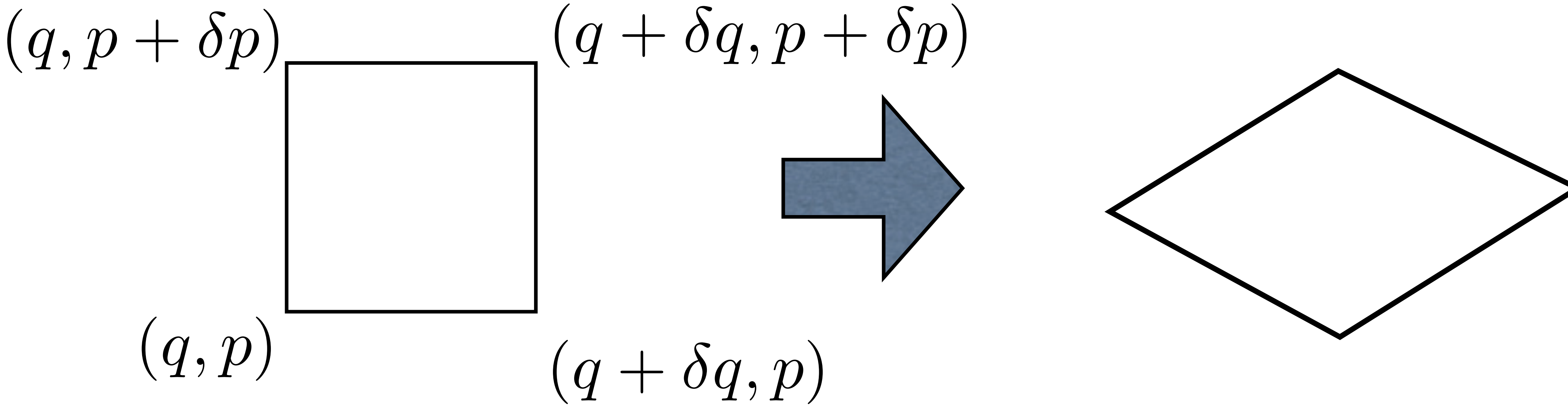
$$(0, \delta p) \to (dt\delta p H_{pp}, \delta p(1 - dtH_{pq})$$




$$(\delta q, 0) \to (\delta q(1 + dtH_{pq}), -\delta q dt H_{qq})$$

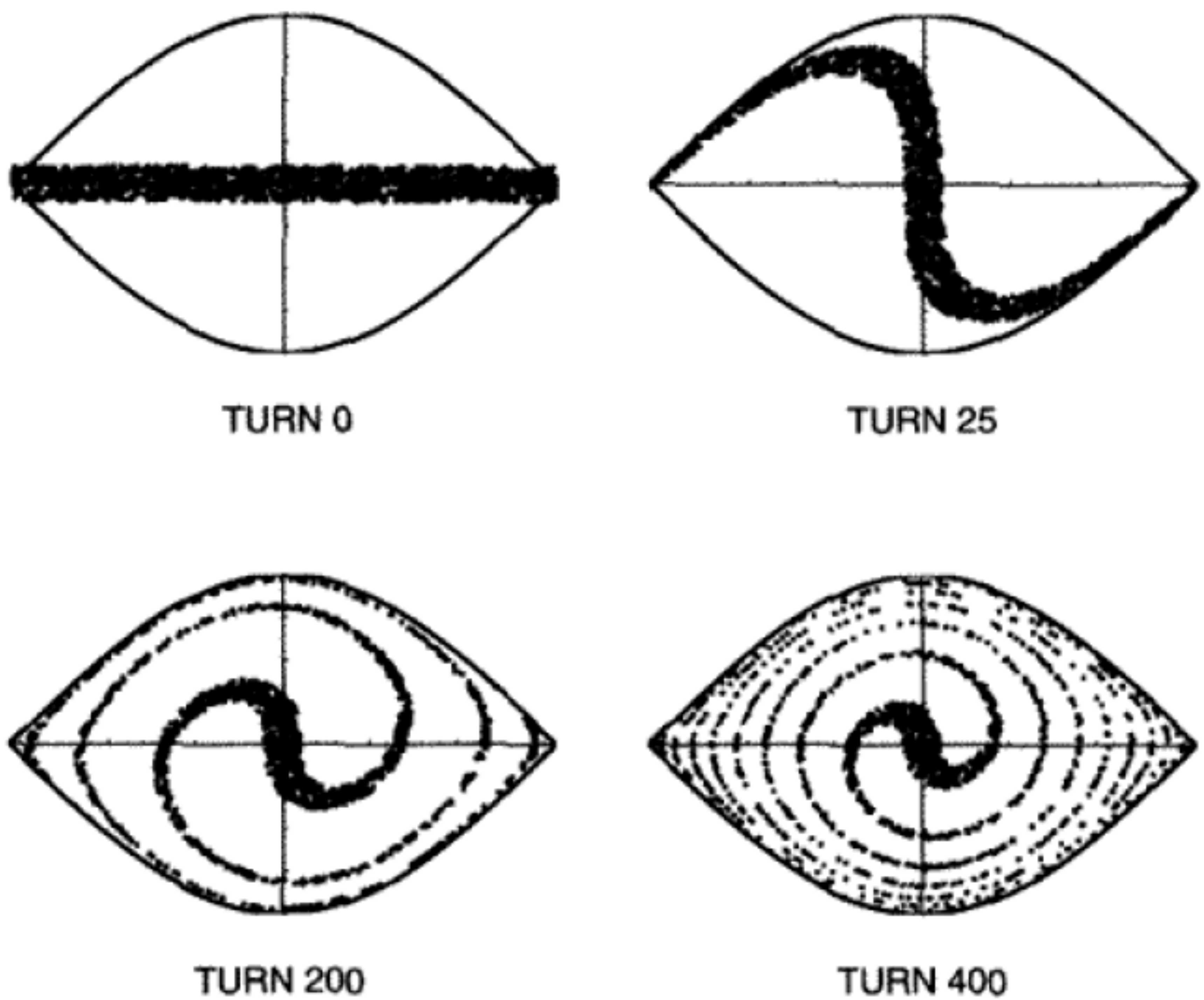
$$(0, \delta p) \to (dt\delta p H_{pp}, \delta p(1 - dtH_{pq})$$

Area : matrix determinant

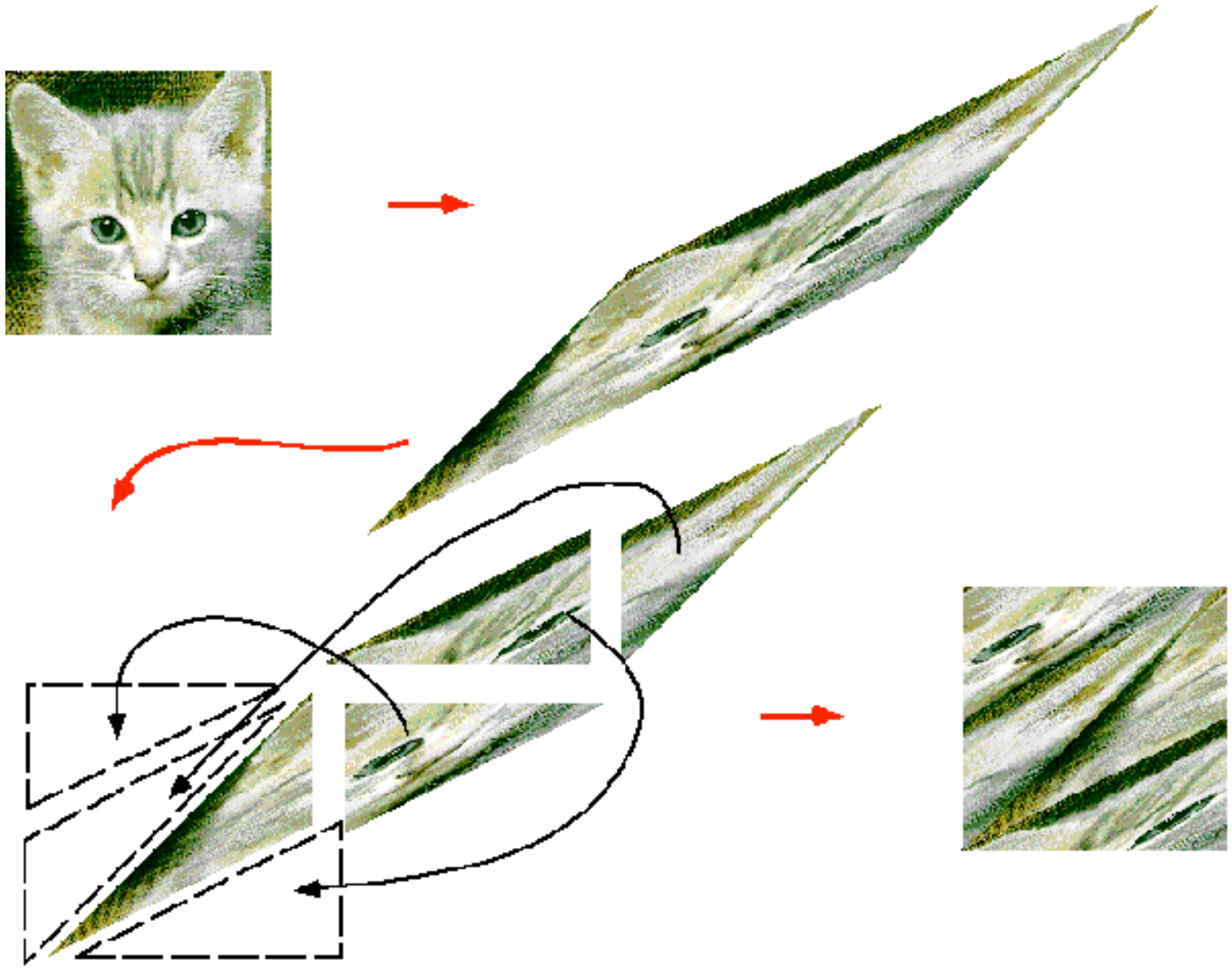
$$\delta q \delta p \to \delta q \delta p(1 + dt^2(H_{qq}H_{pp} - H_{pq}^2))$$

$$\delta A = \mathcal{O}(dt^2) \implies \frac{dA}{dt} = 0$$

The area stays the same BUT the shape can change a lot : "filamentation"



This can even lead to Hamiltonian Chaos : two points very close in the beginning end up being very distant



Arnold's cat map

# "True" form of Liouville theorem

Assume $$\dot{\vec{X}} = \vec{f}(\vec{X}, t)$$

$$\vec{X}(t+dt) = \vec{X}(t) + dt\vec{f}(\vec{X}, t)$$

$$(\vec{X} + d\vec{X})(t+dt) = (\vec{X} + d\vec{X})(t) + dt\vec{f}(\vec{X} + d\vec{X}, t)$$

$$\implies d\vec{X}(t+dt) = d\vec{X}(t) + dt(\vec{f}(\vec{X} + d\vec{X}, t) - \vec{f}(\vec{X}, t))$$

$$J_f(\vec{X})d\vec{X}$$

$$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_N} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_N}{\partial X_1} & \cdots & \frac{\partial f_N}{\partial X_N} \end{pmatrix}$$

$$d\vec{X}(t+dt) = (I + J_f(\vec{X})dt)d\vec{X}(t)$$

# Infinitesimal volume in phase space

$$\delta\Omega(t) \rightarrow \delta\Omega(t+dt) = det(I + J_f(\vec{X})dt)\delta\Omega(t)$$

$$det(I + \epsilon A) \simeq 1 + \epsilon Tr(A) + \mathcal{O}(\epsilon^2)$$

$$\implies det(I + dtJ_f) \simeq 1 + dtTr(J_f) + \mathcal{O}(dt^2)$$

**If** $Tr(J_f) = divf = \sum_i \frac{\partial f_i}{\partial X_i} = 0$ **, then**

$$\delta\Omega(t+dt) = (1 + \mathcal{O}(dt^2))\delta\Omega(t) \qquad \implies \frac{d\Omega}{dt} = 0$$

# "True" form of Liouville theorem

Assume $$\dot{\vec{X}} = \vec{f}(\vec{X}, t)$$

$\Omega(t)$ : volume in phase space

$$Tr(J_f) = div f = \sum_i \frac{\partial f_i}{\partial X_i} = 0$$

$$\Longrightarrow \frac{d\Omega}{dt} = 0$$

"Conservative system"

# Example : Hamiltonian system

$$\frac{d}{dt}\begin{pmatrix} q_i \\ p_i \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p_i} \\ -\frac{\partial H}{\partial q_i} \end{pmatrix}$$

$$J_f = \sum_i \left(\frac{\partial f_{q_i}}{\partial q_i} + \frac{\partial f_{p_i}}{\partial p_i}\right)$$

$$f_{q_i} = \frac{\partial H}{\partial p_i} \qquad f_{p_i} = -\frac{\partial H}{\partial q_i}$$

$$J_f = \sum_i \left(\frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial q_i \partial p_i}\right) = 0$$

## Example 2 :

$$\frac{d}{dt}\begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} = M \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} \qquad M(t) = \begin{pmatrix} 0 & 1 \\ -\omega^2(t) & 0 \end{pmatrix}$$

$Tr(M) = 0$ “Conservative system”

## Counter example : friction

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0$$

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

$Tr(M) = -\gamma \neq 0$ “Dissipative system”